Parameter estimation

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ARTICLE INFO

Keywords:
Supercritical extraction
Parameter estimation
Mathematical modelling

ABSTRACT

Given a system of partial differential equations, $F(t, x, \dot{x}, \Theta, u) = 0$, where x represents state variables, Θ are the parameters, and u are control variables, we describe the supercritical extraction process. The process model describes a partially filled extractor with a fixed bed, which work under constant operating conditions. We assume that the flow is uniform across any cross-section, although the area available for the fluid phase can change along the extractor. We apply the concept of quasi-one-dimensional flow to mimic the modelling of a two-dimensional case. In this work, a distributed-parameter model, based on Reverchon [1] is used to describe a fluid-solid extraction process of caraway oil from caraway seeds with CO_2 as a solvent. The model parameters such as partition factor, internal diffusion coefficient, axial diffusion coefficient and saturation concentration are obtained from parameter estimation. The parameters are estimated based on four experiments performed at $40°C$ and $50°C$ at 200 bar and 300 bar. Given yield data, the model-based parameter estimation uses the maximum likelihood estimation method under assumption of normal error.

1. Introduction

The extraction of natural substances from solid materials and liquids with solvents have been a popular subject of research and development in the last years. Supercritical fluids have multiple applications in an extraction process due to the pressure-dependent dissolving power and both gas- and liquid-like properties (for example fluid-like density and gas-like diffusivity). Among different supercritical fluids, the supercritical CO_2 is one of the most popular due to it is nontoxic, non-flammable and non-corrosive properties. The critical point of CO_2 is relatively low (73.8 bar and $31°C$), compare to other fluids. The supercritical extraction with CO_2 become attractive alternative, to replace traditional extraction techniques.

The extraction of valuable compounds from fixed bed of biomass can be described by one of many mathematical models as presented by Huang et al. [2]. The selection the extraction model is not arbitrary and should be based on the knowledge of phenomenons occurring in the operational unit. Each model has its own assumptions and describe different mass transfer mechanisms and equilibrium relationships.

Based on analogy to heat transfer, Reverchon et al. [3] suggested a hot ball model, where the extraction process is treated analogously to a process in which a hot ball is cooled in a uniform medium. The hot ball model is used to describe an extraction process from solid particles which contains small quantities of solute so that the solubility is not a limiting factor.

Sovova [4] presented the Broken-and-Intact Cell model. The BIC model describes a system where the outer surfaces of particles have been mechanically interrupted. The solute from the broken cells is easily accessible for the solvent. The extraction of easily accessible solute is fast and directly controlled by its diffusion and convection in the solvent. However, the rest of the solute is less accessible because it is closed in the particle's core or intact cells. This solute slowly diffuses through the walls of a cell due to high mass transfer resistance.

Reverchon [1] developed a model, which considers an oil as a single component and assumes that the extraction process is controlled by internal-mass transfer resistance. As a result of these assumptions, the external mass transfer was neglected. The original model of Reverchon does not consider the influence of axial dispersion and does not take into account the change of density and flow rate along the bed.

In our model, the extraction process is assumed to operate in a semi-continuous mode in a cylindrical vessel. The solvent is firstly brought to super-critical condition, pumped through a fixed-bed, of finely chopped biomass, where the solute is extracted from the biomass. Then, the solvent and solute from the extractor are separated in a flush drum, and the extract is collected. The flow rate (F_{in}) and inlet temperature (T_{in}) of the extractor's feed can be measured and manipulated. The pressure (P) in the vessel can be measured and manipulated, while the outlet temperature (T_{out}) can only be measured. A simplified flow diagram is depicted in Figure 1 .

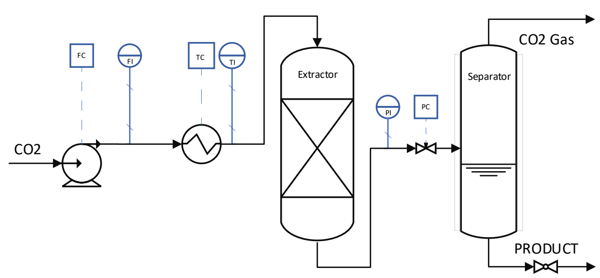

Figure 1: Process flow diagram (Font size)

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2. Materials and methods

2.1. Supercritical fluids

A supercritical fluid (SCF) is any substance at a temperature and pressure above its critical point, where distinct liquid and gas phases do not exist, but below the pressure required to compress it into a solid. It can effuse through porous solids like a gas, overcoming the mass transfer limitations that slow liquid transport through such materials. SCF are much superior to gases in their ability to dissolve materials like liquids or solids. Also, near the critical point, small changes in pressure or temperature result in large changes in density, allowing many properties of a supercritical fluid to be "fine-tuned". By changing the pressure and temperature of the fluid, the properties can be "tuned" to be more liquid-like or more gas-like.

The properties of a fluid can be divided into two kinds, equilibrium properties and transport properties. The equation of state can be used accurately to predict the equilibrium properties, such as density, enthalpy, vapor pressure, fugacity and fugacity coefficient, vapor liquid equilibrium, and all kinds of excess properties.

The thermodynamic properties of supercritical CO_2 such as density, local speed of sound and specific heat capacity vary significantly for slight change in temperature and pressure due to real gas effects. Equation of state (EOS) which account these real gas effects is used to calculate the thermodynamic properties. In order to predict this real gas effects, the Peng-Robinson equation of state (P-R EOS) is used to compute properties of CO_2. Detail information about Peng-Robinson equation of state can be found in the work of Peng and Robinson [5], Elliott [6] or Pratt [7]. The P-R EOS belongs to the specific class of thermodynamic models for modelling the pressure of a gas as a function of temperature and density and can be written as a cubic function of the molar volume (of the density). The P-R EOS is presented by equation 1

$$P = \frac{RT}{V_m - b} - \frac{a\alpha}{V_m^2 + 2bV_m - b^2} \tag{1}$$

where a, b, α are parameters defined as presented in the appendix.

The properties of the CO_2 presented as a function of operating conditions (temperature and pressure) are presented on fig. 2

At standard atmospheric pressure and temperature, the CO_2 behaves as an ideal gas, and its compressibility factor equals to unity. However, at high temperature and pressure, the compressibility factor varies from unity, due to real gas effects. As it is presented on Figure 2a, the compressibility factor obtained from the Peng-Robinson equation of state varies strongly depending on the operating conditions. The compressibility factor can be obtained by given temperature and pressure by solving the polynomial form of the P-R EOS given by equation 2.

$$Z^3 - (1-B)Z^2 + (A - 2B - 3B^2)Z - (AB - B^2 - B^3) = 0 \tag{2}$$

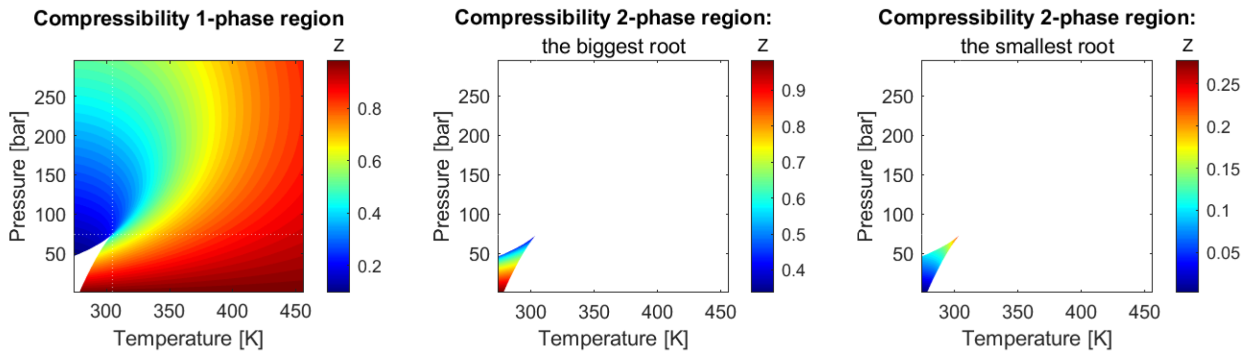
where A and B are parameters as defined in the appendix. The roots of the polynomial can be found iteratively or by Cardano formula. Depends on the operating conditions, one or two roots can be found. In one-phase region the fluid can be describe as gas, liquid or super-critical. In the two-phase region the gas-liquid mixture is present. The biggest root is assigned to the gas phase and smallest root corresponds to the liquid phase.

The real gas effects are also visible on the density plot presented on the Figure 2b. The density can change significantly depends on the operating conditions. By analysing the compressibility and density plots more it can be noticed that very near to the critical point, the fluid can neither be called a liquid nor a gas and has a unique combination of gas-like and liquid-like properties.

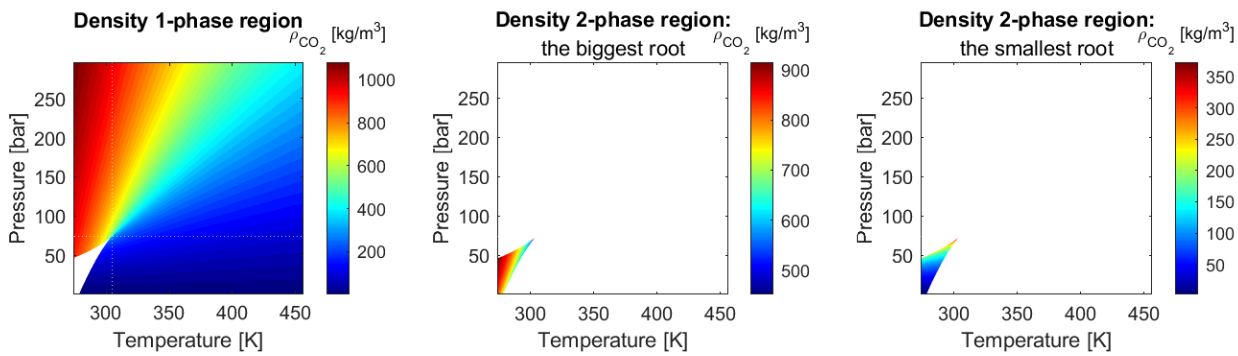
The Figure 2c show behaviour of the heat capacity of a supercritical fluid at constant pressure (C_p). The details of the calculations can be found in the appendix. In contrary to the density which varies monotonically, the specific heat shows very high levels in a narrow region. In the subcritical region, the phase transition is associated with an effective spike in the heat capacity (i.e., the latent heat). Approaching the critical point, the latent heat falls to zero but this is accompanied by a gradual rise in heat capacity in the pure phases near phase transition. At the critical point, the latent heat is zero but the heat capacity shows a diverging singularity. Beyond the critical point, there is no divergence, but rather a smooth peak in the heat capacity; the highest point of this peak identifies the Widom line (as discussed by Simeoni et al. [8] and Banuti [9]).

In order to calculate thermodynamic properties from a real gas, the departure function for that property with respect to the chosen equation of state has to be evaluated. As presented by Elliott [6], the departure function for any thermodynamic property of a real gas is defined as the difference in the value of that property determined from the chosen real gas equation of state and the value of the same property for an ideal gas under the same conditions of temperature and pressure.

