Distribution Theory and Applications — Example Sheet 1

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Question 1. *Construct a non-zero element of $\mathcal{D}(\mathbb{R})$ that vanishes outside $(0,1)$. Construct a non-zero of $\mathcal{D}(\mathbb{R}^n)$ that vanishes outside the ball $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.*

Proof. It is well-known that the function

$$\varphi\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \begin{cases} 0 & \text{if } x \leqslant 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside $(0, \infty)$. The function $\psi(x) := \varphi(x)\varphi(1-x)$ is therefore also smooth and vanishes outside $(0,1)$.

Since ψ vanishes outside $(0,1)$, the function $\psi(x/\varepsilon)$ vanishes outside $(0,\varepsilon)$, and therefore the function $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$ vanishes outside B_ε. □

For both questions 2 and 4, we need the following lemma:

Lemma 1. *Let $K, V \subseteq \mathbb{R}^n$ where K is compact, V is closed, and $K \cap V = \varnothing$. Then there is a nonzero distance between K and V, i.e.,*

$$\inf_{x \in K, v \in V} \|x - v\| > 0.$$

Proof. We know that $K \subseteq V^\complement$ and that $V^\complement$ is open, so for every $x \in K$ there exists an open ball $B(x, \varepsilon_x)$ around x such that $B(x, 2\varepsilon_x) \subseteq V^\complement$. Since $\{B(x, \varepsilon_x)\}$ is an open covering of K, there exist finitely many balls $B(x_1, \varepsilon_1), \dots, B(x_n, \varepsilon_n)$ that cover K. Let $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_n\}$ and $x \in K$, then there is an x_i such that $\|x - x_i\| < \varepsilon$, and since $B(x_i, 2\varepsilon) \subseteq B(x_i, 2\varepsilon_i) \subseteq V^\complement$ it is clear that $B(x, \varepsilon) \subseteq V^\complement$ as well.

We conclude that $B(x,\varepsilon) \subseteq V^\complement$ for any $x \in K$, and therefore that $\inf_{x \in K, v \in V} \|x - v\| \geqslant \varepsilon > 0$. □

Question 2. *Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives*

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h).$$

Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in x for each multi-index α, i.e. prove

$$\lim_{|h| \to 0} \frac{\sup_x |\partial^\alpha R_N(x,h)|}{|h|^N} = 0$$

for each multi-index α.

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x,h) = \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th)\,\mathrm{d}t\,,$$

and note that $(N+1)!\sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} = (h_1+\cdots+h_n)^{N+1}$.

Proof. Let $\varphi \in \mathcal{D}(X)$ with $K = \operatorname{supp}\varphi$, then by lemma 1 we know there exists a nonzero distance $d > 0$ between K and $\mathbb{R}^n\backslash X$. We claim that if $\|h\| \leqslant \frac{d}{2}$, then

$$\operatorname{supp}(R_N) \subseteq \left\{ x \in X \mid d(x,K) \leqslant \frac{d}{2} \right\} =: \hat{K},$$

which is clearly a compact set contained in X. Indeed, if $\|h\| \leqslant \frac{d}{2}$ we have

$$\varphi(x+h) \neq 0 \implies x+h \in K \implies d(x,K) \leqslant \|h\| \leqslant \frac{d}{2}.$$

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x) + R_N(x,h),$$

and since $\sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x)$ vanishes for $x \notin K$, it is clear that $\operatorname{supp}(R_N(\cdot,h))$ must be contained in $\hat{K}$ (for $\|h\| \leqslant \frac{d}{2}$).

