Distribution Theory — Example Sheet 2

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18 November 2020

We will write $\mathcal{R}$ and $\mathcal{F}$ for the reflection and Fourier transform operators.

Question 1. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, one of which has compact support. Show that the convolution $u * v$, defined as in your notes, is uniquely defined and gives rise to an element of $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. The convolution between $u, v \in \mathcal{D}'(\mathbb{R}^n)$ is defined by the formula

$$(u * v) * \varphi = u * (v * \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1}$$

Recall that for all $u \in \mathcal{D}'(\mathbb{R}^n) \varphi \in \mathcal{D}(\mathbb{R}^n)$, we have $\langle u, \varphi \rangle = (u * \check{\varphi})(0)$, and therefore $u * v$ should satisfy

$$\langle u * v, \varphi \rangle = ((u * v) * \check{\varphi})(0) = (u * (v * \check{\varphi}))(0) = \langle u, \mathcal{R}(v * \check{\varphi}) \rangle.$$

Therefore

1. Define $w \colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ by
$$\langle w, \varphi \rangle = \langle u, \mathcal{R}(v * \check{\varphi}) \rangle,$$
then we will show that w satisfies $w * \varphi = u * (v * \varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.
$$\begin{aligned}(w * \varphi)(x) &= \langle w, \tau_x \check{\varphi} \rangle = \langle u, \mathcal{R}(v * \mathcal{R}(\tau_x \check{\varphi})) \rangle = \langle u, \mathcal{R}(v * \tau_{-x} \varphi) \rangle \\ &= \langle u, \mathcal{R} \tau_{-x}(v * \varphi) \rangle = \langle u, \tau_x \mathcal{R}(v * \varphi) \rangle = (u * (v * \varphi))(x).\end{aligned}$$

2. Uniqueness: we have shown in the lectures that if $w * \varphi = w' * \varphi$ for all φ, then $w = w'$. This shows that eq. (1) uniquely defines $u * v$.

Now we prove that $u * v \in \mathcal{D}'(\mathbb{R}^n)$: by the previous equation we have

$$\langle u * v, \varphi \rangle = (u * (v * \check{\varphi}))(0) = \langle u, \widetilde{v * \check{\varphi}} \rangle.$$

Suppose u is compactly supported. Since $\widetilde{v * \check{\varphi}} \in \mathcal{E}(\mathbb{R}^n)$, there exists a compact $K \subseteq X$ and nonnegative C, N such that

$$\begin{aligned}\left| \langle u, \widetilde{v * \check{\varphi}} \rangle \right| &\leqslant C \sum_{\alpha \leqslant N} \sup_{x \in K} \left| \partial^\alpha (\widetilde{v * \check{\varphi}}) \right| = C \sum_{\alpha \leqslant N} \sup_{x \in -K} \left| \partial^\alpha (v * \check{\varphi}) \right| = C \sum_{\alpha \leqslant N} \sup_{x \in -K} \left| v * \partial^\alpha \check{\varphi} \right| \\ &= C \sum_{|\alpha| \leqslant N} \sup_{x \in -K} \left| \langle v, \tau_x \widetilde{\partial^\alpha \check{\varphi}} \rangle \right|.\end{aligned}$$

Note that if $\operatorname{supp} \varphi \subseteq K'$, then $\operatorname{supp} \check{\varphi} \subseteq -K'$, and for $x \in -K$ we find $\operatorname{supp} \tau_x \widetilde{\partial^\alpha \check{\varphi}} \subseteq -K' - K$. Then by the previous equation we find that there exists C', M with

$$|\langle u * v, \varphi \rangle| \leqslant C' \sum_{|\alpha| \leqslant N} \sum_{|\beta| \leqslant M} \sup_{x \in -K' - K} \partial^\beta (\tau_x \widetilde{\partial^\alpha \check{\varphi}}) \leqslant C' \sum_{|\alpha| \leqslant N} \sum_{|\beta| \leqslant M} \sup_x \left| \partial^{\alpha + \beta} \varphi \right| \leqslant C'' \sum_{|\alpha| \leqslant M + N} \sup_x |\partial^\alpha \varphi|,$$

which shows that $u * v \in \mathcal{D}'(\mathbb{R}^n)$. An analogous argument holds if v is compactly supported. $\square$

Question 2. *Show that if $u, v, w \in \mathcal{D}'(\mathbb{R}^n)$ and at least two of them have compact support, then the convolution is associative (i.e., $(u * v) * w) = u * (v * w)$).*

Proof. Note that the convolution between two compactly supported distributions is again compactly supported, which ensures that both expressions 'make sense'. Now, let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then we have

$$((u * v) * w) * \varphi = (u * v) * (w * \varphi) = u * (v * (w * \varphi)) = u * ((v * w) * \varphi) = (u * (v * w)) * \varphi,$$

which proves the theorem. □

Question 3. *Let $\varphi \in \mathcal{D}(\mathbb{R})$ and choose $\varepsilon > 0$ sufficiently small so that* $\operatorname{supp}(\varphi) \subset I_\varepsilon = (-1/\varepsilon, 1/\varepsilon)$. *Given that φ has a uniformly convergent Fourier series on I_ε in the form*

$$\varphi(x) = \sum_{n \in \mathbb{Z}} c_n e^{i\varepsilon\pi n x}, \quad c_n = \frac{\varepsilon}{2} \int_{\mathbb{R}} \varphi(x) e^{-i\varepsilon\pi n x} \, \mathrm{d}x \, ,$$

prove the Fourier inversion theorem on $\mathcal{D}(\mathbb{R})$ by taking a suitable limit.

