SF2812 Project Group: 2B4

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Control of a mobile robot

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1 Abstract

The problem solved in this report is a mixed-integer linear program for a mobile robot moving in 2D-space. The aim is to define the control of a robot.Therefore, we are minimizing the total time steps that steer the robot from rest in the origin to the given set in plane. The constraints in the system are the accelerations in the corresponding coordinate directions at the given time step and binary variables checking whether the robot has reached the given set in plane at time step with the given sampling interval. The problem is modelled in the way that optimal control strategy will be able to steer the robot from the rest in the origin to the given set in minimal time by summing binary variables. Formulating and solving the problem in GAMS results in a minimum time steps of 4. If we decrease the sampling time h and control signal limit α the solution can be changed depending what we keep constant and variable. And when the movement of robot is further constrained in a defined path , then the optimal time steps to reach the given set in plane from origin is 6.

2 Background and problem description

There is a described movement of mobile robot in $2D$ space (plane given by coordinates by z_1 and z_2). Our goal - to find a control of mobile robot.

For the movement description we use 4-dimensional vector $x = (z_1, \dot{z_1}, z_2, \dot{z_2})^T$, where $\dot{z_1}, \dot{z_2}$ are time derivatives of corresponding coordinates.

For the state vector at time step k is used a time-discretized model as following:
$x(k) = (x_1(k), x_2(k), x_3(k), x_4(k))^T, \quad k = 0; 1; ...$

$x_1(k)$ corresponds to the z_1-coordinate at time kh;
$x_2(k)$ - the velocity in the z_1- direction at time kh;
$x_3(k)$ corresponds to the z_2-coordinate at time kh;
$x_4(k)$ - the velocity in the z_2- direction at time kh; here h is the sampling interval.

Therefore, we can write down state equation for the robot considering the time-discrete model:

$$x(k+1) = \Phi x(k) + \Gamma u(k); \quad u(k) \in U;$$

where

$$\Phi = \begin{pmatrix} 1 & h & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} h^2 & 0 \\ h & 0 \\ 0 & h^2 \\ 0 & h \end{pmatrix}$$

where h - the sampling time and U - control signal limitations and given as following :

$$U = \{u = [\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}] : -\alpha \leq u_j \leq \alpha, \quad \alpha > 0, \quad j = 1, 2\}$$

The interpretation given above is the time-discrete control of the system, thus we can rewrite it as following:

$$\ddot{z_1} = u_1, \quad -\alpha \leq u_1 \leq \alpha, \quad \ddot{z_2} = u_2, \quad -\alpha \leq u_2 \leq \alpha.$$

Therefore, the movement of our robot is described by Newton's equation in z_1-direction and z_2-direction respectively.

Based on above information we are assigned to define:
1) A control signal(for $h = 1$ and $\alpha = 2$) which in minimal time heads robot from rest in the origin to the set:

$$X_E = \{x : 5.9 \leq x_1 \leq 6.1; -0.1 \leq x_2 \leq 0.1; 6.9 \leq x_3 \leq 7.1; -0.1 \leq x_4 \leq 0.1\}$$

2) The solution change if we put less sampling time h and the control signal limit α .
3) While steering the robot from the rest in the origin to the set X_E it should move in the corridor defined in Figure 1 all the time.

