

# Topics in Statistical Theory — Summary

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# 1 Basic concepts

## 1.1 Parametric vs nonparametric models

**Definition 1.1.** A *statistical model* is a family of possible data-generating mechanisms. If the parameter space  $\Theta$  is finite-dimensional, we speak of a *parametric model*.

A model is called *well-specified* if there is a  $\vartheta_0 \in \Theta$  for which the data was generated from the distribution with parameter  $\vartheta_0$ , and otherwise it is called *misspecified*.

**Recap 1.2.** Let  $(Y_n)$  be a sequence of random vectors and  $Y$  a random vector.

1. We say that  $(Y_n)$  *converges almost surely* to  $Y$ , notation  $Y_n \xrightarrow{\text{a.s.}} Y$ , if  $\mathbb{P}(Y_n \rightarrow Y) = 1$ .
2. We say that  $(Y_n)$  *converges in probability* to  $Y$ , notation  $Y_n \xrightarrow{\text{P}} Y$ , if for every  $\varepsilon > 0$  we have  $\mathbb{P}(\|Y_n - Y\| > \varepsilon) \rightarrow 0$ .
3. We say that  $(Y_n)$  *converges in distribution* to  $Y$ , notation  $Y_n \xrightarrow{\text{d}} Y$ , if  $\mathbb{P}(Y_n \leq y) \rightarrow \mathbb{P}(Y \leq y)$  for all  $y$  where the distribution function of  $Y$  is continuous.

This is equivalent to the condition that  $\mathbb{E}[f(Y_n)] \rightarrow \mathbb{E}[f(Y)]$  for all bounded Lipschitz functions  $f$ .

It is known that  $Y_n \xrightarrow{\text{a.s.}} Y \implies Y_n \xrightarrow{\text{P}} Y \implies Y_n \xrightarrow{\text{d}} Y$ .

If  $(Y_n)$  is a sequence of random vectors and  $(a_n)$  is a positive sequence, then we write  $Y_n = O_p(a_n)$  if, for all  $\varepsilon > 0$ , there exists  $C > 0$  such that for sufficiently large  $n$  we have

$$\mathbb{P}\left(\frac{\|Y_n\|}{a_n} > C\right) < \varepsilon.$$

We write  $Y_n = o_p(a_n)$  if  $Y_n/a_n \xrightarrow{\text{P}} 0$ .

In a well-specified parametric model, the maximum likelihood estimator (MLE)  $\hat{\vartheta}_n$  typically satisfies  $\hat{\vartheta}_n - \vartheta_0 \in O_p(n^{-1/2})$ . On the other hand, if the model is misspecified, any inference can give very misleading results. To circumvent this problem, we consider *nonparametric models*, which make much weaker assumptions. Such infinite-dimensional models are much less vulnerable to model misspecification, however we will typically pay a price in terms of a slower convergence rate than in well-specified parametric models.

**Example 1.3.** Examples of nonparametric models include:

1. Assume  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F$  for some unknown distribution function  $F$ .
2. Assume  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} f$  for some unknown density  $f$  belonging to a smoothness class.
3. Assume  $Y_i = m(x_i) + \varepsilon_i$  ( $i = 1, \dots, n$ ), where the  $x_i$  are known,  $m$  is unknown and belongs to some smoothness class, and the  $\varepsilon_i$  are i.i.d. with  $\mathbb{E}(\varepsilon_i) = 0$  and  $\text{Var}(\varepsilon_i) = \sigma^2$ .

## 1.2 Estimating an arbitrary distribution function

**Definition 1.4.** Let  $\mathcal{F}$  denote the class of all distribution functions on  $\mathbb{R}$  and suppose  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F \in \mathcal{F}$ . The *empirical distribution function*  $\hat{F}_n$  of  $X_1, \dots, X_n$  is defined as

$$\hat{F}_n(x) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}}.$$

**Recap 1.5.** The *strong law of large numbers* tells us that if  $(Y_n)$  are i.i.d. with finite mean  $\mu$ , then  $\bar{Y} := \frac{1}{n} \sum_{i=1}^n Y_i \xrightarrow{\text{a.s.}} \mu$ .

Note that the strong law of large numbers immediately implies that  $\hat{F}_n(x)$  converges almost surely to  $F(x)$  as  $n \rightarrow \infty$ . However, the following stronger result states that this convergence holds uniformly in  $x$ :

**Theorem 1.6** (Glivenko-Cantelli). *Let  $X_1, X_2, \dots \stackrel{\text{iid}}{\sim} F$ . Then we have*

$$\sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F(x) \right| \xrightarrow{\text{a.s.}} 0.$$

*Proof.* See lecture notes. The main idea of the proof is to “control”  $\hat{F}_n$  in a finite number of points  $x_1, \dots, x_k$ , and then deduce what happens between those points using the fact that distributions are increasing and right-continuous. On [Wikipedia](#), a simplified proof can be found assuming that  $F$  is continuous, which still encapsulates the main idea.  $\square$

**Theorem 1.7** (Dvoretzky-Kiefer-Wolfowitz). *Under the conditions of theorem 1.6, for every  $\varepsilon > 0$  it holds that*

$$\mathbb{P}_F \left( \sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F(x) \right| > \varepsilon \right) \leq 2e^{-2n\varepsilon^2},$$

*and this is a tight bound.*

We will not prove this theorem, however, we will explore a few consequences. One of these consequences is the following:

**Corollary 1.8** (Uniform Glivenko-Cantelli theorem). *Under the conditions of theorem 1.6, for every  $\varepsilon > 0$ , it holds that*

$$\sup_{F \in \mathcal{F}} \mathbb{P}_F \left( \sup_{m \geq n} \sup_{x \in \mathbb{R}} \left| \hat{F}_m(x) - F(x) \right| > \varepsilon \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Proof.* By a union bound, the DKW inequality, and convergence of the geometric series we have

$$\begin{aligned} \sup_{F \in \mathcal{F}} \mathbb{P}_F \left( \sup_{m \geq n} \sup_{x \in \mathbb{R}} \left| \hat{F}_m(x) - F(x) \right| > \varepsilon \right) &\leq \sup_{F \in \mathcal{F}} \sum_{m=n}^{\infty} \mathbb{P}_F \left( \sup_{x \in \mathbb{R}} \left| \hat{F}_m(x) - F(x) \right| > \varepsilon \right) \\ &\leq 2 \sum_{m=n}^{\infty} e^{-2m\varepsilon^2}, \end{aligned}$$

which converges to 0 as it is the tail of a converging sum.  $\square$

For another consequence, we consider the problem of finding a confidence band for  $F$ . Given  $\alpha \in (0, 1)$ , set  $\varepsilon_n := \sqrt{-\frac{1}{2n} \log(\alpha/2)}$ . Then the DKW inequality tells us that

$$\mathbb{P}_F \left( \sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F(x) \right| > \varepsilon_n \right) \leq \alpha,$$

or equivalently, that

$$\mathbb{P}_F \left( \hat{F}_n(x) - \varepsilon_n \leq F(x) \leq \hat{F}_n(x) + \varepsilon_n \text{ for all } x \in \mathbb{R} \right) \geq 1 - \alpha.$$

We can say even more.

