

Inverse Problems — Example Sheet 1

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Note: when writing a norm of a vector $v \in V$, I will simply write $\|v\|$ and not $\|v\|_V$, unless it is unclear in which space v lives. The same holds for inner products.

Question 1. For $\Omega = [0, 1]^2$ and $\mathcal{X} \in L^2(\Omega)$, we consider the integral operator $A: \mathcal{X} \rightarrow \mathcal{X}$ with

$$(Au)(y) := \int_{\Omega} k(x, y)u(x) \, dx,$$

for $k \in L^2(\Omega \times \Omega)$. Show that

- (a) A is linear with respect to u ,
- (b) A is a bounded linear operator, i.e. $\|Au\|_{\mathcal{X}} \leq \|A\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})} \|u\|_{\mathcal{X}}$. Give also an estimate for $\|A\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})}$,
- (c) the adjoint A^* is given via

$$(A^*v)(y) = \int_{\Omega} k(y, x)v(x) \, dx.$$

- (d) A is a compact operator, i.e. $A \in \mathcal{K}(\mathcal{X}, \mathcal{X})$.

Hint: you may use the fact that if an operator A can be written as a limit (in the operator norm) of finite-rank operators then A is compact. An operator B is called finite-rank if $\dim(B) < \infty$.

Solution. (a) Let $\alpha, \beta \in \mathbb{R}$, $u, v \in L^2(\Omega)$ and $y \in \Omega$. Then we have

$$\begin{aligned} (A(\alpha u + \beta v))(y) &= \int_{\Omega} k(x, y)(\alpha u + \beta v)(x) \, dx \\ &= \int_{\Omega} k(x, y)(\alpha u(x) + \beta v(x)) \, dx \\ &= \alpha \int_{\Omega} k(x, y)u(x) \, dx + \beta \int_{\Omega} k(x, y)v(x) \, dx \\ &= (\alpha Au)(y) + (\beta Av)(y) = (\alpha Au + \beta Av)(y). \end{aligned}$$

Since equality holds for all $y \in \Omega$ we find $A(\alpha u + \beta v) = \alpha Au + \beta Av$, which proves that A is linear.

- (b) Let $u \in L^2(\Omega)$, then we have

$$\|Au\|^2 = \int_{\Omega} ((Au)(y))^2 \, dy = \int_{\Omega} \left(\int_{\Omega} k(x, y)u(x) \, dx \right)^2 \, dy = \int_{\Omega} \langle k(\cdot, y), u(\cdot) \rangle^2 \, dy.$$

Now we apply Cauchy-Schwarz and find

$$\int_{\Omega} \langle k(\cdot, y), u(\cdot) \rangle^2 \, dy \leq \int_{\Omega} \|k(\cdot, y)\|^2 \|u\|^2 \, dy \stackrel{\star}{=} \|u\|^2 \iint_{\Omega^2} k^2(x, y) \, dx \, dy = \|u\|^2 \|k\|^2,$$

where \star follows from Fubini's theorem since the integrand is nonnegative. Taking square roots on both sides we find that $\|Au\| \leq \|k\| \|u\|$, so A is bounded with $\|A\| \leq \|k\|$.

- (c) We know that the adjoint is the unique operator that satisfies $\langle Au, v \rangle = \langle u, A^*v \rangle$ for all $u, v \in \mathcal{X}$. Let $u, v \in \mathcal{X}$, then we compute

$$\begin{aligned}\langle Au, v \rangle &= \int_{\Omega} (Au)(y) \cdot v(y) \, dy = \int_{\Omega} \left(\int_{\Omega} k(x, y) u(x) \, dx \right) v(y) \, dy \\ &= \int_{\Omega} \int_{\Omega} k(x, y) u(x) v(y) \, dx \, dy \stackrel{*}{=} \int_{\Omega} \int_{\Omega} k(x, y) u(x) v(y) \, dy \, dx \\ &= \int_{\Omega} u(x) \left(\int_{\Omega} k(x, y) v(y) \, dy \right) \, dx = \langle u, A^*v \rangle\end{aligned}$$

where $(A^*v)(x) = \int_{\Omega} k(x, y) v(y) \, dy$ as required. Here \star follows from Fubini's theorem.