Now let β be a multi-index and define $C := \frac{1}{N!}\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|$ (note that C exists and is finite since all partial derivatives of φ have compact support), then we have

$$\begin{aligned}
\left|\partial^\beta R_N(x,h)\right| &= \left|\partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N (\partial^\alpha\varphi)(x+th)\,\mathrm{d}t\right| \\
&\overset{\star}{=} \left|\sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N (\partial^{\alpha+\beta}\varphi)(x+th)\,\mathrm{d}t\right| \\
&\leqslant \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1)\int_0^1 (1-t)^N \left|(\partial^{\alpha+\beta}\varphi)(x+th)\right|\mathrm{d}t \\
&\leqslant \left[\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|\right] \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1) \\
&= C(N+1)!\sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} = C(|h_1|+\cdots+|h_n|)^{N+1}.
\end{aligned}$$

Here, $\star$ follows from differentiation under the integral sign since the integrand is bounded.

Since this upper bound does not depend on x, we also have

$$\sup_x \left|\partial^\beta R_N(x,h)\right| \leqslant C(|h_1|+\cdots+|h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x \left|\partial^\beta R_N(x,h)\right|}{\|h\|^N} \leqslant \frac{C(|h_1|+\cdots+|h_n|)^{N+1}}{\|h\|^N} \leqslant \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \to 0,$$

and therefore that $\partial^\beta R_N(x,h) = o(\|h\|^n)$ for all multi-indices β. $\square$

Question 3. *Which elements of $\mathcal{D}(X)$ can be represented as a power series on X?*

Solution. It is known that if two power series agree on an open set, they agree on the entire space. Since every $\varphi \in \mathcal{D}(X)$ is identically zero on some open set (outside its support), the only element of $\mathcal{D}(X)$ with a power series representation is the zero function.

Question 4. *Prove the C^∞ Urysohn lemma: if K is a compact subset of $X \subseteq \mathbb{R}^n$, show that one can find a $\varphi \in \mathcal{D}(X)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on a neighborhood of K.*

Solution. Let $K \subseteq X$, then by lemma 1, there is a nonzero distance d from K to $\mathbb{R}^n \backslash X$, so we have neighbourhoods

$$U_1 := \left\{ u \in X \mid d(u, K) < \frac{d}{4} \right\}, \quad U_2 := \left\{ u \in X \mid d(u, K) < \frac{3d}{4} \right\}$$

which is an open neighbourhood of K with $\overline{U} \subseteq X$.

Let $\chi = \mathbb{1}_{U_2}$. Now let $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi \, \mathrm{d}x = 1$ and $\operatorname{supp} \psi \subseteq B(0, \frac{d}{4})$. The we compute

$$(\chi * \psi)(x) = \int_{\mathbb{R}^n} \chi(y)\psi(x-y) \, \mathrm{d}y = \int_{U_2} \psi(x-y) \, \mathrm{d}y \,.$$

Clearly, $\chi * \psi \in \mathcal{D}(X)$, and furthermore, we have for $x \in U_1$ that

$$\int_{U_2} \psi(x-y) \, \mathrm{d}y = \int_{U_2 - x} \psi(z) \, \mathrm{d}z \overset{\star}{=} 1,$$

since $\operatorname{supp} \psi \subseteq B(0, \frac{d}{4}) \subseteq U_2 - x$. This proves the claim.

Question 5. *Given $T \in \mathcal{D}'(X)$, the derivative $\partial^\alpha T$ is defined by*

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Show that $\partial^\alpha T \in \mathcal{D}'(X)$. If $\operatorname{ord}(T) = m$ what can you say about $\operatorname{ord}(\partial^\alpha T)$?

Proof. Let $K \subseteq X$ be compact and $\varphi \in \mathcal{D}(X)$. Since T is a distribution, we know that there exists constants C, N such that

$$|\langle T, \varphi \rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left| \partial^\beta \varphi \right|.$$

Letting $M := |\alpha|$, we find

$$|\langle \partial^\alpha T, \varphi \rangle| = |\langle T, \partial^\alpha \varphi \rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left| \partial^{\alpha+\beta} \varphi \right| \leqslant C \sum_{|\beta| \leqslant M+N} \sup \left| \partial^\beta \varphi \right|.$$

We conclude that $\partial^\alpha T$ is a distribution, and that if $\operatorname{ord}(T) = m$, $\operatorname{ord}(\partial^\alpha T) \leqslant m + |\alpha|$. □

Question 6. *Given $T \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$, prove that for each multi-index α*

$$\partial^\alpha (Tf) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha - \beta} T$$

in $\mathcal{D}'(X)$.