**Recap 1.9.** For any distribution function  $F$ , its *quantile function* is defined as

$$F^{-1}: (0, 1] \rightarrow \mathbb{R} \cup \{\infty\}: p \mapsto \inf \{x \in \mathbb{R} \mid F(x) \geq p\}.$$

When necessary, we also define  $F^{-1}(0) := \sup \{x \in \mathbb{R} \mid F(x) = 0\}$ .

If  $U \sim U(0, 1)$  and  $X \sim F$ , then for any  $x \in \mathbb{R}$  we have

$$\mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x) = \mathbb{P}(X \leq x).$$

This can be written simply as  $F^{-1}(U) \stackrel{d}{=} X$ .

Let  $U_1, \dots, U_n \stackrel{\text{iid}}{\sim} U(0, 1)$  with empirical distribution function  $\hat{G}_n$ , and let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F$ . Then, we have

$$\hat{G}_n(F(x)) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{U_i \leq F(x)\}} \stackrel{d}{=} \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} = \hat{F}_n(x),$$

where  $\stackrel{d}{=}$  means equality in distribution. It follows that

$$\sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F(x) \right| \stackrel{d}{=} \sup_{x \in \mathbb{R}} \left| \hat{G}_n(F(x)) - F(x) \right| \leq \sup_{t \in [0, 1]} \left| \hat{G}_n(t) - t \right|,$$

with equality if  $F$  is continuous. We conclude that if  $F$  is continuous, the distribution of  $\sup_{x \in \mathbb{R}} \left| \hat{F}_n(x) - F(x) \right|$  does not depend on  $F$ .

Other generalisations of theorem 1.6 include Uniform Laws of Large Numbers. Let  $X, X_1, \dots, X_n$  be i.i.d. on a measurable space  $(\mathcal{X}, \mathcal{A})$ , and  $\mathcal{G}$  a class of measurable functions on  $\mathcal{X}$ . We say that  $\mathcal{G}$  satisfies a ULLN if

$$\sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(X_i) - \mathbb{E}[g(X)] \right| \xrightarrow{\text{a.s.}} 0.$$

In theorem 1.6, we showed that  $\mathcal{G} = \{\mathbb{1}_{\{ \cdot \leq x \}} \mid x \in \mathbb{R}\}$  satisfies a ULLN.

**Recap 1.10.** We recall the central limit theorem: if  $X_1, \dots, X_n$  are i.i.d. random variables with mean  $\mu$  and variance  $\sigma^2 < \infty$ , then  $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$ .

Dividing by  $\sigma$  yields

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0, 1),$$

and multiplying both sides by  $n$  and writing  $V_i = \sum_{j=1}^i X_j$  we obtain

$$\frac{V_i - \mathbb{E}V_i}{\sqrt{\text{Var}(V_i)}} \xrightarrow{d} N(0, 1).$$

Another extension starts with the observation that  $\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} N(0, \sigma^2)$ , where

$$\sigma^2 = \text{Var}(\mathbb{1}_{\{X \leq x\}}) = \mathbb{E}[\mathbb{1}_{\{X \leq x\}}^2] - \mathbb{E}[\mathbb{1}_{\{X \leq x\}}]^2 = F(x) - F(x)^2 = F(x)(1 - F(x)).$$

This can be strengthened by considering  $(\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} N(0, \sigma^2) \mid x \in \mathbb{R})$  as a stochastic process.

### 1.3 Order statistics and quantiles

**Definition 1.11.** Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F \in \mathcal{F}$ . The *order statistics* are the ordered samples  $X_{(1)} \leq \dots \leq X_{(n)}$  (where the original order is preserved in case of a tie).

The order statistics of the uniform distribution can be computed explicitly:

**Proposition 1.12.** Let  $U_1, \dots, U_n \stackrel{\text{iid}}{\sim} U(0, 1)$ , let  $Y_1, \dots, Y_{n+1} \stackrel{\text{iid}}{\sim} \text{Exp}(1)$ , and write  $S_j := \sum_{i=1}^j Y_i$  ( $j = 1, \dots, n+1$ ). Then

$$U_{(j)} \stackrel{d}{=} \frac{S_j}{S_{n+1}} \sim \text{Beta}(j, n-j+1) \quad \text{for } j = 1, \dots, n.$$

*Proof.* See example sheet 1, question 1. □

**Definition 1.13.** Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F$ . Then the *sample quantile function* is defined as

$$\hat{F}_n^{-1}(p) = \inf \left\{ x \in \mathbb{R} \mid \hat{F}_n(x) \geq p \right\}.$$

**Proposition 1.14.** It holds that  $\hat{F}_n^{-1}(p) = X_{(\lceil np \rceil)}$ .

*Proof.* By definition,  $\hat{F}_n^{-1}(p)$  is the smallest value of  $x$  for which  $\hat{F}_n(x)$  is larger than  $p$ . Note that

$$\hat{F}_n(x) \geq p \iff \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} \geq p \iff \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} \geq np \iff \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}} \geq \lceil np \rceil.$$

The smallest value of  $x$  for which this occurs is the smallest value of  $x$  such that exactly  $\lceil np \rceil$  of the variables  $X_1, \dots, X_n$  satisfy  $X_i \leq x$ . We conclude that  $\hat{F}_n^{-1}(p) = X_{(\lceil np \rceil)}$ . □

For  $p = \frac{1}{2}$  for example, this proposition tells us that  $\hat{F}_n^{-1}(p) = X_{(\lceil n/2 \rceil)}$ , the median of the data. We now explore the distribution of  $X_{(\lceil np \rceil)}$ .

**Recap 1.15.** We recall two theorems. The first is *Slutsky's theorem*:

**Theorem 1.16.** Let  $(Y_n)$  and  $(Z_n)$  be sequences of random vectors with  $Y_n \xrightarrow{d} Y$  and  $Z_n \xrightarrow{p} c$  for some constant  $c$ . If  $g$  is a continuous real-valued function, then  $g(Y_n, Z_n) \xrightarrow{d} g(Y, c)$ .