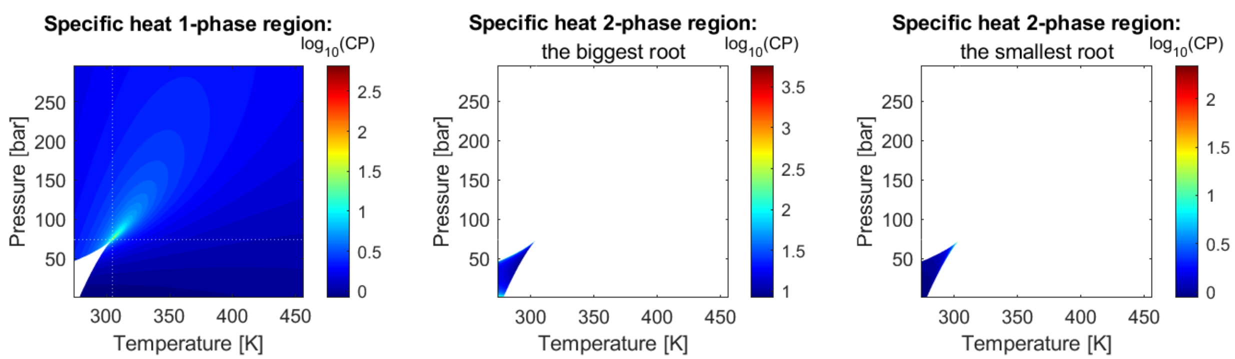
On the other hand, transport properties (viscosity and conductivity) are also important quantities required in engineering design for production, fluid transportation, and processing. According to Sheng et al. [10], there is no satisfactory theory of transport properties of real dense gases and liquids. The main difficulties in the study of transport properties are twofold: one is the inherent difficulties involved in accurate measurements, and the other is the complexity involved in theoretical treatments. Therefore, the generally used correlations of transport coefficient are either empirical or based on some theoretical foundation. Enskog () developed a popular theory for the transport properties of Add ref dense gas based on the distribution function. However, the Enskog theory was proposed for rigid spherical molecules. For real gases, some modification is needed. Following the Enskog theory, many correlations have been proposed in the form of the reduced density and reduced temperature. The correlations of Fenghour et al. [11] and Laesecke and Muzny [12] implemented and compared as presented on Figure 3. The viscosity formulation proposed by NIST consists of four



(a) The compressibility factor based on the Peng-Robinson equation of state



(b) The fluid density based on the Peng-Robinson equation of state



(c) The specific heat of the CO_2 based on the Peng-Robinson equation of state

Figure 2: Properties of CO_2 based on the equation of state

contributions: (i) for the limit of zero density, (ii) for the initial density dependence, (iii) for the residual viscosity, and (iv) for the singularity of the viscosity at the critical point. The NIST correlation covers temperatures from 100 to 2000 K for gaseous CO_2 and from 220 to 700 K with pressures along the melting line up to 8000 MPa for compressed and supercritical liquid states.

Similarly, several correlations for thermal conductivity of CO_2 were compared on Figure 4. The presented figures are focused around critical point, where the singularity is present. Similarities between specific heat and the thermal conductivity can be observed. The NIST correlation (Huber et al. [13]) captures the singular behaviour of thermal conductivity around the critical point. The correlations are

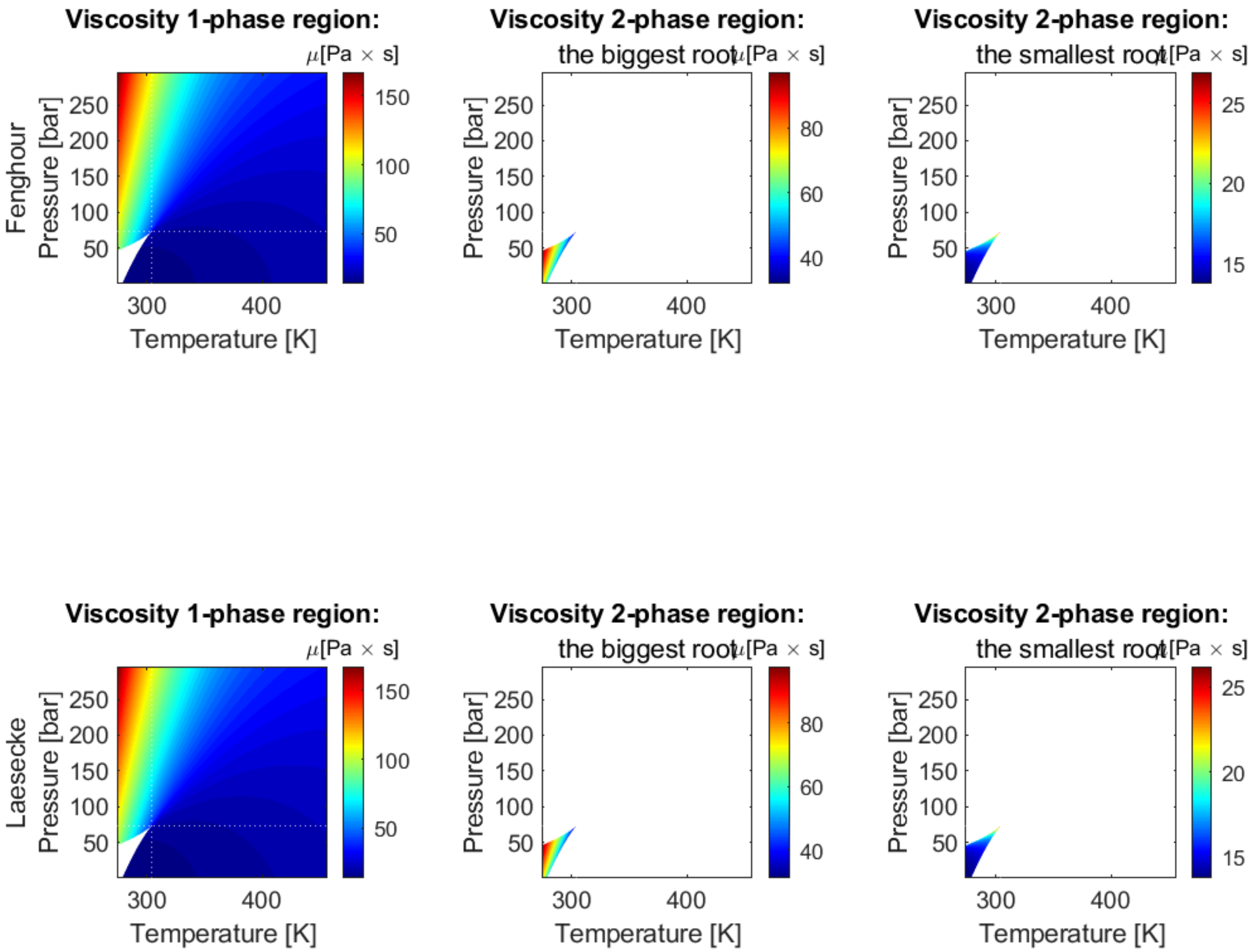


Figure 3: Viscosity obtained based on different correlations (Decrease space between figures)

applicable for the temperature range from the triple point to 1100 K and pressures up to 200 MPa.

2.2. Governing equations

The detail derivation of the governing equation can be found in the appendix (A.2.1) as well as in the work of Anderson [14]. Let's assume that any properties of the flow are unifrom across any given cross-section of an channel (or any device like an extractor). Such a flow is called quasi-one-dimensional. The variation of the cross-section might be an result of its irregular shape (or partial filling of an extractor).

The quasi-one-dimensional compressible Navier-Stokes equations in Cartesian coordinates is given be equations 3 to 5.

$$\frac{\partial\left(\rho_f A_f\right)}{\partial t}+\frac{\partial\left(\rho_f A_f v\right)}{\partial z}=0 \tag{3}$$

$$\frac{\partial\left(\rho_f v A_f\right)}{\partial t}+\frac{\partial\left(\rho_f A_f v^2\right)}{\partial z}=-A_f \frac{\partial P}{\partial z} \tag{4}$$

$$\frac{\partial\left(\rho_f e A_f\right)}{\partial t}+\frac{\partial\left(\rho_f A_f v e\right)}{\partial z}=-P \frac{\left(A_f v\right)}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) \tag{5}$$

where ρ_f is the density of the fluid, A_f is the function which describe change of the cross-section, v is the velocity, P is the total pressure, e is the internal energy of the fluid, t is time and z is the spacial direction.

Based on governing equations, the small discontinuity in flow properties, shown on figure 5, can be analysed. The analysis follow the work of Schreier [15].

The discontinuity is presumed to be at rest relative and the balance equations become

Fix the notation of pressure perturbation section to be coherent with governing equations

$$\rho\delta v+v\delta\rho_f+\delta\rho_f\delta v=0$$

$$\delta P=\delta v\delta\rho_f$$

These relations are equally valid if the two regions are separated by a regions of finite width rather than a discontinuity.

$$\lim_{\rho_f v\to 0}\rho_f\delta v+v\delta\rho_f+\delta\rho_f\delta v=0/\delta\rho_f\to\frac{dv}{d\rho_f}=-\frac{v}{\rho_f}$$

By combining momentum equation with above equation we get

$$\frac{dv}{d\rho_f}=-\frac{dv}{dP}\frac{dP}{d\rho_f}=-\frac{1}{\rho v}\frac{dP}{d\rho_f}=-\frac{v}{\rho_f}$$

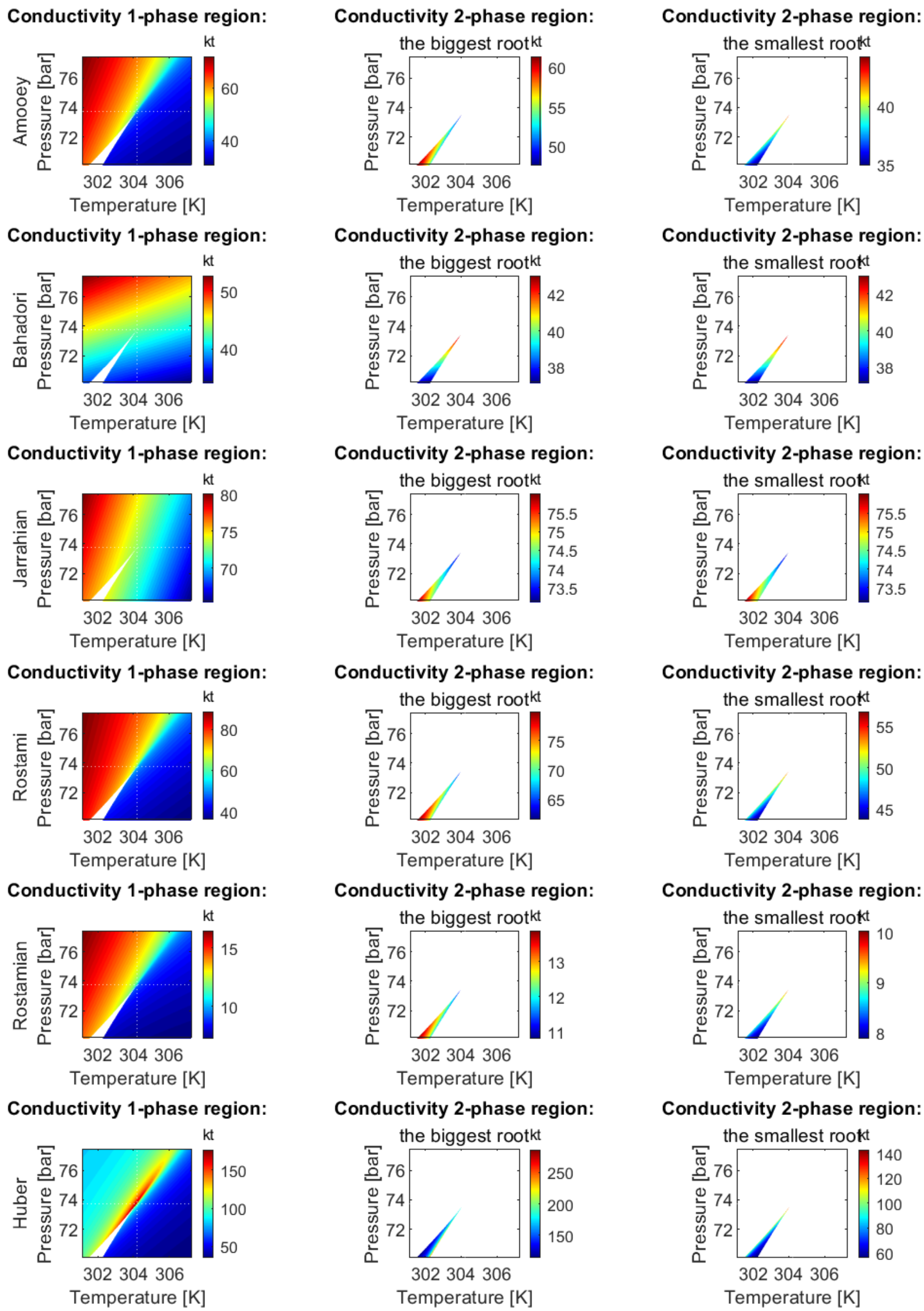


Figure 4: Thermal conductivity obtained based on different correlations

If the flow is presumed to be isentropic, $dP/d\rho = c^2$, so $V^2 = c^2$, where c is the speed of sound. This can be interpreted that a small pressure wave propagates with the speed of sound relative to the flow.

