Making Curry with Rice

An Optimizing Curry Compiler

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Contents

Introduction . . . . . . **v**
0.0.1 Why Curry? v
0.0.2 Current Compilers vi
0.0.3 The Need for Optimizations vii

1 Mathematical Background **1**
1.1 Rewriting 1
1.2 Term Rewriting 3
1.3 Narrowing 8
1.4 Rewriting Strategies 10
1.5 Narrowing Strategies 13
1.6 Graph Rewriting 17
1.7 Previous Work 21

2 The Curry Language **23**
2.1 The Curry Language 23
2.2 Semantics 27
2.2.1 FlatCurry 29
2.2.2 Evaluation 29
2.2.3 Non-determinism 32
2.2.4 Free Variables 33
2.2.5 Higher Order Functions 35
2.2.6 Backtracking Performance 37
2.2.7 Collapsing Functions 39
2.3 Generated Code 41
2.3.1 Higher Order Functions 43
2.3.2 Implementing Non-determinism 46

2.3.3 Fast Backtracking 49

3 Generating and Altering Subexpressions 53

3.0.1 Building Optimizations 53

3.0.2 The Structure of an Optimization 55

3.0.3 An Initial Attempt 60

3.0.4 A Second Attempt: Multiple Transformations Per Pass 62

3.0.5 Adding More Information 62

3.0.6 Reconstruction 65

3.0.7 Optimizing the Optimizer 66

3.1 The Compiler Pipeline 68

3.1.1 Canonical FlatCurry 68

3.1.2 ICurry 75

3.1.3 C 75

4 Basic Optimizations 83

4.0.1 A-Normal Form 83

4.1 Case Canceling 86

4.2 Inlining 89

4.2.1 Partial Applications 92

4.2.2 The Function Table 93

4.2.3 Function Ordering 95

4.2.4 Inlining 96

4.2.5 Reduce 97

5 Memory Optimizations 103

5.1 Unboxing 103

5.1.1 The Transformation 104

5.1.2 Primitive Conditions 105

5.1.3 Strictness Analysis 107

5.2 Shortcutting 110

5.2.1 Non-deterministic RET Nodes 111

5.2.2 RET hnf Functions 113

5.2.3 Shortcutting Results 116

5.3 Deforestation 117
5.3.1 The Original Scheme 118
5.3.2 The Combinator Problem 120
5.3.3 Solution build_fold 120
5.3.4 Implementation 122
5.3.5 Correctness 125

6 Conclusion 131
6.1 Results 131
6.1.1 Tests 131
6.1.2 Results 133
6.2 Conclusion 133
6.2.1 Future Work 136
6.2.2 Conclusion and Related Work 137

References 141

Chapter 0

INTRODUCTION

With all of the chaos in the world today, sometimes it's nice to just relax and make a nice Curry. But people today are impatient. They can't wait; they want their Curry fast. This is a problem, because Curry has historically been considered slow. Some have considered it unusably slow, which is a shame, because Curry is actually a great languguage, and can solve many problems well. In this dissertation we aim to recitify the problem of Curry taking too long. We present the RICE Curry compiler, and show how it can deliver a fast, satisfying, Curry.

0.0.1 Why Curry?

Functional logic programming is a very powerful technique for expressing complicated ideas in a simple form. Curry implements these ideas with a clean, easy to read syntax, which is similar to Haskell, a well known functional programming language. It's also lazy, so evaluation of Curry programs is similar to Haskell as well. Curry extends Haskell with two new concepts. First, there are non-deterministic functions, such as "?". Semantically $a \mathbin{?} b$ will evaluate a and b and will return both answers to the user. Second, there are free, or logic, variables, which are also called logic variable by some authors. A free variable is a variable that is not in the scope of the current function. The value of a free variable is not defined, but it may be constrained.

Consider the following Curry code for solving n-queens:

$$\begin{aligned}
&\mathit{queens} = \mathit{isEmpty}\ (\mathit{set1}\ \mathit{unsafe}\ p) = p\\
&\quad \textbf{where}\ p = \mathit{permute}\ [1\mathinner{..}n]\\
&\mathit{unsafe}\ (\mathit{xs} \mathbin{+\!\!+} [\,a\,] \mathbin{+\!\!+} \mathit{ys} \mathbin{+\!\!+} [\,b\,] \mathbin{+\!\!+} \mathit{zs}) = \mathit{abs}\ (a - b) =:= \mathit{length}\ \mathit{ys}
\end{aligned}$$

In the *unsafe* function the input list is broken into 5 pieces. Two of the pieces, a and b, are lists with a single element. The sublists, xs, ys, and zs are free to be as long as they want. However, We've constrained the total list $\mathit{xs} \mathbin{+\!\!+} [\,a\,] \mathbin{+\!\!+} \mathit{ys} \mathbin{+\!\!+} [\,b\,] \mathbin{+\!\!+} \mathit{zs}$ to be the same as the argument.

The effect is that a and b are arbitrary elements in the list, and ys is the list of elements between a and b.

Free variables are given concrete values in Curry programs through narrowing. The semantics of narrowing and non-determinism in Curry are given by Antoy et al. [17]

0.0.2 Current Compilers

There are currently two mature Curry compilers, Pakcs [50] and Kics2 [25]. Pakcs compiles Curry to Prolog in an effort to leverage Prolog's non-determinism and free variables. Kics2 compiles Curry to Haskell in an effort to leverage Haskell's higher order functions and optimizing compiler. Both compilers have their advantages. Pakcs tends to perform better on non-deterministic expressions with free variables, where Kics2 tends to perform much better on deterministic expressions. Unfortunately neither of these compilers perform well in both circumstances.

Sprite [20], an experimental compiler, aims to fix these inefficiencies. The strategy is to compile to a virtual assembly language, known as LLVM. So far, Sprite has shown promising improvements over both Pakcs and Kics2 in performance, but it is not a mature compiler.

One major disadvantage of all three compilers is that they all attempt to pass off optimization to another compiler. Pakcs attempts to have Prolog optimize the non-deterministic code; Kics2 attempts to use Haskell to optimize deterministic code; and Sprite attempts to use LLVM to optimize the low level code. Unfortunately none of these approaches works very well. While some implementations of Prolog can optimize non-deterministic expressions, they have no concept of higher order functions, so there are many optimizations that cannot be applied. Kics2 is in a similar situation. In order to incorporate non-deterministic computations in Haskell, a significant amount of code must be threaded through each computation. This means that any non-deterministic expression cannot be optimized in Kics2. Finally, since LLVM doesn't know about either higher order functions or non-determinism, it loses many easy opprotunities for optimization.

Curry programs have one last hope for efficient execution. Recently, many scientists [76, 84] have developed a strong theory of partial evaluation for functional logic programs. While these results are interesting, partial evaluation is not currently automatic in Curry. Guidance is required from the programmer to run the optimization. Furthermore, the optimization fails to optimize several common programs.

0.0.3 The Need for Optimizations

So far, none of these approaches have included the large body of work on program optimizations. [5, 6, 67, 57, 3, 22, 21, 2, 7, 95, 91, 43, 83, 37, 44, 42, 94, 39, 79] This leads to the inescapable conclusion that Curry needs an optimizer. We propose a new compiler environment for developing and testing optimizations. The Reduction Inspired Compiler Environment (RICE) Curry compiler. This compiler is intended to make developing new optimizations for Curry as simple as possible. We test this idea by developing several common optimizations for the RICE compiler. Furthermore we implement three specific optimizations for Curry, Unboxing [55], Shortcutting [18], and Deforestation [39]. We chose these optimizations specifically because they focus on reducing the amount of memory consumed by programs, which is a common problem for Curry programs [65].

The rest of this dissertation is organized as follows: chapter 2 presents the mathematical background of Term and Graph Rewriting; chapter 3 presents the Curry Language and its semantics; chapter 4 introduces the GAS system for implementing optimizations; chapter 5 discusses the implementation of several common optimizations; chapter 6 discusses the implementation of Unboxing, Shortcutting, and Deforestation; and chapter 7 concludes by discussing results and future work.

Chapter 1

MATHEMATICAL BACKGROUND

When cooking, it is very important to follow the rules. You don't need to stick to an exact recipe, but you do need to know the how ingredients will react to temperature and how different combinations will taste. Otherwise you might get some unexpected reactions.

Similarly, there isn't a single way to compile Curry programs, however we do need to know the rules of the game. Throughout this compiler, we'll be transforming Curry programs in many different ways, and it's important to make sure that all of these transformations respect the rules of Curry. As we'll see, if we break these rules, then we may get some unexpected results.

We introduce the concept of Rewriting, along with the more specific Term and Graph Rewriting. We give a basic intuition about how to apply these topics, and show several examples using a small, but not trivial, example of a rewrite system for Peano Arithmetic 1.9.

1.1 REWRITING

In programming language terms, the rules of Curry are its semantics. The semantics of Curry are generally given in terms of rewriting. [46, 13, 17] While there are other semantics [4, 41, 92], rewriting is a common formalism for many functional languages, and the general theory of Curry grew out of this discipline [17], a good fit for Curry [49]. We'll give a definition of rewrite systems, then we'll look at two distinct types of rewrite systems: Term Rewrite Systems, which are used to implement transformations and optimizations on the Curry syntax trees; and Graph Rewrite Systems, which define the operational semantics for Curry programs. This mathematical foundation will help us justify the correctness of our transformations even in the presence of laziness, non-determinism, and free variables.

An Abstract Rewriting System (ARS) is a set A along with a relation $\to$. We write $a \to b$ instead of $(a, b) \in \to$, and we have several modifiers on our relation.

- $a \to^n b$ iff $a = x_0 \to x_1 \to \ldots x_n = b$.

$$
\begin{array}{lll}
 & (x \cdot x + 1)(2 + x) & \\
\rightarrow & (x \cdot x + 1)(x + 2) & \text{by commutativity of addition} \\
\rightarrow & (x^2 + 1)(x + 2) & \text{by definition of } x^2 \\
\rightarrow & x^2 \cdot x + 2 \cdot x^2 + 1 \cdot x + 1 \cdot 2 & \text{by FOIL} \\
\rightarrow & x^2 \cdot x + 2x^2 + x + 2 & \text{by identity of multiplication} \\
\rightarrow & x^3 + 2x^2 + x + 2 & \text{by definition of } x^3
\end{array}
$$

Figure 1.1: reducing $(x \cdot x + 1)(2 + x)$ using the standard rules of algebra

- $a \rightarrow^{\leq n} b$ b iff $a \rightarrow^i b$ and $i \leqslant n$.
- reflexive closure: $a \rightarrow^= b$ iff $a = b$ or $a \rightarrow b$.
- symmetric closure: $a \leftrightarrow b$ iff $a \rightarrow b$ or $b \rightarrow a$.
- transitive closure: $a \rightarrow^+ b$ iff $\exists n \in \mathbb{N}. a \rightarrow^{\leq n} b$.
- reflexive transitive closure: $a \rightarrow^* b$ iff $a \rightarrow^= b$ or $a \rightarrow^+ b$.
- rewrite derivation: a sequence of rewrite steps $a_0 \rightarrow a_1 \rightarrow \dots a_n$.
- a is in Normal Form (NF) if no rewrite rules can apply.

A rewrite system is meant to invoke the feeling of algebra. In fact, rewrite system are much more general, but they can still retain the feeling. If we have an expression $(x \cdot x + 1)(2 + x)$, we might reduce this with the reduction in figure 1.1.

We can conclude that $(x \cdot x + 1)(x + 2) \rightarrow^+ x^3 + 2x^2 + x + 2$. This idea of rewriting invokes the feel of algebraic rules. The mechanical process of rewriting allows for a simple implementation on a computer.

It's worth understanding the properties and limitations of these rewrite systems. Traditionally there are two important questions to answer about any rewrite system. Is it *confluent*? Is it *terminating*?

A *confluent* system is a system where the order of the rewrites doesn't change the final result. For example, consider the distributive rule. When evaluating $3 \cdot (4 + 5)$ we could either evaluate the addition or multiplication first. Both of these reductions arrived at the same answer as can be seen in figure 1.2.

$$\begin{array}{ll} & 3 \cdot (4+5) \\ \to & 3 \cdot 4 + 3 \cdot 5 \\ \to & 12 + 15 \\ \to & 27 \end{array}$$

(a) distributing first

$$\begin{array}{ll} & 3 \cdot (4+5) \\ \to & 3 \cdot 9 \\ \to & 27 \end{array}$$

(b) reducing $4 + 5$ first

Figure 1.2: Two possible reductions of $3 \cdot (4 + 5)$. Since this is a confluent system, they both can rewrite to 27.

In a *terminating* system every derivation is finite. That means that eventually there are no rules that can be applied. The distributive rule is terminating, whereas the commutative rule is not terminating. See figure 1.3.

$$\begin{array}{ll} & a \cdot (b+c) \\ \to & a \cdot b + a \cdot c \end{array}$$

$$\begin{array}{ll} & x + y \\ \to & y + x \\ \to & x + y \\ \dots & \end{array}$$

Figure 1.3: A system with a single rule for distribution is terminating, but any system with a commutative rule is not. Note that $x + y \to^2 x + y$

Confluence and termination are important topics in rewriting, but we will largely ignore them. After all, Curry programs are neither confluent nor terminating. However, there will be a few cases where these concepts will be important. For example, if our optimizer isn't terminating, then we'll never actually compile a program.

Now that we have a general notation for rewriting, we can introduce two important rewriting frameworks: term rewriting and graph rewriting, where we are transforming trees and graphs respectively.

1.2 TERM REWRITING

As mentioned previously, one application of term rewriting is to transform terms representing syntax trees. This will be useful in optimizing the Abstract Syntax Trees (ASTs) of Curry programs. Term rewriting is a special case of abstract rewriting. Therefore everything from abstract rewriting will apply to term rewriting.

A term is made up of signatures and variables. [74][Def 3.1.2] We let Σ and V be two arbitrary alphabets, but we require that V be countably infinite, and $\Sigma \cap V = \emptyset$ to avoid name conflicts. A *signature* $f^{(n)}$ consists of a name $f \in \Sigma$ and an arity $n \in \mathbb{N}$. A *variable* $v \in V$ is just a name. Finally a *term* is defined inductively. The term t is either a variable v, or it's a signature $f^{(n)}$ with children $t_1, t_2, \ldots t_n$, where $t_1, t_2, \ldots t_n$ are all terms. We write the set of terms all as $T(\Sigma, V)$. If $t \in T(\Sigma, V)$ then we write $Var(t)$ to denote the set of variables in t. By definition $Var(t) \subseteq V$. We say that a term is *linear* if no variable appears twice in the term [74][Def. 3.2.4].

This inductive definition gives us a tree structure for terms. As an example consider Peano arithmetic $\Sigma = \{+^2, *^2, -^2, <^2, 0^0, S^1, True^0, False^0\}$. We can define the term $*(+(0, S(0)), +(S(0), 0))$. This gives us the tree in figure 1.4. Every term can be converted into a tree like this and vice versa. The symbol at the top of the tree is called the root of the term.

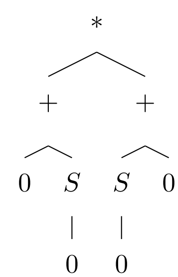


Figure 1.4: Tree representation of the term $*(+(0, S(0)), +(S(0), 0))$.

A *child* c of term $f(t_1, t_2, \ldots t_n)$ is one of $t_1, t_2, \ldots t_n$. A *subterm* s of t is either t itself, or it is a subterm of a child of t. We write $s = t|_p$ where $p = [i_1, i_2, \ldots i_n]$ to denote that t has child t_{i_1} which has child t_{i_2} and so on until $t_{i_n} = s$. Note that we can define this recursively as $t|_{[i_1, i_2, \ldots i_n]} = t_{i_1}|_{[i_2, \ldots i_n]}$, which matches our definition for subterm. We call $[i_1, i_2, \ldots i_n]$ the *path* from t to s [74][Def 3.1.5]. We write ϵ for the empty path, and $i{:}p$ for the path starting with the number i and followed by the path p, and $p \cdot q$ for concatenation of paths p and q.

In our previous term $S(0)$ is a subterm in two different places. One occurrence is at path $[0, 1]$, and the other is at path $[1, 0]$.

We write $t[p \to r]$ to denote replacing subterm $t|_p$ with r. We define the algorithm for this in figure 1.5.

$t[\epsilon \to r] = r$

$f(t_1, \ldots t_i, \ldots t_n)[i{:}p \to r] = f(t_1, \ldots t_i[p \to r], \ldots t_n)$

Figure 1.5: algorithm for finding a subterm of t.

In our above example $t = *(+(0, S(0), +(S(0), 0)))$, We can compute the rewrite $t[[0,1] \to *(S(0), S(0))]$, and we get the term $*(+(0, *(S(0), S(0))), +(S(0), 0))$, with the tree in figure 1.6.

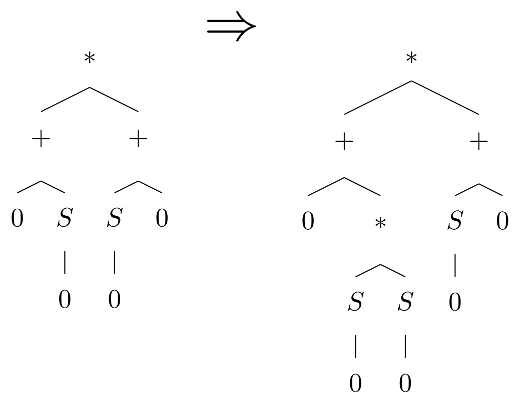


Figure 1.6: The result of the computation $t[[0,1] \to S(0)]$

A substitution replaces variables with terms. Formally, a *substitution* is a mapping from $\sigma{:}V \to T(\Sigma, V)$, such that $\sigma(x) \neq x$ [74][Def. 3.1.7]. We write $\sigma = \{v_1 \mapsto t_1, \ldots v_n \mapsto t_n\}$ to denote the substitution where $s(v_i) = t_i$ for $i \in \{1 \ldots n\}$, and $s(v) = v$ otherwise. We can uniquely extend σ to a function on terms by figure 1.7

$\sigma'(v) = \sigma(v)$

$\sigma'(f(t_1, \ldots t_n) = f(\sigma'(t_1) \ldots \sigma'(t_n))$

Figure 1.7: Algorithm for applying a substitution.

Since this extension is unique, we will just write σ instead of σ'. Term t_1 *matches* term t_2 if there exists some substitution σ such that $t_1 = \sigma(t_2)$ [74][3.1.8],. We call σ a *matcher*. Two terms t_1 and t_2 *unify* if there exists some substitution σ such that $\sigma(t_1) = \sigma(t_2)$ [74][3.1.8],. In this case σ is called a *unifier* for t_1 and t_2.

We can order substitutions based on what variables they define. A substitution $\sigma \leqslant \tau$, iff, there is some substitution ν such that $\tau = \nu \circ \sigma$. The relation $\sigma \leqslant \tau$ should be read as σ is more general than τ, and it is a quasi-order on the set of substitutions. A unifier u for two terms is *most general* (or an mgu), iff, for all unifiers v, $v \leq u$. Mgu's are unique up to renaming of variables. That is, if u_1 and u_2 are mgu's for two terms, then $u_1 = \sigma_1 \circ u_2$ and $u_2 = \sigma_2 \circ u_1$. This can only happen if σ_1 and σ_2 just rename the variables in their terms.

As an example $+(x, y)$ matches $+(0, S(0))$ with $\sigma = \{x \mapsto 0, y \mapsto S(0)\}$. The term $+(x, S(0))$ unifies with term $+(0, y)$ with unifier $\sigma = \{x \mapsto 0, y \mapsto S(0)\}$. If $\tau = \{x \mapsto 0, y \mapsto S(z)\}$, then $\tau \leq \sigma$. We can define $\nu = \{z \mapsto 0\}$, and $\{\sigma = \nu \circ \tau\}$

Now that we have a definition for a term, we need to be able to rewrite it. A *rewrite rule* $l \to r$ is a pair of terms. However this time we require that $Var(r) \subseteq Var(l)$, and that $l \notin V$. A *Term Rewriting System (TRS)* is the pair $(T(\Sigma, V), R)$ where R is a set of rewrite rules.

Definition 1.2.1. Rewriting: Given terms t, s, path p, and rule $l \to r$, we say that t rewrites to s if, l matches $t|_p$ with matcher σ, and $t[p \to \sigma(r)] = s$. The term $\sigma(l)$ is the *redex*, and the term $\sigma(r)$ is the *contractum* of the rewrite.

There are a few important properties of rewrite rules $l \to r$. A rule is left or right linear if l or r is linear respectively [74][Def. 3.2.4]. A rule is *collapsing* if $r \in V$. A rule is *duplicating* if there is an $x \in V$ that occurs more often in r than in l [74][Def. 3.2.5].

Two terms s and t are *overlapping* if t unifies with a subterm of s, or s unifies with a subterm of t at a non-variable position [74][Def. 4.3.3]. Two rules $l_1 \to r_1$ and $l_2 \to r_2$ if l_1 and l_2 overlap. A rewrite system is *overlapping* if, and only if, any two rules overlap. Otherwise it's non-overlapping. Any non-overlapping left linear system is *orthogonal* [74][Def.4.3.4]. Orthogonal systems have several nice properties, such as the following theorem [74][Thm. 4.3.11].

Theorem 1. *Every orthogonal TRS is confluent.*

As an example, in figure 1.8 examples (b) and (c) both overlap. It's clear that these systems aren't confluent, but non-confluence can arise in more subtle ways. The converse to theorem 2.1 isn't true. There can be overlapping systems which are confluent.

When defining rewrite systems we usually follow the constructor discipline; we separate the set $\Sigma = C \uplus F$. C is the set of *constructors*, and F is the set of *function symbols*. Furthermore, for every rule $l \to r$, the root of l is a function symbol, and every other symbol is a constructor

$g(0, y) \to 0$ $g(1, y) \to 1$	$g(0, y) \to 0$ $g(x, 1) \to 1$	$f(g(x, y)) \to 0$ $g(x, y) \to 1$
(a) A non-overlapping system	(b) A system that overlaps at the root	(c) A system that overlaps at a subterm

Figure 1.8: Three TRSs demonstrating how rules can overlap. In (a) they don't overlap at all, In (b) both rules overlap at the root, and in (c) rule 2 overlaps with a subterm of rule 1.

or variable. We call such systems *constructor systems*. As an example, the rewrite system for Peano arithmetic is a constructor system.

$$
\begin{array}{lcllll}
R_1 & : & 0 + y & \to & y \\
R_2 & : & S(x) + y & \to & S(x + y) \\
R_3 & : & 0 * y & \to & 0 \\
R_4 & : & S(x) * y & \to & y + (x * y) \\
R_5 & : & 0 - y & \to & 0 \\
R_6 & : & S(x) - 0 & \to & S(x) \\
R_7 & : & S(x) - S(y) & \to & x - y \\
R_8 & : & 0 \leq y & \to & True \\
R_9 & : & S(x) \leq 0 & \to & False \\
R_{10} & : & S(x) \leq S(y) & \to & x < y \\
R_{11} & : & 0 = 0 & \to & True \\
R_{12} & : & S(x) = 0 & \to & False \\
R_{13} & : & 0 = S(y) & \to & False \\
R_{14} & : & S(x) = S(y) & \to & x = y
\end{array}
$$

Figure 1.9: The rewrite rules for Peano Arithmetic with addition, multiplicaton, subtraction, and comparison. All operators use infix notation.

The two sets are $C = \{0, S, True, False\}$ and $F = \{+, *, -, \leq\}$, and the root of the left hand side of each rule is a function symbol. In contrast, the SKI system is not a constructor system. While S, K, I can all be constructors, the Ap symbol appears in both root and non-root positions of the left hand side of rules. This example will become important for us in Curry. We will do something similar to implement higher order functions. This means that Curry programs won't directly follow the constructor discipline. Therefore, we must be careful when specifying the semantics of function application.

$$
\begin{array}{lll}
Ap(I, x) & \rightarrow & x \\
Ap(Ap(K, x), y) & \rightarrow & x \\
Ap(Ap(Ap(S, x), y), z) & \rightarrow & Ap(Ap(x, z), Ap(y, z))x
\end{array}
$$

Figure 1.10: The SKI system from combinatorial logic.

Constructor systems have several nice properties. They are usually easy to analyze for confluence and termination. For example, if the left hand side of two rules don't unify, then they cannot overlap. We don't need to check if subterms overlap. Furthermore, any term that consists entirely of constructors and variables is in normal form. For this reason, it's not surprising that most functional languages are based on constructor systems.

1.3 NARROWING

Narrowing was originally developed to solve the problem of semantic unification. The goal was, given a set of equations $E = \{a_1 = b_2, a_2 = b_2, \ldots a_n = b_n\}$ how do you solve the $t_1 = t_2$ for arbitrary terms t_1 and t_2. Here a solution to $t_1 = t_2$ is a substitution σ such that $\sigma(t_1)$ can be transformed into $\sigma(t_2)$ by the equations in E.

As an example let $E = \{*(x + (y, z)) = +(*(x, y), *(x, z))\}$ Then the equation $*(1, +(x, 3)) = +(+(*(1, 4), *(y, 5)), *(z, 3))$ is solved by $\sigma = \{x \mapsto +(4, 5), y \mapsto 1, z \mapsto 1\}$. The derivation is in figure 1.11.

$$
\begin{array}{ll}
\sigma(*(1, +(x, 3))) & = \\
*(1, +(+(4, 5), 3)) & = \\
+(*(1, +(4, 5)), *(1, 3)) & = \\
+(+(*(1, 4), *(1, 5)), *(1, 3)) & = \\
\sigma(+(+(*(1, 4), *(y, 5)), *(z, 3)))
\end{array}
$$

Figure 1.11: Derivation of $*(1, +(x, 3)) = +(+(*(1, 4), *(y, 5)), *(z, 3))$ with $\sigma = \{x \mapsto +(4, 5), y \mapsto 1, z \mapsto 1\}$.

Unsurprisingly, there is a lot of overlap with rewriting. One of the earlier solutions to this problem was to convert the equations into a confluent, terminating rewrite system. [60] Unfortunately, this only works for ground terms, that is, terms without variables. However, this idea still has merit. So we want to extend it to terms with variables.

Before, when we rewrote a term t with rule $l \to r$, we assumed it was a ground term, then we could find a substitution σ that would match a subterm $t|_p$ with l, so that $\sigma(l) = t|_p$. To extend this idea to terms with variables in them, we look for a unifier σ that unifies $t|_p$ with l. This is really the only change we need to make [74]. However, now we record σ, because it is part of our solution.

Definition 1.3.1. Narrowing: Given terms t, s, path p, and rule $l \to r$, we say that t narrows to s if, l unifies with $t|_p$ with unifier σ, and $t[p \to \sigma(r)] = s$. We write $t \rightsquigarrow_{p,l\to r,\sigma} s$. We may write $t \rightsquigarrow_\sigma s$ if p and $l \to r$ are clear.

Notice that this is almost identical to the definition of rewriting. The only difference is that σ is a unifier instead of a matcher.

Narrowing was first developed to solve equations for automated theorem provers [89]. However, for our purposes it's more important that narrowing allows us to rewrite terms with free variables. [45]

At this point, rewrite systems are a nice curiosity, but they are completely impractical. This is because we don't have a plan for solving equations in them. In the definition for both rewriting and narrowing, we did not specify how to find σ the correct rule to apply, or even what subterm to apply the rule.

In confluent terminating rewrite systems, we could simply try every possible rule at every possible position with every possible substitution. Since the system is confluent, we could choose the first rule that could be successfully applied, and since the system is terminating, we'd be sure to find a normal form. In a narrowing system, this is still not guaranteed to halt, because there could be an infinite number of substitutions. This is the best possible case for rewrite systems, and we still can ensure that our algorithm will finish. We need a systematic method for deciding what rule should be applied, what subterm to apply it to, and what substitution to use. This is the role of a strategy.

1.4 REWRITING STRATEGIES

Our goal with a rewriting strategy is to be able to find a normal form for any term. Similarly our goal for narrowing will be to find a normal form and substitution. However, we want to be efficient when rewriting. We would like to use only local information when deciding what rule to select. We would also like to avoid unnecessary rewrites. Consider the following term from the SKI system defined in figure 1.10 $Ap(Ap(K,I), Ap(Ap(S,Ap(I,I)), Ap(S,Ap(I,I))))$. It would be pointless to reduce $Ap(Ap(S,Ap(I,I)), Ap(S,Ap(I,I))))$ since $Ap(Ap(K,I,z)$ rewrites to I no matter what z is. In this particular case, since $Ap(Ap(S,Ap(I,I)), Ap(S,Ap(I,I))))$ reduces to itself, we have turned a potentially non-terminating reduction to a terminating one.

A *Rewriting Strategy* $\mathcal{S}{:}T(\Sigma, V) \to Pos \times R$ is a function that takes a term, and returns a position to rewrite, and a rule to rewrite with [59]. Furthermore we require that if $(p, l \to r) = \mathcal{S}(t)$, then $t|_p$ is a redex that matches l. The idea is that $S(t)$ should give us a position to rewrite, and the rule to rewrite with.

For orthogonal rewriting systems, there are two common rewriting strategies that do not run in parallel,[1] innermost and outermost rewriting[59, 68]. Innermost rewriting corresponds to eager evaluation in functional programming. We rewrite the term that matches a rule that is the furthest down the tree. Outermost rewriting correspond roughly to lazy evaluation. We rewrite the highest possible term that matches a rewrite rule.

A strategy is *normalizing* if, when a term t has a normal form, then the strategy will eventually find it. While outermost rewriting isn't normalizing in general, it is for left-normal systems, which is a large subclass of orthogonal rewrite systems [59]. This matches the intuition from programming languages. Lazy languages can perform computations that would run forever with an eager language.

While both of these strategies are well understood, we can actually make a stronger guarantee. We want to reduce only the redexes that are necessary to find a normal form. To formalize this we need to understand what can happen when we rewrite a term. Specifically for a redex s that is a subterm of t, how can s change as we rewrite t. If we're rewriting at position p with rule $l \to r$, then there are 3 cases to consider.

Case 1: we are rewriting s itself. That is, s is the subterm $t|_p$. Then s disappears entirely.

[1] we avoid discussing parallel strategies, because our work is focused on sequential execution of Curry programs. That has been a lot of work done on parallel execution of Curry programs elsewhere [47, 48].

Case 2: s is either above $t|_p$, or they are completely disjoint. In this case s doesn't change.

Case 3: s is a subterm of $t|_p$. In this case s may be duplicated, or erased, moved, or left unchanged. It depends on whether the rule is duplicating, erasing, or right linear.

These cases can be seen in figure 1.12 We can formalize this with the notion of descendants with the following definition from [74][Def. 4.3.6].

Definition 1.4.1. Descendant: Let $s = t|_v$, and $A = l \to_{p,\sigma,R} r$ be a rewrite step in t. The set of descendants of s is given by $Des(s, A)$

$$Des(s, A) = \begin{cases} \emptyset & \text{if } v = u \\ \{s\} & \text{if } p \not\leq v \\ \{t|_{u \cdot w \cdot q} : r|_w = x\} & \text{if } p = u \cdot v \cdot q \text{ and } t|_v = x \text{ and } x \in V \end{cases}$$

This definition extends to derivation $t \to_{A_1} t_1 \to_{A_2} t_2 \to_{A_2} \ldots \to_{A_n}, t_{n+1}$. $Des(s, A_1, A_2 \ldots A_n) = \bigcup_{s' \in Des(s,A_1)} Des(s', A_2, \ldots A_n)$.

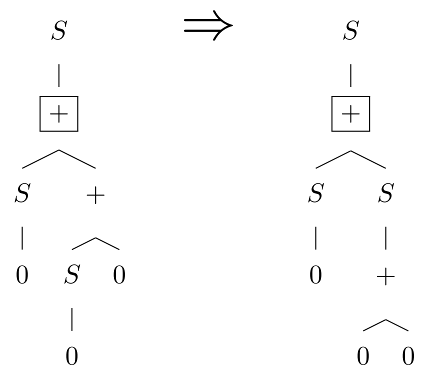
The first part of the definition is formalizing the notion of descendant. The second part is extending it to a rewrite derivation. The extension is straightforward. Calculate the descendants for the first rewrite, then for each descendant, calculate the descendants for the rest of the rewrites. With the idea of a descendant, we can talk about what happens to a term in the future. This is necessary to describing our rewriting strategy. Now we can formally define what it means for a redex to be necessary for computing a normal form.

Definition 1.4.2. Needed: A redex s that is a subterm of t is *needed* in t if, for every derivation of t to a normal form, a descendant of s is the root of a rewrite.

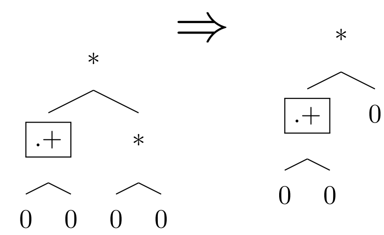
This definition is good because it's immediately clear that, if we're going to rewrite a term to normal form, we need to rewrite all of the needed redexes. In fact, we can guarantee more than that with the following theorem [56].

Theorem 2. *For an orthogonal TRS, any term that is not in normal form contains a needed redex. Furthermore, a rewrite strategy that rewrites only needed redexes is normalizing.*

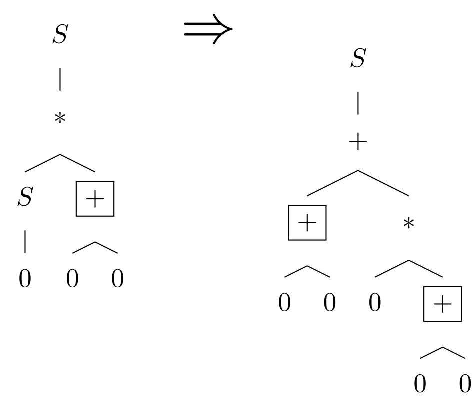
This is a very powerful result. We can compute normal forms by rewriting needed redexes. This is also, in some sense, the best possible strategy. Every needed redex needs to be rewritten. Now we just need to make sure our strategy only rewrites needed redexes. There's only one



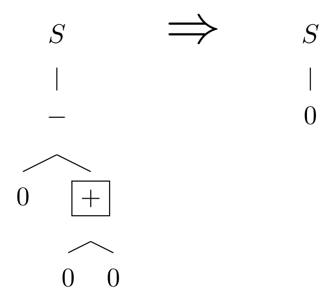
(a) rewrite R_1 at position $[0,1]$ doesn't affect $t|_{[0]}$.



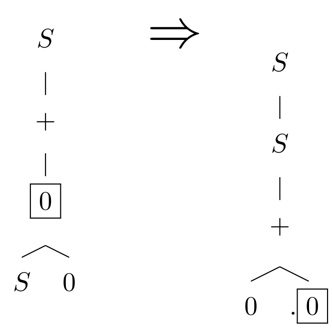
(b) rewrite R_4 at position $[1]$ doesn't affect $t|_{[0]}$.



(c) rewrite R_3 at position $[0]$ duplicates $t|_{[0,1]}$.



(d) rewrite R_5 at position $[0]$ erases term at $t|_{[0,1]}$.



(e) rewrite R_2 at position $[0]$ moves $t|_{[0,1]}$ to position $[0,0,1]$.

Figure 1.12: four cases for the descendants for a term after a single rewrite. The boxed term is either left alone, duplicated, or erased, or moved. The rules are defined in figure 1.9

problem with this plan. Determining if a redex is needed is undecidable in general. However, with some restrictions, there are rewrite systems where this is possible [59][def. 3.3.7].[2]

Definition 1.4.3. Sequential A rewrite system is *sequential* if, given a term t with n variables $v_1, v_2 \dots v_n$, such that t is in normal form, then there is an i such that for every substitution σ from variables to redexes, $\sigma(v_i)$ is needed in $\sigma(t)$.

