Inverse Problems — Summary

Lucas Riedstra

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A *direct problem* is a problem where given an object or *cause*, we must determine the data or *effect*. In an *inverse problem*, we observe data and wish to recover the object.

A problem is called *well-posed* if a unique solution exists that depends continuously on the data. Most inverse problems are, unfortunately, ill-posed.

1 Generalised Solutions

Recap 1.1. 1. An linear operator $A\colon \mathcal{X} \to \mathcal{Y}$ is called *bounded* if

$$\|A\|_{\mathcal{B}(\mathcal{X},\mathcal{Y})} \coloneqq \sup_{u\neq 0} \frac{\|Au\|_{\mathcal{Y}}}{\|u\|_{\mathcal{X}}} = \sup_{\|u\|_{\mathcal{X}}\leq 1} < \infty.$$

It is known that a linear operator between normed spaces is continuous if and only if it is bounded. The set of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denotes $\mathcal{B}(\mathcal{X},\mathcal{Y})$.

2. We let $\mathcal{D}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the domain, null space, and range of A respectively.

3. We will assume $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, so there is an inner product $\langle\cdot,\cdot\rangle$ and any bounded operator $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ has a unique adjoint $A^* \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ which satisfies

$$\langle Au, v\rangle_{\mathcal{Y}} = \langle u, A^*v\rangle_{\mathcal{X}} \quad \text{for all } u \in \mathcal{X},\, v \in \mathcal{Y}.$$

4. For any $\mathcal{X}' \subseteq \mathcal{X}$ we define the *orthogonal complement* of $\mathcal{X}'$ as

$$(\mathcal{X}')^\perp \coloneqq \{u \in \mathcal{X} \mid \langle u, v\rangle_{\mathcal{X}} = 0 \ \forall v \in \mathcal{X}'\}.$$

It is known that $(\mathcal{X}')^\perp$ is a closed subspace of $\mathcal{X}$ and that $\mathcal{X}' \subseteq ((\mathcal{X}')^\perp)^\perp$, where equality holds if and only if $\mathcal{X}'$ is a closed subspace of $\mathcal{X}$. For a non-closed subspace $\mathcal{X}'$ we have $((\mathcal{X}')^\perp)^\perp = \overline{\mathcal{X}'}$.

5. If $\mathcal{X}'$ is a closed subspace of $\mathcal{X}$, then for any $u \in \mathcal{X}$ there exist unique $x_u \in \mathcal{X}'$, $x_u^\perp \in (\mathcal{X}')^\perp$ such that $u = x_u + x_u^\perp$. The map $u \mapsto x_u$ is denoted $P_{\mathcal{X}'}$ and is called the *orthogonal projection* on $\mathcal{X}'$. Properties are:

   (a) $P_{\mathcal{X}'}$ is bounded and self-adjoint with norm 1;

   (b) $P_{\mathcal{X}'} + P_{(\mathcal{X}')^\perp} = I$;

   (c) $P_{\mathcal{X}'}u$ minimises the distance from u to $\mathcal{X}'$;

   (d) $x = P_{\mathcal{X}'}u$ if and only if $x \in \mathcal{X}'$ and $u - x \in (\mathcal{X}')^\perp$.

6. For any $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ we have

$$\mathcal{R}(A)^\perp = \mathcal{N}(A^*) \quad \text{and} \quad \mathcal{N}(A)^\perp = \overline{\mathcal{R}(A^*)}.$$

Lemma 1.2. *For any $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ we have $\overline{\mathcal{R}(A^*A)} = \overline{\mathcal{R}(A^*)}$.*

Proof. It is trivial that $\overline{\mathcal{R}(A^*A)} \subseteq \overline{\mathcal{R}(A^*)}$.

Now, suppose $u \in \overline{\mathcal{R}(A^*)}$ and let $\varepsilon > 0$. Then there exists $v \in \mathcal{X}$ such that $\|A^*v - u\| < \varepsilon/2$. Writing $v = e + f$ with $e \in \mathcal{N}(A^*)$, $f \in \mathcal{N}(A^*)^\perp = \overline{\mathcal{R}(A)}$, we see that $\|A^*f - u\| < \varepsilon/2$.

Since $f \in \overline{\mathcal{R}(A)}$, there exists $x \in \mathcal{X}$ such that $\|Ax - f\| < \varepsilon/(2\|A\|)$. We now compute

$$\|A^*Ax - u\| \leq \|A^*Ax - A^*f\| + \|A^*f - u\| < \|A^*\|\frac{\varepsilon}{2\|A\|} + \frac{\varepsilon}{2} = \varepsilon,$$

and conclude that $u \in \overline{\mathcal{R}(A^*A)}$. This shows that $\overline{\mathcal{R}(A)} \subseteq \overline{\mathcal{R}(A^*A)}$. ∎

1.1 Generalised inverses

We consider the equation

$$Au = f, \tag{1}$$

where $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and f are known, and we wish to find u.

Definition 1.3. An element $u \in \mathcal{X}$ is called a *least-squares solution* of eq. (1) if u is a minimiser of the function $v \mapsto \|Av - f\|_{\mathcal{Y}}$. It is called a *minimal-norm solution* of eq. (1) if it has minimal norm among all least-squares solutions.

Note that a least-squares solution may not exist. If a least-squares solution u exists, then the affine subspace of all least-squares solutions is given by $u + \mathcal{N}(A)$. By writing $u = u^\dagger + v$ for $u^\dagger \in \mathcal{N}(A)^\perp$, $v \in \mathcal{N}(A)$, we find that the space of least-squares solutions is given by $u^\dagger + \mathcal{N}(A)$, and it is now clear that $u^\dagger$ is the unique minimum-norm solution.

Theorem 1.4. *Let $f \in \mathcal{Y}$ and $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then the following are equivalent:*

1. *$u \in \mathcal{X}$ satisfies $Au = P_{\overline{\mathcal{R}(A)}} f$;*
2. *u is a least-squares solution of eq. (1);*
3. *u solves the normal equation*

$$A^* f = A^* Au. \tag{2}$$

Proof. "(1) $\implies$ (2)": We have

$$\|Au - f\|_{\mathcal{Y}} = \left\| P_{\overline{\mathcal{R}(A)}} f - f \right\| = \inf_{g \in \overline{\mathcal{R}(A)}} \|g - f\| \leq \inf_{g \in \mathcal{R}(A)} \|g - f\| = \inf_{u \in \mathcal{X}} \|Au - f\|.$$

"(2) $\implies$ (3)": Let $u \in \mathcal{X}$ be a least-squares solution and $v \in \mathcal{X}$ arbitrary. Define the quadratic polynomial

$$\begin{aligned} F\colon \mathbb{R} \to \mathbb{R}\colon \lambda \mapsto & \|A(u + \lambda v) - f\|^2 \\ &= \langle Au + \lambda Av - f, Au + \lambda Av - f \rangle \\ &= \lambda^2 \|Av\|^2 - 2\lambda \langle Av, f - Au \rangle + \|f - Au\|^2. \end{aligned}$$

As u is a least-squares solution, we know that F attains a minimum in $\lambda = 0$ and therefore that

$$0 = F'(0) = 2\langle Av, f - Au \rangle = 2\langle v, A^*(f - Au) \rangle.$$

Since v is arbitrary, we must have $A^*(f - Au) = 0$, so u satisfies *eq.* (2).

"(3) $\implies$ (1)": From the normal equation we know that $A^*(f - Au) = 0$. For any $x \in \mathcal{X}$, we have

$$\langle Ax, f - Au \rangle = \langle x, A^*(f - Au) \rangle = \langle x, 0 \rangle = 0,$$

so $f - Au \in \mathcal{R}(A)^\perp$.

