Fibred Categories and Stacks

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1 Introduction

In a 2013 interview with *Radiolab* physicist Brian Greene expressed his "deep faith that the universe is coherent" [Gre13]. This conviction in universal coherence compels mathematicians to explain recurring patterns in mathematics through a nested series of overarching theories. For example, vector spaces were introduced in the mid to late 1800s in order to formalize the behavior of coordinates in Euclidean space. Later in 1921, Emmy Noether defined modules over a ring of which vectors spaces over a field is a specific case [Kle07]. Category theory is another such overarching theory, developed to systematize the abundance and expository power of commutative diagrams in mathematics [ML78]. As we will see, the theory of fibrations in turn governs the theory of ordinary categories.

Grothendieck first developed fibrations in [Gro64] in order to systematize ideas of stacks and decent in algebraic geometry. From this origin the theory of fibrations was applied to categorical logic and type theory (discussed in [Jac99]) and as a foundation of category theory. This foundational perspective was developed in the 1970s largely due to Lawvere's explorations of families of objects and indexed categories as related to category theory over arbitrary base toposes, Paré and Schumacher's work on indexed categories in [PS78], Celeyrette's thesis completed in 1974, and Bénabou's own lectures given in Montreal in 1974. The philosophical implications of these works was presented by Bénabou in [Bén85] and is the jumping point for this paper.

At its most abstract, category theory captures the notion of stuff, structures, and properties, which together characterize many mathematical ideas (see Section 2.4 of [BS10]). For example, a group has stuff (the underlying set), structure (the operation), and properties (equations that hold). Categories concretely realize the abstract notion of stuff, structure, and properties. A useful analogy is that a group concretely realizes the abstract notion of elements along with a method of combining elements and undoing a combination. The features of a category which are necessary to identify the stuff, structure, and properties that the category realizes is referred to as the content of the category. As seen in [BS10] a category's content is composed of its categorical features which are preserved by equivalence, and equivalent categories realize the same stuff, structure, and properties.

In ordinary category theory content is constructed via explicitly defining objects andmorphisms along with basic rules about identity morphisms, composition, and associativity. This formulation generalizes recurring patterns in mathematics since categories of familiar mathematical concepts such as sets, groups, and topological spaces are obvious to define in this way. However, this constructive approach obscures an important nuance of ordinary category theory, namely that the stuff, structure, and properties able to be realized by an ordinary category is limited by a specific relationship to the category **Set** which is made explicit in Section 2. For this reason, ordinary category theory is sometimes called naive category theory, although it is far from naive in the usual sense. We refer to stuff, structure, and properties with this limitation as being "over **Set**".

Given mathematical coherence, we are driven to develop a comprehensive theory of "categories" that governs the theory of categories by generalizing stuff, structure, and properties over **Set** to stuff, structure, and properties over an arbitrary base category $\mathcal{B}$, where a "category" is an instance of stuff, structure, and properties over any base category. Fibred category theory makes the abstract theory of "categories" concrete in the same way that ordinary category theory makes the abstract theory of stuff, structure, and properties over **Set** concrete.

The goals of this essay are:

	naive	**general**
abstract	stuff, structure, and properties over **Set**	stuff, structure, and properties over a base category $\mathcal{B}$
concrete	categories	fibrations

Table 1: Moving downwards in the table is the result of realizing an abstract concept. Moving upwards in the table is the result of abstracting a concrete concept. Moving rightwards is the result of generalizing a naive concept.

naive and concrete ($\diamond$)	**general and abstract** ("$\diamond$")	**general and concrete** (fibered $\diamond$)
Set	base category	$\mathcal{B}$
set	"set"	object of $\mathcal{B}$
Fam $\mathcal{C}$	collection of fibres	$\mathcal{F}$
re-indexing morphisms	glue between fibres	cartesian morphisms
category	"category"	fibration
functor	"functor"	fibred functor
natural transformation	"natural transformation"	fibred natural transformaion
Cat	2-category of "categories"	**Fib**
category with equality	"category" with "equality"	spit fibration

Table 2: For linguistic clarity, we use quotes to distinguish notions and structures which are naive and concrete from those which are general and abstract for example "categories", "sets", etc. are the generalized abstractions of categories, sets, etc. that are familiar from ordinary category theory.

1. *Extract and realize a notion of "category".* In Section 2 we identify how **Set** limits the stuff, structure, and properties realized by ordinary categories and extrapolate a generalization of the naive and abstract to the general and abstract. In Section 3 we analyze categories as a realization of stuff, structure, and properties over **Set** and mimic this relationship to define fibrations as the realization of "categories". We repeat this process for other important categorical notions in Section 4. Table 2 previews this work. Lastly in Section 5, we give an equivalent realization of "categories" and discuss our preference for fibrations.

2. *Give evidence that fibrations recapture the richness and complexity of categories.* As Benabou states in [Bén85], "the only 'proof' of such a claim can be found in actual work with fibrations, showing that the major notions and results of naive category theory do translate in the context of fibrations" and in Section 7 we give a general strategy for doing so. As specific examples, in Section 6 we consider the major notion of equality, and in Section 7 we interpret the categorical property of having coproducts as a fibrational property.

In this paper, our exploration of high-level expectations about "categories" and fibrations are based on the work of Bénabou in [Bén85]. The exposition of these ideas in terms of definitions and results is primarily a compilation of the work in [Bor94], [Str99], and [Jac99] each of which makes the ideas in [Bén85] concrete.

2 Stuff, structure, and properties

This section examines how the category **Set** defines a framework for ordinary category theory. We begin with the recurring notion of set-indexed families in ordinary category theory.

Definition 1. Let $\mathcal{C}$ be a category. There is a category Fam $\mathcal{C}$ with

objects Set-indexed families $(X_i)_{i\in I}$ where X_i is an object of $\mathcal{C}$ and I is a set.

morphisms A morphism between families $(X_i)_{i\in I} \to (Y_j)_{j\in J}$ is a pair $(u, (f_i)_{i\in I})$ where $u : I \to J$ is a morphism in **Set** and for each $i \in I$, $f_i : X_i \to Y_{u(i)}$ is a morphism in $\mathcal{C}$.

F is ...	F *only* forgets ...
fully faithful	properties
faithful and essentially surjective	structure
full and essentially surjective	stuff

Table 3: Since stuff, structure, and properties are abstract notions, it is an abuse of vocabulary to say that F forgets stuff (respectively structure and properties). In fact, we mean that F forgets the particular aspects of the category's content that realizes stuff (respectively structure and properties). A factorization of the functor $!_{\mathcal{C}}$ by forgetful functors of these types illuminates the entire content of $\mathcal{C}$.

Let $p_{\mathcal{C}} : \mathrm{Fam}\,\mathcal{C} \to \mathbf{Set}$ be the projection functor defined on objects by $(X_i)_{i\in I} \mapsto I$ and on morphisms by $(u, (f_i)_{i\in I}) \mapsto u$. We say that families $(X_i)_{i\in I}$ are indexed by I and that morphisms $(u, (f_i)_{i\in I})$ are morphisms over u. Suggestively, we make the following definition.

Definition 2. For each I in $\mathbf{Set}$, the fibre over I (denoted $\mathrm{Fam}\,\mathcal{C}_I$) is the subcategory of $\mathrm{Fam}\,\mathcal{C}$ whose objects map to I under $p_{\mathcal{C}}$ and whose morphisms map to id_I under $p_{\mathcal{C}}$. Equivalently, the fibre over I is the subcategory of $\mathrm{Fam}\,\mathcal{C}$ consisting of I-indexed families and morphisms over id_I.

In Section 3, fibrations are defined by taking $p_{\mathcal{C}}$ to be the prototypical example of a fibration where the fibration $p_{\mathcal{C}}$ generalizes the ordinary category $\mathcal{C}$, i.e. $p_{\mathcal{C}}$ and $\mathcal{C}$ realize the same stuff, strucutre, and properties (over $\mathbf{Set}$). The remainder of this section justifies this premise.

In contrast to the standard constructive definition of a category, [BS10] examines a deconstructive approach to detecting the content of a category. From this perspective, it is clear that $p_{\mathcal{C}}$ realizes the same the stuff, structure, and properties as $\mathcal{C}$.

Let $\mathbf{pt}$ be the terminal object of $\mathbf{Cat}$. The category $\mathbf{pt}$ consists of a single object $*$ and a single morphism id_*. For every category $\mathcal{C}$ there is a unique morphism $!_{\mathcal{C}} : \mathcal{C} \to \mathbf{pt}$ which forgets all of the information about $\mathcal{C}$. A systematic factorization of $!_{\mathcal{C}}$ gradually loses pieces of information about $\mathcal{C}$ revealing the content of $\mathcal{C}$.

For example, consider the factorization of $!_{\mathbf{Ab}} : \mathbf{Ab} \to \mathbf{pt}$ via the obvious forgetful functors,

$$\mathbf{Ab} \to \mathbf{Grp} \to \mathbf{Set} \to \mathbf{pt}.$$

The first functor loses the abelian *property* of $\mathbf{Ab}$. The second functor loses the group *structure*, and the last functor loses the underlying *stuff*.

The first functor is fully faithful but it is not essentially surjective on objects since there are groups which are not abelian. The second functor is faithful but it is not full because there are set maps which are not group homomorphisms. However, it is essentially surjective on objects since any set can be given a group structure. The last functor is full and essentially surjective on objects but not faithful. These observations are formalized in Table 3.

The relationship between functor properties and stuff, structure, and properties shows that if a functor $\mathbf{Cat} \to \mathbf{Cat}$ preserves full, faithful, and essentially surjective functors, then it preserves categorical content. The notion of set-indexed families provides the foundation for such a 2-functor $\mathrm{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ that is defined on objects by $\mathcal{C} \mapsto \mathrm{Fam}\,\mathcal{C}$. The following definition gives the action of Fam on functors and natural transformations.

Definition 3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $\mathrm{Fam}\,F : \mathrm{Fam}\,\mathcal{C} \to \mathrm{Fam}\,\mathcal{D}$ be the functor that acts on objects by $(X_i)_{i\in I} \mapsto (FX_i)_{i\in I}$ and on morphisms by $(u, (f_i)_{i\in I}) \mapsto (u, (Ff_i)_{i\in I})$. Functoriality of $\mathrm{Fam}\,F$ follows directly from the functoriality of F.

Let $\alpha : F \Rightarrow G$ be a natural transformation. Let $\mathrm{Fam}\,\alpha : \mathrm{Fam}\,F \Rightarrow \mathrm{Fam}\,G$ be the natural transformation with components

$$\alpha(X_i)_{i\in I} = (\mathrm{id}_I, (\alpha_{X_i})_{i\in I}).$$

Naturality follows from the naturality of α.

Proposition 4. *The above definitions defines a 2-functor* $\mathrm{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ *that preserves full, faithful, and essentially surjective functors. It immediately follows that* Fam *preserves categorical content.*

Proof. Straightforward. ☐

Recall that the functor $!_{\mathcal{C}} : \mathcal{C} \to \mathbf{pt}$ deconstructs the content of $\mathcal{C}$, so the functor $\mathrm{Fam}!_{\mathcal{C}} : \mathrm{Fam}\,\mathcal{C} \to \mathrm{Fam}\,\mathbf{pt}$ does as well. Since $\mathrm{Fam}\,\mathbf{pt}$ is equivalent to the category $\mathbf{Set}$ and the functor $p_{\mathcal{C}}$ corresponds to $\mathrm{Fam}!_{\mathcal{C}}$ under this equivalence, $p_{\mathcal{C}}$ also deconstructs the content of $\mathcal{C}$. In particular, this means that $p_{\mathcal{C}}$ represents the stuff, structure, and properties realized by the content of $\mathcal{C}$. Furthermore, if stuff, structure, and properties can be realized by an ordinary category, then it can be realized by a functor of this form and so it is said to be "over **Set**". As a concrete example of the correspondence between $\mathcal{C}$ and $p_{\mathcal{C}}$, consider the following formulations of coproducts.