2.3. Low Mach number expansion

As discussed by Lions [16], the low Mach number equations are a subset of the fully compressible equations of motion (continuity, momentum and energy). Such a

	ρ	$\rho + \delta\rho$	
$v \rightarrow$	P	$P + \delta P$	$v + \delta v \rightarrow$
	T	$T + \delta T$	

Figure 5: Small discontinuity in one-dimensional flow

set of equations allow for large variations in gas density but it is consider to be acoustically incompressible. The low Mach number equations are preferred over the full compressible equations for low speed flow problems $\left(M_a = \frac{|V|}{\sqrt{\partial P/\partial \rho}} \ll 1\right)$ to avoids the need to resolve fast-moving acoustic signals. The equations are derived from the compressible equations based on the perturbation theory. The perturbation theory develops an expression for the desired solution in terms of a formal power series known as a perturbation series in some "small" parameter ζ, that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem.

The equations 3 to 5 describe the fully compressible equations of motion (respectively the transport of mass, momentum and energy) for quasi-one-dimensional case. We rescale the time variable, considering finally

$$\rho_\zeta = \rho\left(z, t/M_a\right), \quad v_\zeta = \frac{1}{\zeta} v\left(z, t/M_a\right)$$
$$T_\zeta = T\left(z, t/M_a\right), \quad k_\zeta = \zeta k(\rho_f, T)$$

The conservative non-dimensional equations of motion becomes

$$\frac{\partial\left(\rho_\zeta A_f\right)}{\partial t} + \frac{\partial\left(\rho_\zeta A_f v_\zeta\right)}{\partial z} = 0$$
$$\frac{\partial\left(\rho_\zeta A_f v_\zeta\right)}{\partial t} + \frac{\partial\left(\rho_\zeta v_\zeta A_f v_\zeta\right)}{\partial z} + \frac{A_f}{M_a^2}\frac{\partial P_\zeta}{\partial z} = 0$$
$$\frac{\partial\left(\rho_\zeta e_\zeta A_f\right)}{\partial t} + \frac{\partial\left(\rho_\zeta e_\zeta v_\zeta A_f\right)}{\partial z} - \frac{\partial}{\partial z}\left(k\frac{\partial T_\zeta}{\partial z}\right) + P_\zeta \frac{\partial A_f v_\zeta}{\partial z} = 0$$

Let's define $\zeta = M_a^2$ and assume small Mach numbers, $M_a \ll 1$, then the kinetic energy, viscous work, and gravity work terms can be neglected in the energy equation since those terms are scaled by the square of the Mach number. The inverse of Mach number squared remains in the momentum equations, suggesting singular behaviour. In order to explore the singularity, the pressure, velocity and temperature are expanded as asymptotic series in terms of the parameter ζ

$$P_\zeta = P_0 + P_1\zeta + P_2\zeta^2 + \mathcal{O}(\zeta^3)$$
$$\rho_\zeta = \rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)$$
$$v_\zeta = v_0 + v_1\zeta + \mathcal{O}(\zeta^2)$$
$$T_\zeta = T_0 + T_1\zeta + \mathcal{O}(\zeta^2)$$
$$e_\zeta = e_0 + e_1\zeta + \mathcal{O}(\zeta^2)$$

By expanding performing power expansion on the continuity equation and taking the limit of ζ from the positive side we get

$$\lim_{\zeta\to 0_+} \frac{\partial\left(\left(\rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)\right) A_f\right)}{\partial t} + \\ + \frac{\partial\left(\left(\rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)\right) A_f \left(v_0 + v_1\zeta + \mathcal{O}(\zeta^2)\right)\right)}{\partial z} = 0$$

The continuity equation become

$$\frac{\partial\left(\rho_0 A_f\right)}{\partial t} + \frac{\partial\left(\rho_0 A_f v_0\right)}{\partial z} = 0 \tag{6}$$

The form of the continuity equation stays the same. Considering the momentum equation, it can be seen that the inverse of Mach number squared remains which suggests singular behavior.

$$\lim_{\zeta\to 0_+} \frac{\partial\left(\left(\rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)\right) A_f \left(v_0 + v_1\zeta + \mathcal{O}(\zeta^2)\right)\right)}{\partial t} + \\ + \frac{\partial\left(\left(\rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)\right) A_f \left(v_0 + v_1\zeta + \mathcal{O}(\zeta^2)\right)\left(v_0 + v_1\zeta + \mathcal{O}(\zeta^2)\right)\right)}{\partial z} + \\ + A_f \frac{\partial}{\partial z}\left(\frac{P_0}{M_a^2} + \frac{P_1\zeta}{M_a^2} + \frac{P_2\zeta^2}{M_a^2} + \mathcal{O}(\zeta^3)\right)$$

The first two terms stays the same, but the third one become different in structure. By further investigation of the pressure term in the momentum equation it can be observed

$$\lim_{\zeta\to 0_+} \frac{\partial}{\partial z}\left(\frac{P_0}{M_a^2} + \frac{P_1\zeta}{M_a^2} + \frac{P_2\zeta^2}{M_a^2} + \mathcal{O}(\zeta^3)\right) =$$
$$= \lim_{\zeta\to 0_+} \frac{\partial}{\partial z}\left(\frac{P_0}{M_a^2}\right) + \frac{\partial}{\partial z}\left(\frac{P_1\zeta^2}{M_a^2}\right) + \frac{\partial}{\partial z}\left(\frac{P_2\zeta^2}{M_a^2}\right)$$
$$= \lim_{\zeta = M_a^2\to 0_+} \frac{\partial}{\partial z}\left(\frac{P_0}{M_a^2}\right) + \frac{\partial}{\partial z}\left(\frac{P_1 M_a^2}{M_a^2}\right) + \frac{\partial}{\partial z}\left(\frac{P_2 M_a^4}{M_a^2}\right)$$
$$= \lim_{\zeta = M_a^2\to 0_+} 0 + \frac{\partial P_1}{\partial z} + 0$$

The simplification of the P_0 in the momentum equation comes from the fact that P_0 is independent of z. As it was presented above, the thermodynamical pressure moves with speed of sound, and any perturbation propagates instantaneously. The term related to P_2 and higher order terms become zero at the limit of $M_a \rightarrow 0$. The momentum equation become

$$\frac{\partial\left(\rho_0 A_f v_0\right)}{\partial t} + \frac{\partial\left(\rho_0 v_0 A_f v_0\right)}{\partial z} + A_f \frac{\partial P_1}{\partial z} = 0$$

By expanding performing power expansion on the energy equation and taking the limit of ζ from the positive side we get

$$\lim_{\zeta\to 0_+} \frac{\partial\left(\left(\rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)\right) A_f \left(e_0 + e_1\zeta + \mathcal{O}(\zeta^2)\right)\right)}{\partial t} + \\ + \frac{\partial\left(\left(\rho_0 + \rho_1\zeta + \mathcal{O}(\zeta^2)\right) A_f \left(v_0 + v_1\zeta + \mathcal{O}(\zeta^2)\right)\left(e_0 + e_1\zeta + \mathcal{O}(\zeta^2)\right)\right)}{\partial z} + \\ + \frac{\partial}{\partial z}\left(k\frac{\partial}{\partial z}\left(T_0 + T_1\zeta + \mathcal{O}(\zeta^2)\right)\right) + \\ - \left(P_0 + P_1\zeta + P_2\zeta^2 + \mathcal{O}(\zeta^3)\right)\frac{\partial\left(A_f\left(v_0 + v_1\zeta + \mathcal{O}(\zeta^2)\right)\right)}{\partial z} = 0$$

The form of the energy equation stays the same.

$$\frac{\partial\left(\rho_0 e_0 A_f\right)}{\partial t}+\frac{\partial\left(\rho_0 e_0 v_0 A_f\right)}{\partial z}-\frac{\partial}{\partial z}\left(k\frac{\partial T_0}{\partial z}\right)+P_0\frac{\partial A_f v_0}{\partial z}=0$$

where $e_0 = e(\rho_0, T_0)$ and $k = k(\rho_0, T_0)$.

The expansion results in two different types of pressure and they are considered to be split into a thermodynamic component (P_0) and a dynamic component (P_1). The thermodynamic pressure is constant in space, but can change in time. The thermodynamic pressure is used in the equation of state. The dynamic pressure only arises as a gradient term in the momentum equation and acts to enforce continuity.

The resulting unscaled low Mach number equations are:

$$\frac{\partial\left(\rho_f A_f\right)}{\partial t}+\frac{\partial\left(\rho_f A_f v\right)}{\partial z}=0$$
$$\frac{\partial\left(\rho_f A_f v\right)}{\partial t}+\frac{\partial\left(\rho_f v A_f v\right)}{\partial z}+A_f\frac{\partial P_1}{\partial z}=0$$
$$\frac{\partial\left(\rho_f e A_f\right)}{\partial t}+\frac{\partial\left(\rho_f e v A_f\right)}{\partial z}-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_0\frac{\partial A_f v}{\partial z}=0$$

The energy equation can be expanded through the chain rule to obtain

$$\rho A_f\left(\frac{\partial e}{\partial t}+v\frac{\partial e}{\partial z}\right)+e\underbrace{\left(\frac{\partial\left(\rho_f A_f\right)}{\partial t}+\frac{\partial\left(\rho_f v A_f\right)}{\partial z}\right)}_{Continutiy}-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_0\frac{\partial A_f v}{\partial z}=0$$

The non-conservative form of the energy equation become

$$\rho A_f\left(\frac{\partial e}{\partial t}+v\frac{\partial e}{\partial z}\right)-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_0\frac{\partial A_f v}{\partial z}=0$$

If the calorically perfect gas is assumed then $e = C_v T$, where C_v is the constant specific heat. The energy equation can derived in terms of temperature T

$$\rho A_f C_v\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial z}\right)-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_0\frac{\partial A_f v}{\partial z}=0$$

If isothermal case is assumed then, the energy equation becomes

$$\lim_{\Delta T\to 0_+}\rho A_f C_v\left(\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial z}\right)-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_0\frac{\partial A_f v}{\partial z}=0$$

which leads to

$$\frac{\partial A_f v}{\partial z}=0 \tag{7}$$

In one-dimensional case, the equation 7 become equivalent of $\mathrm{div}(A_f v) = 0$, which known as the incompressibility condition (Lions [16]).

A general formulation the internal energy for a real gas is:

$$de = C_v dT-\left[P-T\left(\frac{\partial P}{\partial T}\right)_{V_m}\right]dv_m$$

where v_m is the molar volume.

