

# Distribution Theory and Applications — Summary

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# 1 Distributions

## 1.1 Test functions and distributions

**Definition 1.1.** Let  $X \subseteq \mathbb{R}^n$  be open, then we define the set of *test functions* on  $X$  as

$$\mathcal{D}(X) := C_0^\infty(X) = \{f: X \rightarrow \mathbb{C} \mid f \text{ is smooth with compact support}\}.$$

**Definition 1.2.** Let  $(\varphi_m) \subseteq \mathcal{D}(X)$ . We say that  $(\varphi_m) \rightarrow 0$  in  $\mathcal{D}(X)$  if

1. there exists a compact  $K \subseteq X$  such that  $\text{supp } \varphi_m \subseteq K$  for all  $m$ ;
2.  $\partial^\alpha \varphi_m \rightarrow 0$  uniformly for each multi-index  $\alpha$ .

Note that, for any  $\varphi, \psi \in \mathcal{D}(X)$  and any multi-index  $\alpha$  we have

$$\int_X \varphi \cdot \partial^\alpha \psi \, dx = (-1)^{|\alpha|} \int_X \psi \cdot \partial^\alpha \varphi \, dx,$$

which follows from partial integration and the fact that all boundary terms vanish since  $\varphi$  and  $\psi$  have compact support.

Also, by Taylor's theorem, for any  $\varphi \in \mathcal{D}(X)$ ,  $x, h \in X$  and  $N \in \mathbb{N}$  we have

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h) \quad \text{where } R_N(x, h) = o(|h|^N) \text{ uniformly in } x.$$

**Definition 1.3.** A *distribution* on  $X$  is a linear map  $u: \mathcal{D}(X) \rightarrow \mathbb{C}$  if for every compact set  $K \subseteq X$  there exist constants  $C, N$  such that for all  $\varphi \in \mathcal{D}(X)$  with  $\text{supp } \varphi \subseteq K$  we have

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi|. \quad (1)$$

Condition 1 is called the *seminorm condition*. If, in the seminorm condition, the same  $N$  can be used for every compact set  $K \subseteq X$ , then the least such  $N$  is called the *order* of  $u$ , written  $\text{ord}(u)$ .

The set of all distributions in  $X$  is denoted  $\mathcal{D}'(X)$ .

If  $u \in \mathcal{D}'(X)$  and  $\varphi \in \mathcal{D}(X)$ , then instead of  $u(\varphi)$  we usually write  $\langle u, \varphi \rangle$ .

**Recap 1.4.** A function  $f: X \rightarrow \mathbb{C}$  is called *locally integrable* if  $\int_K |f| \, dx < \infty$  for all compact  $K \subseteq X$ .

The set of locally integrable functions on  $X$  is denoted  $L_{\text{loc}}^1(X)$ .

**Example 1.5.** Let  $M \in \mathbb{N}$  and let  $f_\alpha \in L_{\text{loc}}^1(X)$  for all  $|\alpha| \leq M$ . Define the linear map  $T: \mathcal{D}(X) \rightarrow \mathbb{C}$  by

$$\langle T, \varphi \rangle := \sum_{|\alpha| \leq M} \int_X f_\alpha \cdot \partial^\alpha \varphi \, dx.$$

It is trivial that  $T$  is linear, and we verify that  $T$  is a distribution as follows: take  $\varphi \in \mathcal{D}(X)$  with  $\text{supp } \varphi \subseteq K$ . Then we have

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \sum_{|\alpha| \leq M} \int_K |f_\alpha| \cdot |\partial^\alpha \varphi| \, dx \\ &\leq \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi| \cdot \int_K |f_\alpha| \, dx \\ &\leq \left( \max_\alpha \int_K |f_\alpha| \, dx \right) \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi|. \end{aligned}$$

Therefore, the seminorm condition is satisfied with  $N = M$ . From this, it also follows that  $\text{ord}(T) \leq M$ .

A special case of the previous example is the case  $M = 0$ : in this case the distribution simply becomes

$$\langle \tau_f, \varphi \rangle = \int_X f \varphi \, dx.$$

Henceforth we will abuse notation: if  $f \in L^1_{\text{loc}}(X)$ , then we will write  $f$  instead of  $\tau_f$ , i.e.,  $\langle f, \varphi \rangle = \int_X f \varphi \, dx$ .

**Lemma 1.6** (Sequential continuity). *Let  $u: \mathcal{D}(X) \rightarrow \mathbb{C}$  be a linear map. Then  $u$  is a distribution if and only if, for every sequence  $(\varphi_m) \subseteq \mathcal{D}(X)$  with  $\varphi_m \rightarrow 0$  as in definition 1.2, we have  $\langle u, \varphi_m \rangle \rightarrow 0$ .*

*Proof.* ‘ $\implies$ ’ If  $u$  is a distribution and  $(\varphi_m) \rightarrow 0$ , then  $\text{supp } \varphi_m \subseteq K$  for some compact  $K$ , and by eq. (1) there exist  $C, N$  such that

$$|\langle u, \varphi_m \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_m| \rightarrow 0.$$

‘ $\impliedby$ ’ Suppose there is a compact set  $K$  such that eq. (1) is not valid for any  $C, N$ . Let  $m \in \mathbb{N}$  and  $C = N = m$ , then there is some  $\varphi_m$  with  $\text{supp } (\varphi_m) \subseteq K$ , and

$$|\langle u, \varphi_m \rangle| > m \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m|.$$