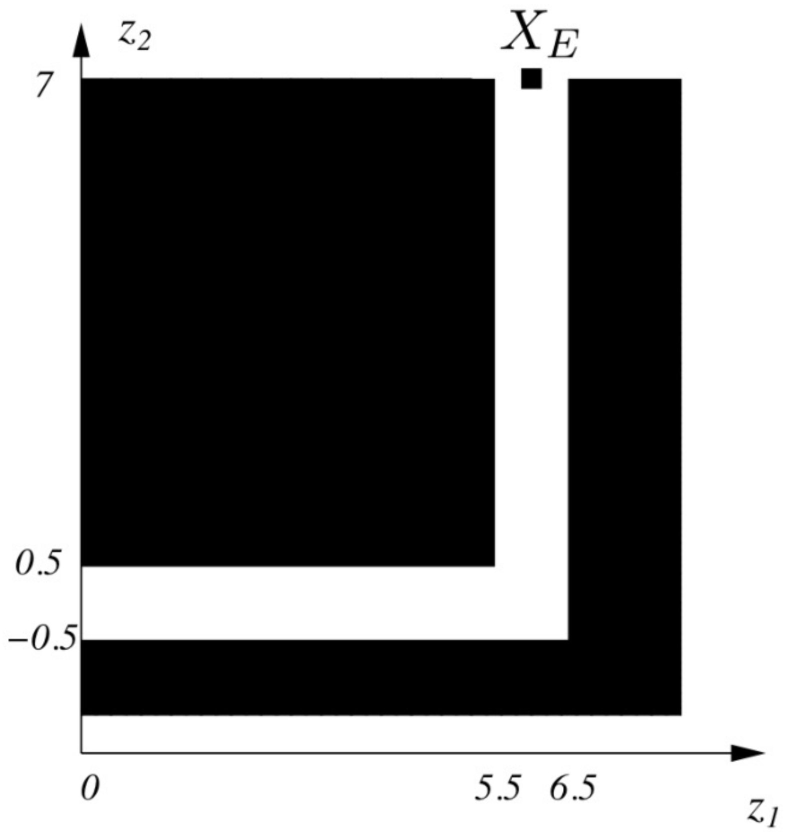

Figure 1: The robot has to stay in the corridor until reaching X_E

3 Mathematical formulation

This problem is formulated as a mixed integer program. An optimal control strategy should be made to steer the robot from rest in the origin:

$$X_O = \{x : x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0\}$$

to the set:

$$X_E = \{x : 5.9 \leq x_1 \leq 6.1, -0.1 \leq x_2 \leq 0.1, 6.9 \leq x_3 \leq 7.1, -0.1 \leq x_4 \leq 0.1\}$$

in the minimal time(i.e., in a minimal number of time steps).

A binary variable $r(k)$ is used to determine whether the robot has reached X_E at time kh (h is the sampling interval):

$$r(k) = \begin{cases} 0 & \text{the robot has reached } X_E \\ 1 & \text{the robot need more steps to reach } X_E \end{cases}$$

The total time steps should be:

$$T = \sum_k r(k)$$

In addition, the total time: $t = T \cdot h$. When h is fixed, to minimize t is to minimize T.

3.1 Constraints of basic exercise

From the problem description, the relationships between $x_i(k)$ (i = 1, 2, 3, 4; k = 1, 2, ...) can be modeled as:

$$\begin{aligned}
& x_1(k+1) = x_1(k) + x_2(k) \cdot h + u_1(k) \cdot \frac{h^2}{2} \\
& x_2(k+1) = x_2(k) + u_1(k) \cdot h \\
& x_3(k+1) = x_3(k) + x_4(k) \cdot h + u_2(k) \cdot \frac{h^2}{2} \\
& x_4(k+1) = x_4(k) + u_2(k) \cdot h \\
\text{where} \qquad & -\alpha \leq u_1(k) \leq \alpha \\
& -\alpha \leq u_2(k) \leq \alpha \\
& (k = 1, 2, ...)
\end{aligned}$$

When the robot is in X_E or the robot has once reached X_E but then go outside, $r(k) = 0$. To achieve that, $r(k)$ should be positive when $x(k) \notin X_E$ in the precondition that the robot has never reached X_E before (i.e. $r(k+1) \leq r(k)$).
The constraints of $r(k)$ can be modeled as:

$$
\begin{aligned}
&r(k) \geq \frac{x_{iloE} - x_i(k)}{cons} + r(k-1) - 1 \\
&r(k) \geq \frac{-x_{iupE} + x_i(k)}{cons} + r(k-1) - 1 \\
&(i = 1, 2, 3, 4; k = 1, 2, ...)
\end{aligned}
$$

Variable or index	Description
h	the sampling interval
α	a positive number
$cons$	a large enough constant
$x_1(k)$	the z_1-coordinate at time kh
$x_2(k)$	the velocity in the z_1-direction at time kh
$x_3(k)$	the z_2-coordinate at time kh
$x_4(k)$	the velocity in the z_2-direction at time kh
$u_1(k)$	the acceleration in the z_1-direction at time kh
$u_2(k)$	the acceleration in the z_2-direction at time kh
x_{iloE}	the lower limit of X_E for x_i
x_{iupE}	the upper limit of X_E for x_i
$r(k)$	whether the robot has reached X_E at time kh
T	the number of total time steps
t	the total time