Proof. Let $\varphi \in \mathcal{D}(X)$, then by definition we have $\langle \partial^\alpha(Tf), \varphi\rangle = \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi\rangle$. Approximate T by a sequence $(\psi_n) \subseteq \mathcal{D}'(X)$, then we find

$$\begin{aligned}
\langle T, (-1)^{|\alpha|} f\partial^\alpha\varphi\rangle &= \lim_{n\to\infty} \langle \psi_n, (-1)^{|\alpha|} f \partial^\alpha \varphi\rangle = \lim_{n\to\infty} (-1)^{|\alpha|} \int_X \psi_n(x) f(x) \partial^\alpha \varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \partial^\alpha(\psi_n(x) f(x))\varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \left(\sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \partial^\beta f(x)\cdot \partial^{\alpha-\beta}\psi_n(x)\right)\varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta}\psi_n, \varphi\rangle = \lim_{n\to\infty} \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle \psi_n, \partial^\beta f \partial^{\alpha-\beta}\varphi\rangle \\
&= \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|}\langle T, \partial^\beta f\partial^{\alpha-\beta}\varphi\rangle = \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\langle \partial^\beta f \partial^{\alpha-\beta} T, \varphi\rangle \\
&= \left\langle \sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T, \varphi \right\rangle.
\end{aligned}$$

□

Question 7. *Let (x_k) be a sequence in X with no limit point in X. Consider the family of linear maps $u_\alpha \colon \mathcal{D}(X) \to \mathbb{C}$ defined by*

$$\langle u_\alpha, \varphi\rangle = \sum_{k=1}^\infty \partial^\alpha \varphi(x_k)$$

for each multi-index α. For what α is $u_\alpha \in \mathcal{D}'(X)$? What is $\operatorname{ord}(u_\alpha)$*?*

Solution. Let $K \subseteq X$ be compact. Since (x_k) does not have a limit point, only finitely many of the x_k lie in K (otherwise (x_k) would have a subsequence contained in K which would have a convergent subsequence). Without loss of generality, assume that $x_1, \dots, x_n \in K$, and $x_{n+1}, x_{n+2}, \dots, \notin K$. Now, for any $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$ we find

$$|\langle u, \varphi\rangle| = \left|\sum_{k=1}^\infty \partial^\alpha\varphi(x_k)\right| = \left|\sum_{k=1}^n \partial^\alpha \varphi(x_k)\right| \leqslant \sum_{k=1}^n |\partial^\alpha\varphi(x_k)| \leqslant n\cdot \sup|\partial^\alpha\varphi| \leqslant n \cdot \sum_{|\beta|\leqslant|\alpha|} \sup\left|\partial^\beta\varphi\right|.$$

This shows that $u_\alpha \in \mathcal{D}'(X)$ for any α, with $\operatorname{ord}(u_\alpha) \leqslant |\alpha|$. We claim that this is an equality, i.e., $\operatorname{ord}(u_\alpha) = |\alpha|$. TODO: How to show??

Question 8. *Find the most general solution to the equations*

(a) $u' = 1$,

(b) $xu' = \delta_0$,

(c) $(e^{2\pi i x} - 1)u' = 0$

in $\mathcal{D}'(\mathbb{R})$.

Solution. Let $\varphi \in \mathcal{D}(X)$.

(a) If $u' = 1$ then we find

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = \langle 1, \varphi\rangle = \langle u', \varphi\rangle = -\langle u, \varphi'\rangle,$$

so for any $c \in \mathbb{R}$ we find by partial integration $\langle u, \varphi'\rangle = -\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} (x+c)\varphi'(x)\,\mathrm{d}x$. From this we deduce that $u = x + c$ for some c.