By dividing  $\varphi_m$  by  $\langle u, \varphi_m \rangle \neq 0$ , we may assume w.l.o.g. that  $\langle u, \varphi_m \rangle = 1$ . We now have a sequence  $(\varphi_m)$  such that

$$\frac{1}{m} > \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m| \implies |\partial^\alpha \varphi_m| < \frac{1}{m} \quad \text{for } |\alpha| \leq m \implies \partial^\alpha \varphi_m \rightarrow 0 \text{ uniformly for all } \alpha.$$

Since each  $\varphi_m$  also satisfies  $\text{supp } \varphi_m \subseteq K$ , by definition 1.2 we have that  $\varphi_m \rightarrow 0$ , but also  $\langle u, \varphi_m \rangle \rightarrow 1$ , a contradiction.  $\square$

## 1.2 Limits in the distribution space

**Definition 1.7.** We say that a sequence  $(u_m) \subseteq \mathcal{D}'(X)$  converges to  $u \in \mathcal{D}'(X)$  and write  $u_m \rightarrow u$  if

$$\langle u_m, \varphi \rangle \rightarrow \langle u, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(X).$$

The following theorem is non-examinable but interesting:

**Theorem 1.8.** *Let  $(u_m)$  be a sequence in  $\mathcal{D}'(X)$  such that  $\lim_{m \rightarrow \infty} \langle u_m, \varphi \rangle$  exists for all  $\varphi \in \mathcal{D}(X)$ . Then the map  $\langle u, \varphi \rangle := \lim_{m \rightarrow \infty} \langle u_m, \varphi \rangle$  is a distribution in  $X$ .*

*Proof.* This is a direct application of the uniform boundedness principle.  $\square$

**Example 1.9.** Let  $X = \mathbb{R}$  and consider the sequence of functions  $u_m \in L^1_{\text{loc}}(\mathbb{R})$  defined by  $u_m(x) = \sin(mx)$ . Then, for all  $\varphi \in \mathcal{D}(\mathbb{R})$ , we have

$$\langle u_m, \varphi \rangle = \int_{\mathbb{R}} \sin(mx) \varphi(x) \, dx = \frac{1}{m} \int_{\mathbb{R}} \cos(mx) \varphi'(x) \, dx \leq \frac{1}{m} \int_{\mathbb{R}} |\varphi'(x)| \, dx \rightarrow 0.$$

Therefore, it holds that  $u_m \rightarrow 0$  in  $\mathcal{D}'(\mathbb{R})$ . With our abuse of notation we write this as  $\lim_{m \rightarrow \infty} \sin(mx) = 0$  in  $\mathcal{D}'(\mathbb{R})$ .

### 1.3 Basic operations

#### 1.3.1 Differentiation and multiplication by smooth functions

For  $u \in C^\infty(X)$  and  $\varphi \in \mathcal{D}(X)$ , we have noted that

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \partial^\alpha u \cdot \varphi \, dx = (-1)^{|\alpha|} \int_X u \cdot \partial^\alpha \varphi \, dx = \langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle.$$

Since the RHS makes sense for any distribution  $u$ , we define

**Definition 1.10.** For  $f \in C^\infty(X)$ ,  $u \in \mathcal{D}'(X)$ , we define  $\partial^\alpha(fu)$  by

$$\langle \partial^\alpha(fu), \varphi \rangle := \langle u, (-1)^{|\alpha|} f \cdot \partial^\alpha \varphi \rangle$$

*Remark.* This definition outlines a more general pattern when working with distributions: first we take some well-defined operator on the collection of smooth maps, then we rewrite it to a form that is sensible for any distribution, and then we *define* that new form as the operator on distributions. This process is called *extending the definition by duality*.

**Example 1.11.** Let  $u = \delta_x$ , then we have

$$\langle \partial^\alpha \delta_x, \varphi \rangle = \langle \delta_x, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x)$$

Furthermore, consider the Heaviside function  $H(x) = \mathbb{1}_{x \geq 0}$ . We have

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{\mathbb{R}} H(x) \varphi'(x) \, dx = -\int_0^\infty \varphi'(x) \, dx = \varphi(0) = \langle \delta_0, \varphi \rangle,$$

so we write  $H' = \delta_0$  in the distributional sense.

**Lemma 1.12.** Suppose  $u' \in \mathcal{D}'(\mathbb{R})$  satisfies  $u' = 0$ . Then  $u$  is constant (i.e.,  $\langle u, \varphi \rangle = \langle c, \varphi \rangle = c \int_{\mathbb{R}} \varphi \, dx$  for some  $c$ ).

*Proof.* Fix any  $\vartheta \in \mathcal{D}(\mathbb{R})$  with  $\langle 1, \vartheta \rangle = 1$ . Let  $\varphi \in \mathcal{D}(\mathbb{R})$  and define

$$\varphi_A := \varphi - \langle 1, \varphi \rangle \vartheta, \quad \varphi_B := \langle 1, \varphi \rangle \vartheta \quad \text{such that } \varphi = \varphi_A + \varphi_B.$$

Note that  $\langle 1, \varphi_A \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle \langle 1, \vartheta \rangle = 0$ .

We claim that the function  $\Phi_A(x) := \int_{-\infty}^x \varphi_A(y) \, dy$  has compact support: since  $\text{supp } \varphi_A \subseteq [a, b]$  for some  $a, b \in \mathbb{R}$ , clearly  $\Phi_A(x) = 0$  for  $x < a$ , while for  $x > b$  we have  $\Phi_A(x) = \langle 1, \varphi_A \rangle = 0$ . Obviously, it holds that  $\Phi'_A = \varphi_A$ . Now we compute

$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle = \langle u, \Phi'_A \rangle + \langle 1, \varphi \rangle \langle u, \vartheta \rangle = -\langle u', \Phi_A \rangle + c \langle 1, \varphi \rangle = \langle c, \varphi \rangle.$$

Since  $\varphi$  was chosen arbitrarily this shows that  $u$  is constant. □

#### 1.3.2 Reflection and translation

For  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ ,  $h \in \mathbb{R}^n$ , define the *translation of  $\varphi$  by  $h$*  by

$$(\tau_h \varphi)(x) := \varphi(x - h),$$

and the *reflection of  $\varphi$*  by  $\check{\varphi}(x) := \varphi(-x)$ .

