

# Distribution Theory and Applications — Summary

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## Contents

<b>1</b>	<b>Distributions</b>	<b>2</b>
1.1	Test functions and distributions . . . . .	2
1.2	Limits in $\mathcal{D}'(X)$ . . . . .	3
1.3	Basic operations . . . . .	4
1.3.1	Differentiation and multiplication by smooth functions . . . . .	4
1.3.2	Reflection and translation . . . . .	4
1.3.3	Convolution . . . . .	5

# 1 Distributions

## 1.1 Test functions and distributions

**Definition 1.1.** Let  $X \subseteq \mathbb{R}^n$  be open, then we define the set of *test functions* on  $X$  as

$$\mathcal{D}(X) := C_0^\infty(X) = \{f: X \rightarrow \mathbb{C} \mid f \text{ is smooth with compact support}\}.$$

**Definition 1.2.** Let  $(\varphi_m) \subseteq \mathcal{D}(X)$ . We say that  $(\varphi_m) \rightarrow 0$  in  $\mathcal{D}(X)$  if

1. there exists a compact  $K \subseteq X$  such that  $\text{supp } \varphi_m \subseteq K$  for all  $m$ ;
2.  $\partial^\alpha \varphi_m \rightarrow 0$  uniformly for each multi-index  $\alpha$ .

Note that, for any  $\varphi, \psi \in \mathcal{D}(X)$  and any multi-index  $\alpha$  we have

$$\int_X \varphi \cdot \partial^\alpha \psi \, dx = (-1)^{|\alpha|} \int_X \psi \cdot \partial^\alpha \varphi \, dx,$$

which follows from partial integration and the fact that all boundary terms vanish since  $\varphi$  and  $\psi$  have compact support.

Also, by Taylor's theorem, for any  $\varphi \in \mathcal{D}(X)$ ,  $x, h \in X$  and  $N \in \mathbb{N}$  we have

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h) \quad \text{where } R_N(x, h) = o(|h|^N) \text{ uniformly in } x.$$

**Definition 1.3.** A *distribution* on  $X$  is a linear map  $u: \mathcal{D}(X) \rightarrow \mathbb{C}$  if for every compact set  $K \subseteq X$  there exist constants  $C, N$  such that for all  $\varphi \in \mathcal{D}(X)$  with  $\text{supp } \varphi \subseteq K$  we have

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi|. \quad (1)$$

Condition 1 is called the *seminorm condition*. If, in the seminorm condition, the same  $N$  can be used for every compact set  $K \subseteq X$ , then the least such  $N$  is called the *order* of  $u$ , written  $\text{ord}(u)$ .

The set of all distributions in  $X$  is denoted  $\mathcal{D}'(X)$ .

If  $u \in \mathcal{D}'(X)$  and  $\varphi \in \mathcal{D}(X)$ , then instead of  $u(\varphi)$  we usually write  $\langle u, \varphi \rangle$ .

**Recap 1.4.** A function  $f: X \rightarrow \mathbb{C}$  is called *locally integrable* if  $\int_K |f| \, dx < \infty$  for all compact  $K \subseteq X$ .

The set of locally integrable functions on  $X$  is denoted  $L_{\text{loc}}^1(X)$ .

**Example 1.5.** Let  $M \in \mathbb{N}$  and let  $f_\alpha \in L_{\text{loc}}^1(X)$  for all  $|\alpha| \leq M$ . Define the linear map  $T: \mathcal{D}(X) \rightarrow \mathbb{C}$  by

$$\langle T, \varphi \rangle := \sum_{|\alpha| \leq M} \int_X f_\alpha \cdot \partial^\alpha \varphi \, dx.$$

It is trivial that  $T$  is linear, and we verify that  $T$  is a distribution as follows: take  $\varphi \in \mathcal{D}(X)$  with  $\text{supp } \varphi \subseteq K$ . Then we have

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \sum_{|\alpha| \leq M} \int_K |f_\alpha| \cdot |\partial^\alpha \varphi| \, dx \\ &\leq \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi| \cdot \int_K |f_\alpha| \, dx \\ &\leq \left( \max_{|\alpha| \leq M} \int_K |f_\alpha| \, dx \right) \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi|. \end{aligned}$$

Therefore, the seminorm condition is satisfied with  $N = M$ . From this, it also follows that  $\text{ord}(T) \leq M$ .