Definition 5 (Coproducts 1)**.** $\mathcal{C}$ has coproducts if for each family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$ indexed by a set I, there exists an object $\coprod_{i\in I} X_i$ of $\mathcal{C}$ equipped with a family of morphisms $(\iota_i : X_i \to \coprod_{i\in I} X_i)_{i\in I}$ such that any family of morphisms $(\gamma_i : X_i \to C)_{i\in I}$ defines a unique morphism $g : \coprod_{i\in I} X_i \to C$ such that $g \circ \iota_i = \gamma_i$ for all $i \in I$.

$$\begin{tikzcd}
& \coprod_{i\in I} X_i \arrow[dl, "g"'] \\
C & X_i \arrow[l, "\gamma_i"] \arrow[u, "\iota_i"']
\end{tikzcd}$$

The references to families of objects and morphisms in this definition suggests that $\mathcal{C}$ having coproducts is in fact a property of $\mathrm{Fam}\,\mathcal{C}$. The next definition translates Definition 5 into the language of $\mathrm{Fam}\,\mathcal{C}$, $p_{\mathcal{C}}$, and fibres.

Definition 6 (Coproducts 2)**.** $\mathcal{C}$ has coproducts if for each object X of $\mathrm{Fam}\,\mathcal{C}$ there exists an object $\coprod X$ in $\mathrm{Fam}\,\mathcal{C}$ indexed by 1 (the terminal object of **Set**) and morphism $\iota : X \to \coprod X$ over $!_I$ where $I = p_{\mathcal{C}}X$ such that any object C in $\mathrm{Fam}\,\mathcal{C}$ indexed by 1 and morphism $\gamma : X \to C$ over $!_I$ defines a unique morphism $g : \coprod X \to C$ over id_1 such that $g \circ \iota = \gamma$.

$$\begin{tikzcd}
& \coprod X \arrow[dl, "g"'] & & & 1 \arrow[dl, "\mathrm{id}_1"'] \\
C & X \arrow[l, "\gamma"] \arrow[u, "\iota"'] & & 1 & I \arrow[l, "!_I"] \arrow[u, "!_I"'] \\
\mathrm{Fam}\,\mathcal{C} \arrow[rrr, "p_{\mathcal{C}}"] & & & \mathbf{Set} &
\end{tikzcd}$$

This definition can be simplified even further.

Definition 7 (Coproducts 3)**.** For each set I the diagonal functor

$$\Delta_I : \mathrm{Fam}\,\mathcal{C}_1 \to \mathrm{Fam}\,\mathcal{C}_I$$

maps the single object family (C) the I-indexed family $(C)_{i\in I}$. $\mathcal{C}$ has coproducts if Δ_I has a left adjoint for every I.

The equivalence of Definition 5 and Definition 6 follows directly from unraveling definitions. Equivalence of Definition 7 is given in [ML78], noticing that the categories $\mathrm{Fam}\,\mathcal{C}_I$ and $\mathcal{C}^I$ are isomorphic.

Comparing Definition 5 and Definition 7 clarifies that the categorical notion of having coproducts is naturally related to $\mathrm{Fam}\,\mathcal{C}$, although it can be stated independently of the categorical organization of set-indexed families. The wealth of such examples, several of which are summarized in Section 7, supports the claim that $p_{\mathcal{C}} : \mathrm{Fam}\,\mathcal{C} \to \mathbf{Set}$ realizes the stuff, structure, and properties captured by the content of $\mathcal{C}$ and thus represents the category $\mathcal{C}$. For further justification see §4 in [Bén85]. This reformulation reveals that the content of a category is inextricable from set-indexing and alludes to a more comprehensive notion of "category" meaning stuff, structure, and properties which depend on $\mathcal{B}$-indexing for an arbitrary base category $\mathcal{B}$. Furthermore, it suggests a method of realizing "categories" by taking $p_{\mathcal{C}}$ to be the canonical example of a realized "category" and extrapolating under the justified assumption that $p_{\mathcal{C}}$ generalizes the ordinary category $\mathcal{C}$.

3 Fibrations

The goal of this section is to fill in the general and concrete cell of Table 1 to be compatible with the rest of the table. To this effect, we must define fibrations so that they realize stuff, structure, and properties over arbitrary base categories $\mathcal{B}$ and so that they generalize ordinary categories. The latter is achieved by taking $p_\mathcal{C}$ to be the prototypical example of a fibration. The former is achieved by systematically replacing **Set** by an arbitrary category $\mathcal{B}$. We begin with the features of $p_\mathcal{C}$ which capture the content of $\mathcal{C}$ and so must be reflected in the definition of fibrations.

1. $p_\mathcal{C}$ elucidates how **Set** defines the framework for the stuff, structure, and properties realized by an ordinary category $\mathcal{C}$.
2. The category $\operatorname{Fam}\mathcal{C}$ and the functor $p_\mathcal{C}$ together express the stuff, structure, and properties realized by $\mathcal{C}$.

Analgously the constitutent parts of a "category" and its realization will be

1. A base category $\mathcal{B}$, whose objects will represent the "sets" which define the framework for a "category".
2. A category $\mathcal{F}$ and a functor $p : \mathcal{F} \to \mathcal{B}$ which together express a "category".

Assumptions about $\mathcal{B}$ will define the behavior of our "sets", while necessary assumptions about p and $\mathcal{F}$ will be made clear from an analysis of $p_\mathcal{C}$ and $\operatorname{Fam}\mathcal{C}$. Section 3.1 focuses on how $p_\mathcal{C}$ organizes set-indexed families of objects in $\mathcal{C}$. Section 3.2 focuses on on the behavior of morphisms in $\operatorname{Fam}\mathcal{C}$.

3.1 The fibres

Let $p : \mathcal{F} \to \mathcal{B}$ be a functor. In order to understand $\mathcal{F}$ in the role of $\operatorname{Fam}\mathcal{C}$, the features of $\operatorname{Fam}\mathcal{C}$ capturing the content of $\mathcal{C}$ (in the sense described in the previous section) must be identified and reformulated to be independent of the property specific to **Set**. In particular, they must avoid the recurring notation $i \in I$ that is used to define the objects and morphisms of $\operatorname{Fam}\mathcal{C}$.

Recall that Definition 6 expresses the property "$\mathcal{C}$ has coproducts" (originally defined in Defintion 5) in the language of $p_\mathcal{C}$ and $\operatorname{Fam}\mathcal{C}$. In the translated definition, objects X in the fibre over I (respectively objects $\coprod X$ and $\mathcal{C}$ in the fibre over 1) replace families $(X_i)_{i\in I}$ (respectively objects $\coprod_{i\in I} X_i$ and $\mathcal{C}$) in the original definition. This example indicates that in order to express content of $\mathcal{C}$ in the language of $p_\mathcal{C}$ and $\operatorname{Fam}\mathcal{C}$, it is sufficient to understand $\operatorname{Fam}\mathcal{C}$ as the collection of fibres of $p_\mathcal{C}$. Thus we need to generalize the notion of fibre defined in Definition 2 to a notion of fibre of an arbitrary functor $p : \mathcal{F} \to \mathcal{B}$.

Definition 8 (Fibre of p)**.** For objects I in $\mathcal{B}$, define $\mathcal{F}_I$ to be the sub-category of $\mathcal{F}$ with

objects X such that $pX = I$.

morphisms f such that $pf = \mathrm{id}_I$.

As desired, this definition is compatible with the notion of fibre defined in Definition 2, and as before we say that $\mathcal{F}_I$ is the fibre over I and an object X in $\mathcal{F}_I$ is indexed by I.

3.2 The glue

The category $\mathcal{F}$ is now organized as the collection of fibres of p over objects of $\mathcal{B}$, just as $\operatorname{Fam}\mathcal{C}$ is can be viewed as the collection of fibres of $p_\mathcal{C}$ over honest sets. Moreover, the fibres of $p_\mathcal{C}$ are glued together so that they respect re-indexing.

Example 9. Let $(Y_j)_{j\in J}$ be a set-indexed family of objects in a category $\mathcal{C}$, and let $u : I \to J$ be a set map. The family $(Y_j)_{j\in J}$ re-indexed by u is the family $(X_i)_{i\in I}$ with $X_i = Y_{u(i)}$ and equipped with the obvious morphism

$$(u, (\mathrm{id}_{X_i})_{i\in I}) : (X_i)_{i\in I} \to (Y_j)_{j\in J}$$

in $\operatorname{Fam}\mathcal{C}$.

Pick a morphism $(v, (g_k)_{k\in K}) : (Z_k)_{k\in K} \to (Y_j)_{j\in J}$ in $\operatorname{Fam}\mathcal{C}$ and a factorization $v = u \circ w$. For each k, $h_k := g_k$ is the unique morphism making the diagram below commute.

$$\begin{tikzcd}
Z_k \arrow[d, dashed, "h_k"'] \arrow[dr, "g_k"] & \\
X_{w(k)} \arrow[r, "\mathrm{id}_{X_{w(k)}}"'] & Y_{v(k)} = Y_{u(w(k))}
\end{tikzcd}$$

Thus, $(w, (h_k)_{k\in K}) : (Z_k)_{k\in K} \to (X_i)_{i\in I}$ is the unique morphism in $\operatorname{Fam}\mathcal{C}$ mapping to w under $p_\mathcal{C}$ such that

$$(v, (g_k)_{k\in K}) = (u, (\mathrm{id}_{X_i})_{i\in I}) \circ (w, (h_k)_{k\in K}).$$

$$\begin{tikzcd}
(Z_k)_{k\in K} \arrow[d, dashed, "{(w, (h_k)_{k\in K})}"'] \arrow[dr, "{(v, (g_k)_{k\in K})}"] & & K \arrow[d, "w"'] \arrow[dr, "v"] & \\
(X_i)_{i\in I} \arrow[r, "{(u, (\mathrm{id}_{X_i})_{i\in I})}"'] & (Y_j)_{j\in J} & I \arrow[r, "u"'] & J \\
\operatorname{Fam}\mathcal{C} \arrow[rr, "p_\mathcal{C}"] & & \mathbf{Set} &
\end{tikzcd}$$

The fibres I and J are glued together by the existence of morphisms $(u, (\mathrm{id}_{X_i})_{i\in I}) : (X_i)_{i\in I} \to (Y_j)_{j\in J}$ for each family $(Y_j)_{j\in J}$ and set map $u : I \to J$. Thus in general, the collection of fibres of $p : \mathcal{F} \to b$ must be glued together in a way that captures the idea of re-indexing. In particular, the glue between fibres will be represented by the inter-fibre morphisms in $\mathcal{F}$. Using Example 9 as a guide, we generalize the morphisms $(u, (f_i)_{i\in I}) : (X_i)_{i\in I} \to (Y_j)_{j\in J}$ to the scenario $p : \mathcal{F} \to \mathcal{B}$.

Definition 10 (Cartesian morphism)**.** A morphism $f : X \to Y$ in $\mathcal{F}$ is cartesian over $u : I \to J$ in $\mathcal{B}$ if

(i) $pf = u$.

(ii) Each morphism $g : Z \to Y$ in $\mathcal{F}$ and factorization $w \circ u$ of $v = pg$ in $\mathcal{B}$ defines a unique morphism $h : Z \to X$ such that $ph = w$ and $f \circ h = g$.

$$\begin{tikzcd}
Z \arrow[d, dashed, "h"'] \arrow[dr, "g"] & & K \arrow[d, "w"'] \arrow[dr, "v"] & \\
X \arrow[r, "f"'] & Y & I \arrow[r, "u"'] & J \\
\mathcal{F} \arrow[rr, "p"] & & \mathcal{B} &
\end{tikzcd}$$

In $\operatorname{Fam}\mathcal{C}$, re-indexing a J-indexed family by a set map $u : I \to J$ defines an I-indexed family $(X_i)_{i\in I}$ with $X_i = Y_{u(i)}$. Example 9 shows that the resulting morphism $(u, (\mathrm{id}_{X_i})_{i\in I})$ is cartesian over u. Therefore, $p_\mathcal{C} : \operatorname{Fam}\mathcal{C} \to \mathbf{Set}$ satisfies the following axiom.

