

# Distribution Theory and Applications — Example Sheet 1

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**Question 1.** Construct a non-zero element of  $\mathcal{D}(\mathbb{R})$  that vanishes outside  $(0, 1)$ . Construct a non-zero element of  $\mathcal{D}(\mathbb{R}^n)$  that vanishes outside the ball  $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$ .

*Proof.* It is well-known that the function

$$\varphi: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \begin{cases} 0 & \text{if } x \leq 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside  $(0, \infty)$ . The function  $\psi(x) := \varphi(x)\varphi(1-x)$  is therefore also smooth and vanishes outside  $(0, 1)$ .

Since  $\psi$  vanishes outside  $(0, 1)$ , the function  $\psi(x/\varepsilon)$  vanishes outside  $(0, \varepsilon)$ , and therefore the function  $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$  vanishes outside  $B_\varepsilon$ .  $\square$

For both questions 2 and 4, we need the following lemma:

**Lemma 1.** Let  $K, V \subseteq \mathbb{R}^n$  where  $K$  is compact,  $V$  is closed, and  $K \cap V = \emptyset$ . Then there is a nonzero distance between  $K$  and  $V$ , i.e.,

$$\inf_{x \in K, v \in V} \|x - v\| > 0.$$

*Proof.* We know that  $K \subseteq V^c$  and that  $V^c$  is open, so for every  $x \in K$  there exists an open ball  $B(x, \varepsilon_x)$  around  $x$  such that  $B(x, 2\varepsilon_x) \subseteq V^c$ . Since  $\{B(x, \varepsilon_x)\}$  is an open covering of  $K$ , there exist finitely many balls  $B(x_1, \varepsilon_1), \dots, B(x_n, \varepsilon_n)$  that cover  $K$ . Let  $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_n\}$  and  $x \in K$ , then there is an  $x_i$  such that  $\|x - x_i\| < \varepsilon$ , and since  $B(x_i, 2\varepsilon) \subseteq B(x_i, 2\varepsilon_i) \subseteq V^c$  it is clear that  $B(x, \varepsilon) \subseteq V^c$  as well.

We conclude that  $B(x, \varepsilon) \subseteq V^c$  for any  $x \in K$ , and therefore that  $\inf_{x \in K, v \in V} \|x - v\| \geq \varepsilon > 0$ .  $\square$

**Question 2.** Given  $\varphi \in \mathcal{D}(X)$ , Taylor's theorem gives

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h).$$

Prove that  $\text{supp}(R_N)$  is contained in some fixed compact  $K \subseteq X$  for  $|h|$  sufficiently small. Show also that  $\partial^\alpha R_N = o(|h|^N)$  uniformly in  $x$  for each multi-index  $\alpha$ , i.e. prove

$$\lim_{|h| \rightarrow 0} \frac{\sup_x |\partial^\alpha R_N(x, h)|}{|h|^N} = 0$$

for each multi-index  $\alpha$ .

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x, h) = \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th) dt,$$

and note that  $(N+1)! \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \dots + h_n)^{N+1}$ .

*Proof.* Let  $\varphi \in \mathcal{D}(X)$  with  $K = \text{supp } \varphi$ , then by lemma 1 we know there exists a nonzero distance  $d > 0$  between  $K$  and  $\mathbb{R}^n \setminus X$ . We claim that if  $\|h\| \leq \frac{d}{2}$ , then

$$\text{supp}(R_N) \subseteq \left\{ x \in X \mid d(x, K) \leq \frac{d}{2} \right\} =: \hat{K},$$

which is clearly a compact set contained in  $X$ . Indeed, if  $\|h\| \leq \frac{d}{2}$  we have

$$\varphi(x+h) \neq 0 \implies x+h \in K \implies d(x, K) \leq \|h\| \leq \frac{d}{2}.$$

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h),$$

and since  $\sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x)$  vanishes for  $x \notin K$ , it is clear that  $\text{supp}(R_N(\cdot, h))$  must be contained in  $\hat{K}$  (for  $\|h\| \leq \frac{d}{2}$ ).

