Distribution Theory and Applications — Example Sheet 1

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Question 1. *Construct a non-zero element of $\mathcal{D}(\mathbb{R})$ that vanishes outside $(0,1)$. Construct a non-zero of $\mathcal{D}(\mathbb{R}^n)$ that vanishes outside the ball $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.*

Proof. It is well-known that the function

$$\varphi\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \begin{cases} 0 & \text{if } x \leqslant 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside $(0,\infty)$. The function $\psi(x) := \varphi(x)\varphi(1-x)$ is therefore also smooth and vanishes outside $(0,1)$.

Since ψ vanishes outside $(0,1)$, the function $\psi(x/\varepsilon)$ vanishes outside $(0,\varepsilon)$, and therefore the function $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$ vanishes outside B_ε. $\square$

Question 2. *Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives*

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h).$$

Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in x for each multi-index α, i.e. prove

$$\lim_{|h| \to 0} \frac{\sup_x |\partial^\alpha R_N(x,h)|}{|h|^N} = 0$$

for each multi-index α.

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x,h) = \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!}(N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th)\,\mathrm{d}t\,,$$

and note that $(N+1)! \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \cdots + h_n)^{N+1}$.

Proof. Since $\varphi \in \mathcal{D}(X)$, we know that $\operatorname{supp}\varphi \subseteq \overline{B_N}$ for some $N \in \mathbb{N}$. Now, suppose $\|h\| < 1$, then

$$\varphi(x+h) \neq 0 \implies \|x+h\| \leqslant N \implies \|x\| \leqslant \|x+h\| + \|h\| \leqslant N+1,$$

so if we define $\psi_h(x) = \varphi(x+h)$ then we know that $\operatorname{supp}\varphi_h \subseteq \overline{B_{N+1}}$.

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h),$$

and since $\sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x)$ vanishes for $x \notin \overline{B_N}$, it is clear that $\operatorname{supp}(R_N(\cdot,h))$ must also be contained in $\overline{B_{N+1}}$ (again, for $\|h\| \leqslant 1$). This shows that $\operatorname{supp}(R_N)$ is contained in $\overline{B_{N+1}}$ for $|h|$ sufficiently small.

Now let β be a multi-index and define $C \coloneqq \frac{1}{N!}\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|$ (note that C exists and is finite since all partial derivatives of φ have compact support), then we have

$$\begin{aligned}
\left|\partial^\beta R_N(x,h)\right| &= \left|\partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^\alpha\varphi)(x+th)\,\mathrm{d}t\right| \\
&\overset{\star}{=} \left|\sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^{\alpha+\beta}\varphi)(x+th)\,\mathrm{d}t\right| \\
&\leqslant \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1)\int_0^1 (1-t)^N\left|(\partial^{\alpha+\beta}\varphi)(x+th)\right|\mathrm{d}t \\
&\leqslant \left[\max_{|\alpha|=N+1, x\in\mathbb{R}^n} \left|(\partial^{\alpha+\beta})\varphi(x)\right|\right] \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!}(N+1) \\
&\leqslant C(N+1)! \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} = C(|h_1|+\cdots+|h_n|)^{N+1}.
\end{aligned}$$

Since this upper bound does not depend on x, we also have

$$\sup_x \left|\partial^\beta R_N(x,h)\right| \leqslant C(|h_1|+\cdots+|h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x \left|\partial^\beta R_N(x,h)\right|}{\|h\|^N} \leqslant \frac{C(|h_1|+\cdots+|h_n|)^{N+1}}{\|h\|^N} \leqslant \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \to 0,$$

and therefore that $\partial^\beta R_N(x,h) = o(\|h\|^n)$ for all multi-indices β. □

Question 3. *Which elements of $\mathcal{D}(X)$ can be represented as a power series on X?*

Solution. It is known that if two power series agree on an open set, they agree on the entire space. Since every $\varphi \in \mathcal{D}(X)$ is identically zero on some open set (outside its support), the only element of $\mathcal{D}(X)$ with a power series representation is the zero function.

Question 4. *Prove the C^∞ Urysohn lemma: if K is a compact subset of $X \subseteq \mathbb{R}^n$, show that one can find a $\varphi \in \mathcal{D}(X)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on a neighborhood of K.*

Solution. Let $K \subseteq U_1$. Define $U_2 \coloneqq U_1 + B(0,1)$ and let $\chi = \mathbb{1}_{U_2}$. Now let $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\psi\,\mathrm{d}x = 1$ and $\operatorname{supp}\psi \subseteq B(0,1)$. The we compute

$$(\chi * \psi)(x) = \int_{\mathbb{R}^n} \chi(y)\psi(x-y)\,\mathrm{d}y = \int_{U_2} \psi(x-y)\,\mathrm{d}y\,.$$

Clearly, $\chi * \psi \in \mathcal{D}(X)$, and furthermore, we have for $x \in U_1$ that

$$\int_{U_2} \psi(x-y)\,\mathrm{d}y = \int_{U_2-x} \psi(z)\,\mathrm{d}z \overset{\star}{=} 1,$$

since $B(0,1) \subseteq U_2 - x$. This proves the claim.