- (d) It is known that for any compact set $X \subseteq \mathbb{R}^n$, polynomials lie dense in $L^2(X)$. Therefore, there exists a sequence of polynomials p_n such that $p_n \rightarrow k$ in $L^2([0, 1]^4)$. It is easily seen that for any polynomial p , the operator

$$(A_p u)(y) := \int_{\Omega} p(x, y) u(x) \, dx$$

has finite rank: let $p(z) = \sum_{|\alpha| \leq n} c_{\alpha} z^{\alpha}$ (where $z \in [0, 1]^4$ and α is a multi-index), then we find

$$(A_p u)(y) = \sum_{|\alpha| \leq n} c_{\alpha} \int_{\Omega} x_1^{\alpha_1} x_2^{\alpha_2} y_1^{\alpha_3} y_2^{\alpha_4} u(x) \, dx = \sum_{|\alpha| \leq n} c_{\alpha} \left(\int_{\Omega} x_1^{\alpha_1} x_2^{\alpha_2} u(x) \, dx \right) y_1^{\alpha_3} y_2^{\alpha_4},$$

so $A_p u$ lies in the Span $\{y_1^{\alpha_1} y_2^{\alpha_2} \mid \alpha_1 + \alpha_2 \leq n\}$, and therefore has finite rank. By (b), we find that $\|A - A_n\| \leq \|k - p_n\| \rightarrow 0$, which shows that $A_n \rightarrow A$ in operator norm. We conclude that A is compact.

Question 2. We consider the problem of differentiation, formulated as the inverse problem of finding u from $Au = f$ with the integral operator $A: L^2([0, 1]) \rightarrow L^2([0, 1])$ defined as

$$(Au)(y) := \int_0^y u(x) \, dx.$$

- (a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(A)}$.

- (b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$. Hint: Consider the Picard criterion.

- (c) Prove or falsify: “The Moore-Penrose inverse of A is continuous.”

Solution. (a) We want to show that we can approximate f by a sequence (Au_n) for some $(u_n) \subseteq L^2[0, 1]$. To this end, define for $n \geq 2$

$$u_n(x) = \begin{cases} 0 & |x - \frac{1}{2}| > \frac{1}{n}, \\ \frac{n}{2} & |x - \frac{1}{2}| \leq \frac{1}{n}. \end{cases}$$

Clearly $u \in L^2[0, 1]$, and we have

$$f_n(y) := (Au_n)(y) = \int_0^y u_n(x) \, dx = \begin{cases} 0 & \text{if } y < \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}(y - \frac{1}{2} + \frac{1}{n}) & \text{if } \frac{1}{2} - \frac{1}{n} \leq y \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } y > \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Therefore we find

$$\begin{aligned}\|f_n - f\|^2 &= \int_0^1 (f_n - f)^2(x) \, dx \\ &= 2 \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2}} \frac{n^2}{4} \left(x - \frac{1}{2} - \frac{1}{n}\right)^2 \, dx \\ &= \frac{n^2}{2} \int_0^{1/n} x^2 \, dx = \frac{1}{6n} \rightarrow 0,\end{aligned}$$

so $f_n \rightarrow f$ in $L^2[0, 1]$. Since $f_n \in \mathcal{R}(A)$ this shows $f \in \overline{\mathcal{R}(A)}$.

(b) In example 2.2.12, it is shown that for this operator, the Picard criterion is

$$2 \sum_{j=1}^{\infty} \sigma_j^{-2} \left(\int_0^1 f(s) \sin(\sigma_j^{-1} s) \, ds \right)^2, \quad (1)$$

where $\sigma_j = \frac{2}{(2j-1)\pi}$.

We compute

$$\int_0^1 f(s) \sin(\sigma_j^{-1} s) \, ds = \int_{1/2}^1 \sin(\sigma_j^{-1} s) \, ds = \sigma_j \left[\cos\left(\frac{1}{2}\sigma_j^{-1}\right) - \cos(\sigma_j^{-1}) \right].$$

We have

$$\cos(\sigma_j^{-1}) = \cos\left(\frac{(2j-1)\pi}{2}\right) = 0 \quad \text{and} \quad \cos\left(\frac{1}{2}\sigma_j^{-1}\right) = \cos\left(\frac{(2j-1)\pi}{4}\right) = \pm \frac{1}{\sqrt{2}}.$$

Plugging this into eq. (1) gives that

$$2 \sum_{j=1}^{\infty} \sigma_j^{-2} \left(\int_0^1 f(s) \sin(\sigma_j^{-1} s) \, ds \right)^2 = 2 \sum_{j=1}^{\infty} \sigma_j^{-2} (\sigma_j^2/2) = \sum_{j=1}^{\infty} 1 = \infty,$$

so f does not satisfy the Picard criterion and therefore $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$.

- (c) The Moore-Penrose inverse of A is discontinuous. This can be seen by theorem 2.1.11: we have in
(b) an element $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$, so $\mathcal{R}(A)$ is not closed, so A^\dagger is discontinuous.