Axiom A: *For every morphism $u : I \to J$ in $\mathcal{B}$ and object Y in the fibre over J, there exists an object X in the fibre over I and a morphism $f : X \to Y$ that is cartesian over u.*

Definition 11 (Fibration)**.** A functor $p : \mathcal{F} \to \mathcal{B}$ is a fibration if it satisfies Axiom A. In this setup, $\mathcal{F}$ is called the total category and $\mathcal{B}$ is called the base category.

Axiom A is a necessary assumption about $p : \mathcal{F} \to \mathcal{B}$ because re-indexing is an important property of $p_{\mathcal{C}}$. Sections 6 and 7 give evidence that Axiom A is also a sufficient assumption. Thus, a fibration realizes a "category".

Before we continue with examples of fibrations, we give two useful lemmas about cartesian morphisms.

Lemma 12. *Let $p : \mathcal{F} \to \mathcal{B}$ be a functor. If $f : W \to X$ is cartesian over $u : I \to J$ and $g : X \to Y$ is cartesian over $v : J \to K$, then $g \circ f$ is cartesian over $v \circ u$.*

Proof. If $h : Z \to Y$ and $w : pZ \to I$ are such that $v \circ u \circ w = ph$, then there exists a unique morphisms $h' : Z \to X$ such that

$$ph' = u \circ w, \quad h = g \circ h'$$

and a unique morphism $h'' : Z \to W$ such that

$$ph'' = w, \quad h' = f \circ h''.$$

Thus h'' is the unique morphism such that $(g \circ f) \circ h'' = h$ and $ph'' = w$. See Lemma 8.1.4 in [Bor94].

$$\begin{tikzcd}
Z \arrow[d, dashed, "h''"'] \arrow[dr, dashed, "h'"] \arrow[drr, bend left, "h"] & & & pZ \arrow[d, "w"'] \arrow[dr, "u \circ w"] \arrow[drr, bend left, "ph"] & & \\
W \arrow[r, "f"'] & X \arrow[r, "g"'] & Y & I \arrow[r, "u"'] & J \arrow[r, "v"'] & K \\
& \mathcal{F} \arrow[rrr, "p"] & & & \mathcal{B} &
\end{tikzcd}$$

□

Lemma 13. *Let $p : \mathcal{F} \to \mathcal{B}$ be a functor. If $f : X \to Y$ and $f' : X' \to Y$ are both cartesian over $u : I \to J$, then X and X' are isomorphic.*

Proof. There exist unique functors $g : X \to X'$ and $g' : X' \to X$ such that $f = f' \circ g$, $f' = f \circ g'$, and $pg = pg' = \mathrm{id}_I$. Uniqueness implies that g and g' are mutually inverse and thus constitute isomorphisms between X and X'. See Lemma 8.1.4 in [Bor94].

$$\begin{tikzcd}
X' \arrow[d, dashed, bend right, "g'"'] \arrow[dr, "f'"] & & I \arrow[d, bend right, "\mathrm{id}_I"'] \arrow[dr, "u"] & \\
X \arrow[u, dashed, bend right, "g"'] \arrow[r, "f"'] & Y & I \arrow[u, bend right, "\mathrm{id}_I"'] \arrow[r, "u"'] & J \\
\mathcal{F} \arrow[rr, "p"] & & \mathcal{B} &
\end{tikzcd}$$

□

3.3 Examples of fibrations

Example 14. As expected, $p_{\mathcal{C}} : \operatorname{Fam}\mathcal{C} \to \mathbf{Set}$ is a fibration, since given a morphism $u : I \to J$ in **Set** and an object $(Y_j)_{j\in J}$ in the fibre over J, the morphism

$$(u, (\mathrm{id}_{Y_{u(i)}})_{i\in I}) : (Y_{u(i)})_{i\in I} \to (Y_j)_{j\in J}$$

in $\operatorname{Fam}\mathcal{C}$ is cartesian over u.

The next example describes how a category $\mathcal{B}$ can be viewed as a fibration over itself.

Example 15. Let $\operatorname{Ar}\mathcal{B}$ be the category with

objects Triples (X, x, I) such that $x : X \to I$ is a morphism in $\mathcal{B}$.

morphisms A morphism $(X, x, I) \to (Y, y, J)$ is a pair (f, u) such that $f : X \to Y$ and $u : I \to J$ are morphisms in $\mathcal{C}$ such that $y \circ f = u \circ x$.

Proposition 16. *If $\mathcal{B}$ has pullbacks, then the codomain functor $\mathrm{cod} : \operatorname{Ar}\mathcal{B} \to \mathcal{B}$ which acts on objects by $(X, x, I) \mapsto I$ and on morphisms by $(f, u) \mapsto u$ is a fibration.*

Proof. Given a morphism $u : I \to J$ in $\mathcal{B}$ and an object (Y, y, J) take a pullback of u and y. Let $f : X \to Y$ be the pullback of u along y and let $x : X \to I$ be the pullback of y along u. Then $u \circ x = y \circ f$, so $(f, u) : (X, x, I) \to (Y, y, J)$ is a morphism in $\operatorname{Ar}\mathcal{B}$.

$$\begin{tikzcd}
X \arrow[r, dashed, "f"] \arrow[d, dashed, "x"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "y"] \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

Given an object (Z, z, K) of $\operatorname{Ar}\mathcal{B}$, a morphism $(g, v) : (Z, z, K) \to (Y, y, J)$, and a factorization $v = u \circ w$ in $\mathcal{B}$, the morphisms $g : Z \to Y$ and $w \circ z : Z \to I$ uniquely define a morphism $h : Z \to X$ making the following diagram commute. Thus (f, u) is cartesian over u.

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "g"] \arrow[rd, dashed, "h"'] \arrow[dd, "z"'] & & \\
& X \arrow[r, "f"'] \arrow[dd, "x"] & Y \arrow[dd, "y"] \\
K \arrow[rrd, dotted, "v"] \arrow[rd, "w"'] & & \\
& I \arrow[r, "u"'] & J
\end{tikzcd}$$

□

Given an object I of $\mathcal{B}$, the fibre cod_I is isomorphic to the slice category $\mathcal{B}/I$ which has

objects Pairs (X, x) such that $x : X \to I$ is a morphism in $\mathcal{B}$.

morphisms A morphism $(X, x) \to (Y, y)$ in $\mathcal{B}/I$ is a morphism $f : X \to Y$ in $\mathcal{B}$ satisfying $x = y \circ f$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
I \arrow[r, equal] & I
\end{tikzcd}$$

For notational simplicity we sometimes denote objects (X, x, I) of $\operatorname{Ar}\mathcal{B}$ and objects (X, x) of $\mathcal{B}/I$ by $x : X \to I$ or by $X \xrightarrow{x} I$.

Next we show how to derive new fibrations from existing fibrations.

Proposition 17. *If $p : \mathcal{F} \to \mathcal{E}$ and $q : \mathcal{E} \to \mathcal{B}$ are fibrations then $q \circ p : \mathcal{F} \to \mathcal{B}$ is a fibration.*

Proof. Let $u : I \to J$ in $\mathcal{B}$ and suppose that $(q \circ p)(Y) = J$. Since q is a fibration there is a morphism $g : W \to p(Y)$ cartesian over u. Since p is a fibration there is a morphism $f : X \to Y$ cartesian over g. It is straightforward to check that f is also cartesian over u. See Proposition 8.1.12 in [Bor94]. ☐

Proposition 18 (Change of base)**.** *The pullback of a fibration $p' : \mathcal{F}' \to \mathcal{B}'$ and a functor $B : \mathcal{B} \to \mathcal{B}'$, defines a fibration $p : \mathcal{F} \to \mathcal{B}$.*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, dashed, "F"] \arrow[d, dashed, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}' \arrow[d, "p'"] \\
\mathcal{B} \arrow[r, "B"'] & \mathcal{B}'
\end{tikzcd}$$

Proof. Let $u : I \to J$ be a morphism in $\mathcal{B}$, and let Y be an object in the fibre $\mathcal{F}_J$. Since $p'(FY) = B(pY) = BJ$, there exists a morphism $f' : X' \to FY$ in $\mathcal{F}'$ that is cartesian over $Bu : BI \to BJ$. Since $p'(X') = BI$ and $p'(f') = Bu$ there exists a unique object X of $\mathcal{F}$ and a unique morphism $f : X \to Y$ such that $pf = u$ and $Ff = f'$ by the universal property of pullbacks. It is straightforward to check that f is cartesian over u. See Lemma 1.5.1 in [Jac99]. ☐

4 Fibrations in a 2-Category

The collection of all categories can be organized into a 2-category **Cat** with

0-cells categories

1-cells functors

2-cells natural transformations

The goal of this section is to construct a 2-category $\mathbf{Fib}(\mathcal{B})$ that has fibrations over $\mathcal{B}$ as 0-cells. A generalized abstraction of categories should be endowed with some notion of "functors" between "categories" and "natural transformations" between "functors". Using $p_{\mathcal{C}}$ as our guiding example, we realize "functors" and "natural transformations" as fibred functors and fibred natural transformations which will be the 1-cells and 2-cells of $\mathbf{Fib}(\mathcal{B})$, respectively.

4.1 Fibred functors

Recall that given a functor $F : \mathcal{C} \to \mathcal{D}$, the 2-functor $\operatorname{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ maps F to a functor $\operatorname{Fam} F : \operatorname{Fam}\mathcal{C} \to \operatorname{Fam}\mathcal{D}$, which we take to be the prototype for a fibred functor. Thus, we are interested in the necessary and sufficient conditions for a functor $\operatorname{Fam}\mathcal{C} \to \operatorname{Fam}\mathcal{D}$ to be in the image of $\operatorname{Fam} : \mathbf{Cat} \to \mathbf{Cat}$.

Proposition 19. *Let $F : \operatorname{Fam}\mathcal{C} \to \operatorname{Fam}\mathcal{D}$ be a functor. The following are equivalent.*

(a) F satisfies

  (i) F maps J-indexed families in $\operatorname{Fam}\mathcal{C}$ to J-indexed families in $\operatorname{Fam}\mathcal{D}$.

  (ii) F commutes with re-indexing.

(b) $F = \operatorname{Fam} F_1$ where $F_1 : \mathcal{C} \to \mathcal{D}$ is the unique functor such that for all single object families (X), $F(X) = (F_1 X)$. Recall that the fibre over 1 of $\operatorname{Fam}\mathcal{C}$ naturally corresponds to $\mathcal{C}$. F_1 is the restriction of F to the fibre over 1 under this correspondence.

Proof. First we show that (a) implies (b). Let $(Y_j)_{j\in J}$ be an object of $\mathrm{Fam}\,\mathcal{C}$. Since F satisfies condition (i), we can let $(X_j)_{j\in J}$ denote the image of $(Y_j)_{j\in J}$ under F. To show that $F = \mathrm{Fam}\,F_1$, it is sufficient to show that $X_j = F_1 Y_j$ for all $j \in J$. Let $u : 1 \to J$ be the map $* \mapsto j$. Since F commutes with re-indexing, the following diagram commutes.

$$\begin{tikzcd}
(\mathrm{Fam}\,\mathcal{C})_J \arrow[r, "F"] \arrow[d, "\text{re-index by } u"'] & (\mathrm{Fam}\,\mathcal{D})_J \arrow[d, "\text{re-index by } u"] \\
(\mathrm{Fam}\,\mathcal{C})_1 \arrow[r, "F"'] & (\mathrm{Fam}\,\mathcal{D})_1
\end{tikzcd}$$

Start with $(Y_j)_{j\in J}$ in $(\mathrm{Fam}\,\mathcal{C})_J$. Going around the diagram clockwise,

$$(Y_j)_{j\in J} \mapsto F(Y_j)_{j\in J} = (X_j)_{j\in J} \mapsto (X_j).$$

Going around the diagram counter-clockwise,

$$(Y_j)_{j\in J} \mapsto (Y_j) \mapsto (F_1 Y_j)$$

implying the desired equality.