The second is the *delta method*:

**Theorem 1.17.** Let  $(Y_n)$  be a sequence of random vectors such that  $\sqrt{n}(Y_n - \mu) \xrightarrow{d} Z$ . If  $g: \mathbb{R}^d \rightarrow \mathbb{R}$  is differentiable at  $\mu$ , then

$$\sqrt{n}(g(Y_n) - g(\mu)) \xrightarrow{d} g'(\mu)Z.$$

**Lemma 1.18.** If  $U_1, \dots, U_n \stackrel{\text{iid}}{\sim} U(0, 1)$  and  $p \in (0, 1)$ , then  $\sqrt{n}(U_{\lceil np \rceil} - p) \xrightarrow{d} N(0, p(1-p))$ .

*Proof.* Let  $Y_1, \dots, Y_{n+1} \stackrel{\text{iid}}{\sim} \text{Exp}(1)$ ,  $V_n := \sum_{i=1}^{\lceil np \rceil} Y_i$  and  $W_n := \sum_{i=\lceil np \rceil+1}^{n+1} Y_i$ . Then  $V_n$  and  $W_n$  are independent, and we have seen that  $U_{\lceil np \rceil} \sim \frac{V_n}{V_n + W_n}$ .

Noting that  $\mathbb{E}V_n = \text{Var}(V_n) = \lceil np \rceil$  we find

$$\begin{aligned} \sqrt{n} \left( \frac{V_n}{n} - p \right) &= \frac{\sqrt{\lceil np \rceil}}{\sqrt{n}} \left( \frac{V_n - \lceil np \rceil}{\sqrt{\lceil np \rceil}} \right) + \frac{\lceil np \rceil - np}{\sqrt{n}} \\ &= \frac{\sqrt{\lceil np \rceil}}{\sqrt{n}} \left( \frac{V_n - \mathbb{E}V_n}{\sqrt{\text{Var}(V_n)}} \right) + \frac{\lceil np \rceil - np}{\sqrt{n}}. \end{aligned}$$

Now, by the central limit theorem, the term between brackets converges to a standard  $N(0, 1)$  distribution. The term  $\sqrt{\lceil np \rceil} \sqrt{n}$  converges to  $\sqrt{p}$  and the term  $(\lceil np \rceil - np)/\sqrt{n}$  converges to 0, so by Slutsky's lemma, we find

$$\sqrt{n} \left( \frac{V_n}{n} - p \right) \xrightarrow{d} \sqrt{p} N(0, 1) = N(0, p).$$

An analogous calculation shows that  $\sqrt{n} \left( \frac{W_n}{n} - (1 - p) \right) \rightarrow N(0, 1 - p)$ .

Now we define  $g: (0, \infty)^2 \rightarrow (0, \infty)$  by  $g(x, y) := x/(x + y)$ , which is differentiable at  $(p, 1 - p)$ . Note that the distribution of  $(V_n, W_n)$  is an  $N(0, \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix})$  distribution. By the delta method we find

$$\begin{aligned} \sqrt{n}(U_{\lceil np \rceil} - p) &\stackrel{d}{=} \sqrt{n} \left( g \left( \frac{V_n}{n}, \frac{W_n}{n} \right) - g(p, q) \right) \\ &\stackrel{d}{\rightarrow} g'(p, 1 - p) N \left( 0, \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right) \\ &= N \left( 0, g'(p, 1 - p) \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} g'(p, 1 - p)^\top \right) \\ &= N(0, p(1 - p)). \end{aligned}$$

□

We now relate what we know about the uniform distribution to the quantile function:

**Theorem 1.19.** *Let  $p \in (0, 1)$  and let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} F$ . Suppose that  $F$  is differentiable at  $\xi_p := F^{-1}(p)$  with derivative  $f(\xi_p)$ . Then*

$$\sqrt{n}(X_{(\lceil np \rceil)} - \xi_p) \xrightarrow{d} N \left( 0, \frac{p(1 - p)}{f(\xi_p)^2} \right).$$

*Proof.* Let  $U_1, \dots, U_n \stackrel{\text{iid}}{\sim} U(0, 1)$ , then we know that  $F^{-1}(U_i) \stackrel{d}{=} X_i$  and thus  $F^{-1}(U_{(\lceil np \rceil)}) \stackrel{d}{=} X_{(\lceil np \rceil)}$ . Applying the delta method with  $g = F^{-1}$ , together with the previous theorem yields

$$\sqrt{n}(X_{(\lceil np \rceil)} - \xi_p) = \sqrt{n}(F^{-1}(U_{(\lceil np \rceil)}) - F^{-1}(p)) \xrightarrow{d} (F^{-1})'(p) \cdot N(0, p(1 - p)).$$

Noting that  $(F^{-1})'(p) = \frac{1}{f(\xi_p)}$  yields the result. □

## 1.4 Concentration inequalities

We turn our attention to concentration inequalities, with a focus on finite-sample results (instead of results that only hold for  $n \rightarrow \infty$ ).

**Definition 1.20.** A random variable  $X$  with mean 0 is called *sub-Gaussian* with parameter  $\sigma^2$  if

$$M_X(t) = \mathbb{E}(e^{tX}) \leq e^{t^2 \sigma^2 / 2}$$

for every  $t \in \mathbb{R}$ .

Note that equality holds when  $X \sim N(0, \sigma^2)$ , since the MGF of an  $N(\mu, \sigma^2)$  distribution is given by  $t \mapsto \exp(\mu t + \sigma^2 t^2 / 2)$ .

**Recap 1.21.** Recall the *tail bound formula* for the expectation: if  $X$  is a nonnegative random variable, then

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x) dx.$$

Furthermore, recall that the *gamma function* is defined for  $z \in (0, \infty)$  by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

and satisfies  $\Gamma(n) = (n-1)!$  for all  $n \in \mathbb{N}$ .

Finally, recall the following inequality: for all  $a, b \in \mathbb{R}$  and  $p \geq 1$

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

This follows from the convexity of the function  $x \mapsto x^p$ .

**Proposition 1.22.** *We consider some characterisations of sub-Gaussianity:*

(a) *Let  $X$  be sub-Gaussian with parameter  $\sigma^2$ . Then*

$$\max\{\mathbb{P}(X \geq x), \mathbb{P}(X \leq -x)\} \leq e^{-x^2/(2\sigma^2)} \quad \text{for every } x \geq 0. \quad (1)$$

(b) *Let  $X$  be a random variable which satisfies  $\mathbb{E}(X) = 0$  and eq. (1). Then for every  $q \in \mathbb{N}$  it holds that*

$$\mathbb{E}(X^{2q}) \leq 2 \cdot q!(2\sigma^2)^q \leq q!(2\sigma)^{2q}.$$

(c) *If  $X$  is a random variable with  $\mathbb{E}(X) = 0$  and  $\mathbb{E}(X^{2q}) \leq q!C^{2q}$  for all  $q \in \mathbb{N}$ , then  $X$  is sub-Gaussian with parameter  $4C^2$ .*

*Proof.* (a) We first consider  $\mathbb{P}(X \geq x)$ . By Markov's inequality, we have for all  $t \in \mathbb{R}$  that

$$\mathbb{P}(X \geq x) = \mathbb{P}(e^{tX} \geq e^{tx}) \leq e^{-tx} \mathbb{E}(e^{tX}) \leq e^{-tx+t^2\sigma^2/2}.$$

Since the LHS is independent of  $t$ , we can take the infimum over  $t$  on the RHS and obtain

$$\mathbb{P}(X \geq x) \leq \inf_{t \in \mathbb{R}} e^{-tx+t^2\sigma^2/2} = e^{-x^2/(2\sigma^2)},$$

since the infimum of  $t^2\sigma^2/2 - tx$  is attained at  $t = x/\sigma^2$  (this method is called *Chernoff bounding*).