Question 3. (a) Let $m, n \in \mathbb{N}$ with $m \geq n \geq 2$. Compute the Moore-Penrose inverses of the following matrices:

- (i) $A = (1, 1, \dots, 1) \in \mathbb{R}^{1 \times n}$;
- (ii) $A = \text{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n \times n}$ with $a_j \in \mathbb{R}$ for $j = 1, \dots, n$;
- (iii) $A \in \mathbb{R}^{m \times n}$ with $A^\top A = I_n$.

- (b) Let $a, b \in \mathbb{R}$ with $a < b$. Compute the Moore-Penrose inverse of the operator $A: L^2([a, b]) \rightarrow \mathbb{R}$ with $Au = \int_a^b u(x) \, dx$.

Solution. (a) (i) Clearly $\mathcal{R}(A) = \mathbb{R}$ and $\mathcal{N}(A)^\perp = \text{Span}\{\mathbf{e}\}$ where \mathbf{e} is the all-ones vector. So A^\dagger must map $x \in \mathbb{R}$ to \mathbf{e}/n , and therefore we have

$$A^\dagger = \mathbf{e}/n \in \mathbb{R}^{n \times 1}.$$

- (ii) Clearly we have $\mathcal{R}(A) = \text{Span}\{\mathbf{e}_j \mid a_j \neq 0\} = \mathcal{N}(A)^\perp$ while $\mathcal{R}(A)^\perp = \text{Span}\{\mathbf{e}_j \mid a_j = 0\} = \mathcal{N}(A)$.

It is easily seen that

$$A^\dagger = \text{diag}(b_1, \dots, b_n) \in \mathbb{R}^{n \times n}, \quad \text{where } b_i = \begin{cases} a_i^{-1} & \text{if } a_i \neq 0; \\ 0 & \text{if } a_i = 0. \end{cases}$$

- (iii) We claim $A^\dagger = A^\top$. For this, we check the first two Moore-Penrose equations: for the first one, we have $A^\top A = I_n = P_{\mathbb{R}^n} = P_{\mathcal{N}(A)^\perp}$ since $\mathcal{N}(A) = \{0\}$. For the second one, let $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{R}^m$ be the columns of A (which are orthonormal), then we have

$$AA^\top = \sum_{i=1}^n \mathbf{u}_i \mathbf{u}_i^\top = P_{\text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}} = P_{\mathcal{R}(A)}.$$

- (b) Clearly we have $\mathcal{R}(A) = \mathbb{R}$ while $\mathcal{N}(A) = \left\{u \mid \int_a^b u(x) \, dx = 0\right\}$. It is also easily seen that

$$\mathcal{N}(A)^\perp = \left\{v \mid \int_a^b v(x)u(x) \, dx = 0 \text{ if } \int_a^b u(x) \, dx = 0\right\} = \text{Span}\{1\}.$$

Therefore we simply have that A^\dagger maps a constant $c \in \mathbb{R}$ to the constant function $\frac{c}{b-a}$.

Question 4. Many forward problems are either modelled as convolutions or they are modelled as the composition of several components, one of which is a convolution. Therefore convolutions play an important role in inverse problems. As in Exercise 1, let $\Omega = [0, 1]^2$ be the unit square and let $\mathcal{X} = L^2(\Omega)$. A convolution is the special case of an integral operator $A: \mathcal{X} \rightarrow \mathcal{X}$ where the kernel has a simple structure:

$$(Au)(y) := \int_{\Omega} k(y-x)u(x) \, dx,$$

for $k \in L^2(\Omega)$. It follows easily from Exercise 1 that A is linear and bounded.

- (a) Although shown in general in Exercise 1, give an explicit form for the adjoint of the convolution.
(b) Let $f = Au$. It follows from the convolution theorem that a convolution can be inverted by means of the Fourier transform

$$u = (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1} \left(\frac{\mathcal{F}(f)}{\mathcal{F}(k)} \right), \quad (2)$$

where \mathcal{F} is the Fourier transform and \mathcal{F}^{-1} its inverse. Implement this formula in MATLAB to deblur the blurry tree image f generated by the script `ex4b_generate_data.m`. Note that the script also outputs $\mathcal{F}(k)$. Add some noise to the data and show that the inversion formula is ill-conditioned.

Hint: Make use of the MATLAB commands `fft2` and `ifft2`.

- (c) Reformulate eq. (2) so that the denominator is non-negative and give a stable approximation of this formula. Implement this formula in MATLAB and empirically show that it is stable.

Hint: Make use of the MATLAB command `conj`.

Proof. (a) We have by question 1 that

$$(A^*v)(y) = \int_{\Omega} k(x-y)v(x) \, dx.$$

(b) See matlab code.

(c) We can write $\mathcal{F}(f)/\mathcal{F}(k) = \mathcal{F}(f)\overline{\mathcal{F}(k)}/|\mathcal{F}(k)|^2$.

□