The internal energy is a function of two intensive properties, in this case T and $v_m = 1/\rho_f$. But, in the case of an ideal gas, the equation of state is such that the second term in this equation is identically equal to zero. So the ideal gas is a special case in which the molar internal energy is a function only of temperature. For Peng-Robinson equation of state, the internal energy is defined as

$$e = C_v T+\frac{a\left(\alpha-T\frac{d\alpha}{dT}\right)}{2\sqrt{2}b}\ln\left[\frac{1+b\left(1-\sqrt{2}\rho\right)}{1+b\left(1+\sqrt{2}\rho\right)}\right]$$

Assuming constant temperature and pressure along space and in time:

$$\lim_{\Delta T,\Delta P\to 0_+}\rho A_f\left(\frac{\partial e}{\partial t}+v\frac{\partial e}{\partial z}\right)-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+P_0\frac{\partial A_f v}{\partial z}=0\to\frac{\partial A_f v}{\partial z}=0$$

It can be deduce that the continuity equation becomes $\frac{\partial\rho_f}{\partial t} = 0$ at constant temperature and pressure. Moreover, the incompressibility condition $div(A_f V) = 0$ is obtained.

Assuming an arbitrary function $\hat{P}$, which describe the total pressure, the dimensional momentum equation can be written as

$$V\underbrace{\left(\frac{\partial\rho_f A_f}{\partial t}+\frac{\rho_f v A_f}{\partial z}\right)}_{Continuity\ equation}+\rho_f A_f\frac{\partial v}{\partial t}+\rho_f v A_f\frac{\partial v}{\partial z}=-A_f\frac{\partial\hat{P}}{\partial z}$$

From the incompressibility conditions we can deduce that

$$\frac{\partial A_f v}{\partial z}=0\to A_f\frac{\partial v}{\partial z}=-v\frac{\partial A_f}{\partial z}$$

By combining both above equations, assuming that $\partial v/\partial t = 0$:

$$\frac{\rho_f v^2}{A_f}\frac{\partial A_f}{\partial z}=\frac{\partial\hat{P}}{\partial z}\to\int\frac{\rho_f v^2}{A_f}\frac{\partial A_f}{\partial z}dz=\int\frac{\partial\hat{P}}{\partial z}dz$$

The l.h.s integral can be solved by assuming ρ_f is constant and introducing superficial velocity $u_s = A_f v$

$$\int\frac{\rho_f v^2}{A_f}\frac{\partial A_f}{\partial z}dz=\int\frac{\rho_f v^2}{A_f}\frac{A_f^2}{A_f^2}\frac{\partial A_f}{\partial z}dz$$
$$=\rho_f u_s^2\int\frac{1}{A_f^3}\frac{\partial A_f}{\partial z}dz=-\frac{\rho_f u_s^2}{2\Delta A_f^2}=-\frac{\rho_f\Delta v^2}{2}$$
$$\int\frac{\partial\hat{P}}{\partial z}dz=\Delta\hat{P}$$

The final form of the momentum equation corresponds to the Bernoulli's principle

$$\Delta\hat{P}=-\frac{\rho_f\Delta v^2}{2}\xrightarrow{P_0=\text{const}}\Delta M_a^2 P_1=-\frac{\rho_f\Delta v^2}{2}$$

The Bernoulli's principle can be used to find the hydrodynamic pressure caused by varying cross-section at steady-state. Moreover, if the flow velocity is relatively low and so all pressure changes are hydrodynamic (due to velocity motion) rather than thermodynamic. The effect of this is that $\partial\rho/\partial P = 0$. In other words, the small changes in pressure due to flow velocity changes do not change the density. This has a secondary effect – the speed of sound in the fluid is $\partial P/\partial\rho = \infty$ in this instance. So there is an infinite speed of sound, which makes the equations elliptic in nature. It can be deduced that at the isothermal conditions the density in the system propagates with the same speed as pressure due to the fact that they are both connected through the equation of state.

2.4. Extraction model

For the sake of clarity of the process model, different colors have been used in the equations to indicate: control variables, state variables, variables and parameters.

2.4.1. Continuity equation

The details of derivation can be found in the appendix A.2.1. The continuity equation for the fluid phase is given by equation 3. If A_f is specified to be a function of void fraction $A_f = A\varepsilon(z)$, where ε is void fraction of the bed and A is the cross-section of the empty extractor. The continuity equation become

$$\frac{\partial\left(\rho_f \varepsilon\right)}{\partial t} + \frac{\partial\left(\rho_f v \varepsilon\right)}{\partial z} = 0$$

Assuming that the mass flow-rate is constant in time, the temporal derivative become zero and the spacial derivative can be integrated along z.

$$\int \frac{\partial\left(\rho_f v \varepsilon\right)}{\partial z} dz = 0 \rightarrow F = \rho_f v \varepsilon \tag{8}$$

where F is a constant obtained from the integration and it is understood as the mass flux per unit area which is assumed to be constant along z. F can treated as a control variable, which can change in time but it is constant in space. Such a assumption allows to find the velocity profile which satisfy the mass continuity based on F, ε and ρ_f. The fluid density ρ_f density is connected through equation of state with thermodynamical pressure, which is assumed to be constant along z (due to the low-Mach number) and temperature . The variation in density might be caused by the accumulation of the fluid in the system (which happen instantaneously along z) or by temperature change. To simplify the dynamic of the system, it is assumed that $F = F(t)$ is a control variable and affects the whole system instantaneously.

Elaborate, the pressure is know (a control variable) and temperature can be found from energy equation

Introduce relation between accumulation, pressure and energy equation

2.4.2. Mass balance for the fluid phase

The detail derivation of the mass balance equation for the fluid phase can be found in the appendix (A.2.1). The movement of the mobile pseudo-homogeneous phase (equation 9) is considered only in the axial direction. The properties of the system in the radial direction are assumed to be uniform. In addition, it is considered that the boundary layer adjacent to the inner wall of the extractor does not exist. Therefore, the velocity profile is constant across any cross-section of the extractor perpendicular to the axial direction. As a result, the plug flow model can be introduced. The particle size distribution and the void fraction of the solid phase cab change along an extractor but they remain constant in time. Moreover, the thermodynamic pressure is considered to be constant along the device, as discussed above. The amount of solute in the solvent is considered negligible. Therefore, the fluid phase can be described as pseudo-homogenous, and its properties are assumed to be the same as the solvent. The mass balance for the fluid phase consists of convection, diffusion, and kinetic terms.

Double check if we can call it a plug flow

$$\frac{\partial c_f(t,z)}{\partial t} = \underbrace{-\frac{u_s}{\varepsilon}\frac{\partial c_f(t,z)}{\partial z}}_{\text{Convective term}} - \underbrace{\frac{c_f(t,z)}{\varepsilon}\frac{\partial u(t,z)}{\partial z}}_{\text{Convective term}} + \underbrace{\frac{D_e^M}{\varepsilon}[T(t,z), P(t), F(t)]\frac{\partial^2 c_f(t,z)}{\partial z^2}}_{\text{Diffusive term}} + \underbrace{\frac{1-\varepsilon}{\varepsilon} r_e(t,z)}_{\text{Kinetic term}} \tag{9}$$

where $c_f(t,z), c_s(t,z), T(t,z)$ correspond to concentration of solute in the fluid phase, concentration of solute in the solid phase and the temperature, respectively. $r_e(t,z)$ is a mass transfer kinetic term. $F(t)$ is the mass flow rate, $P(t)$ is the pressure, ϵ is the void fraction of the bed, $\rho(T(t,z), P(t))$ is the fluid's density, ρ_s is the solids density, $D_e^M(T(t,z), P(t), F(t))$ is the axial mass diffusion coefficient and u is the superficial velocity.

Remove symbols which have been introduced before

2.4.3. Mass balance for the solid phase

Considering the solid phase to be fixed, the convection and diffusion terms in the corresponding mass balance (equation 10) are both assumed to be negligible or absent. Therefore, the only term present in this equation is the kinetic term (defined as presented in equation 11), which links solid and fluid phases. A single pseudo-component is used to represent the extract collectively.

$$\frac{\partial c_s(t,z)}{\partial t} = \underbrace{r_e(t,z)}_{\text{Kinetics}} \tag{10}$$

2.4.4. Kinetic term

The kinetic term is based on two-film theory and follow work of Reverchon [1]. The mass transfer kinetic (equation 11) consists of the overall diffusion coefficient and the concentration gradient, which acts as a driving force for the process.

As the solvent flows through the bed, the CO_2 molecules diffuse into the pores and adsorb on the particle surface to for an external fluid film around the solid particles through the solvent–solid matrix interactions. Assuming that the mean free path of the molecule is much smaller than the pore diameter, the effect of Knudsen diffusion is small and can be neglected. The dissolved solute diffuses from the particle's core through the solid-fluid interface, the pore, and the film into the bulk. The graphical representation of the mass transfer mechanism is shown in Figure 6. The mean solute concentration in the solid phase is denoted as c_s. At the solid-fluid interface, the equilibrium concentrations are given as c_s^* and c_P^*, respectively for solid and fluid phases. The concentration of the solutes in the fluid phase in the centre of the pore is denoted as c_P. As the solute diffuses through the pore, its concentration changes and reaches c_{Pf} at the opening of the pore. The solute diffuses through the film around the particle and reaches a concentration in the bulk c_f. It can be assumed that the two-film theory describes the solid-fluid interface inside the pore. The overall mass transfer coefficient can be introduced if the relation between the solute concentration in one phase and its equilibrium concentration is known.

Bulley et al. [17] suggest a process where the driving force for extraction is given by the difference between concentration of the solute in the bulk, c_f, and in the centre of the pore, c_P^*. The concentration c_P^* is in equilibrium with

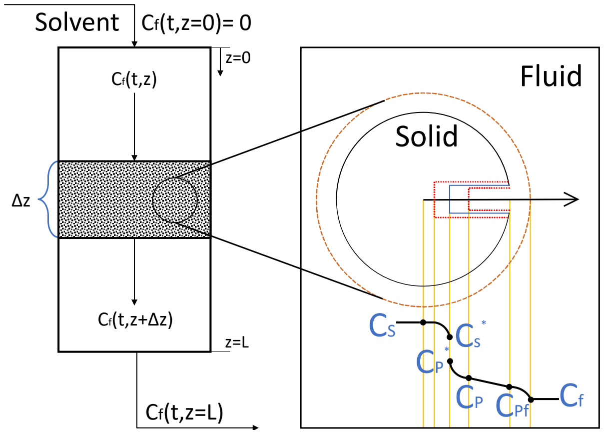


Figure 6: The extraction mechanism

c_s according to an equilibrium relationship. The rate of extraction is thus $r_e\left(c_f - c_P^*(c_s)\right)$.

On the other hand, Reverchon [1] proposes a driving force given by the difference between c_s and c_P^*. c_P^* is determined by an equilibrium relationship with c_f and the extraction rate is $r_e\left(c_s - c_P^*(c_f)\right)$ or more precisely

$$r_e(t,z) = \frac{D_i(T(t,z),P(t))}{\mu l^2}\left(c_s(t,z) - c_P^*(t,z)\right) \tag{11}$$

where μ is sphericity, l a characteristic dimension of particles and can be defined as $l = r/3$, r is the mean particle radius, ρ_s is the solid density, $D_i(T(t,z))$ corresponds to the overall diffusion coefficient and $c_P^*(t,z)$ is a concentration at the solid-fluid interface (which according to the internal resistance model is supposed to be at equilibrium with the fluid phase).

