Inverse Problems — Example Sheet 1

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Note: when writing a norm of a vector $v \in V$, I will simply write $\|v\|$ and not $\|v\|_V$, unless it is unclear in which space v lives. The same holds for inner products.

Question 1. *For $\Omega = [0,1]^2$ and $\mathcal{X} \in L^2(\Omega)$, we consider the integral operator $A\colon \mathcal{X} \to \mathcal{X}$ with*

$$(Au)(y) \coloneqq \int_\Omega k(x,y)u(x)\,\mathrm{d}x\,,$$

for $k \in L^2(\Omega \times \Omega)$. Show that

(a) A is linear with respect to u,

(b) A is a bounded linear operator, i.e. $\|Au\|_{\mathcal{X}} \le \|A\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}\|u\|_{\mathcal{X}}$. Give also an estimate for $\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}$,

(c) the adjoint A^* is given via

$$(A^*v)(y) = \int_\Omega k(y,x)v(x)\,\mathrm{d}x\,.$$

(d) A is a compact operator, i.e. $A \in \mathcal{K}(\mathcal{X},\mathcal{X})$.

Hint: *you may use the fact that if an operator A can be written as a limit (in the operator norm) of finite-rank operators then A is compact. An operator B is called finite-rank if $\dim(B) < \infty$.*

Solution. (a) Let $\alpha, \beta \in \mathbb{R}$, $u, v \in L^2(\Omega)$ and $y \in \Omega$. Then we have

$$\begin{aligned}
(A(\alpha u + \beta v))(y) &= \int_\Omega k(x,y)(\alpha u + \beta v)(x)\,\mathrm{d}x \\
&= \int_\Omega k(x,y)(\alpha u(x) + \beta v(x))\,\mathrm{d}x \\
&= \alpha \int_\Omega k(x,y)u(x)\,\mathrm{d}x + \beta \int_\Omega k(x,y)v(x)\,\mathrm{d}x \\
&= (\alpha Au)(y) + (\beta Av)(y) = (\alpha Au + \beta Av)(y).
\end{aligned}$$

Since equality holds for all $y \in \Omega$ we find $A(\alpha u + \beta v) = \alpha Au + \beta Av$, which proves that A is linear.

(b) Let $u \in L^2(\Omega)$, then we have

$$\|Au\|^2 = \int_\Omega ((Au)(y))^2\,\mathrm{d}y = \int_\Omega \left(\int_\Omega k(x,y)u(x)\,\mathrm{d}x\right)^2 \mathrm{d}y = \int_\Omega \langle k(\cdot,y), u(\cdot)\rangle^2\,\mathrm{d}y\,.$$

Now we apply Cauchy-Schwarz and find

$$\int_\Omega \langle k(\cdot,y), u(\cdot)\rangle^2\,\mathrm{d}y \le \int_\Omega \|k(\cdot,y)\|^2\|u\|^2\,\mathrm{d}y \overset{\star}{=} \|u\|^2 \iint_{\Omega^2} k^2(x,y)\,\mathrm{d}x\,\mathrm{d}y = \|u\|^2\|k\|^2,$$

where $\star$ follows from Fubini's theorem since the integrand is nonnegative. Taking square roots on both sides we find that $\|Au\| \le \|k\|\|u\|$, so A is bounded with $\|A\| \le \|k\|$.

(c) We know that the adjoint is the unique operator that satisfies $\langle Au, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in \mathcal{X}$. Let $u, v \in \mathcal{X}$, then we compute

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega (Au)(y)\cdot v(y)\,\mathrm{d}y = \int_\Omega \left(\int_\Omega k(x,y)u(x)\,\mathrm{d}x\right)v(y)\,\mathrm{d}y \\ &= \int_\Omega\int_\Omega k(x,y)u(x)v(y)\,\mathrm{d}x\,\mathrm{d}y \overset{\star}{=} \int_\Omega\int_\Omega k(x,y)u(x)v(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega u(x)\left(\int_\Omega k(x,y)v(y)\,\mathrm{d}y\right)\mathrm{d}x = \langle u, A^*v\rangle\end{aligned}$$

where $(A^*v)(x) = \int_\Omega k(x,y)v(y)\,\mathrm{d}y$ as required. Here $\star$ follows from Fubini's theorem (TODO: justify).