Question 5. *Given $T \in \mathcal{D}'(X)$, the derivative $\partial^\alpha T$ is defined by*

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Show that $\partial^\alpha T \in \mathcal{D}'(X)$. If $\operatorname{ord}(T) = m$ what can you say about $\operatorname{ord}(\partial^\alpha T)$?

Proof. Let $K \subseteq X$ be compact and $\varphi \in \mathcal{D}(X)$. Since T is a distribution, we know that there exists constants C, N such that

$$|\langle T, \varphi \rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left| \partial^\beta \varphi \right|.$$

Letting $M := |\alpha|$, we find

$$|\langle \partial^\alpha T, \varphi \rangle| = |\langle T, \partial^\alpha \varphi \rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left| \partial^{\alpha+\beta} \varphi \right| \leqslant C \sum_{|\beta| \leqslant M+N} \sup \left| \partial^\beta \varphi \right|.$$

We conclude that $\partial^\alpha T$ is a distribution, and that if $\operatorname{ord}(T) = m$, $\operatorname{ord}(\partial^\alpha T) \leqslant m + |\alpha|$. □

Question 6. *Given $T \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$, prove that for each multi-index α*

$$\partial^\alpha (Tf) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

in $\mathcal{D}'(X)$.

Proof. Let $\varphi \in \mathcal{D}(X)$, then by definition we have $\langle \partial^\alpha (Tf), \varphi \rangle = \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle$. Approximate T by a sequence $(\psi_n) \subseteq \mathcal{D}'(X)$, then we find

$$\begin{aligned}
\langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle &= \lim_{n\to\infty} \langle \psi_n, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle = \lim_{n\to\infty} (-1)^{|\alpha|} \int_X \psi_n(x) f(x) \partial^\alpha \varphi(x) \, \mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \partial^\alpha (\psi_n(x) f(x)) \varphi(x) \, \mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \left(\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \cdot \partial^{\alpha-\beta} \psi_n(x) \right) \varphi(x) \, \mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} \psi_n, \varphi \rangle = \lim_{n\to\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle \psi_n, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle \\
&= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle T, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} T, \varphi \rangle \\
&= \left\langle \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T, \varphi \right\rangle.
\end{aligned}$$

□

Question 7. *Let (x_k) be a sequence in X with no limit point in X. Consider the family of linear maps $u_\alpha \colon \mathcal{D}(X) \to \mathbb{C}$ defined by*

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)$$

for each multi-index α. For what α is $u_\alpha \in \mathcal{D}'(X)$? What is $\operatorname{ord}(u_\alpha)$?

Solution. Let $K \subseteq X$ be compact. Since (x_k) does not have a limit point, only finitely many of the x_k lie in K (otherwise (x_k) would have a subsequence contained in K which would have a convergent subsequence). Without loss of generality, assume that $x_1, \ldots, x_n \in K$, and $x_{n+1}, x_{n+2}, \ldots, \notin K$. Now, for any $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$ we find

$$|\langle u, \varphi \rangle| = \left| \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k) \right| = \left| \sum_{k=1}^{n} \partial^\alpha \varphi(x_k) \right| \leqslant \sum_{k=1}^{n} |\partial^\alpha \varphi(x_k)| \leqslant n \cdot \sup |\partial^\alpha \varphi| \leqslant n \cdot \sum_{|\beta| \leqslant |\alpha|} \sup \left|\partial^\beta \varphi\right|.$$

This shows that $u_\alpha \in \mathcal{D}'(X)$ for any α, with $\operatorname{ord}(u_\alpha) \leqslant |\alpha|$. We claim that this is an equality, i.e., $\operatorname{ord}(u_\alpha) = |\alpha|$. TODO: How to show??

Question 8. *Find the most general solution to the equations*

(a) $u' = 1$,

(b) $xu' = \delta_0$,

(c) $(e^{2\pi i x} - 1)u' = 0$

in $\mathcal{D}'(\mathbb{R})$.

Solution. Let $\varphi \in \mathcal{D}(X)$.

(a) If $u' = 1$ then we find

$$\int_{\mathbb{R}} \varphi(x) \, \mathrm{d}x = \langle 1, \varphi \rangle = \langle u', \varphi \rangle = -\langle u, \varphi' \rangle,$$

so for any $c \in \mathbb{R}$ we find by partial integration $\langle u, \varphi' \rangle = -\int_{\mathbb{R}} \varphi(x) \, \mathrm{d}x = \int_{\mathbb{R}} (x + c)\varphi'(x) \, \mathrm{d}x$. From this we deduce that $u = x + c$ for some c.