If we have a sequential rewrite system, then this leads to an efficient algorithm for reducing terms to normal form. Unfortunately, sequential is also an undecidable property. There is still hope. As we'll see in the next section, with certain restrictions we can ensure the our rewrite systems are sequential. Actually we can make a stronger guarantee. The rewrite system will admit a narrowing strategy that only narrows needed redexes.

1.5 NARROWING STRATEGIES

Similar to rewriting strategies, narrowing strategies attempt to compute a normal form for a term using narrowing steps. However, a narrowing strategy must also compute a substitution for that term. There have been many narrowing strategies including basic [53], innermost [38], outermost [96], standard [35], and lazy [70]. Unfortunately, each of these strategies are too restrictive on the rewrite systems they allow.

$(x + x) + x = 0$

(a) This fails for eager narrowing, because evaluating $x + x$ can produce infinitely many answers. However This is fine for lazy narrowing. We will get $(0 + 0) + 0 = 0, \{x = 0\}$ or $S(S(y + S(y)) + S(y)) = 0\{x = S(y)\}$ and the second one will fail.

$x \leq y + y$

(b) With a lazy narrowing strategy we may end up computing more than is necessary. If x is instantiated to 0, then we don't need to evaluate $y + y$ at all.

Figure 1.13: examples of where eager and lazy narrowing can fail using the rewrite system if figure 1.9.

Fortunately there exists a narrowing strategy that's defined on a large class of rewrite systems, only narrows needed expressions, and is sound and complete. However this strategy requires a new construct called a definitional tree.

[2] The original definition used the notion of a context in normal form.

The idea is that since we are working with constructor rewrite systems, we can group all of the rules defined for the same function symbol together. We'll put them together in a tree structure defined below, and then we can compute a narrowing step by traversing the tree for the defined symbol.

Definition 1.5.1. T is a *partial definitional tree* if T is one of the following.
$T = exempt(\pi)$ where π is a pattern.
$T = leaf(\pi \to r)$ where π is a pattern, and $\pi \to r$ is a rewrite rule.
$T = branch(\pi, o, T_1, \dots T_k)$, where π is a pattern, o is a path, $\pi|_o$ is a variable, $c_1, \dots c_k$ are constructors, and T_i is a pdt with pattern $\pi[c_i(X_1, \dots X_n)]_o$ where n is the arity of c_i, and $X_1, \dots X_n$ are fresh variables.

Given a constructor rewrite system R, T is a *definitional tree* for function symbol f if T is a partial definitional tree, and each leaf in T corresponds to exactly one rule rooted by f. A rewrite system is *inductively sequential* if there exists a definitional tree for every function symbol.

The name "inductively sequential" is justified because there is a narrowing strategy that only reduces needed redexes for any of these systems. We show an example to clarify the definition. In figure 1.14 we show the definitional tree for the $+, \leq$, and $=$ rules. The idea is that, at each branch, we decide which variable to inspect. Then we decide what child to follow based on the constructor of that branch. This gives us a simple algorithm for outermost rewriting with definitional trees. However, we need to extend this to narrowing.

In order to extend the strategy from rewriting to narrowing we need to figure out how to compute a substitution, and we need to define what it means for a narrowing step to be needed. The earliest definition involved finding a most general unifier for the substitution. This has some nice properties. There is a well known algorithm for computing mgu's, which are unique up to renaming of variables. However, this turned out to be the wrong approach. Computing mgu's is too restrictive. Consider the step $x \leq y + z \rightsquigarrow_{2\cdot\epsilon, R_1, \{y \mapsto 0\}} x \leq z$. Without further substitutions $x \leq z$ is a normal form, and $\{y \mapsto 0\}$ is an mgu. Therefore this should be a needed step. But if we were to instead narrow x, we have $x \leq y+z \rightsquigarrow_{\epsilon, R_8, \{x \mapsto 0\}} True$. This step never needs to compute a substitution for y. Therefore we need a definition that isn't dependent on substitutions that might be computed later.

Definition 1.5.2. A narrowing step $t \rightsquigarrow_{p,R,\sigma} s$ is needed, iff, for every $\eta \geq \sigma$, there is a needed

$$
\begin{array}{lcl}
0 + y & \to & y \\
S(x) + y & \to & S(x + y)
\end{array}
$$

$$
\begin{array}{lcl}
0 \leq y & \to & True \\
S(x) \leq 0 & \to & False \\
S(x) \leq S(y) & \to & x < y
\end{array}
$$

$$
\begin{array}{lcl}
0 = 0 & \to & True \\
S(x) = 0 & \to & False \\
0 = S(y) & \to & False \\
S(x) = S(y) & \to & x = y
\end{array}
$$

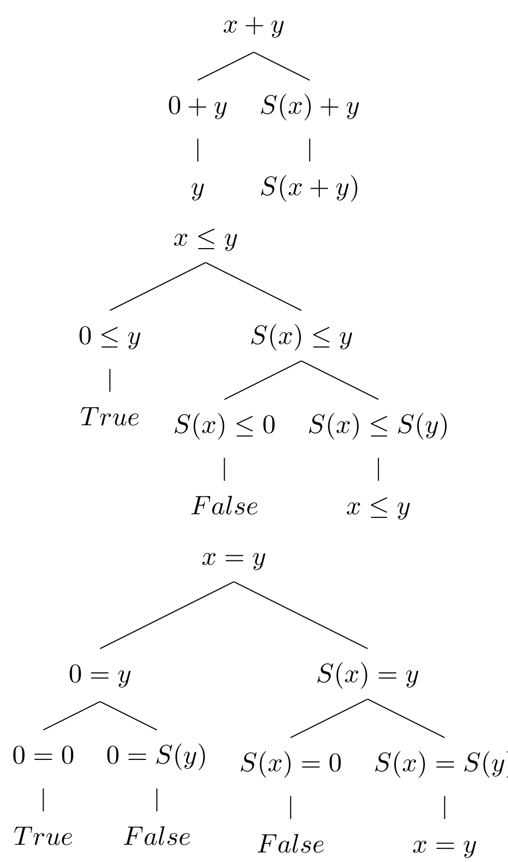


Figure 1.14: Definitional trees for $+$, $\leq$, and $=$.

redex at p in $\eta(t)$.

Here we don't require that σ be an mgu, but, for any less general substitution, it must be the case that we're rewriting a needed redex. So our example, $x \leq y + z \rightsquigarrow_{2\dot{\epsilon}, R_1, \{y \mapsto 0\}} x \leq z$, isn't a needed narrowing step because $x \leq y + z \rightsquigarrow_{2\dot{\epsilon}, R_1, \{x \mapsto 0, y \mapsto 0\}} 0 \leq z$, Isn't a needed rewriting step.

Unfortunately, this definition raises a new problem. Since we are no longer using mgu's for our unifiers, we may not have a unique step for an expression. For example, $x < y \rightsquigarrow_{\epsilon, R_8, \{x \mapsto 0\}} True$, and $x < y \rightsquigarrow_{\epsilon, R_9, \{x \to S(u), t \mapsto S(v)\}} u \leq v$ are both possible needed narrowing steps.

Therefore we define a *Narrowing Strategy* $\mathcal{S}$ as a function from terms to a set of triples of a position, rule, and substitution, such that if $(p, R, \sigma) \in \mathcal{S}(t)$, then $\sigma(t)|_p$ is a redex for rule R.

At this point we have everything we need to define a needed narrowing strategy.

Definition 1.5.3. Let t be a term rooted by function symbol f, T be the definitional tree for f, and "?" be a distinguished symbol to denote that no rule could be found.

$$\lambda(t, T) \in \begin{cases} (\epsilon, R, mgu(t, \pi)) & \text{if } T = rule(\pi, R) \\ (\epsilon, ?, mgu(t, \pi)) & \text{if } T = exempt(\pi) \\ (p, R, \sigma) & \text{if } T = branch(\pi, o, T_1, \dots T_n) \\ & t \text{ unifies with } T_i \\ & (p, R, \sigma) \in \lambda(t, T_i) \\ (o{:}p, R, \sigma \circ \tau) & \text{if } T = branch(\pi, o, T_1, \dots T_n) \\ & t \text{ does not unifies with any } T_i \\ & \tau = mgu(t, \pi) \\ & T' \text{ is the definitional tree for } t|_o \\ & (p, R, \sigma) \in \lambda(t|_o, T') \end{cases}$$

The function λ is a narrowing strategy. It takes an expression rooted by f, and the definition tree for f, and it returns a position, rule and substitution for a narrowing step. If we reach a rule node, then we can just rewrite; if we reach an exempt node, then there is no possible rewrite; if we reach a branch node, then we match a constructor; but if the subterm we're looking at isn't a constructor, then we need to narrow that subterm first.

Theorem 3. *λ is a needed narrowing strategy. Furthermore, λ is sound and complete.*

It should be noted that while λ is complete with respect to finding substitutions and selecting rewrite rules [17], this says nothing about the underlying completeness of the rewrite system we're narrowing. We may still have non-terminating derivations.

This needed narrowing strategy is important in developing the evaluation strategy for Curry programs. In fact, one of the early stages of a Curry compiler is to construct definitional trees for each function defined. However, if we were to implement our compiler using terms, it would be needlessly inefficient. We solve this problem with graph rewriting.

1.6 GRAPH REWRITING

As mentioned above term rewriting is too inefficient to implement Curry. Consider the rule $double(x) = x + x$. Term rewriting requires this rule to make a copy of x, no matter how large it is, whereas we can share the variable if we use a graph. In programming languages, this distinction moves the evaluation strategy from "call by name" to "call by need", and it is what we mean when we refer to "lazy evaluation".

As a brief review of relevant graph theory: A *graph* $G = (V, E)$ is a pair of vertices V and edges $E \subseteq V \times V$. We will only deal with directed graphs, so the order of the edge matters. A *rooted graph* is a graph with a specific vertex r designated as the *root*. The *neighborhood* of v, written $N(v)$ is the set of vertices adjacent to v. That is, $N(v) = \{u \mid (v, u) \in E\}$. A *path* p from vertex u to vertex v is a sequence $u = p_1, p_2 \ldots p_n = v$ where $(p_i, p_{i+1}) \in E$. A rooted graph is *connected* if there is a path from the root to every other vertex in the graph. A graph is *strongly connected* if, for each pair of vertices (u, v), there is a path from u to v and a path form v to u. A path p is a cycle [3] if its endpoints are the same. A graph is acyclic if it contains no cycles. Such graphs are referred to as Directed Acyclic Graphs, or DAG's. A graph H is a *subgraph* of G, $H \subseteq G$ if, and only if, $V_H \subseteq V_G$ and $E_H \subseteq E_G$. A strongly connected component S of G is a subgraph that is strongly connected. We will use the well-known facts that strongly connected components partition a graph. The component graph, which is obtained by shrinking the strongly connected components to a single vertex, is a DAG. To avoid confusion with variables, we will refer to vertices of graphs as nodes.

We define term graphs in a similar way to terms. Let $\Sigma = C \uplus F$ be an alphabet of constructor and function names respectively, and V be a countably infinite set of variables. A *term graph*

---

[3] Some authors will use walk and tour and reserve path and cycle for the cases where there are no repeated vertices. This distinction isn't relevant for our work.

is a rooted graph G with nodes in N where each node n has a label in $\Sigma \cup V$. We'll write $L(n)$ to refer to the label of a node. If $(n, s) \in E$ is an edge, then s is a successor of n. In most applications the order of the outgoing edges doesn't matter, however it is very important in term graphs. So, we will refer to the first successor, second successor and so on. We denote this the same way we did with terms n_i is the ith successor of n. The arity of a node is the number of successors. Finally, no two nodes can be labeled by the same variable.

While the nodes in a term graph are abstract, in reality, they connected using pointers in the implementation. It can be helpful to keep this in mind. As we define more operations on our term graphs, there exists a natural implementation using pointers.

We will often use a linear notation to represent graphs. This has two advantages. The first is that it is exact. There are many different ways to draw the same graph, but there is only one way to write it out a linear representation[34] The second is that this representation corresponds closely to the representation in a computer. The notation these graphs is given by the grammar

$$\begin{aligned} Graph &\rightarrow Node \\ Node &\rightarrow n : L\ (Node, \ldots Node) \\ &\mid\ n \end{aligned}$$

We start with the root node, and for each node in the graph, If we haven't encountered it yet, then we write down the node, the label, and the list of successors. If we have seen it, then we just write down the node. If a node doesn't have any successors, then we'll omit the parentheses entirely, and just write down the label.

A few examples are shown in figure 1.15. Example 1 shows an expression where a single variable is shared several times. Example 2 shows how a rewrite can introduce sharing. Example 3 shows an example of an expression with a loop. These examples would require an infinitely large term, so they cannot be represented in term rewrite systems. Example 4 shows how reduction changes from terms to graphs. In a term rewrite system, if a node is in the pattern of a redex, then it can safely be discarded. However, in graph rewriting this is no longer true.

Definition 1.6.1. Let p be a node in G, then the *subgraph* $G|_p$ is a new graph rooted by p. The nodes are restricted to only those reachable from p.

Notice that we don't define subgraphs by paths like we did with subterms. This is because there may be more than one path to the node p. It may be the case that $G|_p$ and G have the same nodes, such as if the root of G is in a loop.

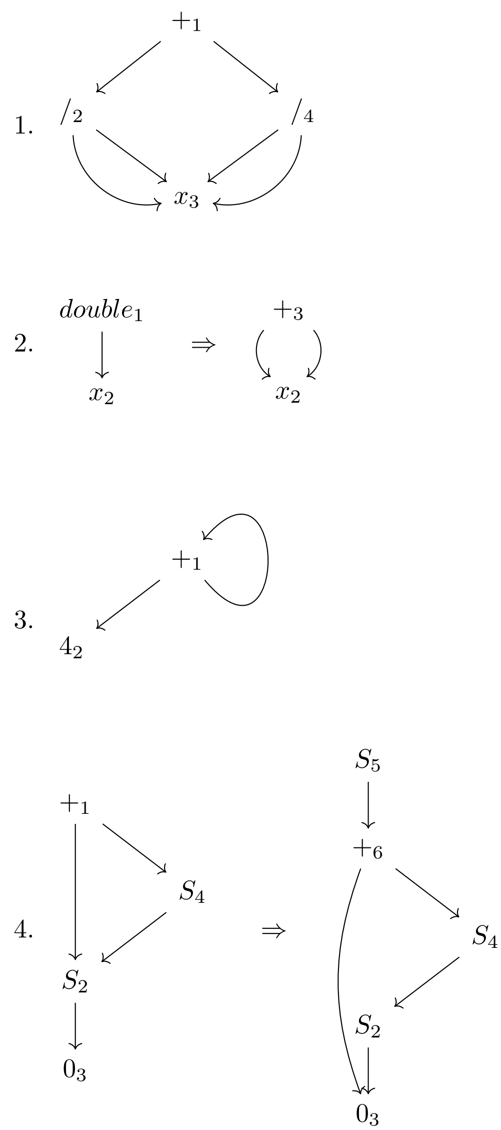


Figure 1.15: 1. $1{:}+(2{:}/(3{:}x,3),4{:}/(3,3))$,

2. $1{:}double(2{:}x) \Rightarrow 3{:}+(2{:}x,2)$

3. $1{:}+(2{:}4,1)$

4. $1{:}+(2{:}S(3{:}0),4{:}S(2)) \Rightarrow 5{:}S(6{:}+(3{:}0),4{:}S(2{:}S(3)))$

Definition 1.6.2. A *replacement* of node p by graph u in g (written $g[p \to u]$) is given by the following procedure. For each edge $(n, p) \in E_g$ replace it with an edge $(n, root_u)$. Add all other edges from E_g and E_u. If p is the root of g, then $root_u$ is now the root.

It should be noted that when implementing Curry, we don't actually change any of the pointers when doing a replacement. Traversing the graph to find all of the pointers to p would be horribly inefficient. Instead we change the contents of p to be the contents of u.

We can define matching in a similar way to terms, but we need to be more careful. When matching terms the structure of the term needed to be the same, however when matching graphs the structure can be wildly different. Consider the following graph.

$$\begin{tikzcd} and \arrow[d, bend right] \arrow[d, bend left] \\ True \end{tikzcd}$$

Here the graph should match the rule $and(True, True) \to True$. But $and(True, True)$ is a term, so they no longer have the same structure. Therefore we must be more careful about what we mean by matching. We define matching inductively on the structure of the term.

Definition 1.6.3. A graph K *matches* a term T if, and only if, T is a variable, or $T = l(T_1, T_2 \ldots T_n)$, the root of K is labeled with l, and for each $i \in \{1 \ldots n\}$, K_i matches T_i.

Now, it may be the case that we have multiple successors pointing to the same node when checking if a graph matches a pattern, but this is OK. As long as the node matches each sub pattern, then the graph will match. We extend substitutions to graphs in the obvious way. A substitution σ maps variables to Nodes. In this definition for matching σ may have multiple variables map to the same node, but this doesn't cause a problem.

Definition 1.6.4. A *rewrite rule* is a pair $L \to R$ where L is a term, and R is a term graph. A graph G matches the rule if there exists subgraph K where K matches L with matcher σ. A *rewrite* is a triple $(K, L \to R, \sigma)$, and we apply the rewrite with $G[K \to \sigma(R)]$.

From here we can define narrowing similarly to how we did for terms. We do not give the definitions here, because they are similar to the definitions in term rewriting. At this point we have discussed the difference between graphs and terms, and how a replacement can be done in a graph. For our purposes in this compiler, that is all that is needed, but the definition of narrowing and properties about inductively sequential GRS's can be found in Echaned and Janodet [34]. They also show that the needed narrowing strategy is still valid for graph rewriting systems.

1.7 PREVIOUS WORK

This was not meant to be an exhaustive examination of rewriting, but rather an introduction to the concepts, since they form this theoretical basis of the Curry language. Most work on term rewriting up through 1990 has been summarized by Klop [59], and Baader and Nipkow [23]. The notation and ideas in this section largely come from Ohlebusch [74], although they are very similar to the previous two summaries. The foundations of term rewriting were laid by Church, Rosser, Curry, Feys, Newman. [30, 32, 73] Most of the work on rewriting has centered on confluence and termination. [59] Narrowing has been developed by Slagle [89]. Sequential strategies were developed by Huet and Levy [52], who gave a decidable criteria for a subset of sequential systems. This led to the work of Antoy on inductively sequential systems [16]. The needed narrowing strategy came from Hanus, Antoy, and Echahed [17]. Graph rewriting is a bit more disconnected. Currently there isn't a consensus on how to represent graphs mathematically. We went with the presentation in [34], but there are also alternatives in [59, 23, 74]

Here we saw how we can rewrite terms and graphs. We'll use this idea in the next chapter to rewrite entire programs. This will become the semantics for our language. Now that we have some tools, It's time to find out how to make Curry!

Chapter 2

THE CURRY LANGUAGE

The Curry language grew out of the efforts to combine the functional and logic programming paradigms [45]. Originally there were two approaches to combine these paradigms, adding functional features to logic languages, and adding logic features to functional languages. The latter approach was very popular and spawned several new languages including Ciao-Prolog [51], Mercury [90], HAL [33], and Oz [87]. The extension of functional languages lead to fewer new languages, but it did lead to libraries like the logict monad in Haskell [58].

Ultimately the solution came from the work on automated theorem proving [89]. Instead of adding features from one paradigm to another, it was discovered that narrowing was a good abstraction for combining the features from both paradigms. This spawned the Curry [46] and Toy [28] languages.

In this chapter we explore the Curry language syntax and semintics. We give example programs to show how programming in Curry differs from Prolog and Haskell. Then we discuss the choices we made in our implementation compared to previous implementations. Finally we give an example of generated code to demonstrate how we compile Curry programs.

2.1 THE CURRY LANGUAGE

In order to write a compiler for Curry, we need to understand how Curry works. We'll start by looking at some examples of Curry programs. We'll see how Curry programs differ from Haskell and Prolog programs. Then we'll move on to defining a small interpreter for Curry. Finally we'll use this interpreter to define equivalent C code.

Curry combines the two most popular paradigms of declarative programming: Functional languages and logic languages. Curry programs are composed of defining equations like Haskell or ML, but we are allowed to have non-deterministic expressions and free variables like Prolog. This will not be an introduction to modern declarative programming languages. The reader is expected to be familiar with functional languages such as Haskell or ML, and logic languages such as Prolog. For an introduction to programming in Curry see [14]. For an exhaustive explanation of the syntax and semantics of Curry see [49].

To demonstrate the features of Curry, we will examine a small Haskell program to permute a list. Then we will simplify the program by adding features of Curry. This will demonstrate the features of Curry that we need to handle in the compiler, and also give a good basis for how we can write the compiler.

First, let's consider an example of a permutation function. This is not the only way to permute a list in Haskell, and you could easily argue that it's not the most elegant way, but I chose it for two reasons. There is no syntactic sugar, and the only two library functions are *concat* and *map*, both very common functions. and the algorithm for permuting a list is similar to the algorithm we will use in Curry.

$$
\begin{aligned}
&\mathit{perms} && :: [a] \to [[a]] \\
&\mathit{perms}\ [\,] && = [[\,]] \\
&\mathit{perms}\ (x : \mathit{xs}) && = \mathit{concat}\ (\mathit{map}\ (\mathit{insert}\ x)\ (\mathit{perms}\ \mathit{xs})) \\
&\quad \textbf{where} && \\
&\qquad \mathit{insert}\ x\ [\,] && = [[x]] \\
&\qquad \mathit{insert}\ x\ (y : \mathit{ys}) && = (x : y : \mathit{ys}) : \mathit{map}\ (y:)\ (\mathit{insert}\ x\ \mathit{ys})
\end{aligned}
$$

The algorithm itself is broken into two parts. The *insert* function will return a list of lists, where x is inserted into ys at every possible position. For example: *insert* $1\ [2, 3]$ returns $[[1, 2, 3], [2, 1, 3], [2, 3, 1]]$. The *perms* function splits the list into a head x and tail xs. First, it computes all permutations of xs, then it will insert x into every possible position of every permutation.

While this algorithm is not terribly complex, it's really more complex than it needs to be. The problem is that we need to keep track of all of the permutations we generate. This doesn't seem like a big problem here. We just put each permutation in a list, and return the whole list of permutations. However, now every part of the program has to deal with the entire list of results. As our programs grow, we will need more data structures for this plumbing, and this problem will grow too. This is not new. Many languages have spent a lot of time trying to resolve this issue. In fact, several of Haskell's most successful concepts, such as monads, arrows, and lenses, are designed strictly to reduce this sort of plumbing.

We take a different approach in Curry. Instead of generating every possible permutation, and searching for the right one, we will non-deterministically generate a single permutation. This seems like a trivial difference, but its really quite substantial. We offload generating all of the possibilities onto the language itself.

We can simplify our code with the non-deterministic *choice* operator ?. Choice is defined by the rules:

$$x \mathbin{?} y = x$$
$$x \mathbin{?} y = y$$

Now our permutation example becomes a little easier. We only generate a single permutation, and when we insert x into ys, we only insert into a single arbitrary position.

$$
\begin{aligned}
&\mathit{perm} && :: [\,a\,] \to [\,a\,] \\
&\mathit{perm}\ [\,] && = [\,] \\
&\mathit{perm}\ (x : \mathit{xs}) && = \mathit{insert}\ x\ (\mathit{perm}\ \mathit{xs}) \\
&\quad \textbf{where} \\
&\qquad \mathit{insert}\ x\ [\,] && = [\,x\,] \\
&\qquad \mathit{insert}\ x\ (y : \mathit{ys}) && = x : y : \mathit{ys} \mathbin{?} y : \mathit{insert}\ x\ \mathit{ys}
\end{aligned}
$$

In many cases functions that return multiple results can lead to much simpler code. Curry has another feature that's just as useful. We can declare a *free variable* in Curry. This is a variable that hasn't been assigned a value. We can then constrain the value of a variable later in the program. In the following example *begin*, x, and *end* are all free variables, but they're constrained by the guard so that $\mathit{begin} +\!\!+ [\,x\,] +\!\!+ \mathit{end}$ is equal to xs. Our algorithm then becomes: pick an arbitrary x in the list, move it to the front, and permute the rest of the list.

$$
\begin{aligned}
&\mathit{perm} \quad :: [\,a\,] \to [\,a\,] \\
&\mathit{perm}\ [\,] = [\,] \\
&\mathit{perm}\ \mathit{xs} \\
&\quad \mid \mathit{xs}{=}{=}(\mathit{begin} +\!\!+ [\,x\,] +\!\!+ \mathit{end}) = x : \mathit{perm}\ (\mathit{begin} +\!\!+ \mathit{end}) \\
&\quad \textbf{where}\ \mathit{begin}, x, \mathit{end}\ \textbf{free}
\end{aligned}
$$

Look at that. We've reduced the number of lines of code by 25%. In fact, this pattern of declaring free variables, and then immediately constraining them is used so often in Curry that we have syntactic sugar for it. A *functional pattern* is any pattern that contains a function that is not at the root.[1] We can use functional patterns to simplify our *perm* function even further.

$$
\begin{aligned}
&\mathit{perm} && :: [\,a\,] \to [\,a\,]\\
&\mathit{perm}\;[\,] && = [\,]\\
&\mathit{perm}\;(\mathit{begin} +\!\!+ [\,x\,] +\!\!+ \mathit{end}) && = x : \mathit{perm}\;(\mathit{begin} +\!\!+ \mathit{end})
\end{aligned}
$$

Now the real work of our algorithm is a single line. Even better, it's easy to read what this line means. Decompose the list into *begin*, x, and *end*, then put x at the front, and permute *begin* and *end*. This is almost exactly how we would describe the algorithm in English.

There is one more important feature of Curry. We can let expressions fail. In fact we've already seen it, but a more explicit example would be helpful. We've shown how we can generate all permutations of a list by generating an arbitrary permutation, and letting the language take care of the exhaustive search. However, we usually don't need, or even want, every permutation. So, how do we filter out the permutations we don't want? The answer is surprisingly simple. We just let expressions fail. An expression fails if it cannot be reduced to a constructor form. The common example here is *head* [ ], but a more useful example might be sorting a list. We can build a sorting algorithm by permuting a list, and only keeping the permutation that's sorted.

$$
\begin{aligned}
&\mathit{sort} :: (\mathit{Ord}\;a) \Rightarrow [\,a\,] \to [\,a\,]\\
&\mathit{sort}\;\mathit{xs} \mid \mathit{sorted}\;\mathit{ys} = \mathit{ys}\\
&\quad \textbf{where}\\
&\qquad \mathit{ys} = \mathit{perm}\;\mathit{xs}\\
&\qquad \mathit{sorted}\;[\,] \qquad\qquad = \mathit{True}\\
&\qquad \mathit{sorted}\;[\,x\,] \qquad\quad\;\; = \mathit{True}\\
&\qquad \mathit{sorted}\;(x : y : \mathit{ys}) = x \leqslant y \wedge \mathit{sorted}\;(y : \mathit{ys})
\end{aligned}
$$

In this example every permutation of *xs* that isn't sorted will fail in the guard. Once an expression has failed, computation on it stops, and other alternatives are tried. As we'll see later on, this ability to conditionally execute a function will become crucial when developing optimizations.

These are some of the useful programming constructs in Curry. While they are convenient for programming, we need to understand how they work if we are going to implement them in a compiler.

[1]This isn't completely correct. While the above code would fully evaluate the list, a functional pattern is allowed to be more lazy. Since the elements don't need to be checked for equality, they can be left unevaluated.

2.2 SEMANTICS

As we've seen, the syntax of Curry is very similar to Haskell. Functions are declared by defining equations, and new data types are declared as algebraic data types. Function application is represented by juxtaposition, so $f\ x$ represents the function f applied to the variable x. Curry also allows for declaring new infix operators. In fact, Curry really only adds two new pieces of syntax to Haskell, **fcase** and **free**. However, the main difference between Curry and Haskell is not immediately clear from the syntax. Curry allows for overlapping rules and free variables. Specifically Curry programs are represented as *Limited Overlapping Inductively Sequential (LOIS)* Rewrite systems. These are is indicatively sequential systems with a single overlapping rule. On the other hand, Haskell programs are transformed into non-overlapping systems.

To see the difference consider the usual definition of factorial.

$$
\begin{aligned}
&\mathit{fac} :: \mathit{Int} \to \mathit{Int}\\
&\mathit{fac}\ 0 = 1\\
&\mathit{fac}\ n = n * \mathit{fac}\ (n - 1)
\end{aligned}
$$

This seems like an innocuous Haskell program, however It's non-terminating for every possible input for Curry. The reason is that *fac* 0 could match either rule. In Haskell all defining equations are ordered sequentially, which results in control flow similar to the following C implementation.

```
int fac(int n)
{
    if(n == 0)
    {
        return 1;
    }
    else
    {
        return n * fac(n-1);
    }
}
```

In fact, every rule with multiple defining equations follows this pattern. In the following equations let p_i be a pattern and E_i be an expression.

$$f\ p_1 = E_1$$
$$f\ p_2 = E_2$$
$$\ldots$$
$$f\ p_n = E_n$$

Then this is semantically equivalent to the following.

$$f\ p_1 \qquad\qquad\qquad = E_1$$
$$f\ \mathit{not}\ p_1 \wedge p_2 \qquad\quad = E_2$$
$$\ldots$$
$$f\ \mathit{not}\ p_1 \wedge \mathit{not}\ p_2 \wedge p_n = E_n$$

Here $\mathit{not}\ p_i$ means that we don't match pattern i. This ensures that we will only ever reduce to a single expression. Specifically we reduce to the first expression where we match the pattern.

Curry rules, on the other hand, are unordered. If we could match multiple patterns, such as in the case of *fac*, then we non-deterministically return both expressions. This means that $\mathit{fac}\ 0$ reduces to both 1 and $\mathit{fac}\ (-1)$. Exactly how Curry reduces an expression non-deterministically will be discussed throughout this dissertation, but for now we can think in terms of sets. If the expression $e \to e_1$ and $e \to e_2$, $e_1 \to^* v_1$ and $e_2 \to^* v_2$, then $e \to^* \{v_1, v_2\}$.[2]

This addition of non-determinism can lead to problems if we're not careful. Consider the following example:

$$\mathit{coin} = 0\ ?\ 1$$
$$\mathit{double}\ x = x + x$$

We would expect that for any x, $\mathit{double}\ x$ should be an even number. However, if we were to rewrite *double coin* using ordinary term rewriting, then we could have the derivation.

$$\mathit{double}\ \mathit{coin} \Rightarrow \mathit{coin} + \mathit{coin} \Rightarrow (0\ ?\ 1) + (0\ ?\ 1) \Rightarrow 0 + (0\ ?\ 1) \Rightarrow 0 + 1 \Rightarrow 1$$

This is clearly not the derivation we want. The problem here is that when we reduced *double coin*, we made a copy of the non-deterministic expression *coin*. This ability to clone non-deterministic expressions to get different answers is known as run-time choice semantics. [54].

[2]This should really be thought of as a multiset, since it's possible for v_1 and v_2 to be the same value.

The alternative to this is call-time choice semantics. When a non-deterministic expression is reduced, all instances of the expression take the same value. One way to enforce this is to represent expressions as graphs instead of terms. Since no expressions are ever duplicated, all instances of *coin* will reduce the same way. This issue of run-time choice semantics will appear throughout the compiler.

2.2.1 FlatCurry

The first step in the compiler pipeline is to parse a Curry program into FlatCurry. The definition is given in figure 2.1. The FlatCurry language is the standard for representing Curry programs in compilers [50, 25, 24, 20], and has been used to define the semantics of Curry programs [4].

The semantics of Curry have already been studied extensively [4], so we informally recall some of the more important points. A FlatCurry program consists of datatype and function definitions. For simplicity we assume that all programs are self contained, because the module system isn't relevant to our work. However, the Rice compiler does support modules. A FlatCurry function contains a single rule, which is responsible for pattern matching and rewriting an expression. Pattern matching is converted into case and choice expressions as defined in [4]. A function returns a new expression graph constructed out of **let** , **free**, f_k, C_k, ?, l, v expressions.

Our presentation of FlatCurry differs from [4] in three notable ways. First, function and constructor applications contain a count of the arguments they still need in order to be fully applied. The application $f_k\ e_1\ e_2 \ldots e_n$ means that f is applied to n arguments, but it needs k more to be fully applied, so the arity of f is $n + k$. Second, we include **let** $\{v\}$ **free** to represent free variables. This was not needed in [4, 25] because free variables we translated to non-deterministic generators. Since we narrow free variables instead of doing this transformation, we must represent free variables in FlatCurry. Finally, we add an explicit failure expression $\perp$ to represent a branch that is not present in the definitional tree. While this is meant to simply represent a failing computation, we've also occasionally found it useful in optimization.

2.2.2 Evaluation

Each program contains a special function *main* that takes no arguments. The program executes by reducing the expression *main* to a *Constructor Normal Form*[3] as defined in figure 2.2. Similar

[3] This is constructor normal form, and not simply a normal form, because a failing expression, like *head* [], is a normal form, since it can't be rewritten, but it contains a function at the root.

$$
\begin{array}{lll}
f \Rightarrow f \; \{\,v\,\} = e & & \\
e \Rightarrow v & & \textit{Variable} \\
\quad \mid \; l & & \textit{Literal} \\
\quad \mid \; e_1 \;?\; e_2 & & \textit{Choice} \\
\quad \mid \; \bot & & \textit{Failed} \\
\quad \mid \; f_k \; \{\,e\,\} & & \textit{Function Application} \\
\quad \mid \; C_k \; \{\,e\,\} & & \textit{Constructor Application} \\
\quad \mid \; \textbf{let} \; \{\,v = e\,\} \; \textbf{in} \; e & & \textit{Variable Declaration} \\
\quad \mid \; \textbf{let} \; \{\,v\,\} \; \textbf{free in} \; e & & \textit{Free Variable Declaration} \\
\quad \mid \; \textbf{case} \; e \; \textbf{of} \; \{\,p \to e\,\} & & \textit{Case Expression} \\
p \Rightarrow C \; \{\,v\,\} & & \textit{Constructor Pattern} \\
\quad \mid \; l & & \textit{Literal Pattern}
\end{array}
$$

Figure 2.1: Syntax definition for FlatCurry

This is largely the same as other presentations [4, 15] but we have elected to add more information that will become relevant for optimizations later. The notation $\{\,e\,\}$ refers to a variable length list $e_1 \; e_2 \ldots e_n$.

to Kics2, Pakcs, and Sprite, [25, 50, 20] we compute constructor normal form by first reducing the *main* to *Head Constructor Form*. That is where the expression is rooted by a constructor. Then each child of the root is reduced to constructor normal form.

Most of the work of evaluation is reducing an expression to head constructor form. Kics2 and Pakcs are able to transform FlatCurry programs into an equivalent rewrite system, and reduce expressions using graph rewriting [25, 50]. The transformation simply created a new function for every nested case expression. This created a series of tail calls for larger functions.

To see this transformation in action, we can examine the FlatCurry function == on lists 2.3. This function is inductively sequential, however both Pakcs and Kics2 will transform it into a series of flat function calls with a single case at the root. Since this would drastically increase the number of function calls, we avoid this transformation. It would also defeat much of the purpose of an optimizing compiler if we weren't allowed to inline functions.

$$
\begin{array}{lll}
n \Rightarrow l & & \textit{literal} \\
\quad \mid \ C_k \ n_1 \ldots n_k & & \textit{constructor}
\end{array}
$$

Figure 2.2: constructor normal forms in FlatCurry.

A CNF is an expression that contains only constructor and literal symbols. All CNFs are normal forms in our system.