According to Bulley et al. [17], a linear equilibrium relationship (equation 12) can be used to find an equilibrium concentration of the solute in the fluid phase $c_f^*(t,z)$ is based on concentration of the solute in the solid phase $c_s(t,z)$

$$c(t,z) = k_p(T(t,z),P(t))q^*(t,z) \tag{12}$$

The volumetric partition coefficient $k_p(T(t,z),P(t))$ behaves as an equilibrium constant between the solute concentration in one phase and the corresponding equilibrium concentration at the solid-fluid interphase. According to Spiro and Kandiah [18], the term $k_p(T(t,z),P(t))$ can be expressed as the function of mass partition factor $k_m(T(t,z))$.

$$k_m(T(t,z)) = \frac{k_p(T(t,z),P(t))\rho_s}{\rho(T(t,z),P(t))} \tag{13}$$

Equation 14 represents of the kinetic term according to Reverchon [1]

$$r_e(t,z) = -\frac{D_i(T(t,z),P(t))}{\mu l^2}\left(c_s(t,z) - \frac{\rho_s}{k_m(T(t,z))\rho(T(t,z),P(t))}c_f(t,z)\right) \tag{14}$$

The above model does not take into account the saturation of fluid, which can be introduced by multiplying the gradient by a function $\gamma(c_f)$ (equation 15). $\gamma(c_f)$ describe the reverse logistic function, which is equal to unity below the c_{sat}, the saturation concentration, and equal to zero, above the c_{sat}. The $\gamma(c_f)$ for $c_{sat} = 5$ is shown on figure 7.

$$\gamma(c_f) = \frac{1}{1 + \exp\left(-k_{sat}\left(c_f - c_{sat}\right)\right)} \tag{15}$$

where k_{sat} is the growth rate, and it is defined as $k_{sat} = 2c_{sat}$.

Explain why the growth rate is defined as it is

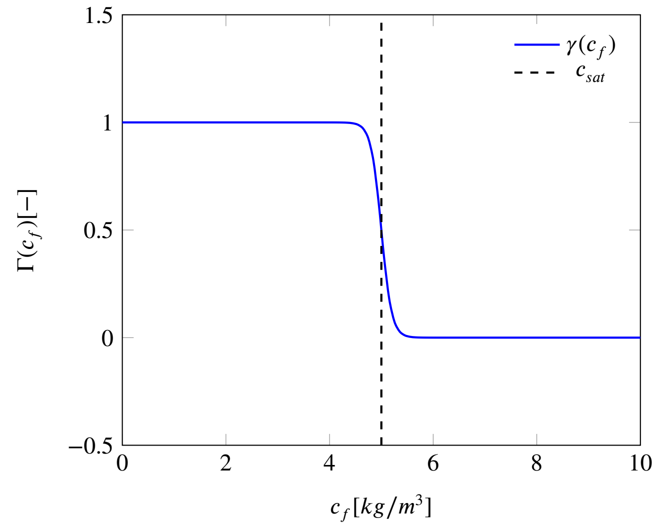


Figure 7: $\gamma(c_f)$ function under assumption of $c_{sat} = 5[kg/m^3]$

The final form of the extraction kinetic equation is given by equation 16.

$$r_e(t,z) = -\frac{D_i(T(t,z),P(t))}{\mu l^2}\gamma(c_f)\left(c_s(t,z) - \frac{\rho_s}{k_m(T(t,z))\rho(T(t,z),P(t))}c_f(t,z)\right) \tag{16}$$

2.4.5. Heat balance

I will be back. The energy equation is not given as a in terms of temperature any more.

The heat balance (equation 17) consists of the convective and diffusive terms. It follows the assumption of a pseudo-homogeneous phase, which properties are the mean between fluid and solid phases. We consider no heat loss through the wall, and there is no heat generation in the system. Therefore, the temperature of the extractor can be changed only by manipulating the temperature of the inlet stream

We assume that at a given section, where the cross-sectional area is A, the flow properties are uniform across that section. Hence, although the area of an extractor changes as a function of a distance along an extractor (e.g. if a fixed fill an extractor partially), z, and therefore in reality the flow field is two-dimensional (the flow varies in the two-dimensional space), we make the assumption that the flow properties vary only with z; this is tantamount to assuming uniform properties across any given cross section. Such flow is defined as quasi-one-dimensional flow.

~~The heat balance equation (equation 17) was developed, assuming the existence of a pseudo-homogeneous phase, which properties are the mean between fluid and solid phases (the amount of solute is considered small enough not to~~

~~affect the overall heat balance). Equation 17 contains the convection and the diffusion terms. It is considered that there is no heat loss through the wall, and there is no heat generation in the system. The temperature of the extractor can be changed only by increasing the temperature of the inlet stream. The pseudo-homogenous phase is assumed to flow only in the axial direction. The numerator of the factor in front of the convection term of the heat equation contains only the specific heat of the fluid~~ $\cancel{C_p(T(t,z),P(t))}$ ~~because the solid phase is stationary. Therefore, this factor can be understood as the fraction of the fluid's total heat through convection. On the other hand, the axial heat diffusion is calculated based on the definition of thermal diffusivity for the fluid, as explained in the appendix.~~

The heat balance (equation 17), is based on Srinivasan and Depcik [19], and consists of convective and diffusive terms. It follows the assumption of a pseudo-homogeneous phase, which properties are the mean between fluid and solid phases. It is considered that there is no heat loss through the walls, and there is no heat generation in the system. The temperature of the extractor can be changed only by manipulating the temperature of the inlet stream $T_{Inlet}(t)$.

$$\frac{\partial T(t,z)}{\partial t} = \underbrace{-\frac{F(t)C_p(T(t,z),P(t))}{A[(1-\epsilon)\rho(T(t,z),P(t))C_p(T(t,z),P(t))+\epsilon\rho_s C_{ps}]}\frac{\partial T(t,z)}{\partial z}}_{\text{Convection}} + \underbrace{D_e^T(T(t,z),P(t))\frac{\partial^2 T(t,z)}{\partial z^2}}_{\text{Diffusion}} \tag{17}$$

where $D_e^M(T(t,z),P(t),F(t))$ is the axial mass diffusion coefficient, $C_p(T(t,z),P(t))$ is the fluid's specific heat, C_{ps} is the specific heat of the solid phase, $D_e^T(T(t,z),P(t),F(t))$ is the axial heat diffusion coefficient.

The heat equation was introduced in the previous chapter. The heat balance describe movement of the internal energy in the system. For real gases it is complicated to write the heat balance in terms of temperature. Alternatively, the temperature can be obtained based on the equation of state if the internal energy and pressure are know. A rootfinder can be used to find a value of temperature, which satisfy the equation of state given values of internal energy and pressure. The temperature needs to be reconstructed from the internal energy in every time-step.

Give reference for measurement function

2.4.6. Extraction yield

The efficiency of the process (the yield) is calculated according to equations 18 to 19, which evaluate the mass of solute at the exit of the extraction unit and sums it. The integral form of the measurement equation can be transformed into the differential form, and augmented with model equations to be solved simultaneously.

$$y(t)\,[kg] = \int_{t_0}^{t_f} \frac{F(t)}{\rho(t,z)}\left[\frac{kg}{s}\left(\frac{kg}{m^3}\right)^{-1}\right] c_f(t,z)\Big|_{z=L}\left[\frac{kg}{m^3}\right] dt\,[s] \tag{18}$$

$$\frac{dy}{dt}\left[\frac{kg}{s}\right] = \frac{F(t)}{\rho(t,z)}\left[\frac{kg}{s}\left(\frac{kg}{m^3}\right)^{-1}\right] c_f(t,z)\Big|_{z=L}\left[\frac{kg}{m^3}\right] \tag{19}$$

2.5. Parameter estimation

The same symbol as void fraction

Conceptually, the unobservable error $\epsilon(t)$ is added to the deterministic model output, $y(t)$ (equation 18), to give the observable dependent variable $Y(t)$ (for example results of an experiments). For discrete observations:

$$Y(t_i) = y(t_i) + \epsilon(t_i)$$

and for continuos variables:

$$Y(t) = y(t) + \epsilon(t)$$

The objective in parameter estimation is to obtain the "best" estimate of θ based on the continuous observations $Y(t)$ or the discrete observations $Y(t_i)$. The above equation can be written in general form as:

$$Y(t_i) = y(\theta, t_i) + \epsilon(t_i) \tag{20}$$

where θ is set of parameters to be estimated from the data. θ is a subset of parameter space Θ, which contain all parameters of the model.

Because of the difficulty of obtaining analytical solutions to the deterministic process model, experiments have been arranged whereby the vector of derivatives $dY(t_i)/dt$ is measured rather than $Y(t_i)$ itself. In such cases, it is assumed that the unobservable error is added to the deterministic derivative $dy(\theta,t_i)/dt$ (equation 19) as follows:

$$\frac{dY(t_i)}{dt} = \frac{dy(\theta,t_i)}{dt} + \epsilon(t_i)$$

If unobservable error in the first observation is designated as ϵ_1, the second error observation ϵ_2' includes ϵ_1 plus a random component introduced side from ϵ_1, or $\epsilon_2' = \epsilon_1 + \epsilon_2$. The error in the third observation is $\epsilon_3' = \epsilon_1 + \epsilon_2 + \epsilon_3$, and so forth. Mandel [20] distinguished between the usually assumed type of independent measuring error in the dependent variable and a 'cumulative' or interval error in which each new observation includes the error of the previous observations. Cumulative errors, arising because of the fluctuations as a function of time in the process itself due to small changes in operating, are not independent - only the differences in measurement from one period to the next are independent.

2.6. Least squares estimation

A least squares parameter estimation does not require prior knowledge of the distribution of unobservable errors, yield unbiased estimates, and results in the minimum variance among all linear unbiased estimators. If the observations Y for the model responses are continuous functions of time from $t = 0$ to $t = t_f$, the Markov (or "rigorous least squares") criterion is to minimize:

$$\phi = \frac{1}{2}\int_0^{t_f} [Y(t) - y(\theta,t)]^T\, \Gamma^{-1}\, [Y(t) - y(\theta,t)]\, dt \tag{21}$$

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \phi \tag{22}$$

where Γ is the covariance matrix (or perhaps a matrix of appropriate weights), $\hat{\theta}$ is the parameter estimate, Θ is the parameter space, and ϕ is the time integrated value of the error squared ("integral squared error"). If the observations are made at discrete instants of time, t_i, $i = 1, 2, ..., n$, the Markov criterion is to minimize:

$$\phi = \frac{1}{2}\sum_{i=1}^{n}\left[Y(t_i) - y(\theta, t_i)\right]^T \Gamma^{-1}\left[Y(t_i) - y(\theta, t_i)\right] \tag{23}$$

$$\hat{\theta} = \arg\min_{\theta\in\Theta} \phi \tag{24}$$

If Γ is a diagonal matrix, ϕ becomes a "weighted least squares" criterion; if $\Gamma = \sigma^2\mathbf{I}$ where σ is a standard deviation, ϕ is the "ordinary least squares" criterion.

2.7. Maximum likelihood estimation

Maximum likelihood estimation (MLE) is a method of estimating the parameters of an assumed probability distribution, given some observed data. This is achieved by maximizing a likelihood function so that, under the assumed statistical model, the observed data is most probable. The MLE has the desirable characteristics of asymptotic efficiency and normality. Each time it has been associated with the (joint) normal distribution because of mathematical convenience. Consider the joint probability density function (the likelihood function) $p\left(\theta|y(t_1), y(t_2), ..., y(t_n)\right)$ for parameter θ. If a maximum of this function over all choices of θ can be found, the estimates so obtained are maximum likelihood estimates. The conditions at the maximum can be evolved incorporating prior information as follows.