(b) If $xu' = \delta_0$ then we have

$$\varphi(0) = \langle \delta_0, \varphi \rangle = \langle xu', \varphi \rangle = \langle u', x\varphi \rangle = \langle u, -(x\varphi)' \rangle = \langle u, -x\varphi' - \varphi \rangle$$

Note that the above is satisfied for $u = -\delta_0 + c$ for any constant c. TODO: is this the most general solution?

(c) Since $e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$, it is clear that $\operatorname{supp}(u') \subseteq \mathbb{Z}$. We will show that this is also sufficient, i.e., that any distribution u with $\operatorname{supp}(u') \subseteq \mathbb{Z}$ yields a solution.

It is easily seen that

$$\begin{aligned} \operatorname{supp}(u') \subseteq \mathbb{Z} &\iff u' = \sum_{n \in \mathbb{Z}} \alpha_n \delta_n \quad \text{for some } (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C} \\ &\iff u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geqslant n} \quad \text{for some } c \in \mathbb{C},\ (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}. \end{aligned}$$

Indeed, if $u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geqslant n}$ then

$$\langle u', \varphi \rangle = -\langle u, \varphi' \rangle \overset{\star}{=} -\alpha_n \sum_{n \in \mathbb{Z} \cap \operatorname{supp} \varphi} \int_n^\infty \varphi'(x) \, \mathrm{d}x = \sum_{n \in \mathbb{Z} \cap \operatorname{supp} \varphi} \alpha_n \varphi(n) = \langle \sum_{n \in \mathbb{Z}} \alpha_n \delta_n, \varphi \rangle,$$

where $\star$ follows from the fact that there are only finitely many n in $\mathbb{Z} \cap \operatorname{supp} \varphi$ (since φ has compact support).

Finally, we compute that

$$\langle (e^{2\pi i x} - 1)u', \varphi \rangle = \langle u', (e^{2\pi i x} - 1)\varphi \rangle = \sum_{n \in \mathbb{Z}} \alpha_n (e^{2\pi i n} - 1)\varphi(n) = 0,$$

so u satisfies the equation.

Question 9. *Define the distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ by the locally integrable function $u(x, y) = \mathbb{1}_{x \geqslant y}$. Show that $\partial_x^2 u - \partial_y^2 u = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Can you give a physical interpretation of this result?*

Proof. Let $f \in \mathcal{D}(\mathbb{R}^2)$, then we have

$$\begin{aligned} \langle \partial_x^2 u - \partial_y^2 u, f \rangle &= \langle \partial_x^2 u, f \rangle - \langle \partial_y^2 u, f \rangle = \langle u, \partial_x^2 f \rangle - \langle u, \partial_y^2 f \rangle = \langle u, \partial_x^2 f - \partial_y^2 f \rangle \\ &\overset{\star}{=} \int_{-\infty}^\infty \int_y^\infty \partial_x^2 f(x, y) \, \mathrm{d}x \, \mathrm{d}y - \int_{-\infty}^\infty \int_{-\infty}^x \partial_y^2 f(x, y) \, \mathrm{d}y \, \mathrm{d}x \\ &= -\int_{-\infty}^\infty \partial_x f(y, y) \, \mathrm{d}y - \int_{-\infty}^\infty \partial_y f(x, x) \, \mathrm{d}x \\ &= -\int_{-\infty}^\infty (\partial_x f + \partial_y f)(x, x) \, \mathrm{d}x \,. \end{aligned}$$

Here, $\star$ follows from Fubini's theorem. Define $g(x) = f(x, x)$, then it is easily seen that $g'(x) = \partial_x f(x, x) + \partial_y f(x, x)$, so we find that

$$\langle \partial_x^2 u - \partial_y^2 u, f \rangle = -\int_{-\infty}^\infty g'(x) \, \mathrm{d}x = \lim_{x \to -\infty} g(x) - \lim_{x \to \infty} g(x) = 0 - 0 = 0.$$