Now we show that (b) implies (a). Let $F_1 : \mathcal{C} \to \mathcal{D}$ be any functor. By definition, $\mathrm{Fam}\,F_1$ satisfies condition (i). $\mathrm{Fam}\,F_1$ commutes with re-indexing since for every set map $u : I \to J$ in **Set** and family $(Y_j)_{j\in J}$ the following mappings are equal.

$$\begin{tikzcd}
(Y_j)_{j\in J} \arrow[r, mapsto, "\mathrm{Fam}\,F_1"] \arrow[d, mapsto, "\text{re-index by } u"'] & (F_1 Y_j)_{j\in J} \arrow[d, mapsto, "\text{re-index by } u"] \\
(Y_{u(i)})_{i\in I} \arrow[r, mapsto, "\mathrm{Fam}\,F_1"'] & (F_1 Y_{u(i)})_{i\in I}
\end{tikzcd}$$

□

This characterization of functors in the image of $\mathrm{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ induces the following defintion of fibred functor which is essentially a generalization of (a) to the fibred scenario $p : \mathcal{F} \to \mathcal{B}$. In order to generalize, notice that F commutes with re-indexing if and only if F preserves the re-indexing morphisms $(u, (\mathrm{id}_{Y_{u(i)}})_{i\in I})$ and recall that re-indexing morphisms are a specific case of cartesian morphisms.

Definition 20 (Fibred Functor)**.** Given fibrations $p : \mathcal{F} \to \mathcal{B}$ and $p' : \mathcal{F}' \to \mathcal{B}$ over the same base $\mathcal{B}$. A fibred functor between p and p' is a ordinary functor $H : \mathcal{F} \to \mathcal{F}'$ such that

(i) $p' \circ H = p$.

(ii) If f is cartesian over u then Hf is also cartesian over u.

Under this definition fibred functors only realize "functors" between "categories" over the same base category. The idea of expanding the notion of fibred functors to realize "functors" between "categories" over arbitrary bases is explored in Section 1.7 of [Jac99].

4.2 Fibred natural transformations

Given a natural transformation $\alpha : F \Rightarrow G$ in $\mathcal{C}$, the 2-functor $\mathrm{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ maps α to a natural transformation $\mathrm{Fam}\,\alpha : \mathrm{Fam}\,F \Rightarrow \mathrm{Fam}\,G$ which we take to be the prototype for fibred natural transformations. As in the case of fibred functors, the goal is to find necessary and sufficient conditions for a natural transformation to be in the image of Fam.

Proposition 21. *Let $F, G : \mathcal{C} \to \mathcal{D}$, and let $\alpha : \operatorname{Fam} F \Rightarrow \operatorname{Fam} G$ be a natural transformation. The following are equivalent.*

(a) Each component $\alpha(X_i)_{i\in I} : (FX_i)_{i\in I} \to (GX_i)_{i\in I}$ is of the form $(\mathrm{id}_I, (f_i)_{i\in I})$.

(b) $\alpha = \operatorname{Fam} \alpha_1$ where $\alpha_1 : F \Rightarrow G$ is the unique natural transformation such that for all single object families (X), $\alpha(X) = (\mathrm{id}_1, (\alpha_1 X))$. In other words, α_1 is the restriction of α to the fibre over 1 under the correspondence between $\operatorname{Fam} \mathcal{C}_1$ and $\mathcal{C}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha_1"] & \mathcal{D} \end{tikzcd} \xrightarrow{\operatorname{Fam}} \begin{tikzcd} \operatorname{Fam}\mathcal{C} \arrow[r, bend left, "\operatorname{Fam} F"] \arrow[r, bend right, "\operatorname{Fam} G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \operatorname{Fam}\mathcal{D} \end{tikzcd}$$

Proof. (b) implies (a) follows immediately from the action of Fam on natural transformations.

To show that (a) implies (b), it is sufficient to show that for each i, $f_i = \alpha_1 X_i$. Let $u : 1 \to I$ be the map $* \mapsto i$. Then $(u, (\mathrm{id}_{X_i})) : (X_i)_{i\in I} \to (X_i)$ is a morphism in $\operatorname{Fam}\mathcal{C}$. By naturality the following diagram commutes.

$$\begin{tikzcd} (FX_i)_{i\in I} \arrow[r, "\alpha(X_i)_{i\in I} = (\mathrm{id}_I, (f_i)_{i\in I})"] \arrow[d, "(u, (\mathrm{id}_{FX_i}))"'] & (GX_i)_{i\in I} \arrow[d, "(u, (\mathrm{id}_{GX_i}))"] \\ (FX_i) \arrow[r, "\alpha(X_i) = (\mathrm{id}_1, (\alpha_1 X_i))"'] & (GX_i) \end{tikzcd}$$

So, $(u, (f_i)) = (u, (\alpha_1 X_i))$ implying the desired equality.

□

This characterization of natural transformations in the image of $\operatorname{Fam} : \mathbf{Cat} \to \mathbf{Cat}$ induces the following definition of fibred natural transformations.

Definition 22 (Fibred natural transformation)**.** Let $H, G : \mathcal{F} \to \mathcal{F}'$ be fibred functors between fibrations $p : \mathcal{F} \to \mathcal{B}$ and $p' : \mathcal{F}' \to \mathcal{B}$. A fibred natural transformation between H and G is an ordinary natural transformation $\alpha : H \Rightarrow G$ such that each component $\alpha_X : HX \to GX$ is a morphism in the fibre $\mathcal{F}'_{pX}$.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, bend left, "H"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\Downarrow \alpha"] \arrow[dr, "p"'] & & \mathcal{F}' \arrow[dl, "p'"] \\ & \mathcal{B} & \end{tikzcd}$$

4.3 The 2-category of fibrations

Proposition 23. $\mathbf{Fib}(\mathcal{B})$ *is a 2-category with*

0-cells *fibrations over $\mathcal{B}$*

1-cells *fibred functors*

2-cells *fibred natural transformations*

See [Bor94], for example.

Section 1.7 of [Jac99] defines a 2-category **Fib** whose 0-cells are all fibrations and with extended notions of fibred functors and fibred natural transformations as 1-cells and 2-cells, respectively. The functor **Fib** $\to$ **Cat** defined by mapping a fibration to its base category is itself a fibration. Cartesian morphisms exist by change of base defined by Proposition 18. **Fib**$(\mathcal{B})$ is the fibre over $\mathcal{B}$ of this fibration.

The richness of the 2-categorical structure immediately endows **Fib**$(\mathcal{B})$ with a generalization of important categorical notions defined by **Cat** and thus a realization of expected "categorical" notions such as "adjunctions" and "equivalence of categories", e.g. see [Her99]

Definition 24 (Fibred adjunction)**.** Given fibrations $p : \mathcal{F} \to \mathcal{B}$ and $p' : \mathcal{F}' \to \mathcal{B}$ and fibred functors $F : \mathcal{F} \to \mathcal{F}'$, $G : \mathcal{F}' \to \mathcal{F}$. We say that F is fibred left adjoint to G if there exist fibred natural transformations $\eta : 1_{\mathcal{F}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{F}'}$ such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd}$$

The quadruple $(F, G, \eta, \varepsilon)$ is called a fibred adjunction.

Definition 25 (Fibred equivalence)**.** Fibrations $p : \mathcal{F} \to \mathcal{B}$ and $p : \mathcal{F}' \to \mathcal{B}$ are equivalent if there exists an adjunction $(F, G, \eta, \varepsilon)$ such that η and ε are isomorphisms.

5 Indexed Categories

In this section, we define an alternate but equivalent realization of "categories", which called indexed categories. Multiple realizations of a single abstract concept is common in mathematics. For example, groups realize the abstract notion of elements, a way to combine elements, and a way to undo a combination of elements. Group presentations also realize this abstraction. Although, mathematicians tend to prefer the relative sleekness of groups, group presentations are a powerful tool for understanding groups. Analogously, we generally take fibrations to be the primary realization of "categories", but indexed categories prove useful in some circumstances.

Table 2 indicates that any realization of "categories" must incorporate some notion of fibres glued together. In order to define an alternate realization of fibres and glue, we analyze an alternative representation of the fibres and the glue of the fibration $\mathrm{cod} : \mathrm{Ar}\,\mathcal{B} \to \mathcal{B}$ from Example 15. Indexed categories will generalize this alternate representation. Table 4 previews this work.

general and abstract ("$\diamond$")	**general and concrete** (fibred $\diamond$)	**general and concrete** (indexed $\diamond$)
base category	$\mathcal{B}$	$\mathcal{B}$
"set"	object of $\mathcal{B}$	object of $\mathcal{B}$
collection of fibres	$\mathcal{F}$	map $\mathrm{ob}\,\mathcal{B} \to \mathrm{ob}\,$**Cat**
glue between fibres	cartesian morphisms	contravariant map $\mathrm{morph}\,\mathcal{B} \to \mathrm{morph}\,$**Cat**
"category"	fibration	indexed category
"functor"	fibred functor	indexed functor
"natural transformation"	fibred natural transformation	indexed natural transformation
2-category of "categories"	**Fib**	**Idx**
"category" with "equality"	spit fibration	functors $\mathcal{B}^{\mathrm{OP}} \to$ **Cat**

Table 4: An expansion of Table 2 to include the alternate realization of stuff, structure, and properties over a base category $\mathcal{B}$ as indexed categories.

5.1 The fibres

Recall that the fibres of cod are familiar, namely the slice categories $\mathcal{B}/I$. So the fibration cod induces a method of identifying the slice categories since the slice category $\mathcal{B}/I$ is the preimage of I (along with the identity morphism id_I) under cod. Analogously, a method for identifying people aged 100 is to use a list of people and their ages. We can think of the list as a map from people to ages and then the set of people aged 100 is the preimage of 100 under this map. Alternatively, we might organize people and their ages by keeping the names of all people aged i in a box labeled i. Then the box labeled 100 identifies the people aged 100. These two organizations motivate the following definition.

pointwise indexing A family $\{X_i\}_{i\in I}$ of sets X_i represented by a map $I \to \mathrm{ob}\,\mathbf{Set}$ defined by $i \mapsto X_i$.

display indexing A function $\varphi : X \to I$.

The list organization of people and ages corresponds to display indexing by the function $\varphi : \{\text{people}\} \to \mathbb{N}$ defined by mapping a person to his/her age. The box organization of people and ages corresponds to pointwise indexing where $I = \mathbb{N}$ and X_i is the set of people aged i.

Pointwise and display indexing are equivalent via the following correspondence (see Section 1.1 in [Jac99])

$$\{X_i\}_{i\in I} \quad \mapsto \quad \varphi : \coprod_{i\in I} X_i \to I \text{ defined by } (i, x \in X_i) \mapsto i \tag{1}$$

$$\{\varphi^{-1}(i)\}_{i\in I} \quad \leftarrow\!\shortmid \quad \varphi : X \to I. \tag{2}$$

The fibration cod is a display indexing of the slice categories $\mathcal{B}/I$ over objects of $\mathcal{B}$, so applying the above correspondence defines a pointwise indexing of the slice categories via a map of objects

$$\begin{gathered}\mathrm{ob}\,\mathcal{B} \to \mathrm{ob}\,\mathbf{Cat}\\ I \mapsto \mathcal{B}/I.\end{gathered}$$

Generalizing to arbitrary "categories", an indexed category will represent fibres by a map of objects $\mathrm{ob}\,\mathcal{B} \to \mathrm{ob}\,\mathbf{Cat}$ and the image of I under this map will be suggestively be called the fibre over I.