So we have $Au \in \overline{\mathcal{R}(A)}$ and $f - Au \in \mathcal{R}(A)^\perp = \overline{\mathcal{R}(A)}^\perp$, from which it follows that $Au = P_{\overline{\mathcal{R}(A)}} f$. □

The following lemma gives a precise condition for when a least-squares solution exists:

Lemma 1.5. *Equation (1) has a least-squares solution if and only if $f \in \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$.*

Proof. " $\implies$ " Suppose u is a least-squares solution. Then $f - Au \in \mathcal{R}(A)^\perp$, so $f = Au + (f - Au) \in \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$.

" $\impliedby$ " Suppose $f = Au + g$ for some $u \in \mathcal{X}$, $g \in \mathcal{R}(A)^\perp = \overline{\mathcal{R}(A)}^\perp$. Then by the previous theorem, $Au = P_{\overline{\mathcal{R}(A)}} f$, so u is a least-squares solution. □

Corollary 1.6. *If $\mathcal{R}(A)$ is closed, then eq. (1) always has a least-squares solution.*

In particular, this holds if $\mathcal{R}(A)$ is finite-dimensional. Therefore, if either $\mathcal{X}$ or $\mathcal{Y}$ is finite-dimensional, eq. (1) has a least-squares solution for any A.

We have already seen that if a least-squares solution u exists, then the affine subspace of all least-sqares solutions is $u + \mathcal{N}(A)$, and the unique minimum-norm solution is the projection of 0 onto this affine subspace, which is the unique element of $u + \mathcal{N}(A)$ that lies in $\mathcal{N}(A)^\perp$.

Definition 1.7. Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and define

$$\tilde{A} \coloneqq A\restriction_{\mathcal{N}(A)^\perp} \colon \mathcal{N}(A)^\perp \to \mathcal{R}(A).$$

Clearly $\tilde{A}$ is bijective and we define the *Moore-Penrose inverse*

$$A^\dagger \colon \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \to \mathcal{N}(A)^\perp \colon f \mapsto \tilde{A}^{-1} P_{\overline{\mathcal{R}(A)}} f.$$

Remark. Note that $\overline{\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp} = \overline{\mathcal{R}(A)} \oplus \mathcal{R}(A)^\perp = \overline{\mathcal{R}(A)} \oplus \overline{\mathcal{R}(A)}^\perp = \mathcal{Y}$, and therefore the operator $\tilde{A}$ is *densely defined*, and it is defined on all of $\mathcal{Y}$ if and only if $\mathcal{R}(A)$ is closed.

We will not prove the following theorem, but it is interesting:

Theorem 1.8. *The Moore-Penrose inverse $A^\dagger$ is continuous if and only if $\mathcal{R}(A)$ is closed.*

The following characterises all important facts about the Moore-Penrose inverse:

Theorem 1.9 (Moore-Penrose equations)**.** *The operator $A^\dagger$ satisfies the following equations:*

(1) $A^\dagger A = P_{\mathcal{N}(A)^\perp}$*;*

(2) $AA^\dagger = P_{\overline{\mathcal{R}(A)}}\restriction_{\mathcal{D}(A^\dagger)}$*;*

(3) $AA^\dagger A = A$*;*

(4) $A^\dagger A A^\dagger = A^\dagger$*.*

Conversely, if any linear operator $B \colon \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \to \mathcal{N}(A)^\perp$ satisfies $BA = P_{\mathcal{N}(A)^\perp}$ and $AB = P_{\overline{\mathcal{R}(A)}}\restriction_{\mathcal{D}(A^\dagger)}$ then $B = A^\dagger$.

Proof. We have

$$A^\dagger A u = \tilde{A}^{-1} A P_{\mathcal{N}(A)^\perp} u = P_{\mathcal{N}(A)^\perp} u,$$

which proves (1). Furthermore, we have

$$AA^\dagger f = A\tilde{A}^{-1} P_{\overline{\mathcal{R}(A)}} f = P_{\overline{\mathcal{R}(A)}} f,$$

which proves (2). Finally, (3) follows from (1) and (4) follows from (2).

Now, suppose B satisfies (1) and (2). First we show that $B\restriction_{\mathcal{R}(A)} = \tilde{A}^{-1}$, then we show that $B\restriction_{\mathcal{R}(A)^\perp} = 0$. This shows that $B = A^\dagger$. Let $f = Au \in \mathcal{R}(A)$ with $u \in \mathcal{N}(A)^\perp$, then

$$Bf = BAu = P_{\mathcal{N}(A)^\perp} u = u = \tilde{A}^{-1} f, \quad \text{so } B\restriction_{\mathcal{R}(A)} = \tilde{A}^{-1}.$$

Finally, let $f \in \mathcal{R}(A)^\perp$, then $ABf = P_{\overline{\mathcal{R}(A)}} f = 0$, and since $Bf \in \mathcal{N}(A)^\perp$ this implies $Bf = 0$. We conclude that $B\restriction_{\mathcal{R}(A)^\perp} = 0$, and this concludes the proof. □

The Moore-Penrose inverse has the important property that it maps every f in its domain to the corresponding minimum-norm least-squares solution:

Theorem 1.10. *For every $f \in \mathcal{D}(A^\dagger)$, the minimum-norm solution $u^\dagger$ to eq. (1) is given by $u^\dagger = A^\dagger f$.*

Proof. Since $f \in \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$, we know that there exists a unique minimum-norm solution $u^\dagger \in \mathcal{N}(A)^\dagger$. We write

$$u^\dagger = P_{\mathcal{N}(A)^\perp}(u^\dagger) = A^\dagger A u^\dagger = A^\dagger P_{\overline{\mathcal{R}(A)}} f = A^\dagger A A^\dagger f = A^\dagger f.$$

□

Remark. We can also consider the normal equation $A^* f = A^* A u$ as a least-squares problem, whose minimum-norm solution is $(A^* A)^\dagger A^* f$. It is clear that this expression must equal the minimum-norm solution $u^\dagger$ from eq. (1).

1.2 Compact operators

Definition 1.11. Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then A is called *compact* if for any bounded $B \subseteq \mathcal{X}$, the image $A(B)$ is precompact in $\mathcal{Y}$. The set of compact operators in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is denoted $\mathcal{K}(\mathcal{X}, \mathcal{Y})$.

Lemma 1.12. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then A is compact if and only if, for every bounded sequence $(x_n) \subseteq X$, the sequence $(Ax_n) \subseteq Y$ has a convergent subsequence.*

Theorem 1.13. *Let $A \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ with $\dim(\mathcal{R}(A)) = \infty$. Then $A^\dagger$ is discontinuous.*

Proof. If $\dim \mathcal{R}(A) = \infty$, then $\mathcal{X}$ and $\mathcal{N}(A)^\perp$ are infinite-dimensional as well. Chose an orthonormal sequence $(x_n) \subseteq \mathcal{N}(A)^\perp$, then after taking a subsequence if necessary, we can assume that $f_n := Ax_n$ converges. However, we have

$$\left\|A^\dagger(f_n - f_m)\right\|^2 = \left\|A^\dagger A(x_n - x_m)\right\|^2 = \left\|P_{\mathcal{N}(A)^\perp}(x_n - x_m)\right\|^2 = \|x_n - x_m\|^2 = 2,$$

and in particular the sequence $(A^\dagger f_n)$ does not converge. This shows that $A^\dagger$ is discontinuous. □

In particular, combining this with theorem 1.8 shows that the range of a compact operator is always open, and that not every element in $\mathcal{Y}$ has a least-squares solution.