Original FlatCurry representation of == on lists.

$$
\begin{array}{l}
(==)\ v_2\ v_3 = \textbf{case}\ v_2\ \textbf{of} \\
\quad [\,] \to \textbf{case}\ v_3\ \textbf{of} \\
\qquad\qquad\quad [\,] \to \textit{True} \\
\qquad\qquad\quad v_4 : v_5 \to \textit{False} \\
\quad v_6 : v_7 \to \textbf{case}\ v_3\ \textbf{of} \\
\qquad [\,] \to \textit{False} \\
\qquad v_8 : v_9 \to v_6 {==} v_8 \wedge v_7 {==} v_9
\end{array}
$$

Transformed FlatCurry representation of == on lists.

$$
\begin{array}{l}
(==)\ v_2\ v_3 = \textbf{case}\ v_2\ \textbf{of} \\
\quad [\,] \to \textit{eqListNil}\ v_3 \\
\quad v_6 : v_7 \to \textit{eqListCons}\ v_3\ v_6\ v_7 \\
\textit{eqListNil}\ v_3 = \textbf{case}\ v_3\ \textbf{of} \\
\qquad\qquad\qquad\qquad [\,] \to \textit{True} \\
\qquad\qquad\qquad\qquad v_4 : v_5 \to \textit{False} \\
\textit{eqListCons}\ v_3\ v_6\ v_7 = \textbf{case}\ v_3\ \textbf{of} \\
\quad [\,] \to \textit{False} \\
\quad v_8 : v_9 \to v_6 {==} v_8 \wedge v_7 {==} v_9
\end{array}
$$

Figure 2.3: Transformation of FlatCurry == function into a flat representation for Pakcs and Kics2.

$weird = \textbf{let } x = \mathit{False} \mathrel{?} \mathit{True}$

$\quad \textbf{in case } x \textbf{ of}$

$\qquad\qquad \mathit{False} \rightarrow \mathit{True}$

$\qquad\qquad \mathit{True} \rightarrow \mathit{False}$

Figure 2.4: The function *weird*

This can't be expressed as rewrite rules, because the expression we're pattern matching on is defined locally.

2.2.3 Non-determinism

Currently there are three approaches to evaluating non-deterministic expression in Curry: *backtracking*, *Pull-Tabbing*[8], and *Bubbling*[9]. At this time there are complete strategies for evaluating Curry programs, so we have elected to use backtracking. It is the simplest to implement, and it's well understood.

In our system, backtracking is implemented in the usual way. When an expression rooted by a node n with label by f is rewritten to an expression rooted by e, we push the rewrite $(n, n_f, \mathit{Continue})$ onto a backtracking stack, where n_f is a copy of the original node labeled by f. If the expression is labeled by a choice $e_1 \mathrel{?} e_2$, and it is rewritten to the left hand side e_1, then we push $(n, n_?, \mathit{Stop})$ onto the backtracking stack to denote that this was an alternative, and we should stop backtracking.

Unfortunately, while backtracking is well defined for rewriting systems, our representation of FlatCurry programs isn't a graph rewrite system. This is because we don't flatten our FlatCurry functions like Pakcs and Kics2. As an example of why FlatCurry programs aren't a graph rewriting system, consider the FlatCurry function *weird* 2.4. This function defines a local variables x which is used in a case expression. If this were a rewrite system, then we would be able to translate the **case** expression into pattern matching, but a rule can't pattern match on a locally defined variable. We show the reduction of *wierd* in figure 2.5.

We've entered an infinite loop of computing the same rewrite. The problem is that when we're backtracking, and replacing nodes with their original versions, we're going too far back in the computation. In this example, when backtracking *weird*, we want to backtrack to a point where x has been created, and we just want to evaluate the case again.

- We start with a root r labeled by *weird*.
- Node n_1 labeled by ? is created with children $[\mathit{False}, \mathit{True}]$.
- n_1 is rewritten to *False* and $(n_1, \mathit{True}, \mathit{Stop})$ is pushed on the backtracking stack.
- r is rewritten to *True* and $(r, \mathit{weird}, \mathit{Continue})$ is pushed on the backtracking stack.
- r is a constructor normal form.
- backtracking to the closest alternative.
- The backtracking stack is $[(r, \mathit{weird}, \mathit{Continue}), (n_1, \mathit{True}, \mathit{Stop})]$.
- reduce r.
- Node n_2 labeled by ? is created with children $[\mathit{False}, \mathit{True}]$.
- ...

Figure 2.5: Evaluation of the function *weird*.

We solve this problem by creating a new function for each case expression in our original function. Figure 2.6 show an example for *weird* and == which were defined above. This is actually very similar to how Pakcs and Kics2 transformed their programs into rewrite systems by flattening them. The difference is that we don't need to make any extra function calls unless we are already backtracking. There is no efficiency cost in either time or space with our solution. The only cost is a little more complexity in the code generator, and an increase in the generated code size. This seems like an acceptable trade off, since our programs are still similar in size to equivalent programs compiled with GHC.

As far as we're aware, this is a novel approach for improving the efficiency of backtracking in rewriting systems. The correctness of this method follows from the redex contraction theorem, which is proved later.

2.2.4 Free Variables

Free variables are similar to non-deterministic expressions. In fact, in both Kics2 and Sprite [25, 20] they are replaced by non-deterministic generators of the appropriate type [11]. However,

$$
\begin{aligned}
&\mathit{weird} = \textbf{let}\ x = \mathit{False}\ ?\ \mathit{True} \\
&\qquad\quad \textbf{in case}\ x\ \textbf{of} \\
&\qquad\qquad\qquad \mathit{False} \rightarrow \mathit{True} \\
&\qquad\qquad\qquad \mathit{True} \rightarrow \mathit{False}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{weird}_1\ x = \textbf{case}\ x\ \textbf{of} \\
&\qquad\qquad\qquad \mathit{False} \rightarrow \mathit{True} \\
&\qquad\qquad\qquad \mathit{True} \rightarrow \mathit{False}
\end{aligned}
$$

$$
\begin{aligned}
&(==)\ v_2\ v_3 = \textbf{case}\ v_2\ \textbf{of} \\
&\qquad\qquad\qquad [\,] \qquad \rightarrow \textbf{case}\ v_3\ \textbf{of} \\
&\qquad\qquad\qquad\qquad\qquad\qquad [\,] \rightarrow \mathit{True} \\
&\qquad\qquad\qquad\qquad\qquad\qquad v_4 : v_5 \rightarrow \mathit{False} \\
&\qquad\qquad\qquad v_6 : v_7 \rightarrow \textbf{case}\ v_3\ \textbf{of} \\
&\qquad\qquad\qquad\qquad\qquad\qquad [\,] \rightarrow \mathit{False} \\
&\qquad\qquad\qquad\qquad\qquad\qquad v_8 : v_9 \rightarrow v_6{==}v_8 \wedge v_7{==}v_9
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{eqList}_1\ v_3 = \textbf{case}\ \ v_3\ \textbf{of} \\
&\qquad\qquad\qquad [\,] \rightarrow \mathit{True} \\
&\qquad\qquad\qquad v_4 : v_5 \rightarrow \mathit{False}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{eqList}_2\ v_6\ v_7\ v_3 = \textbf{case}\ v_3\ \textbf{of} \\
&\qquad\qquad\qquad\quad [\,] \rightarrow \mathit{False} \\
&\qquad\qquad\qquad\quad v_8 : v_9 \rightarrow v_6{==}v_8 \wedge v_7{==}v_9
\end{aligned}
$$

Figure 2.6: Functions at case for *weird* and == for lists.

$$\textbf{data}\ \mathit{Light} = \mathit{Red} \mid \mathit{Yellow} \mid \mathit{Green}$$

$$\begin{aligned}
&\mathit{change}\ x = \textbf{case}\ x\ \textbf{of} \\
&\qquad \mathit{Red} \quad\ \to \mathit{Green} \\
&\qquad \mathit{Green} \ \to \mathit{Yellow} \\
&\qquad \mathit{Yellow} \to \mathit{Red}
\end{aligned}$$

Figure 2.7: A simple traffic light program

in Rice, free variables are instantiated by narrowing. If a free variable is the scrutinee of a case expression, then we push copies of the remaining patterns onto the stack along with another copy of the variable. If the free variable is replaced by a constructor with arguments, such as *Just*, then we instantiate the arguments with free variables.

This is easier to see with an example. Consider the traffic light function in figure 2.7. The *change* function moves the light from *Red* to *Green* to *Yellow*. When calling this function with a free variable, we have the derivation below in figure 2.8.

2.2.5 Higher Order Functions

Now that we have a plan for the logic features of Curry, we move on to higher order functions. This subject has been extensively studied by the function languages community, and we take the approach of [77]. Higher order functions are represented using defunctionalization [86]. Recall that in FlatCurry, an expression f_k represents a partial application that is missing k arguments. We introduce an *apply* function that has an unspecified arity. where $\mathit{apply}\ f_k\ e_1\ e_2 \ldots e_n$ applys f_k to the arguments $e_1\ e_2 \ldots e_n$.

The behavior of *apply* is specified below.

$$\begin{aligned}
&\mathit{apply}\ f_k\ x_1 \ldots x_n \\
&\quad \mid k > n \ \ = f_{k-n}\ x_1 \ldots x_n \\
&\quad \mid k == n = f\ x_1 \ldots x_n \\
&\quad \mid k < n \ \ = \mathit{apply}\ (f\ x_1 \ldots x_k)\ x_{k+1} \ldots x_n
\end{aligned}$$

If the first argument f of *apply* is not partially applied, then evaluate f until it is, and proceed as above. In the case that f is a free variable, then we return $\perp$, because we don't support higher

- We start with root r labeled by *change*, with a child x labeled by *free*.
- x is rewritten to *Red* and $(x, \mathit{Green}, \mathit{Stop}), (x, \mathit{Yellow}, \mathit{Stop}), (x, \mathit{free}, \mathit{Continue})$ are all pushed on the stack
- r is rewritten to *Green*, and $(r, \mathit{change}, \mathit{Continue})$ is pushed on the stack
- r is a constructor normal form
- backtracking to the closest alternative
- backtracking stack is $[(r, \mathit{change}, \mathit{Continue}), (x, \mathit{Green}, \mathit{Stop}), (x, \mathit{Yellow}, \mathit{Stop}), (x, \mathit{free}, \mathit{Continue})]$.
- reduce r
- x is labeled by *Green*
- r is rewritten to *Yellow*, and $(r, \mathit{change}, \mathit{Continue})$ is pushed on the stack
- r is a constructor normal form
- backtracking to the closest alternative
- backtracking stack is $[(r, \mathit{change}, \mathit{Continue}), (x, \mathit{Yellow}, \mathit{Stop}), (x, \mathit{free}, \mathit{Continue})]$
- reduce r
- x is labeled by *Yellow*
- r is rewritten to *Red*, and $(r, \mathit{change}, \mathit{Continue})$ is pushed on the stack
- r is a constructor normal form
- backtracking to the closest alternative
- backtracking stack is $[(r, \mathit{change}, \mathit{Continue}), (x, \mathit{free}, \mathit{Continue})]$
- The stack is empty, and there are no alternatives.

Figure 2.8: Evaluation of *change* x **where** x **free**

order narrowing.

2.2.6 Backtracking Performance

Now that we've established a method for implementing non-determinism, we would like to improve the performance. Currently we push nodes on the backtracking stack for every rewrite. Often, we don't need to push most of these rewrites. Consider the following code for computing Fibonacci numbers:

$$
\begin{aligned}
&\mathit{fib}\ n = \textbf{case}\ n < 2\ \textbf{of} \\
&\qquad\qquad\quad \mathit{True}\ \to n \\
&\qquad\qquad\quad \mathit{False} \to \mathit{fib}\ (n-1) + \mathit{fib}\ (n-2) \\
&\mathit{main} = \textbf{case}\ \mathit{fib}\ 20{=}{=}(1\ ?\ 6765)\ \textbf{of} \\
&\qquad\qquad\quad \mathit{True} \to \mathit{putStrLn}\ \texttt{"found answer"}
\end{aligned}
$$

This program will compute *fib* 20, pushing all of those rewrites onto the stack as it does, and then, when it discoverers that $\mathit{fib}\ 20 \not\equiv 1$, it will undo all of those computations, only to redo them immediately afterwards! This is clearly not what we want. Since *fib* is a deterministic function, we would like to avoid pushing these rewrites onto the stack. Unfortunately, this isn't as simple as it would first seems for two reasons. First, determining if a function is non-deterministic in general is undecidable, so any algorithm we developed would push rewrites for some deterministic computations. Second, a function may have a non-deterministic argument. For example, we could easily change the above program to:

$$
\begin{aligned}
&\mathit{main} = \textbf{case}\ \mathit{fib}\ (1\ ?\ 20){=}{=}6765\ \textbf{of} \\
&\qquad\qquad\quad \mathit{True} \to \mathit{putStrLn}\ \texttt{"found answer"}
\end{aligned}
$$

Now the expression with *fib* is no longer deterministic. We sidestep the whole issue by noticing that while it's impossible to tell if an expression is non-deterministic at compile time, it's very easy to tell if it is at run time.

As far as we're aware, this is another novel solution. Each expression contains a Boolean flag that marks if it is non-deterministic. We called these *nondet* flags, and we refer to an expression whose root node is marked with a nondet flag as nondet. The rules for determining if an expression e is nondet are: if e is labeled by a choice, then e is nondet; if e is labeled by a function that has a case who's scrutinee is nondet, or is a forward to a nondet, then e is nondet; if $e' \to^* e$ and e' is nondet, then e is nondet.

Any node not marked as nondet doesn't need to be pushed on the stack because it's not part of a choice, all of its case statements scrutinized deterministic nodes, and it's not forwarding to a non-deterministic node. However proving this is a more substantial problem.

We prove this for the class of limited overlapping inductively sequential graph rewriting systems, with the understanding our system is equivalent. This proof is based on a corresponding proof for set functions in Curry [12][Lemma 2]. The original proof was concerned with a deterministic derivation from an expression to a value. While the idea is similar, we don't want to necessarily derive an expression to a value. Instead we define a deterministic redex, and deterministic step below, and show that there is an analogous theorem for a derivation of deterministic steps, even if it doesn't compute a value.

Definition 2.2.1. Given a rewrite system R with fixed strategy ϕ, a *computation space* [12] of expression e, $C(e)$ is finitely branching tree defined inductively the rule $C(e) = \langle e, C(e_1), C(e_2) \dots C(e_n) \rangle$.

We now need the notions of a deterministic redex and a deterministic rewrite. Ultimately we want to show that if we have a deterministic reduction, then we can perform that computation at any point without affecting the results. One implication of this would be that performing a deterministic computation before a non-deterministic choice was made would be the same as performing the computation after the choice. This would justify our fast backtracking scheme, because it would be equivalent to performing the computation before the choice was made.

Definition 2.2.2. A redex n in expression e is deterministic if there is at most one rewrite rule that could apply to $e|_n$. A rewrite $e \rightarrow_n e'$ is deterministic if n is a deterministic redex.

Next we rephrase our notion of nondet for a LOIS system.

Definition 2.2.3. let $e \rightarrow e_1 \rightarrow \dots v$ be a derivation for e to v. A node n in e_i is *nondet* iff

1. n is labeled by a choice.
2. A node in the redex pattern [10] of n is *nondet*.
3. There exists some $j < i$ where n is a subexpression of e_j and n is *nondet*.

The first property is that all choice nodes are nondet. The second property is equivalent to the condition that any node that scrutinizes a nondet node should be nondet. Finally, the third property is that nondet should be a persistent attribute. This corresponds to the definition we gave for nondet nodes above.

If n is a redex that isn't marked as nondet, then n can't be labeled by a choice. Since choice is the only rule in a LOIS system that is non-deterministic, n must be a deterministic redex. We recall a theorem used to prove the correctness of set function.[12][Def. 1, Lemma 1]

Lemma 4. *Given an expression e where $e \rightarrow_{n_1} e_1$ and $e \rightarrow_{n_2} e_2$, if $n_1 \neq n_2$, then there exists a u_1 and u_2 where $t_1 \rightarrow^= u_1$ and $t_2 \rightarrow^= u_2$ and $u_1 = u_2$ up to renaming of nodes.*

This leads directly to our first important theorem. If n is a deterministic redex in a derivation, then we can move it earlier in the derivation.

Theorem 5 (Redex Compression Theorem)**.** *if n is a deterministic redex of e where $n \rightarrow n'$, and $e \rightarrow e_1 \rightarrow_n e_2$. Then there exists a derivation $e[n \rightarrow n'] \rightarrow^= e'$ where $e_2 = e'$ up to renaming of nodes.*

Proof. By definition of rewriting $e \rightarrow_n e[n \rightarrow n']$. Since n is a deterministic redex, it must be the case that the redex in $e \rightarrow e_1$ was not n. So by the previous lemma, we can swap the order of the rewrites. $\square$

Finally we show that if a is a subexpression of e and $a \rightarrow^* b$ using only deterministic redexes, then $e[a \rightarrow b]$ rewrites to the same values.

Theorem 6 (Path Compression Theorem)**.** *if a is a subexpression of e and $a \rightarrow^* b$ using only deterministic rewrites, and $e \rightarrow e_1 \rightarrow \ldots e_n$ is a derivation where b is a subexpression of e_n, then there is a derivation $e[a \rightarrow b] \rightarrow^* e_n$.*

Proof. This follows by induction on the length of the derivation. In the base case $a = b$, and there is nothing to prove. In the inductive case $a \rightarrow_p a' \rightarrow^* b$. Since $a \rightarrow_p a'$ is deterministic by assumption, we can apply the path compression theorem and say that $e[a \rightarrow a'] \rightarrow^* e_n$. By the inductive hypotheses we can say that $e[a \rightarrow a'][a' \rightarrow b] \rightarrow^* e_n$. Therefore $e[a \rightarrow b] \rightarrow^* e_n$. This establishes our result. $\square$

2.2.7 Collapsing Functions

While this result is great, and it allows us to avoid creating a large number of stack frames, there's a subtle aspect of graph rewriting that gets in the way. If a node n_1 labeled by function f is rewritten to n_2, then the definition of applying a rewrite rule [34][Def. 8, Def. 10, Def. 19] would require us to traverse the graph, and find every node that has n_1 as a child, and redirect that pointer to n_2. This is clearly inefficient, so this isn't done in practice. A much faster method is to simply replace the contents, the label and children, of n_1 with the contents of n_2. This works most of the time, but we run into a problem when function a rewrites to a single variable, such as the *id* function. We call these functions *collapsing functions*. One option to solve this problem is to evaluate the contractum to head constructor form, and copy the constructor to the root [62]. This is commonly used in lazy functional languages, however it does not work for Curry programs. Consider the expression following expression.

$$
\begin{aligned}
& f = \textbf{let } x = \mathit{True} \mathbin{?} \mathit{False} \\
& \quad y = \mathit{id}\ x \\
& \quad \textbf{in } \mathit{not}\ y
\end{aligned}
$$

When y is evaluated, then it will evaluate x, and x will evaluate to *True*. If we then copy the *True* constructor to y, then we have two copies of *True*. But, since y is deterministic, we don't need to undo y when backtracking. So, y will remain *True* after backtracking, instead of returning to $id\ x$. While constructor copying is definitely invalid with fast backtracking, it's unclear if it would be valid with a normal backtracking algorithm.

We can solve this problem by using forwarding nodes, sometimes called indirection nodes [82]. The idea is that when we rewrite an expression rooted by a collapsing function, instead of copying the constructor, we just replace the root with a special forwarding node, $FORWARD(x)$, where x is the variable that the function collapses to.

There is one more possibility to address before we move on. One performance optimization with forwarding nodes is *path compression*. If we have a chain of forwarding nodes $FORWARD(FORWARD(FORWARD($ we want to collapse this to simply $FORWARD(x)$. This is unequivocally invalid in non-deterministic backtracking systems. Consider the following function.

$$
\begin{aligned}
& f = \textbf{let } x = \mathit{True} \mathbin{?} \mathit{False} \\
& \qquad\quad y = \mathit{id}\ x \\
& \qquad \textbf{in } \ \textbf{case } y \textbf{ of} \\
& \qquad\qquad\qquad \mathit{False} \rightarrow \textbf{case } x \textbf{ of} \\
& \qquad\qquad\qquad\qquad\qquad \mathit{False} \rightarrow ()
\end{aligned}
$$

When reducing this function, we create two forwarding nodes that are represented by the variables x and y. We refer to these nodes as $FORWARD_x$ and $FORWARD_y$ respectively. While evaluating y in the case expression, we create two forwarding nodes that are represented by the variables

x and y respectively. We refer to these nodes as $FORWARD_x$ and $FORWARD_y$ respectively. So x is reduced to $FORWARD_x(True)$, and y is reduced to $FORWARD_y(FORWARD_x(True))$. If we contract y to $FORWARD_y(True)$, then when we backtrack we replace x with $FORWARD_x(False)$, and y is replaced with $id(True)$. The reason that y doesn't change to $id(False)$ is because y has lost its reference to x. Now, not only do we fail to find a solution for f, we've ended up in a state where x and y have different values.

2.3 GENERATED CODE

Now that we've examined all of the different choices to make in constructing a compiler, we can start to design the generated code and runtime system for the compiler. In this section we give examples of generated code to implement Curry functions, and discuss the low level details of the Rice runtime. We start with a first order deterministic subset of Curry, then we add higher order function, finally we add non-determinism and free variables. Throughout this section we will use `teletype font` to represent generated C code to distinguish it from Curry or FlatCurry code.

We will introduce the generated code by looking at the *not* function defined below. We choose this function, because it's small enough to be understandable, but it still demonstrates most of the decisions in designing the generated code and runtime system.

$$
\begin{aligned}
&\mathit{not}\ x = \textbf{case}\ x\ \textbf{of} \\
&\qquad\qquad \mathit{False} \rightarrow \mathit{True} \\
&\qquad\qquad \mathit{True}\ \ \rightarrow \mathit{False}
\end{aligned}
$$

Before we discuss generated code, we need to discuss expressions and the runtime system for programs.

When a FlatCurry module is compiled, it's translated into a C program. Every function f defined in the FlatCurry module is compiled into a C function that can reduce an expression, rooted by a node labeled with f, into head constructor form. These functions are called `f_hnf` for historical reasons [50].

An expression in our compiled code is a rooted labeled graph. nodes in the graph are given the following definition.

```
typedef struct Node
```

```
{
    int missing;
    bool nondet;
    Symbol* symbol;
    field children[4];
} Node;
```

A `field` is a union of a `Node*` and the representations of the primitive types `Int`, `Float`, and `Char`, as well as a `field*` to be described shortly. The `children` field contains an array of children for this node. If a node could have more than three children, such as a node representing the (, , , , , ,) constructor, then `children[3]` holds a pointer to a variable length array that holds the rest of the children. This leads to non-uniform indexing into nodes. For example `n->children[1]` returns the second child of the node, but the sixth child must be retrieved with `n->children[3].a[2]`. We use a `child_at` macro to simplify the code, so `child_at(n,5)` returns the sixth child. The `symbol` field is a pointer to the static information of the node. This includes the name, arity, and `tag` for the node, as well as a function pointer responsible for reducing the node to head constructor from. We include a `TAG` macro to access the tag of a node. This is purely for convenience. For a node labeled by function f, this is a pointer to `f_hnf`. Because the calling convention is complicated, we hide this detail with an `HNF` macro, so `HNF(f)` evaluated the node labeled by f to head constructor from. The `missing` field represents a partial function application. If `missing` is greater than 0, then `f` is partially applied. The `nondet` field represents the nondet marker described in the fast backtracking algorithm.

Each function and constructor generates a `set` and `make` function. For the *not* function, we would generate

```
void set_not(field root, field x);
field make_not(field x);
```

The `set_not` function sets the `root` parameter to be a *not* node. This is accomplished by changing the symbol and children for `root`. The `make_not` function allocates memory for a new *not* node.

Each program in our language defines an expression *main*, and runs until *main* is evaluated to constructor normal form. This evaluation is broken up into two pieces. The primary driver of a program is the `nf` function, which is responsible for evaluating the main expression to constructor normal form. The `nf` function computes this form by first evaluating an expression to head constructor form. When an expression is in head constructor from, `nf` evaluates each subexpression to constructor normal from, producing the following execution loop:

```
void nf(field expr)
{
    HNF(expr);

    for(int i = 0; i < expr.n->symbol->arity; i++)
    {
        nf(child_at(expr, i));
    }
}
```

All that's missing here is the `hnf` functions. We give a simplified version of the `not_hnf` function in 2.9, and we will fill in details as we progress.

We can see that the main driver of this function is the `while(true)` loop. The loop looks up the tag of `x`, and if it's a function tag, when we evaluate it to head constructor form. If the tag for `x` is `FAIL`, which represents an exempt node, then we set the root to `FAIL` and return. If the tag is `Prelude_True` or `Prelude_False`, we set the root to the corresponding expression, and return from the loop. Finally, in order to implement collapsing functions, we introduce a `FORWARD` tag. If the tag is `FORWARD`, then we traverse the forwarding chain, and continue evaluating the `x`.

Finally, while we are evaluating the node stored in the local variable `x`, we introduce a new variable `scrutenee`. This is because if `x` evaluates to a forwarding node, we need to evaluate the child of `x`. If we were to update `x`, and then return an expression containing `x` later, then we would have compressed the forwarding path. As mentioned previously, this is not valid.

At this point we have a strategy for how to compile first order deterministic Curry functions. Next we show how we handle partial application and higher order functions.

2.3.1 Higher Order Functions

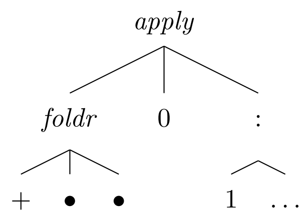
Earlier we gave an interpretation of how to handle *apply* nodes, but there are still a few details to work out. Recall the semantics we gave for apply nodes:

```
void Prelude_not_hnf(field root)
{
  field x = child_at(root, 0);

  field scrutenee = x;
  while(true)
  {
    switch(TAG(scrutenee))
    {
      case FAIL_TAG:
      {
        fail(root);
        return;
      }
      case FORWARD_TAG:
      {
        scrutenee = child_at(scrutenee,0);
        break;
      }
      case FUNCTION_TAG:
      {
        HNF(scrutenee);
        break;
      }
      case Prelude_True_TAG:
      {
        set_Prelude_False(root, 0);
        return;
      }
      case Prelude_False_TAG:
      {
        set_Prelude_True(root, 0);
        return;
      }
    }
  }
}
```

$$
\begin{aligned}
&\mathit{apply}\ f_k\ [x_1, \ldots x_n] \\
&\quad \mid k > n\ \ = f_{k-n}\ x_1 \ldots x_n \\
&\quad \mid k{=}{=}n = f\ x_1 \ldots x_n \\
&\quad \mid k < n\ \ = \mathit{apply}\ (f\ x_1 \ldots x_k)\ [x_{k+1}, \ldots x_n]
\end{aligned}
$$

If f is missing any arguments, then we call f a partial application. Let's look at a concrete example. In the expression $foldr_2\ (+_2)$, *foldr* is a partial application that is missing 2 arguments. We will write this as *foldr* $(+_2) \bullet \bullet$ where $\bullet$ denotes a missing argument. Now, suppose that we want to apply the following expression.



Remember that each node represents either a function or a constructor, and each node has a fixed arity. For example, + has an arity of 2, and *foldr* has an arity of 3. This is true for every + or *foldr* node we encounter. However, it's not true for *apply* nodes. In fact, an *apply* node may have any positive arity. Furthermore, by definition, an *apply* node can't be missing any arguments. For this reason, we use the `missing` field to hold the number of arguments the node is applied to.[4]

The algorithm for reducing apply nodes is straightforward, but brittle. There are several easy mistakes to make here. The major problem with function application is getting the arguments in the correct positions. To help alleviate this problem we make a non-obvious change to the structure of nodes. We store the arguments in reverse order. To see why this is helpful, let's consider the *foldr* example above. But this time, let's decompose it into 3 apply nodes, so we have $\mathit{apply}\ (\mathit{apply}\ (\mathit{apply}\ foldr_3\ (+_2))\ 0)\ [1, 2, 3]$. In our innermost apply node, which will be evaluated first, we apply $foldr_3$ to $+_2$ to get $foldr_2\ (+_2) \bullet \bullet$. This is straightforward. We simply put + as the first child. However, when we apply $foldr_2\ (+_2) \bullet \bullet$ to 0, we need to put 0 in the second child slot. In general, when we apply an arbitrary partial application f to x, what child do we put x in? Well, if we're storing the arguments in reverse order, then we get a really handy result. Given function f_k that is missing k arguments, then $\mathit{apply}\ f_k\ x$ reduces to $f_{k-1}\ x$ where x

[4]In reality we set missing to the negative value of the arity to distinguish an apply node from a partial application.

is the $k-1$ child. The missing value for a function tells us exactly where to put the arguments. This is completely independent of the arity of the function.

$$
\begin{aligned}
&\mathit{apply}\ (\mathit{apply}\ (\mathit{apply}\ (foldr_3 \bullet \bullet\bullet)\ (+_2))\ 0)\ [1,2,3] \\
&\Rightarrow \mathit{apply}\ (\mathit{apply}\ (foldr_2 \bullet \bullet (+_2))\ 0)\ [1,2,3] \\
&\Rightarrow \mathit{apply}\ (foldr_1 \bullet 0\ (+_2))\ [1,2,3] \\
&\Rightarrow foldr_0\ [1,2,3]\ 0\ (+_2))
\end{aligned}
$$

The algorithm is given in figure 2.10. There are a few more complications to point out. To avoid complications, we assume arguments that a function is being applied to are stored in the array at `children[3]` of the apply node. That gives us the structure $\mathit{apply}\ f \bullet \bullet a_n \ldots a_1$. This isn't done in the runtime system because it would be inefficient, but it simplifies the code for this presentation. We also make use of the `set_child_at` macro, which simplifies setting child nodes and is similar to `child_at`. Finally, the loop to put the partial function in head constructor form uses `while(f.n->missing <= 0)` instead of `while(true)`. This is because our normal form is a partial application, which does not have its own tag.

We reduce an apply node in two steps. First get the function `f`, which is the first child of an apply node. Then, reduce it to a partial application. If `f` came from a non-deterministic expression, the save the apply node on the stack. We split the second step into two cases. If `f` is under applied, or has exactly the right number of arguments, then copy the contents of `f` into the root, and move the arguments over and reduce. If `f` is over applied, then make a new copy of `f`, and copy arguments into it until it's fully applied. Finally we reduce the fully applied copy of `f` and apply the rest of the arguments.

2.3.2 Implementing Non-determinism

Now, we that we can reduce a higher order functional language, we would like to extend our implementation to handle features from logic languages.

The implementation doesn't change too much. First we add two new tags `CHOICE` and `FREE` to represent non-deterministic choice and free variable nodes respectively. The choice nodes are treated in a similar manner to a function. We call the `choose` function to reduce a choice to HCF, and push the alternative on the stack.

The choose function in 2.11 reduces a choice node to head constructor form. Since choice is a collapsing rule, we return a forwarding node. The function is also responsible for keeping track

```
void apply_hnf(field root)
{
    field f = child_at(root,0);
    field* children = root.n->children[3].a;

    while(f.n->missing <= 0)
    {
        // Normal HNF loop
    }

    int nargs = -root.n->missing;
    int missing = f.n->missing;

    if(missing <= nargs)
    {
        set_copy(root, f);
        for(int i = nargs; i > 0; i--, missing--)
        {
            set_child_at(root, missing-1, children[i-1]);
        }

        root.n->missing = missing;

        if(missing == 0)
        {
            HNF(root);
        }
    }
    else
    {
        field newf = copy(f);

        while(missing > 0)
        {
            set_child_at(newf, missing-1, children[nargs-1]);
            nargs--;
            missing--;
        }
```

```
void choose(field root)
{
    field choices[2] = {child_at(root,0), child_at(root,1)};
    int side = child_at(root,2).i;

    field saved;
    saved.n = (Node*)alloc(sizeof(Node));
    memcpy(saved.n, root.n, sizeof(Node));

    child_at_i(saved,2) = !side;
    stack_push(bt_stack, root, saved, side == 0);

    set_forward(root,choices[side]);
    root.n->nondet = true;

}
```

Figure 2.11: Implementation of the `choose` function.

of which branch of the choice we should reduce, and pushing the alternative on the backtracking stack. We accomplish this by keeping a marker in the second child of a choice node. This marker is 0 if we should reduce to the left hand side, and 1 if we should reduce to the righ hand side.

Free variables are more interesting. To narrow a free variable we pick a possible constructor, and replace the `scrutinee` node with that constructor. All arguments to the constructor are instantiated with free variables. Then, we push a rewrite on the stack to replace `scrutinee` with a free variable using the `push_frame` function. This is because after each possible choice has been exhausted, we want to reset this node back to a free variable in case it is used in another non-deterministic branch of the computation. Finally, for every other constructor, we push an alternative on the backtracking stack using the `push_choice` function.

The only other necessary change is to push a rewrite onto the backtracking stack when we reach either a fail, or constructor case. The `Prelude_not_1` function is a function at a case expression discussed in section 2.2.3. The changes to the *not* function are give in figure 2.12.

Due to space constraints not all sections are show. The pieces of code that don't changed are omitted and replaced with `...`.

2.3.3 Fast Backtracking

Finally we show how we implement the fast backtracking technique described earlier. The implementation actually doesn't change much, we simply make use of the `nondet` flag in each node. While we're evaluating `scrutinee`, we keep track of whether or not we've seen a non-deterministic node in a local variable, and if we have, we push the root on the backtracking stack. If we haven't seen a non-deterministic node, then we can simply avoid pushing this rewrite. The generated code for *not* is given in figure 2.13.

In this chapter we've discussed the Curry language, and overviewed the semantics of Curry programs. We've shown different approaches to implementing a system for running Curry programs, and we've discussed the choices we've made. When a decision needed to be made, we prioritized correctness, then efficient execution, and then ease of implementation.

In some sense, we've given a recipe of how to translate Curry into C. In the next chapter we introduce the tools to make this recipe. We introduce a system for implementing transformations as rewrite rules. We then show how this system can simplify the construction of a compiler, and use it to transform FlatCurry programs into a form that is easier to optimize and compile to C.

```
void Prelude_not_hnf(field root) {
  field x = child_at(root, 0);
  field scrutenee = x;
  while(true) {
    switch(TAG(scrutenee)) {
      case FAIL_TAG:
        push_frame(root, make_Prelude_not_1(x));
        fail(root);
        return;
      ...
      case CHOICE_TAG:
        choose(scrutenee);
        break;

      case FREE_TAG:
        push(bt_stack(free_var())choose(scrutenee);
        push_frame(scrutenee, free_var());
        push_choice(scrutenee, make_Prelude_False(0));
        set_Prelude_True(scrutenee, 0);
        break;

      case Prelude_True_TAG:
        push_frame(root, make_Prelude_not_1(x));
        set_Prelude_False(root, 0);
        return;
      ...
    }
  }
}
```

Figure 2.12: implementation of *not* with possible non-deterministic values.

```
void Prelude_not_hnf(field root) {
  field x = child_at(root, 0);
  field scrutenee = x;
  bool nondet = false;

  while(true) {
    nondet |= scrutenee->nondet;
    switch(TAG(scrutenee)) {
      case FAIL_TAG:
        if(nondet) push_frame(root, make_Prelude_not_1(x));
        fail(root);
        return;

      ...
      case CHOICE_TAG:
        choose(scrutenee);
        nondet = true;
        break;

      case FREE_TAG:
        push_frame(scrutenee, free_var());
        push_choice(scrutenee, make_Prelude_False(0));
        set_Prelude_True(scrutenee, 0);
        nondet = true;
        break;

      case Prelude_True_TAG:
        if(nondet) push_frame(root, make_Prelude_not_1(x));
        set_Prelude_False(root, 0);
        return;

      ...
    }
  }
}
```

Figure 2.13: implementation of *not* with possible non-deterministic values.