Extending the definitions of translation and reflection by duality yields the following:

**Definition 1.13.** For  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $h \in \mathbb{R}^n$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , define

$$\langle \tau_h u, \varphi \rangle := \langle u, \tau_{-h} \varphi \rangle \quad \text{and} \quad \langle \check{u}, \varphi \rangle := \langle u, \check{\varphi} \rangle.$$

**Lemma 1.14.** For  $u \in \mathcal{D}'(\mathbb{R}^n)$ , define  $V_h \in \mathcal{D}'(\mathbb{R}^n)$  for  $0 \neq h \in \mathbb{R}^n$  by

$$V_h := \frac{\tau_{-h}u - u}{\|h\|}$$

If  $(h_j) \subseteq \mathbb{R}^n$  is a sequence for which  $\lim_{j \rightarrow \infty} \frac{h_j}{\|h_j\|} = m \in S^{n-1}$ , then  $V_{h_j} \rightarrow \sum_i m_i \frac{\partial u}{\partial x_i}$  in  $\mathcal{D}'(\mathbb{R}^n)$ .

*Proof.* By definition, we can write  $\langle V_h, \varphi \rangle = \frac{1}{\|h\|} \langle u, \tau_h \varphi - \varphi \rangle$ . Now Taylor's theorem tells us that

$$(\tau_h \varphi - \varphi)(x) = \varphi(x - h) - \varphi(x) = - \sum_i h_i \frac{\partial \varphi}{\partial x_i} + R(x, h),$$

where  $R(x, h) = o(\|h\|)$  in  $D(\mathbb{R}^n)$  (exercise sheet 1, question 2).

By sequential continuity, we have

$$\lim_{j \rightarrow \infty} \langle V_{h_j}, \varphi \rangle = \langle u, - \sum_i m_i \frac{\partial \varphi}{\partial x_i} \rangle = \langle \sum_i m_i \frac{\partial u}{\partial x_i}, \varphi \rangle,$$

which shows that  $V_{h_j} \rightarrow \sum_i m_i \frac{\partial u}{\partial x_i}$  in  $\mathcal{D}'(\mathbb{R}^n)$ . □

### 1.3.3 Convolution

For  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , note that  $(\tau_x \check{\varphi})(y) = \check{\varphi}(y - x) = \varphi(x - y)$ .

**Definition 1.15.** For  $u \in C^\infty(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , we define the *convolution*  $u * \varphi : \mathbb{R}^n \rightarrow \mathbb{C}$  as

$$(u * \varphi)(x) := \int_{\mathbb{R}^n} u(x - y) \varphi(y) dy = \int_{\mathbb{R}^n} u(y) \varphi(x - y) dy = \langle u, \tau_x \check{\varphi} \rangle.$$

Since the RHS makes sense for any  $u \in \mathcal{D}'(\mathbb{R}^n)$ , we extend the definition this way: for  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , we define the convolution  $u * \varphi$  as

$$(u * \varphi)(x) := \langle u, \tau_x \check{\varphi} \rangle.$$

**Lemma 1.16.** Let  $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$  and define  $\Phi_x(y) := \varphi(x, y)$ . Suppose for any  $x \in \mathbb{R}^n$  there exists a neighbourhood  $N(x)$  and a compact  $K \subseteq \mathbb{R}^m$  such that  $\varphi(x, y)$  for all  $x \in N(x)$ ,  $y \notin K$ .

Then  $x \mapsto \langle u, \Phi_x \rangle$  is differentiable with

$$\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi_x \rangle$$

for any  $u \in \mathcal{D}'(\mathbb{R}^m)$ .

*Proof.* Fix  $x \in \mathbb{R}^n$ , then by Taylor's formula we have

$$\Phi_{x+h}(y) = \Phi_x(y) + \sum_{i=1}^n h_i \frac{\partial \varphi}{\partial x_i} + R(x, y, h),$$

where  $\partial_y^\alpha R(x, y, h) = o(\|h\|)$ , uniformly in  $y$ , for any multi-index  $\alpha$ . Furthermore, by assumption there exists a compact  $K$  such that for  $h$  small enough,  $\text{supp } R(x, \cdot, h) \subseteq K$ . Therefore,  $R(x, \cdot, h)$  is a test function for  $h$  small enough.

Combining the previous two facts shows that  $R(x, \cdot, h) = o(\|h\|)$  in  $\mathcal{D}(\mathbb{R}^m)$  as  $h \rightarrow 0$ .

Let  $u \in \mathcal{D}'(\mathbb{R}^m)$ , then we find by sequential continuity that  $\langle u, R(x, \cdot, h) \rangle$  is also  $o(\|h\|)$ , and therefore

$$\langle u, \Phi_{x+h} \rangle = \langle u, \Phi_x \rangle + \sum_{i=1}^n h_i \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle + o(\|h\|).$$

This shows that  $x \mapsto \langle u, \Phi_x \rangle$  is differentiable with

$$\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle.$$

From this the result follows. □

**Corollary 1.17.** *If  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , then  $u * \varphi$  is differentiable with  $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi$ .*

*Proof.* Apply the previous lemma with  $\Phi_x(y) := \varphi(x - y)$ .  $\square$

Due to the previous corollary, we often call  $u * \varphi$  a *regularisation* of  $u$ .