We will need the following theorem, an infinite-dimensional analogue of the spectral theorem:

Theorem 1.14 (Eigenvalue decomposition of self-adjoint compact operators)**.** *Let $\mathcal{X}$ be a Hilbert space, and $A \in \mathcal{K}(\mathcal{X}, \mathcal{X})$ self-adjoint. Then there exists an orthonormal basis (x_j) of $\overline{\mathcal{R}(A)}$ and a sequence of eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots > 0$ such that for all $u \in \mathcal{X}$ we have*

$$Au = \sum_{j=1}^{\infty} \lambda_j \langle u, x_j \rangle x_j.$$

The sequence (λ_j) is either finite or converges to 0.

The previous theorem gives rise to an infinite-dimensional analogue of the SVD:

Theorem 1.15. *Let $A \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$. Then there exists a (not necessarily infinite) sequence $\sigma_1 \geq \sigma_2 \geq \cdots > 0$ converging to 0, and orthonormal bases (x_j), (y_j) of $\mathcal{N}(A)^\perp$ and $\overline{\mathcal{R}(A)}$ respectively, such that*

$$Ax_j = \sigma_j y_j, \quad A^* y_j = \sigma_j x_j \quad \textit{for all } j \in \mathbb{N},$$

and such that for all $u \in \mathcal{X}$ and $f \in \mathcal{Y}$ we have

$$Au = \sum_{j=1}^{\infty} \sigma_j \langle u, x_j \rangle y_j, \quad A^* f = \sum_{j=1}^{\infty} \sigma_j \langle f, y_j \rangle x_j.$$

The sequence $\{(\sigma_j, x_j, y_j)\}$ is called the singular value decomposition (SVD) of A.

Proof. Define $B \coloneqq A^*A$ and $C \coloneqq AA^*$, which are both compact, self-adjoint, and positive semi-definite operators. By the previous theorem, we can write

$$Cf = \sum_{j=1}^{\infty} \sigma_j^2 \langle f, y_j \rangle y_j,$$

where (y_j) is a basis of $\overline{\mathcal{R}(AA^*)} = \overline{\mathcal{R}(A)}$ and (σ_j) is a positive decreasing sequence converging to 0.

Note that

$$BA^*y_j = A^*AAy_j = A^*Cy_j = A^*\sigma^2 y_j = \sigma_j^2 A^* y_j,$$

so A^*y_j is an eigenvector of B with eigenvector σ_j^2.

We show that $\left(\frac{A^*y_j}{\sigma_j}\right)$ is an orthonormal basis of $\mathcal{R}(A)^\perp$. is an orthonormal basis of $\mathcal{N}(A)^\perp$: their inner product is given by

$$\left\langle \frac{A^*y_j}{\sigma_j}, \frac{A^*y_k}{\sigma_k} \right\rangle = \frac{1}{\sigma_j \sigma_k} \langle y_j, Cy_k \rangle = \frac{\sigma_k}{\sigma_j} \langle y_j, y_k \rangle = 0,$$

and since the (y_j) are a basis of $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^\perp$ it is clear that the span of (A^*y_j) is dense in $\overline{\mathcal{R}(A^*)} = \mathcal{N}(A)^\perp$.

If we choose $x_j = \frac{A^*y_j}{\sigma_j}$, we find by construction that $A^*y_j = \sigma_j x_j$ and

$$Ax_j = \frac{AA^*y_j}{\sigma_j} = \frac{Cy_j}{\sigma_j} = \sigma_j y_j.$$

Finally, we see that

$$Au = \sum_{j=1}^{\infty} \langle u, x_j \rangle Ax_j = \sum_{j=1}^{\infty} \sigma_j \langle u, x_j \rangle y_j \quad \text{and} \quad A^*f = \sum_{j=1}^{\infty} \langle f, y_j \rangle A^* y_j = \sum_{j=1}^{\infty} \sigma_j \langle f, y_j \rangle x_j.$$

□

Theorem 1.16. *Let $A \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ with SVD $\{(\sigma_j, x_j, y_j)\}$ and let $f \in \mathcal{D}(A^\dagger)$. Then*

$$A^\dagger f = \sum_{j=1}^{\infty} \frac{1}{\sigma_j} \langle f, y_j \rangle x_j.$$

Remark. Note that this is comparable to $A^*f = \sum_{j=1}^{\infty} \sigma_j \langle f, y_j \rangle x_j$, except that A^* is a smoothing operator (since $\sigma_j \to 0$), while $A^\dagger$ does the opposite. Furthermore, $A^\dagger$ amplifies the right singular vectors corresponding to small singular values the most — intuitively, the corresponding left singular vectors are vectors where A doesn't "see much".

Proof. Define $Bf = \sum_{j=1}^{\infty} \frac{1}{\sigma_j} \langle f, y_j \rangle x_j$. Then by theorem 1.9, we must check that $BA = P_{\mathcal{N}(A)^\perp}$ and $AB = P_{\overline{\mathcal{R}(A)}} \restriction_{\mathcal{D}(A^\perp)}$.

For the first equation, we compute

$$BAu = \sum_{j=1}^{\infty} \frac{1}{\sigma_j} \left\langle \sum_{i=1}^{\infty} \sigma_i \langle u, x_i \rangle y_i, y_j \right\rangle x_j = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{\sigma_i}{\sigma_j} \langle u, x_i \rangle \langle y_i, y_j \rangle x_j = \sum_{j=1}^{\infty} \langle u, x_j \rangle x_j.$$

Since (x_j) is a basis of $\mathcal{N}(A)^\perp$, this proves that $BA = P_{\mathcal{N}(A)^\perp}$.

For the second equation, an analogous computation gives $ABf = \sum_{i=1}^{\infty} \langle f, y_i \rangle y_i$, and since (y_i) is a basis of $\overline{\mathcal{R}(A)}$, this proves that $AB = P_{\overline{\mathcal{R}(A)}} \restriction_{\mathcal{D}(A^\dagger)}$. □

Definition 1.17. Let $A \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ have SVD $\{(\sigma_j, x_j, y_j)\}$. We say that $f \in \mathcal{Y}$ satisfies the *Picard criterion* if

$$\sum_j \frac{|\langle f, y_j \rangle|^2}{\sigma_j^2} < \infty.$$

Note that the expression on the left corresponds to $\|A^\dagger f\|^2$ if $f \in \mathcal{D}(A^\dagger)$.

Theorem 1.18. *Let $f \in \overline{\mathcal{R}(A)}$. Then $f \in \mathcal{R}(A)$ if and only if f satisfies the Picard criterion.*

Proof. '$\implies$' Write $f = Au$, then

$$\sum_j \frac{|\langle f, y_j \rangle|^2}{\sigma_j^2} = \sum_j \frac{|\langle Au, y_j \rangle|^2}{\sigma_j^2} = \sum_j \frac{|\langle u, A^* y_j \rangle|^2}{\sigma_j^2} = \sum_j |\langle u, x_j \rangle|^2 < \infty.$$

'$\impliedby$' Define $u \coloneqq \sum_{j=1}^\infty \frac{1}{\sigma_j} \langle f, y_j \rangle x_j$ (note that by assumption this sum converges). Then

$$Au = A \sum_{j=1}^\infty \frac{1}{\sigma_j} \langle f, y_j \rangle x_j = \sum_{j=1}^\infty \frac{1}{\sigma_j} \langle f, y_j \rangle A x_j = \sum_{j=1}^\infty \langle f, y_j \rangle y_j = P_{\overline{\mathcal{R}(A)}} f = f,$$

so $Au = f$ which implies $f \in \mathcal{R}(A)$. □

We have seen that the stability of $A^\dagger$ depends on the speed of decay of the singular values (σ_j). We formalise this:

Definition 1.19. Let $A \in \mathcal{K}(\mathcal{X}, \mathcal{Y})$ have singular values (σ_j). Then the ill-posed inverse problem $Au = f$ is called *mildly ill-posed* if the σ_j decay polynomially (i.e., $\frac{1}{\sigma_n} \leq Cn^\gamma$ for some C, γ) and *severely ill-posed* otherwise.

