

Topics — Example Sheet 2

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Question 2. For $\beta \in (0, 1]$ and $L > 0$, let $\mathcal{F}_2(\beta, L)$ denote the class of densities on \mathbb{R}^2 that satisfy

$$|f(x, y) - f(x_0, y_0)| \leq L(|x - x_0|^\beta + |y - y_0|^\beta)$$

for all $(x, y), (x_0, y_0) \in \mathbb{R}^2$. Let K be a non-negative kernel on \mathbb{R} with $\mu_\beta(K)$ and $R(K)$ finite. Given i.i.d. pairs $(X_1, Y_1), \dots, (X_n, Y_n)$, consider the kernel density estimator \hat{f}_n obtained using a product kernel, i.e.,

$$\hat{f}_n(x_0, y_0) := \frac{1}{nh^2} \sum_{i=1}^n K\left(\frac{x_0 - X_i}{h}\right) K\left(\frac{y_0 - Y_i}{h}\right).$$

Find a bound on $\text{MSE}\left\{\hat{f}_n(x_0, y_0)\right\}$ that holds uniformly for all $f \in \mathcal{F}_2(\beta, L)$ and $(x_0, y_0) \in \mathbb{R}^2$.

Proof. First we compute a variance bound following the proof of proposition 19: noting that $\hat{f}_n(x, y) = \frac{1}{n} \sum_{i=1}^n K_h(x - X_i) K_h(y - Y_i)$, we have

$$\begin{aligned} \text{Var } \hat{f}_n(x, y) &= \frac{1}{n} \text{Var}(K_h(x - X_i) K_h(y - Y_i)) \leq \frac{1}{n} \mathbb{E}[K_h^2(x - X_i) K_h^2(y - Y_i)] \\ &= \frac{1}{nh^4} \iint_{\mathbb{R}^2} K^2\left(\frac{x-w}{h}\right) K^2\left(\frac{y-z}{h}\right) f(w, z) dw dz \\ &\leq \frac{\|f\|_\infty}{nh^2} \iint_{\mathbb{R}^2} K^2(s) K^2(t) ds dt = \frac{\|f\|_\infty R(K)^2}{nh^2}. \end{aligned}$$

Next, we compute a bias bound following the proof of proposition 22. Note that we have

$$\begin{aligned} \text{Bias } \hat{f}_n(x, y) &= \frac{1}{h^2} \iint_{\mathbb{R}^2} K\left(\frac{x-w}{h}\right) K\left(\frac{y-z}{h}\right) f(w, z) dw dz - f(x, y) \\ &= \iint_{\mathbb{R}^2} K(s) K(t) \{f(x - sh, y - th) - f(x, y)\} ds dt, \end{aligned}$$

and taking absolute values gives

$$\begin{aligned} \left| \text{Bias } \hat{f}_n(x, y) \right| &\leq \iint_{\mathbb{R}^2} K(s) K(t) |f(x - sh, y - th) - f(x, y)| ds dt \\ &\leq L \iint_{\mathbb{R}^2} K(s) K(t) (|sh|^\beta + |th|^\beta) ds dt \\ &= 2Lh^\beta \int_{\mathbb{R}} K(s) \int_{\mathbb{R}} |t| K(t) dt ds \\ &= 2Lh^\beta \mu_\beta(K) \int_{\mathbb{R}} K(s) ds = 2Lh^\beta \mu_\beta(K). \end{aligned}$$

We therefore have

$$\text{MSE } \hat{f}_n(x, y) \leq \frac{1}{nh^2} \|f\|_\infty R(K)^2 + 4L^2 \mu_\beta(K)^2 h^{2\beta}.$$

Completely analogous to the proof of theorem 23, we can show that $\|f\|_\infty$ is bounded uniformly over $\mathcal{F}_2(\beta, L)$, and the minimiser of MSE is of order $n^{-1/(2\beta+2)}$. Plugging this into the expression gives

$$\sup_{(x,y)} \sup_{f \in \mathcal{F}} \text{MSE } \hat{f}_n(x, y) \leq C n^{-2\beta/(2\beta+2)},$$

for some C depending only on β, L, K . □

Question 10. Let $n \geq 3$, let $a \leq x_1 < \dots < x_n \leq b$, and let $\mathbf{g} \in \mathbb{R}^n$. Prove that the natural cubic spline interpolant to \mathbf{g} at x_1, \dots, x_n is the unique minimiser of $R(\tilde{g}) = \int_a^b \tilde{g}''(x)^2 dx$ over all $\tilde{g} \in \mathcal{S}_2[a, b]$ that interpolate \mathbf{g} at x_1, \dots, x_n .

Hint: Let g be the natural cubic spline interpolant, $h := \tilde{g} - g$, and consider $\int_a^b g''(x)h''(x) dx$.

Solution. As in the hint we let g be the natural cubic spline interpolant, $\tilde{g} \in \mathcal{S}_2[a, b]$, and $h := \tilde{g} - g$. We know that g is a cubic polynomial p_i on each interval $[x_i, x_{i+1}]$ (denote its leading coefficient by c_i), that g'' is continuous, and that $g'' = 0$ on $[a, x_1]$ and on $[x_n, b]$. Furthermore, we know that $h(x_i) = 0$ for all i . Using this, we can write

$$\begin{aligned} \int_a^b g''(x)h''(x) dx &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} p_i''(x)h''(x) dx \\ &= \sum_{i=1}^{n-1} \left([p_i''(x)h'(x)]_{x=x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} p_i'''(x)h'(x) dx \right) \\ &= - \sum_{i=1}^{n-1} c_i \int_{x_i}^{x_{i+1}} h'(x) dx \\ &= - \sum_{i=1}^{n-1} c_i (h(x_{i+1}) - h(x_i)) = 0. \end{aligned}$$

Now we find

$$R(g) = \int_a^b g''(x)^2 dx = \int_a^b g''(x)(\tilde{g}''(x) - h''(x)) dx = \int_a^b g''(x)\tilde{g}''(x) dx \stackrel{\text{CS}}{\leq} \sqrt{R(g)}\sqrt{R(\tilde{g})},$$

with equality if and only if $g'' = \tilde{g}''$ (by Cauchy-Schwarz). Rearranging gives $\sqrt{R(g)} \leq \sqrt{R(\tilde{g})} \implies R(g) \leq R(\tilde{g})$. Since $\tilde{g} \in \mathcal{S}_2[a, b]$ was arbitrary, we deduce that g is a minimiser of R over all function in \mathcal{S}_2 .

For uniqueness: we already know that any other minimiser $h \in \mathcal{S}_2$ must satisfy $g'' = h''$ a.e., so $g - h$ is a polynomial of degree 1 a.e., and by continuity we know that $g - h$ is a polynomial of degree 1. However, since g and h must agree on $n \geq 3$ points it follows that $g = h$.