# Distribution Theory and Applications — Example Sheet 1

Lucas Riedstra

...

**Question 1.** *Construct a non-zero element of $\mathcal{D}(\mathbb{R})$ that vanishes outside $(0, 1)$. Construct a non-zero of $\mathcal{D}(\mathbb{R}^n)$ that vanishes outside the ball $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.*

*Proof.* It is well-known that the function

$$\varphi\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \begin{cases} 0 & \text{if } x \leqslant 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside $(0, \infty)$. The function $\psi(x) := \varphi(x)\varphi(1 - x)$ is therefore also smooth and vanishes outside $(0, 1)$.

Since $\psi$ vanishes outside $(0, 1)$, the function $\psi(x/\varepsilon)$ vanishes outside $(0, \varepsilon)$, and therefore the function $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$ vanishes outside $B_\varepsilon$. □

**Question 2.** *Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives*

$$\varphi(x + h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h).$$

*Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in $x$ for each multi-index $\alpha$, i.e. prove*

$$\lim_{|h| \to 0} \frac{\sup_x |\partial^\alpha R_N(x, h)|}{|h|^N} = 0$$

*for each multi-index $\alpha$.*

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x, h) = \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!} (N + 1) \int_0^1 (1 - t)^N (\partial^\alpha \varphi)(x + th) \, \mathrm{d}t\,,$$

and note that $(N + 1)! \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \cdots + h_n)^{N+1}$.

*Proof.* Since $\varphi \in \mathcal{D}(X)$, we know that $\operatorname{supp} \varphi \subseteq \overline{B_N}$ for some $N \in \mathbb{N}$. Now, suppose $\|h\| < 1$, then

$$\varphi(x + h) \neq 0 \implies \|x + h\| \leqslant N \implies \|x\| \leqslant \|x + h\| + \|h\| \leqslant N + 1,$$

so if we define $\psi_h(x) = \varphi(x + h)$ then we know that $\operatorname{supp} \varphi_h \subseteq \overline{B_{N+1}}$.

By Taylor's theorem, we have

$$\varphi(x + h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h),$$

and since $\sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x)$ vanishes for $x \notin \overline{B_N}$, it is clear that $\operatorname{supp}(R_N(\cdot,h))$ must also be contained in $\overline{B_{N+1}}$ (again, for $\|h\| \leqslant 1$). This shows that $\operatorname{supp}(R_N)$ is contained in $\overline{B_{N+1}}$ for $|h|$ sufficiently small.

Now let $\beta$ be a multi-index and define $C := \frac{1}{N!}\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|$ (note that $C$ exists and is finite since all partial derivatives of $\varphi$ have compact support), then we have

$$\begin{aligned}
\left|\partial^\beta R_N(x,h)\right| &= \left|\partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^\alpha\varphi)(x+th)\,\mathrm{d}t\right| \\
&\overset{\star}{=} \left|\sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^{\alpha+\beta}\varphi)(x+th)\,\mathrm{d}t\right| \\
&\leqslant \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1)\int_0^1 (1-t)^N\left|(\partial^{\alpha+\beta}\varphi)(x+th)\right|\mathrm{d}t \\
&\leqslant \left[\max_{|\alpha|=N+1, x\in\mathbb{R}^n}\left|(\partial^{\alpha+\beta})\varphi(x)\right|\right]\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!}(N+1) \\
&\leqslant C(N+1)!\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!} = C(|h_1|+\cdots+|h_n|)^{N+1}.
\end{aligned}$$

Since this upper bound does not depend on $x$, we also have

$$\sup_x \left|\partial^\beta R_N(x,h)\right| \leqslant C(|h_1|+\cdots+|h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x\left|\partial^\beta R_N(x,h)\right|}{\|h\|^N} \leqslant \frac{C(|h_1|+\cdots+|h_n|)^{N+1}}{\|h\|^N} \leqslant \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \to 0,$$

and therefore that $\partial^\beta R_N(x,h) = o(\|h\|^n)$ for all multi-indices $\beta$. $\square$

**Question 3.** *Which elements of $\mathcal{D}(X)$ can be represented as a power series on $X$?*

*Solution.* It is known that if two power series agree on an open set, they agree on the entire space. Since every $\varphi \in \mathcal{D}(X)$ is identically zero on some open set (outside its support), the only element of $\mathcal{D}(X)$ with a power series representation is the zero function.