For  $\mathbb{P}(X \leq -x) = \mathbb{P}(-X \geq x)$  we can use the fact that  $-X$  is also sub-Gaussian with parameter  $\sigma^2$ .

(b) By the previous part, we have  $\mathbb{P}(|X| \geq x) \leq 2e^{-x^2/(2\sigma^2)}$ . Some calculations give

$$\begin{aligned} \mathbb{E}(X^{2q}) &= \int_0^\infty \mathbb{P}(X^{2q} \geq x) dx = \int_0^\infty \mathbb{P}(|X| \geq x^{1/(2q)}) dx \\ &= 2q \int_0^\infty x^{2q-1} \mathbb{P}(|X| \geq x) dx \\ &\leq 4q \int_0^\infty x^{2q-1} e^{-x^2/(2\sigma^2)} dx. \end{aligned}$$

Now set  $t = x^2/2\sigma^2$ , so that  $x = \sigma(2t)^{1/2}$  and thus  $dx = \sigma(2t)^{-1/2} dt$ . Plugging that in we get

$$\begin{aligned} \mathbb{E}(X^{2q}) &\leq 4q \int_0^\infty (\sigma(2t)^{1/2})^{2q-1} e^{-t} \sigma(2t)^{-1/2} dt = 2^{q+1} q \sigma^{2q} \int_0^\infty t^{q-1} e^{-t} dt \\ &= 2^{q+1} q \sigma^{2q} \Gamma(q) = 2 \cdot q!(2\sigma)^q. \end{aligned}$$

- (c) Note that  $x \mapsto e^{-tx}$  is convex for every  $t \in \mathbb{R}$ , so  $\mathbb{E}(e^{-tX}) \geq e^{-t\mathbb{E}(X)} = e^0 = 1$  by Jensen's inequality. Let  $X'$  denote an independent copy of  $X$ : then  $X - X'$  has a symmetric distribution, so all its odd moments vanish. Therefore we find

$$\begin{aligned} \mathbb{E}[e^{tX}] &\leq \mathbb{E}[e^{-tX'}]\mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(X-X')}] = \mathbb{E}\sum_{q=0}^{\infty} \left[ \frac{t^{2q}(X-X')^{2q}}{(2q)!} \right] \\ &= \sum_{q=0}^{\infty} \frac{t^{2q}\mathbb{E}[(X-X')^{2q}]}{(2q)!} \leq \sum_{q=0}^{\infty} \frac{2^{2q-1}t^{2q}(\mathbb{E}[X^{2q}] + \mathbb{E}[(X')^{2q}])}{(2q)!} \\ &\leq \sum_{q=0}^{\infty} \frac{2^{2q-1}t^{2q}2q!C^{2q}}{(2q)!} = \sum_{q=0}^{\infty} \frac{(2tC)^{2q}q!}{(2q)!} = \sum_{q=0}^{\infty} \frac{(2tC)^{2q}}{\prod_{j=1}^q (q+j)} \\ &\leq \sum_{q=0}^{\infty} \frac{(2tC)^{2q}}{\prod_{j=1}^q (2j)} = \sum_{q=1}^{\infty} \frac{(2t^2C^2)^q}{q!} = e^{2t^2C^2}. \end{aligned}$$

This shows that  $X$  is sub-Gaussian with parameter  $4C^2$ . □

Note that the proposition is not an “if and only if”-type theorem: suppose we start with a sub-Gaussian variable  $X$  with parameter  $\sigma^2$ . Then by (b), we have  $\mathbb{E}[X^{2q}] \leq q!(2\sigma)^{2q}$ , and (c) then implies that  $X$  is sub-Gaussian with parameter  $16\sigma^2$ .

**Theorem 1.23** (Hoeffding's inequality). *Let  $X_1, \dots, X_n$  be independent sub-Gaussian random variables, with  $X_i$  having parameter  $\sigma_i^2$ . Then  $\bar{X}$  is sub-Gaussian with parameter  $\bar{\sigma}^2$ . In particular, we have*

$$\max\{\mathbb{P}(\bar{X} \geq x), \mathbb{P}(\bar{X} \leq -x)\} \leq e^{-nx^2/(2\bar{\sigma}^2)}.$$

*Proof.* For  $t \in \mathbb{R}$ , we have

$$\mathbb{E}[e^{t\bar{X}}] = \mathbb{E}[e^{(t/n)\sum_i X_i}] = \prod_{i=1}^n \mathbb{E}[e^{(t/n)X_i}] \leq \prod_{i=1}^n e^{t^2\sigma_i^2/(2n^2)} = e^{t^2\bar{\sigma}^2/(2n)},$$

which shows  $\bar{X}$  is sub-Gaussian with parameter  $\bar{\sigma}^2/n$ . Applying part (a) of the previous proposition shows the second result. □

*Remark.* A direct consequence of Hoeffding's inequality is that

$$\mathbb{P}(|\bar{X}| \geq x) \leq 2e^{-nx^2/(2\bar{\sigma}^2)}.$$

The inequality is often stated in this weaker way.

**Lemma 1.24** (Hoeffding's lemma). *Let  $X$  be a random variable with  $\mathbb{E}X = 0$  that satisfies  $a \leq X \leq b$ . Then  $X$  is sub-Gaussian with parameter  $(b-a)^2/4$ .*

*Proof.* See Example Sheet 1, question 2. □

**Corollary 1.25.** *Let  $X_1, \dots, X_n$  be independent random variables where  $\mathbb{E}[X_i] = \mu_i$  and  $a_i \leq X_i \leq b_i$ . Then we have*

$$\mathbb{P}(\bar{X} - \bar{\mu} \geq x) \leq \exp\left(-\frac{2n^2x^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

*Proof.* By Hoeffding's lemma,  $X_i - \mu_i$  is sub-Gaussian with parameter  $(b_i - a_i)^2/4$  for each  $i$ . The result now follows from theorem 1.23. □



Note that when  $X$  takes values in  $[a, b]$ , its variance is at most  $(b - a)^2$ . However, when  $\text{Var}(X_i) \ll (b_i - a_i)^2$ , Hoeffding's inequality can be loose (for example, when  $X_i \sim \text{Bern}(p_i)$  with  $p_i$  small). In such circumstances, Bennett's or Bernstein's inequality may give better results.