Example 1.20. Consider the heat equation with initial conditions and boundary values:

$$\begin{cases} v_t - v_{xx} = 0 & (x,t) \in (0,\pi) \times \mathbb{R}_{>0}, \\ v(0,t) = v(\pi,t) = 0 & t \geq 0, \\ v(x,0) = u(x) & x \in (0,\pi), \\ v(x,T) = f(x) & x \in (0,\pi). \end{cases}$$

Then the forward problem is to determine f given u, while the inverse problem is to determine u given f. The solution for the foward problem is given by

$$f = Au \coloneqq \sum_{j=1}^\infty e^{-j^2 T} \langle u, \sin(jx) \rangle \sin(jx),$$

and the eigenvalues are therefore $\sigma_j = e^{-j^2 T}$. Since these clearly decay exponentially, this problem is severely ill-posed.

2 Classical regularisation theory

Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that $\mathcal{R}(A)$ is not closed (this happens for example when A is compact and does not have finite rank), and consider the inverse problem $Au = f$. Suppose we measure not f, but noisy data f_δ such that $\|f_\delta - f\| \leq \delta$. Then since $A^\dagger$ is discontinuous, we cannot expect that $A^\dagger f_\delta \to A^\dagger f$ as $\delta \to 0$. Therefore, we must replace $A^\dagger$ by operators that approximate it.

Definition 2.1. Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. A family $(R_\alpha)_{\alpha>0}$ of continuous operators is called a *regularisation* of $A^\dagger$ if

$$\lim_{\alpha \to 0} R_\alpha f = A^\dagger f \quad \text{for all } f \in \mathcal{D}(A^\dagger).$$

If all R_α are linear (TODO: and bounded?), then we speak of a *linear regularisation* of $A^\dagger$.

Theorem 2.2 (Banach-Steinhaus)**.** *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces and $\{A_\alpha\} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{Y})$ a family of pointwise bounded operators. Then $\{A_\alpha\}$ is bounded in norm.*

Corollary 2.3. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces and $(A_j) \subseteq \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then (A_j) converges pointwise to some $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ if and only if $\{A_j\}$ is norm-bounded and converges pointwise on some dense subset $\mathcal{X}' \subseteq \mathcal{X}$.*

Theorem 2.4. *Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces, $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $(R_\alpha)_{\alpha>0}$ a linear regularisation. If $A^\dagger$ is not continuous, (R_α) is not norm-bounded. In particular, there exists $f \in \mathcal{Y}$ with $\|R_\alpha f\| \to \infty$.*

Proof. Suppose (R_α) is norm-bounded. Let $\alpha_j \to 0$, then we know that $R_{\alpha_j} \to A^\dagger$ pointwise on $\mathcal{D}(A^\dagger)$. Since $\mathcal{D}(A^\dagger)$ is dense in $\mathcal{Y}$, corollary 2.3 then tells us that $A^\dagger$ is bounded and therefore continuous, a contradiction.

By the Banach-Steinhaus theorem, if (R_α) is not norm-bounded, it is not pointwise bounded, so there must exist $f \in \mathcal{Y}$ such that $\{\|R_\alpha f\|\}$ is not bounded. □

Recap 2.5. Recall that any bounded sequence in a Hilbert space has a weakly convergent subsequence.

Theorem 2.6. *Let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and (R_α) a linear regularisation of $A^\dagger$. If $\{\|AR_\alpha\|\}_{\alpha>0}$ is bounded, then $\|R_\alpha f\| \to \infty$ as $\alpha \to 0$ for every $f \notin \mathcal{D}(A^\dagger)$.*

Proof. Define $u_\alpha := R_\alpha f$ for $f \notin \mathcal{D}(A^\dagger)$, and assume there exists a sequence $\alpha_k \to 0$ such that $\{\|u_{\alpha_k}\|\}$ is bounded. After taking a subsequence if necessary, we may assume that $u_{\alpha_k} \rightharpoonup u$ for some $u \in \mathcal{X}$, and therefore we also have $Au_{\alpha_k} \rightharpoonup Au$.

We also have $\lim_{\alpha\to 0} AR_\alpha f = AA^\dagger f = P_{\overline{\mathcal{R}(A)}} f$ for $f \in \mathcal{D}(A^\dagger)$, and since we assumed $\{AR_\alpha\}$ was norm-bounded, by corollary 2.3 we have $\lim_{\alpha\to 0} AR_\alpha f = P_{\overline{\mathcal{R}(A)}} f$ for all $f \in \mathcal{Y}$.

Since Au_{α_k} is convergent and has weak limit Au, it must also have limit Au, so we find $Au = P_{\overline{\mathcal{R}(A)}} f$ so $f \in \mathcal{D}(A^\dagger)$, a contradiction. □

We need some process to choose a parameter. To this end, note that we have

$$\left\|R_\alpha f_\delta - A^\dagger f\right\| \leq \|R_\alpha(f_\delta - f)\| + \left\|(R_\alpha - A^\dagger)f\right\| \leq \delta\|R_\alpha\| + \left\|(R_\alpha - A^\dagger)f\right\|. \tag{3}$$

The first term is called the *data error* and is unbounded for $\alpha \to 0$, and the second term is called the *approximation error* which does vanish for $\alpha \to 0$. Therefore, we want to choose α small enough to have a low approximation error, while keeping the data error at bay.

2.1 Parameter choice rules

Definition 2.7. A function $\alpha\colon \mathbb{R}_{>0} \times \mathcal{Y} \to \mathbb{R}_{>0}\colon (\delta, f_\delta) \mapsto \alpha(\delta, f_\delta)$ is called a *parameter choice rule* (PCR). We distinguish three types:

1. An *a priori* PCR depends only on δ;
2. An *a posteriori* PCR depends on both δ and f_δ;
3. A *heuristic* PCR depends only on f_δ.

Definition 2.8. Let $(R_\alpha)_{\alpha>0}$ be a regularisation of $A^\dagger$ and α a parameter choice rule. We call (R_α, α) a *convergent regularisation* if

$$\lim_{\delta\to 0} \sup_{f_\delta : \|f-f_\delta\|\le\delta} \left\| R_\alpha f_\delta - A^\dagger f \right\| = 0$$

and

$$\lim_{\delta\to 0} \sup_{f_\delta : \|f-f_\delta\|\le\delta} \alpha(\delta, f_\delta) = 0. \tag{4}$$

2.1.1 A priori parameter choice rules

We will not prove the following theorem, which guarantees the existence of a priori PCRs:

Theorem 2.9. *Let $(R_\alpha)_{\alpha>0}$ be a regularisation of $A^\dagger$. Then there exists an a priori PCR $\alpha = \alpha(\delta)$ such that (R_α, α) is convergent.*

We can characterise PCRs in the following way:

Theorem 2.10. *Let $(R_\alpha)_{\alpha>0}$ be a linear regularisation of $A^\dagger$, and $\alpha = \alpha(\delta)$ an a priori PCR. Then (R_α, α) is convergent if and only if*

$$\lim_{\delta\to 0} \delta \left\| R_{\alpha(\delta)} \right\| = 0 \quad \textit{and} \quad \lim_{\delta\to 0} \alpha(\delta) = 0.$$

Proof. “ $\implies$ ” Suppose (R_α, α) is convergent. It is clear that $\lim_{\delta\to 0} \alpha(\delta) = 0$ by eq. (4). Suppose $\lim_{\delta\to 0} \delta \left\| R_{\alpha(\delta)} \right\| \neq 0$. Then there exists a sequence $(\delta_k) \to 0$ and a constant $C > 0$ such that $\delta_k \left\| R_{\alpha(\delta_k)} \right\| \ge C$ for all k. This implies we can find a sequence $(g_k) \subseteq \mathcal{Y}$ with $\|g_k\| = 1$ and $\delta_k \left\| R_{\alpha(\delta_k)} g_k \right\| \ge C$ for all k.