**Question 4.** *Prove the $C^\infty$ Urysohn lemma: if $K$ is a compact subset of $X \subseteq \mathbb{R}^n$, show that one can find a $\varphi \in \mathcal{D}(X)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on a neighborhood of $K$.*

*Solution.* Let $K \subseteq U_1$. Define $U_2 := U_1 + B(0,1)$ and let $\chi = \mathbb{1}_{U_2}$. Now let $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\psi\,\mathrm{d}x = 1$ and $\operatorname{supp}\psi \subseteq B(0,1)$. The we compute

$$(\chi * \psi)(x) = \int_{\mathbb{R}^n}\chi(y)\psi(x-y)\,\mathrm{d}y = \int_{U_2}\psi(x-y)\,\mathrm{d}y\,.$$

Clearly, $\chi * \psi \in \mathcal{D}(X)$, and furthermore, we have for $x \in U_1$ that

$$\int_{U_2}\psi(x-y)\,\mathrm{d}y = \int_{U_2-x}\psi(z)\,\mathrm{d}z \overset{\star}{=} 1,$$

since $B(0,1) \subseteq U_2 - x$. This proves the claim.

**Question 5.** *Given $T \in \mathcal{D}'(X)$, the derivative $\partial^\alpha T$ is defined by*

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

*Show that $\partial^\alpha T \in \mathcal{D}'(X)$. If $\operatorname{ord}(T) = m$ what can you say about $\operatorname{ord}(\partial^\alpha T)$?*

*Proof.* Let $K \subseteq X$ be compact and $\varphi \in \mathcal{D}(X)$. Since $T$ is a distribution, we know that there exists constants $C, N$ such that

$$|\langle T, \varphi \rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left| \partial^\beta \varphi \right|.$$

Letting $M := |\alpha|$, we find

$$|\langle \partial^\alpha T, \varphi \rangle| = |\langle T, \partial^\alpha \varphi \rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left| \partial^{\alpha+\beta} \varphi \right| \leqslant C \sum_{|\beta| \leqslant M+N} \sup \left| \partial^\beta \varphi \right|.$$

We conclude that $\partial^\alpha T$ is a distribution, and that if $\operatorname{ord}(T) = m$, $\operatorname{ord}(\partial^\alpha T) \leqslant m + |\alpha|$. □

**Question 6.** *Given $T \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$, prove that for each multi-index $\alpha$*

$$\partial^\alpha (Tf) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

*in $\mathcal{D}'(X)$.*

*Proof.* Let $\varphi \in \mathcal{D}(X)$, then by definition we have $\langle \partial^\alpha (Tf), \varphi \rangle = \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle$. Approximate $T$ by a sequence $(\psi_n) \subseteq \mathcal{D}'(X)$, then we find

$$\begin{aligned}
\langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle &= \lim_{n\to\infty} \langle \psi_n, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle = \lim_{n\to\infty} (-1)^{|\alpha|} \int_X \psi_n(x) f(x) \partial^\alpha \varphi(x) \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \partial^\alpha (\psi_n(x) f(x)) \varphi(x) \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \left( \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \cdot \partial^{\alpha-\beta} \psi_n(x) \right) \varphi(x) \,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} \psi_n, \varphi \rangle = \lim_{n\to\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle \psi_n, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle \\
&= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle T, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} T, \varphi \rangle \\
&= \left\langle \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T, \varphi \right\rangle.
\end{aligned}$$

□

**Question 7.** *Let $(x_k)$ be a sequence in $X$ with no limit point in $X$. Consider the family of linear maps $u_\alpha \colon \mathcal{D}(X) \to \mathbb{C}$ defined by*

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)$$

*for each multi-index $\alpha$. For what $\alpha$ is $u_\alpha \in \mathcal{D}'(X)$? What is $\operatorname{ord}(u_\alpha)$?*

*Solution.* Let $K \subseteq X$ be compact. Since $(x_k)$ does not have a limit point, only finitely many of the $x_k$ lie in $K$ (otherwise $(x_k)$ would have a subsequence contained in $K$ which would have a convergent subsequence). Without loss of generality, assume that $x_1, \dots, x_n \in K$, and $x_{n+1}, x_{n+2}, \dots, \notin K$. Now, for any $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$ we find

$$|\langle u, \varphi\rangle| = \left|\sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)\right| = \left|\sum_{k=1}^{n} \partial^\alpha \varphi(x_k)\right| \leqslant \sum_{k=1}^{n} |\partial^\alpha \varphi(x_k)| \leqslant n \cdot \sup |\partial^\alpha \varphi| \leqslant n \cdot \sum_{|\beta| \leqslant |\alpha|} \sup \left|\partial^\beta \varphi\right|.$$

This shows that $u_\alpha \in \mathcal{D}'(X)$ for any $\alpha$, with $\operatorname{ord}(u_\alpha) \leqslant |\alpha|$. We claim that this is an equality, i.e., $\operatorname{ord}(u_\alpha) = |\alpha|$. TODO: How to show??