*Convention.* If  $u \in \mathcal{D}'(X)$  and  $\varphi \in \mathcal{D}(X)$ , then instead of  $\langle u, \varphi \rangle$  we also write  $\langle u(t), \varphi(t) \rangle$  (or with any other dummy variable) when the variable used for  $\varphi$  is not directly clear.

## 1.4 Density of test functions in distribution space

**Lemma 1.18.** *If  $u \in \mathcal{D}'(\mathbb{R}^n)$  and  $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ , then*

$$(u * \varphi) * \psi = u * (\varphi * \psi).$$

*Proof.* Fix  $x \in \mathbb{R}^n$ . Now we write

$$\begin{aligned} ((u * \varphi) * \psi)(x) &= \int_{\mathbb{R}^n} \langle u(z), (\tau_y \check{\varphi})(z) \rangle \cdot \psi(x - y) \, dy \\ &= \int_{\mathbb{R}^n} \langle u(z), (\tau_{x-y} \check{\varphi})(z) \rangle \cdot \psi(y) \, dy \\ &= \int_{\mathbb{R}^n} \langle u(z), \psi(y) (\tau_{x-y} \check{\varphi})(z) \rangle \, dy. \end{aligned}$$

We would like to interchange integral and application of  $u$ , and we will have to justify this using Riemann sums:

$$\begin{aligned} \int_{\mathbb{R}^n} \langle u(z), \psi(y) (\tau_{x-y} \check{\varphi})(z) \rangle \, dy &= \lim_{\varepsilon \downarrow 0} \sum_{m \in \mathbb{Z}^n} \langle u(z), \psi(\varepsilon m) \varphi(x - z - \varepsilon m) \rangle \varepsilon^n \\ &\stackrel{*}{=} \lim_{\varepsilon \downarrow 0} \langle u(z), \sum_{m \in \mathbb{Z}^n} \psi(\varepsilon m) \varphi(x - z - \varepsilon m) \varepsilon^n \rangle \\ &\stackrel{**}{=} \left\langle u(z), \int_{\mathbb{R}^n} \psi(y) \varphi(x - z - y) \, dy \right\rangle \\ &= \langle u(z), (\varphi * \psi)(x - z) \rangle = \langle u(z), \widetilde{(\tau_x \varphi * \psi)}(z) \rangle = (u * (\varphi * \psi))(x). \end{aligned}$$

Here,  $*$  is by the fact that the sum is finite since  $\psi$  has compact support, while  $**$  is by sequential continuity of  $u$  and the fact that the Riemann sum converges to the convolution integral *in the space of test functions* (non-examinable fact).  $\square$

We will use the following trick many times:

**Proposition 1.19.** *For any  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$  we have  $\langle u, \varphi \rangle = (u * \check{\varphi})(0)$ .*

*Proof.* We have  $(u * \check{\varphi})(0) = \langle u, \tau_0 \varphi \rangle = \langle u, \varphi \rangle$ .  $\square$

For example, from this trick it follows that if  $u * \varphi = 0$  for all  $\varphi$ , then  $u = 0$ .

**Theorem 1.20.** *If  $u \in \mathcal{D}'(\mathbb{R}^n)$ , there exists a sequence  $(\varphi_k) \subseteq \mathcal{D}(\mathbb{R}^n)$  such that  $\varphi_k \rightarrow u$  in  $\mathcal{D}'(\mathbb{R}^n)$ .*

*Proof.* Fix  $\psi \in \mathcal{D}(\mathbb{R}^n)$  with  $\int_{\mathbb{R}^n} \psi \, dx = 1$ , and set  $\psi_k(x) := k^n \psi(kx)$ . Note that  $\int_{\mathbb{R}^n} \psi_k \, dx = 1$ .

Now, fix any  $\chi \in \mathcal{D}(\mathbb{R}^n)$  with  $\chi \equiv 1$  on  $\{\|x\| < 1\}$  and  $\chi \equiv 0$  on  $\{\|x\| < 2\}$ . Define  $\chi_k(x) := \chi(x/k)$ , so that  $\lim_{k \rightarrow \infty} \chi_k(x) = 1$  for all  $x$ . We will set

$$\varphi_k(x) := \chi_k(x)(u * \psi_k)(x).$$

Clearly we have  $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$  since each  $\chi_k$  has compact support.

Now, take any  $\vartheta \in \mathcal{D}(\mathbb{R}^n)$ , then we have

$$\begin{aligned}\langle \varphi_k, \vartheta \rangle &= \langle \chi_k(u * \psi_k), \vartheta \rangle = \langle u * \psi_k, \chi_k \vartheta \rangle = \left[ (u * \psi_k) * \widetilde{\chi_k \vartheta} \right](0) \\ &= \left[ u * (\psi_k * \widetilde{\chi_k \vartheta}) \right](0).\end{aligned}$$

Now we compute  $\psi_k * \widetilde{\chi_k \vartheta}$ : note that

$$\begin{aligned}(\psi_k * \widetilde{\chi_k \vartheta})(x) &= \int_{\mathbb{R}^n} k^n \psi(k(x-y)) \chi\left(-\frac{y}{k}\right) \vartheta(-y) \, dy \\ &= \int_{\mathbb{R}^n} \psi(y) \chi\left(\frac{y}{k^2} - \frac{x}{k}\right) \vartheta\left(\frac{y}{k} - x\right) \, dy \\ &= \vartheta(-x) + R_k(-x) = (\vartheta + \widetilde{R_k})(x)\end{aligned}$$

where  $R_k(x) = \int_{\mathbb{R}^n} \psi(y) \left[ \chi\left(\frac{y}{k^2} + \frac{x}{k}\right) \vartheta\left(\frac{y}{k} + x\right) - \vartheta(x) \right] \, dy$ .