Now let  $\beta$  be a multi-index and define  $C := \frac{1}{N!} \max_{|\alpha|=N+1, x \in \mathbb{R}^n} |(\partial^{\alpha+\beta})\varphi(x)|$  (note that  $C$  exists and is finite since all partial derivatives of  $\varphi$  have compact support), then we have

$$\begin{aligned} |\partial^\beta R_N(x, h)| &= \left| \partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th) dt \right| \\ &\stackrel{*}{=} \left| \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^{\alpha+\beta} \varphi)(x+th) dt \right| \\ &\leq \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} (N+1) \int_0^1 (1-t)^N |(\partial^{\alpha+\beta} \varphi)(x+th)| dt \\ &\leq \left[ \max_{|\alpha|=N+1, x \in \mathbb{R}^n} |(\partial^{\alpha+\beta})\varphi(x)| \right] \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} (N+1) \\ &= C(N+1)! \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} = C(|h_1| + \dots + |h_n|)^{N+1}. \end{aligned}$$

Here,  $\star$  follows from differentiation under the integral sign since the integrand is bounded.

Since this upper bound does not depend on  $x$ , we also have

$$\sup_x |\partial^\beta R_N(x, h)| \leq C(|h_1| + \dots + |h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x |\partial^\beta R_N(x, h)|}{\|h\|^N} \leq \frac{C(|h_1| + \dots + |h_n|)^{N+1}}{\|h\|^N} \leq \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \rightarrow 0,$$

and therefore that  $\partial^\beta R_N(x, h) = o(\|h\|^n)$  for all multi-indices  $\beta$ . □

**Question 3.** Which elements of  $\mathcal{D}(X)$  can be represented as a power series on  $X$ ?

*Solution.* It is known that if two power series agree on an open set, they agree on the entire space. Since every  $\varphi \in \mathcal{D}(X)$  is identically zero on some open set (outside its support), the only element of  $\mathcal{D}(X)$  with a power series representation is the zero function.

**Question 4.** Prove the  $C^\infty$  Urysohn lemma: if  $K$  is a compact subset of  $X \subseteq \mathbb{R}^n$ , show that one can find a  $\varphi \in \mathcal{D}(X)$  such that  $0 \leq \varphi \leq 1$  and  $\varphi = 1$  on a neighborhood of  $K$ .

*Solution.* Let  $K \subseteq X$ , then by lemma 1, there is a nonzero distance  $d$  from  $K$  to  $\mathbb{R}^n \setminus X$ , so we have neighbourhoods

$$U_1 := \left\{ u \in X \mid d(u, K) < \frac{d}{4} \right\}, \quad U_2 := \left\{ u \in X \mid d(u, K) < \frac{3d}{4} \right\}$$

which is an open neighbourhood of  $K$  with  $\overline{U} \subseteq X$ .

Let  $\chi = \mathbb{1}_{U_2}$ . Now let  $\psi \in \mathcal{D}(\mathbb{R}^n)$  satisfy  $\int_{\mathbb{R}^n} \psi \, dx = 1$  and  $\text{supp } \psi \subseteq B(0, \frac{d}{4})$ . Then we compute

$$(\chi * \psi)(x) = \int_{\mathbb{R}^n} \chi(y) \psi(x - y) \, dy = \int_{U_2} \psi(x - y) \, dy.$$

Clearly,  $\chi * \psi \in \mathcal{D}(X)$ , and furthermore, we have for  $x \in U_1$  that

$$\int_{U_2} \psi(x - y) \, dy = \int_{U_2 - x} \psi(z) \, dz \stackrel{*}{=} 1,$$

since  $\text{supp } \psi \subseteq B(0, \frac{d}{4}) \subseteq U_2 - x$ . This proves the claim.

**Question 5.** Given  $T \in \mathcal{D}'(X)$ , the derivative  $\partial^\alpha T$  is defined by

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Show that  $\partial^\alpha T \in \mathcal{D}'(X)$ . If  $\text{ord}(T) = m$  what can you say about  $\text{ord}(\partial^\alpha T)$ ?