Table 1: Definition of variables or indices

3.2 Mathematical model of basic exercise

The goal is to minimize the total time steps. Thus, the optimization problem can be formulated as (MIP):

$$
\begin{aligned}
(MIP) \qquad & \min \qquad T = \sum_k r(k) \\
& s.t. \qquad x_1(k+1) = x_1(k) + x_2(k) \cdot h + u_1(k) \cdot \frac{h^2}{2} \\
& \qquad\qquad x_2(k+1) = x_2(k) + u_1(k) \cdot h \\
& \qquad\qquad x_3(k+1) = x_3(k) + x_4(k) \cdot h + u_2(k) \cdot \frac{h^2}{2} \\
& \qquad\qquad x_4(k+1) = x_4(k) + u_2(k) \cdot h \\
& \qquad\qquad -\alpha \leq u_1(k) \leq \alpha \\
& \qquad\qquad -\alpha \leq u_2(k) \leq \alpha \\
& \qquad\qquad r(k) \geq \frac{x_{iloE} - x_i(k)}{cons} + r(k-1) - 1 \quad (i = 1, 2, 3, 4) \\
& \qquad\qquad r(k) \geq \frac{-x_{iupE} + x_i(k)}{cons} + r(k-1) - 1 \quad (i = 1, 2, 3, 4) \\
& \qquad\qquad (k = 1, 2, ...)
\end{aligned}
$$

$r(k)$ is a binary variable in (MIP).
The initial conditions are:

$$x_1(0) = 0, x_2(0) = 0, x_3(0) = 0, x_4(0) = 0$$

3.3 Constraints of advanced exercise

In the advanced exercise, more constraints are added to $x_1(k)$ and $x_3(k)$ because the robot can only walk in the corridor marked in Figure 1. The corridor can be divided into two parts:

$$X_A = \{x : 0.0 \leq x_1 \leq 6.5, -0.5 \leq x_3 \leq 0.5\}$$

$$X_B = \{x : 5.5 \leq x_1 \leq 6.5, -0.5 \leq x_3 \leq 7.1\}$$

The intersection of X_A and X_B is:

$$X_C = \{x : 5.5 \leq x_1 \leq 6.5, -0.5 \leq x_3 \leq 0.5\}$$

First, the robot should move to the set X_C in constraint of X_A, and then it will move to the set X_E in constraint of X_B. To achieve that, two more binary variables

are introduced which named $p_1(k)$ and $p_2(k)$.
$p_1(k)$ is used to determine whether the robot is moving in X_A and need more steps to reach X_C at time kh (h is the sampling interval):

$$p_1(k) = \begin{cases} 0 & \text{the robot is not in } X_A \text{ or has reached } X_C \\ 1 & \text{the robot is in } X_A \text{ and need more steps to reach } X_C \end{cases}$$

$p_2(k)$ is used to determine whether the robot is moving in X_B and need more steps to reach X_E at time kh (h is the sampling interval):

$$p_2(k) = \begin{cases} 0 & \text{the robot is not in } X_B \text{ or has reached } X_E \\ 1 & \text{the robot is in } X_B \text{ and need more steps to reach } X_E \end{cases}$$

From what has been discussed above:

$$r(k) = p_1(k) + p_2(k)$$

When the robot is in X_C or the robot has once reached X_C but then go outside, $p_1(k) = 0$. To achieve that, $p_1(k)$ should be positive when $p_1(k) \notin X_C$ in the precondition that the robot has never reached X_C before (i.e. $p_1(k+1) \leq p_1(k)$). The constraints of $p_1(k)$ can be modeled as:

$$\begin{aligned} & p_1(k) \geq \frac{x_{iloC} - x_i(k)}{cons} + p_1(k-1) - 1 \\ & p_1(k) \geq \frac{-x_{iupC} + x_i(k)}{cons} + p_1(k-1) - 1 \\ & (i = 1, 2, 3, 4; k = 1, 2, ...) \end{aligned}$$

Because the shape of X_A is rectangle, the robot will not go outside of X_A in the minimal time condition. When the robot is in $\{X_A - X_C\}$, $p_2(k)$ should be 0, so:

$$p_2(k) \leq \frac{x_1(k) - x_{1loC}}{cons} + 1$$

New variable or index	Description
$p_1(k)$	whether the robot has reached X_C in X_A at time kh
$p_2(k)$	whether the robot has reached X_E in X_B at time kh
x_{iloC}	the lower limit of X_C for x_i
x_{iupC}	the upper limit of X_C for x_i
x_{3lo}	the lower limit of corridor for x_3
x_{1up}	the upper limit of corridor for x_1