5.2 The glue

In a fibration, cartesian morphisms represent the glue between fibres. In the case of the fibration cod, a cartesian morphism (f, u) over $u : I \to J$ is induced by taking the pullback of $u : I \to J$ and $y : Y \to J$ and letting f be the pullback of u along y. The proccess of pulling back u along morphisms with codomain J also induces a functor between slice categories $\mathcal{B}/J \to \mathcal{B}/I$ (see Example 1.4.2 in [Jac99], for example). The correspondence between cartesian morphisms in cod and naturally arising functors between slice categories suggests that in indexed categories the glue will be represented by functors between fibres.

Definition 26. Given a morphism $u : I \to J$ in $\mathcal{B}$ there is a functor $u^* : \mathcal{B}/J \to \mathcal{B}/I$ defined on

objects by $(Y, y, J) \mapsto (X, x, I)$ where x is the pullback of y along u as in Proposition 16.

morphisms by $h : (Y, y, J) \to (Y', y', J)$ maps to the unique morphism $\tilde{h} : X \to X'$ defined by $h \circ f$ and x under the universal property of pullbacks. $\tilde{h}$ is indeed a morphism $(X, x, I) \to (X', x', I)$ in $\mathcal{B}/I$ since $x = x' \circ \tilde{h}$.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, dashed, "\tilde{h}"'] \arrow[ddr, bend right, "x"'] & & Y \arrow[dr, "h"] \arrow[ddr, dotted, "y"] & \\
& X' \arrow[rr] \arrow[d, "x'"'] & & Y' \arrow[d, "y'"] \\
& I \arrow[rr, "u"'] & & J
\end{tikzcd}$$

In general, there is no functorial way of taking pullbacks. So in general we do not have equality

$$(v \circ u)^* = u^* \circ v^*$$

but rather an isomorphsim

$$(v \circ u)^* \simeq u^* \circ v^*.$$

Thus the fibration cod induces the “almost functor” slice : $\mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ defined on objects by $I \mapsto \mathcal{B}/I$ and on morphisms by $u \mapsto u^*$ which we take to be the canonical example of an indexed category.

This example suggests that the glue between fibres in indexed categories is realized as functors induced by morphisms in the base category with composition and identity preserved only up to isomorphism.

Definition 27 (Indexed category)**.** An indexed category $H : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ is composed of maps

$$\begin{aligned} \operatorname{ob} \mathcal{B}^{\mathrm{OP}} &\to \operatorname{ob} \mathbf{Cat} \\ I &\mapsto HI \end{aligned} \qquad\qquad \begin{aligned} \operatorname{morph} \mathcal{B}^{\mathrm{OP}} &\to \operatorname{morph} \mathbf{Cat} \\ (u : I \to J) &\mapsto (Hu : HJ \to HI) \end{aligned}$$

equipped with isomorphisms

$$\begin{aligned} \eta_{u,v} &: Hu \circ Hv \simeq H(v \circ u) \\ \varepsilon_I &: H(\mathrm{id}_I) \simeq \mathrm{id}_{H(I)}. \end{aligned}$$

such that for all $I \xrightarrow{u} J \xrightarrow{v} K \xrightarrow{w} L$ the following coherence conditions are satisfied.

$$\begin{tikzcd}
& Hu & \\
H\mathrm{id}_I \circ Hu \arrow[ur, "\varepsilon_I(Hu)"] \arrow[r, "\eta_{\mathrm{id}_I,u}"'] & Hu \arrow[u, equal] & Hu \circ H\mathrm{id}_J \arrow[ul, "(Hu)\varepsilon_J"'] \arrow[l, "\eta_{u,\mathrm{id}_J}"]
\end{tikzcd}$$

$$\begin{tikzcd}
Hu \circ Hv \circ Hw \arrow[r, "(Hu)\eta_{v,w}"] \arrow[d, "\eta_{u,v}(Hw)"'] & Hu \circ H(w \circ v) \arrow[d, "\eta_{u,w \circ v}"] \\
H(v \circ u) \circ Hw \arrow[r, "\eta_{v \circ u,w}"'] & H(w \circ v \circ u)
\end{tikzcd}$$

Indexed categories are also called psuedo-functors.

Proposition 28. slice : $\mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ *is an indexed category. See [Str99] Example 1.6 for details.*

Definition 29. There is a 2-cateogry $\mathbf{Idx}(\mathcal{B})$ with

0-cells indexed categories $H : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$

1-cells indexed functors

2-cells indexed natural transformations

For definitions of indexed functors and indexed natural transformations refer to Definition 1.3.4 of [Her93].

5.3 Equivalence of fibrations and indexed categories

There is an equivalence between the $\mathbf{Fib}(\mathcal{B})$ and $\mathbf{Idx}(\mathcal{B})$ that corresponds to the equivalence between pointwise and display indexing. First, we generalize the map defined in Equation (1) so that the sets X_i are arbitrary categories and the set map φ is a fibration. This generalization defines a map $\int : \operatorname{ob} \mathbf{Idx} \to \operatorname{ob} \mathbf{Fib}$ called the Grothendieck construction and is due to Grothendieck [Gro64].

Given an indexed category $H : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ define an category $\mathcal{F}$ with

objects Pairs (I, X) composed of an object I of $\mathcal{B}$ and an object X of HI. Compare to the set $\coprod_{i \in I} X_i$ which is composed of pairs $(i \in I, x \in X_i)$.

morphisms Pairs $(u, f) : (I, X) \to (J, Y)$ composed of a morphism $u : I \to J$ in $\mathcal{B}$ and a morphism $f : X \to (Hu)(Y)$ in HI.

The composition of morphisms $(w, h) : (K, Z) \to (I, X)$ and $(u, f) : (I, X) \to (J, Y)$ is the pair $(u \circ w, g)$ such that g is the composition

$$Z \xrightarrow{h} (Hw)(X) \xrightarrow{(Hw)(f)} (Hw \circ Hu)(Y) \xrightarrow{\simeq} H(u \circ w)(Y)$$

where the isomorphism is given by $\eta_{w,u}$.

Define $\int H : \mathcal{F} \to \mathcal{B}$ to be the projection functor acting on objects by $(I, X) \mapsto I$ and on morphisms by $(u, f) \mapsto u$.

Proposition 30. *$\int H$ is a fibration.*

Proof. Let $u : I \to J$ be a morphism in $\mathcal{B}$ and let (J, Y) be an object in $\mathcal{F}$. So Y is an object of HJ and $Hu : HJ \to HI$ is a functor.

Claim. $(u, \eta_{\mathrm{id}_I,u}(Y)) : (I, (H\mathrm{id}_I \circ Hu)(Y)) \to (J, Y)$ is cartesian over u.

For convenience let X denote the object $(H\mathrm{id}_I \circ Hu)(Y)$ of HI and let f denote the isomorphism $\eta_{\mathrm{id}_I,u}(Y)$ throughout. Let $(v, g) : (K, Z) \to (J, Y)$ and fix a factorization $v = u \circ w$. Since $(Hw)(f)$ is an isomorphism, there is a unique morphism $h : Z \to (Hw)(Z)$ such that $g = f \circ h$. So $(w, h) : (K, Z) \to (I, X)$ is the unique morphism such that $(u, f) \circ (w, h) = (v, g)$.

$$\begin{tikzcd}
Z \arrow[d, dashed, "h"'] \arrow[rrrr, bend left, "g"] & & & & \\
(Hw)(X) \arrow[rr, "\simeq", "(Hw)(f)"'] & & (Hw \circ Hu)(Y) \arrow[rr, "\simeq", "\eta_{w,u}(Y)"'] & & (Hv)(Y)
\end{tikzcd}$$

□

The fibres of $\int H$ are equivalent to the fibres of H and the cartesian morphisms of $\int H$ are determined by the behavior of H on morphisms. Referring to Table 4, this correspondence suggests that H and $\int H$ represent the same "category". For example, the fibrations $\int$ slice and cod are equivalent, and the indexed category slice was constructed to represent the same stuff, structure, and properties as the fibration cod.

Next, we want to analyze the construction of slice from cod described in the beginning of this section, in order to construct an indexed category from an arbitrary fibration so that they represent the same "category".

The definition of fibration stipulates that for every morphism $u : I \to J$ in $\mathcal{B}$ and object Y in the fibre over J there *exists* a cartesian lift $f : X \to Y$ over u but it does not give a standard method for choosing a specific cartesian lifting amongst possibly many options. For example, the functor cod is a fibration because a pullback *exists* for every pair of morphism $u : I \to J$ and $y : Y \to J$ but cod is not a priori equipped with a specific choice of pullbacks. By contrast, the indexed category slice defines a specific choice of pullbacks as the image of (Y, y, J) under the functor $u^* : \mathcal{B}/J \to \mathcal{B}/I$. The fibration $\int$ slice also defines a specific choice of pullbacks as the image of (Y, y, J) under the composition

$$\mathcal{B}/J \xrightarrow{u^*} \mathcal{B}/I \xrightarrow{\mathrm{id}_I^*} \mathcal{B}/I.$$

Assuming that $\mathrm{id}_I^* = \mathrm{id}_{\mathcal{B}/I}$, the indexed category slice and the fibration $\int$ slice define the same choice of pullback.

In general, the proof of Proposition 30 defines a canonical choice of cartesian morphisms for the fibration $\int H$ induced by the isomorphism η. Thus, fibrations constructed from indexed categories are equipped with more information than arbitrary fibrations, namely specific choices of cartesian morphisms, suggesting that indexed categories can only be constructed from fibrations equipped with these specific choices. We introduce the notion of a cleavage as a method of bookkeeping these choices.

Definition 31 (Cleavage)**.** A cleavage for a fibration $p : \mathcal{F} \to \mathcal{B}$ is a choice

$$\mathrm{Cart}(u, Y) : u^*Y \to Y$$

of cartesian lifting for each pair $u : I \to J$ and Y in the fibre over J.

The relationship between fibrations and cleavages is analogous to the relationship between vector spaces and vector space bases. In order to talk about a basis of a vector space, the vector space must come equipped with a basis or we must use the Axiom of Choice to define a basis. Similarly, in order to talk about a cleavage of a fibration, the fibration must come equipped with a cleavage or we must use the Axiom of Choice to define a cleavage. Just as $\mathbb{R}^2$ has a standard basis $\{(1,0),(0,1)\}$, the fibration $\int H$ has a standard cleavage defined by

$$\begin{aligned} u^*(J, Y) &= (I, (H\mathrm{id}_I \circ Hu)(Y)) \\ \mathrm{Cart}(u, (J, Y)) &= (u, \eta_{\mathrm{id}_I, u}(Y)) : u^*(J, Y) \to (J, Y). \end{aligned}$$

Next we analyze the behavior of these cartesian morphisms with respect to identity and composition morphisms.

Let $u : I \to J$ and $v : J \to K$ be morphisms in $\mathcal{B}$ and let Z be an object of the category HK. Then,

$$\begin{aligned} u^*(v^*(K, Z)) &= u^*(J, (H\mathrm{id}_J \circ Hv)(Z)) \\ &= (I, (H\mathrm{id}_I \circ Hu \circ H\mathrm{id}_J \circ Hv)(Z)) \\ &\simeq (I, (H\mathrm{id}_I \circ H(v \circ u))(Z)) \\ &= (v \circ u)^*(K, Z) \end{aligned}$$

where the isomorphism follows from two applications of η and

$$\mathrm{id}_I^*(K, Z) = (I, (H\mathrm{id}_I \circ H\mathrm{id}_K)(Z)) \simeq (K, Z)$$

where the isomorphism follows from two applications of ε.

These relationships generalize to arbitrary cleavages as shown in the following proposition.