This shows that $\partial_x u - \partial_y u = 0$, or equivalently, that u satisfies the wave equation. $\square$

Question 10. *Compute* $\Delta(\|x\|^{2-n})$ *in* $\mathcal{D}'(\mathbb{R}^n)$ *for* $n \geqslant 3$, *i.e. compute*

$$\langle \Delta(\|x\|^{2-n}), \varphi \rangle = \langle \|x\|^{2-n}, \Delta\varphi \rangle = \int \frac{\Delta\varphi}{\|x\|^{n-2}} \, \mathrm{d}x$$

for arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^n)$. *Note that* $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ *and* $\Delta = \sum_i (\frac{\partial}{\partial x_i})^2$.

Hint: *use* $\int \mathrm{d}x = \int_{\|x\| \leqslant \varepsilon} \mathrm{d}x + \int_{\|x\| > \varepsilon} \mathrm{d}x$ *and treat each integral separately.*

Solution. We follow the hint: let $\varepsilon > 0$, then we first compute

$$\int_{\|x\|>\varepsilon} \frac{\Delta\varphi}{\|x\|^{n-2}} \, \mathrm{d}x = \sum_{i=1}^n \int_{\|x\|>\varepsilon} \frac{\frac{\partial^2}{\partial x_i^2}\varphi}{\|x\|^{n-2}} \, \mathrm{d}x = \sum_{i=1}^n \int_{\|x\|>\varepsilon} \varphi \cdot \frac{\partial^2}{\partial x_i^2} \|x\|^{2-n} \, \mathrm{d}x \,.$$

Now, it is easily computed that $\frac{\partial}{\partial x_i}\|x\| = \frac{x_i}{\|x\|}$, and therefore

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2}\|x\|^{2-n} &= \frac{\partial}{\partial x_i}(2-n)x_i\|x\|^{-n} = (2-n)\Big(\|x\|^{-n} - n x_i^2 \|x\|^{-n-2}\Big) \\ &= (n-2)\|x\|^{-n}\left(n\left(\frac{x_i}{\|x\|}\right)^2 - 1\right). \end{aligned}$$

TODO: finish

Question 11. *Let* (f_k) *be the sequence of smooth functions defined by* $f_k(x) = \frac{1}{\pi}\frac{k}{(kx)^2+1}$. *Prove that* $f_k \to \delta_0$ *in* $\mathcal{D}'(\mathbb{R})$. *Compute the limits*

(a) $\lim_{k\to\infty} k^2 x e^{-k^2x^2}$,

(b) $\lim_{k\to\infty} k^3 e^{ikx}$,

(c) $\lim_{k\to\infty} \frac{\sin(kx)}{\pi x}$,

in $\mathcal{D}'(\mathbb{R})$.

Solution. It is easily seen that every f_k is locally integrable. Furthermore, noting that $\arctan(kx)' = \frac{k}{(kx)^2+1}$, we see by the dominated convergence theorem that

$$\begin{aligned} \lim_{k\to\infty}\langle f_k, \varphi\rangle &= \frac{1}{\pi}\lim_{k\to\infty}\int_{\mathbb{R}} \frac{k}{(kx)^2+1}\varphi(x)\,\mathrm{d}x = -\frac{1}{\pi}\lim_{k\to\infty}\int_{\mathbb{R}} \arctan(kx)\varphi'(x)\,\mathrm{d}x \\ &= -\frac{1}{\pi}\int_{\mathbb{R}}\lim_{k\to\infty}\arctan(kx)\varphi'(x)\,\mathrm{d}x = -\frac{1}{\pi}\left(2\frac{\pi}{2}\int_0^\infty \varphi'(x)\,\mathrm{d}x\right) = \varphi(0), \end{aligned}$$

which proves $f_k \to \delta_0$.