**Theorem 1.26** (Bennett's inequality). *Let  $X_1, \dots, X_n$  be independent random variables with  $\mathbb{E}[X_i] = 0$ ,  $\sigma_i^2 := \text{Var}(X_i) < \infty$ , and  $X_i \leq b$  for some  $b > 0$ . Define  $S := \sum_{i=1}^n X_i$ ,  $\nu := \overline{\text{Var}(X_i)}$  and  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  by  $\varphi(u) := e^u - 1 - u = \sum_{k=2}^{\infty} \frac{u^k}{k!}$ , then for every  $t > 0$  we have*

$$\log \mathbb{E}[e^{tS}] \leq \frac{n\nu}{b^2} \varphi(bt).$$

Defining  $h: (0, \infty) \rightarrow [0, \infty)$  by  $h(u) := (1 + u) \log(1 + u) - u$ , we have for every  $x > 0$  that

$$\mathbb{P}(\bar{X} \geq x) \leq \exp \left( -\frac{n\nu}{b^2} h\left(\frac{bx}{\nu}\right) \right).$$

*Proof.* Define  $g: \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(u) := \sum_{k=0}^{\infty} \frac{u^k}{(k+2)!} = \begin{cases} \frac{\varphi(u)}{u^2} & \text{if } u \neq 0, \\ \frac{1}{2} & \text{if } u = 0. \end{cases}$$

Then one can check that  $g$  is increasing on  $\mathbb{R}$ , so

$$e^{tX_i} - 1 - tX_i = t^2 X_i^2 g(tX_i) \leq t^2 X_i^2 g(tb) = X_i^2 \frac{\varphi(bt)}{b^2},$$

and therefore

$$e^{tX_i} \leq 1 + tX_i + X_i^2 \frac{\varphi(bt)}{b^2} \implies \mathbb{E}[e^{tX_i}] \leq 1 + \mathbb{E}[X_i^2] \frac{\varphi(bt)}{t^2} = 1 + \text{Var}(X_i) \frac{\varphi(bt)}{b^2}.$$

Hence for  $t > 0$  we have

$$\begin{aligned} \log \mathbb{E}[e^{tS}] &= \sum_{i=1}^n \log \mathbb{E}[e^{tX_i}] \leq n \cdot \frac{1}{n} \sum_{i=1}^n \log \left( 1 + \text{Var}(X_i) \frac{\varphi(bt)}{b^2} \right) \\ &\stackrel{*}{\leq} n \log \left( 1 + \frac{\nu \varphi(bt)}{b^2} \right) \stackrel{**}{\leq} \frac{n\nu}{b^2} \varphi(bt). \end{aligned}$$

Here, (\*) follows from the fact that  $\log$  is a concave function while (\*\*) follows from the fact that  $\log(1 + u) \leq u$  for all  $u \geq 0$ . This concludes the proof for the first part of the theorem.

Now, we apply the method of Chernoff bounding and find

$$\mathbb{P}(\bar{X} \geq x) = \mathbb{P}(S \geq nx) \leq \inf_{t>0} e^{-ntx} \mathbb{E}[e^{tS}] \leq \inf_{t>0} e^{-ntx + n\nu \varphi(bt)/b^2} = \exp \left( -\frac{n\nu}{b^2} h\left(\frac{bx}{\nu}\right) \right),$$

since one can check that the infimum is attained at  $t = b^{-1} \log(1 + bx/\nu)$ .  $\square$

**Definition 1.27.** A random variable  $X$  with  $\mathbb{E}X = 0$  is called *sub-Gamma in the right tail* with variance factor  $\sigma^2 > 0$  and scale  $c > 0$  if

$$\mathbb{E}[e^{tX}] \leq \exp \left( \frac{\sigma^2 t^2}{2(1 - ct)} \right)$$

for all  $t \in [0, 1/c)$ .

Note that this definition looks like that of sub-Gaussianity, except that  $e^{tX}$  can explode as  $t$  approaches  $1/c$ . We give some characteristics of sub-Gamma distributions:

**Definition 1.28.** For any  $x \in \mathbb{R}$  we define  $x_+ := \max(x, 0)$ .

**Proposition 1.29.** (a) Let  $X$  be sub-Gamma in the right tail with variance factor  $\sigma^2$  and scale  $c$ . Then

$$\mathbb{P}(X \geq x) \leq \exp\left(-\frac{x^2}{2(\sigma^2 + cx)}\right)$$

for all  $x \geq 0$ .

(b) Let  $X$  be a random variable with  $\mathbb{E}X = 0$ ,  $\mathbb{E}[X^2] \leq \sigma^2$  and  $\mathbb{E}[(X_+)^q] \leq q!\sigma^2 c^{q-2}/2$  for all  $q \geq 3$ . Then  $X$  is sub-Gamma in the right tail with variance factor  $\sigma^2$  and scale parameter  $c$ .

*Proof.* (a) Again, we apply a Chernoff bound: we have

$$\begin{aligned} \mathbb{P}(X \geq x) &\leq \inf_{t \in [0, 1/c)} e^{-tx} \mathbb{E}[e^{tX}] \leq \inf_{t \in [0, 1/c)} \exp\left(-tx + \frac{\sigma^2 t^2}{2(1-ct)}\right) \\ &\leq \exp\left(-\frac{x^2}{2(\sigma^2 + cx)}\right), \end{aligned}$$

where we have set  $t = x/(\sigma^2 + cx) \in [0, 1/c)$  in the final step.

(b) Recall from the proof of Bennett's inequality that  $g$  is increasing and therefore for  $u \leq 0$  we have  $\varphi(u) = u^2 g(u) \leq u^2 g(0) = \frac{u^2}{2}$ . Therefore, for every  $u \in \mathbb{R}$  we have

$$\varphi(u) \leq \frac{u^2}{2} + \sum_{q=3}^{\infty} \frac{(u_+)^q}{q!}.$$

We deduce that for  $t \in [0, 1/c)$  we have (note  $\log(x) \leq x - 1$  for all  $x$ ):

$$\log \mathbb{E}[e^{tX}] \leq \mathbb{E}(e^{tX}) - 1 = \mathbb{E}[\varphi(tX)] \leq \mathbb{E}\left[\frac{t^2 X^2}{2} + \sum_{q=3}^{\infty} \frac{t^q X_+^q}{q!}\right].$$