Proposition 32. *Given $I \xrightarrow{u} J \xrightarrow{v} K$ in $\mathcal{B}$ and Z in the fibre over K, we have isomorphisms*

$$\begin{aligned} u^*(v^*Z) &\xrightarrow{\simeq} (v \circ u)^*Z \\ \mathrm{id}_K^*Z &\xrightarrow{\simeq} Z \end{aligned}$$

over id_I and id_K respectively such that the following diagrams commute.

$$\begin{tikzcd} u^*(v^*Z) \arrow[dd, dashed, "\simeq"'] \arrow[dr, "\mathrm{Cart}(u, v^*Z)"] & & \\ & v^*Z \arrow[dr, "\mathrm{Cart}(v, Z)"] & \\ (v \circ u)^*Z \arrow[rr, "\mathrm{Cart}(v \circ u, Z)"'] & & Z \end{tikzcd} \qquad \begin{tikzcd} I \arrow[dd, "\mathrm{id}_I"'] \arrow[dr, "u"] & & \\ & J \arrow[dr, "v"] & \\ I \arrow[rr, "v \circ u"'] & & K \end{tikzcd}$$

$$
\begin{tikzcd}
\mathrm{id}_K^* Z \arrow[d, dashed, "\simeq"'] \arrow[dr, "\mathrm{Cart}(\mathrm{id}_K, Z)"] & & K \arrow[d, "\mathrm{id}_K"'] \arrow[dr, "\mathrm{id}_K"] & \\
Z \arrow[r, "\mathrm{id}_Z"'] & Z & K \arrow[r, "\mathrm{id}_K"'] & K
\end{tikzcd}
$$

$$
\mathcal{F} \xrightarrow{p} \mathcal{B}
$$

Proof. Since the composition of cartesian morphisms is again cartesian, $\mathrm{Cart}(v \circ u, Z)$ and $\mathrm{Cart}(v, Z) \circ \mathrm{Cart}(u, v^*Z)$ are both cartesian over $v \circ u$. The proof of Lemma 13 defines desired isomorphism $u^*(v^*Z) \to (v \circ u)^*Z$.

Since id_Z and $\mathrm{Cart}(\mathrm{id}_K, Z)$ are both cartesian over id_K, the proof of Lemma 13 defines the desired isomorphism $\mathrm{id}_K^* Z \to Z$. For additional details see Section 3 in [Str99] ☐

Using the language of cleavages, the correspondence between the indexed category slice and the fibration cod can be summarized by:

1. Let I be an object in the base category $\mathcal{B}$. Then the slice category $\mathcal{B}/I$ is both the image of I under slice and the fibre of cod over I.
2. Let $u : I \to J$ be a morphism in the base category $\mathcal{B}$. We can define a cleavage of cod so that $\mathrm{Cart}(u, (Y, y, J)) : u^*(Y, y, J) \to (Y, y, J)$ is a pullback of u along y. These choices of pullbacks also define a functor $u^* : \mathcal{B}/J \to \mathcal{B}/I$ acting on objects by $(Y, y, J) \mapsto u^*(Y, y, J)$. Thus in order to construct slice from cod it is necessary to make specific choices of pullbacks or equivalently equip cod with a cleavage.

Generalizing, given a fibration $p : \mathcal{F} \to \mathcal{B}$ equipped with a cleavage, let $H_p : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ be the indexed category defined on

objects by $I \mapsto \mathcal{F}_I$

morphisms by $u : I \to J$ maps to the functor $u^* : \mathcal{F}_J \to \mathcal{F}_I$ (called the reindexing functor) such that

> **objects** $Y \mapsto u^*Y$ as defined by the cleavage.
>
> **morphisms** for $f : Z \to Y$, $u^*f : u^*Z \to u^*Y$ is the unique morphism over id_I such that
>
> $$\mathrm{Cart}(u, Y) \circ u^*f = f \circ \mathrm{Cart}(u, Z).$$

And with isomorphisms η and ε defined by the isomorphisms in Proposition 32.

$$
\begin{tikzcd}
u^*Z \arrow[dd, dashed, "u^*f"'] \arrow[dr, "\mathrm{Cart}(u, Z)"] & & & I \arrow[dd, "\mathrm{id}_I"'] \arrow[dr, "u"] & & \\
& Z \arrow[dr, "f"] & & & J \arrow[dr, "\mathrm{id}_J"] & \\
u^*Y \arrow[rr, "\mathrm{Cart}(u, Y)"'] & & Y & I \arrow[rr, "u"'] & & J
\end{tikzcd}
$$

$$
\mathcal{F} \xrightarrow{p} \mathcal{B}
$$

This definition exactly mimics the construction of the indexed category slice from the fibration cod equipped with a choice of pullback, and as expected the indexed categories H_{cod} and slice are equivalent (see [Str99], for example). In general, H_p and p realize the same "category" since they represent the same collection of fibres and glue.

Assuming the axiom of choice every fibration can be equipped with a cleavage. Therefore, we have maps

$$\operatorname{ob}\mathbf{Fib}(\mathcal{B}) \xrightarrow{H_\square} \operatorname{ob}\mathbf{Idx}(\mathcal{B})$$
$$\operatorname{ob}\mathbf{Fib}(\mathcal{B}) \xleftarrow{\int} \operatorname{ob}\mathbf{Idx}(\mathcal{B})$$

which can be extended to equivalences between $\mathbf{Fib}(\mathcal{B})$ and $\mathbf{Idx}(\mathcal{B})$. See Proposition 1.3.6 in [Her93].

5.4 A comparison of fibrations and indexed categories

A group presentation provides a convenient structure for understanding the group it presents. However, even though groups and group presentations realize the same abstraction, mathematicians tend to prefer working with the standard definition of groups. For example, a first course in abstract algebra covers Lagrange's Theorem not the group presentation interpretation of Lagrange's Theorem. Analogously, Bénabou remarks in §12 of [Bén85] that indexed categories present fibrations and have corresponding advantages and disadvantages when compared with fibrations.

Advantages of fibrations

1. *Properties over structure.* A fibration is a functor equipped with an extra property, namely Axiom A. On the other hand an indexed category is a functor equipped with an extra structure, namely the isomorphisms η and ε. Since the Grothendieck construction defines a correspondence between indexed categories and fibrations equipped with a cleavage, a comparison of fibrations and indexed categories corresponds to a comparison of fibrations and fibrations equipped with a cleavage. Immediately, this precisely illuminates the relationship between structure and properties in the context of fibrations and cleavages. A fibration is a functor $p : \mathcal{F} \to \mathcal{B}$ with the extra property "a cleavage must exist" (assuming the Axiom of Choice) and a fibration equipped with a cleavage is a functor $p : \mathcal{F} \to \mathcal{B}$ with the extra structure "a specific cleavage".

   Mathematicians prefer properties over structure because structure may have it's own properties which are not implicit to the underlying object. For example, in the statement "a fibration p equipped with a cleavage satisfies $\diamond$" it is unclear where $\diamond$ is implicit to the fibration or if it depends on the specific choice of cleavage. Interpreting $\diamond$ as a property of the corresponding indexed category H_p even further obscures this difference. Conversely, there is no analogous confusion in the statement "a fibration p satisfies $\diamond$". Thus working with fibrations as opposed to indexed categories avoids the question "does a property depend on the choice of cleavage or soley on the fibration itself?" just as working with groups as opposed to group presentations avoids the question "does a property depend on the choice of presentation or solely on the group it presents?". For further discussion, see Section 1.10 in [Jac99].

2. *Transparent realization of fibres glued together.* Given a fibration $p : \mathcal{F} \to \mathcal{B}$, the category $\mathcal{F}$ explicity realizes the abstract notion of fibres glued together, which is conversely obscured in the indexed category $\int p$. Analogously, the pointwise indexing map $I \to \mathbf{Set}$ obscures the set $\coprod_{i\in I} X_i$ which is explicit in display indexing. As a consequence, results such as composition of fibrations is again a fibration (Proposition 17) are tricky to state and prove as a result about indexed categories. Again see Section 1.10 in [Jac99] for details.

3. *Generalizability.* Fibrations are directly realizable in any 2-category (see [Str74, Joh93, Her93], for example). This advantage is discussed further in the Conclusion.

Advantages of indexed categories

In spite of its disadvantages, group presentations provide a useful framework for understanding groups. Similarly, some "categorical" notions are more expediently and expressively realized as results of indexed categories than as results of fibrations. Notably, it is easier to generalize the opposite of a category to the opposite of an indexed category than to the opposite of a fibration.

Definition 33 (The opposite of an indexed category)**.** Let $H : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ be an indexed category. Then the opposite of H is the indexed category $H^{\mathrm{OP}} : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ which is the composition $(-)^{\mathrm{OP}} \circ H$, i.e.

$$I \mapsto (HI)^{\mathrm{OP}} \quad \text{and} \quad (u : I \to J) \mapsto \left((Hu)^{\mathrm{OP}} : (HJ)^{\mathrm{OP}} \to (HI)^{\mathrm{OP}}\right).$$

Then we define the opposite of a fibration so that the realizations of "opposite of a category" as fibrations and indexed categories to correspond under the Grothendieck construction,

Definition 34 (The opposite of a fibration)**.** Let $p : \mathcal{F} \to \mathcal{B}$ be a fibration. The opposite of p is the fibration $p^{\mathrm{OP}} := \int((H_p)^{\mathrm{OP}}) : \mathcal{F} \to \mathcal{B}$.

The equivalence $p_{\mathcal{C}^{\mathrm{OP}}} \simeq (p_{\mathcal{C}})^{\mathrm{OP}}$ evidences that the opposite of a fibration generalizes the opposite of a category. Defining the opposite of a fibration without translating through the language of indexed categories is significantly more convoluted as shown in Section 5 of [Str99].

6 Fibrational equality

Naive category theory noticeably lacks statements of the form "objects X and Y are equal", and the ubiquity of isomorphism statements (i.e. statements of the form "objects X and Y are isomorphic") compensates for the absence of equality statements (see §1 in [Bén85]).

From a purely categorical perspective, the lack of equality statements is unsurprising because equality of objects is not preserved by equivalence of categories (see §8 in [Bén85]). In light of Section 2, equality of objects is therefore extraneous to the content of a category $\mathcal{C}$. If such extra information exists it is information about $\mathcal{C}$ *not* about the equivalence class of categories to which $\mathcal{C}$ belongs. Furthermore, $\mathcal{C}$ may not emit such information and even if it does it may differ from the equality of objects emitted by an equivalent category. Thus, it should not be surprising that there is no universal way to detect equality of objects in an arbitrary category.

On the other hand, the lack of equality statements may be disconcerting to algebraists and topologists who often state equality between isomorphic objects. For example, topology textbooks are rife with statements such as $\pi_1(S^1) = \mathbb{Z}$ when in fact we only have $\pi_1(S^1) \simeq \mathbb{Z}$ the category **Grp**. This abuse of notation reflects faith in Vodevsky's Univalence Axiom [Uni13] and the Principle of Structuralism [Awo14] both of which formalize the statement:

Isomorphism is equivalent to equality.

In this section, we justify this faith by proving the fibred Yoneda Lemma and realizing an "equivalence" between "categories" and "categories equipped with equality" as a corollary.

First we must realize "categories equipped with equality" as fibrations and indexed categories equipped with an extra structure realizing "equality". As usual we begin with an examination of $p_{\mathcal{C}}$.

In §9 of [Bén85], Bénabou claims that the language of Example 9 assumes that $\mathcal{C}$ is a category with equality. This assumption allows us to define a cleavage of $p_{\mathcal{C}}$ by

$$\begin{aligned} u^*(Y_j)_{j\in J} &= (X_i)_{i\in I} && \text{with } X_i = Y_{u(i)} \\ \mathrm{Cart}(u, (Y_j)_{j\in J}) &= (u, (\mathrm{id}_{X_i})_{i\in I}). \end{aligned}$$

Notably, given $I \xrightarrow{u} J \xrightarrow{v} K$ and $(Z_k)_{k\in K}$ as in the setup of Proposition 32, we have

$$\begin{aligned} (v \circ u)^*(Z_k)_{k\in K} &= (X_i)_{i\in I} && \text{with } X_i = Z_{v(u(i))} \\ &= u^*(Y_j)_{j\in J} && \text{with } Y_j = Z_{v(j)} \\ &= u^*(v^*(Z_k)_{k\in K}) \end{aligned}$$

and

$$\mathrm{id}_K^*(Z_k)_{k\in K} = (Z_k)_{k\in K}.$$

Unraveling these equalities shows that for this cleavage the isomorphisms promised by Proposition 32 are in fact identities.