So

$$\langle \varphi_k, \vartheta \rangle = (u * (\vartheta + \widetilde{R_k}))(0) = (u * \check{\vartheta})(0) + (u * \widetilde{R_k})(0) = \langle u, \vartheta \rangle + \langle u, R_k \rangle.$$

We must now only prove that  $R_k \rightarrow 0$  in  $\mathcal{D}(\mathbb{R}^n)$ , and then by sequential continuity it follows that  $\varphi_k \rightarrow u$  in  $\mathcal{D}'(\mathbb{R}^n)$ .  $\square$

## 2 Distributions with compact support

**Definition 2.1.** Let  $Y \subseteq X$  be open and  $u \in \mathcal{D}'(X)$ . We say that  $u$  *vanishes* on  $Y$  if  $\langle u, \varphi \rangle = 0$  for all  $\varphi \in \mathcal{D}(Y)$ .

**Definition 2.2.** For  $u \in \mathcal{D}'(X)$ , we define the *support* of  $u$  as

$$\text{supp } u := X \setminus \bigcup \{Y \subseteq X \mid Y \text{ open, } u \text{ vanishes on } Y\}.$$

For example, the support of  $\delta_x$  is simply  $\{x\}$ .

### 2.1 Test functions and distributions

We will now consider a bigger space of test functions, which yields a smaller space of distributions.

**Definition 2.3.** We define  $\mathcal{E}(X)$  as the space of smooth functions  $\varphi: X \rightarrow \mathbb{C}$ . We say that a sequence  $(\varphi_m) \subseteq \mathcal{E}(X)$  converges to 0 if  $\partial^\alpha \varphi \rightarrow 0$  uniformly on compact subsets of  $X$  for every multi-index  $\alpha$ .

**Definition 2.4.** We define  $\mathcal{E}'(X)$  as the space of linear maps  $u: \mathcal{E}(X) \rightarrow \mathbb{C}$  for which there exists a compact  $K \subseteq X$  and nonnegative constants  $C, N$  such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \quad (2)$$

for all  $\varphi \in \mathcal{E}(X)$ .

**Lemma 2.5** (Sequential continuity). *Let  $u: \mathcal{E}(X) \rightarrow \mathbb{C}$  be a linear map. Then  $u \in \mathcal{E}'(X)$  if and only if, for every sequence  $(\varphi_m) \subseteq \mathcal{E}(X)$  with  $\varphi_m \rightarrow 0$ , we have  $\langle u, \varphi_m \rangle \rightarrow 0$ .*

*Proof.* **TODO:** □

**Lemma 2.6.** *If  $u \in \mathcal{E}'(X)$ , then  $u|_{\mathcal{D}(X)}$  defines an element of  $\mathcal{D}'(X)$  with compact support and finite order.*

*Conversely, for each  $u \in \mathcal{D}'(X)$  with compact support there exists a unique extension  $\tilde{u} \in \mathcal{E}'(X)$  with  $\text{supp}(\tilde{u}) = \text{supp}(u)$  and  $\tilde{u}|_{\mathcal{D}(X)} = u$ .*

*Proof.* Let  $u \in \mathcal{E}'(X)$ , so that there exists a compact  $K \subseteq X$  with  $|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$ . Now, for any compact  $K' \subseteq X$  and any  $\varphi$  with  $\text{supp } \varphi \subseteq K'$ , eq. (1) is clearly satisfied, and we can use the same  $N$  for all compact  $K'$ , so clearly  $u|_{\mathcal{D}(X)}$  is an element of  $\mathcal{D}'(X)$  with finite order. Finally, suppose  $\varphi$  is supported in  $X \setminus K$ , then it is clear that  $\langle u, \varphi \rangle = 0$ , which proves that  $\text{supp } u \subseteq K$  and therefore that  $u$  has compact support.

Now suppose  $u \in \mathcal{D}'(X)$  has compact support, let  $\rho \in \mathcal{D}(X)$  be 1 in a neighbourhood of  $\text{supp } u$ , and define  $\tilde{u} \in \mathcal{E}'(X)$  by

$$\langle \tilde{u}, \varphi \rangle := \langle u, \rho \varphi \rangle \quad \forall \varphi \in \mathcal{E}(X).$$

Clearly  $\tilde{u}$  is an element of  $\mathcal{E}'(X)$  since  $\text{supp}(\rho \varphi) \subseteq \text{supp } \rho$  and

$$|\langle \tilde{u}, \varphi \rangle| = |\langle u, \rho \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{\text{supp}(\rho)} |\partial^\alpha (\rho \varphi)| \stackrel{*}{\leq} C' \sum_{|\alpha| \leq N} \sup_{\text{supp } \rho} |\partial^\alpha \varphi|,$$

where  $\star$  follows from the Leibniz rule. It is also clear that  $\text{supp } \tilde{u} = \text{supp } u$ .