**Question 8.** *Find the most general solution to the equations*

*(a)* $u' = 1$,

*(b)* $xu' = \delta_0$,

*(c)* $(e^{2\pi i x} - 1)u' = 0$

*in* $\mathcal{D}'(\mathbb{R})$.

*Solution.* Let $\varphi \in \mathcal{D}(X)$.

(a) If $u' = 1$ then we find
$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = \langle 1, \varphi\rangle = \langle u', \varphi\rangle = -\langle u, \varphi'\rangle,$$
so for any $c \in \mathbb{R}$ we find by partial integration $\langle u, \varphi'\rangle = -\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} (x+c)\varphi'(x)\,\mathrm{d}x$. From this we deduce that $u = x + c$ for some $c$.

(b) If $xu' = \delta_0$ then we have
$$\varphi(0) = \langle \delta_0, \varphi\rangle = \langle xu', \varphi\rangle = \langle u', x\varphi\rangle = \langle u, -(x\varphi)'\rangle = \langle u, -x\varphi' - \varphi\rangle$$
Note that the above is satisfied for $u = -\delta_0 + c$ for any constant $c$. TODO: is this the most general solution?

(c) Since $e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$, intuitively it must be the case that $u'$ is 0, except "on $\mathbb{Z}$", whatever that may mean. Therefore, we guess that, for any sequence $(\alpha_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ and constant $c \in \mathbb{C}$, the map
$$u = c + \sum_{n\in\mathbb{Z}} \alpha_n \mathbb{1}_{x \geqslant n}.$$
We compute the derivative of $u$. It is easily seen that $u' = \sum_{n\in\mathbb{Z}} \alpha_n \delta_n$ (the infiniteness of the sum does not pose a problem since the test functions are compactly supported, so $\langle u, \varphi\rangle$ will always be a finite sum). From this, we see that
$$\langle (e^{2\pi i x} - 1)u', \varphi\rangle = \langle u', (e^{2\pi i x} - 1)\varphi\rangle = \sum_{n\in\mathbb{Z}} \alpha_n (e^{2\pi i n} - 1)\varphi(n) = 0,$$
so $u$ satisfies the equation. TODO: Why is this the most general solution? Intuitively clear, but how to make this rigorous?

**Question 9.** *Define the distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ by the locally integrable function $u(x, y) = \mathbb{1}_{x \geqslant y}$. Show that $\partial_x^2 u - \partial_y^2 u = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Can you give a physical interpretation of this result?*

*Proof.* Let $f \in \mathcal{D}(\mathbb{R}^2)$, then we have

$$\begin{aligned}
\langle \partial_x^2 u - \partial_y^2 u, f \rangle = \langle \partial_x^2 u, f \rangle - \langle \partial_y^2 u, f \rangle = \langle u, \partial_x^2 f \rangle - \langle u, \partial_y^2 f \rangle = \langle u, \partial_x^2 f - \partial_y^2 f \rangle \\
&\overset{\star}{=} \int_{-\infty}^{\infty} \int_{y}^{\infty} \partial_x^2 f(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{-\infty}^{\infty} \int_{-\infty}^{x} \partial_y^2 f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} \partial_x f(y,y)\,\mathrm{d}y - \int_{-\infty}^{\infty} \partial_y f(x,x)\,\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} (\partial_x f + \partial_y f)(x,x)\,\mathrm{d}x\,.
\end{aligned}$$

Here, $\star$ follows from Fubini's theorem. Define $g(x) = f(x,x)$, then it is easily seen that $g'(x) = \partial_x f(x,x) + \partial_y f(x,x)$, so we find that

$$\langle \partial_x^2 u - \partial_y^2 u, f \rangle = -\int_{-\infty}^{\infty} g'(x)\,\mathrm{d}x = \lim_{x \to -\infty} g(x) - \lim_{x \to \infty} g(x) = 0 - 0 = 0.$$

□

This shows that $\partial_x u - \partial_y u = 0$, or equivalently, that $u$ satisfies the wave equation.