*Proof.* Let  $K \subseteq X$  be compact and  $\varphi \in \mathcal{D}(X)$ . Since  $T$  is a distribution, we know that there exists constants  $C, N$  such that

$$|\langle T, \varphi \rangle| \leq C \sum_{|\beta| \leq N} \sup |\partial^\beta \varphi|.$$

Letting  $M := |\alpha|$ , we find

$$|\langle \partial^\alpha T, \varphi \rangle| = |\langle T, \partial^\alpha \varphi \rangle| \leq C \sum_{|\beta| \leq N} \sup |\partial^{\alpha+\beta} \varphi| \leq C \sum_{|\beta| \leq M+N} \sup |\partial^\beta \varphi|.$$

We conclude that  $\partial^\alpha T$  is a distribution, and that if  $\text{ord}(T) = m$ ,  $\text{ord}(\partial^\alpha T) \leq m + |\alpha|$ . □

**Question 6.** Given  $T \in \mathcal{D}'(X)$  and  $f \in C^\infty(X)$ , prove that for each multi-index  $\alpha$

$$\partial^\alpha(Tf) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

in  $\mathcal{D}'(X)$ .

*Proof.* Let  $\varphi \in \mathcal{D}(X)$ , then by definition we have  $\langle \partial^\alpha(Tf), \varphi \rangle = \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle$ . Approximate  $T$  by a sequence  $(\psi_n) \subseteq \mathcal{D}'(X)$ , then we find

$$\begin{aligned}
\langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle &= \lim_{n \rightarrow \infty} \langle \psi_n, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle = \lim_{n \rightarrow \infty} (-1)^{|\alpha|} \int_X \psi_n(x) f(x) \partial^\alpha \varphi(x) dx \\
&= \lim_{n \rightarrow \infty} \int_X \partial^\alpha (\psi_n(x) f(x)) \varphi(x) dx \\
&= \lim_{n \rightarrow \infty} \int_X \left( \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \cdot \partial^{\alpha-\beta} \psi_n(x) \right) \varphi(x) dx \\
&= \lim_{n \rightarrow \infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} \psi_n, \varphi \rangle = \lim_{n \rightarrow \infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle \psi_n, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle \\
&= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle T, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} T, \varphi \rangle \\
&= \left\langle \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T, \varphi \right\rangle.
\end{aligned}$$

□

**Question 7.** Let  $(x_k)$  be a sequence in  $X$  with no limit point in  $X$ . Consider the family of linear maps  $u_\alpha: \mathcal{D}(X) \rightarrow \mathbb{C}$  defined by

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)$$

for each multi-index  $\alpha$ . For what  $\alpha$  is  $u_\alpha \in \mathcal{D}'(X)$ ? What is  $\text{ord}(u_\alpha)$ ?

*Solution.* Let  $K \subseteq X$  be compact. Since  $(x_k)$  does not have a limit point, only finitely many of the  $x_k$  lie in  $K$  (otherwise  $(x_k)$  would have a subsequence contained in  $K$  which would have a convergent subsequence). Without loss of generality, assume that  $x_1, \dots, x_n \in K$ , and  $x_{n+1}, x_{n+2}, \dots \notin K$ . Now, for any  $\varphi \in \mathcal{D}(X)$  with  $\text{supp}(\varphi) \subseteq K$  we find

$$|\langle u, \varphi \rangle| = \left| \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k) \right| = \left| \sum_{k=1}^n \partial^\alpha \varphi(x_k) \right| \leq \sum_{k=1}^n |\partial^\alpha \varphi(x_k)| \leq n \cdot \sup |\partial^\alpha \varphi| \leq n \cdot \sum_{|\beta| \leq |\alpha|} \sup |\partial^\beta \varphi|.$$

This shows that  $u_\alpha \in \mathcal{D}'(X)$  for any  $\alpha$ , with  $\text{ord}(u_\alpha) \leq |\alpha|$ . We claim that this is an equality, i.e.,  $\text{ord}(u_\alpha) = |\alpha|$ . **TODO:** How to show??

**Question 8.** Find the most general solution to the equations

(a)  $u' = 1$ ,

(b)  $xu' = \delta_0$ ,

(c)  $(e^{2\pi i x} - 1)u' = 0$

in  $\mathcal{D}'(\mathbb{R})$ .

*Solution.* Let  $\varphi \in \mathcal{D}(X)$ .

