Distribution Theory and Applications — Example Sheet 1

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For both question 2, we need the following lemma:

Lemma 1. *Let $K, V \subseteq \mathbb{R}^n$ where K is compact, V is closed, and $K \cap V = \varnothing$. Then there is a nonzero distance between K and V, i.e.,*

$$\inf_{x \in K, v \in V} \|x - v\| > 0.$$

Proof. We know that $K \subseteq V^{\complement}$ and that $V^{\complement}$ is open, so for every $x \in K$ there exists an open ball $B(x, \varepsilon_x)$ around x such that $B(x, 2\varepsilon_x) \subseteq V^{\complement}$. Since $\{B(x, \varepsilon_x)\}$ is an open covering of K, there exist finitely many balls $B(x_1, \varepsilon_1), \ldots, B(x_n, \varepsilon_n)$ that cover K. Let $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $x \in K$, then there is an x_i such that $\|x - x_i\| < \varepsilon$, and since $B(x_i, 2\varepsilon) \subseteq B(x_i, 2\varepsilon_i) \subseteq V^{\complement}$ it is clear that $B(x, \varepsilon) \subseteq V^{\complement}$ as well.

We conclude that $B(x, \varepsilon) \subseteq V^{\complement}$ for any $x \in K$, and therefore that $\inf_{x \in K, v \in V} \|x - v\| \geqslant \varepsilon > 0$. $\square$

Question 2. *Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives*

$$\varphi(x + h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h).$$

Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in x for each multi-index α, i.e. prove

$$\lim_{|h| \to 0} \frac{\sup_x |\partial^\alpha R_N(x, h)|}{|h|^N} = 0$$

for each multi-index α.

Hint: *you may find it convenient to use the following form of the remainder*

$$R_N(x, h) = \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!} (N + 1) \int_0^1 (1 - t)^N (\partial^\alpha \varphi)(x + th)\, \mathrm{d}t\,,$$

and note that $(N + 1)! \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \cdots + h_n)^{N+1}$.

Proof. Let $\varphi \in \mathcal{D}(X)$ with $K = \operatorname{supp} \varphi$, then by lemma 1 we know there exists a nonzero distance $d > 0$ between K and $\mathbb{R}^n \backslash X$. We claim that if $\|h\| \leqslant \frac{d}{2}$, then

$$\operatorname{supp}(R_N) \subseteq \left\{ x \in X \mid d(x, K) \leqslant \frac{d}{2} \right\} =: \hat{K},$$

which is clearly a compact set contained in X. Indeed, if $\|h\| \leqslant \frac{d}{2}$ we have

$$\varphi(x + h) \neq 0 \implies x + h \in K \implies d(x, K) \leqslant \|h\| \leqslant \frac{d}{2}.$$

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x) + R_N(x,h),$$

and since $\sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x)$ vanishes for $x \notin K$, it is clear that $\operatorname{supp}(R_N(\cdot,h))$ must be contained in $\hat{K}$ (for $\|h\| \leqslant \frac{d}{2}$).

Now let β be a multi-index and define $C := \frac{1}{N!}\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|$ (note that C exists and is finite since all partial derivatives of φ have compact support), then we have

$$\begin{aligned}
\left|\partial^\beta R_N(x,h)\right| &= \left|\partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^\alpha\varphi)(x+th)\,\mathrm{d}t\right| \\
&\overset{\star}{=} \left|\sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^{\alpha+\beta}\varphi)(x+th)\,\mathrm{d}t\right| \\
&\leqslant \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1)\int_0^1 (1-t)^N\left|\left(\partial^{\alpha+\beta}\varphi\right)(x+th)\right|\mathrm{d}t \\
&\leqslant \left[\max_{|\alpha|=N+1, x\in\mathbb{R}^n}\left|(\partial^{\alpha+\beta})\varphi(x)\right|\right]\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!}(N+1) \\
&= C(N+1)!\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!} = C(|h_1|+\cdots+|h_n|)^{N+1}.
\end{aligned}$$

Here, $\star$ follows from differentiation under the integral sign since the integrand is bounded.

Since this upper bound does not depend on x, we also have

$$\sup_x \left|\partial^\beta R_N(x,h)\right| \leqslant C(|h_1|+\cdots+|h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x\left|\partial^\beta R_N(x,h)\right|}{\|h\|^N} \leqslant \frac{C(|h_1|+\cdots+|h_n|)^{N+1}}{\|h\|^N} \leqslant \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \to 0,$$

and therefore that $\partial^\beta R_N(x,h) = o(\|h\|^n)$ for all multi-indices β. □

Question 8. *Find the most general solution to the equations*

(a) $u' = 1$,

(b) $xu' = \delta_0$,

(c) $(e^{2\pi i x}-1)u' = 0$

in $\mathcal{D}'(\mathbb{R})$.

Solution. Let $\varphi \in \mathcal{D}(X)$.

(a) It is clear that $x' = 1$ in the distributional sense, since for any test function φ we have

$$\langle x', \varphi\rangle = -\langle x, \varphi'\rangle = -\int_{\mathbb{R}} x\varphi'(x)\,\mathrm{d}x = \int_{\mathbb{R}}\varphi(x)\,\mathrm{d}x = \langle 1, \varphi\rangle.$$

Therefore, the equation $u' = 1$ is equivalent to the equation $(u-x)' = 0$. We know that this implies that $u - x = c$ for some constant $c \in \mathbb{C}$, so the most general solution is $u = x + c$.

(b) If $xu' = \delta_0$ then we have

$$\varphi(0) = \langle \delta_0, \varphi \rangle = \langle xu', \varphi \rangle = \langle u', x\varphi \rangle = \langle u, -(x\varphi)' \rangle = \langle u, -x\varphi' - \varphi \rangle$$

Note that the above is satisfied for $u = -\delta_0 + c$ for any constant $c \in \mathbb{C}$.

I do not know if this is the most general solution and/or how one would show this.

(c) Since $e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$, it is clear that $\operatorname{supp}(u') \subseteq \mathbb{Z}$. We will show that this is also sufficient, i.e., that any distribution u with $\operatorname{supp}(u') \subseteq \mathbb{Z}$ yields a solution.

It is easily seen that

$$\begin{aligned} \operatorname{supp}(u') \subseteq \mathbb{Z} &\iff u' = \sum_{n \in \mathbb{Z}} \alpha_n \delta_n \quad \text{for some } (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C} \\ &\iff u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geqslant n} \quad \text{for some } c \in \mathbb{C},\, (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}. \end{aligned}$$

Indeed, if $u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geqslant n}$ then

$$\langle u', \varphi \rangle = -\langle u, \varphi' \rangle \overset{\star}{=} -\alpha_n \sum_{n \in \mathbb{Z} \cap \operatorname{supp} \varphi} \int_n^\infty \varphi'(x) \, \mathrm{d}x = \sum_{n \in \mathbb{Z} \cap \operatorname{supp} \varphi} \alpha_n \varphi(n) = \langle \sum_{n \in \mathbb{Z}} \alpha_n \delta_n, \varphi \rangle,$$

where $\star$ follows from the fact that there are only finitely many n in $\mathbb{Z} \cap \operatorname{supp} \varphi$ (since φ has compact support).

Finally, we compute that

$$\langle (e^{2\pi i x} - 1)u', \varphi \rangle = \langle u', (e^{2\pi i x} - 1)\varphi \rangle = \sum_{n \in \mathbb{Z}} \alpha_n (e^{2\pi i n} - 1)\varphi(n) = 0,$$

so u satisfies the equation.