(d) It is known that for any compact set $X \subseteq \mathbb{R}^n$, polynomials lie dense in $L^2(X)$. Therefore, there exists a sequence of polynomials p_n such that $p_n \to k$ in $L^2([0,1]^4)$. It is easily seen that for any polynomial p, the operator

$$(A_pu)(y) := \int_\Omega p(x,y)u(x)\,\mathrm{d}x$$

has finite rank: let $p(z) = \sum_{|\alpha|\le n} c_\alpha z^\alpha$ (where $z \in [0,1]^4$ and α is a multi-index), then we find

$$(A_pu)(y) = \sum_{|\alpha|\le n} c_\alpha \int_\Omega x_1^{\alpha_1}x_2^{\alpha_2}y_1^{\alpha_3}y_2^{\alpha_4}u(x)\,\mathrm{d}x = \sum_{|\alpha|\le n} c_\alpha\left(\int_\Omega x_1^{\alpha_1}x_2^{\alpha_2}\,\mathrm{d}x\right)y_1^{\alpha_3}y_2^{\alpha_4},$$

so A_pu lies in the Span $\{y_1^{\alpha_1}y_2^{\alpha_2} \mid \alpha_1+\alpha_2 \le n\}$, and therefore has finite rank. By (b), we find that $\|A - A_n\| \le \|k - p_n\| \to 0$, which shows that $A_n \to A$ in operator norm. We conclude that A is compact.

Question 2. *We consider the problem of differentiation, formulated as the inverse problem of finding u from $Au = f$ with the integral operator $A\colon L^2([0,1]) \to L^2([0,1])$ defined as*

$$(Au)(y) := \int_0^y u(x)\,\mathrm{d}x\,.$$

(a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(A)}$.

(b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$. Hint: Consider the Picard criterion.

(c) Prove or falsify: "The Moore-Penrose inverse of A is continuous."

Solution. (a) We want to show that we can approximate f by a sequence (Au_n) for some $(u_n) \subseteq L^2[0,1]$. To this end, define for $n \ge 2$

$$u_n(x) = \begin{cases} 0 & \left|x - \frac{1}{2}\right| > \frac{1}{n}, \\ \frac{n}{2} & \left|x - \frac{1}{2}\right| \le \frac{1}{n}. \end{cases}$$

Clearly $u \in L^2[0,1]$, and we have

$$f_n(y) := (Au_n)(y) = \int_0^y u_n(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } y < \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}(y - \frac{1}{2} + \frac{1}{n}) & \text{if } \frac{1}{2} - \frac{1}{n} \le y \le \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } y > \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Therefore we find

$$
\begin{aligned}
\|f_n - f\|^2 &= \int_0^1 (f_n - f)^2(x)\,\mathrm{d}x \\
&= 2\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \frac{n^2}{4}(x - \frac{1}{2} - \frac{1}{n})^2\,\mathrm{d}x \\
&= \frac{n^2}{2}\int_0^{1/n} x^2\,\mathrm{d}x = \frac{1}{6n} \to 0,
\end{aligned}
$$

so $f_n \to f$ in $L^2[0,1]$. Since $f_n \in \mathcal{R}(A)$ this shows $f \in \overline{\mathcal{R}(A)}$.

(b) Note that A is compact, since we can write $(Au)(y) = \int_{[0,1]} k(x,y)u(x)\,\mathrm{d}x$ with $k(x,y) = \mathbb{1}_{x\leq y}$.