(a) If  $u' = 1$  then we find

$$\int_{\mathbb{R}} \varphi(x) dx = \langle 1, \varphi \rangle = \langle u', \varphi \rangle = -\langle u, \varphi' \rangle,$$

so for any  $c \in \mathbb{R}$  we find by partial integration  $\langle u, \varphi' \rangle = -\int_{\mathbb{R}} \varphi(x) dx = \int_{\mathbb{R}} (x+c) \varphi'(x) dx$ . From this we deduce that  $u = x + c$  for some  $c$ .

(b) If  $xu' = \delta_0$  then we have

$$\varphi(0) = \langle \delta_0, \varphi \rangle = \langle xu', \varphi \rangle = \langle u', x\varphi \rangle = \langle u, -(x\varphi)' \rangle = \langle u, -x\varphi' - \varphi \rangle$$

Note that the above is satisfied for  $u = -\delta_0 + c$  for any constant  $c$ . **TODO:** is this the most general solution?

(c) Since  $e^{2\pi ix} = 1 \iff x \in \mathbb{Z}$ , it is clear that  $\text{supp}(u') \subseteq \mathbb{Z}$ . We will show that this is also sufficient, i.e., that any distribution  $u$  with  $\text{supp}(u') \subseteq \mathbb{Z}$  yields a solution.

It is easily seen that

$$\begin{aligned} \text{supp}(u') \subseteq \mathbb{Z} &\iff u' = \sum_{n \in \mathbb{Z}} \alpha_n \delta_n \quad \text{for some } (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C} \\ &\iff u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geq n} \quad \text{for some } c \in \mathbb{C}, (\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}. \end{aligned}$$

Indeed, if  $u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geq n}$  then

$$\langle u', \varphi \rangle = -\langle u, \varphi' \rangle \stackrel{\star}{=} -\alpha_n \sum_{n \in \mathbb{Z} \cap \text{supp } \varphi} \int_n^\infty \varphi'(x) dx = \sum_{n \in \mathbb{Z} \cap \text{supp } \varphi} \alpha_n \varphi(n) = \langle \sum_{n \in \mathbb{Z}} \alpha_n \delta_n, \varphi \rangle,$$

where  $\star$  follows from the fact that there are only finitely many  $n$  in  $\mathbb{Z} \cap \text{supp } \varphi$  (since  $\varphi$  has compact support).

Finally, we compute that

$$\langle (e^{2\pi ix} - 1)u', \varphi \rangle = \langle u', (e^{2\pi ix} - 1)\varphi \rangle = \sum_{n \in \mathbb{Z}} \alpha_n (e^{2\pi in} - 1)\varphi(n) = 0,$$

so  $u$  satisfies the equation.

**Question 9.** Define the distribution  $u \in \mathcal{D}'(\mathbb{R}^2)$  by the locally integrable function  $u(x, y) = \mathbb{1}_{x \geq y}$ . Show that  $\partial_x^2 u - \partial_y^2 u = 0$  in  $\mathcal{D}'(\mathbb{R}^2)$ . Can you give a physical interpretation of this result?

*Proof.* Let  $f \in \mathcal{D}(\mathbb{R}^2)$ , then we have

$$\begin{aligned} \langle \partial_x^2 u - \partial_y^2 u, f \rangle &= \langle \partial_x^2 u, f \rangle - \langle \partial_y^2 u, f \rangle = \langle u, \partial_x^2 f \rangle - \langle u, \partial_y^2 f \rangle = \langle u, \partial_x^2 f - \partial_y^2 f \rangle \\ &\stackrel{\star}{=} \int_{-\infty}^\infty \int_y^\infty \partial_x^2 f(x, y) dx dy - \int_{-\infty}^\infty \int_{-\infty}^x \partial_y^2 f(x, y) dy dx \\ &= - \int_{-\infty}^\infty \partial_x f(y, y) dy - \int_{-\infty}^\infty \partial_y f(x, x) dx \\ &= - \int_{-\infty}^\infty (\partial_x f + \partial_y f)(x, x) dx. \end{aligned}$$