By Fubini's theorem, since the infinite sum has only positive terms we may interchange sum and expectation to obtain

$$\mathbb{E}\left[\frac{t^2 X^2}{2} + \sum_{q=3}^{\infty} \frac{t^q \mathbb{E}[X_+^q]}{q!}\right] = \frac{t^2 \text{Var}[X]}{2} + \sum_{q=3}^{\infty} \frac{t^q \mathbb{E}[X_+^q]}{q!} \leq \frac{\sigma^2 t^2}{2} \sum_{q=2}^{\infty} t^{q-2} c^{q-2} = \frac{\sigma^2 t^2}{2(1-ct)}.$$

□

Following this proposition, we can prove Bernstein's inequality:

**Theorem 1.30** (Bernstein's inequality). Let  $X_1, \dots, X_n$  be independent random variables with  $\mathbb{E}[X] = 0$ ,  $\frac{1}{n} \sum_{i=1}^n \text{Var}(X_i) \leq \sigma^2$  and  $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[(X_i)_+^q] \leq q!\sigma^2 c^{q-2}/2$  some  $\sigma, c > 0$  and for all  $q \geq 3$ . Then  $S := \sum_{i=1}^n X_i$  is sub-Gamma in the right tail with variance factor  $n\sigma^2$  and scale parameter  $c$ . In particular we have

$$\mathbb{P}(\bar{X} \geq x) \leq \exp\left(-\frac{nx^2}{2(\sigma^2 + cx)}\right),$$

for all  $x \geq 0$ .

*Proof.* We have by part (b) of the previous proposition

$$\log \mathbb{E}[e^{tS}] = \sum_{i=1}^n \log \mathbb{E}[e^{tX_i}] \leq n \frac{\sigma^2 t^2}{2(1-ct)},$$

and the second claim follows from part (a) of the previous proposition:

$$\mathbb{P}(\bar{X} \geq x) = \mathbb{P}(S \geq nx) \leq \exp\left(-\frac{nx^2}{2(\sigma^2 + cx)}\right).$$

□

## 2 Kernel density estimation

Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} f$ , and suppose we wish to estimate the density function  $f$ . The oldest way to do this is with a histogram: we divide  $\mathbb{R}$  into equally sized intervals or *bins*, and let  $I_x$  denote the bin containing  $x \in \mathbb{R}$ .

**Definition 2.1.** The *histogram density estimator*  $\hat{f}_n^H$  with bin width  $b > 0$  is given by

$$\hat{f}_n^H(x) := \frac{1}{nb} \sum_{i=1}^n \mathbb{1}_{X_i \in I_x}.$$

There are a few major drawbacks to using histograms: it is difficult to choose  $b$  and the positioning of bin edges, the theoretical performance is suboptimal (mostly due to their discontinuity) and graphical display in the multivariate case is difficult.

### 2.1 The univariate kernel density estimator

**Definition 2.2.** A Borel measurable function  $K: \mathbb{R} \rightarrow \mathbb{R}$  is called a *kernel* if it satisfies  $\int_{\mathbb{R}} K(x) dx = 1$ . A *univariate kernel density estimator* of  $f$  with kernel  $K$  and *bandwidth*  $h > 0$  is defined as

$$\hat{f}_n(x) := \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - X_i}{h}\right).$$

Defining  $K_h(x) := \frac{1}{h} K\left(\frac{x}{h}\right)$ , we can rewrite this as

$$\hat{f}_n(x) := \frac{1}{n} \sum_{i=1}^n K_h(x - X_i).$$

Usually  $K$  is chosen to be non-negative (which ensures that  $K$  itself and  $\hat{f}_n$  are themselves density functions), and  $K$  is often chosen to be symmetric about 0. Generally, the choice of kernel  $K$  is much less important than the choice of bandwidth  $h$ .

If we consider  $\hat{f}_n(x)$  as a point estimator of  $f(x)$ , we typically wish to minimise the *mean squared error*

$$\text{MSE}(\hat{f}_n(x)) := \mathbb{E}\left[(\hat{f}_n(x) - f(x))^2\right].$$

Other possibilities include the mean absolute error which (unlike the MSE) is scale-invariant. However, the MSE has an appealing decomposition into variance and bias terms:

$$\text{MSE}(\hat{f}_n(x)) = \text{Var}(\hat{f}_n(x)) + \text{Bias}^2(\hat{f}_n(x)).$$

We can express the MSE in terms of convolutions:

**Definition 2.3.** Let  $g_1, g_2: \mathbb{R} \rightarrow \mathbb{R}$  be measurable. Then the *convolution* of  $g_1$  and  $g_2$ , denoted  $g_1 * g_2$ , is defined by

$$(g_1 * g_2)(x) := \int_{\mathbb{R}} g_1(x - z)g_2(z) dz.$$

We can compute

$$\begin{aligned} \text{Bias } \hat{f}_n(x) &= \mathbb{E}[\hat{f}_n(x)] - f(x) = \mathbb{E}[K_h(x - X_1)] - f(x) = \int_{\mathbb{R}} K_h(x - z)f(z) dz \\ &= (K_h * f)(x) - f(x). \end{aligned} \tag{2}$$

Analogously, letting  $\xi_i := K_h(x - X_i)$  (note that these are i.i.d. random variables), we have

$$\text{Var } \hat{f}_n(x) = \frac{1}{n} \text{Var}(\xi_1) = \frac{1}{n} (\mathbb{E}[\xi_1^2] - \mathbb{E}^2[\xi_1]) = \frac{1}{n} [(K_h^2 * f)(x) - (K_h * f)^2(x)]. \quad (3)$$

To assess performance of  $h$  and  $K$ , we want to assess the performance of  $\hat{f}_n$  as an estimation of  $f$  as a function. This gives the following definition:

**Definition 2.4.** We define the *mean integrated squared error* or MISE as

$$\text{MISE}(\hat{f}_n) := \mathbb{E} \left( \int_{\mathbb{R}} (\hat{f}_n(x) - f(x))^2 dx \right) \stackrel{*}{=} \int_{\mathbb{R}} \text{MSE}(\hat{f}_n(x)) dx,$$

where  $\star$  follows from Fubini's theorem since the integrand is nonnegative.

We now aim to find bounds on the bias and the variance of  $\hat{f}_n$  in order to choose  $h$  and  $K$  appropriately.