Now let $f \in \mathcal{D}(A^\dagger)$ and define $f_k \coloneqq f + \delta_k g_k$, then clearly we have $f_k \to f$, but also

$$C \le \left\| R_{\alpha(\delta_k)}(\delta_k g_k) \right\| = \left\| R_{\alpha(\delta_k)}(f_{\delta_k} - f) \right\| \le \left\| R_{\alpha(\delta_k)} f_{\delta_k} - A^\dagger f \right\| + \left\| (R_{\alpha(\delta_k)} - A^\dagger) f \right\|.$$

In particular we find that $\left\| (R_{\alpha(\delta_k)} - A^\dagger) f \right\| \ge C$, so clearly R_α is not convergent.

“ $\impliedby$ ” This follows immediately from eq. (3). □

A problem with a priori PCRs is that they are scale-invariant: if $\alpha = \alpha(\delta)$ gives a convergent regularisation, then $\hat\alpha = \alpha(k\delta)$ also gives a convergent regularisation for any k. In practice, it is not always clear which scale should be chosen.

2.1.2 A posteriori parameter choice rules

Let $f \in \mathcal{D}(A^\dagger)$ and f_δ s.t. $\|f - f_\delta\| \le \delta$. Letting $u^\dagger$ denote the minimum-norm solution of the problem $Au = f$, and defining $\mu \coloneqq \left\| Au^\dagger - f \right\| = \inf_{u\in\mathcal{X}} \|Au - f\|$, we see that

$$\left\| Au^\dagger - f_\delta \right\| \le \left\| Au^\dagger - f \right\| + \|f - f_\delta\| \le \mu + \delta.$$

Therefore, it is not useful to choose $\alpha(\delta, f_\delta)$ with $\|Au_\alpha - f_\delta\| < \mu + \delta$: if this is the case, we are most likely overfitting.

This motivates *Morozov's discrepancy principle*:

Definition 2.11. Let (R_α) be a (TODO: linear?) regularisation of $A^\dagger$ and assume $\mathcal{R}(A)$ is dense in $\mathcal{Y}$. Fix $\eta > 1$, and define

$$\alpha(\delta, f_\delta) = \sup\{\alpha > 0 : \|AR_\alpha f_\delta - f_\delta\| \leq \eta\delta\}.$$

Then $\alpha(\delta, f_\delta)$ is said to satisfy *Morozov's discrepancy principle.*

It can be shown that the above α indeed gives a convergent regularisation.

2.1.3 Heuristic parameter choice rules

Heuristic parameter choice rules unfortunately only work if the original problem was well-posed:

Theorem 2.12 (Bakushinskii)**.** *Let (R_α) be a regularisation of $A^\dagger$ and suppose there exists a heuristic parameter choice rule α such that (R_α, α) is convergent. Then $A^\dagger$ is continuous from $\mathcal{Y}$ to $\mathcal{X}$.*

2.2 Spectral regularisation

We will now start with specific examples of regularisations. Spectral regularisations are derived from the spectral decomposition

$$A^\dagger f = \sum_{j=1}^{\infty} \sigma_j^{-1} \langle f, y_j\rangle x_j.$$

We construct a regularisation by replacing σ_j^{-1} by some function $g_\alpha(\sigma_j)$, i.e.,

$$R_\alpha f = \sum_{j=1}^{\infty} g_\alpha(\sigma_j) \langle f, y_j\rangle x_j. \tag{5}$$

Let us explore which conditions g_α must satisfy:

Theorem 2.13. *Let, for $\alpha > 0$, the function $g_\alpha \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ satisfy*

1. $\lim_{\alpha\to 0} g_\alpha(\sigma) = \frac{1}{\sigma}$ *for all* $\sigma > 0$*;*
2. $g_\alpha(\sigma) \leq C_\alpha$ *for some* $C_\alpha > 0$*;*
3. $\sup_{\alpha,\sigma} \sigma g_\alpha(\sigma) \leq \gamma$ *for some* $\gamma > 0$*.*

Then collection (R_α) defined by eq. (5) is a linear regularisation of $A^\dagger$, and in particular, we have $\|R_\alpha\| \leq C_\alpha$.

Proof. From condition 2 it follows that all R_α are bounded. Since

$$\langle f, y_j\rangle = \langle P_{\overline{\mathcal{R}(A)}} f, y_j\rangle = \langle AA^\dagger f, y_j\rangle = \langle A^\dagger f, A^* y_j\rangle = \sigma_j \langle u^\dagger, x_j\rangle,$$

we compute

$$(R_\alpha - A^\dagger)f = \sum_j (g_\alpha(\sigma_j) - \sigma_j)^{-1} \langle f, y_j\rangle x_j = \sum_j (\sigma_j g_\alpha(\sigma_j) - 1)\langle u^\dagger, x_j\rangle x_j,$$

and since $\sigma g_\alpha(\sigma_j) \leq \gamma$, we have $(\sigma_j g_\alpha(\sigma_j) - 1)^2 \leq 1 + \gamma^2$, so that

$$\left\|(R_\alpha - A^\dagger)f\right\|^2 \leq (1+\gamma^2)\left\|u^\dagger\right\|^2 < \infty.$$

Since $\left\|(R_\alpha - A^\dagger)f\right\|$ is finite, we may apply the reverse Fatou lemma to the sum and obtain

$$\limsup_{\alpha\to 0} \left\|(R_\alpha - A^\dagger)f\right\|^2 \leq \sum_j \left(\sigma_j \limsup_{\alpha\to 0} g_\alpha(\sigma_j) - 1\right)^2 \langle u^\dagger, x_j\rangle^2 = 0,$$

and therefore $R_\alpha f \to A^\dagger f$ as $\alpha \to 0$. $\square$

Example 2.14. The first, very simple example is the *truncated SVD*: we simply define

$$g_\alpha(\sigma) = \begin{cases} 1/\sigma & \sigma \geq \alpha, \\ 0 & \sigma < \alpha. \end{cases}$$

It is easy to check that g_α satisfies the conditions of theorem 2.13, and that all R_α are finite-rank operators with $\|R_\alpha\| \leq \frac{1}{\alpha}$. Therefore, if we choose $\alpha = \alpha(\delta)$ such that $\delta/\alpha(\delta) \to 0$, then we obtain a convergent regularisation.

This also highlights the problem with this method: as δ gets smaller, we need more and more singular vectors which are generally expensive to compute.

Example 2.15. The second example is *Tikhonov regularisation*. Here, we define $g_\alpha(\sigma) = \frac{\sigma}{\sigma^2+\alpha}$, and again it is easily checked that the conditions of theorem 2.13 are satisfied, noting that

$$\frac{\sigma}{\sigma^2 + \alpha} \leq \frac{\sigma}{2\sigma\sqrt{\alpha}} = \frac{1}{2\sqrt{\alpha}} =: C_\alpha.$$

Therefore, if $\delta/\sqrt{\alpha(\delta)} \to 0$, the regularisation is convergent.