Definition 35 (Split fibration)**.** A cleavage is split if the isomorphisms in Proposition 32 are identities. A fibration is split if it has a split cleavage. Let **SplitFib**($\mathcal{B}$) be the full subcategory of **Fib**($\mathcal{B}$) whose objects are split fibrations.

Split fibrations generalize categories with equality and thus constitute a compatible realization of "category with equality". So the the goal of this section reduces to showing that every fibration is equivalent to a split fibration. Unfortunately, not every fibration is splitable, so this fact is nontrivial.

Example 36 (An unsplittable fibration)**.** Let $\mathbb{Z}$ and $\mathbb{Z}/2$ be the single object categories representing the groups $\mathbb{Z}$ and $\mathbb{Z}/2$, respectively. The functor $p : \mathbb{Z} \to \mathbb{Z}/2$ defined on objects by $* \mapsto *$ and on morphisms by $n \mapsto n \mod 2$ is a fibration, since

$$\mathrm{Cart}(0, *) = 0 \quad \text{and} \quad \mathrm{Cart}(1, *) = 1$$

is a cleavage of p. The cartesianness of 0 and 1 as morphisms in $\mathbb{Z}$ follows directly from the fact that $\mathbb{Z}$ has unique inverses. However, this cleavage is not split since $\mathrm{Cart}(1, *) \circ \mathrm{Cart}(1, *) = 1 \circ 1 = 2$ while $\mathrm{Cart}(0, *) = 0$. In fact, there is no split cleavage of p.

Proof. Suppose toward contradiction that p is equipped with a split cleavage. Then,

$$\mathrm{Cart}(1, *) \circ \mathrm{Cart}(1, *) = \mathrm{Cart}(1 \circ 1, *) = \mathrm{Cart}(0, *) = 0$$

implying that $\mathrm{Cart}(1, *) = 0$ since all nontrivial elements of $\mathbb{Z}$ have infinite order. However, this is a contradiction, since $p(\mathrm{Cart}(1, *)) = 1$ while $p(0) = 0$. $\square$

The equivalence $H_\Box : \mathbf{Fib}(\mathcal{B}) \to \mathbf{Idx}(\mathcal{B})$, induces a realization of "category with equality" in the world of indexed categories.

Definition 37 (Split indexed category)**.** An indexed category is split if the isomorphisms η and ε are identities. Thus, an indexed category is split if and only if it is an honest functor. The category $\hat{\mathcal{B}} = [\mathcal{B}^{\mathrm{OP}}, \mathbf{Cat}]$ of functors $\mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ is the full subcategory of $\mathbf{Idx}(\mathcal{B})$ whose objects are split indexed categories.

Example 38. An important split indexed category is the functor $yB = \mathcal{B}(-, B) : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ defined such that on

objects I maps to the discrete category $\mathcal{B}(I, B)$

morphisms a morphism $u : I \to J$ in $\mathcal{B}$ maps to the functor $\mathcal{B}(u, B)$ defined by precomposition with u, i.e.

$$(-) \circ u : \mathcal{B}(J, B) \to \mathcal{B}(I, B).$$

By the (standard) Yoneda Lemma, there exists a bijection between categories HB and $\hat{\mathcal{B}}(yB, H) = \mathbf{Idx}(\mathcal{B})(yB, H)$ that is natural in B and in H for any split indexed category H. The fibred Yoneda Lemma extends this result to an equivalence between cateogories HB and $\mathbf{Idx}(\mathcal{B})(yB, H)$ for any indexed category H [ML78]. First, we translate these notions into the world of fibrations.

Under the Grothendieck construction the split indexed category $\mathcal{B}(-, B)$ corresponds to the fibration $\underline{B} : \mathcal{B}/B \to \mathcal{B}$ (see the discussion preceding Theorem 3.1 in [Str99], for example) defined such that on

objects $(I \xrightarrow{i} B)$ maps to I.

morphisms a morphism $f : (I \xrightarrow{i} B) \to (J \xrightarrow{j} B)$ in $\mathcal{B}/B$ maps to the underlying morphism $f : I \to J$ in $\mathcal{B}$.

The Grothendieck construction also equips $\underline{B}$ with a canonical cleavage such that for $u : I \to J$ in $\mathcal{B}$

$$u^*(J \xrightarrow{j} B) = (I \xrightarrow{u} J \xrightarrow{j} B)$$
$$\mathrm{Cart}(u, J \xrightarrow{j} B) = u.$$

This cleavage is split so $\underline{B}$ is a split fibration.

Proposition 39. *Every morphism in $\mathcal{B}/B$ is cartesian with respect to the fibration $\underline{B}$.*

Proof. Let $u : I \to J$ and $v : J \to K$ constitute morphisms $(I \xrightarrow{i} B) \to (J \xrightarrow{j} B)$ and $(K \xrightarrow{k} B) \to (J \xrightarrow{j} B)$ in $\mathcal{B}/B$ respectively. So $i = j \circ u$ and $k = j \circ v$. Fix a factorization $v = u \circ w$ in $\mathcal{B}$.

$$\begin{tikzcd}
K \arrow[dr, dashed, "w"'] \arrow[drr, "v"] \arrow[ddr, bend right, "k"'] & & \\
& I \arrow[r, "u"'] \arrow[d, "i"] & J \arrow[dl, "j"] \\
& B &
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[d, "w"'] \arrow[dr, "v"] & \\
I \arrow[r, "u"'] & J
\end{tikzcd}$$

$$\mathcal{B}/B \xrightarrow{\quad \underline{B} \quad} \mathcal{B}$$

Since

$$i \circ w = j \circ u \circ w = j \circ v = k,$$

w is a morphism $(K \xrightarrow{k} B) \to (I \xrightarrow{i} B)$ in $\mathcal{B}/B$. Furthermore, any such morphism $(K \xrightarrow{k} B) \to (I \xrightarrow{i} B)$ mapping to w under $\underline{B}$ must be w. Thus w is the unique morphism such that $u \circ w = v$ in $\mathcal{B}/B$, proving that u (viewed as an arbitrary morphism in $\mathcal{B}/B$) is cartesian. $\square$

Now we are prepared to state the fibred Yoneda Lemma.

Lemma 40 (Fibered Yoneda Lemma)**.** *For any fibration $p : \mathcal{F} \to \mathcal{B}$ and object B in $\mathcal{B}$, the categories $\mathcal{F}_B$ and $\mathbf{Fib}(\mathcal{B})(\underline{B}, p)$ are equivalent.*

Translating this result into the world of indexed categories immediately proves the desired extension of the standard Yoneda Lemma, and thus justifies its name (see Section 8.3 of [Bor94], for example).

Recall that the categories $\mathcal{F}_B$ are the fibres of the indexed category H_p. The equivalence in the fibred Yoneda Lemma indicates the existence of a corresponding indexed category $\mathbf{Fib}(\mathcal{B})(\underline{\square}, p)$ with fibres $\mathbf{Fib}(\mathcal{B})(\underline{B}, p)$. We need to define the glue that makes $\mathbf{Fib}(\mathcal{B})(\underline{\square}, p)$ an indexed category. This strategy described in Section 3 of [Str99].

Let $u : I \to J$ be a morphism in $\mathcal{B}$. Define $\mathbf{Fib}(\mathcal{B})(\underline{u}, p) : \mathbf{Fib}(\mathcal{B})(\underline{J}, p) \to \mathbf{Fib}(\mathcal{B})(\underline{I}, p)$ to be the functor such that on

objects the fibred functor $H : \mathcal{B}/J \to \mathcal{F}$ between $\underline{J}$ and p maps to the fibred functor $H \circ \underline{u}$ where $\underline{u} : \mathcal{B}/I \to \mathcal{B}/J$ is defined by postcomposition with u, i.e.

$$\underline{u}(X \xrightarrow{x} I) = (X \xrightarrow{x} I \xrightarrow{u} J).$$

morphisms the natural transformation $\alpha : G \Rightarrow H$ maps to the natural transformation $\alpha\underline{u} : G \circ \underline{u} \Rightarrow H \circ \underline{u}$ obtained via whiskering.

$$\begin{tikzcd}
\mathcal{B}/J \arrow[rr, bend left, "G"{name=U}] \arrow[rr, bend right, "H"'{name=D}] \arrow[dr, "\underline{J}"'] & & \mathcal{F} \arrow[dl, "p"] \\
& \mathcal{B} &
\arrow[from=U, to=D, phantom, "\Downarrow \alpha"]
\end{tikzcd}
\quad \xmapsto{\ \mathbf{Fib}(\mathcal{B})(\underline{u}, p)\ } \quad
\begin{tikzcd}
\mathcal{B}/I \arrow[r, "\underline{u}"] \arrow[dr, "\underline{I}"'] & \mathcal{B}/J \arrow[rr, bend left, "G"{name=U}] \arrow[rr, bend right, "H"'{name=D}] \arrow[d, "\underline{J}"] & & \mathcal{F} \arrow[dll, bend left, "p"] \\
& \mathcal{B} & &
\arrow[from=U, to=D, phantom, "\Downarrow \alpha"]
\end{tikzcd}$$

Then, $\mathbf{Fib}(\mathcal{B})(\underline{\square}, p) : \mathcal{B}^{\mathrm{OP}} \to \mathbf{Cat}$ defined by

$$I \mapsto \mathbf{Fib}(\mathcal{B})(\underline{I}, p) \qquad (u : I \to J) \mapsto (\mathbf{Fib}(\mathcal{B})(\underline{u}, p) : \mathbf{Fib}(\mathcal{B})(\underline{J}, p) \to \mathbf{Fib}(\mathcal{B})(\underline{I}, p))$$

is an honest functor and so a split indexed category (for details see Propositions 8.25 and 8.3.4 in [Bor94]).

Theorem 41. *Every fibration is equivalent to a split fibration.*

Proof. Let $p : \mathcal{F} \to \mathcal{B}$ be a fibration. For each object I in $\mathcal{B}$, the categories $H_p I = \mathcal{F}_I$ and $\mathbf{Fib}(\mathcal{B})(\underline{I}, p)$ are equivalent by the fibred Yoneda Lemma. Thus, H_p and $\mathbf{Fib}(\mathcal{B})(\underline{\square}, p)$ are equivalent as indexed categories implying that p and $\int \mathbf{Fib}(\mathcal{B})(\underline{\square}, p)$ are equivalent as fibrations. Since $\mathbf{Fib}(\mathcal{B})(\underline{\square}, p)$ is a split indexed category, $\int \mathbf{Fib}(\mathcal{B})(\underline{\square}, p)$ is a split fibration. Thus, every fibration is equivalent to a split fibration. (Proof details are inspired by Proposition 8.3.4 in [Bor94] and for further exposition see Theorem 3.1 in [Str99]). $\square$

The meat of this argument is contained in the fibred Yoneda Lemma, which we now prove.

Proof of the fibred Yoneda Lemma. Define a functor $\mathbf{Fib}(\mathcal{B})(\underline{B}, p) \to \mathcal{F}_B$ such that on

objects the fibred functor $H : \mathcal{B}/B \to \mathcal{F}$ between fibrations $\underline{B}$ and p maps to the object $H(B \xrightarrow{\mathrm{id}_B} B)$ in $\mathcal{F}_B$.

morphisms Given fibred functors $G, H : \mathcal{B}/B \to \mathcal{F}$ between fibrations $\underline{B}$ and p, a fibred natural transformation $\alpha : G \Rightarrow H$ maps to the morphism $\alpha(B \xrightarrow{\mathrm{id}_B} B)$

For simplicity we sometimes write $H(\mathrm{id}_B)$ and $\alpha(\mathrm{id}_B)$. It is sufficient to show that this functor is fully faithful and essentially surjective on objects.