## 2.2 Bounds on variance and bias

**Definition 2.5.** For a kernel  $K$ , define  $R(K) := \int_{\mathbb{R}} K^2(u) du$ .

**Proposition 2.6.** Let  $\hat{f}_n$  be the kernel density estimator with kernel  $K$  and bandwidth  $h > 0$  constructed from  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} f$ . Then for any  $x \in \mathbb{R}, h > 0, n \in \mathbb{N}$  we have

$$\text{Var } \hat{f}_n(x) \leq \frac{1}{nh} \|f\|_{\infty} R(K).$$

*Proof.* By eq. (3) we have

$$\begin{aligned} \text{Var } \hat{f}_n(x) &\leq \frac{1}{n} (K_h^2 * f)(x) = \frac{1}{nh^2} \int_{\mathbb{R}} K^2\left(\frac{x-z}{h}\right) f(z) dz = \frac{1}{nh} \int_{\mathbb{R}} K^2(u) f(x-uh) du \\ &\leq \frac{1}{nh} \|f\|_{\infty} \int_{\mathbb{R}} K^2(u) du = \frac{1}{nh} \|f\|_{\infty} R(K). \end{aligned} \quad (4)$$

□

Obtaining a bound on the bias is not at all straightforward: we will need to introduce conditions on both the density  $f$  and the kernel  $K$ .

**Definition 2.7.** Let  $I \subseteq \mathbb{R}$  be an interval, fix  $\beta, L > 0$ , and let  $m := \lceil \beta \rceil - 1$ . A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is said to belong to the *Hölder class*  $\mathcal{H}(\beta, L)$  if  $f$  is  $m$  times differentiable on  $I$  and

$$\left| f^{(m)}(x) - f^{(m)}(y) \right| \leq L|x-y|^{\beta-m} \quad \text{for all } x, y \in I.$$

If  $I$  is unspecified, we let  $I = \mathbb{R}$ .

The densities in  $\mathcal{H}(\beta, L)$  are denoted by

$$\mathcal{F}(\beta, L) := \left\{ f \in \mathcal{H}(\beta, L) \mid f \geq 0 \text{ and } \int_{\mathbb{R}} f dx = 1 \right\}.$$

**Definition 2.8.** Fix  $\ell \in \mathbb{N}$ . We say a kernel  $K$  is of *order*  $\ell$  if  $\int_{\mathbb{R}} x^j k(x) dx = 0$  for  $j = 1, \dots, \ell - 1$ .

*Remark.* Most kernels used in practice are of order 2. Note that a kernel of order  $\geq 3$  cannot be nonnegative, since we have  $\int_{\mathbb{R}} x^2 K(x) dx = 0$ . Therefore, the kernels are not themselves densities and the corresponding kernel density estimate is not guaranteed to be a density.

**Proposition 2.9.** Assume that  $f \in \mathcal{F}(\beta, L)$  and that  $K$  is a kernel of order  $\ell := \lceil \beta \rceil$ , and furthermore assume that

$$\mu_\beta(K) := \int_{\mathbb{R}} |u|^\beta |K(u)| \, du < \infty.$$

Then the kernel density estimate with bandwidth  $h$  and kernel  $K$  based on  $X_1, \dots, X_n \sim f$  satisfies

$$\left| \text{Bias } \hat{f}_n(x) \right| \leq \frac{L}{(\ell-1)!} \mu_\beta(K) h^\beta \quad \text{for all } x \in \mathbb{R}, h > 0, n \in \mathbb{N}.$$

*Proof.* By eq. (2), we have

$$\text{Bias } \hat{f}_n(x) = \frac{1}{h} \int_{\mathbb{R}} K\left(\frac{x-z}{h}\right) f(z) \, dz - f(x) = \int_{\mathbb{R}} K(u) (f(x-uh) - f(x)) \, du.$$

By applying Taylor's theorem with the Lagrange remainder we obtain, with  $m = \lceil \beta \rceil - 1$ , that

$$f(x-uh) - f(x) = \sum_{j=1}^{m-1} \frac{(-uh)^j}{j!} f^{(j)}(x) + \frac{(-uh)^m}{m!} f^{(m)}(x - \tau uh) \quad \text{for some } \tau \in [0, 1].$$

Since  $\int_{\mathbb{R}} u^j K(u) \, du = 0$  for all  $j \leq m$ , plugging the sum into the integral will give 0. Therefore, we find

$$\text{Bias } \hat{f}_n(x) = \frac{(-h)^m}{m!} \int_{\mathbb{R}} u^m K(u) f^{(m)}(x - \tau uh) \, du = \frac{(-h)^m}{m!} \int_{\mathbb{R}} u^m K(u) \left[ f^{(m)}(x - \tau uh) - f^{(m)}(x) \right] \, du,$$

where the last inequality follows again from the fact that  $K$  is of order  $m+1$ .

Now we use that  $f \in \mathcal{F}(\beta, L)$ , and conclude

$$\left| \text{Bias } \hat{f}_n(x) \right| \leq \frac{Lh^m}{m!} \int_{\mathbb{R}} |u|^m |K(u)| |\tau uh|^{\beta-m} \, du \leq \frac{Lh^\beta}{m!} \int_{\mathbb{R}} |u|^\beta |K(u)| \, du = \frac{L}{(\ell-1)!} \mu_\beta(K) h^\beta,$$

which concludes the proof.  $\square$

Combining propositions 2.6 and 2.9, we find that

$$\text{MSE } \hat{f}_n(x) \leq \frac{1}{nh} R(K) \|f\|_\infty + \frac{L^2}{((\ell-1)!)^2} \mu_\beta^2(K) h^{2\beta}.$$

By minimising this w.r.t.  $h$ , we find that the optimal  $h$  is given by

$$h_n^* = \left( \frac{((\ell-1)!)^2 \|f\|_\infty R(K)}{2\beta L^2 \mu_\beta^2(K)} \right)^{1/(2\beta+1)} n^{-1/(2\beta+1)},$$

and the corresponding MSE is given by

$$\text{MSE } \hat{f}_n(x) \leq \left( \frac{\|f\|_\infty^{2\beta} R(K)^{2\beta} L^2 \mu_\beta^2(K) [(2\beta)^{2\beta+1} + 1]}{((\ell-1)!)^2 (2\beta)^{2\beta}} \right) n^{-2\beta/(2\beta+1)},$$

This  $O(n^{-2\beta/(2\beta+1)})$  bound on the rate is slower than the  $O(1/n)$  rate found in parametric problems, but such a rate is only obtained when the assumed model is correct.

We can strengthen this as follows:

**Theorem 2.10.** Assume that  $K$  is a kernel of order  $\ell := \lceil \beta \rceil$  and that  $\mu_\beta(K)$  and  $R(K)$  are both finite. Fix  $\alpha > 0$  and let  $h = \alpha n^{-1/(2\beta+1)}$ . Then there exists  $C > 0$ , independent of  $n$ , such that

$$\sup_{x \in \mathbb{R}} \sup_{f \in \mathcal{F}(\beta, L)} \text{MSE } \hat{f}_n(x) \leq C n^{-2\beta/(2\beta+1)}.$$

*Proof.* We will show that the class  $\mathcal{F}(\beta, L)$  is uniformly bounded in supremum norm. Let  $K^*$  be a bounded kernel of order  $\ell$  (see example sheet **TODO:** ), then by the previous proposition with  $h = 1$  we have by nonnegativity of  $f$  that

$$\begin{aligned} f(x) &\leq \left| f(x) - \int_{-\infty}^{\infty} K^*(x-z)f(z) dz \right| + \left| \int_{-\infty}^{\infty} K^*(x-z)f(z) dz \right| \\ &\leq \left| \text{Bias } \hat{f}_{n,K^*}(x) \right| + \|K^*\|_{\infty} \int_{-\infty}^{\infty} f(z) dz \\ &\leq \frac{L}{(\ell-1)!} \mu_{\beta}(K^*) + \|K^*\|_{\infty}, \end{aligned}$$

and this bound is independent of  $f$  and  $x$ .