This method does not require computing the SVD of A: it is easily shown that $u_\alpha \coloneqq R_\alpha f$ is the unique solution to the *regularised normal equation*

$$(A^*A + \alpha I)u_\alpha = A^*f.$$

While $A^*A + \alpha I$ is always invertible, computing the inverse is expensive, so we usually use some approximation of the inverse.

Finally, it can also be shown that

$$u_\alpha = \min_{u \in \mathcal{X}} \|Au - f\|^2 + \alpha\|u\|^2,$$

so we can also view u_α as the solution of an optimisation problem.

3 Variational regularisation

3.1 Background

3.1.1 Banach spaces and weak convergence

A *Banach space* $\mathcal{X}$ is a complete normed vector space. We define the *dual space* $\mathcal{X}^* := \mathcal{L}(X, \mathbb{R})$, and for $p \in \mathcal{X}^*, u \in \mathcal{X}$ we usually write $\langle p, u \rangle$ instead of $p(u)$. For any $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ we define the *adjoint* $A^* \colon \mathcal{Y}^* \to \mathcal{X}^*$ by $\langle A^* p, u \rangle := \langle p, Au \rangle$ for all $p \in \mathcal{X}^*, u \in \mathcal{X}$. The dual space $\mathcal{X}'$ is equipped with the norm

$$\|p\|_{\mathcal{X}^*} := \sup_{\|u\| \leq 1} \langle p, u \rangle,$$

and with this norm $\mathcal{X}^*$ is a Banach space.

The *bi-dual space* is defined as $\mathcal{X}^{**} := (\mathcal{X}^*)^*$. The mapping $E \colon \mathcal{X} \to (\mathcal{X})^{**}$ defined by $\langle E(u), p \rangle := \langle p, u \rangle$ is a continuous linear isometry, and we will regard $\mathcal{X}$ as a subspace of $\mathcal{X}^{**}$ using this isometry. If $\mathcal{X} = \mathcal{X}^{**}$ (i.e., E is surjective), the space $\mathcal{X}$ is called *reflexive*. A space $\mathcal{X}$ is called *separable* if $\mathcal{X}$ has a countable dense subset.

A sequence $(u_k) \subseteq \mathcal{X}$ is said to *converge weakly* to $u \in \mathcal{X}$, denoted $u_k \rightharpoonup u$, if $\langle p, u_k \rangle \to \langle p, u \rangle$ for all $p \in \mathcal{X}^*$.

A sequence $(p_k) \subseteq \mathcal{X}^*$ is said to *converge weakly-** to $p \in \mathcal{X}'$, denoted $p_k \overset{*}{\rightharpoonup} p$, if $\langle p_k, u \rangle \to \langle p, u \rangle$ for all $u \in \mathcal{X}$.

Theorem 3.1. *Let $\mathcal{X}$ be Banach, then the unit ball is compact in $\mathcal{X}^*$ w.r.t. the weak-$*$ topology. If $\mathcal{X}$ is separable, then the weak-$*$ topology is metrisable and every bounded sequence in $\mathcal{X}^*$ has a weakly-$*$ convergent subsequence.*

Theorem 3.2. *Let $\mathcal{X}$ be reflexive, then every bounded sequence in $\mathcal{X}$ has a weakly convergent subsequence.*

We define $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Definition 3.3. Let $\mathcal{X}$ be a Banach space with topology τ_X. A functional $E \colon \mathcal{X} \to \overline{\mathbb{R}}$ is said to be *sequentially lower-semicontinuous with respect to $\tau_{\mathcal{X}}$* or simply $\tau_{\mathcal{X}}$-LSC if

$$E(u) \leq \liminf_{n \to \infty} E(u_n) \quad \text{if } u_n \xrightarrow{\tau} u.$$

Specifically, if $\tau_{\mathcal{X}}$ is the weak topology, then E is called *weakly* LSC. If $\tau_{\mathcal{X}}$ is the topology induced by the norm on $\mathcal{X}$, then E is called *strongly* LSC or simply LSC.

Lemma 3.4. *Let $E \colon \mathcal{X} \to \overline{\mathbb{R}}$ be $\tau_{\mathcal{X}}$-LSC and $a \in \mathbb{R}$. Then $\{u \in \mathcal{X} \mid E(u) \leq a\}$ is $\tau_{\mathcal{X}}$-closed.*

Proof. Let $(u_n) \xrightarrow{\tau} u$ and suppose $E(u_n) \leq a$ for all n. Then clearly $E(u) \leq \liminf E(u_n) \leq a$. □

3.1.2 Convex analysis

Definition 3.5. Let $C \subseteq \mathcal{X}$. Then the *characteristic function* of C is defined as

$$\chi_C(u) := \begin{cases} 0, & u \in C, \\ \infty, & u \notin C. \end{cases}$$

Using characteristic functions, we have $\min_{u \in C} E(u) = \min_{u \in \mathcal{X}} E(u) + \chi_C(u)$.

Definition 3.6. Let $E \colon \mathcal{X} \to \overline{\mathbb{R}}$, then the *effective domain* is $\operatorname{dom}(E) := \{u \mid E(u) < \infty\}$.
The functional E is called *proper* if $\operatorname{dom}(E) \neq \varnothing$.

Definition 3.7. A functional $E \colon \mathcal{X} \to \overline{\mathbb{R}}$ is called:

1. *convex* if for all $u \neq v \in \mathcal{X}$ and $\lambda \in (0,1)$ we have $E(\lambda u + (1-\lambda)v) \leq \lambda E(u) + (1-\lambda)E(v)$;
2. *strictly convex* if the above inequality is strict;
3. *strongly convex* with constant $\vartheta > 0$ if $u \mapsto E(u) - \vartheta\|u\|^2$ is convex.

Note that $C \subseteq \mathcal{X}$ is a convex set if and only if χ_C is a convex function.

Lemma 3.8. *Nonnegative linear combinations of convex functionals are convex. If one of the components is strictly convex, then the nonnegative linear combination is also strictly convex.*

Definition 3.9. Let $E\colon \mathcal{X} \to \overline{\mathbb{R}}$ be a functional. We define the *Fenchel conjugate*

$$E^*\colon \mathcal{X}^* \to \overline{\mathbb{R}}\colon p \mapsto \sup_{u \in \mathcal{X}} [\langle p, u\rangle - E(u)].$$

Theorem 3.10. *For any $E\colon \mathcal{X} \to \mathbb{R}$ we have $E^{**}\restriction_{\mathcal{X}} \leq E$. If E is proper and LSC, then $E^{**}\restriction_{\mathcal{X}} = E$.*

Definition 3.11. A functional $E\colon \mathcal{X} \to \overline{\mathbb{R}}$ is called *subdifferentiable* at $u \in \mathcal{X}$ if there exists a $p \in \mathcal{X}^*$ such that

$$E(v) \geq E(u) + \langle p, v-u\rangle \quad \text{for all } v \in \mathcal{X}.$$

In this case, we call p a *subgradient* of E at position u. The collection of all subgradients of E at u is denoted by $\partial E(u)$ and is called the *subdifferential* of E at u.