The posterior probability density $p(\theta|y)$ can be expressed as the ratio of two probability densities if we make use of the analogue for continuous variables of Equation **??**:

$$p\left(\theta|y(t_n), ..., y(t_1)\right) = \frac{p\left(\theta, y(t_n), ..., y(t_1)\right)}{p\left(y(t_n), ..., y(t_1)\right)} \tag{25}$$

Notation is a bit unclear

The numerator of the right-hand side od Equation 25 using Equation **??** becomes

$$p\left(\theta, y(t_n), ..., y(t_1)\right) = p\left(y(t_n)|\theta, y(t_{n-1}), ..., y(t_1)\right) \cdot p\left(\theta, y(t_{n-1}), ..., y(t_1)\right) \tag{26}$$

These operations can be continued repetitively until we get

$$p\left(\theta, y(t_n), ..., y(t_1)\right) = p(\theta)\prod_{i=1}^{n} p\left(y(t_i)|\theta, y(t_{i-1}), ..., y(t_1)\right) \tag{27}$$

Examination of Equation 20 shows that $Y(t_i)$ depends only on t_i, θ and $\epsilon(t_i)$ and is not conditioned by any previous measurement. Consequently, we can write

$$p\left(y(t_i)|\theta, y(t_{i-1}), ..., y(t_1)\right) = p\left(y(t_i)|\theta\right) \tag{28}$$

provided Equation 20 is observed as a constraint. The desired joint conditional probability function is thus

$$p\left(\theta|y(t_n), ..., y(t_1)\right) = \frac{p(\theta)\prod_{i=1}^{n} p\left(y(t_i)|\theta\right)}{p\left(y(t_n), ..., y(t_1)\right)} \tag{29}$$

We can get rid of the evidence term $p\left(y(t_n), ..., y(t_1)\right)$ because it's constant with respect to the maximization. Moreover, if we are lacking a prior distribution over the quantity we want to estimate, then $p(\theta)$ can be omitted. In such a case:

$$p\left(\theta|y(t_n), ..., y(t_1)\right) = \prod_{i=1}^{n} p\left(y(t_i)|\theta\right) = \prod_{i=1}^{n} L\left(\theta|y(t_i)\right) \tag{30}$$

By collecting a values of y and selecting the values of θ that maximize the likelihood function $L(\theta|y)$, a function described in chapter **??** in connection with Bayes' theorem. Such estimators, $\hat{\theta}$, are known as maximum likelihood estimators. In effect, the methods selects those values of θ that are at least as likely to generate the observed sample as any other set of values of the parameters if the probability density of y were to be extensively simulated through use of the probability density $p(y|\theta)$. In making a maximum likelihood estimate, we assume that the form of the probability density (~~only the θ need be determined~~) and that all possible values of θ are equally likely before experimentation.

Is the reference it previous chapter needed?

Needed?

The likelihood function for the parameters based on several observations is the product of the individual functions if the observations are independent.

$$L\left(\theta|y(t_n), ..., y(t_1)\right) = \prod_{i=1}^{n} L\left(\theta|y(t_i)\right) = p\left(y(t_1)|\theta\right)p\left(y(t_2)|\theta\right)...p\left(y(t_n)|\theta\right) \tag{31}$$

In choosing as estimates of θ the values that maximize L for the given values $\left(y(t_i)\right)$, it turns out that it is more convenient to work with the $\ln L$ than with L itself:

$$\ln L = \ln p\left(y(t_1)|\theta\right) + \ln p\left(y(t_2)|\theta\right) + ... + \ln p\left(y(t_n)|\theta\right) = \sum_{i=1}^{n} \ln p\left(y(t_i); \theta\right) \tag{32}$$

By assuming that the conditional distribution of $\bar{Y}_i$, given y_i, is normal, then we form the likelihood function based on the probability density:

$$\begin{aligned} p\left(\theta, \sigma|y(t_n), ..., y(t_1)\right) &= \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma}}\exp\left[\frac{1}{2\sigma}\left(Y(t_i) - y(\theta, t_i)\right)^2\right] \\ L\left(\theta, \sigma|y(t_n), ..., y(t_1)\right) &= \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\sigma}}\exp\left[\frac{1}{2\sigma}\left(Y(t_i) - y(\theta, t_i)\right)^2\right] \end{aligned} \tag{33}$$

where σ is the variance

By taking the natural logarithm of the Equation 33, the final form of the objective function can be obtained:

$$\ln L = -\frac{n}{2}\left(\ln\sqrt{2\pi} + \ln\sigma^2\right) - \frac{\sum_{i=1}^{n}\left[Y(t_i) - y(\theta, t_i)\right]^2}{2\sigma^2} \tag{34}$$

The parameter estimation problem can be formulated as follow:

$$\begin{aligned} \hat{\theta}_{MLE} \quad &= \arg\max_{\sigma,\theta\in\Theta} \ln L = \arg\max_{\sigma,\theta\in\Theta} p(\theta|y) \\ \text{subject to} \quad & \dot{x} = f(t, x, \theta) \\ & \dot{\theta} = 0 \\ & y = y(x) \end{aligned} \tag{35}$$

x has not been defined

Based on the first order optimality condition, the $\ln L$ can be maximized with respect to the vector θ by equating to zero the partial derivatives of $\ln L$ with respect to each of the parameters:

$$\frac{\partial \ln L}{\partial\theta} = \frac{\partial\sum_{i=1}^{n}\ln p\left(y(t_i)|\theta\right)}{\partial\theta} = 0 \tag{36}$$

Solution of Equations 36 yield the desired estimates $\hat{\theta}$. For some models, these equations can be explicitly solved for $\hat{\theta}$ but in general no closed-form solution to the maximization problem is known or available, and an MLE can only be found via numerical optimization.

3. Results

4. Conclusions

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A. Appendix

A.1. Thermodynamic

A.1.1. Equation of state and properties of the fluid phase

We consider equations of state in the general form $P(t)V_{CO_2}[T(t,z),P(t)] = Z\,RT(t,z)$, where V_{CO_2} denotes the molar volume of CO_2, Z represents its compressibility factor, and R is the universal gas constant. More specifically, we are interested in these equations because of the possibility to express the compressibility Z as an explicit function of temperature and pressure. This is the case when Z is obtained as one of the physically meaningful roots of a polynomial equation, like the Peng-Robinson's and stuff with REFS to equations of state.

reference to Cardano formula

In Peng-Robinson's equation, the compressibility $\overline{Z}[T(t,z),P(t)]$ solves the third-order polynomial equation

$$Z^3 - [1 - B[T(t,z),P(t)]]\,Z^2 + \left[A[T(t,z),P(t)] - 2B[T(t,z),P(t)] - 3B[T(t,z),P(t)]^2\right] Z = 0, \tag{37}$$

where $A[T(t,z),P(t)]$ and $B[T(t,z),P(t)]$ are functions of time and space defined on the attraction parameter, $a[T(t,z)] = a^c_{CO_2}\,\alpha[T(t,z)]$ with $a^c_{CO_2} \approx 0.45724 R^2 {T^c_{CO_2}}^2 / P^c_{CO_2}$, and the repulsion parameter, $b_{CO_2} \approx 0.07780 RT^c_{CO_2}/P^c_{CO_2}$, both functions of the critical temperature $T^c_{CO_2}$ and pressure $P^c_{CO_2}$. Specifically, we have

$$A[T(t,z),P(t)] = \frac{\alpha[T(t,z)]\,a^c_{CO_2}P(t)}{R^2T^2(t,z)}; \tag{38a}$$

$$B[T(t,z),P(t)] = \frac{b_{CO_2}P(t)}{RT(t,z)}. \tag{38b}$$

The quantity $\alpha[T(t,z)] = \left[1 + \kappa_{CO_2}\left[1 - \sqrt{T(t,z)/T^c_{CO_2}}\right]\right]^2$, with constant $\kappa_{CO_2} = 0.37464 + 1.54226\omega_{CO_2} - 0.26992{\omega_{CO_2}}^2$, is a dimensionless correction term defined on the acentric factor $\omega_{CO_2} = 0.239$ of CO_2 molecules.

By denoting the physical constants as $\varphi_Z = \left(R, T^c_{CO_2}, P^c_{CO_2}, \kappa_{CO_2}\right)$, we obtain a spatio-temporal representation of the compressibility $\overline{Z}\left[T(t,z),P(t) \mid \varphi_Z\right]$ with its complete set of functional dependencies and parameters.

Density of the fluid phase

The density ρ_F of the fluid phase is assumed to be equal to the density of solvent, at given temperature and pressure. Because temperature $T(t,z)$ of the fluid phase is a modelled variable, we allow for the density to vary along the bed and in time. From an equation of state of the form $P(t)V_{CO_2}[T(t,z),P(t)] = Z\,RT(t,z)$, we get

$$\rho_F\left[T(t,z),P(t) \mid \varphi_{\rho_F}\right] = \frac{P(t)M_{CO_2}}{RT(t,z)\overline{Z}\left[T(t,z),P(t) \mid \varphi_Z\right]}, \tag{39}$$

where M_{CO_2} denotes the molar mass of CO_2 and $\overline{Z}[T(t,z),P(t)]$ is the compressibility factor that solves Eq. (37). The density of the fluid is thus a function of space and time, due to its dependence on temperature and pressure, and it is expressed in terms of the set of physical constants $\varphi_{\rho_F} = \left(R, M_{CO_2}, T^c_{CO_2}, P^c_{CO_2}, \omega_{CO_2}\right)$.

Heat capacity of the fluid phase

The specific heat C^F_p can be calculated from the equation of state, again under the assumption that the fluid phase consists of pure carbon dioxide and that the specific heat of real fluids can be calculated from an ideal contribution plus a residual term REF. In the following, we report only the main steps in the derivation of $C^{CO_2}_p[T(t,z),P(t)]$: Step-by-step derivations using the Peng-Robinson's equation are given in Appendix REF.

At given temperature and pressure, for CO_2 we have

$$C^{CO_2}_v[T(t,z),P(t)] = C^I_v[T(t,z),P(t)] + C^R_v[T(t,z),P(t)]; \tag{40a}$$

$$C^{CO_2}_p[T(t,z),P(t)] = \underbrace{C^I_p[T(t,z),P(t)]}_{\text{Eq. (41)}} + \underbrace{C^R_p[T(t,z),P(t)]}_{\text{Eq. (42)}}. \tag{40b}$$

$C^{CO_2}_v[T(t,z),P(t)]$ and $C^{CO_2}_p[T(t,z),P(t)]$ are the specific heat of CO_2 at constant volume and pressure, respectively. $C^I_v[T(t,z),P(t)]$ and $C^I_p[T(t,z),P(t)]$, with $C^I_p(T(t,z)) - C^I_v(T(t,z)) = R$, are the specific heat of an ideal gas at constant volume and pressure. $C^R_v[T(t,z),P(t)]$ and $C^R_p(T(t,z),P(t))$ are the correction terms.

For CO_2 REF, we have the ideal gas contribution to the specific heat at constant $P(t)$, as function of $T(t,z)$,

$$C^I_p[T(t,z),P(t)] = C_{P0} + C_{P1}T(t,z) + C_{P2}T^2(t,z) + C_{P3}T^3(t,z) \tag{41}$$

where the coefficients of the expansion are $C_{P0} = 4.728$, $C_{P1} = 1.75 \times 10^{-3}$, $C_{P2} = -1.34 \times 10^{-5}$, and $C_{P3} = 4.10 \times 10^{-9}$. For the correction term $C^R_p[T(t,z)P(t)]$ at constant pressure $P(t)$, we have

$$C^R_p[T(t,z),P(t)] = \underbrace{C^R_v[T(t,z),P(t)]}_{\text{Eq. (46)}} + T(t,z)\underbrace{\left(\frac{\partial P(t)}{\partial T}\right)_{V_{CO_2}(t,z)}}_{\text{Eq. (45)}}\underbrace{\left(\frac{\partial V_{CO_2}[T(t,z),P(t)]}{\partial T}\right)_{P(t)}}_{\text{Eq. (43)}} \tag{42}$$

The braced terms are obtained from the chosen equation of state $P(t)V[T(t,z),P(t)] = Z[T(t,z),P(t)]\,RT(t,z)$.