Here,  $\star$  follows from Fubini's theorem. Define  $g(x) = f(x, x)$ , then it is easily seen that  $g'(x) = \partial_x f(x, x) + \partial_y f(x, x)$ , so we find that

$$\langle \partial_x^2 u - \partial_y^2 u, f \rangle = - \int_{-\infty}^\infty g'(x) dx = \lim_{x \rightarrow -\infty} g(x) - \lim_{x \rightarrow \infty} g(x) = 0 - 0 = 0.$$

This shows that  $\partial_x u - \partial_y u = 0$ , or equivalently, that  $u$  satisfies the wave equation.  $\square$

**Question 10.** Compute  $\Delta(\|x\|^{2-n})$  in  $\mathcal{D}'(\mathbb{R}^n)$  for  $n \geq 3$ , i.e. compute

$$\langle \Delta(\|x\|^{2-n}), \varphi \rangle = \langle \|x\|^{2-n}, \Delta\varphi \rangle = \int \frac{\Delta\varphi}{\|x\|^{n-2}} dx$$

for arbitrary  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ . Note that  $\|x\| = (x_1^2 + \dots + x_n^2)^{1/2}$  and  $\Delta = \sum_i (\frac{\partial}{\partial x_i})^2$ .

Hint: use  $\int dx = \int_{\|x\| \leq \varepsilon} dx + \int_{\|x\| > \varepsilon} dx$  and treat each integral separately.

*Solution.* We follow the hint: let  $\varepsilon > 0$ , then we first compute

$$\int_{\|x\| > \varepsilon} \frac{\Delta\varphi}{\|x\|^{n-2}} dx = \sum_{i=1}^n \int_{\|x\| > \varepsilon} \frac{\frac{\partial^2}{\partial x_i^2} \varphi}{\|x\|^{n-2}} dx = \sum_{i=1}^n \int_{\|x\| > \varepsilon} \varphi \cdot \frac{\partial^2}{\partial x_i^2} \|x\|^{2-n} dx.$$

Now, it is easily computed that  $\frac{\partial}{\partial x_i} \|x\| = \frac{x_i}{\|x\|}$ , and therefore

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} \|x\|^{2-n} &= \frac{\partial}{\partial x_i} (2-n)x_i \|x\|^{-n} = (2-n) \left( \|x\|^{-n} - nx_i^2 \|x\|^{-n-2} \right) \\ &= (n-2) \|x\|^{-n} \left( n \left( \frac{x_i}{\|x\|} \right)^2 - 1 \right). \end{aligned}$$

**TODO:** finish

**Question 11.** Let  $(f_k)$  be the sequence of smooth functions defined by  $f_k(x) = \frac{1}{\pi} \frac{k}{(kx)^2 + 1}$ . Prove that  $f_k \rightarrow \delta_0$  in  $\mathcal{D}'(\mathbb{R})$ . Compute the limits

(a)  $\lim_{k \rightarrow \infty} k^2 x e^{-k^2 x^2},$

(b)  $\lim_{k \rightarrow \infty} k^3 e^{ikx},$

(c)  $\lim_{k \rightarrow \infty} \frac{\sin(kx)}{\pi x},$

in  $\mathcal{D}'(\mathbb{R})$ .

*Solution.* It is easily seen that every  $f_k$  is locally integrable. Furthermore, noting that  $\arctan(kx)' = \frac{k}{(kx)^2 + 1}$ , we see by the dominated convergence theorem that

$$\begin{aligned} \lim_{k \rightarrow \infty} \langle f_k, \varphi \rangle &= \frac{1}{\pi} \lim_{k \rightarrow \infty} \int_{\mathbb{R}} \frac{k}{(kx)^2 + 1} \varphi(x) dx = -\frac{1}{\pi} \lim_{k \rightarrow \infty} \int_{\mathbb{R}} \arctan(kx) \varphi'(x) dx \\ &= -\frac{1}{\pi} \int_{\mathbb{R}} \lim_{k \rightarrow \infty} \arctan(kx) \varphi'(x) dx = -\frac{1}{\pi} \left( 2 \frac{\pi}{2} \int_0^\infty \varphi'(x) dx \right) = \varphi(0), \end{aligned}$$

which proves  $f_k \rightarrow \delta_0$ .

