Distribution Theory — Example Sheet 2

Lucas Riedstra

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Question 1. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, one of which has compact support. Show that the convolution $u * v$, defined as in your notes, is uniquely defined and gives rise to an element of $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. The convolution between $u, v \in \mathcal{D}'(\mathbb{R}^n)$ is defined by the formula

$$(u * v) * \varphi = u * (v * \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1}$$

To show that this is uniquely defined, recall that for all $u \in \mathcal{D}'(\mathbb{R}^n)\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have $\langle u, \varphi \rangle = (u * \check{\varphi})(0)$. Therefore, we have

$$\langle u * v, \varphi \rangle = ((u * v) * \check{\varphi})(0) = (u * (v * \check{\varphi}))(0),$$

which shows that the formula eq. (1) uniquely defines $\langle u * v, \varphi \rangle$ for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and therefore that $u * v$ is well-defined.

Now we prove that $u * v \in \mathcal{D}'(\mathbb{R}^n)$: by the previous equation we have

$$\langle u * v, \varphi \rangle = (u * (v * \check{\varphi}))(0) = \langle u, \widecheck{v * \check{\varphi}} \rangle.$$

Suppose u is compactly supported. Since $\widecheck{v * \check{\varphi}} \in \mathcal{E}(\mathbb{R}^n)$, there exists a compact $K \subseteq X$ and nonnegative C, N such that

$$\begin{aligned}\left|\langle u, \widecheck{v * \check{\varphi}} \rangle\right| &\leqslant C \sum_{\alpha \leqslant N} \sup_{x \in K} \left|\partial^\alpha (\widecheck{v * \check{\varphi}})\right| = C \sum_{\alpha \leqslant N} \sup_{x \in -K} \left|\partial^\alpha (v * \check{\varphi})\right| = C \sum_{\alpha \leqslant N} \sup_{x \in -K} \left|v * \partial^\alpha \check{\varphi}\right| \\ &= C \sum_{|\alpha| \leqslant N} \sup_{x \in -K} \left|\langle v, \tau_x \widecheck{\partial^\alpha \check{\varphi}} \rangle\right|.\end{aligned}$$

Note that if $\operatorname{supp} \varphi \subseteq K'$, then $\operatorname{supp} \check{\varphi} \subseteq -K'$, and for $x \in -K$ we find $\operatorname{supp} \tau_x \widecheck{\partial^\alpha \check{\varphi}} \subseteq -K' - K$. Then by the previous equation we find that there exists C', M with

$$|\langle u * v, \varphi \rangle| \leqslant C' \sum_{|\alpha| \leqslant N} \sum_{|\beta| \leqslant M} \sup_{x \in -K' - K} \partial^\beta (\tau_x \widecheck{\partial^\alpha \check{\varphi}}) = C' \sum_{|\alpha| \leqslant N} \sum_{|\beta| \leqslant M} \sup_x \left|\partial^{\alpha + \beta} \varphi\right| \leqslant C'' \sum_{|\alpha| \leqslant M + N} \sup_x |\partial^\alpha \varphi|,$$

which shows that $u * v \in \mathcal{D}'(\mathbb{R}^n)$. An analogous argument holds if v is compactly supported. □

Question 2. *Show that if $u, v, w \in \mathcal{D}'(\mathbb{R}^n)$ and at least two of them have compact support, then the convolution is associative (i.e., $(u * v) * w) = u * (v * w)$).*

Proof. Note that the convolution between two compactly supported distributions is again compactly supported, which ensures that both expressions 'make sense'. Now, let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then we have

$$((u * v) * w) * \varphi = (u * v) * (w * \varphi) = u * (v * (w * \varphi)) = u * ((v * w) * \varphi) = (u * (v * w)) * \varphi,$$

which proves the theorem. □

Question 3. *Let* $\varphi \in \mathcal{D}(\mathbb{R})$ *and choose* $\varepsilon > 0$ *sufficiently small so that* $\operatorname{supp}(\varphi) \subset I_\varepsilon = (-1/\varepsilon, 1/\varepsilon)$. *Given that* φ *has a uniformly convergent Fourier series on* I_ε *in the form*

$$\varphi(x) = \sum_{n\in\mathbb{Z}} c_n e^{i\varepsilon\pi nx}, \quad c_n = \frac{\varepsilon}{2}\int_{\mathbb{R}} \varphi(x) e^{-i\varepsilon\pi nx}\,\mathrm{d}x\,,$$

prove the Fourier inversion theorem on $\mathcal{D}(\mathbb{R})$ *by taking a suitable limit.*

Proof. Let $\psi \in \mathcal{D}(\mathbb{R})$, then we want to prove that

$$\psi(x) = \frac{1}{(2\pi)^n}\iint e^{i\lambda(x-y)}\psi(y)\,\mathrm{d}y\,\mathrm{d}\lambda\,.$$

?? □

Question 4. *For* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *prove that* $\sum_m \varphi(m) = \sum_n \hat{\varphi}(2\pi n)$. *This is the famous Poisson summation formula.*

Proof. We have

$$\sum_m \varphi(m) = \frac{1}{(2\pi)^n}\sum_m \int e^{i\lambda m}\hat{\varphi}(\lambda)\,\mathrm{d}\lambda = \sum_m \int e^{2\pi i\lambda m}\hat{\varphi}(2\pi\lambda)\,\mathrm{d}\lambda$$

□