Lemma 3.12. *Let $E\colon \mathcal{X} \to \overline{\mathbb{R}}$ be convex, then E is subdifferentiable at all points $u \in \operatorname{dom}(E)$. If E is also proper, then E is not subdifferentiable at any $u \notin \operatorname{dom}(E)$.*

Theorem 3.13. *Let $E\colon \mathcal{X} \to \overline{\mathbb{R}}$ be proper and convex and $u \in \operatorname{dom}(E)$. Then $\partial E(u)$ is convex and weakly-$*$ compact in $\mathcal{X}^*$.*

Theorem 3.14. *Let E, F be proper LSC convex functionals and $u \in \operatorname{dom}(E) \cap \operatorname{dom}(F)$ such that at least one of E and F is continuous at u. Then $\partial(E+F)(u) = \partial E(u) + \partial F(u)$.*

Theorem 3.15. *Let E be convex. Then u is a global minimiser of E if and only if $0 \in \partial E(u)$.*

Definition 3.16. Let E be convex, $u, v \in \mathcal{X}$, $E(v) < \infty$ and $q \in \partial E(v)$. Then the *Bregman distance* of E between u and v is defined as

$$D_E^q(u,v) \coloneqq E(u) - E(v) - \langle q, u-v\rangle \geq 0.$$

If we also have $E(u) < \infty, p \in \partial E(u)$, then we define the *symmetric Bregman distance*

$$D_E^{p,q}(u,v) \coloneqq D_E^p(v,u) + D_E^q(u,v) = \langle p-q, u-v\rangle.$$

Definition 3.17. For $p > 0$, a functional E is called *absolutely p-homogeneous* if $E(\lambda u) = |\lambda|^p E(u)$ for all $\lambda \in \mathbb{R}, u \in \mathcal{X}$.

Proposition 3.18. *Let E be a convex, proper and absolutely one-homogeneous, and $p \in \partial E(u)$. Then:*

1. *$E(u) = \langle p, u\rangle$;*
2. *$D^p(v,u) = E(v) - \langle p, v\rangle$ for all $v \in \mathcal{X}$;*
3. *$E^*(p) = \chi_{\partial E(0)}(p)$.*

Furthermore, we have the following:

Proposition 3.19. *Let E be proper, convex, and absolutely one-homogeneous, and let $u \in \mathcal{X}$. Then $p \in \partial E(u)$ if and only if $p \in \partial E(0)$ and $\langle p, u\rangle = E(u)$.*

3.1.3 Minimisers

Definition 3.20. We say that $u^* \in \mathcal{X}$ is a *minimiser* of a functional E if u minimises E and $E(u) < \infty$.

Definition 3.21. A functional E is called *coercive* if $\|u_j\| \to \infty \implies |E(u_j)| \to \infty$.

Lemma 3.22. *Let E be proper, coercive and bounded from below. Then $\inf_{u \in \mathcal{X}} E(u) > -\infty$ and there exists a (bounded) minimising sequence (u_j) with $E(u_j) \to \inf_u E(u)$.*

Theorem 3.23 (Direct method)**.** *Let $\mathcal{X}$ be Banach and $\tau_{\mathcal{X}}$ a topology on $\mathcal{X}$ such that any bounded sequence in $\mathcal{X}$ has a $\tau_{\mathcal{X}}$ convergent subsequence. Then any proper, bounded from below, coercive, τ_X-LSC functional has a minimiser.*

Proof. Since E is bounded from below, we have $\inf_u E(u) > -\infty$, so there exists a bounded minimising sequence (u_j), which we can assume is $\tau_{\mathcal{X}}$ convergent with limit u^* after taking a subsequence if necessary. By lower-semicontinuity of E we have

$$E(u^*) \leq \liminf_{k \to \infty} E(u_j) = \lim_{j \to \infty} E(u_j) = \inf_u E(u),$$

so u^* is a minimiser. □

Theorem 3.24. *If a strictly convex functional has a minimiser, it is unique.*

Proof. Suppose $u \neq v$ are two minimisers, then by strict convexity, we have $E(\frac{1}{2}u + \frac{1}{2}v) < E(u)$, a contradiction. □

3.1.4 Duality in convex optimisation

Consider the *primal* optimisation problem

$$(P) \coloneqq \inf_{u \in \mathcal{X}} E(Au) + F(u),$$

where E, F are proper, convex and LSC, and $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Since E is convex and LSC, we have $E = E^{**}$ so we can rewrite the primal problem as the *saddle point problem*

$$\inf_{u \in \mathcal{X}} \sup_{\eta \in \mathcal{Y}^*} \langle \eta, Au \rangle - E^*(\eta) + F(u).$$

Since $\inf \sup \geq \sup \inf$ always holds we have

$$(P) \geq \sup_{\eta \in \mathcal{Y}^*} \inf_{u \in \mathcal{X}} \langle \eta, y \rangle - E^*(\eta) + F(u) = \sup_{\eta \in \mathcal{Y}^*} -E^*(\eta) - F^*(-A^*\eta) =: (D).$$

The problem (D) is called the *dual problem*, and the fact that $(D) \leq (P)$ is called *weak duality.* The value $(P) - (D)$ is called the *duality gap*, and if $(P) = (D)$, we speak of *strong duality.*

We have the following:

Theorem 3.25. *Suppose the function $E(Au) + F(u)$ is proper, convex, LSC and coercive. Suppose also that there exists $u_0 \in \mathcal{X}$ s.t. $F(u) < \infty$, $E(Au_0) < \infty$, and $E(y)$ is continuous at $y = Au_0$. Then:*

1. *The dual problem (D) has at least one solution $\hat{\eta}$;*
2. *There is no duality gap;*
3. *If (P) has an optimal solution $\hat{u}$, then we have*

$$A^*\hat{\eta} \in \partial F(\hat{u}), \quad -\hat{\eta} \in \partial E(A\hat{u}).$$

3.2 Regularisation properties

We consider the problem $Au = f$ where $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and where $\mathcal{Y}$ is a Banach space and $\mathcal{X}$ is the dual of a separable Banach space. Recall that in Tychonoff regularisation, we have

$$R_\alpha(f_\delta) = \arg\min_{u \in \mathcal{X}} \|Au - f_\delta\|^2 + \alpha \|u\|^2.$$

Here, the term $\|u\|^2$ is known as the *regularisation term*, and penalising large values of $\|u\|$ ensures regularity of the solution. Inspired by this, suppose we have any function $\mathcal{J}(u)$ as regulariser, then the *variational regularisation problem* is given by

$$R_\alpha f_\delta \in \arg\min_{u \in \mathcal{X}} \frac{1}{2}\|Au - f_\delta\|^2 + \alpha \mathcal{J}(u),$$

Definition 3.26. Let $u^\dagger_{\mathcal{J}}$ be a least-squares solution that minimises $\mathcal{J}$ over all least-squares solutions. Then we call $u^\dagger_{\mathcal{J}}$ a *$\mathcal{J}$-minimising solution* of the problem $Au = f$.

Convention. We will assume there exists t least one least-squares solution with a finite value of $\mathcal{J}$.