Now we have

$$\text{MSE } \hat{f}_n(x) \leq \frac{R(K)\|f\|_{\infty}}{nh} + \frac{L^2}{((\ell-1)!)^2} \mu_{\beta}^2(K) h^{2\beta} \leq C n^{-2\beta/(2\beta+1)}.$$

□

### 2.3 Bounds on the integrated variance and bias

To bound the MISE, we will give bounds on the integrated variance and bias.

**Proposition 2.11.** Let  $\hat{f}_n$  denote the kernel density estimate with bandwidth  $h$  and kernel  $K$  based on  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} P$  (where  $P$  is a distribution on  $\mathbb{R}$ ). Then

$$\int_{-\infty}^{\infty} \text{Var } \hat{f}_n(x) dx = \frac{1}{nh} R(K).$$

*Proof.* We have by Fubini and eq. (4) that

$$\begin{aligned} \int_{-\infty}^{\infty} \text{Var } \hat{f}_n(x) dx &\leq \frac{1}{nh^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K^2\left(\frac{x-z}{h}\right) f(z) dz dx = \frac{1}{nh^2} \int_{-\infty}^{\infty} f(z) \int_{-\infty}^{\infty} K^2\left(\frac{x-z}{h}\right) dx dz \\ &= \frac{1}{nh} R(K) \int_{-\infty}^{\infty} f(z) dz = \frac{1}{nh} R(K). \end{aligned}$$

□

**Recap 2.12.** Let  $[a, b] = I \subseteq \mathbb{R}$  be an interval, then  $f: I \rightarrow \mathbb{R}$  is called *absolutely continuous* if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, whenever  $(x_1, y_1), \dots, (x_m, y_m)$  are disjoint subintervals of  $I$  with  $\sum_{i=1}^m (y_i - x_i) < \delta$ , we have  $\sum_{i=1}^m |f(y_i) - f(x_i)| < \varepsilon$ .

It is known that absolute continuity is equivalent to being differentiable Lebesgue almost everywhere with a so-called *weak derivative*  $f'$  that satisfies  $f(x) = f(a) + \int_a^x f'(t) dt$  for all  $x \in [a, b]$ .

**Recap 2.13.** The *generalised Minkowski inequality* states that any Borel measurable function  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  we have that

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} g(u, x) du \right)^2 dx \leq \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} g^2(u, x) dx \right)^{1/2} du \right)^2.$$

---

To obtain bounds on the integrated squared bias, we will require smoothness conditions w.r.t. the  $L^2(\mathbb{R})$  norm.

**Definition 2.14.** Fix  $\beta, L > 0$  and let  $m := \lceil \beta \rceil - 1$ . The *Nikolski class*  $\mathcal{N}(\beta, L)$  consists of functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  that are  $(m - 1)$  times differentiable and for which  $f^{(m-1)}$  is absolutely continuous with weak derivative  $f^{(m)}$  satisfying

$$\left\{ \int_{-\infty}^{\infty} \left[ f^{(m)}(x+t) - f^{(m)}(x) \right]^2 dx \right\}^{1/2} \leq L|t|^{\beta-m} \quad \text{for all } t \in \mathbb{R}.$$

The densities in  $\mathcal{N}(\beta, L)$  are denoted by  $\mathcal{F}_{\mathcal{N}}(\beta, L)$ .

**Proposition 2.15.** Fix  $\beta, L > 0$  and let  $K$  be a kernel of order  $\ell := \lceil \beta \rceil$ . Let  $\hat{f}_n$  denote the KDE with kernel  $K$  and bandwidth  $h$  based on  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} f \in \mathcal{F}_{\mathcal{N}}(\beta, L)$ . Then we have

$$\int_{-\infty}^{\infty} \text{Bias}^2 \hat{f}_n(x) dx \leq \frac{L^2}{((\ell - 1)!)^2} \mu_{\beta}^2(K) h^{2\beta}.$$

*Proof.* **TODO:** write this out (integration + taylor expansion + 2x minkowski). □

Putting everything together, we obtain the following:

**Theorem 2.16.** Fix  $\beta, L > 0$ , and let  $K$  be a kernel of order  $\ell = \lceil \beta \rceil$  with  $R(K)$  and  $\mu_{\beta}(K)$  finite. Let  $\hat{f}_n$  be the KDE with kernel  $K$  and bandwidth  $h$  based on  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} f \in \mathcal{F}_{\mathcal{N}}(\beta, L)$ . Then we have

$$\text{MISE } \hat{f}_n \leq \frac{R(K)}{nh} + \frac{L^2}{((\ell - 1)!)^2} \mu_{\beta}^2(K) h^{2\beta}.$$

In particular, fixing  $\alpha > 0$  and taking  $h = \alpha n^{-1/(2\beta+1)}$ , there exists  $C > 0$  independent of  $n$  such that

$$\sup_{f \in \mathcal{F}_{\mathcal{N}}(\beta, L)} \text{MISE } \hat{f}_{n,h,K} \leq C n^{-2\beta/(2\beta+1)}.$$

## 2.4 Bandwidth selection

The choice of bandwidth in the previous theorem is not practical since we have not specified  $\alpha$  and  $\beta$  is typically unknown. One possible approach is *least squares cross validation*. For this, note that minimising the MISE is equivalent to minimising

$$\text{MISE}(\hat{f}_n) - \int_{\mathbb{R}} f^2(x) dx = \mathbb{E} \left[ \int_{\mathbb{R}} \hat{f}_n^2(x) dx \right] - 2\mathbb{E} \left[ \int_{\mathbb{R}} \hat{f}_n(x) f(x) dx \right],$$

and it can be checked that an unbiased estimator for the above is given by

$$\text{LSCV}(h) := \int_{\mathbb{R}} \hat{f}_n^2(x) dx - \frac{2}{n} \sum_{i=1}^n \hat{f}_{n,-i}(X_i),$$

where  $\hat{f}_{n,-i}$  is the KDE based on all observations except  $X_i$ . We now choose  $h$  such that LSCV is minimised.