A special case of the previous example is the case  $M = 0$ : in this case the distribution simply becomes

$$\langle T_f, \varphi \rangle = \int_X f \varphi \, dx.$$

Henceforth we will abuse notation: if  $f \in L^1_{\text{loc}}(X)$ , then we will write  $f$  instead of  $T_f$ , i.e.,  $\langle f, \varphi \rangle = \int_X f \varphi \, dx$ .

**Lemma 1.6.** *Let  $u: \mathcal{D}(X) \rightarrow \mathbb{C}$  be a linear map. Then  $u$  is a distribution if and only if, for every sequence  $(\varphi_j) \subseteq \mathcal{D}(X)$  with  $\varphi_j \rightarrow 0$  as in definition 1.2, we have  $\langle u, \varphi_m \rangle \rightarrow 0$ .*

*Proof.* ‘ $\implies$ ’ If  $u$  is a distribution and  $(\varphi_m) \rightarrow 0$ , then  $\text{supp } \varphi_m \subseteq K$  for some compact  $K$ , and by eq. (1) there exist  $C, N$  such that

$$|\langle u, \varphi_m \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_m| \rightarrow 0.$$

‘ $\impliedby$ ’ Suppose there is a compact set  $K$  such that eq. (1) is not valid for any  $C, N$ . Let  $m \in \mathbb{N}$  and  $C = N = m$ , then there is some  $\varphi_m$  with  $\text{supp}(\varphi_m) \subseteq K$ , and

$$|\langle u, \varphi_m \rangle| > m \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m|.$$

By dividing  $\varphi_m$  by  $\langle u, \varphi_m \rangle \neq 0$ , we may assume w.l.o.g. that  $\langle u, \varphi_m \rangle = 1$ . We now have a sequence  $(\varphi_m)$  such that

$$\frac{1}{m} > \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m| \implies |\partial^\alpha \varphi_m| < \frac{1}{m} \quad \text{for } |\alpha| \leq m \implies \partial^\alpha \varphi_m \rightarrow 0 \text{ uniformly for all } \alpha.$$

Since each  $\varphi_m$  also satisfies  $\text{supp } \varphi_m \subseteq K$ , by definition 1.2 we have that  $\varphi_m \rightarrow 0$ , but also  $\langle u, \varphi_m \rangle \rightarrow 1$ , a contradiction.  $\square$

## 1.2 Limits in $\mathcal{D}'(X)$

**Definition 1.7.** We say that a sequence  $(u_m) \subseteq \mathcal{D}'(X)$  converges to  $u \in \mathcal{D}'(X)$  and write  $u_m \rightarrow u$  if

$$\langle u_m, \varphi \rangle \rightarrow \langle u, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(X).$$

The following theorem is non-examinable but interesting:

**Theorem 1.8.** *Let  $(u_m)$  be a sequence in  $\mathcal{D}'(X)$  such that  $\lim_{m \rightarrow \infty} \langle u_m, \varphi \rangle$  exists for all  $\varphi \in \mathcal{D}(X)$ . Then the map  $\langle u, \varphi \rangle := \lim_{m \rightarrow \infty} \langle u_m, \varphi \rangle$  is a distribution in  $X$ .*

*Proof.* This is a direct application of the uniform boundedness principle.  $\square$

**Example 1.9.** Let  $X = \mathbb{R}$  and consider the sequence of functions  $u_m \in L^1_{\text{loc}}(\mathbb{R})$  defined by  $u_m(x) = \sin(mx)$ . Then, for all  $\varphi \in \mathcal{D}(\mathbb{R})$ , we have

$$\langle u_m, \varphi \rangle = \int_{\mathbb{R}} \sin(mx) \varphi(x) \, dx = \frac{1}{m} \int_{\mathbb{R}} \cos(mx) \varphi'(x) \, dx \leq \frac{1}{m} \int_{\mathbb{R}} |\varphi'(x)| \, dx \rightarrow 0.$$

Therefore, it holds that  $u_m \rightarrow 0$  in  $\mathcal{D}'(\mathbb{R})$ . With our abuse of notation we write this as  $\lim_{m \rightarrow \infty} \sin(mx) = 0$  in  $\mathcal{D}'(\mathbb{R})$ .