Chapter 3

GENERATING AND ALTERING SUBEXPRESSIONS

In this chapter we introduce our engine for Generating and Altering Subexpressions, of the GAS system. This system proves to be incredibly versatile and is the main workhorse of the compiler and optimizer. We show how to construct, combine, and improve the efficiency of transformations, as well as how the system in implemented. We then show an extended example of using the GAS system to transform FlatCurry programs into a canonical form so that we can compile them to C code, as discussed in the last chapter.

3.0.1 Building Optimizations

Optimization is usually considered the most difficult aspect of writing a modern compiler. It's easy to see why. There are dozens of small optimizations to make, and each one needs to be written, shown correct, and tested.

Furthermore, there are several levels where an optimization can be applied. Some optimizations apply to a programs AST, some to another intermediate representation, some to the generated code, and even some to the runtime system. There are even optimizations that are applied during transformations between representations. For this chapter, we will be describing a system to apply optimizations to FlatCurry programs. While this is not the only area of the compiler where we applied optimizations, it is by far the most extensive, so it's worth understanding how our optimization engine works.

Generally speaking, most optimizations have the same structure. Find an area in the AST where the optimization applies, and then replace it with the optimized version. As an example, consider the code for the absolute value function defined below.

$$
\begin{aligned}
&\mathit{abs}\ x \\
&\quad \mid x < 0 \qquad\ \ = -x \\
&\quad \mid \mathit{otherwise} = x
\end{aligned}
$$

This will be translated into FlatCurry as

$$
\begin{aligned}
abs\ x = \textbf{case}\ (x < 0)\ \textbf{of} & \\
& \mathit{True} \rightarrow -x \\
& \mathit{False} \rightarrow \textbf{case}\ \mathit{otherwise}\ \textbf{of} \\
& \qquad\qquad \mathit{True} \rightarrow x \\
& \qquad\qquad \mathit{False} \rightarrow \bot
\end{aligned}
$$

While this transformation is obviously inefficient, it is general and has a straightforward implementation. A good optimizer should be able to recognize that *otherwise* reduces to *True*, and reduce the case-expression. So for this one example, we have two different optimizations we need to implement. We need to reduce *otherwise* to *True*, then we can reduce the second case expression to x.

There are two common approaches to solving this problem. The first is to make a separate function for each optimization. Each function will traverse the AST and try to apply its optimization. The second option is to make a few large functions that attempt to apply several optimizations at once. There are trade-offs for each.

The first option has the advantage that each optimization is easy to write and understand. However, is suffers from a lot of code duplication, and it's not very efficient. We must traverse the entire AST every time we want to apply an optimization. Both LLVM and the JVM fall into this category. [63, 75] The second option is more efficient, and there is less code duplication, but it leads to large functions that are difficult to maintain or extend.

Using these two options generally leads to optimizers that are difficult to maintain. To combat this problem, many compilers will provide a language to describe optimization transformation. Then the compiler writer can use this domain specific language to develop their optimizations. With the optimization descriptions, the compiler can search the AST of a program to find any places where optimizations apply. However, It is difficult or impossible to write many common optimizations in this style. [81]

The aim of our solution is to try to get the best of all three worlds. We've developed an approach to simplify Generating and Altering Subexpressions (GAS) . Our approach was to do optimization entirely by rewriting. This has several advantages, and might be the most useful result of this work. First, developing new optimizations is simple. We can write down new optimizations in this system within minutes. It was often easier to write down the optimization and test it, than it was to try to describe the optimization in English. Second, any performance improvement we made to the optimization engine would apply to every optimization. Third, optimizations were easy to maintain and extend. If one optimization didn't work, we could look at it and test it in isolation. Fourth, this code is much smaller than a traditional optimizer. This isn't really a fair comparison given the relative immaturity of our compiler, but we were able to implement 16 optimizations and code transformations in under 150 lines of code. This gives a sense of scale of how much easier it is to implement optimizations in this system. Fifth, Since We're optimizing by rewrite rules, the compiler can easily output what rule was used, and the position where it was used. This is enough information to entirely reconstruct the optimization derivation. We found this very helpful in debugging. Finally, optimizations are written in Curry. We didn't need to develop a DSL to describe the optimizations, and there are no new ideas for programmers to learn if they want to extend the compiler.

We should note that there are some potential disadvantages to the GAS system as well. The first disadvantage is that there are some optimizations and transformations that are not easily described by rewriting. The second is that, while we've improved the efficiency of the algorithm considerably, it still takes longer to optimize programs then we'd like.

The first problem isn't really a problem at all. If there is an optimization that doesn't lend itself well to rewriting, we can always write it as a traditional optimization. Furthermore, as we'll see later, we don't have to stay strictly in the bounds of rewriting. The second problem is actually more fundamental to Curry. Our implementation relies on finding a single value from a set generated by a non-deterministic function. Current implementations are inefficient, but there are new implementations being developed. [1] We also believe that an optimizing compiler would help with this problem [66].

3.0.2 The Structure of an Optimization

The goal with GAS is to make optimizations simple to implement and easily readable. While this is a challenging problem, we can actually leverage Curry here. Remember that the semantics of Curry are already non-deterministic rewriting.

Each optimization is going to be a function from a FlatCurry expression to another FlatCurry expression.

$$\textbf{type}\ \mathit{Opt} = \mathit{Expr} \to \mathit{Expr}$$

For readability we use the FlatCurry syntax defined in figure 2.1, While this version of FlatCurry is easier to read, we will need the actual representation of FlatCurry programs to implement the compiler. This representation is given in figure 3.1, and the transformation from the FlatCurry syntax to the FlatCurry representation is given in figure 3.2. We can describe an optimization by simply describing what it does to each expression. As an example consider the definition for floating let-expressions:

$$\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [\mathit{Let}\ \mathit{vs}\ e] \mathbin{+\!\!+} \mathit{bs}) = \mathit{Let}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [e] \mathbin{+\!\!+} \mathit{bs}))$$

This optimization tells us that, if an argument to a function application is a **let** expression, then we can move the let-expression outside. This works for let-expressions, but what if there's a free variable declaration inside of a function? Well, we can define that case with another rule.

$$\begin{aligned}
&\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [\mathit{Let}\ \mathit{vs}\ e] \mathbin{+\!\!+} \mathit{bs}) &&= \mathit{Let}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [e] \mathbin{+\!\!+} \mathit{bs}))\\
&\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [\mathit{Free}\ \mathit{vs}\ e] \mathbin{+\!\!+} \mathit{bs}) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [e] \mathbin{+\!\!+} \mathit{bs}))
\end{aligned}$$

This is where the non-determinism comes in. Suppose we have an expression:

$$f\ (\mathbf{let}\ x = 1\ \mathbf{in}\ x)\ (\mathbf{let}\ r\ \mathbf{free\ in}\ 2)$$

This could be matched by either rule. The trick is that we don't care which rule matches, as long as they both do eventually. This will be transformed into one of the following:

$$\begin{aligned}
&\mathbf{let}\ r\ \mathbf{free\ in\ let}\ x = 1\ \mathbf{in}\ f\ x\ 2\\
&\mathbf{let}\ x = 1\ \mathbf{in\ let}\ r\ \mathbf{free\ in}\ f\ x\ 2
\end{aligned}$$

Either of these options is acceptable. In fact, we could remove the ambiguity by making our rules a confluent system, as shown by the code below. However, we will not worry about confluence for most optimizations.

$$\begin{aligned}
&\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [\mathit{Let}\ \mathit{vs}\ e] \mathbin{+\!\!+} \mathit{bs}) &&= \mathit{Let}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [e] \mathbin{+\!\!+} \mathit{bs}))\\
&\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [\mathit{Free}\ \mathit{vs}\ e] \mathbin{+\!\!+} \mathit{bs}) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ f\ (\mathit{as} \mathbin{+\!\!+} [e] \mathbin{+\!\!+} \mathit{bs}))\\
&\mathit{float}\ (\mathit{Let}\ \mathit{vs}\ (\mathit{Free}\ \mathit{ws}\ e)) &&= \mathit{Free}\ \mathit{ws}\ (\mathit{Let}\ \mathit{vs}\ e)
\end{aligned}$$

Great, now we can make an optimization. It was easy to write, but it's not a very complex optimization. In fact, most optimizations we write won't be very complex. The power of optimization comes from making small improvements several times.

$\textbf{type}\ \mathit{QName} = (\mathit{String}, \mathit{String})$

$\textbf{type}\ \mathit{Arity} = \mathit{Int}$

$\textbf{type}\ \mathit{VarIndex} = \mathit{Int}$

$\textbf{data}\ \mathit{Visibility} = \mathit{Public} \mid \mathit{Private}$

$\textbf{data}\ \mathit{FuncDecl} = \mathit{Func}\ \mathit{QName}\ \mathit{Arity}\ \mathit{Rule}$

$\textbf{data}\ \mathit{Rule}$

$\quad = \mathit{Rule}\ [\mathit{VarIndex}]\ \mathit{Expr}$

$\quad \mid \mathit{External}\ \mathit{String}$

$\textbf{data}\ \mathit{CombType} = \mathit{FuncCall} \mid \mathit{ConsCall} \mid \mathit{FuncPartCall}\ \mathit{Arity} \mid \mathit{ConsPartCall}\ \mathit{Arity}$

$\textbf{data}\ \mathit{Expr}$

$\quad = \mathit{Var}\ \mathit{VarIndex}$

$\quad \mid \mathit{Lit}\ \mathit{Literal}$

$\quad \mid \mathit{Comb}\ \mathit{CombType}\ \mathit{QName}\ [\mathit{Expr}]$

$\quad \mid \mathit{Let}\ [(\mathit{VarIndex}, \mathit{Expr})]\ \mathit{Expr}$

$\quad \mid \mathit{Free}\ [\mathit{VarIndex}]\ \mathit{Expr}$

$\quad \mid \mathit{Or}\ \mathit{Expr}\ \mathit{Expr}$

$\quad \mid \mathit{Case}\ \mathit{Expr}\ [\mathit{BranchExpr}]$

$\textbf{data}\ \mathit{BranchExpr} = \mathit{Branch}\ \mathit{Pattern}\ \mathit{Expr}$

$\textbf{data}\ \mathit{Pattern}$

$\quad = \mathit{Pattern}\ \mathit{QName}\ [\mathit{VarIndex}]$

$\quad \mid \mathit{LPattern}\ \mathit{Literal}$

$\textbf{data}\ \mathit{Literal}$

$\quad = \mathit{Intc}\ \ \mathit{Int}$

$\quad \mid \mathit{Floatc}\ \mathit{Float}$

$\quad \mid \mathit{Charc}\ \mathit{Char}$

Figure 3.1: Curry representation of FlatCurry programs

This is the standard representation of FlatCurry programs as defined in [**?**], We have removed *CaseType* and *Typed* from *Expr*, and *TypeExpr* and *Visibility* from *FuncDecl*, because they are not used in our translation.

$$[\![f\ v_1, v_2 \dots v_n = e]\!] = \mathit{FuncDecl}\ f\ n\ (\mathit{Rule}\ [v_1, v_2 \dots v_n]\ [\![e]\!])$$

$$\begin{array}{lll}
[\![v]\!] & & = \mathit{Var}\ v \\
[\![l]\!] & & = \mathit{Lit}\ [\![l]\!] \\
[\![e_1\ ?\ e_2]\!] & & = \mathit{Or}\ [\![e_1]\!]\ [\![e_2]\!] \\
[\![\bot]\!] & & = \mathit{Comb}\ \mathit{ConsCall}\ (\texttt{""}, \texttt{"FAIL"})\ [\,] \\
[\![f_k\ e_1\ e_2 \dots e_n]\!] & \mid k{=}{=}0 & = \mathit{Comb}\ \mathit{FuncCall}\ f\ [[\![e_1]\!], [\![e_2]\!] \dots [\![e_n]\!]] \\
 & \mid \mathit{otherwise} & = \mathit{Comb}\ (\mathit{FuncPartCall}\ k)\ f\ [[\![e_1]\!], [\![e_2]\!] \dots [\![e_n]\!]] \\
[\![C_k\ e_1\ e_2 \dots e_n]\!] & \mid k{=}{=}0 & = \mathit{Comb}\ \mathit{ConsCall}\ f\ [[\![e_1]\!], [\![e_2]\!] \dots [\![e_n]\!]] \\
 & \mid \mathit{otherwise} & = \mathit{Comb}\ (\mathit{ConsPartCall}\ k)\ f\ [[\![e_1]\!], [\![e_2]\!] \dots [\![e_n]\!]] \\
\multicolumn{2}{l}{[\![\ \mathbf{let}\ v_1 = e_1, \dots v_n = e_n\ \mathbf{in}\ e]\!]} & = \mathit{Let}\ [(v_1, [\![e_1]\!]) \dots (v_n, [\![e_n]\!])]\ [\![e]\!] \\
\multicolumn{2}{l}{[\![\ \mathbf{let}\ v_1, v_2 \dots v_n\ \mathbf{free\ in}\ e]\!]} & = \mathit{Free}\ [v_1, v_2, \dots v_n]\ [\![e]\!] \\
\multicolumn{2}{l}{[\![\ \mathbf{case}\ e\ \mathbf{of}\ \mathit{branches}]\!]} & = \mathit{Case}\ e\ [\![\mathit{branches}]\!]
\end{array}$$

$$\begin{array}{ll}
[\![l \to e]\!] & = \mathit{LPattern}\ [\![l]\!]\ [\![e]\!] \\
[\![C\ v_1\ v_2 \dots v_n \to e]\!] & = \mathit{Pattern}\ C\ [v_1, v_2 \dots v_n]\ [\![e]\!]
\end{array}$$

$$\begin{array}{lll}
[\![l]\!] = & \mid \mathit{isInt}\ l & = \mathit{Intc}\ l \\
 & \mid \mathit{isFloat}\ l & = \mathit{Floatc}\ l \\
 & \mid \mathit{isChar}\ l & = \mathit{Charc}\ l
\end{array}$$

Figure 3.2: Translation from FlatCurry syntax to the Curry representation of FlatCurry.

Now that we can do simple examples, let's look at a more substantial transformation. Let-expressions are deceptively complicated. They allow us to make arbitrarily complex, mutually recursive, definitions. However, most of the time a large let expression could be broken down into several small let expressions. Consider the definition below:

$$\begin{array}{l} \textbf{let } a = b \\ \quad\; b = c \\ \quad\; c = d + e \\ \quad\; d = b \\ \quad\; e = 1 \\ \textbf{in } \; a \end{array}$$

This is a perfectly valid definition, but we could also break it up into the three nested let expressions below.

$$\begin{array}{l} \textbf{let } e = 1 \\ \textbf{in } \; \textbf{let } b = c \\ \qquad\quad c = d + e \\ \qquad\quad d = b \\ \quad\; \textbf{in } \; \textbf{let } a = b \\ \qquad\; \textbf{in } \; a \end{array}$$

It's debatable which version is better coding style, but the second version is inarguably more useful for the compiler. There are several optimizations that can be safely performed on a single let bound variable. Unfortunately, splitting the let expression into blocks isn't trivial. The algorithm involves making a graph out of all references in the let block, then finding the strongly connected components of that reference graph, and, finally, rebuilding the let expression from the component graph. The full algorithm is given below in figure 3.3

While this optimization is significantly more complicated then the *float* example, We can still implement it in our system. Furthermore, we're able to factor out the code for building the graph and finding the strongly connected components. This is the advantage of using Curry functions as opposed to strict rewrite rules. We have much more freedom in constructing the right-hand side of our rules.

Now that we can create optimizations, what if we want both *blocks* and *float* to be able to run? This is an important part of the compilation process to get expressions into a canonical

$$
\begin{array}{l}
\mathit{blocks}\ (\mathit{Let}\ \mathit{vs}\ e) \mid \mathit{numBlocks} > 1 = e' \\
\quad \textbf{where}\ (e', \mathit{numBlocks}) = \mathit{makeBlocks}\ \mathit{es}\ e
\end{array}
$$

$$
\begin{array}{l}
\mathit{makeBlocks}\ \mathit{vs}\ e = (\mathit{letExp}, \mathit{length}\ \mathit{comps}) \\
\quad \textbf{where}\ \mathit{letExp} = \mathit{foldr}\ \mathit{makeBlock}\ e\ \mathit{comps} \\
\qquad\qquad \mathit{makeBlock}\ \mathit{comp} = \lambda \mathit{exp} \rightarrow \mathit{Let}\ (\mathit{map}\ \mathit{getExp}\ \mathit{comp})\ \mathit{exp} \\
\qquad\qquad \mathit{getExp}\ (\_ \mathbin{+\!\!+} [(n, \mathit{exp})] + + \_) = (n, \mathit{exp}) \\
\qquad\qquad \mathit{comps} = \mathit{scc}\ (\mathit{vs} \gg\!= \mathit{makeEdges}) \\
\qquad\qquad \mathit{makeEdges}\ (v, \mathit{exp}) = [(v, f) \mid f \leftarrow \mathit{freeVars}\ \mathit{exp} \cap \mathit{map}\ \mathit{fst}\ \mathit{vs}]
\end{array}
$$

Figure 3.3: Transformation for splitting let expressions into mutually recursive blocks.

form. It turns out that combining two optimizations is simple. We just make a non-deterministic choice between them.

$$\mathit{floatBlocks} = \mathit{float}\ ?\ \mathit{blocks}$$

This is a new optimization that will apply either *float* or *blocks*. The ability to compose optimizations with ? is the heart of the GAS system. Each optimization can be developed and tested in isolation, then they can be combined for efficiency.

3.0.3 An Initial Attempt

Our first attempt is quite simple, really. We pick an arbitrary subexpression with *subExpr* and apply an optimization. We can then use a non-deterministic fix point operator to find all transformations that can be applied to the current expression. We can define the non-deterministic fix point operator using either the Findall library, or Set Function [26, 12] The full code is given in figure 3.4

While this attempt is short and readable, there's a problem with it. It is unusably slow. While looking at the code, it's pretty clear to see what the problem is. Every time we traverse the expression, we can only apply a single transformation. This means that if we need to apply 100 transformations, which is not uncommon, then we need to traverse the expression 100 times.

$$
\begin{array}{l}
\mathit{fix} :: (a \to a) \to a \to a \\
\mathit{fix}\ f\ x \\
\quad \mid f\ x{=}{=}\emptyset \quad = x \\
\quad \mid \mathit{otherwise} = \mathit{fix}\ f\ (f\ x)
\end{array}
$$

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathit{subExpr} :: \mathit{Expr} \to \mathit{Expr}} \\
\mathit{subExpr}\ e & = e \\
\mathit{subExpr}\ (\mathit{Comb}\ \mathit{ct}\ f\ \mathit{vs}) & = \mathit{subExpr}\ (\mathit{foldr1}\ (?)\ \mathit{es}) \\
\mathit{subExpr}\ (\mathit{Let}\ \mathit{vs}\ e) & = \mathit{subExpr}\ (\mathit{foldr1}\ (?)\ (e : \mathit{map}\ \mathit{snd}\ \mathit{es})) \\
\mathit{subExpr}\ (\mathit{Free}\ \mathit{vs}\ e) & = \mathit{subExpr}\ e \\
\mathit{subExpr}\ (\mathit{Or}\ e_1\ e_2) & = \mathit{subExpr}\ e_1\ ?\ \mathit{subExpr}\ e_2 \\
\mathit{subExpr}\ (\mathit{Case}\ e\ \mathit{bs}) & = \mathit{subExpr}\ (e : \mathit{map}\ \mathit{branchExpr}\ \mathit{bs}) \\
\multicolumn{2}{l}{\quad \mathbf{where}\ \mathit{branchExpr}\ (\mathit{Branch}\ \_\ e) = e}
\end{array}
$$

$$
\begin{array}{l}
\mathit{reduce} :: \mathit{Opt} \to \mathit{Expr} \to \mathit{Expr} \\
\mathit{reduce}\ \mathit{opt}\ e = \mathit{opt}\ (\mathit{subExpr}\ e)
\end{array}
$$

$$
\begin{array}{l}
\mathit{simplify} :: \mathit{Opt} \to \mathit{Expr} \to \mathit{Expr} \\
\mathit{simplify}\ \mathit{opt}\ e = \mathit{fix}\ (\mathit{reduce}\ \mathit{opt})\ e
\end{array}
$$

Figure 3.4: A first attempt at an optimization engine. Pick an arbitrary subexpression and try to optimize it.

3.0.4 A Second Attempt: Multiple Transformations Per Pass

Our second attempt runs much faster. Instead of picking an arbitrary subexpression, we choose to traverse the expression manually. Now, we can check at each node if an optimization applies. We only need to make two changes. The biggest is that we eliminate *subExpr* and change *reduce* to traverse the entire expression. Now *reduce* can apply an optimization at every step. We've also made *reduce* completely deterministic. The second change is that since *reduce* is deterministic, we can change *fix* to be a more traditional implementation of a fix point operator. The new implementation is given in figure 3.5

This approach is significantly better. Aside from applying multiple rules in one pass, we also limit our search space when applying our optimizations. While there's still more we can do, the new approach makes the GAS library usable on larger Curry programs, like the standard Prelude.

3.0.5 Adding More Information

Rather surprisingly our current approach is actually sufficient for compiling FlatCurry. However, to optimize Curry we're going to need more information when we apply a transformation. Specifically, we'll be able to create new variables. To simplify optimizations, we'll require that each variable name can only be used once. Regardless, we need a way to know what is a safe variable name that we're allowed to use. We may also need to know if we're rewriting the root of an expression. Fortunately, all we need to change is to define *Opt* to accept more parameters. For each optimization, we'll pass in an $n :: Int$ that represents the next variable v_n that is guaranteed to be fresh. We'll also pass in a $top :: Bool$ that tells us if we're at root of a function we're optimizing. We also return a pair of $(Expr, Int)$ to denote the optimized expression, and the number of new variables we used.

$$\textbf{type}\ Opt = (Int, Bool) \rightarrow Expr \rightarrow (Expr, Int)$$

If we later decide that we want to add more information, then we just update the first parameter. The only problem is, how do we make sure we're calling each optimization with the correct n and *top*? We just need to update *reduce* and *runOpt*. In order to keep track of the next available free variable we use the *State* monad. We do need to make minor changes to *fix* and *simplify*, but this is just to make them compatible with *State*. The full implementation is in figure 3.6.

```
fix :: (a → a) → a → a
fix f x
  | f x==x = x
  | otherwise = fix f (f x)

reduce :: Opt → Expr → Expr
reduce opt (Var v)          = runOpts opt (Var v)
reduce opt (Lit l)          = runOpts opt (Lit l)
reduce opt (Comb ct f es)   = runOpts opt (Comb ct f (map (reduce opt) es))
  where es' = map (reduce opt) es
reduce opt (Let vs e)       = runOpts opt Let (map runLet vs) (reduce opt e)
  where runLet (v, e) = (v, reduce opt e)
reduce opt (Free vs e)      = runOpts opt (Free vs (reduce opt e))
reduce opt (Or a b)         = runOpts opt (Or (reduce opt a) (reduce opt a))
reduce opt (Case e bs)      = runOpts opt (Case (reduce opt e) bs')
  where runBranch (Branch p e) = Branch p (reduce opt e)
        bs' = map runBranch bs

runOpts :: Opt → Expr → Expr
runOpts opt e = case oneValue (opt e) of
                  Nothing → e
                  Just e'  → e'

simplify :: Opt → Expr → Expr
simplify opt e = fix (reduce opt) e
```

Figure 3.5: A second attempt. Traverse the expression and, at each node, check if an optimization applies.

```
reduce :: Opt → Bool → Expr → State Int Expr
reduce opt top (Var v)          = runOpts opt top (Var v)
reduce opt top (Lit l)          = runOpts opt top (Lit l)
reduce opt top (Comb ct f es)   = do es′ ← mapM (reduce opt False es)
                                     runOpts opt top (Comb ct f es′)
reduce opt top (Let vs in e)    = do vs′ ← mapM runVar vs
                                     e′  ← mapM reduce opt False e
                                     runOpts opt top (Let vs′ in e′)
  where runVar (v, e) = do e′ ← reduce opt False e
                           return (v, e′)
reduce opt top (Free vs e)      = do e′ ← reduce opt False e
                                     runOpts opt top (Free vs e′)
reduce opt top (Or a b)         = do a′ ← run opt False a
                                     b′ ← run opt False b
                                     runOpts opt (Or a′ b′)
reduce opt top (Case e bs)      = do e′  ← reduce opt False e
                                     bs′ ← mapM runBranch bs
                                     runOpts opt (Case e′ bs′)
  where runBranch (Branch pat e) = do e′ ← reduce opt False e
                                      return (Branch pat e′)

runOpts :: Opt → Bool → Expr → State Int Expr
runOpts opt top e = do v ← get
                       case opt (v, top) e of
                         Nothing      → return e
                         Just (e′, dv) → do put (v + dv)
                                            return e′
fix :: (a → State b a) → a → b → a
fix f x s = let (x′, s′) = runState (f x) s
            in  if x==x′ then x else fix f x′ s′
```

Figure 3.6: A third attempt. keep track of the next fresh variable, and if we're at the root.

3.0.6 Reconstruction

Right now we have everything we need to write all of our optimizations. However, we've found it useful to be able to track which optimizations were applied and where they were applied. This helps with testing, debugging, and designing optimizations, as well as generating optimization derivations that we'll see later in this dissertation. It is difficult to overstate just how helpful this addition was in building this compiler.

If we want to add this, then we need to make a few changes. First, we need to decide on a representation for a rewrite derivation. Traditionally a rewrite derivation is a sequence of rewrite steps, where each step contains the rule and position of the rewrite. We describe paths in figure 3.7. To make reconstruction easier, we also include the expression that is the result of the rewrite. This gives us the type:

$$\begin{aligned}&\textbf{type}\ \mathit{Path} = [\mathit{Int}]\\&\textbf{type}\ \mathit{Step} = (\mathit{String}, \mathit{Path}, \mathit{Expr})\\&\textbf{type}\ \mathit{Derivation} = [\mathit{Step}]\end{aligned}$$

This leads to the last change we need to make to our *Opt* type. We need each optimization to also tell us its name. This is good practice in general, because it forces us to come up with unique names for each optimization.

$$\textbf{type}\ \mathit{Opt} = (\mathit{Int}, \mathit{Bool}) \to \mathit{Expr} \to (\mathit{Expr}, \mathit{String}, \mathit{Int})$$

We only need to make a few change to the algorithm. Instead of using the *State* monad, we use a combination of the *State* and *Writer* monads, so we can keep track of the derivation. We've elected to call this the *ReWriter* monad. We add a function $\mathit{update} :: \mathit{Expr} \to \mathit{Step} \to \mathit{Int} \to \mathit{ReWriter}\ \mathit{Expr}$ that is similar to *put* from *State*. This updates the state variable, and creates a single step. The *reduce* function requires few changes. We change the Boolean variable *top* to a more general *Path*. Because of this change, we need to add the correct subexpression position, instead of just changing *top* to *False*. The *RunOpts* function is similar. We just change *top* to a *Path*, and check if it's null. Finally *fix* and *simplify* are modified to remember the rewrite steps we've already computed. We change the return type of *simplify* so that we have the list of steps. The full implementation is in figure 3.8

Now that we've computed the rewrite steps, it's a simple process to reconstruct them into a string. The *pPrint* function comes from the FlatCurry Pretty Printing Library.

$$
\begin{array}{ll}
\mathit{ndpath}\ e & = [\,] \\
\mathit{ndpath}\ (\mathit{Comb}\ \mathit{ct}\ f\ \mathit{es}) & = \mathit{anymap}\ \mathit{argPath}\ \mathit{es} \\
\quad \mathbf{where}\ \mathit{argPath}\ (i, e) & = i : \mathit{ndpath}\ e_i \\
\mathit{ndpath}\ (\mathit{Or}\ e_0\ e_1) & = 0 : \mathit{ndpath}\ e_0\ ?\ 1 : \mathit{ndpath}\ e_1 \\
\mathit{ndpath}\ (\mathit{Let}\ \mathit{vs}\ e_{-1}) & = \mathit{anymap}\ \mathit{letPath}\ \mathit{es} \\
 & \ \ ?\ -1 : \mathit{ndpath}\ e_{-1} \\
\quad \mathbf{where}\ \mathit{letPath}\ (i, (\_, e)) = i : \mathit{ndpath}\ e & \\
\mathit{ndpath}\ (\mathit{Case}\ e\ \mathbf{of}\ \mathit{alts}) & = -1 : \mathit{ndpath}\ e \\
 & \ \ ?\ \mathit{anymap}\ \mathit{altPath}\ \mathit{alts} \\
\quad \mathbf{where}\ \mathit{altPath}\ (i, \mathit{Branch}\ \_\ e) = i : \mathit{ndpath}\ e &
\end{array}
$$

$$\mathit{anymap}\ f = \mathit{anyof} \circ \mathit{map}\ f \circ \mathit{zip}\ [1\,..]$$

Figure 3.7: The definition of a path for Curry expressions.
This function non-deterministically returns a path to a subexpression.

$$
\begin{array}{l}
\mathit{reconstruct} :: \mathit{Expr} \to [\mathit{Step}] \to \mathit{String} \\
\mathit{reconstruct}\ \_\ [\,] = \texttt{""} \\
\mathit{reconstruct}\ e\ ((\mathit{rule}, p, \mathit{rhs}) : \mathit{steps}) = \mathbf{let}\ e' = e[p \mapsto \mathit{rhs}] \\
\qquad\qquad\qquad \mathbf{in}\ \texttt{"=>\_"} + \mathit{rule} + \texttt{"\ \ "} + (\mathit{show}\ p) + \texttt{"\textbackslash n"} + \\
\qquad\qquad\qquad\quad \mathit{pPrint}\ e' + \texttt{"\textbackslash n"} + \\
\qquad\qquad\qquad\quad \mathit{reconstruct}\ e'\ \mathit{steps}
\end{array}
$$

3.0.7 Optimizing the Optimizer

Remember that our optimizing engine is going to run for every optimization, so it's worth taking the time to tune it to be as efficient as possible. There are a few tricks we can use to make the optimization process faster. The first trick is really simple. We add a Boolean variable *seen* to the ReWriter monad. This variable starts as *False*, and we set it to *True* if we apply any optimization. This avoids the linear time check for every call of *fix* to see if we actually ran any optimizations. The second quick optimization is to notice that variables, literals, and type expressions are never going to run an optimization, so we can immediately return in each of those

$$
\begin{array}{l}
\mathit{reduce} :: \mathit{Opt} \rightarrow \mathit{Path} \rightarrow \mathit{Expr} \rightarrow \mathit{ReWriter}\ \mathit{Expr} \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Var}\ v) \quad = \mathit{return}\ (\mathit{Var}\ v) \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Lit}\ l) \quad = \mathit{return}\ (\mathit{Lit}\ l) \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Comb}\ \mathit{ct}\ f\ \mathit{es}) = \mathbf{do}\ \mathit{es}' \leftarrow \mathit{mapM}\ \mathit{runArg}\ (\mathit{zip}\ [0\,..]\ \mathit{es}) \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{runOpts}\ \mathit{opt}\ p\ (\mathit{Comb}\ \mathit{ct}\ f\ \mathit{es}) \\
\quad \mathbf{where}\ \mathit{runArg}\ (n, e) \quad = \mathit{reduce}\ \mathit{opt}\ (n : p)\ e \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Let}\ \mathit{vs}\ e) \quad = \mathbf{do}\ \mathit{vs}' \leftarrow \mathit{mapM}\ \mathit{runVar}\ (\mathit{zip}\ [0\,..]\ \mathit{vs}) \\
\qquad\qquad\qquad\qquad\qquad\quad e' \ \leftarrow \mathit{mapM}\ \mathit{reduce}\ \mathit{opt}\ (-1 : p)\ e \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{runOpts}\ \mathit{opt}\ p\ (\mathit{Let}\ \mathit{vs}'\ e') \\
\quad \mathbf{where}\ \mathit{runVar}\ (n, (v, e)) = \mathit{fmap}\ (\lambda x \rightarrow (v, x))\ (\mathit{reduce}\ \mathit{opt}\ (n : p)\ e) \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Free}\ \mathit{vs}\ \mathbf{in}\ e) \quad = \mathbf{do}\ e' \leftarrow \mathit{reduce}\ \mathit{opt}\ (0 : p)\ e \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{runOpts}\ \mathit{opt}\ p\ (\mathit{Free}\ \mathit{vs}\ e') \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Or}\ a\ b) \quad = \mathbf{do}\ a \quad \leftarrow \mathit{reduce}\ \mathit{opt}\ (0 : p)\ a \\
\qquad\qquad\qquad\qquad\qquad\quad b \quad \leftarrow \mathit{reduce}\ \mathit{opt}\ (1 : p)\ b \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{runOpts}\ \mathit{opt}\ (\mathit{Or}\ (a'\ ?\ b')) \\
\mathit{reduce}\ \mathit{opt}\ p\ (\mathit{Case}\ e\ \mathit{bs}) \quad = \mathbf{do}\ e' \ \leftarrow \mathit{reduce}\ \mathit{opt}\ (-1 : p)\ e \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{bs}' \leftarrow \mathit{mapM}\ \mathit{runBranch}\ (\mathit{zip}\ [0\,..]\ \mathit{bs}) \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{runOpts}\ \mathit{opt}\ (\mathit{Case}\ e'\ \mathit{bs}') \\
\quad \mathbf{where}\ \mathit{runBranch}\ (n, (\mathit{Branch}\ \mathit{pat}\ e)) = \mathit{fmap}\ (\mathit{Branch}\ \mathit{pat})\ (\mathit{reduce}\ \mathit{opt}\ (n : p)\ e)
\end{array}
$$

$$
\begin{array}{l}
\mathit{runOpts} :: \mathit{Opt} \rightarrow \mathit{Path} \rightarrow \mathit{Expr} \rightarrow \mathit{ReWriter}\ \mathit{Expr} \\
\mathit{runOpts}\ \mathit{opt}\ p\ e = \mathbf{do}\ v \leftarrow \mathit{get} \\
\qquad\qquad\qquad\qquad \mathbf{case}\ \mathit{oneValue}\ (\mathit{opt}\ (v, \mathit{null}\ p))\ \mathbf{of} \\
\qquad\qquad\qquad\qquad\quad \mathit{Nothing} \qquad\qquad \rightarrow \mathit{return}\ e \\
\qquad\qquad\qquad\qquad\quad \mathit{Just}\ (e', \mathit{rule}, \mathit{dv}) \rightarrow \mathbf{do}\ \mathit{update}\ (e', \mathit{rule}, p)\ \mathit{dv} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathit{return}\ e' \\
\mathit{fix} :: (a \rightarrow \mathit{ReWriter}\ a) \rightarrow a \rightarrow \mathit{Int} \rightarrow [\mathit{Step}] \rightarrow (a, [\mathit{Step}]) \\
\mathit{fix}\ f\ x\ n\ \mathit{steps} = \quad \mathbf{let}\ (x', n', \mathit{steps}') = \mathit{runRewriter}\ (f\ x)\ n \\
\qquad\qquad\qquad\qquad \mathbf{in}\ \ \mathbf{if}\ x{=}{=}x'\ \mathbf{then}\ x\ \mathbf{else}\ \mathit{fix}\ f\ x'\ n'\ (\mathit{steps} +\!\!+ \mathit{steps}')
\end{array}
$$

Figure 3.8: The final version of GAS with reconstruction.

cases without calling *runOpt*. This is actually a much bigger deal than it might first appear. All of the leaves are going to either be variables, literals, or constructors applied to no arguments. For expression trees the leaves are often the majority of the nodes in the tree. Finally, we can put a limit on the number of optimizations to apply. If we ever reach that number, then we can immediately return. This can stop our optimizer from taking too much time.