Proof. Since $\mathcal{D}(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, we know that the Fourier inversion formula holds. We only need to show that the Fourier tranform of φ is again an element of $\mathcal{D}(\mathbb{R})$. (??) □

Question 4. *For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ prove that $\sum_m \varphi(m) = \sum_n \hat{\varphi}(2\pi n)$. This is the famous Poisson summation formula.*

Proof. We have

$$\sum_m \varphi(m) = \frac{1}{(2\pi)^n} \sum_m \int e^{i\lambda m} \hat{\varphi}(\lambda) \, \mathrm{d}\lambda = \sum_m \int e^{2\pi i \lambda m} \hat{\varphi}(2\pi\lambda) \, \mathrm{d}\lambda$$

(??) □

Question 5. *If $u \in H^s(\mathbb{R}^n)$ show that $D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$. If $s > t$ show that $H^s(\mathbb{R}^n) \subseteq H^t(\mathbb{R}^n)$.*

Proof. Assuming $u \in H^s(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \langle\lambda\rangle^{2(s-|\alpha|)} \left|\widehat{D^\alpha u}(\lambda)\right|^2 \mathrm{d}\lambda &= \int_{\mathbb{R}^n} \langle\lambda\rangle^{2(s-|\alpha|)} \|\lambda\|^{2|\alpha|} |\hat{u}(\lambda)|^2 \, \mathrm{d}\lambda \\ &\lesssim \int_{\mathbb{R}^n} \langle\lambda\rangle^{2s} |\hat{u}(\lambda)|^2 \, \mathrm{d}\lambda < \infty, \end{aligned}$$

which proves $D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$.

The second claim follows immediately from the fact that $\langle\lambda\rangle^t \leqslant \langle\lambda\rangle^s$ for $s \geqslant t$ and λ sufficiently large. □

Question 6. *Show that $S(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n) = H^0(\mathbb{R}^n)$ and deduce that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.*

Hint: *Use Parseval's theorem.*

Proof. We will show that $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$: since $\mathcal{D}(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$, this shows that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ as well.

(??)

Now, for $u \in H^s(\mathbb{R}^n)$, let $(\varphi_m) \to u$ in L^2. Then we have

$$\|\varphi_m - u\|_{H^s}^2 = \int \langle\lambda\rangle^{2s} |(\varphi_m - u)(\lambda)|^2 \, \mathrm{d}\lambda$$

(??) □

Question 7. *Prove that multiplication by a Schwarz function gives rise to a continuous map from $H^s(\mathbb{R}^n)$ to itself, i.e., $\|\varphi u\|_{H^s} \lesssim \|u\|_{H^s}$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$. You may assume Peetre's inequality: for $\lambda, \nu \in \mathbb{R}^n$ and $s \in \mathbb{R}$*

$$\left(\frac{1+\|\lambda\|^2}{1+\|\mu\|^2}\right)^s \leqslant 2^{|s|}\left(1+\|\lambda-\mu\|^2\right)^{|s|}.$$

Proof. We have

$$\|\varphi u\|_{H^s}^2 = \int |\mathcal{F}[\varphi u](\lambda)|^2 (1+\|\lambda\|)^2 \, \mathrm{d}\lambda$$

(??) (how to bound fourier transform of product??) □

Question 19. *Compute the Fourier transforms of the functions*

(a) $\operatorname{sign}(x)$;

(b) $\arctan(x)$;

(c) $x \log|x| - x$;

(d) $\exp\left(i\omega x^2\right)$

in $\mathcal{S}'(\mathbb{R})$, where $\omega \in \mathbb{R}$.