For the partial derivative of the volume with respect to temperature T at constant pressure $P(t)$, we have

$$\left(\frac{\partial V_{CO_2}\left[T(t,z),P(t)\right]}{\partial T}\right)_{P(t)} = \frac{Z\left[T(t,z),P(t)\right]R}{P(t)} + \frac{RT(t,z)}{P(t)}\underbrace{\left(\frac{\partial Z\left[T(t,z),P(t)\right]}{\partial T}\right)_{P(t)}}_{\text{Eq. (44)}} \tag{43}$$

with partial derivative of the compressibility factor with respect to temperature T at constant pressure $P(t)$

$$\left(\frac{\partial Z\left[T(t,z),P(t)\right]}{\partial T}\right)_{P(t)} = \left(\frac{\partial \dfrac{P(t)V_{CO_2}\left[T(t,z),P(t)\right]}{RT(t,z)}}{\partial T}\right)_{P(t)} \tag{44}$$

Similarly, for the partial derivative of the pressure with respect to temperature at constant volume, we have

$$\left(\frac{\partial P(t)}{\partial T}\right)_{V_{CO_2}(t,z)} = \left(\frac{\partial \dfrac{ZRT(t,z)}{V_{CO_2}\left[T(t,z),P(t)\right]}}{\partial T}\right)_{V(t,z)} \tag{45}$$

The residual specific heat at constant volume is obtained, by definition, by using the residual internal energy

$$C_v^{R}\left[T(t,z),P(t)\right] = \left(\frac{\partial U^{R}\left[T(t,z),P(t)\right]}{\partial T}\right)_{V(t,z)} . \tag{46}$$

By denoting the physical constants as $\varphi_{C_p^F} = \left(R, M_{CO_2}, T^c_{CO_2}, P^c_{CO_2}, \omega_{CO_2}\right)$, we get the spatio-temporal representation $C_p^F\left[T(t,z), P(t) \mid \varphi_{C_p^F}\right]$ of the specific heat of the fluid phase

Add internal energy equation

A.2. Governing equations

A.2.1. Mass continuity

Following the work of Anderson [14], the governing equations for compressible fluid with non-uniform cross-section can be obtained. Let's assume that any properties of the flow are uniform across any given cross-section of an extractor. The variation of the cross-section might be an result of partial filling of an extractor or its irregular shape. In reality, such a flow is two-dimensional, because with the area changing as a function of z, in actuality there will be flow-field variations in both directions. The assumption of quasi-one-dimensional flow dictates that the flow properties are function of z only. The equations described by quasi-one-dimensional assumption hold: (1) mass conservation, (2) Newton's second law, and (3) energy conservation. To ensure that these physical principles are satisfied the modified governing equation can be derived. Let's start with the integral form of the continuity equation:

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_f} \rho_f \, d\mathcal{V}_f + \iint_S \rho_f \mathbf{V}\cdot \mathbf{dS} = 0 \tag{47}$$

We apply this equation to the shaded control volume shown in Fig. 8.

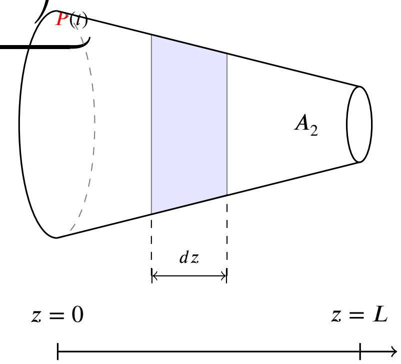

Figure 8: Control volume for deriving partial differential equation for unsteady, quasi-one-dimensional flow

This control volume is a slice of an extractor, where the infinitesimal thickens of the slice is dz. On the left side of the control volume, consistent with the quasi-one-dimensional assumptions, the density, velocity, pressure and internal energy denoted by ρ_f, V, P, and e, respectively, are uniform over the are A. Similarly, on the right side of the control volume, the density, velocity, pressure, and internal energy $\rho_f + d\rho_f$, $V + dV$, $P + dP$, and $e + de$, respectively, are uniform over the area available for fluid phase $A_f + dA_f$. Applied to the control volume in Fig. 8, the volume integral in Eq. 47 becomes, in the limit as dz becomes very small,

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_f} \rho_f \, d\mathcal{V}_f = \frac{\partial}{\partial t}\left(\rho_f A_f dz\right) \tag{48}$$

where $A\,dz$ is the volume of the control volume in the limit of dz becoming vanishingly small. The surface integral in Eq. 47 becomes

$$\iint_S \rho_f \mathbf{V}\cdot\mathbf{dS} = -\rho_f V A_f + (\rho_f + d\rho_f)(V + dV)(A_f + dA_f) \tag{49}$$

where the minus sign on the leading term on the right-hand side is due to the vectors $\mathbf{V}$ and $\mathbf{dS}$ pointing in opposite directions over the left of the control volume, and hence the dot product is negative. Expending the triple product term

$$\begin{aligned}\iint_S \rho_f \mathbf{V}\cdot\mathbf{dS} &= -\rho_f V A_f + \rho_f V A_f + \rho_f V dA_f + \rho_f A_f dV \\ &+ \rho_f dV dA_f + V A_f d\rho_f + V d\rho_f dA + A_f d\rho_f dV + d\rho_f dV dA_f\end{aligned} \tag{50}$$

In the limit as dz becomes very small, the terms involving products of the differential in Eq. 50, sych as $\rho_f dV dA_f$, $d\rho_f dV dA_f$, go to zero much faster than those terms involving only one differential. Hence, all terms involving products of differentials can be dropped, yielding in the limit as dz becomes very small

$$\iint_S \rho_f \mathbf{V}\cdot\mathbf{dS} = \rho_f V dA_f + \rho_f A_f dV + V A_f d\rho_f \tag{51}$$

Substituting Eqs. 48 and 51 into 47, we have

$$\frac{\partial\left(\rho_f A_f\right)}{\partial t}+\frac{\partial\left(\rho_f A_f V\right)}{\partial z}=0 \tag{52}$$

Above partial differential equation form of the continuity equation suitable for unsteady, quasi-one-dimensional flow. It ensures that mass is conserved for this mode of the flow. The $A_f(z)$ is an arbitrary function, which describe change of the cross-section of an extractor. The function $A_f(z)$ can be defined as $A_f(z)=\mathbf{A}\varepsilon(z)$, where ε is the bed porosity and $\mathbf{A}$ is the cross-section of an empty extractor.

$$\frac{\partial\left(\rho_f \mathbf{A}\varepsilon(z)\right)}{\partial t}+\frac{\partial\left(\rho_f \mathbf{A}\varepsilon(z) V\right)}{\partial z}=0 \tag{53}$$

The equation can be simplified by cancel out a constant $\mathbf{A}$

$$\frac{\partial\left(\rho_f \varepsilon(z)\right)}{\partial t}+\frac{\partial\left(\rho_f \varepsilon(z) V\right)}{\partial z}=0 \tag{54}$$

If so called superficial velocity is defined $u=\varepsilon V$, the mass continuity becomes

$$\frac{\partial\left(\rho_f \varepsilon(z)\right)}{\partial t}+\frac{\partial\left(\rho_f u\right)}{\partial z}=0 \tag{55}$$

A.2.2. *Transport of a species*

The transport of a chemical species, in this case a solute, can be described by analogous equation to the Eq. 47 with additional terms on the right-hand side. The first term on the right-hand side describes that a substance goes from high density regions to low density regions and is based on the Fick's law $\left(J_{diff}=D\frac{\partial C_f}{\partial z}\right)$. The other term correspond for the mass transfer between solid and fluid phases, which is treated as a source term.

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_f} C_f d\mathcal{V}_f+\iint_S C_f\mathbf{V}\cdot\mathbf{dS}=\iint_S J_{diff}\cdot\mathbf{n}\,\mathbf{dS}+\frac{\partial}{\partial t}\iiint_{\mathcal{V}_s} C_s d\mathcal{V}_s \tag{56}$$

Similarly to the continuity equation, in the limit as dz becomes very small

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_f} C_f d\mathcal{V}_f=\frac{\partial}{\partial t}\left(C_f A_f dz\right) \tag{57}$$

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_s} C_s d\mathcal{V}_s=\frac{\partial}{\partial t}\left(C_s A_s dz\right) \tag{58}$$

The surface integrals in the limit of dz becomes

$$\iint_S C_f\mathbf{V}\cdot\mathbf{dS}=C_f V dA_f+C_f A_f dV+VA_f dC_f \tag{59}$$

From the Divergence theorem in multi-variable calculus, we have

$$\iint_S J_{diff}\cdot\mathbf{n}\,\mathbf{dS}=\iiint_{\mathcal{V}_f}\nabla J_{diff}d\mathcal{V}_f=\nabla\iiint_{\mathcal{V}_f}J_{diff}d\mathcal{V}_f=\nabla\left(J_{diff}A_f dz\right) \tag{60}$$

By substituting the equations derived above into Eq. 56 we obtain

$$\frac{\partial\left(C_f A_f\right)}{\partial t}+\frac{\partial\left(C_f A_f V\right)}{\partial x}=\frac{\partial\left(C_s A_s\right)}{\partial t}+\frac{\partial\left(J_{diff}A_f\right)}{\partial z} \tag{61}$$

By defining $A_f=A\cdot\varepsilon$, $A_s=A\cdot(1-\varepsilon)$ and $u=V\cdot\varepsilon$, and assuming that A is constant, the above equation becomes

$$\frac{\partial\left(C_f\varepsilon\right)}{\partial t}+\frac{\partial\left(C_f u\right)}{\partial x}=\frac{\partial\left(C_s(1-\varepsilon)\right)}{\partial t}+\frac{\partial\left(J_{diff}\varepsilon\right)}{\partial z} \tag{62}$$

By expanding above equation, splitting variable and assuming that $\frac{\partial\varepsilon}{\partial t}=0$ we get

$$\frac{\partial C_f}{\partial t}+\frac{u}{\varepsilon}\frac{\partial C_f}{\partial z}+\frac{C_f}{\varepsilon}\frac{\partial u}{\partial z}=\frac{1-\varepsilon}{\varepsilon}\frac{\partial C_s}{\partial t}+\frac{D}{\varepsilon}\frac{\partial C_f}{\partial z}\frac{\partial\varepsilon}{\partial z}+\frac{\partial}{\partial z}\left(D\frac{\partial C_f}{\partial z}\right) \tag{63}$$

The equation can be further simplified if $\frac{\partial u}{\partial z}=\frac{\partial\varepsilon}{\partial z}=D=0$, which corresponds to the assumptions of constant velocity along the bed(which might be a case of isothermal and low-mach number flow), constant porosity(which comes from the assumption of constant area for both solid and fluid phase) and no radial diffusion.

$$\frac{\partial C_f}{\partial t}+\frac{u}{\varepsilon}\frac{\partial C_f}{\partial z}=\frac{1-\varepsilon}{\varepsilon}\frac{\partial C_s}{\partial t} \tag{64}$$

The Eq. 64 is equivalent to the equation presented by Reverchon [1].

A.2.3. *Momentum conservation*

Similarly to the mass conservation, the momentum conservation is derived for inviscid fluid with no body forces

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_f}\left(\rho_f V_z\right)d\mathcal{V}_f+\iint_S\left(\rho_f V_z\mathbf{V}\right)\mathbf{dS}=\iint_S(PdS)_z \tag{65}$$

where V_z is the z component of the velocity.

