Notes on Linear Systems

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October 24, 2020

We often face some linear algebra related problems in data science. Here are a collection of my notes.

1 General Cases

Usually, the problem will be reduced to solve a linear system:

$$Ax = b, \text{ with } A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, \tag{1}$$

where m is the number of linear equations and n is the number of variables. There are several cases:

- Case 1: A is full-ranked
  - Case 1.1: If A is square, then (1) has a unique solution $x = A^{-1}b$.
  - Case 1.2: If A is not square, then (1) is not solvable. Instead, we consider the least square problem

    $$\hat{x} = \underset{x\in\mathbb{R}^n}{\arg\max} \|Ax - b\|_2. \tag{2}$$

    In this case, the least square problem (2) has a unique solution and can be solved by solving the equivalent normal equation

    $$A^T A\hat{x} = A^T b.$$

    Since A is full-ranked, $A^T A$ is invertible. So

    $$\hat{x} = (A^T A)^{-1} A^T b.$$

- Case 2: A is not full-ranked. In this case, no matter A is square or not, we can find the solution to the least square problem by using the psudoinverse:

  $$\hat{x} = A^+ b = \underset{x\in\mathbb{R}^n}{\arg\max} \|Ax - b\|_2$$

where A^+ is defined using the SVD of A in the following way: If $A = U \begin{pmatrix} \Sigma_r & 0_{12} \\ 0_{21} & 0_{22} \end{pmatrix} V^T$, then $A^+ = V \begin{pmatrix} \Sigma_r^{-1} & 0_{21} \\ 0_{12} & 0_{22} \end{pmatrix} U^T$. Note that since A is not full-ranked, the least square problem have finite many solutions.