Finally we will show uniqueness: suppose  $\tilde{v}$  is an extension of  $u$  with  $\text{supp } \tilde{v} = \text{supp } u$ , and write any  $\varphi \in \mathcal{E}(X)$  as  $\varphi = \rho \varphi + (1 - \rho) \varphi = \varphi_0 + \varphi_1$ . Then since  $\varphi_0 \in \mathcal{D}(X)$  and  $\varphi_1$  vanishes on a neighbourhood of  $\text{supp } u$ , we find

$$\langle \tilde{v}, \varphi \rangle = \langle \tilde{v}, \varphi_0 + \varphi_1 \rangle = \langle \tilde{v}, \varphi_0 \rangle = \langle u, \varphi_0 \rangle,$$

which is independent of choice of extension. □



## 2.2 Convolution between distributions

**Definition 2.7.** Define for  $u \in \mathcal{E}'(\mathbb{R}^n), \varphi \in \mathcal{E}(\mathbb{R}^n)$  the *convolution*

$$(u * \varphi)(x) = \langle u, \tau_x \tilde{\varphi} \rangle.$$

This convolution satisfies the same properties as the convolution between  $\mathcal{D}'(\mathbb{R}^n)$  and  $\mathcal{D}(\mathbb{R}^n)$ . Also, if  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , then  $u * \varphi \in \mathcal{D}(\mathbb{R}^n)$ .

**Definition 2.8.** Let  $u, v \in \mathcal{D}'(\mathbb{R}^n)$ , at least one of which has compact support, define  $u * v: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$  by

$$(u * v) * \varphi = u * (v * \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We will show (example sheet 2, question 1) that  $u * v$  is uniquely defined and gives rise to an element of  $\mathcal{D}'(\mathbb{R}^n)$  via  $\langle u * v, \varphi \rangle := [(u * v) * \tilde{\varphi}](0)$ .

**Lemma 2.9.** Given  $u, v \in \mathcal{D}'(\mathbb{R}^n)$ , at least one of which has compact support, we have  $u * v = v * u$ .

*Proof.* First we note that  $(u * \varphi) * \psi = u * (\varphi * \psi)$  holds if  $u$  has compact support and at least one of  $\varphi, \psi$  has compact support.

Fix  $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ , we see from our earlier shown properties that

$$(u * v) * (\varphi * \psi) = u * (v * (\varphi * \psi)) = u * ((v * \varphi) * \psi) = u * (\psi * (v * \varphi)) = (u * \psi) * (v * \varphi).$$

If we interchange  $u$  and  $v$  in the above, that is equivalent to interchanging  $\varphi$  and  $\psi$ , which we know must yield the same result. This shows  $u * v$  and  $v * u$  agree on  $\varphi * \psi$  for all  $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ . Defining  $E = u * v - v * u$ , we find that  $0 = E * (\varphi * \psi) = (E * \varphi) * \psi$  for all  $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ , so  $E * \varphi = 0$  for all  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , so  $E = 0$ .  $\square$

## 3 Tempered distributions and Fourier analysis

### 3.1 Functions of rapid decay

**Definition 3.1.** For any  $f: \mathbb{R}^n \rightarrow \mathbb{C}$  and multi-indices  $\alpha, \beta$  we define  $\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f|$ .

We define the *Schwartz space*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha, \beta} < \infty \text{ for all } \alpha, \beta \right\}.$$

We say that a sequence  $(\varphi_n) \subseteq \mathcal{S}(\mathbb{R}^n)$  converges to 0 if  $\|\varphi_n\|_{\alpha, \beta} \rightarrow 0$  for every  $\alpha, \beta$ .

**Example 3.2.** The function  $x \mapsto \exp(-\|x\|^2)$  lies in  $\mathcal{S}(\mathbb{R}^n)$ .

**Proposition 3.3.** For all  $n$  we have that  $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$ .

*Proof.* Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , then for all  $N \in \mathbb{N}$  we have

$$\int_{\mathbb{R}^n} |\varphi(x)| dx = \int_{\mathbb{R}^n} (1 + \|x\|)^{-N} (1 + \|x\|)^N |\varphi(x)| dx \stackrel{?}{\leq} C \sum_{|\alpha| \leq N} \|\varphi\|_{\alpha, 0} \int_{\mathbb{R}^n} (1 + \|x\|)^{-N} dx.$$

Since  $\int_{\mathbb{R}^n} (1 + \|x\|)^{-N} dx$  is finite for  $N$  large enough (??), this proves the claim.  $\square$

**Definition 3.4.** A linear map  $u: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$  is called a *tempered distribution* if there exists constants  $C, N$  such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|, |\beta| \leq N} \|\varphi\|_{\alpha, \beta} \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The space of tempered distributions is denoted by  $\mathcal{S}'(\mathbb{R}^n)$ .

This definition is equivalent to sequential continuity.

### 3.2 The Fourier transform on Schwartz functions

*Convention.* We write  $D := -i\partial$  and  $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$ .

**Definition 3.5.** For  $f \in L^1(\mathbb{R}^n)$ , define the *Fourier transform* of  $f$  by

$$[\mathcal{F}(f)](\lambda) = \hat{f}(\lambda) := \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} f(x) \, dx \quad \text{where } \lambda \in \mathbb{R}^n.$$

**Lemma 3.6.** If  $f \in L^1(\mathbb{R}^n)$ , then  $\hat{f}$  is continuous.

*Proof.* If  $\lambda_m \rightarrow \lambda \in \mathbb{R}^n$ , then by the dominated convergence theorem we have

$$\hat{f}(\lambda_m) = \int_{\mathbb{R}^n} e^{-i\lambda_m \cdot x} f(x) \, dx \stackrel{\text{DCT}}{=} \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} f(x) \, dx = \hat{f}(\lambda),$$

where we were able to use the dominated convergence theorem since the integrand is absolutely bounded by  $|f|$  and  $f \in L^1$ .  $\square$

It turns out that this idea generalises: the faster the function  $f$  decays, the smoother the Fourier transform  $\hat{f}$  is.