Now that the GAS system is in place, we can move onto compiling FlatCurry programs. In the next section we discus the compiler pipeline, and how to transform the FlatCurry into C.

3.1 THE COMPILER PIPELINE

This compiler, unsurprisingly, follows a traditional compiler pipeline. While we start with an AST, there are still five phases left before we can generate C code. First, we normalize FlatCurry to a canonical form. Second we optimize the FlatCurry. Third, we sanitize the FlatCurry to simplify the process of generating C code. Fourth, we compile the FlatCurry to ICurry, an intermediate representation that is closer to C. Finally, we compile the ICurry to C. These steps are referred to as pre-process, optimize, post-process, toICurry, and toC within the compiler.

We give an example of a function as it passes through each of the stages of the compiler in figure 3.9. After pre-processing the let expression has been floated to the root, and the missing branch has been filled in. After optimization, code is organized into blocks, and functions have been reduced. After post processing, let bound variables with a case expression have been factored out into their own functions. At this point the code is ready to be translated into ICurry and then C.

While there are several small details that are important to constructing a working Curry compiler, we'll concern ourselves with the big picture here.

3.1.1 Canonical FlatCurry

The preproces and post-process steps of the compiler make heavy use the of GAS system, and transform the FlatCurry program in to a form that is more amenable to C, such as removing case and let expression from inside function applications. We will discuss the optimization phase in the next section, but for now we can see how transformations work.

Let's start with an example:

$$1 + \textbf{let } x = 3 \textbf{ in } x$$

Curry function

$$f\ \mathit{True} = \mathit{False}$$
$$f\ x = \mathit{not}\ (\mathbf{let}\ y = \mathit{not}\ x\ \mathbf{in}\ \mathit{not}\ y)$$

FlatCurry representation

$$\begin{aligned} f\ v_1 = {} & (\mathbf{case}\ v_1\ \mathbf{of}\ \mathit{True} \to \mathit{False})\ ?\\ & (\mathit{not}\ (\mathbf{let}\ v_2 = \mathit{not}\ v_1\ \mathbf{in}\ \mathit{not}\ v_2)) \end{aligned}$$

After Pre-processing

$$\begin{aligned} f\ v_1 = {} & \mathbf{let}\ v_2 = \mathit{not}\ \mathit{v1\_}\\ & \mathbf{in}\ (\mathbf{case}\ v_1\ \mathbf{of}\\ & \qquad \mathit{True} \to \mathit{False}\\ & \qquad \mathit{False} \to \bot)\ ?\ (\mathit{not}\ (\mathit{not}\ \mathit{v2})) \end{aligned}$$

After Optimization

$$\begin{aligned} & \mathbf{let}\ v_4 = \mathbf{case}\ v_1\ \mathbf{of}\\ & \qquad \mathit{True} \to \mathit{False}\\ & \qquad \mathit{False} \to \bot\\ & \mathbf{in}\ \ \mathbf{let}\ v_5 = \mathbf{case}\ v_1\ \mathbf{of}\\ & \qquad\quad \mathit{True} \to \mathit{False}\\ & \qquad\quad \mathit{False} \to \mathit{True}\\ & \quad \mathbf{in}\ \ \mathit{v4}\ ?\ \mathit{v5} \end{aligned}$$

After Post-processing

$$\begin{aligned} f\ v_1 = {} & \mathbf{let}\ v_4 = f_0\ v_1\\ & \mathbf{in}\ \ \mathbf{let}\ v_5 = f_1\ v_1\\ & \quad \mathbf{in}\ \ v_4\ ?\ v_5\\ f_0\ v_1 = {} & \mathbf{fcase}\ v_1\ \mathbf{of}\\ & \qquad \mathit{True} \to \mathit{False}\\ & \qquad \mathit{False} \to \bot\\ f_1\ v_1 = {} & \mathbf{fcase}\ v_1\ \mathbf{of}\\ & \qquad \mathit{True} \to \mathit{False}\\ & \qquad \mathit{False} \to \mathit{True} \end{aligned}$$

Figure 3.9: f at each stage of the optimizer.

$$
\begin{aligned}
& f\ v_1\ v_2 \ldots v_n = b \\
& s = \textbf{case}\ e\ \textbf{of}\ \{\, C\ v_1 \ldots v_n \to s \,\} \\
& \quad \mid\ \textbf{let}\ \{ v = e \}\ \textbf{in}\ s \\
& \quad \mid\ \textbf{let}\ \{ v \}\ \textbf{free in}\ s \\
& \quad \mid\ e \\
& e = v \\
& \quad \mid\ l \\
& \quad \mid\ f_k\ e_1\ e_2 \ldots e_n \\
& \quad \mid\ C_k\ e_1\ e_2 \ldots e_n \\
& \quad \mid\ e_1\ ?\ e_2
\end{aligned}
$$

Figure 3.10: Canonical FlatCurry.

We split expressions into statement like expressions s, and expressions e. Statement like expressions roughly correspond to control flow, and are translated to variable declaration and control flow statements in C.

This is a perfectly fine Curry program, but C does not allow variable declarations in an expression, so we need to rewrite this Curry expression to:

$$\textbf{let}\ x = 3\ \textbf{in}\ 1 + x$$

We don't reduce **let** $x = 3$ **in** x yet, because that would be an optimization. However, this will be reduced later. We can translate the new expression to C in a direct manner. This is the purpose of the pre-process and post-process steps. We rewrite a Curry expression that doesn't make sense in C to an equivalent Curry expression that we can translate directly to C. Most of the transformations consist of disallowing certain syntactic constructs. Canonical FlatCurry is defined in figure 3.10.

Examples of the preprocessing transformations are presented in figures. 3.12 and 3.13. We use the symbol $\Rightarrow$ for the optimization relation. The implementation is presented in figure 3.11. We only show the initial implementation of an optimiation that excludes the name and path, but it can be extended to the full optimization in a straightforward manner. The full implementation can be found in the `src/Optimize/Preprocess.curry` file at [64].

In practice several of these rules are generalized and optimized. For example let-expressions may have many mutually recursive variables, and when floating a let bound variable inward, we may want to recursively traverse the expression to find the innermost declaration possible. However, these extensions to the rules are also included in the repository.

While most of these transformations are simple, a few require some explanation. The **blocks** transformation takes a let block with multiple variable definitions, and rewrites it to a series of let blocks where all variables are split into strongly connected components. This isn't strictly necessary, but it removes the need to check for mutual recursion during the optimization phase. It will often transform a block of mutually defined variables into a cascading series of let expressions with a single variable, which will allow more optimizations to run throughout the compiler.

The **alias** transformation will remove any aliased variables. If one variables is aliased to another, then it will do the substitution, but if a variable is aliased to itself, then it cannot be reduced to a normal form, so we can replace it with an infinite loop.

Finally the **Fill cases** transformation completes the definitional tree. If we have a case with branches for constructors $C_1, C_2 \ldots C_k$, then we look up the type T that all of these constructor belong to. Then we get the list $Ctrs$ of all constructors that belonging to T. This list will contain $C_1, C_2, \ldots C_n$, but it may contain more. For each constructor not represented in the case-expression, we create a new branch $C_i \to \bot$.

After running all of these transformations, our program is in canonical form and we may choose to optimize it, or we may skip straight to the post-processing phase. At this point we only need two transformations for post processing, however we will need to add more to support some of the optimizations. If we ever have an expression of the form **let** $x =$ **case** $\ldots$, then we need to transform the case-expression into a function call. We don't do this transformation in pre-processing because we don't want to split functions apart during optimizations. The **Let-Case** transformation has a single rule given in figure 3.14.

Every let with a case-expression creates a new function $f\#n$ where n is incremented every time.

Finally, in our post-processing phase we simply factor out the scrutinee of a case-expression into a variable. The transformation is straightforward. An example of a preprecess derivation is given in 3.15. At this point point we are ready to transform the cononicalized FlatCurry into ICurry.

$$
\begin{aligned}
&\mathit{float}\ (\mathit{Let}\ (\mathit{as} +\!\!+ [(x, \mathit{Let}\ \mathit{vs}\ e_1)] +\!\!+ \mathit{bs})\ e_2) &&= \mathit{Let}\ ((x, e_1) : \mathit{vs} +\!\!+ \mathit{as} +\!\!+ \mathit{bs})\ e_2\\
&\mathit{float}\ (\mathit{Let}\ (\mathit{as} +\!\!+ [(x, \mathit{Free}\ \mathit{vs}\ e_1)] +\!\!+ \mathit{bs})\ e_2) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Let}\ ((x, e_1) : \mathit{as} +\!\!+ \mathit{bs})\ e_2)\\
&\mathit{float}\ (\mathit{Or}\ (\mathit{Let}\ \mathit{vs}\ e_1)\ e_2) &&= \mathit{Let}\ \mathit{vs}\ (\mathit{Or}\ e_1\ e_2)\\
&\mathit{float}\ (\mathit{Or}\ e_1\ (\mathit{Let}\ \mathit{vs}\ e_2)) &&= \mathit{Let}\ \mathit{vs}\ (\mathit{Or}\ e_1\ e_2)\\
&\mathit{float}\ (\mathit{Or}\ (\mathit{Free}\ \mathit{vs}\ e_1)\ e_2) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Or}\ e_1\ e_2)\\
&\mathit{float}\ (\mathit{Or}\ e_1\ (\mathit{Free}\ \mathit{vs}\ e_2)) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Or}\ e_1\ e_2)\\
&\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ n\ (\mathit{as} +\!\!+ [\mathit{Let}\ \mathit{vs}\ e] +\!\!+ \mathit{bs})) &&= \mathit{Let}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ n\ (\mathit{as} +\!\!+ [e] +\!\!+ \mathit{bs}))\\
&\mathit{float}\ (\mathit{Comb}\ \mathit{ct}\ n\ (\mathit{as} +\!\!+ [\mathit{Free}\ \mathit{vs}\ e] +\!\!+ \mathit{bs})) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Comb}\ \mathit{ct}\ n\ (\mathit{as} +\!\!+ [e] +\!\!+ \mathit{bs}))\\
&\mathit{float}\ (\mathit{Case}\ (\mathit{Let}\ \mathit{vs}\ e)\ \mathit{alts}) &&= \mathit{Let}\ \mathit{vs}\ (\mathit{Case}\ e\ \mathit{alts})\\
&\mathit{float}\ (\mathit{Case}\ (\mathit{Free}\ \mathit{vs}\ e)\ \mathit{alts}) &&= \mathit{Free}\ \mathit{vs}\ (\mathit{Case}\ e\ \mathit{alts})
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{flatten}\ (\mathit{apply}\ (\mathit{apply}\ f\ \mathit{as})\ \mathit{bs}) \quad = \mathit{applyf}\ f\ (\mathit{as} +\!\!+ \mathit{bs})\\
&\mathit{flatten}\ (\mathit{apply}\ (\mathit{Case}\ e\ \mathit{bs})\ \mathit{xs}) \quad = \mathit{Case}\ e\ \mathit{bs}'\\
&\qquad \textbf{where}\ \mathit{bs}' = [\mathit{Branch}\ p\ (\mathit{applyf}\ e'\ \mathit{xs}) \mid (\mathit{Branch}\ p\ e') \leftarrow \mathit{bs}]\\
&\mathit{flatten}\ (\mathit{Case}\ (\mathit{Case}\ e\ \mathit{alt2})\ \mathit{alt1}) = \mathit{Case}\ e\ \mathit{bs}\ (\mathit{map}\ \mathit{addCase}\ \mathit{alt2})\\
&\qquad \textbf{where}\ \mathit{addCase}\ (\mathit{Branch}\ p\ e') = \mathit{Branch}\ p\ (\mathit{Case}\ e'\ \mathit{b1})
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{blocks}\ \_\ (\mathit{Let}\ \mathit{vs}\ e) \mid \mathit{changed} = e'\\
&\qquad \textbf{where}\ (e', \mathit{changed}) = \mathit{makeBlocks}\ \mathit{vs}\ e
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{alias}\ \_\ (\mathit{Let}\ (\mathit{as} +\!\!+ [(v, \mathit{Var}\ y)] +\!\!+ \mathit{bs})\ e)\\
&\quad \mid v{=}{=}y \qquad = \mathit{Let}\ (\mathit{as} +\!\!+ [(v, \mathit{loop})] +\!\!+ \mathit{bs})\ e\\
&\quad \mid \mathit{otherwise} = \mathit{suby}\ (\mathit{Let}\ (\mathit{as} +\!\!+ \mathit{bs})\ e)\\
&\qquad \textbf{where}\ \mathit{loop} = \mathit{Comb}\ \mathit{FuncCall}\ (\texttt{"Prelude"}, \texttt{"loop"})\ []\\
&\qquad\qquad\quad \mathit{suby} = (\mathit{bodyf})[\cdot]
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{fillCases}\ \mathit{dt}\ \_\ (\mathit{Case}\ e\ \mathit{bs})\\
&\qquad \mid \mathit{not}\ (\mathit{null}\ \mathit{exempts}) = \mathit{Case}\ e\ (\mathit{bs} +\!\!+ \mathit{exempts})\\
&\qquad \textbf{where}\ \mathit{exempts} = [\mathit{Branch}\ (\mathit{Pattern}\ b\ [])\ \mathit{exempt} \mid b \leftarrow \mathit{missingBranches}\ \mathit{dt}\ \mathit{bs}]
\end{aligned}
$$

Figure 3.11: In *fillCases*, *dt* is a *DataTable*, which holds information about data types. The *missingBranches* takes a list of branches and a *DataTable* and returns the names of the branches that aren't present. In *alias* the $(\mathit{bodyf})[\cdot]$ function applies a substitution to an expression.

Let Floating

$$\begin{array}{l} \textbf{let}\ x = \textbf{let}\ y = e_1 \\ \qquad\quad \textbf{in}\ e_2 \\ \textbf{in}\ e_3 \end{array} \quad\Rrightarrow\quad \begin{array}{l} \textbf{let}\ y = e_1 \\ \quad \textbf{in let}\ x = e_2 \\ \qquad \textbf{in}\ \ e_3 \end{array}$$

$$\begin{array}{l} \textbf{let}\ x = \textbf{let}\ y\ \textbf{free} \\ \qquad\quad \textbf{in}\ e_1 \\ \textbf{in}\ e_2 \end{array} \quad\Rrightarrow\quad \begin{array}{l} \textbf{let}\ y\ \textbf{free} \\ \textbf{in let}\ x = e_2 \\ \quad \textbf{in}\ e_3 \end{array}$$

$$(\textbf{let}\ x = e_1\ \textbf{in}\ e_2)\ ?\ e_3 \quad\Rrightarrow\quad \textbf{let}\ x = e_1\ \textbf{in}\ (e_2\ ?\ e_3)$$

$$(\textbf{let}\ x\ \textbf{free in}\ e_1)\ ?\ e_2 \quad\Rrightarrow\quad \textbf{let}\ x\ \textbf{free in}\ (e_1\ ?\ e_2)$$

$$f\ (\textbf{let}\ x = e_1\ \textbf{in}\ e_2) \quad\Rrightarrow\quad \begin{array}{l} \textbf{let}\ x = e_1 \\ \textbf{in}\ f\ \ e_2 \end{array}$$

$$f\ (\textbf{let}\ x\ \textbf{free in}\ e) \quad\Rrightarrow\quad \begin{array}{l} \textbf{let}\ x\ \textbf{free} \\ \textbf{in}\ f\ \ e \end{array}$$

$$\begin{array}{l} \textbf{case let}\ x = e_1 \\ \qquad \textbf{in}\ e_2\ \textbf{of} \\ \qquad \mathit{alts} \end{array} \quad\Rrightarrow\quad \begin{array}{l} \textbf{let}\ x = e_1 \\ \textbf{in case}\ e_2\ \textbf{of}\ \ \mathit{alts} \end{array}$$

$$\begin{array}{l} \textbf{case let}\ x\ \textbf{free} \\ \qquad \textbf{in}\ e\ \textbf{of} \\ \qquad \mathit{alts} \end{array} \quad\Rrightarrow\quad \begin{array}{l} \textbf{let}\ x\ \textbf{free} \\ \textbf{in case}\ e\ \textbf{of}\ \ \mathit{alts} \end{array}$$

Figure 3.12: GAS rules for putting FlatCurry programs into canonical form

Case in Case

$$\begin{array}{lcl}
\textbf{case}\ (\textbf{case}\ e\ \textbf{of} & & \textbf{case}\ e\ \textbf{of}\ \{\, b_2 \rightarrow \\
\qquad\qquad \{\, b_2 \rightarrow e_2 \,\})\ \textbf{of} & \Rrightarrow & \qquad\qquad \textbf{case}\ e_2\ \textbf{of} \\
\quad \{\, b_1 \rightarrow e_1 \,\} & & \qquad\qquad\qquad \{\, b_1 \rightarrow e_1 \,\}\,\}
\end{array}$$

Double Apply

$$\mathit{apply}\ (\mathit{apply}\ f\ [x])\ [y] \quad \Rrightarrow \quad \mathit{apply}\ f\ [x, y]$$

Case Apply

$$\begin{array}{lcl}
\mathit{apply}\ (\textbf{case}\ e\ \textbf{of} & \Rrightarrow & \textbf{case}\ e\ \textbf{of} \\
\qquad\qquad \{\, \mathit{pat} \rightarrow f \,\})\ x & & \quad \{\, \mathit{pat} \rightarrow f\ x \,\}
\end{array}$$

Blocks

$$\begin{array}{lcl}
\textbf{let}\ a = b & & \textbf{let}\ e = 1 \\
\quad b = c & & \textbf{in}\ \ \textbf{let}\ b = c \\
\quad c = d + e & \Rrightarrow & \qquad\quad c = d + e \\
\quad d = b & & \qquad\quad d = b \\
\quad e = 1 & & \qquad \textbf{in}\ \ \textbf{let}\ a = b \\
\textbf{in}\ \ a & & \qquad\quad \textbf{in}\ \ a
\end{array}$$

Alias

$$\begin{array}{lcl}
\textbf{let}\ x = y\ \textbf{in}\ e & \Rrightarrow & e\ [x \rightarrow y] \\
\textbf{let}\ x = x\ \textbf{in}\ e & \Rrightarrow & \textbf{let}\ x = \mathit{loop}\ \textbf{in}\ e
\end{array}$$

Case Fill

$$\begin{array}{lcl}
 & & \textbf{case}\ e\ \textbf{of} \\
\textbf{case}\ e\ \textbf{of} & \Rrightarrow & \quad \mathit{True} \rightarrow e \\
\quad \mathit{True} \rightarrow e & & \quad \mathit{False} \rightarrow \bot
\end{array}$$

Figure 3.13: GAS rules for putting FlatCurry programs into canonical form (continued)

Let Case

$$f\ v_1 \dots v_n = \mathbf{let}\ x = \mathbf{case}\ e\ \mathbf{of}\ \{\, p_i \to e_i \,\}\ \mathbf{in}\ e' \quad \Rightarrow \quad \begin{array}{l} f\ v_1 \dots v_n = \mathbf{let}\ x = f_1\ x_1 \dots x_k\ \mathbf{in}\ e' \\ f_1\ x_1 \dots x_k = \mathbf{case}\ e\ \mathbf{of}\ \{\, p_i \to e_i \,\} \\ \quad \mathbf{where}\ \{x_1 \dots x_k\} {=}{=} \mathit{freeVars}\ [\, e, e_1, e_2 \dots e_n \,] \end{array}$$

Var Case

$$\mathbf{case}\ e\ \mathbf{of}\ \mathit{alts} \quad \Rightarrow \quad \mathbf{let}\ x = e\ \mathbf{in}\ \mathbf{case}\ x\ \mathbf{of}\ \mathit{alts}$$

Figure 3.14: Rule for moving a let bound case out of a function, and eliminating compound expressions in case-expressions.

3.1.2 ICurry

ICurry is meant to be a bridge between Curry code and imperative languages like C, Python, and Assembly. The let and case-expressions have been transformed into statements, and variables have been explicitly declared. All mutually recursive declarations are broken here into two steps. Declare memory for each node, then fill in the pointers. This allows us to create expression graphs with loops in them. Each function is organized into a sequence of blocks, and each block is broken up into declarations, assignments, and a single statement. A statement can either fail, return a new expression graph, or inspect a single variable to choose a case. It should be noted that restricting the scrutinee of the case statement to a single variable will cause efficiency problems, but we'll address this later.

After we've finished transforming the FlatCurry, the transformation to ICurry is much easier to implement. The algorithm from [15], given in figure 3.17, can be applied directly to the translated program. We show an example of translating the function f from figure 3.9 into ICurry in figure 3.18.

3.1.3 C

We already have a good idea of what the C code should look like, and our ICurry structure fits closely with this. This major difference is that we need to be sure to declare and allocate memory for all variables, which leads to a split in the structure of the generated code. The

$$
\begin{array}{l}
\mathit{poweraux}\ \mathit{v1}\ \mathit{v2}\ \mathit{v3} = \mathbf{case}\ (==)\ \mathit{v3}\ 0\ \mathbf{of} \\
\qquad\qquad \mathit{True} \rightarrow \mathit{v1} \\
\qquad\qquad \mathit{False} \rightarrow \mathbf{let}\ \mathit{v4} = \mathit{square}\ \mathit{v2} \\
\qquad\qquad\qquad\qquad \mathit{v5} = \mathit{halve}\ \mathit{v3} \\
\qquad\qquad\qquad \mathbf{in}\ \ \mathbf{case}\ (==)\ \boxed{(\mathit{apply}\ (\mathit{apply}\ \mathit{mod}\ \mathit{v3})\ 2)\ 1}\ \mathbf{of} \\
\qquad\qquad\qquad\qquad \mathit{True} \rightarrow \mathit{powaux}\ ((*)\ \mathit{v1}\ \mathit{v2})\ \mathit{v4}\ \mathit{v5} \\
\qquad\qquad\qquad\qquad \mathit{False} \rightarrow \mathit{powaux}\ \mathit{v1}\ \mathit{v4}\ \mathit{v5}
\end{array}
$$

$\Rrightarrow$ **Double Apply** $[1, -1, -1, 0]$

$$
\begin{array}{l}
\mathit{poweraux}\ \mathit{v1}\ \mathit{v2}\ \mathit{v3} = \mathbf{case}\ (==)\ \mathit{v3}\ 0\ \mathbf{of} \\
\qquad\qquad \mathit{True} \rightarrow \mathit{v1} \\
\qquad\qquad \mathit{False} \rightarrow \mathbf{let}\ \mathit{v4} = \boxed{\mathit{square}\ \mathit{v2}} \\
\qquad\qquad\qquad\qquad \mathit{v5} = \boxed{\mathit{halve}\ \mathit{v3}} \\
\qquad\qquad\qquad \mathbf{in}\ \ \mathbf{case}\ (==)\ (\mathit{apply}\ \mathit{mod}\ \mathit{v3}\ 2)\ 1\ \mathbf{of} \\
\qquad\qquad\qquad\qquad \mathit{True} \rightarrow \mathit{powaux}\ ((*)\ \mathit{v1}\ \mathit{v2})\ \mathit{v4}\ \mathit{v5} \\
\qquad\qquad\qquad\qquad \mathit{False} \rightarrow \mathit{powaux}\ \mathit{v1}\ \mathit{v4}\ \mathit{v5}
\end{array}
$$

$\Rrightarrow$ **Blocks** $[1]$

$$
\begin{array}{l}
\mathit{poweraux}\ \mathit{v1}\ \mathit{v2}\ \mathit{v3} = \mathbf{case}\ ((==)\ \mathit{v3}\ 0)\ \mathbf{of} \\
\qquad\qquad \mathit{True} \rightarrow \mathit{v1} \\
\qquad\qquad \mathit{False} \rightarrow \mathbf{let}\ \mathit{v4} = \mathit{square}\ \mathit{v2} \\
\qquad\qquad\qquad \mathbf{in}\ \ \mathbf{let}\ \mathit{v5} = \mathit{halve}\ \mathit{v3} \\
\qquad\qquad\qquad\quad \mathbf{in}\ \ \mathbf{case}\ (==)\ (\mathit{apply}\ \mathit{mod}\ \mathit{v3}\ 2)\ 1\ \mathbf{of} \\
\qquad\qquad\qquad\qquad\quad \mathit{True} \rightarrow \mathit{powaux}\ ((*)\ \mathit{v1}\ \mathit{v2})\ \mathit{v4}\ \mathit{v5} \\
\qquad\qquad\qquad\qquad\quad \mathit{False} \rightarrow \mathit{powaux}\ \mathit{v1}\ \mathit{v4}\ \mathit{v5}
\end{array}
$$

Figure 3.15: Reducing the *powaux* function defined in the standard Float library. The first reduction occurs in the *False* branch [1] or the in expression [-1] in the scrutinee of the case [-1] in the first argument of the apply node [0], so it has a path of [1,-1,-1,0].

code responsible for creating expression graphs and declaring memory will go in the *.h file, and the code for executing the hnf function will go in the *.c file. This is a common pattern for structuring C and C++ code, so it's not surprising that we take the same approach.

For each Data type D, we generate both a `make_D` function and a `set_D`. The difference is that `make_D` will allocate memory for a new node, while `set_D` takes an existing node as a parameter, and transforms it to the given type of node. We do the same thing for every ICurry function f, and produce a `make_f` and `set_f` function in C. Each node contains a `symbol`, that denotes the type of node, and holds information such as the name, arity, and hnf function of the node. Along with setting the `symbol` from section 2.3, the make and set functions reset the nondet flag to `false`, and set any children that were passed into the node.

As mentioned in the last chapter, for every function f, we need to generate a function for every case expression in f. $f|_p$ where p is a path to a case statement. The translation to C code is where we finally generate these functions. Surprisingly, this doesn't have too much of an effect on the code generator. Instead of generating code for a function f, we collect a list of pairs (f, p), where p is a path to a case statement, then we generate code for each (f, p) pair. In practice, it is useful to keep track of the declared variables, so we actually generate code for a triple $(f, vars, p)$. While generating code for (f, p), we track the current position p' of the code we're generating. When we need to generate code to push f onto the stack, such as when the code evaluates a non-deterministic expression, then we push `f_p'` onto the stack.

Aside from that change, generating the C code proceeds exactly as you would expect. Declarations are turned into C declarations; assignments are turned into calls to make functions; return statements are turned into calls to set functions; and case statements are turned into the while/switch construct from the last chapter.

At this point we can now produce a running program. In fact, the code produced by this compiler is already relatively efficient, As we will see later on, in many cases it outperforms the current state-of-the-art Curry compilers.

In this chapter, we've built up the GAS tool for performing transformations on FlatCurry code in a simple declarative way. We've already seen how this transformation tool eases the process of compiling programs, but it turns out to be much more powerful. As we'll see in the next chapter, when we apply GAS to optimizations, we can produce some complex optimizations, such as inlining, with rather minimal effort. This allows us to implement new optimizations faster, and test them more thoroughly, than if we had written each optimization by hand. In fact, it's often faster to implement an optimization, and just see what it does to the code, than to try to reason it out ourselves. This lets us iterate quickly on designing new optimizations, and produce more complex and powerful optimizations. Now we're cooking with GAS!

$$
\begin{array}{lll}
p & \Rightarrow t^* f^* & \textit{program} \\
t & \Rightarrow C_1\ C_2 \ldots C_n & \textit{datatype} \\
f & \Rightarrow f = b & \textit{function} \\
b & \Rightarrow d\_1 & \textit{block} \\
 & \quad \ldots & \\
 & \quad d\_k & \\
 & \quad a_1 & \\
 & \quad \ldots & \\
 & \quad a_n & \\
 & \quad s & \\
d & \Rightarrow \text{declare } x & \textit{variable declaration} \\
 & \mid \ \textit{declfree } x & \textbf{free} \textit{ variable declaration} \\
a & \Rightarrow v = e & \\
s & \Rightarrow \textit{return } e & \textit{return statement} \\
 & \mid \ \bot & \textit{failure} \\
 & \mid \ \textbf{case } x \textbf{ of} & \textbf{case} \textit{ statement} \\
 & \qquad C_1 \to b_1 & \\
 & \qquad \ldots & \\
 & \qquad C_n \to b_n & \\
e & \Rightarrow v & \textit{variable expression} \\
 & \mid \ \textit{NODE } (l, e_1, \ldots, e_n) & \textit{node creation} \\
 & \mid \ e_1 \ ? \ e_2 & \textit{choice expression} \\
v & \Rightarrow x & \textit{local variable} \\
 & \mid \ v\ [i] & \textit{variable access} \\
 & \mid \ \textit{ROOT} & \textit{root variable} \\
l & \Rightarrow C & \textit{constructor label} \\
 & \mid \ f & \textit{function label} \\
 & \mid \ \textit{LABEL } (v) & \textit{variable label}
\end{array}
$$

Figure 3.16: Abstract syntax of function definitions in ICurry

$\mathcal{F}\ (f\ (x_1, \ldots, x_n) = e) := f = \mathcal{B}\ (x_1, \ldots, x_n, e, ROOT)$

$\mathcal{B}\ (\{x\}, \bot, root) \qquad := \bot$

$\mathcal{B}\ (\{x\}, e, root) \qquad :=$

$\quad \{\text{declare } x\}$

$\quad \mathcal{D}\ (e)$

$\quad \{x\_i = root\ [i]\}$

$\quad \mathcal{A}\ (e)$

$\quad \mathcal{R}\ (e)$

$\mathcal{D}\ (\textbf{let}\ \{x\}\ \textbf{free in}\ e) \qquad := \{\text{free } x\}$

$\mathcal{D}\ (\textbf{let}\ \{x = e\}\ \textbf{in}\ e') \qquad := \{\text{declare } x\}$

$\mathcal{D}\ (\textbf{case}\ e\ \textbf{of}\ \{p \to e\}) := \text{declare } x\_e$

$\mathcal{A}\ (\textbf{let}\ \{x = e\}\ \textbf{in}\ e') \qquad :=$

$\quad \{x = \mathcal{E}\ (e)\}$

$\quad [x\_i\ [p] = x\_j \mid x\_i, x\_j`\ \textbf{in}\ `\{x\}, e_i \mid\_\ p = x\_j]$

$\mathcal{A}\ (\textbf{case}\ e\ \textbf{of}\ \_) := x\_e = \mathcal{E}\ (e)$

$\mathcal{R}\ (\textbf{case}\ e'\ \textbf{of}\ \{\ C\ (\{x\}) \to e\} := \textbf{case}\ E\ (e)\ \textbf{of}\ \mathcal{B}\ (\{x\}, e, x\_e)$

$\mathcal{R}\ (e) \qquad := return\ E\ (e)$

$\mathcal{E}\ (x) \qquad := x$

$\mathcal{E}\ (c\ \{e\}) \qquad := NODE\ (c, \{\mathcal{E}\ (e)\})$

$\mathcal{E}\ (f\ \{e\}) \qquad := NODE\ (f, \{\mathcal{E}\ (e)\})$

$\mathcal{E}\ (e_1\ ?\ e_2) \qquad := \mathcal{E}\ (e_1)\ ?\ \mathcal{E}\ (e_2)$

$\mathcal{E}\ (\textbf{let}\ \{x = e\}\ \textbf{in}\ e') \qquad := \mathcal{E}\ (e)$

$\mathcal{E}\ (\textbf{let}\ \{x\}\ \textbf{free in}\ e) \qquad := \mathcal{E}\ (e)$

Figure 3.17: Algorithm for translating FlatCurry into ICurry

After Post-processing

$$
\begin{aligned}
&f\ v_1 = \textbf{let}\ v_4 = f_0\ v_1\\
&\qquad\quad \textbf{in}\ \ \textbf{let}\ v_5 = f_1\ v_1\\
&\qquad\qquad\quad \textbf{in}\ \ v_4\ ?\ v_5\\
&f_0\ v_1 = \textbf{fcase}\ v_1\ \textbf{of}\\
&\qquad\qquad\quad \mathit{True} \to \mathit{False}\\
&\qquad\qquad\quad \mathit{False} \to \bot\\
&f_1\ v_1 = \textbf{fcase}\ v_1\ \textbf{of}\\
&\qquad\qquad\quad \mathit{True} \to \mathit{False}\\
&\qquad\qquad\quad \mathit{False} \to\ \mathit{True}
\end{aligned}
$$

ICurry

$$
\begin{aligned}
&f\ /\ 1 : \{\\
&\quad \text{declare } x_2\\
&\quad \text{declare } x_3\\
&\quad \text{declare } x_4\\
&\quad x_2 = \mathit{ROOT}\ [0]\\
&\quad x_3 = f_0\ (\mathit{x2})\\
&\quad x_4 = f_1\ (\mathit{x2})\\
&\quad \mathit{return}\ (x_3\ ?\ x_4)\\
&\}\\
&f_0\ /\ 1 : \{\\
&\quad \text{declare } x_2\\
&\quad x_2 = \mathit{ROOT}\ [0]\\
&\quad \textbf{case}\ x_2\ \textbf{of}\\
&\qquad \mathit{True}\ /\ 0 \to \{\\
&\qquad\quad \mathit{return}\ \mathit{False}\ ()\\
&\qquad \}\\
&\qquad \mathit{False}\ /\ 0 \to \{\\
&\qquad\quad \mathit{exempt}\\
&\qquad \}\\
&\}
\end{aligned}
$$

Figure 3.18: translating f and f_0 into icurry.

We omit f_1 for space, and because it's largely the same as f_0.

Chapter 4

BASIC OPTIMIZATIONS

In the last chapter we saw how the GAS tool let us write transformation rules as rewrite rules in Curry. The power of this tool came from two aspects. The first is that it's easy to write rules syntactically. The second is that the rules are written in Curry, so we are not limited by our rewriting system. We'll put this second part to use in optimizating Curry expressions.

In this chapter we outline a number of optimizations that were necessary to implement in order for unboxing, deforestation, and shortcutting to be effective. We start by introducing a new restriction on FlatCurry expressions called Administrative Normal Form, or A-Normal Form. This is a common form for functional program optimizers to take, and it provides several benefits to Curry too. We describe the transformation, and why it's useful, then we detail a few smaller optimizations that move let-expressions around. The goal is to move the let-expression to a position just before the variable is used in the expression. Finally we discuss four optimizations that will do most of the work in the compiler: Case canceling, dead code elimination, inlining, and reduction. These optimization are an important part of any optimizing compiler, but they are often tricky to get right. In fact, with the exception of dead code elimination, It's not clear at all that they are even valid for Curry. We show an effective method to implement them in a way that they remain valid for Curry expressions.

4.0.1 A-Normal Form

Before we discuss any substantial optimizations, we need to deal with a significant roadblock to optimizing Curry. Equational reasoning is not valid for Curry programs. Consider the following program.

