Distribution Theory and Applications — Example Sheet 1

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Question 1. *Construct a non-zero element of $\mathcal{D}(\mathbb{R})$ that vanishes outside $(0,1)$. Construct a non-zero of $\mathcal{D}(\mathbb{R}^n)$ that vanishes outside the ball $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.*

Proof. It is well-known that the function

$$\varphi\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \begin{cases} 0 & \text{if } x \leqslant 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside $(0,\infty)$. The function $\psi(x) := \varphi(x)\varphi(1-x)$ is therefore also smooth and vanishes outside $(0,1)$.

Since ψ vanishes outside $(0,1)$, the function $\psi(x/\varepsilon)$ vanishes outside $(0,\varepsilon)$, and therefore the function $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$ vanishes outside B_ε. □

Question 2. *Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives*

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h).$$

Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in x for each multi-index α, i.e. prove

$$\lim_{|h| \to 0} \frac{\sup_x |\partial^\alpha R_N(x,h)|}{|h|^N} = 0$$

for each multi-index α.

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x,h) = \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th)\,\mathrm{d}t\,,$$

and note that $(N+1)! \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \cdots + h_n)^{N+1}$.

Solution. Since $\varphi \in \mathcal{D}(X)$, we know that $\operatorname{supp} \varphi \subseteq \overline{B_N}$ for some $N \in \mathbb{N}$. Now, suppose $\|h\| < 1$, then

$$\varphi(x+h) \neq 0 \implies \|x+h\| \leqslant N \implies \|x\| \leqslant \|x+h\| + \|h\| \leqslant N+1,$$

so if we define $\psi_h(x) = \varphi(x+h)$ then we know that $\operatorname{supp} \varphi_h \subseteq \overline{B_{N+1}}$.

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h),$$

and since $\sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x)$ vanishes for $x \notin \overline{B_N}$, it is clear that $\operatorname{supp}(R_N(\cdot,h))$ must also be contained in $\overline{B_{N+1}}$ (again, for $\|h\| \leqslant 1$). This shows that $\operatorname{supp}(R_N)$ is contained in $\overline{B_{N+1}}$ for $|h|$ sufficiently small.

Now let β be a multi-index and define $C \coloneqq \frac{1}{N!}\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|$ (note that C exists and is finite since all partial derivatives of φ have compact support), then we have

$$\begin{aligned}
\left|\partial^\beta R_N(x,h)\right| &= \left|\partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^\alpha\varphi)(x+th)\,\mathrm{d}t\right| \\
&\overset{\star}{=} \left|\sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^{\alpha+\beta}\varphi)(x+th)\,\mathrm{d}t\right| \\
&\leqslant \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1)\int_0^1 (1-t)^N\left|(\partial^{\alpha+\beta}\varphi)(x+th)\right|\mathrm{d}t \\
&\leqslant \left[\max_{|\alpha|=N+1, x\in\mathbb{R}^n}\left|(\partial^{\alpha+\beta})\varphi(x)\right|\right]\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!}(N+1) \\
&\leqslant C(N+1)!\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!} = C(|h_1|+\cdots+|h_n|)^{N+1}.
\end{aligned}$$

Since this upper bound does not depend on x, we also have

$$\sup_x \left|\partial^\beta R_N(x,h)\right| \leqslant C(|h_1|+\cdots+|h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x\left|\partial^\beta R_N(x,h)\right|}{\|h\|^N} \leqslant \frac{C(|h_1|+\cdots+|h_n|)^{N+1}}{\|h\|^N} \leqslant \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \to 0,$$

and therefore that $\partial^\beta R_N(x,h) = o(\|h\|^n)$ for all multi-indices β.