(a) We have  $-(\frac{1}{2} e^{-k^2 x^2})' = k^2 x e^{-k^2 x^2}$  and therefore

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}} k^2 x e^{-k^2 x^2} \varphi(x) dx = \frac{1}{2} \lim_{k \rightarrow \infty} \int_{\mathbb{R}} e^{-k^2 x^2} \varphi'(x) dx = \frac{1}{2} \int_{\mathbb{R}} \lim_{k \rightarrow \infty} e^{-k^2 x^2} \varphi'(x) dx = 0,$$

so the sequence converges to 0 in  $\mathcal{D}'(\mathbb{R})$ .

(b) We have

$$\lim_{k \rightarrow \infty} k^3 \int_{\mathbb{R}} e^{ikx} \varphi(x) dx = \lim_{k \rightarrow \infty} 2\pi k^3 \mathcal{F}^{-1}[\varphi](k).$$

Since  $\varphi$  is a Schwarz function, its inverse Fourier transform is also a Schwarz function, and in particular  $k^3 \mathcal{F}^{-1}[\varphi](k) \rightarrow 0$  as  $k \rightarrow \infty$ .

(c) This sequence converges to  $\delta_0$ , although I have no idea how to prove it.

**Question 12.** Compute the limit

$$\lim_{k \rightarrow \infty} \frac{1}{2} + \sum_{m=1}^k \cos(\pi m x)$$

in  $\mathcal{D}'(-1, 1)$ .

*Solution.* Note that

$$f_k(x) := \frac{1}{2} + \sum_{m=1}^k \cos(\pi m x) = \frac{1}{2} (1 + e^{i\pi x} + e^{-i\pi x} + \dots + e^{i\pi k x} + e^{-i\pi k x}) = \frac{1}{2} \sum_{m=-k}^k (e^{i\pi x})^m.$$

By viewing the last term as a geometric series  $\frac{1}{2} e^{-ik\pi x} \sum_{m=0}^{2k+1} (e^{ix})^m$  we can compute that

$$f_k(x) = \frac{\sin((n + \frac{1}{2})\pi x)}{2 \sin(\frac{1}{2}\pi x)}.$$

It can be shown (???) that  $f_k \rightarrow \delta_0$  in  $\mathcal{D}'(\mathbb{R})$ .

**Question 13.** We define the principal value of  $1/x$ , written  $\text{p.v.}(1/x)$ , by

$$\langle \text{p.v.}(1/x), \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|x| > \varepsilon} \frac{\varphi(x)}{x} dx \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

Prove that  $\text{p.v.}(1/x) \in \mathcal{D}'(\mathbb{R})$  and that  $\text{ord}(\text{p.v.}(1/x)) = 1$ . Show that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x - i\varepsilon} = \text{p.v.} \left( \frac{1}{x} \right) + i\pi \delta_0 \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

*Proof.* First we must show that  $\text{p.v.}(1/x)$  is well-defined. ?? □

**Question 14.** ??

*Proof.* ?? □

**Question 15.** ??

*Proof.* ?? □

**Question 16.** Define the distribution  $u \in \mathcal{E}'(\mathbb{R}^3)$  by the locally integrable function  $u(x) = \mathbb{1}_{|x| \leq 1}$ . Prove that  $-\sum_i x_i (\frac{\partial u}{\partial x_i}) = d\sigma_2$  in  $\mathcal{E}'(\mathbb{R}^3)$ , where  $d\sigma_2$  is the surface element on the sphere  $S^2 \subseteq \mathbb{R}^3$ .

*Proof.* We have

$$\begin{aligned} \left\langle -\sum_i x_i \frac{\partial u}{\partial x_i}, \varphi \right\rangle &= -\sum_i \left\langle \frac{\partial u}{\partial x_i}, x_i \varphi \right\rangle = \sum_i \left\langle u, \frac{\partial}{\partial x_i} (x_i \varphi) \right\rangle = \left\langle u, \sum_i \frac{\partial}{\partial x_i} (x_i \varphi) \right\rangle \\ &= \int_{\|x\| \leq 1} \sum_i \frac{\partial}{\partial x_i} (x_i \varphi) dx \end{aligned}$$

□