Proof. (a) We have for $\varphi \in \mathcal{S}(\mathbb{R})$ that

$$\begin{aligned}
\langle \widehat{\operatorname{sign}}, \varphi \rangle &= \langle \operatorname{sign}, \hat{\varphi} \rangle = \int_{\mathbb{R}} \operatorname{sign}(\lambda) \hat{\varphi}(\lambda) \, \mathrm{d}\lambda = \int_{\mathbb{R}} \operatorname{sign}(\lambda) \int_{\mathbb{R}} e^{-i\lambda x} \varphi(x) \, \mathrm{d}x \, \mathrm{d}\lambda \\
&\overset{\star}{=} \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} \operatorname{sign}(\lambda) e^{-i\lambda x} \, \mathrm{d}\lambda \, \mathrm{d}x = \int_{\mathbb{R}} \varphi(x) \lim_{R\to\infty} \left(\int_0^R e^{-i\lambda x} \, \mathrm{d}\lambda - \int_{-R}^0 e^{-i\lambda x} \, \mathrm{d}\lambda \right) \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \lim_{R\to\infty} \int_{|x|>\varepsilon} \varphi(x) \left(\frac{e^{ixR} + e^{-ixR}}{ix} - \frac{2}{ix} \right) \mathrm{d}x \\
&= \lim_{\varepsilon \to 0} \lim_{R\to\infty} \int_{\mathbb{R}} \frac{\varphi(x) \mathbb{1}_{|x|>\varepsilon}}{ix} \cdot e^{ixR} \, \mathrm{d}x + \lim_{\varepsilon \to 0} \lim_{R\to\infty} \int_{\mathbb{R}} \frac{\varphi(x) \mathbb{1}_{|x|>\varepsilon}}{ix} \cdot e^{-ixR} \, \mathrm{d}x + 2i \mathrm{P.V.}\left(\frac{1}{x}\right).
\end{aligned}$$

We claim the first two terms go to 0: this is because the term in the integral is the Fourier transform of $\frac{\varphi(x)\mathbb{1}_{|x|>\varepsilon}}{ix}$ evaluated at $\pm R$, and since the function is in L^1, its Fourier transform decays to 0 as $|R| \to \infty$.

We conclude $\widehat{\operatorname{sign}} = 2i \mathrm{P.V.}(\frac{1}{x})$.

(b) We know that $\arctan'(x) = \frac{1}{1+x^2} =: f(x)$, then we have $\widehat{\arctan}(\lambda) = \frac{1}{i\lambda}\hat{f}(\lambda)$ (in the distributional sense).

We have, using Fubini and the fact that $\langle \hat{f}, \varphi \rangle$ is finite, that

$$\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}} \frac{\hat{\varphi}(\lambda)}{1+\lambda^2} \, \mathrm{d}\lambda = \int_{\mathbb{R}} \varphi(x) \lim_{R\to\infty} \int_{-R}^R \frac{e^{-i\lambda x}}{1+\lambda^2} \, \mathrm{d}\lambda \, \mathrm{d}x \overset{\star}{=} \int_{\mathbb{R}} \varphi(x) \left(\pi e^{-|x|}\right) \mathrm{d}x\,,$$

from which it follows that the Fourier transform of $\frac{1}{1+x^2}$ is given by $\pi e^{-|\lambda|}$, and therefore the Fourier transform of arctan is given by $\frac{\pi}{i\lambda} e^{-|\lambda|}$.

(c) The derivative of this function, outside of 0, is $\log|x|$.

(d) Clearly, if $\omega = 0$, the function is 1 and its Fourier transform is $2\pi\delta_0$, so assume $\omega \neq 0$. We have analogously to (b), with $f(x) = \exp\big(i\omega x^2\big)$, that

$$\langle \widehat{f}, \varphi \rangle = \int_{\mathbb{R}} \hat{\varphi}(\lambda) e^{i\omega\lambda^2} \,\mathrm{d}\lambda = \int_{\mathbb{R}} \varphi(x) \lim_{R\to\infty} \int_{-R}^{R} e^{i\omega\lambda^2 - i\lambda x} \,\mathrm{d}\lambda \,\mathrm{d}x\,.$$

Now, by completing the square we have

$$i(\omega\lambda^2 - x\lambda) = i\left(\sqrt{\omega}\lambda - \frac{x}{2\sqrt{\omega}}\right)^2 - \frac{ix^2}{4\omega},$$

and therefore

$$\begin{aligned}\lim_{R\to\infty} \int_{-R}^{R} e^{i\omega\lambda^2 - i\lambda x} \,\mathrm{d}\lambda = e^{-ix^2/(4\omega)} \lim_{R\to\infty} \int_{-R}^{R} e^{i(\sqrt{\omega}\lambda - \frac{x}{2\sqrt{\omega}})^2} \,\mathrm{d}\lambda = \frac{e^{-ix^2/(4\omega)}}{\sqrt{\omega}} \lim_{R\to\infty} \int_{-R}^{R} e^{i\lambda^2} \,\mathrm{d}\lambda \\ = \frac{e^{-ix^2/(4\omega)}}{\sqrt{\omega}} (1+i)\sqrt{\frac{\pi}{2}},\end{aligned}$$

where we use that the *Fresnel integral* $\int_{-\infty}^{\infty} e^{ix^2} \,\mathrm{d}x$ is known.
Plugging this back into our original equation yields

$$\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}} \varphi(x) \frac{e^{-ix^2/(4\omega)}}{\sqrt{\omega}} (1+i)\sqrt{\frac{\pi}{2}} \,\mathrm{d}x\,,$$

which shows that

$$\hat{f}(\lambda) = (1+i) e^{-i\lambda^2/(4\omega)} \sqrt{\frac{\pi}{2\omega}}.$$

in the distributional sense.

□