Lemma 3.27. *Let $\mathcal{J}(u) \coloneqq \sum_{i=1}^n \mathcal{J}_i(u)$, where each $\mathcal{J}_i(u)$ is convex and absolutely p_i-homogeneous ($p_i > 0$). Then $\mathcal{N}(J)$ is a linear subspace of $\mathcal{X}$.*

Proof. We note that $\mathcal{J}_i$ is nonnegative, since

$$0 = \mathcal{J}_i(0) = \mathcal{J}_i\left(\frac{1}{2}u - \frac{1}{2}u\right) \le \frac{1}{2}\mathcal{J}_i(u) + \frac{1}{2}\mathcal{J}_i(-u) = \mathcal{J}_i(u).$$

Therefore we have $\mathcal{N}(\mathcal{J}) = \cap_i \mathcal{N}(\mathcal{J}_i)$, and we will show that every $\mathcal{N}(\mathcal{J}_i)$ is a subspace. Let $\lambda \in \mathbb{R}, u, v \in \mathcal{N}(\mathcal{J}_i)$, then

$$\mathcal{J}_i(\lambda u + v) = 2^{p_i}\mathcal{J}_i\left(\frac{\lambda u}{2} + \frac{v}{2}\right) \le 2^{p_i - 1}(|\lambda|^{p_i}\mathcal{J}_i(u) + \mathcal{J}_i(v)) = 0.$$

so $\lambda u + v \in \mathcal{N}(\mathcal{J}_i)$. This completes the proof. □

Lemma 3.28. *Let the assumptions of lemma 3.27 be satisfied, and suppose $u \in \mathcal{X}, v \in \mathcal{N}(\mathcal{J})$. Then $\mathcal{J}(u + v) = \mathcal{J}(u)$.*

Proof. Clearly it suffices to prove this lemma for each $\mathcal{J}_i$. It is easily seen that for any $t \in (0, 1)$ we have

$$\mathcal{J}_i(u + v) = \mathcal{J}_i\left(t\frac{u}{t} + (1-t)\frac{v}{1-t}\right) \le t\mathcal{J}_i\left(\frac{u}{t}\right) = t^{1-p_i}\mathcal{J}(u).$$

Letting $t \to 1$, we find $\mathcal{J}_i(u + v) \le \mathcal{J}_i(u)$.

Similarly, we have

$$\mathcal{J}_i(u) = \mathcal{J}_i(u + v - v) = \mathcal{J}_i\left(t\frac{u+v}{t} + (1-t)\frac{-v}{1-t}\right) \le t\mathcal{J}_i\left(\frac{u+v}{t}\right) = t^{1-p_i}\mathcal{J}_i(u+v),$$

and letting $t \to 1$ we obtain $\mathcal{J}_i(u) \le \mathcal{J}_i(u + v)$, so $\mathcal{J}_i(u) = \mathcal{J}_i(u + v)$, so $\mathcal{J}(u) = \mathcal{J}(u + v)$. □

Recap 3.29. Let $\mathcal{X}$ be a Banach space and let $U \subseteq \mathcal{X}$ be a closed subspace of $\mathcal{X}$. Then U is called *complemented* in $\mathcal{X}$ if there exists a closed subspace $V \subseteq \mathcal{X}$ such that $\mathcal{X} = U \oplus V$.

In general, it is difficult to determine which closed subspaces of $\mathcal{X}$ are complemented. However, it is known that all **finite-dimensional** subspaces are complemented.

Lemma 3.30. *Suppose $\mathcal{J}$ is proper, convex, and satisfies the conditions of lemma 3.27. Suppose also that:*

(i) $\dim \mathcal{N}(\mathcal{J}) < \infty$ *(so that $\mathcal{X} = \mathcal{N}(\mathcal{J}) + \mathcal{X}_0$ for some closed subspace $\mathcal{X}_0$) and $\mathcal{J}$ is coercive on $\mathcal{X}_0$;*

(ii) $\mathcal{N}(A) \cap \mathcal{N}(\mathcal{J}) = \{0\}$.

Then the function $\Phi_\alpha(u) \coloneqq \frac{1}{2}\|Au - f\|^2 + \alpha\mathcal{J}(u)$ is coercive on $\mathcal{X}$ for any $\alpha > 0$.

Proof. Let $(u_j) \subseteq \mathcal{X}$ be a sequence such that $(\Phi_\alpha(u_j))$ is bounded. We will prove that (u_j) is bounded.

Decompose every u_j as $u_j = u_j^0 + u_j^{\mathcal{N}}$ with $u_j^0 \in \mathcal{X}_0$ and $u_j^{\mathcal{N}} \in \mathcal{N}(\mathcal{J})$. Now, since $(\Phi_\alpha(u_j))$ is bounded, the sequence $(\mathcal{J}(u_j)) = (\mathcal{J}(u_j^0))$ is bounded as well, and by coercivity of $\mathcal{J}$ on $\mathcal{X}^0$ we conclude that (u_j^0) is bounded.

Now define $\tilde{A} \coloneqq A\restriction_{\mathcal{N}(\mathcal{J})}$. Since $\mathcal{N}(A) \cap \mathcal{N}(\mathcal{J}) = \{0\}$, $\tilde{A}$ has an inverse on $A\mathcal{N}(\mathcal{J})$, and since $\dim A\mathcal{N}(\mathcal{J}) < \infty$, that inverse is bounded. Therefore we find

$$\|u_j^{\mathcal{N}}\| = \left\|\tilde{A}^{-1}(Au_j^{\mathcal{N}})\right\| \leq C\left\|\tilde{A}u_j^{\mathcal{N}}\right\| = C\left\|\tilde{A}u_j - f - (Au_j^0 - f)\right\| \leq C\big(\|Au_j - f\| + \|A\|\|u_j^0\| + \|f\|\big).$$

Since $\Phi_\alpha(u_j)$ is bounded, $\|Au_j - f\|$ is also bounded, and the other terms are also all bounded, so we conclude that $(u_j^{\mathcal{N}})$ is bounded, and therefore that (u_j) is bounded. □

Theorem 3.31. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces with topologies $\tau_\mathcal{X}, \tau_\mathcal{Y}$. Assume that:*

(a) bounded sequences in $\mathcal{X}$ have $\tau_\mathcal{X}$-convergent subsequences;

(b) $\mathcal{J} \colon \mathcal{X} \to [0, \infty]$ is proper, convex, $\tau_\mathcal{X}$-LSC and satisfies the assumptions of lemma 3.30;

(c) A is continuous w.r.t. $\tau_\mathcal{X}$ and $\tau_\mathcal{Y}$;

(d) $\|\cdot\|_\mathcal{Y}$ is $\tau_\mathcal{Y}$-LSC.

Then:

1. *There exists a $\mathcal{J}$-minimising solution $u_\mathcal{J}^\dagger$ of the equation $Au = f$;*
2. *For any $\alpha > 0$, $f \in \mathcal{Y}$ there exists a minimiser*

$$u_\alpha = R_\alpha f \in \arg\min \frac{1}{2}\|Au - f\|^2 + \alpha\mathcal{J}(u).$$

Proof.

1. Let $\mathbb{L}$ be the set of least-squares solutions and $\mu \coloneqq \inf\{\|Au - f\| \mid u \in \mathcal{X}\}$, then we can write
$$\mathbb{L} = \{u \in \mathcal{X} : \|Au - f\| \leq \mu\}.$$
Since $\|\cdot\|_\mathcal{Y}$ is $\tau_\mathcal{Y}$-LSC and $A\colon \tau_\mathcal{X} \to \tau_\mathcal{Y}$ is continuous, we have by lemma 3.4 that $\mathbb{L}$ is $\tau_\mathcal{X}$-closed. Now consider the problem
$$\inf_{u\in\mathbb{L}} \mathcal{J}(u) = \inf_{u\in\mathcal{X}} \mathcal{J}(u) + \chi_\mathbb{L}(u).$$
By assumption there exists $u \in \mathcal{L}$ with $\mathcal{J}(u) < \infty$, and the objective function $\mathcal{J} + \chi_\mathcal{L}$ is bounded from below (by 0). Using similar arguments as in the previous lemma (???), it can be shown that it is coercive. Finally, characteristic functions of closed sets are LSC, so $\chi_\mathbb{L}$ and $\mathcal{J}$ are both $\tau_\mathcal{X}$-LSC, and therefore their sum is $\tau_\mathcal{X}$-LSC as well. By the direct method (theorem 3.23), we conclude that a $\mathcal{J}$-minimising solution exists.

2. The objective function Φ_α is coercive by the previous lemma, and also bounded from below. It is easily seen that Φ_α is $\tau_\mathcal{X}$-LSC, and using the direct method we conclude that Φ_α has a minimiser. □