**Lemma 3.7.** For  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , we have  $\mathcal{F}[D_x^\alpha \varphi](\lambda) = \lambda^\alpha \hat{\varphi}(\lambda)$  and  $\mathcal{F}[x^\beta \varphi](\lambda) = (-1)^{|\beta|} D_\lambda^\beta \hat{\varphi}(\lambda)$ .

*Proof.* Since  $|x^\alpha D^\beta \varphi| \rightarrow 0$  as  $\|x\| \rightarrow \infty$ , we have using integration by parts

$$\begin{aligned} \mathcal{F}[D_\lambda^\alpha \varphi](\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} D_x^\alpha \varphi(x) \, dx \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_x^\alpha (e^{-i\lambda \cdot x}) \varphi(x) \, dx \\ &= \lambda^\alpha \hat{\varphi}(\lambda). \end{aligned}$$

For the second part of the lemma, by differentiation under the integral sign we get

$$\begin{aligned} \mathcal{F}[x^\beta \varphi](\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} x^\beta \varphi(x) \, dx \\ &= \int_{\mathbb{R}^n} ((-D_\lambda)^\beta e^{-i\lambda \cdot x}) \varphi(x) \, dx \\ &= (-1)^{|\beta|} D_\lambda^\beta \hat{\varphi}(\lambda). \end{aligned}$$

$\square$

We define the *inverse Fourier transform* by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \hat{f}(\lambda) \, d\lambda.$$

We will now show that on  $\mathcal{S}(\mathbb{R}^n)$ , the inverse Fourier transform is indeed an inverse:

**Theorem 3.8.** The Fourier transform  $\mathcal{F}$  defines a continuous isomorphism from  $\mathcal{S}(\mathbb{R}^n)$  to itself.

*Proof.* Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ . First, we show that  $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ : by the previous lemma we have for multi-indices  $\alpha, \beta$  that

$$\begin{aligned} |\lambda^\alpha (-D_\lambda)^\beta \hat{\varphi}(\lambda)| &= |\lambda^\alpha \mathcal{F}[x^\beta \varphi](\lambda)| = |\mathcal{F}[D_x^\alpha (x^\beta \varphi)](\lambda)| = \left| \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} D^\alpha (x^\beta \varphi) \, dx \right| \\ &\leq \int_{\mathbb{R}^n} |D^\alpha (x^\beta \varphi)| \, dx, \end{aligned} \tag{3}$$

which is finite since  $D^\alpha(x^\beta\varphi)$  is also a Schwartz function and therefore integrable.

From the previous lemma we also infer that  $\hat{\varphi}$  is smooth, so indeed we have  $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ . From eq. (3) it is also easily seen that if  $\varphi_m \rightarrow 0$  in  $\mathcal{S}(\mathbb{R}^n)$ , then  $\hat{\varphi}_m \rightarrow 0$  in  $\mathcal{S}(\mathbb{R}^n)$  also, which shows that  $\mathcal{F}$  is continuous.

To prove surjectivity and injectivity, we will show that  $\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x)$  (???). Indeed we have

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y)} \varphi(y) \, dy \, d\lambda \\ &= \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y) - \varepsilon \|\lambda\|^2} \varphi(y) \, dy \, d\lambda \\ &\stackrel{*}{=} \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y) - \varepsilon \|\lambda\|^2} \, d\lambda \, dy,\end{aligned}$$

where  $\star$  follows from Fubini's theorem.

Now we note that

$$\int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y) - \varepsilon \|\lambda\|^2} \, d\lambda = \prod_{j=1}^n \int_{\mathbb{R}} e^{i\lambda_j(x_j - y_j) - \varepsilon \lambda_j^2} \, d\lambda \stackrel{**}{=} \prod_{j=1}^n \left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{-\frac{(x_j - y_j)^2}{4\varepsilon}} = \left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-\frac{\|x-y\|^2}{4\varepsilon}}.$$

To explain  $**$ , **TODO:** .

and plugging that into the above yields

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \lim_{\varepsilon \downarrow 0} 2^{-n} (\pi\varepsilon)^{-n/2} \int_{\mathbb{R}^n} \varphi(y) e^{-\|x-y\|^2/(4\varepsilon)} \, dy \\ &\stackrel{***}{=} \pi^{-n/2} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \varphi(x - 2\sqrt{\varepsilon}y') e^{-\|y'\|^2} \, dy \\ &\stackrel{\text{DCT}}{=} \pi^{-n/2} \varphi(x) \int_{\mathbb{R}^n} e^{-\|y'\|^2} \, dy = \varphi(x),\end{aligned}$$

where  $***$  follows from the substitution  $x - y = 2\sqrt{\varepsilon}y'$ .

Finally, continuity of  $\mathcal{F}^{-1}$  is easily shown with an argument analogous to that for continuity of  $\mathcal{F}$  (????).  $\square$

### 3.3 The Fourier transform on tempered distributions

**Lemma 3.9.** For  $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ , we have  $\int_{\mathbb{R}^n} \varphi(x) \hat{\psi}(x) \, dx = \int_{\mathbb{R}^n} \hat{\varphi}(x) \psi(x) \, dx$ .