### 1.3 Basic operations

#### 1.3.1 Differentiation and multiplication by smooth functions

For  $u \in C^\infty(X)$  and  $\varphi \in \mathcal{D}(X)$ , we have noted that

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \partial^\alpha u \cdot \varphi \, dx = (-1)^{|\alpha|} \int_X u \cdot \partial^\alpha \varphi \, dx = \langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle.$$

Since the RHS makes sense for any distribution  $u$ , we define

**Definition 1.10.** For  $f \in C^\infty(X)$ ,  $u \in \mathcal{D}'(X)$ , we define  $\partial^\alpha(fu)$  by

$$\langle \partial^\alpha(fu), \varphi \rangle := \langle u, (-1)^{|\alpha|} f \cdot \partial^\alpha \varphi \rangle$$

*Remark.* This definition outlines a more general pattern when working with distributions: first we take some well-defined operator on the collection of smooth maps, then we rewrite it to a form that is sensible for any distribution, and then we *define* that new form as the operator on distributions. This process is called *extending the definition by duality*.

**Example 1.11.** Let  $u = \delta_x$ , then we have

$$\langle \partial^\alpha \delta_x, \varphi \rangle = \langle \delta_x, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x)$$

Furthermore, consider the Heaviside function  $H(x) = \mathbb{1}_{x \geq 0}$ . We have

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{\mathbb{R}} H(x) \varphi'(x) \, dx = -\int_0^\infty \varphi'(x) \, dx = \varphi(0) = \langle \delta_0, \varphi \rangle,$$

so we write  $H' = \delta_0$  in the distributional sense.

**Lemma 1.12.** Suppose  $u' \in \mathcal{D}'(\mathbb{R})$  satisfies  $u' = 0$ . Then  $u$  is constant (i.e.,  $\langle u, \varphi \rangle = \langle c, \varphi \rangle = c \int_{\mathbb{R}} \varphi \, dx$  for some  $c$ ).

*Proof.* Fix any  $\vartheta \in \mathcal{D}(\mathbb{R})$  with  $\langle 1, \vartheta \rangle = 1$ . Let  $\varphi \in \mathcal{D}(\mathbb{R})$  and define

$$\varphi_A := \varphi - \langle 1, \varphi \rangle \vartheta, \quad \varphi_B := \langle 1, \varphi \rangle \vartheta \quad \text{such that } \varphi = \varphi_A + \varphi_B.$$

Note that  $\langle 1, \varphi_A \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle \langle 1, \vartheta \rangle = 0$ .

We claim that the function  $\Phi_A(x) := \int_{-\infty}^x \varphi_A(y) \, dy$  has compact support: since  $\text{supp } \varphi_A \subseteq [a, b]$  for some  $a, b \in \mathbb{R}$ , clearly  $\Phi_A(x) = 0$  for  $x < a$ , while for  $x > b$  we have  $\Phi_A(x) = \langle 1, \varphi_A \rangle = 0$ . Obviously, it holds that  $\Phi'_A = \varphi_A$ . Now we compute

$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle = \langle u, \Phi'_A \rangle + \langle 1, \varphi \rangle \langle u, \vartheta \rangle = -\langle u', \Phi_A \rangle + c \langle 1, \varphi \rangle = \langle c, \varphi \rangle.$$

Since  $\varphi$  was chosen arbitrarily this shows that  $u$  is constant. □

#### 1.3.2 Reflection and translation

For  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ ,  $h \in \mathbb{R}^n$ , define the *translation of  $\varphi$  by  $h$*  by

$$(T_h \varphi)(x) := \varphi(x - h),$$

and the *reflection of  $\varphi$*  by  $\check{\varphi}(x) := \varphi(-x)$ .

Extending the definitions of translation and reflection by duality yields the following:

**Definition 1.13.** For  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $h \in \mathbb{R}^n$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , define

$$\langle T_h u, \varphi \rangle := \langle u, T_{-h} \varphi \rangle \quad \text{and} \quad \langle \check{u}, \varphi \rangle := \langle u, \check{\varphi} \rangle.$$

**Lemma 1.14.** For  $u \in \mathcal{D}'(\mathbb{R}^n)$ , define  $V_h \in \mathcal{D}'(\mathbb{R}^n)$  for  $0 \neq h \in \mathbb{R}^n$  by

$$V_h := \frac{T_{-h}u - u}{\|h\|}$$

If  $(h_j) \subseteq \mathbb{R}^n$  is a sequence for which  $\lim_{j \rightarrow \infty} \frac{h_j}{\|h_j\|} = m \in S^{n-1}$ , then  $V_{h_j} \rightarrow \sum_i m_i \frac{\partial u}{\partial x_i}$  in  $\mathcal{D}'(\mathbb{R}^n)$ .