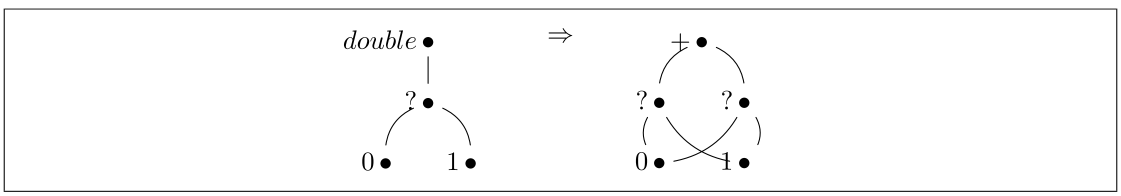
$$\begin{aligned} &double\ x = x + x \\ &main \quad\;\; = double\ (0\ ?\ 1) \end{aligned}$$

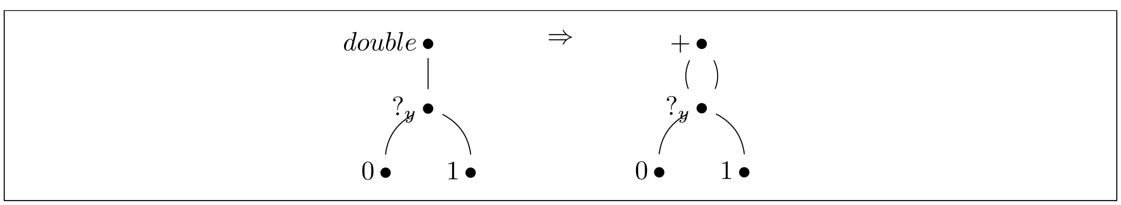
In pure lazy functional languages, it is always safe to replace a function with its definition. So we should be able to rewrite *main* to $(0+1)\,?\,(0+1)$, but this expression will produce a different set of answers. This is the primary problem with optimizing functional logic languages, but exactly why this happens is a bit tricky to pin down. The non-determinism isn't the only problem, for a trivial example inlining $id\ (0\ ?\ 1)$ is fine. We can even duplicate non-deterministic expressions with the following example.

$$\begin{aligned} &double\ x = x + x \\ &main \quad\ = \textbf{let}\ y = (0\ ?\ 1) \\ &\qquad\qquad\ \ \textbf{in}\ double\ y \end{aligned}$$

Here y is a non-deterministic expression, because it produces two answers when evaluated, but the expression **let** $y = (0\,?\,1)$ **in** $y+y$ is just fine. So, what is the problem with our first example?

The real problem is a bit more subtle, and we have to step back into the world of graph rewriting. If we construct the graph for the first expression we see:





Now the real issue comes to light. In the second example, while we copied a non-deterministic expression in the code, we didn't copy the non-deterministic expression in the graph. This gives us a powerful tool when reasoning about Curry expressions. Even if a variable is duplicated in the source code, it is not copied in the graph. Since this duplication of non-deterministic expressions was the main concern for correctness, the solution is pretty straightforward. Every expression that could be copied should be explicitly stored in a variable.

We can enforce this restriction by disallowing any compound expressions. Specifically, all function calls, constructor calls, choices, and case expression must either be applied to literal values or variables. Fortunately we are not the first to come up with this idea. In fact this restricted form is used in many functional compilers, and is known as Administrative Normal Form

$$
\begin{array}{lll}
a \Rightarrow & v & \textit{Variable} \\
| & l & \textit{Literal} \\
| & \bot & \textit{Failed} \\
e \Rightarrow & a_1 \; ? \; a\_2 & \textit{Choice} \\
| & f_k \; a_1 \; a\_2 \ldots a_n & \textit{Function Application} \\
| & C_k \; a_1 \; a\_2 \ldots a_n & \textit{Constructor Application} \\
| & \textbf{let } v_1 = e_2 \ldots v_n = e_n \textbf{ in } e & \textit{Variable Declaration} \\
| & \textbf{let } v_1, v_2, \ldots v_n \textbf{ free in } e & \textit{Free Variable Declaration} \\
| & \textbf{case } a \textbf{ of } \{ p_1 \to e_1; \ldots p_n \to e_n \} & \textit{Case Expression} \\
p \Rightarrow & C \; v_1 \; v_2 \ldots v_n & \textit{Constructor Pattern} \\
| & l & \textit{Literal Pattern}
\end{array}
$$

Figure 4.1: Restricting Curry expressions to A-Normal Form.
An atom is either a variable, a literal, or a failure. Compound expressions are only allowed to contain atoms.

(ANF) [37]. The idea originally was to take CPS, another well known intermediate representation for functional languages, and remove common “administrative redexes”. After removing the administrative redexes, we can remove the continuations, and rewrite the program using let-expressions. Flanagan et al. showed that these transformations can be reduced into a single A-Normal form transformation. We give the definition of A-Normal Form for Curry programs in figure 4.1 and we implement the transformation using GAS in figure 4.2.

As long as we enforce this A-Normal Form structure, we restore equational reasoning for Curry programs. We don’t even need to enforce A-Normal Form strictly here. During optimization, it’s often useful to be able to inline constructors and partial applications. Since a constructor is computationally inert, we can inline one without fear of problems with non-deterministic expressions. This will be referred to as limited A-Normal Form.

In fact, this idea isn’t even new to Curry. In [4] FlatCurry programs are translated into a “Normal Form”, before evaluation begins. Although, it should be noted that this form is more limited because case statements and choice expressions may contain compound expressions. We choose to flatten these expressions as well for two reasons. It produces more uniform, albeit longer,

$$
\begin{array}{lll}
\textbf{case } e \textbf{ of} \ldots & \Rightarrow \textbf{let } x = e & \textbf{in case } x \textbf{ of} \ldots \\
f\ a_1\ a\_2 \ldots e_k \ldots e_n & \Rightarrow \textbf{let } x = e_k & \textbf{in } f\ a_1\ a\_2 \ldots x \ldots e_n \\
e_1\ ?\ e_2 & \Rightarrow \textbf{let } x = e_1 & \textbf{in } x\ ?\ e_2 \\
a_1\ ?\ e_2 & \Rightarrow \textbf{let } x = e_2 & \textbf{in } a_1\ ?\ x
\end{array}
$$

Figure 4.2: Rules for transforming Curry expression to A-Normal Form.
a is used for atoms, e is used for arbitrary expressions, and x is a fresh variable name.

programs, and more optimizing transformations become valid. Some examples of programs in ANF are given in figure 4.3.

4.1 CASE CANCELING

Finally, we come to our first example of an optimization. In fact, this is arguably our most important optimization. It's a very simple optimization, but it proves to be very powerful. Consider the following code:

$$
\begin{aligned}
\mathit{notTrue} = \textbf{case } & \mathit{True} \textbf{ of} \\
& \mathit{True} \rightarrow \mathit{False} \\
& \mathit{False} \rightarrow \mathit{True}
\end{aligned}
$$

While it is very unlikely that a programmer would actually write this code, but as we'll see, this code comes up frequently when inlining. This is fantastic, because it's clear what we should do here. We know that the *True* branch will be taken, so we might as well discard the case expression altogether.

$$\mathit{notTrue} = \mathit{False}$$

This transformation is called Case Canceling, and it's the workhorse of all of our other optimizations. The transformation, given in 4.4 looks intimidating at first, but it's very straightforward. If the scrutinee of a case is a constructor, then we find the appropriate branch, and reduce to that branch. The only real complication is that we need to keep the expression in A-Normal form. However, we can simply add let-expressions for every variable that the constructor binds.

We also include two other optimizations. These optimizations are really about cleaning up

$$
\begin{array}{l}
\mathit{fib}\ n = \mathbf{case}\ n < 1 \\
\qquad\qquad \mathit{True}\ \to n \\
\qquad\qquad \mathit{False} \to \mathit{fib}\ (n-1) + \mathit{fib}\ (n-2)
\end{array}
$$

$$
\begin{array}{l}
\mathit{fib}\ n = \mathbf{let}\ x = n < 1 \\
\qquad\quad \mathbf{in}\ \ \mathbf{case}\ x\ \mathbf{of} \\
\qquad\qquad\qquad \mathit{True}\ \to n \\
\qquad\qquad\qquad \mathit{False} \to \mathbf{let}\ \mathit{n1} = n - 1 \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{n2} = n - 2 \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{f1} = \mathit{fib}\ \mathit{n1} \\
\qquad\qquad\qquad\qquad\qquad\quad \mathit{f2} = \mathit{fib}\ \mathit{n2} \\
\qquad\qquad\qquad\qquad\quad \mathbf{in}\ \ \mathit{f1} + \mathit{f2}
\end{array}
$$

$$
\begin{array}{l}
\mathit{sumPrimes} = \mathit{foldr}\ (+)\ 0 \\
\quad \circ\, \mathit{filter}\ \mathit{isPrime} \\
\quad \circ\ \mathit{enumFromTo}\ 1
\end{array}
$$

$$
\begin{array}{l}
\mathit{sumPrimes} = \mathbf{let}\ \mathit{v1} = (+) \\
\mathbf{in}\ \mathbf{let}\ \mathit{v2} = \mathit{foldr}\ \mathit{v1}\ 0 \\
\mathbf{in}\ \mathbf{let}\ \mathit{v3} = \mathit{isPrime} \\
\mathbf{in}\ \mathbf{let}\ \mathit{v4} = \mathit{filter}\ \mathit{v3} \\
\mathbf{in}\ \mathbf{let}\ \mathit{v5} = \mathit{enumFromTo}\ 1 \\
\mathbf{in}\ \mathbf{let}\ \mathit{v6} = \mathit{v4} \circ \mathit{v5} \\
\mathbf{in}\ \mathit{v2} \circ \mathit{v6}
\end{array}
$$

Figure 4.3: Examples of Curry programs translated to A-Normal Form

after Case Canceling runs. The first is Case Variable elimination. The idea is also simple. Consider the following expression from the optimization of *compare* for *Bool*:

```
...
  in case v2 of
        True  → LT
        False → case v2 of
                    True  → EQ
                    False → case v1 of
                                True  → GT
                                False → EQ
...
⇒ Case Var [−1, 0, −1]
...
  in case v2 of
        True  → LT
        False → case False of
                    True  → EQ
                    False → case v1 of
                                True  → GT
                                False → EQ
...
⇒ Case Cancel [−1, 0, −1, 1]
...
  in case v2 of
        True  → LT
        False → case v1 of
                    True  → GT
                    False → EQ
...
```

The use of Case Variable elimination allows us to set up a situation where a case can cancel later. This occurs a lot in practice, but this optimization may raise red flags for some. If we're replacing a variable, which represents a node in our expression graph, with a completely distinct literal value, how do we know that this replacement is valid? This isn't clear. In fact, in general it's not valid to replace a variable. That variable could be shared, and it could represent a non-deterministic expression. Case Canceling is fine, because we only change a case statement, which only affects flow of control. It doesn't actually affect any of the data in the expression graph. However, we can justify this by looking at the computation space for our expression again. Suppose that we have the expression **case** x **in** e. If x is deterministic, then there is no problem. If x is non-deterministic, then x has already been reduced to head normal form, and been pushed on the backtracking stack. Furthermore, the root of this function has also been pushed on the backtracking stack, since **case** x **of** e depended on a non-deterministic variable. So, if x has changed while backtracking, the current expression has been undone. Therefore replacing x with a constructor in this expression has no effect.

Finally we have Dead Code Elimination. This is a standard optimization. In short, if we have an empty **let** or **free** expression, then we can remove them. Furthermore if a variable is never used, then it can also be removed. Finally, if we have **let** $x = e$ **in** x, then we don't need to create the variable x. These are all clearly correct, as long as we're careful to make sure that our variable definitions aren't recursive.

Now that we've finally created an optimization, we can get back to moving code around in convoluted patterns. In the next section we look at how we can inline functions. Unlike Case Canceling, Inlining isn't obviously correct, and, in fact, we have to do a lot of work to inline functions in Curry.

4.2 INLINING

As mentioned at the start of this chapter, inlining isn't generally valid in Curry. So, we need to establish cases when inlining is valid, determine when it's a good idea to inline, and ensure that our inlining algorithm is correct. This work is largely based on [79, 27].

Similarly to [79], we need to make a distinction between inlining and reduction. When we use the term *inlining* we are referring to replacing a let bound variable with it's definition. For example **let** $x = \mathit{True}$ **in** $\mathit{not}\ x$ could inline to $\mathit{not}\ \mathit{True}$. When we use the term *reduction*, we are referring to replacing a function call with the body of the function. Again, as an example **let** $x = \mathit{True}$ **in** $\mathit{not}\ x$ could reduce to:

Case Cancel

$$
\begin{array}{ll}
\textbf{case}\ (C_i\ e_{i,1} \ldots e_{k_i})\ \textbf{of} & \Rightarrow \textbf{let} \quad x_{i1} = e_1 \\
\quad C_1\ x_{1,1} \ldots x_{1,k_1} \rightarrow e_1 & \qquad\quad \ldots \\
\quad C_2\ x_{2,1} \ldots x_{2,k_2} \rightarrow e_2 & \quad \textbf{in let}\ x_{i,k_i} = e_{k_i} \\
\quad \ldots & \quad \textbf{in} \quad e_i \\
\quad C_i\ x_{i1} \ldots x_{i,k_i} \rightarrow e_i & \\
\quad \ldots & \\
\quad C_n\ x_{n,1} \ldots x_{n,k_n} \rightarrow e_n &
\end{array}
$$

Case Var

$$
\begin{array}{ll}
\textbf{case}\ x\ \textbf{of} & \Rightarrow \textbf{case}\ x\ \textbf{of} \\
\quad (C \ldots) \rightarrow \ldots x \ldots & \qquad (C \ldots) \rightarrow \ldots (C \ldots) \ldots
\end{array}
$$

Dead Code

$$
\begin{array}{ll}
\textbf{let free in}\ e & \Rightarrow e \\
\textbf{let in}\ e & \Rightarrow e \\
\textbf{let}\ v\ \textbf{free in}\ e & \mid v \notin e \Rightarrow e \\
\textbf{let}\ v = \ldots \textbf{in}\ e & \mid v \notin e \Rightarrow e \\
\textbf{let}\ v = e\ \ \textbf{in}\ v & \mid v \notin e \Rightarrow e
\end{array}
$$

Figure 4.4: Case Canceling, Case Variable, and Dead Code Elimination optimizations.

$$
\begin{array}{l}
\textbf{let } x = \mathit{True} \\
\textbf{in } \textbf{case } x \textbf{ of} \\
\qquad \mathit{True} \rightarrow \mathit{False} \\
\qquad \mathit{False} \rightarrow \mathit{True}
\end{array}
$$

The first problem with inlining and reduction we encounter is recursion. Consider the expression:

$$\textbf{let } \mathit{loop} = \mathit{loop} \textbf{ in} \ldots$$

If we were to inline this variable, we could potentially send the optimizer into an infinite loop. So, we need to somehow mark all recursive variables and functions. The next problem follows immediately after that. So far we've done transformations with local information, but reduction is going to require global information. In fact, for reduction to be effective, it will require information from different modules. Consider the function:

$$\mathit{sumPrimes} = \mathit{foldr}\ (+)\ 0 \circ \mathit{filter}\ \mathit{isPrime} \circ \mathit{enumFromTo}\ 1$$

Aside from the fact that *sumPrimes* contains mostly recursive functions, we wouldn't be able to optimize it anyway, because $\circ$ is defined in the standard Prelude. If we can't inline the definition of $\circ$, then we're fighting a losing battle.

This brings us to our third problem with inlining. The *sumPrimes* function is actually partially applied. Its type should be $\mathit{sumPrimes} :: \mathit{Int} \rightarrow \mathit{Int}$, but *sumPrimes* is defined in a point-free style. Point-free programming causes a lot of problems, specifically because FlatCurry is a combinator language. In IR's like Haskell's Core, we could solve this problem by inlining a lambda expression, but it's not clear at all that inlining a lambda expression is valid in Curry. Instead, to solve this problem, we convert functions to be fully applied.

In order to solve these problems, we keep a map from function names to several attributes about the function. This includes: if the function is defined externally; if the function is known to be deterministic; if the function contains cases; the parameters of the function; the current number of variables in a function; the size of the function; and the function definition. This map is updated every time we optimize a new function, so we can inline all functions that we've already optimized.

4.2.1 Partial Applications

Dealing with partial applications is a bit more tricky. In fact, we can't use the GAS system to solve this problem because we may not know if a function is a partial application until we've optimized it. Consider the *sumPrimes* function again. It doesn't look like a partial application because the root function, $\circ$, is fully applied. Let's look at the definition for $\circ$. In Curry it's defined using a lambda expression.

$$f \circ g = \lambda x \rightarrow f\ (g\ x)$$

However, when translated to FlatCurry, this lambda expression is turned into a combinator.

$$f \circ g = compLambda_1\ f\ g$$
$$compLambda\ f\ g\ x = f\ (g\ x)$$

So, when we try to inline *sumPrimes* we end up with the following derivation.

$\textbf{let}\ v1 = (+)$
$\textbf{in let}\ v2 = foldr\ v1\ 0$
$\textbf{in let}\ v3 = isPrime$
$\textbf{in let}\ v4 = filter\ v3$
$\textbf{in let}\ v5 = enumFromTo\ 1$
$\textbf{in let}\ v6 = v4 \circ v5$
$\textbf{in}\ v2 \circ v6$
$REDUCE \Rightarrow [-1, -1, -1, -1, -1, -1]$
$\textbf{let}\ v1 = (+)$
$\textbf{in let}\ v2 = foldr_1\ v1\ 0$
$\textbf{in let}\ v3 = isPrime\_1$
$\textbf{in let}\ v4 = filter\_1\ v3$
$\textbf{in let}\ v5 = enumFromTo\_1\ 1$
$\textbf{in let}\ v6 = v4 \circ v5$
$\textbf{in}\ compLambda_1$
$REDUCE \Rightarrow [-1, -1, -1, -1, -1]$
$\textbf{let}\ v1 = p_2$
$\textbf{in let}\ v2 = foldr_1\ v1\ 0$

in let $v3 = isPrime\_1$

in let $v4 = filter\_1\ v3$

in let $v5 = enumFromTo\_1\ 1$

in let $v6 = compLambda_1\ v4\ v5$

in $compLambda_1\ v2\ v6$

At this point there's no more optimization that can be done, because everything is a partial function. But this is clearly ridiculous. We've created a pipeline, and when we pass it a variable, then everything will be fully applied. So, how do we solve the problem?

The key is to notice that if the root of the function is a partial application, then we can rewrite our definition.

$f\ v_1 \ldots v_k = g_n \ldots$

$\Rightarrow$ **Add Missing Variables**

$f\ v_1 \ldots v_k\ x_1 \ldots x_n = apply\ (g_n \ldots)\ x_1 \ldots x_n$

where $x_1 \ldots x_n$ *are fresh variables*

The *sumPrimes* functions is transformed with the derivation in 4.5 and we can continue to optimize the function.

Unfortunately, since this involves the definition of the function itself, and not just its body, we can't use the GAS system here. However, this does solve our problem. It leads to a new problem though. Since we are changing the arity of functions, any function calls may have the wrong arity.

4.2.2 The Function Table

In order to keep track of all of the functions we've optimized we create a function lookup table called $\mathcal{F}$. The function table is just a map from function names to information about the function. We use the following definitions for lookups into the function table. $I_\mathcal{F}\ f$ returns true if we believe that f is a good candidate for reduction. We have designed the compiler so that the heuristic we use is easy to tweak, but at the very least f should not be external, or a loop breaker, and should not be too big. $U_\mathcal{F}\ x\ f\ e$ attempts to determine if reducing the function f in the expression **let** $x = f \ldots$ **in** e would be useful. Again this heuristic is easily tweakable, but currently, a function is useful if x is returned from the function, it's used as the scrutinee of a case expression, or it's used in a function that's likely to be reduced. $S_\mathcal{F}\ f$ returns True if

$\textbf{let}\ v_1 = (+)$

$\textbf{in let}\ v_2 = \mathit{foldr}\ v_1\ 0$

$\textbf{in let}\ v_3 = \mathit{isPrime}$

$\textbf{in let}\ v_4 = \mathit{filter}\ v_3$

$\textbf{in let}\ v_5 = \mathit{enumFromTo}\ 1$

$\textbf{in let}\ v_6 = \mathit{compLambda}_1\ v_4\ v_5$

$\textbf{in}\ \mathit{compLambda}_1\ v_2\ v_6$

$\Rightarrow$ **Add Missing Variables**

$\mathit{apply}\ (\textbf{let}\ v_1 = (+)$

$\quad\quad \textbf{in let}\ v_2 = \mathit{foldr}\ v_1\ 0$

$\quad\quad \textbf{in let}\ v_3 = \mathit{isPrime}$

$\quad\quad \textbf{in let}\ v_4 = \mathit{filter}\ v_3$

$\quad\quad \textbf{in let}\ v_5 = \mathit{enumFromTo}\ 1$

$\quad\quad \textbf{in let}\ v_6 = \mathit{compLambda}_1\ v_4\ v_5$

$\quad\quad \textbf{in}\ \mathit{compLambda}_1\ v_2\ v_6)\ x_1$

...

$\Rightarrow$ **Let Floating**

$\textbf{let}\ v_1 = (+)$

$\textbf{in let}\ v_2 = \mathit{foldr}\ v_1\ 0$

$\textbf{in let}\ v_3 = \mathit{isPrime}$

$\textbf{in let}\ v_4 = \mathit{filter}\ v_3$

$\textbf{in let}\ v_5 = \mathit{enumFromTo}\ 1$

$\textbf{in let}\ v_6 = \mathit{compLambda}_1\ v_4\ v_5$

$\textbf{in}\ \mathit{apply}\ (\mathit{compLambda}_1\ v_2\ v_6)\ x_1$

$\Rightarrow$ **Unapply**

$\textbf{let}\ v_1 = (+)$

$\textbf{in let}\ v_2 = \mathit{foldr}\ v_1\ 0$

$\textbf{in let}\ v_3 = \mathit{isPrime}$

$\textbf{in let}\ v_4 = \mathit{filter}\ v_3$

$\textbf{in let}\ v_5 = \mathit{enumFromTo}\ 1$

$\textbf{in let}\ v_6 = \mathit{compLambda}_1\ v_4\ v_5$

$\textbf{in}\ \mathit{compLambda}\ v_2\ v_6\ x_1$

Figure 4.5: Adding a missing variable to *sumPrimes*

f is a simple reduction with no case expressions. It's always useful to reduce these functions. $C_{\mathcal{F}}\ f\ [e_1, \ldots e_n]$ returns true if reducing f with $e_1 \ldots e_n$ will likely cause Case Canceling.

4.2.3 Function Ordering

The problem of function ordering seems like it should be pretty inconsequential, but it turns out to be very important. However, this problem has already been well studied [27, 79], and the solutions for other languages apply equally well to Curry.

The problem seems very complicated at the start. We want to know what is the best order to optimize functions. Fortunately there's a very natural solution. If possible we should optimize a function before we optimize any function that calls it. This turns out to be an exercise in Graph Theory.

We define the Call Graph of a set of function $\mathbf{F} = \{f_1, f_2, \ldots f_n\}$ to be the graph $G_{\mathbf{F}} = (\mathbf{F}, \{f_i \rightarrow f_j | f_i \ \ calls \ \ f_j\})$. This problem reduces to finding the topological ordering of $G_{\mathbf{F}}$. Unfortunately, if $\mathbf{F}$ contains any recursion, then the topological ordering isn't defined. So, instead, we split $G_{\mathbf{F}}$ into strongly connected components, and find the topological ordering of those components. Within each component, we pick a "loop breaker" which is removed from the graph, and attempt to find the topological order of each component again. This process repeats until our graph is acyclic.

These loop breakers are marked in $\mathcal{F}$, and they are never allowed to be reduced. Every other function can be reduced, because all functions that it calls, except for possibly the loop breakers, have been optimized.

Consider the program:

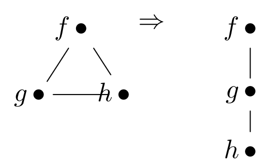
$f\ x = g\ x$

$g\ x = h\ x$

$h\ x = \textbf{case}\ x\ \textbf{of}$

$\quad 0 \rightarrow 0$

$\quad \_ \rightarrow 1 + f\ x$



The graph for this function is a triangle, because f calles g which calls h which calls f. However, if we mark h as a loop breaker, then suddenly this problem is easy. When we optimize h, we are free to reduce f and g.

$h\ x = \textbf{case}\ x\ \textbf{of}$

$\quad 0 \rightarrow 0$

$\quad \_ \rightarrow 1 + f\ x$

$\Rightarrow \mathit{REDUCE}$

$h\ x = \textbf{case}\ x\ \textbf{of}$

$\quad 0 \rightarrow 0$

$\quad \_ \rightarrow 1 + g\ x$

$\Rightarrow \mathit{REDUCE}$

$h\ x = \textbf{case}\ x\ \textbf{of}$

$\quad 0 \rightarrow 0$

$\quad \_ \rightarrow 1 + h\ x$

4.2.4 Inlining

Now that we have everything in order, we can start developing the inlining transformation. As mentioned before, we need to be careful with inlining. In general, unrestricted inlining isn't valid in Curry. This is a large change from lazy languages like Haskell, where it's valid, but not always a good idea. The other major distinction is that FlatCurry is a combinator language. This means that we have no lambda expressions, which limits what we can even do with inlining.

Fortunately for us, these problems actually end up canceling each other out. In Peyton-Jones work [79] most of the focus was on inlining let bound variables, because this is where duplication of computation could occur. However, we have two things working for us. The first is that we can't inline a lambda since they don't exist. The second is that we've translated FlatCurry to A-Normal Form. While Haskell programs are put into A-Normal Form when translating to STG code [83], this is not the case for Core. Certain constraints are enforced, such as the trivial constructor argument invariant, but in general Core is less restricted.

Translating to A-Normal form gives us an important result. If we inline a constructor, a literal or a variable, then we don't affect the computed results.

Theorem 7. *If* **let** $x = e_1$ **in** e *is a Curry expression in limited A-Normal Form, and* e_1 *is a constructor, literal, or variable, or partial application, then* $e[x \mapsto e_1]$ *computes the same results.*

Proof. The cases for literals and variables are trivial, so we omit them here. Also, note that a partial application functions identically to a constructor, since the only place they can be reduced is the constructor case of `apply_hnf`.

Now let's consider the case of **let** $x = C \ldots$ **in** e. Assume that C is applied to either variables or literal values. It may be the case that $e[x \mapsto C \ldots]$ duplicates C. However, since C is already constructor, it is a deterministic node. By the path compression theorem $e[x \mapsto C \ldots]$ computes the same results.

If C is applied to other constructors or partial applications, then the result follows from structural induction. □

Now we have enough information to inline variables as long as we restrict inlining to literals, constructors, variables, and partial applications, although the case for variables is already subsumed by **Alias**. We add two new rules. **Let Folding** allows us to move variable definitions closer to where they're actually used, and **Unapply** allows us to simplify expressions involving *apply*. Both of these are useful for inlining and reduction. The GAS rules are given in figure 4.6. Note that the **Unapply** rule corresponds exactly to the evaluation step for application nodes in our semantics.

4.2.5 Reduce

Finally we come to reduction. While this was a simpler task than inlining in GHC, it becomes a very tricky prospect in Curry. Fortunately, we've already done the hard work. At this point, the only place a function is allowed to appear in our expressions is as the root of the expression, as the root of a branch in a **case** expression, as the root the result of a **let** expression, or as a variable assignment in a let-expression. Furthermore our functions only contain trivial arguments, so it's now valid to reduce any function we come across.

Theorem 8 (reduction)**.** *let* e *be an expression in limited A-Normal Form, and let* $e|_p = f\ e_1 \ldots e_n$*. then* $e[p \mapsto (body f)[v_1 \mapsto e_1, \ldots v_n \mapsto e_n]]$ *computes the same results as* e*.*

Let Folding

$$\textbf{let } v = e_1 \textbf{ in case } e \textbf{ of} \ldots \mid v \notin e \quad \Rightarrow \textbf{case } e \textbf{ of } C_i \ldots \to \textbf{let } v = e_1 \textbf{ in} \ldots$$

Unapply

$$\begin{array}{lll} \textit{apply } f_k \textit{ as} & \mid k < n & \Rightarrow \textit{apply } (\textit{apply } f \textit{ as}_1) \textit{ as}_2 \\ & \mid k == n & \Rightarrow f \textit{ as} \\ & \mid k > n & \Rightarrow f_{k-n} \textit{ as} \end{array}$$

Inline Constructor

$$\textbf{let } x = C\ v_1 \ldots v_n \textbf{ in } e \quad \mid x \notin \{v_1, \ldots v_n\} \Rightarrow e[x \mapsto C\ v_1\ v_2 \ldots v_n]$$

Inline Partial

$$\textbf{let } x = f_k\ v_1 \ldots v_n \textbf{ in } e \quad \mid x \notin \{v_1, \ldots v_n\} \Rightarrow e[x \mapsto f_k\ v_1\ v_2 \ldots v_n]$$

Inline Literal

$$\textbf{let } x = l \quad \textbf{in } e \quad \Rightarrow e[x \mapsto l]$$

Figure 4.6: rules for variable inlining.

We need to ensure that x isn't used recursively before we inline it.

Proof. If f is the root of the expression, or f is the root of a branch or let, then this is equivalent to taking a step in our semantics. If that particular branch of the case is never evaluated at runtime, then reducing f has no effect.

If f is a let bound variable, i.e. $\textbf{let } x = f \ldots \textbf{in} \ldots$, when we replace f it's still bound to a single node in the graph.

Furthermore, only nodes that can be duplicated while reducing f are the arguments. However, the arguments are constrained to be literals, constructors, variables, or partial applications. Therefore, we are never duplicating any choice nodes.

Finally, replacing f with it's definition is a single deterministic step, so we can apply the path compression theorem again to obtain our result.

□

We give the GAS rules for reduction in figure 4.7. The rules are straightforward, and we make sure that $B_{\mathcal{F}}\ f$ replaces the definition with fresh variables. Therefore, we avoid any need to deal with shadowing and name capture. This strategy was taken from [79] and it works very well. Although, since FlatCurry uses numbers exclusively to represent variables, we don't get

Reduce Base

$f\ e_1 \dots e_n \qquad\qquad | \ top\ \&\ I_{\mathcal{F}}\ f \Rightarrow (\mathit{body} f)[v_1 \mapsto e_1, \dots v_n \mapsto e_n]$

Reduce Branch

$\textbf{case}\ e\ \textbf{of}\ \mathit{Ctr} \to f\ e_1 \dots e_n \ | \ I_{\mathcal{F}}\ f \qquad \Rightarrow \textbf{case}\ e\ \textbf{of}\ \mathit{Ctr} \to (\mathit{body} f)[v_1 \mapsto e_1, \dots v_n \mapsto e_n]$

Reduce Let

$\textbf{let} \dots \textbf{in}\ f\ e_1 \dots e_n \qquad | \ I_{\mathcal{F}}\ f \qquad \Rightarrow \textbf{let} \dots \textbf{in}\ (\mathit{body} f)[v_1 \mapsto e_1, \dots v_n \mapsto e_n]$

Reduce Useful

$\textbf{let}\ x = f\ e_1 \dots e_n\ \textbf{in}\ e \qquad | \ U_{\mathcal{F}}\ x\ f\ e \quad \Rightarrow \textbf{let}\ x = (\mathit{body} f)[v_1 \mapsto e_1, \dots v_n \mapsto e_n]\ \textbf{in}\ e$

Reduce Simple

$\textbf{let}\ x = f\ e_1 \dots e_n\ \textbf{in}\ e \qquad | \ S_{\mathcal{F}}\ f \qquad \Rightarrow \textbf{let}\ x = (\mathit{body} f)[v_1 \mapsto e_1, \dots v_n \mapsto e_n]\ \textbf{in}\ e$

Reduce Cancels

$\textbf{let}\ x = f\ e_1 \dots e_n\ \textbf{in}\ e \qquad | \ C_{\mathcal{F}}\ f\ e \qquad \Rightarrow \textbf{let}\ x = (\mathit{body} f)[v_1 \mapsto e_1, \dots v_n \mapsto e_n]\ \textbf{in}\ e$

Figure 4.7: Rules for reduction.

All expressions are kept in A-Normal Form. **Reduce Base** is only run if we are at the root of the body. While the last three rules are effectively the same, tt's useful to keep them separated for debugging reduction derivations.

the same readable code.

We end by giving a couple of examples of reductions to see how they work in practice. The first example returns from the start of this chapter. We see that *double* $(0 \mathbin{?} 1)$ is reduced so we don't make a needless call to *double*, but we've avoided the problem of run time choice semantics.

Our next function comes from a possible implementation of $\leqslant$ for Boolean values. In fact, this is the implementation we chose for the instance of the *Ord* class for *Bool*. The example is a bit long, but it shows how many of these optimizations work together to produce efficient code.

In the next chapter we discuss three more optimizations, Unboxing, Shortcutting, and Deforestation. While Unboxing and Deforestation are in common use in lazy function compilers, they have not been used for functional-logic languages before. Shortcutting is a new optimization to Curry.

$double\ x = x + x$

$main = double\ (0\ ?\ 1)$

$double\ (0\ ?\ 1)$

ANF [1]

let $v_1 = 0\ ?\ 1$

in $double\ v_1$

Reduce Let [1]

let $v_1 = 0\ ?\ 1$

in $v_1 + v_1$

Figure 4.8: Derivation of *double* (0 ? 1) showing that we still arrive at an equivalent expression.

$$
\begin{aligned}
&\mathit{not}\ v_1 = \mathbf{case}\ v_1\ \mathbf{of}\\
&\qquad\qquad \mathit{True} \to \mathit{False}\\
&\qquad\qquad \mathit{False} \to \mathit{True}\\
&v_1 \wedge v_2 = \mathbf{case}\ v_1\ \mathbf{of}\\
&\qquad\qquad \mathit{True} \quad \to v_2\\
&\qquad\qquad \mathit{False} \quad \to \mathit{False}\\
&v_1 \vee v_2 = \mathbf{case}\ v_1\ \mathbf{of}\\
&\qquad\qquad \mathit{True} \quad \to \mathit{True}\\
&\qquad\qquad \mathit{False} \quad \to v_2\\
&v_1 \leqslant v_2 = \mathit{not}\ v_1 \vee v_2
\end{aligned}
$$

$(\mathit{not}\ v_1) \vee v_2$

$\Rightarrow$ **ANF App** $[\,]$

$$
\begin{aligned}
&\mathbf{let}\ v_3 = \mathit{not}\ v_1\\
&\mathbf{in}\ v_3 \vee v_2
\end{aligned}
$$

$\Rightarrow$ **Reduce Useful** $[\,]$

$$
\begin{aligned}
&\mathbf{let}\ v_3 = \mathbf{case}\ \ v_1\ \mathbf{of}\\
&\qquad\qquad\quad \mathit{True} \to \mathit{False}\\
&\qquad\qquad\quad \mathit{False} \to \mathit{True}\\
&\mathbf{in}\ v_3 \vee v_2
\end{aligned}
$$

$\Rightarrow$ **Reduce Let** $[\,]$

$$
\begin{aligned}
&\mathbf{let}\ v_3 = \mathbf{case}\ \ v_1\ \mathbf{of}\\
&\qquad\qquad\quad \mathit{True} \to \mathit{False}\\
&\qquad\qquad\quad \mathit{False} \to \mathit{True}\\
&\mathbf{in\ case}\ v_3\ \mathbf{of}\\
&\qquad \mathit{True} \to v_3\\
&\qquad \mathit{False} \to v_2
\end{aligned}
$$

$\Rightarrow$ **Case Var** $[-1]$

$$
\begin{aligned}
&\mathbf{let}\ v_3 = \mathbf{case}\ \ v_1\ \mathbf{of}\\
&\qquad\qquad\quad \mathit{True} \to \mathit{False}\\
&\qquad\qquad\quad \mathit{False} \to \mathit{True}\\
&\mathbf{in\ case}\ v_3\ \mathbf{of}\\
&\qquad \mathit{True} \to \mathit{True}\\
&\qquad \mathit{False} \to v_2
\end{aligned}
$$

$\Rightarrow$ **Inline Case in Case** $[\,]$

$$
\begin{aligned}
&\mathbf{case}\ (\mathbf{case}\ v_1\ \mathbf{of}\\
&\qquad\qquad \mathit{True} \to \mathit{False}\\
&\qquad\qquad \mathit{False} \to \mathit{True})\ \mathbf{of}\\
&\qquad \mathit{True} \to \mathit{True}
\end{aligned}
$$

$\Rightarrow$ **Inline Constructor** $[0]$

$$
\begin{array}{l}
\textbf{case } v_1 \textbf{ of} \\
\quad \mathit{True} \rightarrow \textbf{case } \mathit{False} \textbf{ of} \\
\qquad\qquad \mathit{True} \rightarrow \mathit{True} \\
\qquad\qquad \mathit{False} \rightarrow v_2 \\
\quad \mathit{False} \rightarrow \textbf{let } \mathit{v8} = \mathit{True} \\
\qquad\quad \textbf{in case } \mathit{v8} \textbf{ of} \\
\qquad\qquad\quad \mathit{True} \rightarrow \mathit{True} \\
\qquad\qquad\quad \mathit{False} \rightarrow v_2
\end{array}
$$

$\Rightarrow$ **Case Cancel Constructor** $[0]$

$$
\begin{array}{l}
\textbf{case } v_1 \textbf{ of} \\
\quad \mathit{True} \rightarrow v_2 \\
\quad \mathit{False} \rightarrow \textbf{let } \mathit{v8} = \mathit{True} \textbf{ in} \\
\qquad\quad \textbf{in case } \mathit{v8} \textbf{ of} \\
\qquad\qquad\quad \mathit{True} \rightarrow \mathit{True} \\
\qquad\qquad\quad \mathit{False} \rightarrow v_2
\end{array}
$$

$\Rightarrow$ **Inline Constructor** $[1]$

$$
\begin{array}{l}
\textbf{case } v_1 \textbf{ of} \\
\quad \mathit{True} \rightarrow v_2 \\
\quad \mathit{False} \rightarrow \textbf{case } \mathit{True} \textbf{ of} \\
\qquad\qquad \mathit{True} \rightarrow \mathit{True} \\
\qquad\qquad \mathit{False} \rightarrow v_2
\end{array}
$$

$\Rightarrow$ **Case Cancel Constructor** $[1]$

$$
\begin{array}{l}
\textbf{case } v_1 \textbf{ of} \\
\quad \mathit{True} \rightarrow v_2 \\
\quad \mathit{False} \rightarrow \mathit{True}
\end{array}
$$

Figure 4.10: Derivation of $\leqslant$ for *Bool* condinued

Chapter 5

MEMORY OPTIMIZATIONS

In this chapter we develop three new optimizations for Curry. First, Unboxing is an attempt to remove boxed values from our language. We discuss our implementation of primitive values and operations, and how explicitly representing the boxes around these values leads to optimizations. Second, Shortcutting is an attempt to remove the node created as the scrutinee of a case expression. In lazy functional languages like Haskell, this isn't a problem, but in Curry the scrutinee may be non-deterministic, and so we need a plan for dealing with that. Finally, Deforestation is a optimization for removing intermediate lists. This has been studied extensively in functional languages, but it has not been shown to be valid in the presence of non-determinism. We prove its validity in Curry, and give a formulation that can apply to combinator languages.