Table 2: Definition of new variables or indices

3.4 Mathematical model of advanced exercise

$$
\begin{aligned}
(MIP_2) \qquad \min \qquad & T = \sum_k r(k) \\
s.t. \qquad & x_1(k+1) = x_1(k) + x_2(k) \cdot h + u_1(k) \cdot \frac{h^2}{2} \\
& x_2(k+1) = x_2(k) + u_1(k) \cdot h \\
& x_3(k+1) = x_3(k) + x_4(k) \cdot h + u_2(k) \cdot \frac{h^2}{2} \\
& x_4(k+1) = x_4(k) + u_2(k) \cdot h \\
& -\alpha \le u_1(k) \le \alpha \\
& -\alpha \le u_2(k) \le \alpha \\
& p_1(k) \ge \frac{x_{iloC} - x_i(k)}{cons} + p_1(k-1) - 1 \quad (i = 1, 3) \\
& p_1(k) \ge \frac{-x_{iupC} + x_i(k)}{cons} + p_1(k-1) - 1 \quad (i = 1, 3) \\
& p_2(k) \le \frac{x_1(k) - x_{1loC}}{cons} + 1 \\
& r(k) = p_1(k) + p_2(k) \\
& r(k) \ge \frac{x_{iloE} - x_i(k)}{cons} + r(k-1) - 1 \quad (i = 1, 2, 3, 4) \\
& r(k) \ge \frac{-x_{iupE} + x_i(k)}{cons} + r(k-1) - 1 \quad (i = 1, 2, 3, 4) \\
& x_1(k) \le x_{1up} \\
& x_3(k) \ge x_{3lo} \\
& (k = 1, 2, ...)
\end{aligned}
$$

$r(k)$, $p_1(k)$, $p_2(k)$ are binary variables in (MIP_2).
The initial conditions are:

$$x_1(0) = 0, x_2(0) = 0, x_3(0) = 0, x_4(0) = 0$$

4 Results and analysis

4.1 Question 1

The mixed integer programming model with sampling time(h) equal to 1 and control signal constraint(α) equal to 2 provides an optimal time of 4 steps to move the robot from origin to the set X_E which implies the minimal time required is 4 and the corresponding control signal is as shown in table 1. The robot is free to move in the 2D space and it takes almost linear path as shown in the figure 2. and this can be analyzed analogous to the shortest distance between two points is a straight line.

Time step	u_1	u_2
1	2.000	1.650
2	0.975	2.000
3	-2.000	-2.000
4	-1.075	-1.075

Table 3: Optimal control signal corresponding to time steps

Time(kh)	x_1	x_2	x_3	x_4
0	0	0	0	0
1	1.000	2.000	0.825	1.650
2	3.487	2.975	3.475	3.650
3	5.463	0.975	6.125	1.650
4	5.900	-0.100	6.900	-0.100