*Proof.* By definition, we can write  $\langle V_h, \varphi \rangle = \frac{1}{\|h\|} \langle u, T_h \varphi - \varphi \rangle$ . Now Taylor's theorem tells us that

$$(T_h \varphi - \varphi)(x) = \varphi(x - h) - \varphi(x) = - \sum_i h_i \frac{\partial \varphi}{\partial x_i} + R(x, h),$$

where  $R(x, h) = o(\|h\|)$  in  $\mathcal{D}(\mathbb{R}^n)$  (exercise sheet 1, question 2).

By sequential continuity, we have

$$\lim_{j \rightarrow \infty} \langle V_{h_j}, \varphi \rangle = \langle u, - \sum_i m_i \frac{\partial \varphi}{\partial x_i} \rangle = \langle \sum_i m_i \frac{\partial u}{\partial x_i}, \varphi \rangle,$$

which shows that  $V_{h_j} \rightarrow \sum_i m_i \frac{\partial u}{\partial x_i}$  in  $\mathcal{D}'(\mathbb{R}^n)$ . □

### 1.3.3 Convolution

For  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , note that  $(T_x \check{\varphi})(y) = \check{\varphi}(y - x) = \varphi(x - y)$ .

**Definition 1.15.** For  $u \in C^\infty(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , we define the *convolution*  $u * \varphi: \mathbb{R}^n \rightarrow \mathbb{C}$  as

$$(u * \varphi)(x) := \int_{\mathbb{R}^n} u(x - y) \varphi(y) dy = \int_{\mathbb{R}^n} u(y) \varphi(x - y) dy = \langle u, T_x \check{\varphi} \rangle.$$

Since the RHS makes sense for any  $u \in \mathcal{D}'(\mathbb{R}^n)$ , we extend the definition this way: for  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , we define the convolution  $u * \varphi$  as

$$(u * \varphi)(x) := \langle u, T_x \check{\varphi} \rangle.$$

**Lemma 1.16.** Let  $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$  and define  $\Phi_x(y) := \varphi(x, y)$ . Suppose for any  $x \in \mathbb{R}^n$  there exists a neighbourhood  $N(x)$  and a compact  $K \subseteq \mathbb{R}^m$  such that  $\varphi(x, y)$  for all  $x \in N(x)$ ,  $y \notin K$ .

Then  $x \mapsto \langle u, \Phi_x \rangle$  is differentiable with

$$\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi_x \rangle$$

for any  $u \in \mathcal{D}'(\mathbb{R}^m)$ .

*Proof.* Fix  $x \in \mathbb{R}^n$ , then by Taylor's formula we have

$$\Phi_{x+h}(y) = \Phi_x(y) + \sum_{i=1}^n h_i \frac{\partial \varphi}{\partial x_i} + R(x, y, h),$$

where  $\partial_y^\alpha R(x, y, h) = o(\|h\|)$ , uniformly in  $y$ , for any multi-index  $\alpha$ . Furthermore, by assumption there exists a compact  $K$  such that for  $h$  small enough,  $\text{supp } R(x, \cdot, h) \subseteq K$ . Therefore,  $R(x, \cdot, h)$  is a test function for  $h$  small enough.

Combining the previous two facts shows that  $R(x, \cdot, h) = o(\|h\|)$  in  $\mathcal{D}(\mathbb{R}^m)$  as  $h \rightarrow 0$ .

Let  $u \in \mathcal{D}'(\mathbb{R}^m)$ , then we find by sequential continuity that  $\langle u, R(x, \cdot, h) \rangle$  is also  $o(\|h\|)$ , and therefore

$$\langle u, \Phi_{x+h} \rangle = \langle u, \Phi_x \rangle + \sum_{i=1}^n h_i \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle + o(\|h\|).$$

This shows that  $x \mapsto \langle u, \Phi_x \rangle$  is differentiable with

$$\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle.$$

From this the result follows. □

**Corollary 1.17.** *If  $u \in \mathcal{D}'(\mathbb{R}^n)$ ,  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , then  $u * \varphi$  is differentiable with  $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi$ .*

*Proof.* Apply the previous lemma with  $\Phi_x(y) := \varphi(x - y)$ . □