We the momentum conservation to the shaded control volume in Fig. 8, the integrals on the left side are evaluated in the same manner as discussed above in the regard to the continuity equation. That is,

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}_f}\left(\rho_f V_z\right)d\mathcal{V}_f=\frac{\partial}{\partial t}\left(\rho_f VA_f dz\right) \tag{66}$$

equation

and

$$\iint_S\left(\rho_f V_z\mathbf{V}\right)dS=-\rho_f V^2+\left(\rho_f+d\rho_f\right)(V+dV)^2(A+dA) \tag{67}$$

The evaluation of the pressure force term on the right side of Eq. 65 can be understood based on the Fig. 9. Here, the z components of the vector PdS are shown on all four side of the control volume. Remember that $\mathbf{dS}$ is assumed to points away from the control volume; hence any z component $(PdS)_z$ that acts toward the left (in the negative z direction) is a negative quantity, and any z component that acts toward the right (in the positive z direction) is a positive quantity. Also note that the z component of $P\mathbf{dS}$ acting on the top and the bottom inclined faces of the control volume in Fig. 9 can be expressed as the pressure P acting on the component of the inclined are projected perpendicular to the z direction, $dA_f/2$; hence, the contribution of each inclined face (top or bottom) to the pressure integral in Eq. 65 is $-P(dA_f/2)$. All together, the right-hand side of Eq. 65 is expressed as follows:

$$\iint(PdS)_z=-PA_f+(P+dP)(A+dA_f)-2P\frac{dA_f}{2} \tag{68}$$

Substituting Eqs. 66 to 68 into Eq. 65, we have

$$\frac{\partial}{\partial t}\left(\rho_f VA_f dz\right)-\rho_f V^2A_f+(\rho_f+d\rho_f)(V+dV)^2(A_f+dA_f)$$

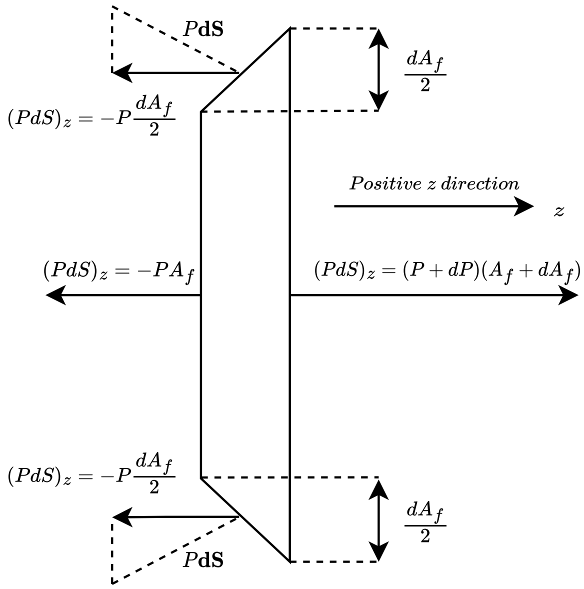

Figure 9: The forces in the z direction acting on the control volume

$$= PA_f - (P + dP)(A + dA_f) + PdA_f \tag{69}$$

Cancelling like terms and ignoring products of differentials, equation above becomes in the limit dz becoming very small

$$\frac{\partial}{\partial t}\left(\rho_f V A_f dz\right) + d\left(\rho_f V^2 A_f\right) = -AdP \tag{70}$$

Dividing above equation by dz and taking the limit as dz goes to zero, we obtain

$$\frac{\partial\left(\rho_f V A_f\right)}{\partial t} + \frac{\partial\left(\rho_f V^2 A_f\right)}{\partial z} = -A_f\frac{\partial P}{\partial z} \tag{71}$$

The Eq. 71 can be expanded further by assuming that $A_f = \mathbf{A}\varepsilon$

$$\frac{\partial\left(\rho_f V \mathbf{A}\varepsilon\right)}{\partial t} + \frac{\partial\left(\rho_f V^2 \mathbf{A}\varepsilon\right)}{\partial z} = -\mathbf{A}\varepsilon\frac{\partial P}{\partial t} \tag{72}$$

The equation can be further simplified by assuming that the cross-section of an extractor $\mathbf{A}$ is constant and cancel out

$$\frac{\partial\left(\rho_f V \varepsilon\right)}{\partial t} + \frac{\partial\left(\rho_f V^2 \varepsilon\right)}{\partial z} = -\varepsilon\frac{\partial P}{\partial t} \tag{73}$$

If the superficial velocity $u = \varepsilon V$ is introduced, then the momentum conservation becomes

$$\frac{\partial\left(\rho_f u\right)}{\partial z} + \frac{\partial\left(\rho_f u^2/\varepsilon\right)}{\partial z} = -\varepsilon\frac{\partial P}{\partial z} \tag{74}$$

Eq. 71 represents the conservative form of the momentum equation for the quasi-one-dimensional flow. The equivalent non-conservative form can be obtained by multiplying the continuity equation by V and subtracting it from Eq. 71

$$\frac{\partial\left(\rho_f V A_f\right)}{\partial t} - V\frac{\partial\left(\rho_f A_f\right)}{\partial t} + \frac{\left(\rho_f V^2 A_f\right)}{\partial z} - V\frac{\left(\rho_f V A_f\right)}{\partial z} = -A_f\frac{\partial P}{\partial z} \tag{75}$$

Expanding the derivatives on the left-hand size of above equation and cancelling like terms, gives

$$\rho_f A_f\frac{\partial V}{\partial t} + \rho_f A_f V\frac{\partial V}{\partial z} = -A_f\frac{\partial P}{\partial z} \tag{76}$$

Dividing above equation by A_f the non-conservative form of the momentum can be obtained

$$\rho_f\frac{\partial V}{\partial t} + \rho_f V\frac{\partial V}{\partial z} = -\frac{\partial P}{\partial z} \tag{77}$$

The Eq. 77 is stylistically the same as the general momentum conservation for one-dimensional flow with no-body forces. The momentum equation can be expressed in terms of superficial velocity $u = V\varepsilon$.

$$\rho_f\frac{\partial(u/\varepsilon)}{\partial t} + \rho_f\frac{u}{\varepsilon}\frac{\partial(u/\varepsilon)}{\partial z} = -\frac{\partial P}{\partial z} \tag{78}$$

By expanding all the terms of equation above, we get

$$\frac{\rho_f}{\varepsilon}\frac{\partial u}{\partial t} + \rho_f u\frac{\partial\varepsilon^{-1}}{\partial t} + \rho_f\frac{u}{\varepsilon}\frac{1}{\varepsilon}\frac{\partial u}{\partial z} + \rho_f\frac{u}{\varepsilon}u\frac{\partial\varepsilon^{-1}}{\partial z} = -\frac{\partial P}{\partial z} \tag{79}$$

If the bed is not compressible and doesn't change its properties during the batch, then $\frac{\partial\varepsilon}{\partial t} = 0$

$$\frac{\rho_f}{\varepsilon}\left(\frac{\partial u}{\partial t} + \frac{u}{\varepsilon}\frac{\partial u}{\partial z} + u^2\frac{\partial\varepsilon^{-1}}{\partial z}\right) = -\frac{\partial P}{\partial z} \tag{80}$$

If the porosity is constant along an extractor, then the momentum conservation equation becomes

$$\frac{\rho_f}{\varepsilon}\left(\frac{\partial u}{\partial t} + \frac{u}{\varepsilon}\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial z} \tag{81}$$

The Eq. 81 represents non-conservative form of the momentum equation for quasi-one-dimensional flow with no body forces and constant porosity.

A.2.4. Energy conservation

Let's consider the integral form of the energy equation for adiabatic flow with no body forces and no viscous effects

$$\frac{\partial}{\partial}\iiint_{\mathcal{V}_f}\rho_f\left(e_f + \frac{V^2}{2}\right)d\mathcal{V}_f + \iint_S \rho_f\left(e_f + \frac{V^2}{2}\right)\mathbf{V}\cdot\mathbf{dS} = -\iint_S (P\mathbf{V})\cdot\mathbf{dS} \tag{82}$$

Applied to the shaded control volume in Fig. 8, and keeping in mind the pressure forces shown in Fig. 9, Eq. 82 becomes

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho_f\left(e_f + \frac{V^2}{2}\right)A_f dz\right] - \rho_f\left(e_f + \frac{V^2}{2}\right)VA_f \\ &+ \left(\rho_f + d\rho_f\right)\left[e_f + de_f + \frac{(V + dV)^2}{2}\right](V + fV)\left(A_f + dA_f\right) \\ &= -\left[-PVA_f + (P + dP)(V + dV)\left(A_f + dA_f\right) - 2\left(PV\frac{dA_f}{2}\right)\right]\end{aligned} \tag{83}$$

Neglecting products of differential and cancelling like terms, the above equation becomes

$$\frac{\partial}{\partial t}\left[\rho_f\left(e_f + \frac{V^2}{2}\right)A_f dz\right] + d\left(\rho_f r_f A_f\right) + \frac{\left(\rho_f V^3 A_f\right)}{2} = -d\left(PA_f V\right) \tag{84}$$

or

$$\frac{\partial}{\partial t}\left[\rho_f\left(e_f + \frac{V^2}{2}\right)A_f dz\right] + d\left[\rho_f\left(e_f\frac{V^2}{2}\right)VA_f\right] = -d\left(PA_f V\right) \tag{85}$$

Taking the limit as dz approaches zero, the equation above becomes the following partial differential equation

$$\frac{\partial\left[\rho_f\left(e_f+V^2/2\right)A\right]}{\partial t}+\frac{\partial\rho_f\left(e_f+V^2/2\right)VA_f}{\partial z}=-\frac{\partial\left(PA_fV\right)}{\partial z} \quad (86)$$

Equation 86 is the conservation form of the energy expressed in terms of the total energy $e+V^2/2$, appropriate for unsteady, quasi-one-dimensional flow. The energy equation can be expressed in terms of internal energy if Eq. 71 is multiplicated by V and then subtracted from Eq. 86

$$\frac{\partial\left(\rho_f e_f A_f\right)}{\partial t}+\frac{\partial\left(\rho_f e_f V A_f\right)}{\partial z}=-P\frac{\partial A_f V}{\partial z} \quad (87)$$

The equation above is the conservation form of the energy equation expressed in terms of internal energy e_f suitable for quasi-one-dimensional flow. The non-conservative for is then obtained by multiplying the continuity equation 52, by e_f and substructing it from 87, yielding

$$\rho_f A_f\frac{\partial e_f}{\partial t}+\rho_f A_f V\frac{\partial e_f}{\partial z}=-P\frac{\partial(A_f V)}{\partial z} \quad (88)$$

Expanding right-hand side and dividing by A_f, the above equation becomes

$$\rho_f\frac{\partial e_f}{\partial t}+\rho_f V\frac{\partial e_f}{\partial z}=-P\frac{V}{A_f}\frac{\partial A_f}{\partial z} \quad (89)$$

or

$$\rho_f\frac{\partial e_f}{\partial t}+\rho_f V\frac{\partial e_f}{\partial z}=-P\frac{\partial V}{\partial z}-PV\frac{\partial(ln\ A_f)}{\partial z} \quad (90)$$

Equation 90 is the non-conservative for of the energy equation expressed in therms of internal energy, appropriate to unsteady quasi-one-dimensional flow. The reason for obtaining the energy equation in the form of Eq. 90 is that, for a calorically perfect gas, it leads directly to a form of the energy equation in therms of temperature T. For calorically perfect gas $e_f = C_v T$

$$\frac{\partial\left(\rho_f e_f A_f\right)}{\partial t}+\frac{\partial\left(\rho_f e_f V A_f\right)}{\partial z}=-P\frac{\partial A_f V}{\partial z}+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \quad (91)$$