Table 4: vector x at time kh

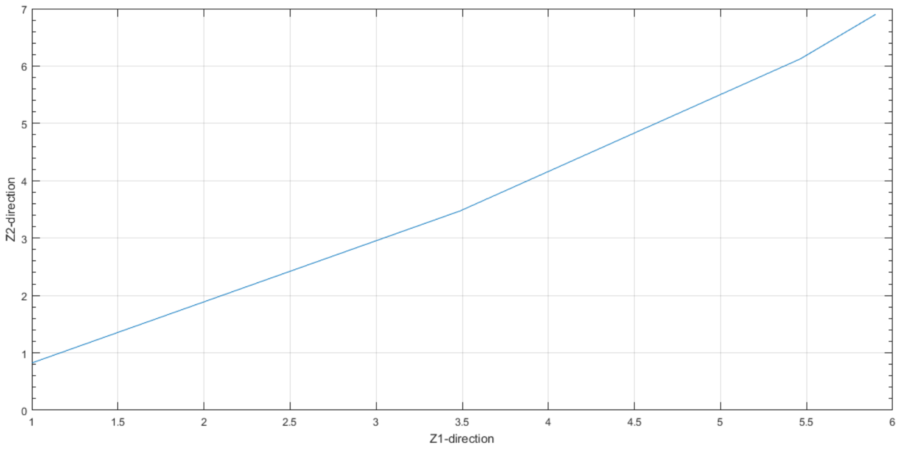

Figure 2: Optimal robot path

4.2 Question 2

When the sampling time h and control signal constraint α are decreased together from 1 and 2 respectively to any other combinations, the optimal time steps T will be unchanged or increased but the change of total time t ($t = T \cdot h$) is uncertain. More smaller the value in h and α requires more no.of steps k set in GAMS to calculate the optimal time or time steps for the problem.
If the sampling time h is decreased and keep α unchanging, the total time steps T will be unchanged or increased, which depends on how much h has been decreased. The total time t will be decreased since the robot's path of movement will be smoother when h is decreased.
If α is decreased and keep h unchanging, the total time steps T and the total time t will both be unchanged or increased, which depends on how much α has been decreased. Because when α is decreased, it will be harder to accelerate the robot.

4.3 Question 3

When the movement of robot is further constrained in a defined path as shown in figure 1. then the optimal time steps required to reach the set X_E from origin is found to be 6 and the control signal as shown in table 3. Now the robot is constrained to travel with in the corridor and the path taken by the robot to reach the destination can be seen in the figure3. The change of time steps required for any decrease in sampling time h and control signal constraint α from 1 and 2 respectively are with the same explanation in question 2 holds for this problem as well.

Time step	u_1	u_2
1	1.400	0.250
2	2.000	-0.750
3	-2.000	2.000
4	-0.800	2.000
5	-1.450	-1.400
6	0.750	-2.000

Table 5: Optimal control signal corresponding to time steps

Time(kh)	x_1	x_2	x_3	x_4
0	0	0	0	0
1	0.700	1.400	0.125	0.250
2	3.100	3.400	0	-0.500
3	5.500	1.400	0.500	1.500
4	6.500	0.600	3.000	3.500
5	6.375	-0.850	5.800	2.100
6	5.900	-0.100	6.900	0.100

Table 6: vector x at time kh

5 Conclusion

The robot movement optimization problem is modelled as a mixed integer linear problem with the given constraints and in the first case the robot is free to move in a 2D space and one can think of a diagonal path in a space or a straight line between two points, the solution to the problem showed the similar pattern of straight path between the 2 points with a minimum time of 4 . When the problem is further investigated for sensitivity of sampling time and the control limit it was found random nature due to the discrete nature of the no.of steps considered in the problem.

In the advanced exercise the robot movement is further constrained to move in a corridor and the problem gets complex with further addition of the constraints and it was found that it takes 2 extra time steps with the interval $h = 1$ than the earlier problem in free space and it can be concluded that the optimization of time required for robot movement increases with increase in constraints on the path.

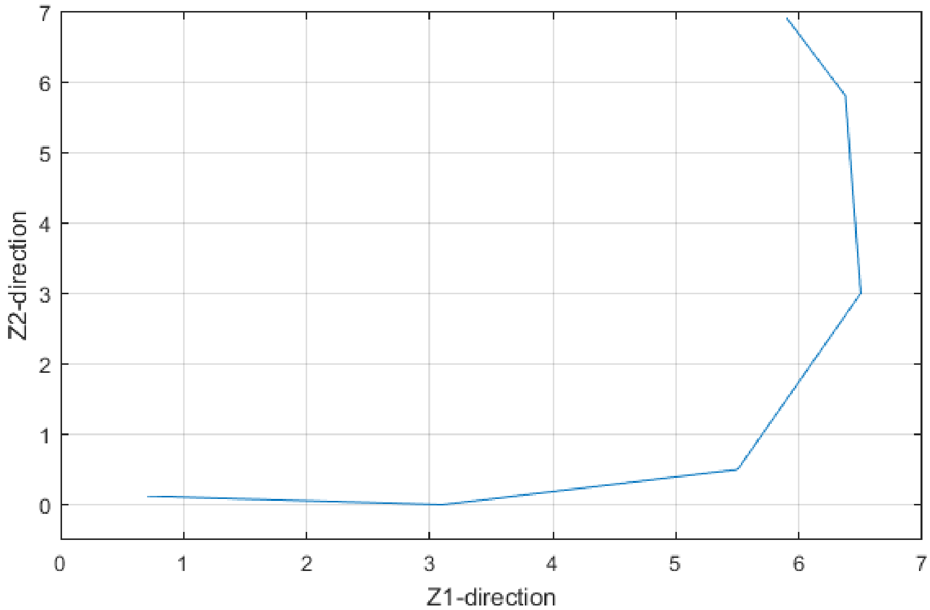

Figure 3: Optimal robot path