To apply the Picard criterion we must find the singular values and right singular vectors of A, which are equal to the square roots of the eigenvalues of AA^* and the eigenvectors of AA^*.

$$
\begin{aligned}
\langle Au, v\rangle &= \int_0^1 (Au)(y)\cdot v(y)\,\mathrm{d}y \\
&= \int_0^1 \int_0^y u(x)\,\mathrm{d}x\, v(y)\,\mathrm{d}y \\
&= \int_0^1 \int_0^y u(x)v(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_x^1 u(x)v(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 u(x) \int_x^1 v(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \langle u, A^*v\rangle
\end{aligned}
$$

where $v(x) = \int_x^1 v(y)\,\mathrm{d}y$. Therefore, we find

$$
(AA^*u)(x) = \int_0^x \int_y^1 u(z)\,\mathrm{d}z\,\mathrm{d}y\,.
$$

Now we solve the eigenequation. Firstly, note that $\mathcal{N}(A) = \{0\}$, so let $\lambda^2 > 0$. Then we have

$$
\int_0^x \int_y^1 u(z)\,\mathrm{d}z\,\mathrm{d}y = \lambda^2 u(x) \tag{1}
$$

$$
\int_x^1 u(z)\,\mathrm{d}z = \lambda^2 u'(x) \tag{2}
$$

$$
-u(x) = \lambda^2 u''(x). \tag{3}
$$

Furthermore, from (1) we infer $u(0) = 0$ while from (2) we infer $u'(1) = 0$. The general solution to (3) is given by $u(x) = a\cos(\lambda^{-1}x) + b\sin(\lambda^{-1}x)$. Plugging $u(0) = 0$ gives $a = 0$, and $u'(1) = 0$ gives

$$
b\lambda^{-1}\cos(\lambda^{-1}) = 0 \implies b = 0 \text{ or } \lambda^{-1} = \pi\left(n - \frac{1}{2}\right) \text{ for some } n \in \mathbb{Z}_{>0}.
$$

Our singular values are therefore $\frac{1}{(n-\frac{1}{2})\pi}$ ($n \in \mathbb{Z}_{>0}$) with right singular vectors $y_n(x) = \sqrt{2}\sin((n-\frac{1}{2})\pi x)$. We now compute

$$
\langle f, y_n\rangle = \sqrt{2}\int_{1/2}^1 \sin\left(\left(n - \frac{1}{2}\right)\pi x\right)\mathrm{d}x = ...
$$

(c) The Moore-Penrose inverse of A is discontinuous. This can be seen by theorem 2.1.11: in (a) we have constructed a sequence $(f_n) \subseteq \mathcal{R}(A)$ that converges, and in (b) we have shown that its limit lies outside of $\mathcal{R}(A)$. Therefore, $\mathcal{R}(A)$ is not closed, so $A^\dagger$ is discontinuous.

Question 3. *(a) Let $m, n \in \mathbb{N}$ with $m \geq n \geq 2$. Compute the Moore-Penrose inverses of the following matrices:*

(i) $A = (1, 1, \dots, 1) \in \mathbb{R}^{1\times n}$;

(ii) $A = \operatorname{diag}(a_1, \dots, a_n) \in \mathbb{R}^{n\times n}$ with $a_j \in \mathbb{R}$ for $j = 1, \dots, n$;

(iii) $A \in \mathbb{R}^{m\times n}$ with $A^\top A = I_n$.

(b) Let $a, b \in \mathbb{R}$ with $a < b$. Compute the Moore-Penrose inverse of the operator $A\colon L^2([a, b]) \to \mathbb{R}$ with $Au = \int_a^b u(x)\,\mathrm{d}x$.