5.1 UNBOXING

So far we've avoided talking about operations in Curry for primitive data types *Int*, *Char*, and *Float*. This is primarily because all primitive values in Curry are boxed, and the choice of how we represent boxes has a pervasive effect on the compiler. Since we knew how we intended to implement Unboxing, we decided to use that representation from the beginning.

We chose to follow the style of Unboxing from Launchbury et al. [55] and represent all boxes explicitly in FlatCurry. This has several advantages, but one of the most important is that we can apply optimizations to the boxes themselves.

Let's look at an example to see how this works. Consider the function to compute Fibonacci numbers. We will work with this example extensively in the next couple of optimizations, in an attempt to see how much we can optimize it.

$$
\begin{aligned}
&\mathit{fib} :: \mathit{Int} \to \mathit{Int}\\
&\mathit{fib}\ n = \mathbf{case}\ n \leqslant 1\ \mathbf{of}\\
&\qquad\qquad \mathit{True} \ \to n
\end{aligned}
$$

$$\mathit{False} \rightarrow \mathit{fib}\ (n-1) + \mathit{fib}\ (n-2)$$

After translating to A-Normal Form we have:

$$
\begin{array}{l}
\mathit{fib} :: \mathit{Int} \rightarrow \mathit{Int} \\
\mathit{fib}\ n = \mathbf{let}\ \mathit{cond} = n \leqslant 1 \\
\qquad\quad \mathbf{in\ case}\ \mathit{cond}\ \mathbf{of} \\
\qquad\qquad\quad \mathit{True}\ \rightarrow n \\
\qquad\qquad\quad \mathit{False} \rightarrow \mathbf{let}\ n_1 = n - 1 \\
\qquad\qquad\qquad\qquad \mathbf{in\ let}\ f_1 = \mathit{fib}\ n_1 \\
\qquad\qquad\qquad\qquad \mathbf{in\ let}\ n_2 = n - 2 \\
\qquad\qquad\qquad\qquad \mathbf{in\ let}\ f_2 = \mathit{fib}\ n_2 \\
\qquad\qquad\qquad\qquad \mathbf{in}\ f_1 + f_2
\end{array}
$$

Unfortunately, this function can't really be optimized. The *fib* function is recursive, so we can't reduce it, and $n - 1$ is a primitive operation. However, we use a lot of memory for this function. We create 8 nodes for each recursive call, and there's really no need for this. The problem is that each of our primitive operations must be represented as a node to have a uniform representation in Curry.

5.1.1 The Transformation

The idea behind the Unboxing transformation is that we represent every box around primitive operations explicitly. The expression $1 + 2$ is transformed into $\mathit{Int}\ 1 + \mathit{Int}\ 2$, and the *fib* function is transformed into:

$$
\begin{array}{l}
\mathit{fib} :: \mathit{Int} \rightarrow \mathit{Int} \\
\mathit{fib}\ n = \mathbf{let}\ \mathit{cond} = n \leqslant \mathit{Int}\ 1 \\
\qquad\quad \mathbf{in\ case}\ \mathit{cond}\ \mathbf{of} \\
\qquad\qquad\quad \mathit{True}\ \rightarrow n \\
\qquad\qquad\quad \mathit{False} \rightarrow \mathbf{let}\ n_1 = n - \mathit{Int}\ 1 \\
\qquad\qquad\qquad\qquad \mathbf{in\ let}\ f_1 = \mathit{fib}\ n_1 \\
\qquad\qquad\qquad\qquad \mathbf{in\ let}\ n_2 = n - \mathit{Int}\ 2 \\
\qquad\qquad\qquad\qquad \mathbf{in\ let}\ f_2 = \mathit{fib}\ n_2 \\
\qquad\qquad\qquad\qquad \mathbf{in}\ f_1 + f_2
\end{array}
$$

This doesn't seem like we've actually helped at all, but the real improvement comes in the implementation of $x + y$, $x - y$ and $x \leqslant y$. Let's look at $x + y$ first. We can implement this operation using a primitive add operation. This will be translated into an add instruction at the C level.

$$
\begin{aligned}
x + y = \textbf{case}\ x\ \textbf{of}\\
&\quad \mathit{Int}\ x_{\text{prim}} \to \textbf{case}\ y\ \textbf{of}\\
&\qquad\qquad \mathit{Int}\ y_{\text{prim}} \to \textbf{let}\ v = +_{\text{prim}}\ x_{\text{prim}}\ y_{\text{prim}}\\
&\qquad\qquad\qquad\qquad \textbf{in}\ \mathit{Int}\ v
\end{aligned}
$$

We can implement $x \leqslant y$ in a similar fashion:

$$
\begin{aligned}
x \leqslant y = \textbf{case}\ x\ \textbf{of}\\
&\quad \mathit{Int}\ x_{\text{prim}} \to \textbf{case}\ y\ \textbf{of}\\
&\qquad\qquad \mathit{Int}\ y_{\text{prim}} \to \leqslant_{\text{prim}}\ x_{\text{prim}}\ y_{\text{prim}}
\end{aligned}
$$

The difference is that $\leqslant_{\text{prim}}$ needs to return a *True* or *False* value. We can implement this as `x_prim <= y_prim ? make_Prelude_True() : make_Prelude_False()`. Now we actually have something we can optimize. We still can't do a lot of with the recursive calls to *fib*, but let's see what happens. We end up with the definition for *fib* in figure 5.1. The full derivation can be seen in the appendix.

As we can see, the code is significantly longer, but now we've included the primitive operations in our code. The variables v_2, n_1, n_2, p_1, p_2 are all primitive values, so we don't need to allocate any memory for them. Unfortunately, there's still a problem. We're still allocating 1 node for $cond, f_1, f_2$ and 2 nodes for the Int constructors. So, we're still allocating 5 nodes. This is an improvement, but we can certainly do better.

5.1.2 Primitive Conditions

The first optimization is that we really don't need to allocate memory for $cond$. $x \leqslant y$ really should be a primitive operation returning a Boolean value, but this doesn't have an equivalent in FlatCurry, so we introduce the **pcase** construct.

$$
\begin{aligned}
&\textbf{pcase}\ \mathit{primCond}\ \textbf{of}\\
&\quad \mathit{True} \to e_t\\
&\quad \mathit{False} \to e_f
\end{aligned}
$$

$$
\begin{array}{l}
\mathit{fib}\ n = \textbf{case}\ n\ \textbf{of} \\
\quad \mathit{Int}\ v_2 \rightarrow \textbf{let}\ \mathit{cond} = \leqslant_{\text{prim}}\ v_2\ 1 \\
\qquad\quad \textbf{in case}\ \mathit{cond} \\
\qquad\qquad\qquad \mathit{True} \rightarrow n \\
\qquad\qquad\qquad \mathit{False} \rightarrow \textbf{let}\ n_1 = -_{\text{prim}}\ v_2\ 1 \\
\qquad\qquad\qquad\qquad\qquad \textbf{in let}\ f_1 = \mathit{fib}\ (\mathit{int}\ n_1) \\
\qquad\qquad\qquad\qquad\qquad \textbf{in case}\ f_1\ \textbf{of} \\
\qquad\qquad\qquad\qquad\qquad\qquad \mathit{Int}\ p_1 \rightarrow \textbf{let}\ n_2 = -_{\text{prim}}\ v_2\ 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \textbf{in let}\ f_2 = \mathit{fib}\ (\mathit{int}\ n_2) \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \textbf{in case}\ f_2\ \textbf{of} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathit{Int}\ p_2 \rightarrow \textbf{let}\ r = +_{\text{prim}}\ p_1\ p_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textbf{in}\ \mathit{Int}\ r
\end{array}
$$

Figure 5.1: Optimized *fib* after Unboxing

The *primCond* must be a primative condition expression, which is either $==_{\text{prim}}$ or $\leqslant_{\text{prim}}$, and the arguments must be primitive values. The semantics of **pcase** are exactly what be expected, but now we can translate it into a simple `if` statement in C.

```
if([compile primCond])
{
    [compile e_t]
}
else
{
    [compile e_f]
}
```

We don't need to worry about the backtracking stack, because a primitive condition can't be non-deterministic. Now, we can eliminate the *cond* node. After implementing this construct, the new version is in figure 5.2. Now we're down to 4 nodes, but we can still do better. The next challenge is Unboxing parameters.

$$
\begin{array}{l}
\mathit{fib}\ n = \textbf{case}\ n\ \textbf{of} \\
\quad \mathit{Int}\ v_2 \to \textbf{pcase}\ \leqslant_{\text{prim}}\ v_2\ 1 \\
\quad\quad \mathit{True}\ \to n \\
\quad\quad \mathit{False} \to \textbf{let}\ n_1 = -_{\text{prim}}\ v_2\ 1 \\
\quad\quad\quad\quad\quad\quad \textbf{in let}\ f_1 = \mathit{fib}\ (\mathit{int}\ n_1) \\
\quad\quad\quad\quad\quad\quad \textbf{in case}\ f_1\ \textbf{of} \\
\quad\quad\quad\quad\quad\quad\quad\quad \mathit{Int}\ p_1 \to \textbf{let}\ n_2 = -_{\text{prim}}\ v_2\ 2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \textbf{in let}\ f_2 = \mathit{fib}\ (\mathit{int}\ n_2) \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \textbf{in case}\ f_2\ \textbf{of} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathit{Int}\ p_2 \to \textbf{let}\ r = +_{\text{prim}}\ p_1\ p_2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \textbf{in}\ \mathit{Int}\ r
\end{array}
$$

Figure 5.2: The *fib* function with primitive cases

5.1.3 Strictness Analysis

Unfortunately, Unboxing parameters is slightly more complicated. Earlier we eliminated boxes from local variables in the function. However, if a parameter is unboxed, then we may need to store a primitive value as a child of a node. This requires a fundamental change to our node datatype. Fortunately, C already has a mechanism for this. We use a `union` to tie integers, characters, floating point numbers, and nodes together. The full definition for a node in our expression graph is given below.

```
typedef union field
{
    struct Node*  n; //normal node child
    union field*  a; //array child (for children[3])
    unsigned long c; //primitive character
    long          i; //primitive int
    double        f; //primitive float
} field ;
```

```
typedef struct Node
{
    Int missing;
    bool nondet;
    Symbol* symbol;
    field children[4];
} Node;
```

Now that we have the ability to store primitive values in a node, we need to figure out when storing a primitive value is actually valid. Fortunately, this is a well studied problem [71, 61, 88].

Lazy functional languages often try to remove laziness for efficiency reasons. We don't want to create an expression for a primitive value if we're only going to deconstruct it, so it becomes useful to know what parameters in a function must be evaluated, or what parameters in a function are strict. This strictness analysis has been a major focus of research in the lazy functional community.

We implemented an earlier form of strictness analysis discovered by Mycroft [71]. This works by determining if a parameter is strict by abstract interpretation, which sounds complicated, but in reality it's actually a easy idea. Each parameter is represented as a variable, and the body of the function is converted into a Boolean expression. The translation is similar to [71], so we don't go through it here. There are newer ideas for strictness analysis [61, 88], but Mycroft's solution is sufficient for our purposes, so better implementations are outside the scope of this research.

Once we know which arguments are strict we, can split the function into a wrapper function and a worker function [88]. We can see this with *fib* in figure 5.3, and the optimized version in figure 5.4. Notice that *fib* is no longer recursive, so we can inline it. After optimization we have the following definition of *fib#worker*.

We're down to allocating 2 nodes. We only need to allocate nodes for the calls to *fib#worker*. This means that we've reduced our memory consumption by 75%. That's a huge improvement, but we can still do better. With the next optimization we look at how to remove the remaining allocations.

$$
\begin{array}{l}
\mathit{fib}\ n = \mathbf{case}\ n\ \mathbf{of} \\
\quad \mathit{Int}\ v_2 \rightarrow \mathit{fib\#worker}\ v_2 \\[1ex]
\mathit{fib\#worker}\ v_1 = \mathbf{let}\ n = \mathit{Int}\ v_1 \\
\quad \mathbf{in\ case}\ n\ \mathbf{of} \\
\quad\quad \mathit{Int}\ v_2 \rightarrow \\
\quad\quad\quad \mathbf{in\ pcase}\ \leqslant_{\mathrm{prim}}\ v_2\ 1 \\
\quad\quad\quad\quad \mathit{True} \rightarrow n \\
\quad\quad\quad\quad \mathit{False} \rightarrow \mathbf{let}\ n_1 = -_{\mathrm{prim}}\ v_2\ 1 \\
\quad\quad\quad\quad\quad \mathbf{in\ let}\ f_1 = \mathit{fib}\ (\mathit{int}\ n_1) \\
\quad\quad\quad\quad\quad \mathbf{in\ case}\ f_1\ \mathbf{of} \\
\quad\quad\quad\quad\quad\quad \mathit{Int}\ p_1 \rightarrow \mathbf{let}\ n_2 = -_{\mathrm{prim}}\ v_2\ 2 \\
\quad\quad\quad\quad\quad\quad\quad \mathbf{in\ let}\ f_2 = \mathit{fib}\ (\mathit{int}\ n_2) \\
\quad\quad\quad\quad\quad\quad\quad \mathbf{in\ case}\ f_2\ \mathbf{of} \\
\quad\quad\quad\quad\quad\quad\quad\quad \mathit{Int}\ p_2 \rightarrow \mathbf{let}\ r = +_{\mathrm{prim}}\ p_1\ p_2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathbf{in}\ \mathit{Int}\ r
\end{array}
$$

Figure 5.3: The *fib* function after strictness analysis.

$$
\begin{array}{l}
\mathit{fib\#worker}\ v_2 = \\
\quad \mathbf{pcase}\ \leqslant_{\mathrm{prim}}\ v_2\ 1\ \mathbf{of} \\
\quad\quad\quad\quad \mathit{True} \rightarrow \mathit{Int}\ v_1 \\
\quad\quad\quad\quad \mathit{False} \rightarrow \mathbf{let}\ n_1 = -_{\mathrm{prim}}\ v_2\ 1 \\
\quad\quad\quad\quad\quad\quad \mathbf{in\ let}\ f_1 = \mathit{fib\#worker}\ n_1 \\
\quad\quad\quad\quad\quad\quad \mathbf{in\ case}\ f_1\ \mathbf{of} \\
\quad\quad\quad\quad\quad\quad\quad\quad \mathit{Int}\ p_1 \rightarrow \mathbf{let}\ n_2 = -_{\mathrm{prim}}\ v_2\ 2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathbf{in\ let}\ f_2 = \mathit{fib\#worker}\ n_2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathbf{in\ case}\ f_2\ \mathbf{of} \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathit{Int}\ p_2 \rightarrow \mathbf{let}\ r = +_{\mathrm{prim}}\ p_1\ p_2 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathbf{in}\ \mathit{Int}\ r
\end{array}
$$

Figure 5.4: The *fib* function after strictness analysis and optimization.

5.2 SHORTCUTTING

In the last section were able to optimize the *fib* function from allocating 8 nodes per recursive call to only allocating 2 nodes per recursive call. However, we were left with a problem that we can't solve by a code transformation.

$$\begin{array}{l}\textbf{let}\ f_1 = \mathit{fib\#worker}\ n_1\\ \textbf{in case}\ f_1\ \textbf{of}\\ \qquad \mathit{Int}\ p_1 \to\ \ldots\end{array}$$

This is unfortunate, because we never use f_1 after the **case** expression. It seems like we should be able to avoid constructing the node, and in fact, implementations of functional languages do avoid this. Unfortunately, we really do need a physical node. The reason is because $\mathit{fib\#worker}\ n_1$ could be non-deterministic.

It is worth looking at an attempt to try to replace the node with a function call. One possibility would be to try to statically analyze *fib#worker* and determine if it's deterministic. This is a reasonable idea, but it has two major drawbacks. First, determining if a function is non-deterministic is undecidable, so the best we could do is an approximation. Second, even if *fib#worker* is deterministic, the expression $\mathit{fib\#worker}\ n_1$ could still be non-deterministic, so any sort of determinism analysis is going to fail for any expression that contains a parameter to the function. This is going to be very restrictive for any possible optimization.

We need a node to hold the value for $\mathit{fib\#worker}\ n_1$, but this value will only be used in the case expression. In fact, it's not possible for this node to be shared with any part of the expression graph. This leads to a new idea. If we need a node for f_1 could we avoid allocating memory for that node? Well, sometimes we can.

The idea here is simple, but the implementation becomes tricky. We want to use a single, statically allocated, node for every variable that's only used as the scrutinee of a case.

There are two steps to the optimization. The first step is marking every node that's only used as the scrutinee, and the second is swaping that variable for the statically allocated node during code generation. We call this node `RET` for return.

Marking the node can be done in FlatCurry using GAS. The only effect of this rule is to mark x' as a variable that can be stored in the `RET` node.

Case Call

$$\textbf{let}\ x = e_1\ \textbf{in case}\ x\ \textbf{of}\ e_2 \mid x \notin e_1, e_2 \Rightarrow \textbf{let}\ x' = e_1\ \textbf{in case}\ x'\ \textbf{of}\ e_2$$

While this transformation is simple enough, we need to determine if it's valid.

First we assume that e_1 is a deterministic expression. In that case, there is only one thing that could go wrong. It's possible that e_1 also reduces an expression that could be stored in `RET`. For example, consider the program:

$$
\begin{aligned}
&f\ x = \textbf{case}\ g\ x\ \textbf{of} \\
&\qquad\qquad \mathit{True} \rightarrow \mathit{False} \\
&\qquad\qquad \mathit{False} \rightarrow \mathit{True} \\
&\mathit{main} = \textbf{case}\ f\ 3\ \textbf{of} \\
&\qquad\qquad \mathit{True} \rightarrow 0 \\
&\qquad\qquad \mathit{False} \rightarrow 1
\end{aligned}
$$

In the evaluation of *main*, we can store $f\ 3$ in the `RET` node, but while we are evaluating $f\ 3$, we store $g\ 3$ in the same `RET` node. While this is concerning, it's not actually a problem. At the beginning of `f_hnf`, we store all of the children of `root` as local variables, and then when we've computed the value, we overwrite the `root` node. Aside from the very start and end of the function, we never interact with the `root` node, so even if we reuse `RET` in the middle of evaluating f, it doesn't actually affect the results.

It seems like we should be able to store these marked variables in the `RET` node, and then just call the appropriate `_hnf` function. In fact this was the first idea we tried. The generated code for main is given in figure 5.5.

This initial version actually works very well. In fact, for *fib#worker* we're able to remove the remaining 2 allocations. This is fantastic, and we'll come back to this point later, but before we celebrate, we need to deal with a looming problem.

5.2.1 Non-deterministic RET Nodes

The problem with the scheme we've developed so far is that if `RET` is non-deterministic, then we may push it, or an expression containing it, onto the backtracking stack. This is a major problem with this optimization, because `RET` will almost certainly have been reused by the time backtracking occurs.

This optimization was built on the idea that `RET` is only ever used in a single case expression. Therefore, it's important that we never put `RET` on the backtracking stack. We need rethink on our idea. Initially, we wanted to avoid allocating a node if a variable is used in a single

```
void main_hnf(Node* root)
{
    set_f(RET, make_int(3));
    Node* RET_forward = RET;
    nondet = false;
    while(true)
    {
        switch(RET_forward->tag)
        {
            ...
            case True:
                if(nondet)
                    ...
                set_int(root, 0);
                return;
                ...
        }
    }
```

Figure 5.5: First attempt at compiling *main* with Shortcutting.

case. Instead, we will only allocate a variable if `RET` is non-deterministic. This means that for deterministic expression, we don't allocate any memory, but for non-deterministic expression, we still have a persistent variable on the stack. This lead to the second implementation in figure 5.6.

5.2.2 RET hnf Functions

While this solution is better, it's still not correct. Three things can still go wrong here. The first problem is that `RET` might have been reduced to a forwarding node, so it might not be responsible for the non-determinism, such as **case** *id* (0 ? 1) **of** .... There's clearly non-determinism here, but the *id* node isn't the cause of it, so it shouldn't be pushed on the backtracking stack.

Another problem is that, if `RET` is a forwarding node, then the node it forwards to might have reused `RET`. This is a much more serious problem, because we would push the wrong value on the backtracking stack.

Finally, we still haven't avoided putting `RET` on the backtracking stack, because if `RET` is non-deterministic, it will be pushed on the stack as the left hand side of a stack frame. While we're reducing `RET`, we need to know what node to push on the stack. This means that both the caller and the callee need to know what node we created.

This is starting to seem hopeless. How can we avoid creating nodes for deterministic expressions, but still only create a single node that the caller and callee agree on if the expression is non-deterministic? The answer is that we need to change how `RET` nodes are reduced. Specifically, we create a new reduction function that only handles nodes stored in `RET`. In the case of f, we would create a `f_hnf`, a `f__hnf` and a `f_RET_hnf`. The third function only reduces f that has been stored in a `RET` node.

The difference between `f_hnf` and `f_RET_hnf` is that instead of passing the root node, we pass `Node* backup`. The `backup` node is where we'll store the contents of `RET` if we discover evaluating f is non-deterministic. At the end of the function, we return `backup`. Now both the caller and callee agree on `backup`. Furthermore, since `backup` is a local variable, it's not affected if f reuses `RET` over the course of its evaluation. This leads to the definition for `f_RET_hnf` in figure 5.7.

Now, we finally have a working function. We only allocate memory if the expression is non-deterministic. If the expression is non-deterministic in multiple places, we reuse that same `backup` node.

```
void main_hnf(Node* root)
{
    set_f(RET, make_int(3));
    Node* RET_forward = RET;
    nondet = false;
    while(true)
    {
        switch(RET_forward->tag)
        {
            ...
            case True:
                if(nondet)
                {
                    Node* backup = copy(RET);
                    stack_push(bt_stack, root, main_(backup));
                }
                set_int(root, 0);
                return;
                ...
        }
    }
```

Figure 5.6: Second attempt at compiling *main* with Shortcutting.

```
Node* f_RET_hnf(Node* backup)
{
    Node* v1 = RET->children[0];
    set_g(RET, v1);
    Node* RET_forward = RET;
    Node* g_backup = g_RET_hnf(NULL);
    bool nondet = g_backup == NULL;
    while(true)
    {
        nondet |= RET_forward->nondet;
        switch(RET_forward->tag)
        {
            ...
            case True:
                if(nondet)
                {
                    if(!backup)
                    {
                        backup = (Node*)calloc(1,sizeof(Node));
                    }
                    set_False(backup);
                    stack_push(bt_stack, backup, g_backup);
                }
                set_False(RET);
                return backup;
            ...
        }
    }
```

Figure 5.7: Compiling f with Shortcutting.

This also works well if we have multiple reductions in a row. Suppose we have the following Curry code:

$$
\begin{aligned}
&\mathit{main} = \textbf{case}\ f\ 4\ \textbf{of}\\
&\qquad\qquad\quad \mathit{True} \rightarrow \mathit{False}\\
&\qquad\qquad\quad \mathit{False} \rightarrow \mathit{False}\\
&f\ n = \textbf{case}\ n\ \textbf{of}\\
&\qquad\qquad 0 \rightarrow \mathit{True}\\
&\qquad\qquad \_ \rightarrow f\ (n-1)
\end{aligned}
$$

In this case f is a recursive function, so when we reduce f 4, we need to immediately reduce f 3. This is no problem at all, because we're reducing f 4 with `f_RET_hnf`. Ignoring the complications of Unboxing for the moment, we can generate the following code for the return of f.

```
Node* v2 = make_int(n-1)
set_f(RET, v2);
return f_RET_hnf(backup);
```

5.2.3 Shortcutting Results

Shortcutting was originally formulated in the context of the Pakcs compiler [19, 18], which handled reduction a little differently then we do here. Instead of calling an hnf function directly, it had a giant lookup table that would dispatch the node to be reduced to the correct reduction predicate. The compiler would then take a single step. Shortcutting was an attempt to circumvent, or shortcut, this lookup table, and it produced code that's actually similar to what we have in Rice. Instead of having a monolithic $\mathbf{H}$ predicate that normalized any expression, the authors split it into several $\mathbf{H}_f$ predicates for each function f. This also had the effect that knowing the argument to be normalized allowed that function to be called directly without ever having to construct the node. While we went about it a different way, we've achieved the same goal as the Shortcutting paper.

Before we move onto our next optimization, we should look back at what we've done so far. Initially, we had a *fib* function that allocated 8 nodes for every recursive call. Then, through Unboxing, we were able to cut that down to only 2 allocations per call. Finally, using Shortcutting, we were able to eliminate those two allocations. We would expect a substantial speedup by reducing memory consumption by 75%, but removing those last two allocations is a difference in kind. The *fib* function runs in exponential time, and since each step allocates some memory, the original *fib* function allocated an exponential amount of memory on the heap. However, our fully optimized *fib* function only allocates a static node at startup. We've moved from exponential memory allocated on the heap to constant space. While *fib* still runs in exponential time, it runs much faster, since it doesn't need to allocate memory. Surprisingly, *fib* is still just as efficient with non-deterministic arguments. If n is non-deterministic, the wrapper function will evaluate n before calling the worker.

Now that we've removed most of the implicitly allocated memory with Unboxing and Short-cutting, we can work on removing explicitly allocated memory with a technique from functional languages.

5.3 DEFORESTATION

We now turn to our final optimization, Deforestation. The goal of this optimization is to remove intermediate data structures. Programmers often write in a pipeline style when writing functional programs. For example, consider the program:

$$\mathit{sumPrimes} = \mathit{sum} \circ \mathit{filter}\ \mathit{isPrime} \circ \mathit{enumFromTo}\ 2$$

While this style is concise and readable, it isn't efficient. First, we create a list of the n integers, then we create a new list of all of the integers that are prime, and finally we sum the values in that list. It would be much more efficient to compute this sum directly.

$$\begin{array}{l}
\mathit{sumPrimes}\ n = \mathit{go}\ 2\ n \\
\quad \textbf{where}\ \mathit{go}\ k\ n \\
\qquad\qquad \mid k \geqslant n \qquad\ \ = 0 \\
\qquad\qquad \mid \mathit{isPrime}\ k = k + \mathit{go}\ (k+1)\ n \\
\qquad\qquad \mid \mathit{otherwise}\ \ = \mathit{go}\ (k+1)\ n
\end{array}$$

This pipeline pattern is pervasive in functional programming, so it's worth understanding and optimizing it. In particular, we want to eliminate the two intermediate lists created here. This is the goal of Deforestation.

5.3.1 The Original Scheme

Deforestation has actually gone through several forms throughout it's history. The original optimization proposed by Wadler [94] was very general, but it required a complicated algorithm, and it could fail to terminate. There have been various attempts to improve this algorithm [93] and [36] that have focused on restricting the form of programs.

An alternative was proposed by Gill in his dissertation [40, 39] called foldr-build Deforestation or short-cut Deforestation. This approach is much simpler, always terminates, and has a nice correctness proof, but it comes at the cost of generality. Foldr-build Deforestation only works with functions that produce and consume lists. Still, lists are common enough in functional languages that this optimization has proven to be effective.

Since then foldr-build Deforestation has been extended to Stream Fusion [31]. While this optimization is able to cover more cases than foldr-build Deforestation, it relies on more advanced compiler technology.

The foldr-build optimization itself is actually very simple. It relies on an observation about the structure of a list. All lists in Curry are built up from cons and nil cells. The list $[1, 2, 3, 4]$ is really $1:2:3:4:[\,]$. One very common list processing technique is a fold, which takes a binary operation and a starting element, and reduces a list to a single value. In Curry, the *foldr* function is defined as:

$$\begin{aligned}
&\mathit{foldr} :: (a \to b \to b) \to b \to [\,a\,] \to b\\
&\mathit{foldr} \oplus z\ [\,] \qquad\quad = z\\
&\mathit{foldr} \oplus z\ (x : xs) = x \oplus \mathit{foldr}\ f\ \ z\ xs
\end{aligned}$$

As an example, we can define the *sum* function as $\mathit{sum}\ xs = \mathit{foldr}\ (+)\ 0$. To see what this is really doing we can unroll the recursion. Suppose we evaluate $\mathit{foldr}\ (+)\ 0\ [1, 2, 3, 4, 5]$, then we have:

$$\begin{aligned}
&\mathit{foldr}\ (+)\ 0\ [1, 2, 3, 4, 5]\\
&\ \Rightarrow 1 + \mathit{foldr}\ (+)\ 0\ [2, 3, 4, 5]\\
&\ \Rightarrow 1 + 2 + \mathit{foldr}\ (+)\ 0\ [3, 4, 5]\\
&\ \Rightarrow 1 + 2 + 3 + \mathit{foldr}\ (+)\ 0\ [4, 5]\\
&\ \Rightarrow 1 + 2 + 3 + 4 + \mathit{foldr}\ (+)\ 0\ [5]\\
&\ \Rightarrow 1 + 2 + 3 + 4 + 5 + \mathit{foldr}\ (+)\ 0\ [\,]\\
&\ \Rightarrow 1 + 2 + 3 + 4 + 5 + 0
\end{aligned}$$

But wait, this looks very similar to our construction of a list.

$$1 \ : \ 2 \ : \ 3 \ : \ 4 \ : \ 5 \ : \ [\,]$$
$$1 + 2 + 3 + 4 + 5 + 0$$

We've just replaced the : with + and the [ ] with 0. If the compiler can find where we will do this replacement, then we don't need to construct the list. On its own, this is a very hard problem, but we can help the compiler along. We just need a standard way to construct a list. This can be done with the *build* function.

$$\mathit{build} :: (\forall\ b\ (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow b) \rightarrow [\,a\,]$$
$$\mathit{build}\ g = g\ (:)\ [\,]$$

The *build* function takes a function that constructs a list. However, instead of construction the list with : and [ ], we abstract this by passing the constructors in as arguments c and n respectively. At that point, we can construct the list by calling *build* on our builder function.

As an example, let's look at the function *enumFromTo* a b that constructs a list of integers from a to b.

$$\begin{array}{l} \mathit{enumFromTo}\ a\ b \\ \quad \mid a > b \qquad\quad = [\,] \\ \quad \mid \mathit{otherwise} = a : \mathit{enumFromTo}\ (a+1)\ b \end{array}$$

We can turn this into a build function.

$$\begin{array}{l} \mathit{enumFromTo}\ a\ b = \mathit{build}\ (\mathit{enumFromTo\_build}\ a\ b) \\ \mathit{enumFromTo\_build}\ a\ b\ c\ n \\ \quad \mid a > b \qquad\quad = n \\ \quad \mid \mathit{otherwise} = a\ \text{`}c\text{`}\ \mathit{enumFromTo\_build}\ (a+1)\ b\ c\ n \end{array}$$

We can create build functions for several list creation functions found in the standard library, so it looks like we're ready to apply Deforestation to Curry. Unfortunately there are two problems we need to solve. The first is an implementation problem, and the second is a theoretical problem. First, while we can apply foldr/build Deforestation, we can't actually optimize the results. Second, we still need to show it's valid for curry.

5.3.2 The Combinator Problem

Let's look back at the motivating example, and see how it could be optimized in Haskell, or any language that can inline lambda expressions. The derivation in figure 5.8 comes from the original paper [39].

This looks good. In fact, we obtained the original expression we were trying for. Unfortunately we don't get the same optimization in Rice. The problem is actually the definition of *filter*.

$$\mathit{filter}\ f = \mathit{build}\ (\lambda c\ n \to \mathit{foldr}\ (\lambda x\ y \to \mathbf{if}\ f\ x\ \mathbf{then}\ x\ \text{`}c\text{`}\ y\ \mathbf{else}\ y)\ n)$$

Functions that transform lists, such as *filter*, *map*, and *concat*, are written as a build applied to a fold. Unfortunately this doesn't work well with our inliner. Since we don't inline lambda expressions, and since reductions can only be applied to let bound variables, we simply can't do this reduction. Instead we need a new solution.

5.3.3 Solution build_fold

Our solution to this problem is to introduce a new combinator for transforming lists. We call this *build_fold* since it is a build applied to a fold.

$$\mathit{build\_fold} :: ((c \to b \to b) \to (a \to b \to b)) \to (b \to b) \to [\,a\,] \to b$$
$$\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs} = \mathit{foldr}\ (\mathit{mkf}\ (:))\ (\mathit{mkz}\ [\,])\ \mathit{xs}$$

The idea behind this combinator is a combination of a build and a fold. This function was designed to be easily composable with both build and fold. Ideally, it could fit in the middle of build and fold and still reduce. As and example:

$$\mathit{foldr}\ (+)\ 0\ (\mathit{build\_fold}\ \mathit{filter\_mkf}\ \mathit{filter\_mkz}\ (\mathit{build}\ \mathit{enumFromTo\_build}))$$

Ideally, this function should reduce into something relatively efficient, Furthermore we wanted *build_fold* to compose nicely with itself. For example, $\mathit{map}\ f \circ \mathit{map}\ g$ should compose to something like $\mathit{map}\ (f \circ g)$.