(a) We have $-(\frac{1}{2}e^{-k^2x^2})' = k^2 x e^{-k^2x^2}$ and therefore

$$\lim_{k\to\infty}\int_{\mathbb{R}} k^2 x e^{-k^2x^2}\varphi(x)\,\mathrm{d}x = \frac{1}{2}\lim_{k\to\infty}\int_{\mathbb{R}} e^{-k^2x^2}\varphi'(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\lim_{k\to\infty} e^{-k^2x^2}\varphi'(x)\,\mathrm{d}x = 0,$$

so the sequence converges to 0 in $\mathcal{D}'(\mathbb{R})$.

(b) We have

$$\lim_{k\to\infty} k^3 \int_{\mathbb{R}} e^{ikx}\varphi(x)\,\mathrm{d}x = \lim_{k\to\infty} 2\pi k^3 \mathcal{F}^{-1}[\varphi](k).$$

Since φ is a Schwarz function, its inverse Fourier transform is also a Schwarz function, and in particular $k^3\mathcal{F}^{-1}[\varphi](k) \to 0$ as $k \to \infty$.

(c) This sequence converges to δ_0, although I have no idea how to prove it.

Question 12. *Compute the limit*

$$\lim_{k\to\infty} \frac{1}{2} + \sum_{m=1}^{k} \cos(\pi m x)$$

in $\mathcal{D}'(-1,1)$.

Solution. Note that

$$f_k(x) \coloneqq \frac{1}{2} + \sum_{m=1}^{k} \cos(\pi m x) = \frac{1}{2}\big(1 + e^{i\pi x} + e^{-i\pi x} + \cdots + e^{i\pi m x} + e^{-i\pi m x}\big) = \frac{1}{2}\sum_{m=-k}^{k} (e^{i\pi x})^m.$$

By viewing the last term as a geometric series $\frac{1}{2}e^{-ik\pi x}\sum_{m=0}^{2k+1}(e^{ix})^m$ we can compute that

$$f_k(x) = \frac{\sin\big((n+\frac{1}{2})\pi x\big)}{2\sin\big(\frac{1}{2}\pi x\big)}.$$

It can be shown (???) that $f_k \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$.

Question 13. *We define the* principal value *of* $1/x$, *written* p.v.$(1/x)$, *by*

$$\langle \text{p.v.}(1/x), \varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

Prove that p.v.$(1/x) \in \mathcal{D}'(\mathbb{R})$ *and that* $\operatorname{ord}(\text{p.v.}(1/x)) = 1$. *Show that*

$$\lim_{\varepsilon\to 0}\frac{1}{x-i\varepsilon} = \text{p.v.}\left(\frac{1}{x}\right) + i\pi\delta_0 \quad \textit{in } \mathcal{D}'(\mathbb{R}).$$

Proof. First we must show that p.v.$(1/x)$ is well-defined. ?? □

Question 14. *??*

Proof. ?? □

Question 15. *??*

Proof. ?? □

Question 16. *Define the distribution* $u \in \mathcal{E}'(\mathbb{R}^3)$ *by the locally integrable function* $u(x) = \mathbb{1}_{|x|\leqslant 1}$. *Prove that* $-\sum_i x_i(\frac{\partial u}{\partial x_i}) = \mathrm{d}\sigma_2$ *in* $\mathcal{E}'(\mathbb{R}^3)$, *where* $\mathrm{d}\sigma_2$ *is the surface element on the sphere* $S^2 \subseteq \mathbb{R}^3$.

Proof. We have

$$\begin{aligned}\left\langle -\sum_i x_i\frac{\partial u}{\partial x_i}, \varphi\right\rangle &= -\sum_i\left\langle \frac{\partial u}{\partial x_i}, x_i\varphi\right\rangle = \sum_i\left\langle u, \frac{\partial}{\partial x_i}(x_i\varphi)\right\rangle = \left\langle u, \sum_i \frac{\partial}{\partial x_i}(x_i\varphi)\right\rangle \\ &= \int_{\|x\|\leqslant 1}\sum_i \frac{\partial}{\partial x_i}(x_i\varphi)\,\mathrm{d}x\end{aligned}$$

□