Faithful. Suppose that fibred natural transformations $\alpha, \beta : G \Rightarrow H$ are such that $\alpha(\mathrm{id}_B) = \beta(\mathrm{id}_B)$. Consider the object $I \xrightarrow{i} B$ in $\mathcal{B}/B$. Since i is cartesian in $\mathcal{B}/B$, Hi is cartesian in $\mathcal{F}$ and thus there is a unique morphism $G(I \xrightarrow{i} B) \to H(I \xrightarrow{i} B)$ in $\mathcal{F}_I$ making the following diagram commute.

$$\begin{tikzcd}
G(I \xrightarrow{i} B) \arrow[r, dashed, "\exists!"] \arrow[d, "Gi"'] & H(I \xrightarrow{i} B) \arrow[d, "Hi"] \\
G(B \xrightarrow{\mathrm{id}_B} B) \arrow[r, "\alpha(\mathrm{id}_B) = \beta(\mathrm{id}_B)"'] & H(B \xrightarrow{\mathrm{id}_B} B)
\end{tikzcd}$$

Since i is a morphism $(I \xrightarrow{i} B) \to (B \xrightarrow{\mathrm{id}_B} B)$ in $\mathcal{B}/B$, naturality of α and β implies that $\alpha(I \xrightarrow{i} B)$ and $\beta(I \xrightarrow{i} B)$ both make the above diagram commute. Thus they are equal, and so $\alpha = \beta$.

Full. Suppose that $f : G(\mathrm{id}_B) \to H(\mathrm{id}_B)$ is a morphism in $\mathcal{F}_B$. Define a fibred natural transformation $\alpha : G \Rightarrow H$ such that for each object $(I \xrightarrow{i} B)$ of $\mathcal{B}/B$, $\alpha(I \xrightarrow{i} B)$ is the unique morphism in $\mathcal{F}_I$ such that the diagram below commutes.

$$\begin{tikzcd}
G(I \xrightarrow{i} B) \arrow[r, dashed, "\alpha(I \xrightarrow{i} B)"] \arrow[d, "Gi"'] & H(I \xrightarrow{i} B) \arrow[d, "Hi"] \\
G(B \xrightarrow{\mathrm{id}_B} B) \arrow[r, "f"'] & H(B \xrightarrow{\mathrm{id}_B} B)
\end{tikzcd}$$

Clearly $\alpha(\mathrm{id}_B) = f$. It is a straightforward exercise to show that α is natural.

Essentially surjective on objects. Equip p with a cleavage using the Axiom of Choice. For an object X in $\mathcal{F}_I$, define a fibred functor $\underline{B} \to p$ to be the functor $H_X : \mathcal{B}/B \to \mathcal{F}$ such that on

objects $H_X(I \xrightarrow{i} B) = i^*X$.

morphisms Given $u : (I \xrightarrow{i} B) \to (J \xrightarrow{j} B)$ define $H_X u : i^*X \to j^*X$ to be the unique morphism such that $\mathrm{Cart}(j, X) \circ H_X u = \mathrm{Cart}(i, X)$ and such that $p(H_X u) = u$.

Functoriality of H follows from uniqueness of the morphisms $H_X u$, and furthermore

$$(p \circ H_X)(I \xrightarrow{i} B) = p(i^*X) = I$$

and

$$\underline{B}(I \xrightarrow{i} B) = I.$$

So to show that H_X is a fibred functor, it suffices to show that it preserves cartesian morphisms.

Let $u : (I \xrightarrow{i} B) \to (J \xrightarrow{j} B)$ be a cartesian morphism in $\mathcal{B}/B$. Recall that u can be any morphism in $\mathcal{B}/B$ since every morphism in $\mathcal{B}/B$ is cartesian. By uniqueness $H_X u$ is the composition of the cartesian morphism $\mathrm{Cart}(u, j^*X)$ and the isomorphism $u^*(j^*X) \to (j \circ u)^*X = i^*X$ over id_I. Since isomorphisms are cartesian and the composition of cartesian morphisms is again cartesian, $H_X u$ is cartesian.

$$\begin{tikzcd}
u^*(j^*X) \arrow[dr, "\mathrm{Cart}(u, j^*X)"] & & \\
 & j^*X \arrow[dr, "\mathrm{Cart}(j, X)"] & \\
i^*X \arrow[uu, dashed, "\simeq"] \arrow[ur, "H_X u"] \arrow[rr, "\mathrm{Cart}(i, X)"'] & & X
\end{tikzcd}$$

As desired, $H_X(B \xrightarrow{\mathrm{id}}_B B) = \mathrm{id}_B^* X$ is isomorphic to X.

$\square$

7 Generalizing categorical notions

The proof that fibrations sufficiently generalize ordinary categories consists of showing that there are compatible fibrational analogues of the major categorical notions. We say that a categorical property or notion $\diamond$ is compatible with its fibrational analogue "fibred $\diamond$" if $p_{\mathcal{C}}$ has fibred $\diamond$ if and only if $\mathcal{C}$ has $\diamond$.

We have already seen several fibrational analogues of categorical notions including opposite of a fibration, fibred adjuntions, and fibred equivalences, and compatibility is discussed below.

Opposite of a fibration. Let $\mathcal{C}$ be an ordinary category. The opposite of the fibration corresponding to $\mathcal{C}$ is equivalent to the fibration corresponding to $\mathcal{C}^{\mathrm{OP}}$. In other words $p_{\mathcal{C}^{\mathrm{OP}}}$ and $(p_{\mathcal{C}})^{\mathrm{OP}}$ are equivalent fibrations (see Section 5 in [Str99], for example).

Fibred adjunction. An ordinary adjunction $F \dashv G : \mathcal{C} \to \mathcal{D}$ lifts to a fibred adjunction $\mathrm{Fam}\, F \dashv \mathrm{Fam}\, G : \mathrm{Fam}\,\mathcal{C} \to \mathrm{Fam}\,\mathcal{D}$ between fibrations $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ (see Example 1.8.7 in [Jac99]). Conversely, a fibred adjunction between fibrations $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ uniquely determines a fibration between $\mathcal{C}$ and $\mathcal{D}$ by restricting to the fibre over 1.

Fibred equivalence. Let $F \dashv G$ be an equivalence between $\mathcal{C}$ and $\mathcal{D}$. Then $\mathrm{Fam}\, F \dashv \mathrm{Fam}\, G$ is a fibred equivalence between $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$. Conversely, restricting a fibred equivalence between $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ to the fibre over 1 induces an equivalence between $\mathcal{C}$ and $\mathcal{D}$. Thus $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories if and only if $p_{\mathcal{C}}$ and $p_{\mathcal{D}}$ are equivalent fibrations.

In general, a fibred adjunction restricts to an ordinary adjunction between individual fibres. Furthermore, adjunctions between fibres obtained in this way satisfy certain naturality requirements known as the Beck-Chevalley Condition (BCC). The converse is also true: a collection of adjunctions between fibres satisfying BCC lifts to a fibred adjunction between the corresponding fibrations (see Lemma 1.8.9 in [Jac99]). In other words, a fibred adjuction can be constructed fibrewise, reflecting the general principle that fibred analogues of categorical notions can be defined fibrewise (see Definition 1.8.1 [Jac99]).

As an example of this strategy for defining fibred analogues of categorical notions, we define fibred coproducts. This definition brings us full-circle since we motivated fibrations by observing that a category $\mathcal{C}$ having coproducts is intrinsically related to the fibration $p_{\mathcal{C}}$. Defining a compatible fibred analogue of ordinary coproducts justifies that original intuition.

Definition 42 (Fibred coproducts)**.** Let $\mathcal{B}$ be a category with pullbacks. A fibration $p : \mathcal{F} \to \mathcal{B}$ has fibred coproducts if the following conditions hold.

(i) For each morphism $u : I \to J$ in the base category there is an adjunction $\coprod_u \dashv u^* : \mathcal{F}_J \to \mathcal{F}_I$. Let η_u be the unit of this adjunction, and let ε_u be the counit.

(ii) (Beck-Chevalley Condition) Let K be a pullback of $u : I \to J$ and $s : L \to J$ in the base category.

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "s"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

For every such setup, the natural transformation

$$\textstyle\coprod_v r^* \Longrightarrow \coprod_v r^* u^* \coprod_u \simeq \coprod_v v^* s^* \coprod_u \Longrightarrow s^* \coprod_u$$

induced by η_u and ε_v is an isomorphism.

Lastly, we must show that fibred coproducts is compatible with ordinary coproducts.

Proposition 43. *Let $\mathcal{C}$ be a category. Then $p_{\mathcal{C}} : \mathrm{Fam}\,\mathcal{C} \to \mathbf{Set}$ has fibred coproducts if and only if $\mathcal{C}$ has coproducts.*

Proof. Suppose that $p_{\mathcal{C}}$ has fibred coproducts. Since Δ_I is the reindexing functor $!_I^*$, (i) immediately implies that $\mathcal{C}$ has coproducts.

Now suppose that $\mathcal{C}$ has coproducts, so each Δ_I has a left adjoint $\coprod_I$. Let $\coprod(X_i)_{i\in I}$ denote $\coprod_I(X_i)_{i\in I}$. For each morphism $u : I \to J$ in **Set**, let $\coprod_u : \mathrm{Fam}\,\mathcal{C}_I \to \mathrm{Fam}\,\mathcal{C}_J$ be the functor defined on objects by

$$(X_i)_{i\in I} \mapsto \left(\textstyle\coprod(X_i)_{i|u(i)=j}\right)_{j\in J}.$$

and with the natural action on morphisms defined by the action of $\coprod$. It is straightforward to show that $\coprod_u$ is left adjoint to u^*. Given the setup of (ii), the Beck-Chevalley Condition holds because

$$\textstyle\coprod_v r^*(X_i)_{i\in I} = \coprod_v (X_{r(k)})_{k\in K} = \left(\coprod(X_{r(k)})_{k|v(k)=\ell}\right)_{\ell\in L}$$

and

$$\textstyle s^* \coprod_u (X_i)_{i\in I} = s^*\left(\coprod(X_i)_{i|u(i)=j}\right)_{j\in J} = \left(\coprod(X_i)_{i|u(i)=s(\ell)}\right)_{\ell\in L}$$

and if $v(k) = \ell$ then $r(k) = i$ if and only if $u(i) = s(\ell)$. □

8 Conclusion

Naive category theory is limited because the stuff, structure, and properties that can be realized by an ordinary category is limited by its relationship to **Set**. The theory of fibrations extends naive category theory by realizing stuff, structure, and properties over an arbitrary base category $\mathcal{B}$. However, this perspective reveals that the theory of fibrations is analogously limited because it does not have the machinery to realize stuff, strucutre, and properties over an arbitrary base "category". If such a theory exists then we might say that it realizes ""categories"" and that it is limited because it does not have the machinery to realize stuff, structure, and peroperties over of an arbitrary base ""category"". And so on.

Notice in Proposition 23 how the fibration $\mathbf{Fib} \to \mathbf{Cat}$ eludicates that the 2-category **Fib** can be considered to be over the 2-category **Cat**. Taking **Fib** and the map $\mathbf{Fib} \to \mathbf{Cat}$ as the prototypical example, [Her99] explores a generalization of ordinary fibrations in **Cat** to fibrations in an arbitrary 2-category which are organized into a 2-category called a fibred 2-category or a 2-fibration [Buc14]. **Fib** is the 2-fibration over **Cat**. In particular, the objects of the 2-fibration over **Cat** realize "categories". Therefore, the objects of the 2-fibration over **Fib** potentially realize ""categories"", etc., although such a claim needs further justification.

The notion of 2-fibrations does not just satsify the mathematicians' eternal quest for universal coherence, but it also has practical applications in type theory [Her93] and toposes [Joh93]. Another direction of generalization follows the idea that bicategories are the "right" foundation for category theory [Str80, Bén67] and that isomorphism (as opposed to equality) is the "right" notion of equivalence [Uni13]. Towards this goal [Buc14] develops the notion of a fibred bicategory. In conclusion we have the following nested series of overarching theories: the theory of fibred bicategories governs the theory of fibred 2-categories governs the theory of fibrations governs ordinary category theory.

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