We achieve this by combining pieces of both *build* and *foldr*. The two functions *mkf* and *mkz* make the f and z functions from fold, however they take c and n as arguments similar to *build*. The idea is that *mkf* takes an f from *foldr* as a parameter, and returns a new f. The *map* and *filter* implementations are given below.

$$sumPrimes\ m = sum\ (filter\ isPrime\ (enumFromTo\ 2\ m))$$

$\Rightarrow$

$$sumPrimes\ m = foldr\ (\lambda x\ y \to x + y)\ 0$$
$$(build\ (\lambda c\ n \to foldr\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ x\ `c`\ y\ \mathbf{else}\ y)\ n)$$
$$(build\ enumFromTo\_build\ 2\ m))$$

$\Rightarrow$

$$sumPrimes\ m = foldr\ (\lambda x\ y \to x + y)\ 0$$
$$(build\ (\lambda c\ n \to (enumFromTo\_build\ 2\ x)\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ x\ `c`\ y\ \mathbf{else}\ y)\ n))$$

$\Rightarrow$

$$sumPrimes\ m = enumFromTo\_build\ 2\ m\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ (\lambda x\ y \to x + y)\ x\ y\ \mathbf{else}\ y)\ 0$$

$\Rightarrow$

$$sumPrimes\ m = enumToFrom\_build\ 2\ m\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ x + y\ \mathbf{else}\ y)\ 0$$
$$\mathbf{where}\ enumToFrom\_build\ k\ m\ c\ z = \mathbf{if}\ k > m$$
$$\mathbf{then}\ z$$
$$\mathbf{else}\ c\ k\ (enumToFrom\_build\ (k+1)\ m\ c\ z)$$

$\Rightarrow$

$$sumPrimes\ m = enumToFrom\_build\ 2\ m\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ x + y\ \mathbf{else}\ y)\ 0$$
$$\mathbf{where}\ enumToFrom\_build\ k\ m\ c\ z = \mathbf{if}\ k > m$$
$$\mathbf{then}\ z$$
$$\mathbf{else}\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ x + y\ \mathbf{else}\ y)$$
$$k\ (enumToFrom\_build\ (k+1)\ m\ c\ z)$$

$\Rightarrow$

$$sumPrimes\ m = enumToFrom\_build\ 2\ m$$
$$\mathbf{where}\ enumToFrom\_build\ k\ m = \mathbf{if}\ k > m$$
$$\mathbf{then}\ z$$
$$\mathbf{else}\ (\lambda x\ y \to \mathbf{if}\ isPrime\ x\ \mathbf{then}\ x + y\ \mathbf{else}\ y)$$
$$k\ (enumToFrom\_build\ (k+1)\ m\ c\ z)$$

$\Rightarrow$

$$sumPrimes\ m = enumToFrom\_build\ 2\ m$$
$$\mathbf{where}\ enumToFrom\_build\ k\ m = \mathbf{if}\ k > m$$
$$\mathbf{then}\ z$$
$$\mathbf{else\ if}\ isPrime\ k$$
$$\mathbf{then}\ x + (enumToFrom\_build\ (k+1)\ m\ c\ z)$$
$$\mathbf{else}\ (enumToFrom\_build\ (k+1)\ m\ c\ z)$$

Figure 5.8: Optimization derivation for for short-cut Deforestation

$$map\ f = build\_fold\ (map\_mkf\ f)\ map\_mkz$$
$$map\_mkf\ f\ c\ x\ y = f\ x\ `c`\ y$$
$$map\_mkz\ n = n$$

$$filter\ p = build\_fold\ (filter\_mkc\ p)\ filter\_mkz$$
$$filter\_mkf\ p\ c\ x\ y = \textbf{if}\ p\ x\ \textbf{then}\ x\ `c`\ y\ \textbf{else}\ y$$
$$filter\_mkz\ n = n$$

The purpose of the convoluted definition of *build_fold* is that it plays nicely with *build* and *foldr*. We have the following three theorems about *build_fold*, which we will prove later. These are analogous to the *foldr* / *build* theorem.

$$build\_fold\ mkf\ mkz\ (build\ g) = build\ (\lambda c\ n \to g\ (mkf\ c)\ (mkz\ n))$$
$$foldr\ f\ z\ (build\_fold\ mkf\ mkz\ xs) = foldr\ (mkf\ f)\ (mkz\ z)\ xs$$
$$build\_fold\ mkf_1\ mkz_1\ (build\_fold\ mkf_2\ mkz_2\ xs) = build\_fold\ (mkf_2 \circ mkf_1)\ (mkz_2 \circ mkz_1)\ xs$$

Now that we've removed all of the lambdas from our definitions, we can look at the implementation.

5.3.4 Implementation

Deforestation turned out to be one of the easiest optimizations to implement. The implementation is entirely in GAS, and it proceeds in two steps. First we find any case where a *build* or *build_fold* is used exclusively in either a *build_fold* or *fold*. If this is the case, we inline the variable into it's single use. This temporarily takes our expression out of A-Normal Form, but we will restore that with the second step, which is the actual Deforestation transformation. It simply applies either the *foldr* / *build* theorem, or one of the three *build_fold* theorems from above. The definitions are given in figure 5.9 The optimization derivation for *sumPrimes* is in figure 5.10.

So far we've done a decent job. It's not as efficient as the Haskell version, but that's not surprising. However, we can still improve this. The main problem here is that we can't optimize a partial application. This is unfortunate, because the *build_fold* function tends to create large expressions of partially applied functions. Fortunately we've already solved this problem earlier in our compiler. We already have a way to detect if an expression is partially applied, so, in the post processing phase, we do a scan for any partially applied functions. If we find one, then we outline

Inline foldr/build

$$\begin{array}{lll}
\textbf{let } x = \mathit{build}\ g \textbf{ in } e & \mid e|_p = \mathit{foldr}\ _\ _\ x & \Rightarrow e[[p,2] \mapsto \mathit{build}\ g] \\
\textbf{let } x = \mathit{build}\ g \textbf{ in } e & \mid e|_p = \mathit{build_fold}\ _\ _\ x & \Rightarrow e[[p,2] \mapsto \mathit{build}\ g] \\
\textbf{let } x = \mathit{build_fold}\ \mathit{mkf}\ \mathit{mkz} \textbf{ in } e & \mid e|_p = \mathit{foldr}\ f\ z\ x & \Rightarrow e[[p,2] \mapsto \mathit{build_fold}\ \mathit{mkf}\ \mathit{mkz}] \\
\textbf{let } x = \mathit{build_fold}\ \mathit{mkf}\ \mathit{mkz} \textbf{ in } e & \mid e|_p = \mathit{build_fold}\ _\ _\ x & \Rightarrow e[[p,2] \mapsto \mathit{build_fold}\ \mathit{mkf}\ \mathit{mkz}]
\end{array}$$

Deforest foldr/build

$$\mathit{foldr}\ f\ z\ (\mathit{build}\ g) \Rightarrow g\ f\ z$$

Deforest build_fold/build

$$\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (\mathit{build}\ g) \Rightarrow \mathit{build}\ (\lambda c\ n \to g\ (\mathit{mkf}\ c)\ (\mathit{mkz}\ n))$$

Deforest foldr/build_fold

$$\begin{array}{ll}
\mathit{foldr}\ f\ z\ (\mathit{build_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs}) & \Rightarrow \textbf{let } f_1 = \mathit{mkf}\ f \\
& \quad \textbf{in let } \mathit{z_1} = \mathit{mkz}\ z \\
& \quad \textbf{in } \mathit{foldr}\ f_1\ \mathit{z_1}\ \mathit{xs}
\end{array}$$

Deforest build_fold/build_fold

$$\begin{array}{ll}
\mathit{build_fold}\ \mathit{mkf}_1\ \mathit{mkz}_1\ (\mathit{build_fold}\ \mathit{mkf}_2\ \mathit{mkz}_2\ \mathit{xs}) & \Rightarrow \textbf{let } f_1 = \mathit{mkf}_2 \circ \mathit{mkf}_1 \\
& \quad \textbf{in let } \mathit{z_1} = \mathit{mkz}_2 \circ \mathit{mkz}_1 \\
& \quad \textbf{in } \mathit{build_fold}\ f_1\ \mathit{z_1}\ \mathit{xs}
\end{array}$$

Figure 5.9: The Deforestation optimization.

The lambda in the build rule is a call to a known function.

The lets are added to keep the expression in A-Normal Form.

$\textbf{let } v_1 = \textit{enumFromTo}\ 2\ n$

$\textbf{in let } v_2 = \textit{filter isPrime}\ v_1$

$\textbf{in } \textit{sum}\ v_2$

$\Rightarrow$ **Reduce Useful**

$\textbf{let } v_1 = \textit{build enumFromTo\_build}\ 2\ n$

$\textbf{in let } v_2 = \textit{build\_fold}\ (\textit{filter\_mkf isPrime})\ \textit{id}\ v_1$

$\textbf{in } \textit{foldr}\ (+)\ 0\ v_2$

$\Rightarrow$ **Inline foldr/build**

$\textbf{let } v_1 = \textit{build enumFromTo\_build}\ 2\ n$

$\textbf{in let } v_2 =$

$\textbf{in } \textit{foldr}\ (+)\ 0\ (\textit{build\_fold}\ (\textit{filter\_mkf isPrime})\ \textit{id}\ v_1)$

$\Rightarrow$ **Deforest build_fold/build**

$\textbf{let } v_1 = \textit{build enumFromTo\_build}\ 2\ n$

$\textbf{in let } v_2 = \textit{build\_fold}\ (\textit{filter\_mkf isPrime})\ \textit{id}\ v_1$

$\textbf{in } \textit{foldr}\ (+)\ 0$

$\quad (\textit{build}\ (\textit{mk\_build}\ (\textit{enumFromTo\_build}\ 2\ n)\ (\textit{filter\_mkf isPrime})\ \textit{id})$

$\Rightarrow$ **Deforest build_fold/build**

$\textit{mk\_build}\ (\textit{enumFromTo\_build}\ 2\ n)\ (\textit{filter\_mkf isPrime})\ \textit{id}\ (+)\ 0$

$\Rightarrow \textit{ANF}$

$\textbf{let } f = \textit{filter\_mkf isPrime}\ (+)$

$\quad\ z = \textit{id}\ 0$

$\textbf{in } \textit{enumFromTo\_build}\ 2\ n\ f\ z$

$\Rightarrow$ **Inline Literal**

$\textbf{let } f = \textit{filter\_mkf isPrime}\ (+)$

$\textbf{in } \textit{enumFromTo\_build}\ 2\ n\ f\ 0$

Figure 5.10: Derivation for *sumPrimes*

it and attempt to optimize it. If we can't optimize it at all, then we do nothing. Otherwise, we make a new outlined function, and replace the call to the partially applied function with a call to the outlined function. This would actually be worth doing even if we didn't implement Deforestation. With function outlining our final optimized code is given below.

$$
\begin{aligned}
&\mathit{sumPrimes}\ n = \mathit{enumFromTo\_build}\ 2\ n\ f'\ 0 \\
&f'\ x\ y = \mathbf{if}\ \mathit{isPrime}\ x\ \mathbf{then}\ x + y\ \mathbf{else}\ y \\
&\mathit{enumFromTo\_build}\ a\ b\ c\ n \\
&\quad \mid a > b \qquad\quad = n \\
&\quad \mid \mathit{otherwise} = a\ \text{`}c\text{`}\ \mathit{enumFromTo\_build}\ (a+1)\ b\ c\ n
\end{aligned}
$$

This certainly isn't perfect, but it's much closer to what we were hoping for. Combining this with Unboxing and Shortcutting gives us some very efficient code. While these results are very promising, we still need to know if Deforestation is even valid for Curry.

5.3.5 Correctness

First we show that the *build_fold* theorems are valid for a deterministic subset of Curry using the same reasoning as the original foldr-build rule.

Theorem 9. *For any deterministic f, z, g, mkf, and mkz, the following equations hold.*

$$
\begin{aligned}
&\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (\mathit{build}\ g) = \mathit{build}\ (\lambda c\ n \rightarrow g\ (\mathit{mkf}\ c)\ (\mathit{mkz}\ n)) \\
&\mathit{foldr}\ f\ z\ (\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs}) = \mathit{foldr}\ (\mathit{mkf}\ f)\ (\mathit{mkz}\ z)\ \mathit{xs} \\
&\mathit{build\_fold}\ \mathit{mkf}_1\ \mathit{mkz}_1\ (\mathit{build\_fold}\ \mathit{mkf}_2\ \mathit{mkz}_2\ \mathit{xs}) = \mathit{build\_fold}\ (\mathit{mkf}_2 \circ \mathit{mkf}_1)\ (\mathit{mkz}_2 \circ \mathit{mkz}_1)\ \mathit{xs}
\end{aligned}
$$

Proof. Recall that the free theorem for *build* is

$$
\begin{aligned}
&(\forall\ (a : A)\ (\forall\ (b : B)\ h\ (f\ a\ b) = f'\ a\ (h\ b))) \Rightarrow \\
&\forall\ (b : B)\ h\ (g_B\ f\ b) = g\_B'\ f'\ (h\ b)
\end{aligned}
$$

We substitute $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}$ for h, $(:)$ for f and $\mathit{mkf}\ (:)$ for f'. From the definition of *build_fold* we have $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (a : b) = (\mathit{mkf}\ (:))\ a\ (\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ b)$
and $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ [\,] = \mathit{mkz}\ [\,]$. Therefore we have $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (g\ (:)\ b) = g\ (\mathit{mkf}\ (:))\ (\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ b)$
This gives us the following result.

$$\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (\mathit{build}\ g) = g\ (\mathit{mkf}\ (:))\ (\mathit{mkz}\ [\,])$$

Finally, working backwards from the definition of *build* we have our theorem.

$$\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (\mathit{build}\ g) = \mathit{build}\ (\lambda c\ n \to g\ (\mathit{mkf}\ c)\ (\mathit{mkz}\ n))$$

Again with *foldr*

if $\forall\ (a : A)\ (\forall\ (b : B)\ b\ (x\ \oplus\ y) = (a\ x)\ \otimes\ (b\ y)$ and $b\ u = u'$

then $b \circ \mathit{fold}\ \oplus\ u = \mathit{fold}\ \otimes\ u' \circ (\mathit{map}\ a)$

Here we take $b = \mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}$, $\oplus = f$, and $\otimes = \mathit{mkf}\ f\ \ a = \mathit{id}$

then the statment becomes:

if $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (f\ x\ y) = (\mathit{mkf}\ f)\ x\ (\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ y)$

and $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ [\,] = \mathit{mkz}\ [\,]$

then $\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz} \circ \mathit{fold}\ f\ z = \mathit{fold}\ (\mathit{mkf}\ f)\ (\mathit{mkz}\ z)$

Since both conditions follow directly from the definition of *build_fold* we are left with

$$\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz} \circ \mathit{fold}\ f\ z = \mathit{fold}\ (\mathit{mkf}\ f)\ (\mathit{mkz}\ z)$$

which is exactly what we wanted. Free theorems are fun!

Finally for *build_fold* / *build_fold* rule suppose we have the expression

$$\mathit{foldr}\ f\ z\ (\mathit{build\_fold}\ \mathit{mkf}_1\ \mathit{mkz}_1\ (\mathit{build\_fold}\ \mathit{mkf}_2\ \mathit{mkz}_2\ \mathit{xs}))$$

From the previous result we have:

$$\mathit{foldr}\ (\mathit{mkf}_1\ f)\ (\mathit{mkz}_1\ z)\ (\mathit{build\_fold}\ \mathit{mkf}_2\ \mathit{mkz}_2\ \mathit{xs})$$
$$\Rightarrow \mathit{foldr}\ (\mathit{mkf}_2\ (\mathit{mkf}_1\ f))\ (\mathit{mkz}_2\ (\mathit{mkz}_1\ z))\ \mathit{xs}$$
$$\Rightarrow \mathit{foldr}\ ((\mathit{mkf}_2 \circ \mathit{mkf}_1)\ f)\ ((\mathit{mkz}_2 \circ \mathit{mkz}_1)\ z)\ \mathit{xs}$$
$$\Rightarrow \mathit{foldr}\ f\ z\ (\mathit{build\_fold}\ (\mathit{mkf}_2 \circ \mathit{mkf}_1)\ (\mathit{mkz}_2 \circ \mathit{mkz}_1)\ \mathit{xs})$$

which establishes our result:

$$\mathit{build\_fold}\ \mathit{mkf}_1\ \mathit{mkz}_1\ (\mathit{build\_fold}\ \mathit{mkf}_2\ \mathit{mkz}_2) = \mathit{build\_fold}\ (\mathit{mkf}_2 \circ \mathit{mkf}_1)\ (\mathit{mkz}_2 \circ \mathit{mkz}_1)$$

$\square$

While this gives us confidence that Deforestation is a useful optimization, we've already seen that equational reasoning doesn't always apply in Curry. In fact, as they are currently stated, it's not surprising that these rules don't hold in Curry. However, with a few assumptions, we can remedy this problem. First, we need to rewrite our rules so that the reduced expression is in A-Normal form.

$$
\begin{aligned}
&\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (\mathit{build}\ g) = \textbf{let}\ g' = (\lambda c\ n \rightarrow \textbf{let}\ f = \mathit{mkf}\ c \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z = \mathit{mkz}\ n \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \textbf{in}\ \ g\ f\ z) \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \textbf{in}\ \mathit{build}\ g' \\
&\mathit{foldr}\ f\ z\ (\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs}) = \textbf{let}\ f' = \mathit{mkf}\ f \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad z' = \mathit{mkz}\ z \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \ \textbf{in}\ \ \mathit{foldr}\ f'\ z'\ \mathit{xs} \\
&\mathit{build\_fold}\ \mathit{mkf1}\ \mathit{mkz2}\ (\mathit{build\_fold}\ \mathit{mkf2}\ \mathit{mkz2}\ \mathit{xs}) = \textbf{let}\ \mathit{mkf} = \mathit{mkf1} \circ \mathit{mkf2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathit{mkz} = \mathit{mkz1} \circ \mathit{mkz2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \textbf{in}\ \ \mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs}
\end{aligned}
$$

Now we are ready to state our result.

Theorem 10. *suppose f, z, g, mkf, and mkz are all functions who's right had side is an expression in A-Normal form, then the following reductions are valid.*

$$
\begin{aligned}
&\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ (\mathit{build}\ g) = \textbf{let}\ g' = (\lambda c\ n \rightarrow \textbf{let}\ f = \mathit{mkf}\ c \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad z = \mathit{mkz}\ n \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \textbf{in}\ \ g\ f\ z) \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \textbf{in}\ \mathit{build}\ g' \\
&\mathit{foldr}\ f\ z\ (\mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs}) = \textbf{let}\ f' = \mathit{mkf}\ f \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad z' = \mathit{mkz}\ z \\
&\qquad\qquad\qquad\qquad\qquad\quad\ \ \textbf{in}\ \ \mathit{foldr}\ f'\ z'\ \mathit{xs} \\
&\mathit{build\_fold}\ \mathit{mkf1}\ \mathit{mkz2}\ (\mathit{build\_fold}\ \mathit{mkf2}\ \mathit{mkz2}\ \mathit{xs}) = \textbf{let}\ \mathit{mkf} = \mathit{mkf1} \circ \mathit{mkf2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathit{mkz} = \mathit{mkz1} \circ \mathit{mkz2} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ \textbf{in}\ \ \mathit{build\_fold}\ \mathit{mkf}\ \mathit{mkz}\ \mathit{xs}
\end{aligned}
$$

Proof. We show the result for foldr-build, and the rest are similar calculations. We intend to show that for any f, z, and g that

$$\textit{fold}\ f\ z\ (\textit{build}\ g\ (:)\ [\,])\texttt{==}g\ f\ z$$

We proceed in a manner similar to [29]. First, notice that *build* g (:) [ ] is constructing a list. However, since g is potentially non-deterministic, and it might fail, we may have a non-deterministic collection of lists when normalizing this expression. Let's make this explicit.

$$\begin{aligned}
&\textit{build}\ g\ (:)\ [\,] = g_{1,1} : g_{1,2} : g_{1,3} : \ldots \textit{end}_1\\
&\quad ?\ g_{2,1} : g_{2,2} : g_{2,3} : \ldots \textit{end}_2\\
&\quad\quad \ldots\\
&\quad ?\ g_{k,1} : g_{k,2} : g_{k,3} : \ldots \textit{end}_k\\
&\quad \textbf{where}\ \textit{end}_i = [\,]\ ?\ \bot
\end{aligned}$$

Here we have a collection of k lists, and each list ends either with the empty list, or the computation may have failed along the way. Therefore, $\textit{end}_i$ may be either [ ] or $\bot$. In fact, it might be the case that an entire list is $\bot$, but this is fine, because that would still fit this form.

Now, let's see what happens when we normalize the entire expression. Recall that function application distributes over choice. That is, $f\ (a\ ?\ b) = f\ \ a\ ?\ f\ \ b$.

$$\begin{aligned}
&\textit{foldr} \oplus z\ (\textit{build}\ g\ (:)\ [\,])\\
&\quad = \textit{foldr} \oplus z\ (g_{1,1} : g_{1,2} : g_{1,3} : \ldots \textit{end}_1\ ?\\
&\quad\quad\quad\quad g_{2,1} : g_{2,2} : g_{2,3} : \ldots \textit{end}_2\ ?\\
&\quad\quad\quad\quad \ldots\\
&\quad\quad\quad\quad g_{k,1} : g_{k,2} : g_{k,3} : \ldots \textit{end}_k)\\
&\quad = \textit{foldr} \oplus z\ (g_{1,1} : g_{1,2} : g_{1,3} : \ldots \textit{end}_1)\ ?\\
&\quad\quad \textit{foldr} \oplus z\ (g_{2,1} : g_{2,2} : g_{2,3} : \ldots \textit{end}_2)\ ?\\
&\quad\quad\quad\quad \ldots\\
&\quad\quad \textit{foldr} \oplus z\ (g_{k,1} : g_{k,2} : g_{k,3} : \ldots \textit{end}_k)\\
&\quad = (g_{1,1} \oplus g_{1,2} \oplus g_{1,3} \oplus \ldots z_{\textit{end}_1})\ ?\\
&\quad\quad (g_{2,1} \oplus g_{2,2} \oplus g_{2,3} \oplus \ldots z_{\textit{end}_2})\ ?\\
&\quad\quad\quad \ldots\\
&\quad\quad (g_{k,1} \oplus g_{k,2} \oplus g_{k,3} \oplus \ldots z_{\textit{end}_k})\\
&\quad\quad\quad \textbf{where}\ z_{\textit{end}_i} = \textbf{if}\ \textit{end}_i\texttt{==}\textit{failed}\ \textbf{then}\ \textit{failed}\ \textbf{else}\ z\\
&\quad = g \oplus z
\end{aligned}$$

This proves the result.

□

Note that while this does prove the result, there are still some interesting points here. First, we never made any assumptions about f or z. In fact, we didn't really make any assumptions about g, but we did at least give an explicit form for its values. This form is guaranteed by the type. This line of reasoning looks like a promising direction for future explorations into parametricity for functional-logic programming.

Second, it should be noted that branches in g that produce $\bot$ don't necessarily fail when evaluated. If f is strict, then any failure in the list will cause the entire branch to fail. Consider the following expression:

$$\mathit{foldr}\ (\lambda x\ y \to 1)\ 0\ (\mathit{build}\ (\lambda c\ n \to 0\ `c`\ 1\ `c`\ \bot))$$

Evaluating *build* to normal form would produce a failure, since the tail of the list is $\bot$. However, since the f in *foldr* never looks at either of it's arguments, this branch of the computation can still return a result.

In this chapter we've developed three optimizations to help reduce the memory allocated by Curry programs. These optimizations seem effective, and we've shows why they're correct, but we still need to find out how effective they are. In the next chapter we show a small bechmarking suite to test the efficacy of these optimizations, and to show the results of each. We then discuss possible future directions for this research.

Chapter 6

CONCLUSION

6.1 RESULTS

Now that we've finally implemented all of the optimizations, we need to see if they were actually effective. Before we can look at the results, we need to discuss methodology. The tests we ran aren't extensive. We're not trying to characterize every area of this compiler's performance, we just want a general idea of the time and memory consumption. While a more extensive test that looks at finer details like pipeline stalls and cache misses would be interesting, and no doubt informative, we are really only concerned with two characteristics. How long did the program take to run? How many allocations were made by the program?

There are many ways to measure memory allocation and execution time. For memory allocation, we could estimate the memory using the operating system, or we could use a tool like Valgrind to find the number of allocations. However, an even simpler solution is to keep a count of the number of times we create a node. This is simple to implement, and doesn't effect the run-time performance noticeably. Therefore, we measure memory allocation by instrumenting the run time system to report how many nodes are created. To measuring execution time, we took the approach from [72], and ran the programs multiple times, while taking the fastest result. We also ran the program in multiple different environments on multiple machines.

6.1.1 Tests

In order to determine how effective our optimizations are, we've developed a small suite of test programs that are meant to test both deterministic and non-deterministic programs. This suite is loosely based on the one used to test the Kics2 compiler [25]. However, we've made a few alterations. We've added several programs to test non-determinism, and we've removed or modified programs using functional patterns, because that was not part of the RICE compiler.

We split the functions into tree groups: Numeric computations meant to test Unboxing; list computations meant to test Deforestation; and non-deterministic computations.

- **Numeric computations:**
  - **fib** is the Fibonacci program from chapter 5. We test it with both deterministic and non-deterministic input.
  - **tak** computes a long, mutually recursive, function with many numeric calculations.
- **Non-deterministic computations:**
  - **cent** attempts to find all expressions containing the numbers 1 to 5 that evaluate to 100.
  - **half** computes half of a number defined using piano arithmetic.

    $$\begin{aligned}&\mathit{half}\ n \mid x + x {=}{=} n = x\\&\quad \textbf{where}\ x\ \textbf{free}\end{aligned}$$

  - **perm** computes all of the permutations of a list.
  - **queens_perm** is the program from the introduction. It computes solutions to the n-queens problem by permuting a list, and checking if it's a valid solution.
  - **sort** sorts a list by finding a sorted permutation.
- **Deforestation:**
  - **queens_det** computes solutions to the n-queens problem using a backtracking solution and list comprehension.
  - **reverse_builtin** reverses a list without using functions or data types defined in the standard Prelude.
  - **reverse_foldr** reverses a list using a reverse function written as a fold.
  - **reverse_prim** reverses a list using the built-in reverse function and primitive numbers.
  - **sum_squares** computes $\mathit{sum} \circ \mathit{map}\ \mathit{squre} \circ \mathit{filter}\ \mathit{odd} \circ \mathit{enumFromTo}\ 1$.
  - **buildFold** computes a long chain of list processing functions.
  - **primes** computes a list of primes.
  - **sieve** computes *sumPrimes* from Chapter 5.

6.1.2 Results

The results for running the tests are given if figure 6.1 for timing, and 6.2 for memory. While we attempted to be as fair in our assessments as possible, there are some cases where we couldn't include a compiler, either because it was taking too long, or it used too much memory, and was killed by the operation system. For the timing results we include Kics2 and Pakcs when we can, however neither compiler offers a way to see how much memory was used during computation, so they are not included in the memory results.

There is also a much more interesting problem. In some cases we simply can't compare the optimized code. The optimized code runs so fast that the we can't accurately time it.

We'll look at *fib*, because it illustrates this problem particularly well. The algorithm for *fib* is an exponential time algorithm. Specifically, it runs in $O(1.61^n)$. This means that *fib* n runs roughly 1.61 times as fast as *fib* $(n+1)$. In running our example with a non-deterministic input, both Kics2 and Pakcs were only able to run up to *fib* 27. In contrast, fully optimized RICE was able to run *fib* 42 in about the same amount of time. Using this as a very rough approximation, we find that our optimized code is running $1.61^{15} \approx 1000$ times as fast.

This is a very impressive speedup, but we've already discussed the reason for it. After we applied Unboxing and Shortcutting, we were able eliminate all but a constant number of heap allocations from the program. This would be a great result on it's own, but it gets even better when we compare it to GHC. Compiling the same *fib* algorithm on GHC produced code that ran about twice as fast as our optimized RICE code, and when we turned off Optimizations for GHC we ran faster by a factor of 8. It's not surprising to us that our code ran slower than GHC. In fact, we would be shocked if it managed to keep up. What is surprising, and encouraging, is that we were competitive at all. It suggests that Curry isn't inherently slower than Haskell. We believe that a more mature Curry compiler could run as fast as GHC on most, if not all, deterministic functions.

6.2 CONCLUSION

These results were honestly significantly better than we ever expected with this project. Initially, we hoped to compete with Kics2, since it was leveraging GHC's optimizer to produce efficient code. However, we found that not only could we beat Kics2 in all cases, and in many cases the results were simply incomparable, and in some cases we were even able to compete with GHC

	fib	fib_nondet	tak
pakcs	25.900	30.244	6867.545
kics2	0.060	3.370	6.670
unopt			
basic			
unbox			
shortcut			
deforest			
all	0.004	0.004	0.945

	cent	half	perm	queens_perm	sort
pakcs	65.630	983.572	45.102	568.812	212.331
kics2	26.740	25.270	4.000	2.980	8.070
unopt					
basic					
unbox					
shortcut					
deforest					
all	0.433	0.531	0.625	0.160	0.241

	queens_det	rev	rev_fold	rev_prim	sum_squares	buildFold	primes	sieve
pakcs	1051.782	OoM	262.144	27.963	33.486	OoM	6480.616	2845.308
kics2	1.070	0.790	0.170	0.180	0.080	13.980	31.360	3.170
unopt								
basic								
unbox								
shortcut								
deforest								
all	0.323	0.288	0.028	0.025	0.011	0.580	0.625	1.030

Figure 6.1: Results for running time

	fib	fib_nondet	tak
unopt			
basic			
unbox			
shortcut			
deforest			
all	11	4	4

	cent	half	perm	queens_perm	sort
unopt					
basic					
unbox					
shortcut					
deforest					
all	18276630	25120033	1880786	6099651	11949170

	queens_det	rev	rev_fold	rev_prim	sum_squares	buildFold	primes	sieve
unopt								
basic								
unbox								
shortcut								
deforest								
all	17858521	12624889	1310732	1310734	600007	30000008	16373478	63922711

Figure 6.2: Results for memory usage

itself. Furthermore, we've shown that the memory optimizations really were effective for Curry programs. This isn't much of a surprise. Allocating less memory is a good strategy for improving run-time performance. It is good to know that the presence of non-determinism doesn't affect this commonly head belief.

It's a little more surprising that these optimizations all turned out to be valid in Curry. In fact, a surprising number of optimizations are valid in Curry under suitable conditions. This might not seem very significant, until we look at what optimizations aren't valid. For example, common sub-expression elimination was not included in this compiler, because is that it simply isn't a valid Curry transformation. It introduces sharing where none existed. If the common sub-expression is non-deterministic, then we will change the set of results. On the other hand, common sub-expression elimination is fairly innocuous in most other languages.

6.2.1 Future Work

Most currys are made form curry powder and coconut milk, however our Curry was mostly made from low hanging fruit. As nice as our results are, we would like to see this work extended in the future. We believe that a better inliner and strictness analyzer would go a long way to producing even more efficient code.

In fact, a general theory of inlining in Curry would be hugely beneficial. One of the biggest drawbacks to this compiler is that we can't represent lambda expressions in FlatCurry, and inline them. Before we could even attempt this, we would need to know when it's safe to inline a lambda in Curry.

We would also like to move from short-cut Deforestation to Stream Fusion. This should be possible, but it would require a more sophisticated strictness analyzer, and we may not be able to get away with our combinator approach.

We would also like to see the development of new, Curry specific, optimizations. Right now the ? operator acts as a hard barrier. We can move let bound variables outside of it, but we can't move the choice itself. However, there may be an option for using pull-tabbing or bubbling to move the choice to make room for more optimizations.

Finally, developing a better run-time system would also be an important improvement. While we did work to make sure our run time system was efficient, it could certainly be better. Integrating this work with the Sprite [20] compiler might solve this issue.

6.2.2 Conclusion and Related Work

We have presented the RICE Optimizing Curry compiler. The compiler was primarily built to test the effectiveness of various optimizations on Curry programs. While testing these optimizations, we've also built an efficient evaluation method for backtracking Curry programs, as well as a general system for describing and implementing optimizations. The compiler itself is written in Curry.

This system incorporated a lot of work from the functional language community, and the Haskell community in particular. The work on general optimizations [80], Inlining [79], Unboxing [55], Deforestation [39], and the STG-machine [83, 78] were all instrumental in the creation of this compiler, as well as the work by Appel and Peyton-Jones about functional compiler construction [22, 21, 82].

While there has been some work on optimizations for functional-logic programs, there doesn't seem to be a general theory of optimization. Peemöller and Ramos et al. [76, 85] have developed a theory of partial evaluation for Curry programs, and Moreno [69] has worked on the Fold/Unfold transformation from Logic programming. We hope that our work can help bridge the gap to traditional compiler optimizations.

The implementation of the GAS system was instrumental in developing optimizations for this compiler. It not only allowed us to implement optimizations more efficiently, but also to test new optimizations, and through optimization derivations, discover which optimizations were effective, which were never used, and which were wrong. This greatly simplified debugging optimizations, but it also allowed us to test more complicated optimizations. Often we would just try an idea to see what code it produced, and if it fired in unintended places. It's difficult to overstate just how useful this system was in the compiler.

While the run-time system was not the primary focus of this dissertation, we were able to produce some useful results. The path compression theorem, and the resulting improvement to backtracking, is a significant improvement to the current state-of-the-art for backtracking Curry programs.

When starting this project, Shortcutting was already known to be valid for functional logic programs. It was developed for them specifically. However, it was a nice surprise to find that Unboxing and Deforestation were both valid in Curry. It was even more remarkable that, with some simple restrictions, we could make inlining and reduction valid in Curry as well.

We believe that this work is a good start for optimizing Curry compilers, and we would like to see it continue. After having a taste of optimized Curry, we want to turn up the heat, and deliver an even hotter dish. But for now, we've made a tasty Curry with RICE.

Index

$C(e)$, 38

$FORWARD$, 40

$T(\Sigma, V)$, 4

$\leftrightarrow$, 2

σ, 5

$\rightarrow$, 1

$\rightarrow^* b$, 2

$\rightarrow^+$, 2

$\rightarrow^=$, 2

$\rightarrow^n$, 1

$\rightarrow^{\leq n}$, 2

f_k, 35

t, $_p$4

$t[p \rightarrow r]$, 4

`children`, 42

`f_hnf`, 41

`field`, 42

`missing`, 45

`node`, 41

`tag`, 42

Abstract Rewriting System, 1

backtracking, 32

bubbling, 32

child, 4

collapsing, 6

collapsing functions, 40

computation space, 38

confluent, 2

connected, 17

Constructor Normal Form, 29

contractum, 6

definitional tree, 14

descendant, 11

duplicating, 6

FlatCurry, 29

forwarding node, 40

Generating and Altering Subexpressions (GAS), 54

graph, 17

Head Constructor Form, 30

inductively sequential, 14

Limited Overlapping Inductively Sequential, 27

linear, 4

matcher, 5

Narrowing, 9

narrowing strategy, 16
needed, 11
neighborhood, 17
nondet, 37
normal form, 2
normalizing, 10

orthogonal, 6
overlapping, 6

path, 4, 17
path compression, 40
Path Compression Theorem, 39
pull-tabbing, 32

redex, 6
Redex Compression Theorem, 39
replacement, 20
rewrite derivation, 2
rewrite rule, 6, 20
Rewrite Strategy, 10
Rewriting, 6
root, 17
rooted graph, 17

sequential, 13
signature, 4
strongly connected, 17
subgraph, 17
substitution, 5
subterm, 4

term, 4
term graph, 17
Term Rewriting System (TRS), 6
terminating, 3

unifier, 5

variable, 4

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