*Proof.* This follows from Fubini's theorem:

$$\int_{\mathbb{R}^n} \varphi(x) \hat{\psi}(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(x) \psi(\lambda) e^{-i\lambda \cdot x} \, d\lambda \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(\lambda) \varphi(x) e^{-i\lambda \cdot x} \, dx \, d\lambda = \psi(\lambda) \psi(\lambda) \hat{\varphi}(\lambda) \, d\lambda.$$

$\square$

The above result can be rewritten as  $\langle \hat{\varphi}, \psi \rangle = \langle \varphi, \hat{\psi} \rangle$ , which motivates the definition of the Fourier transform for tempered distributions:

**Definition 3.10.** For  $u \in \mathcal{S}'(\mathbb{R}^n)$ , we define its Fourier transform by

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Using sequential continuity and theorem 3.8, it is easily seen that  $\hat{u}$  is indeed an element of  $\mathcal{S}'(\mathbb{R}^n)$ .

**Example 3.11.** It is easily checked that  $\delta_0 \in \mathcal{S}'(\mathbb{R}^n)$ , and we can compute

$$\langle \hat{\delta}_0, \varphi \rangle = \langle \delta_0, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x) dx = \langle 1, \varphi \rangle,$$

so we can write  $\hat{\delta}_0 = 1$ . Analogously, by the Fourier inversion theorem we have

$$\langle \hat{1}, \varphi \rangle = \langle 1, \hat{\varphi} \rangle = \int_{\mathbb{R}^n} \hat{\varphi}(\lambda) d\lambda = (2\pi)^n \varphi(0) = \langle (2\pi)^n \delta_0, \varphi \rangle,$$

so we write  $\hat{1} = (2\pi)^n \delta_0$ .

We can easily generalise lemma 3.7 to the Fourier transform on  $\mathcal{S}'(\mathbb{R}^n)$ , so

$$\mathcal{F}[D^\alpha u] = \lambda^\alpha \hat{u}, \quad \mathcal{F}[x^\beta u] = (-D^\beta) \hat{u}.$$

**Theorem 3.12.** *The Fourier transform  $\mathcal{F}$  extends to a continuous isomorphism on  $\mathcal{S}'(\mathbb{R}^n)$ .*

*Proof.* We claim that  $\check{u} = (2\pi)^{-n} \mathcal{F}[\mathcal{F}[u]]$ . To check this, note that by the Fourier inversion theorem we have for  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  that

$$\check{\varphi}(x) = \varphi(-x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} \hat{\varphi}(\lambda) d\lambda = (2\pi)^{-n} \mathcal{F}[\mathcal{F}[\varphi]](x),$$

and therefore

$$\langle \check{u}, \varphi \rangle = \langle u, \check{\varphi} \rangle = \langle u, (2\pi)^{-n} \mathcal{F}[\mathcal{F}[\varphi]] \rangle = \langle (2\pi)^{-n} \mathcal{F}[\mathcal{F}[u]], \varphi \rangle.$$

This shows that  $\mathcal{F}$  is bijective (since  $\mathcal{F} \circ \mathcal{F}$  is bijective). For continuity of  $\mathcal{F}$  and its inverse: using theorem 3.8, we find

$$\begin{aligned} & u_m \rightarrow 0 \text{ in } \mathcal{S}'(\mathbb{R}^n) \\ \iff & \langle u_m, \varphi \rangle \rightarrow 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \\ \iff & \langle u_m, \hat{\varphi} \rangle \rightarrow 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \\ \iff & \langle \hat{u}_m, \varphi \rangle \rightarrow 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \\ \iff & \hat{u}_m \rightarrow 0 \text{ in } \mathcal{S}'(\mathbb{R}^n). \end{aligned}$$

□

### 3.4 Sobolev spaces

*Convention.* We write  $\langle \lambda \rangle := (1 + \|\lambda\|)^2$  for  $\lambda \in \mathbb{R}^n$ . Note that  $\langle \lambda \rangle \sim 1$  as  $\|\lambda\| \rightarrow 0$  and  $\langle \lambda \rangle \rightarrow \|\lambda\|$  as  $\|\lambda\| \rightarrow \infty$ .

**Definition 3.13.** For  $s \in \mathbb{R}$ , define the *Sobolev space*  $H^s(\mathbb{R}^n)$  to be the set of tempered distributions  $u \in \mathcal{S}'(\mathbb{R}^n)$  for which  $\hat{u}$  can be identified with a measurable function  $\hat{u}: \mathbb{R}^n \rightarrow \mathbb{C}$  such that  $\langle \lambda \rangle^s \hat{u} \in L^2(\mathbb{R}^n)$ .

**Lemma 3.14** (Sobolev lemma). *If  $u \in H^s(\mathbb{R}^n)$  with  $s > \frac{n}{2}$ , then  $u$  is continuous.*

*Proof.* We will show that  $\hat{u}$  is integrable. By Cauchy-Schwarz, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{u}(\lambda)| d\lambda &= \int_{\mathbb{R}^n} \langle \lambda \rangle^{-s} \langle \lambda \rangle^s |\hat{u}(\lambda)| d\lambda \\ &\leq \left( \int_{\mathbb{R}^n} \langle \lambda \rangle^{-2s} d\lambda \right)^{1/2} \|u\|_{H^s} \\ &= C \|u\|_{H^s} \left( \int_0^\infty r^{n-1} (1+r^2)^{-s} dr \right)^{1/2}, \end{aligned}$$

where the last line follows from using polar coordinates and  $C$  is the area of the  $(n-1)$ -sphere.

Writing  $s = \frac{n}{2} + \varepsilon$ , we find that the integrand  $r^{n-1}(1+r^2)^{-s}$  is of order  $O(r^{-1-2\varepsilon})$  as  $r \rightarrow \infty$ , and therefore the integral is finite, so indeed we have  $\hat{u} \in L^1(\mathbb{R}^n)$ .

By applying theorem 3.8 to a test function, we can show that  $u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \hat{u}(\lambda) \, d\lambda$ , which is continuous by the dominated convergence theorem.  $\square$