Solution. (a) (i) Clearly $\mathcal{R}(A) = \mathbb{R}$ and $\mathcal{N}(A)^\perp = \operatorname{Span}\{\mathbf{e}\}$ where $\mathbf{e}$ is the all-ones vector. So $A^\dagger$ must map $x \in \mathbb{R}$ to $\mathbf{e}/n$, and therefore we have

$$A^\dagger = \mathbf{e}/n \in \mathbb{R}^{n\times 1}.$$

(ii) Clearly we have $\mathcal{R}(A) = \operatorname{Span}\{\mathbf{e}_j \mid a_j \neq 0\} = \mathcal{N}(A)^\perp$ while $\mathcal{R}(A)^\perp = \operatorname{Span}\{\mathbf{e}_j \mid a_j = 0\} = \mathcal{N}(A)$.

It is easily seen that

$$A^\dagger = \operatorname{diag}(b_1, \dots, b_n) \in \mathbb{R}^{n\times n}, \quad \text{where } b_i = \begin{cases} a_i^{-1} & \text{if } a_i \neq 0; \\ 0 & \text{if } a_i = 0. \end{cases}$$

(iii) If $A^\top A = I$, then the columns $\mathbf{u}_1, \dots, \mathbf{u}_n$ of A are orthogonal. In particular, A is injective (so $\mathcal{N}(A) = \{0\}$) and $\mathcal{R}(A) = \operatorname{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$. Extend $\mathbf{u}_1, \dots, \mathbf{u}_n$ to an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_m$ of $\mathbb{R}^m$, then we must have

$$A^\dagger\left(\sum_{i=1}^m \alpha_i \mathbf{u}_i\right) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n.$$

From this expression it is easily seen that $A^\dagger = A^\top$.

(b) Clearly we have $\mathcal{R}(A) = \mathbb{R}$ while $\mathcal{N}(A) = \left\{u \mid \int_a^b u(x)\,\mathrm{d}x = 0\right\}$. It is also easily seen that

$$\mathcal{N}(A)^\perp = \left\{v \mid \int_a^b v(x)u(x)\,\mathrm{d}x = 0 \text{ if } \int_a^b u(x)\,\mathrm{d}x = 0\right\} = \operatorname{Span}\{1\}.$$

Therefore we simply have that $A^\dagger$ maps a constant $c \in \mathbb{R}$ to the constant function $\frac{c}{b-a}$.

Question 4. *Many forward problems are either modelled as convolutions or they are modelled as the composition of several components, one of which is a convolution. Therefore convolutions play an important role in inverse problems. As in Exercise 1, let $\Omega = [0, 1]^2$ be the unit square and let $\mathcal{X} = L^2(\Omega)$. A convolution is the special case of an integral operator $A\colon \mathcal{X} \to \mathcal{X}$ where the kernel has a simple structure:*

$$(Au)(y) := \int_\Omega k(y - x)u(x)\,\mathrm{d}x\,,$$

for $k \in L^2(\Omega)$. It follows easily from Exercise 1 that A is linear and bounded.

(a) Although shown in general in Exercise 1, give an explicit form for the adjoint of the convolution.

(b) Let $f = Au$. It follows from the convolution theorem that a convolution can be inverted by means of the Fourier transform

$$u = (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}\left(\frac{\mathcal{F}(f)}{\mathcal{F}(k)}\right), \tag{4}$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ its inverse. Implement this formula in MATLAB *to deblur the blurry tree image f generated by the script* `ex4b_generate_data.m`*. Note that the script also outputs $\mathcal{F}(k)$. Add some noise to the data and show that the inversion formula is ill-conditioned.*

Hint: Make use of the MATLAB *commands* `fft2` *and* `ifft2`*.*

(c) Reformulate eq. (4) so that the denominator is non-negative and give a stable approximation of this formula. Implement this formula in MATLAB and empirically show that it is stable.

Hint: Make use of the MATLAB *command* `conj`*.*

Proof. (a) We have by question 1 that

$$(A^*v)(y) = \int_\Omega k(x-y)v(x)\,\mathrm{d}x\,.$$

(b)

$\square$