Distribution Theory and Applications — Summary

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October 29, 2020

Contents

1 Distributions

1.1 Test functions and distributions

Definition 1.1. Let $X \subseteq \mathbb{R}^n$ be open, then we define the set of *test functions* on X as

$$\mathcal{D}(X) := C_0^\infty(X) = \{f \colon X \to \mathbb{C} \mid f \text{ is smooth with compact support}\}.$$

Definition 1.2. Let $(\varphi_m) \subseteq \mathcal{D}(X)$. We say that $(\varphi_m) \to 0$ in $\mathcal{D}(X)$ if

1. there exists a compact $K \subseteq X$ such that $\operatorname{supp} \varphi_m \subseteq K$ for all m;
2. $\partial^\alpha \varphi_m \to 0$ uniformly for each multi-index α.

Note that, for any $\varphi, \psi \in \mathcal{D}(X)$ and any multi-index α we have

$$\int_X \varphi \cdot \partial^\alpha \psi \, \mathrm{d}x = (-1)^{|\alpha|} \int_X \psi \cdot \partial^\alpha \varphi \, \mathrm{d}x \, ,$$

which follows from partial integration and the fact that all boundary terms vanish since φ and ψ have compact support.

Also, by Taylor's theorem, for any $\varphi \in \mathcal{D}(X)$, $x, h \in X$ and $N \in \mathbb{N}$ we have

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h) \quad \text{where } R_N(x,h) = o(|h|^N) \text{ uniformly in } x.$$

Definition 1.3. A *distribution* on X is a linear map $u \colon \mathcal{D}(X) \to \mathbb{C}$ if for every compact set $K \subseteq X$ there exist constants C, N such that for all $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp} \varphi \subseteq K$ we have

$$|u(\varphi)| \leqslant C \sum_{|\alpha| \leqslant N} \sup |\partial^\alpha \varphi|. \tag{1}$$

Condition 1 is called the *seminorm condition.* If, in the seminorm condition, the same N can be used for every compact set $K \subseteq X$, then the least such N is called the *order* of u, written $\operatorname{ord}(u)$.

The set of all distributions in X is denoted $\mathcal{D}'(X)$.

If $u \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}(X)$, then instead of $u(\varphi)$ we usually write $\langle u, \varphi \rangle$.

Recap 1.4. A function $f \colon X \to \mathbb{C}$ is called *locally integrable* if $\int_K |f| \, \mathrm{d}x < \infty$ for all compact $K \subseteq X$.

The set of locally integrable functions on X is denoted $L^1_{\mathrm{loc}}(X)$.

Example 1.5. Let $M \in \mathbb{N}$ and let $f_\alpha \in L^1_{\mathrm{loc}}(X)$ for all $|\alpha| \leqslant M$. Define the linear map $T \colon \mathcal{D}(X) \to \mathbb{C}$ by

$$\langle T, \varphi \rangle := \sum_{|\alpha| \leqslant M} \int_X f_\alpha \cdot \partial^\alpha \varphi \, \mathrm{d}x \, .$$

It is trivial that T is linear, and we verify that T is a distribution as follows: take $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp} \varphi \subseteq K$. Then we have

$$\begin{aligned} |\langle T, \varphi \rangle| &\leqslant \sum_{|\alpha| \leqslant M} \int_K |f_\alpha| \cdot |\partial^\alpha \varphi| \, \mathrm{d}x \\ &\leqslant \sum_{|\alpha| \leqslant M} \sup |\partial^\alpha \varphi| \cdot \int_K |f_\alpha| \, \mathrm{d}x \\ &\leqslant \left(\max_\alpha \int_K |f_\alpha| \, \mathrm{d}x \right) \sum_{|\alpha| \leqslant M} \sup |\partial^\alpha \varphi|. \end{aligned}$$

Therefore, the seminorm condition is satisfied with $N = M$. From this, it also follows that $\operatorname{ord}(T) \leqslant M$.

A special case of the previous example is the case $M = 0$: in this case the distribution simply becomes

$$\langle \tau_f, \varphi \rangle = \int_X f\varphi \, \mathrm{d}x \, .$$

Henceforth we will abuse notation: if $f \in L^1_{\mathrm{loc}}(X)$, then we will write f instead of τ_f, i.e., $\langle f, \varphi \rangle = \int_X f\varphi \, \mathrm{d}x$.

Lemma 1.6 (Sequential continuity)**.** *Let $u\colon \mathcal{D}(X) \to \mathbb{C}$ be a linear map. Then u is a distribution if and only if, for every sequence $(\varphi_m) \subseteq \mathcal{D}(X)$ with $\varphi_m \to 0$ as in definition 1.2, we have $\langle u, \varphi_m \rangle \to 0$.*

Proof. '$\implies$' If u is a distribution and $(\varphi_m) \to 0$, then $\operatorname{supp} \varphi_m \subseteq K$ for some compact K, and by eq. (1) there exist C, N such that

$$|\langle u, \varphi_m \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup |\partial^\alpha \varphi_m| \to 0.$$

'$\impliedby$' Suppose there is a compact set K such that eq. (1) is not valid for any C, N. Let $m \in \mathbb{N}$ and $C = N = m$, then there is some φ_m with $\operatorname{supp}(\varphi_m) \subseteq K$, and

$$|\langle u, \varphi_m \rangle| > m \sum_{|\alpha| \leqslant m} \sup |\partial^\alpha \varphi_m|.$$

By dividing φ_m by $\langle u, \varphi_m \rangle \neq 0$, we may assume w.l.o.g. that $\langle u, \varphi_m \rangle = 1$. We now have a sequence (φ_m) such that

$$\frac{1}{m} > \sum_{|\alpha| \leqslant m} \sup |\partial^\alpha \varphi_m| \implies |\partial^\alpha \varphi_m| < \frac{1}{m} \quad \text{for } |\alpha| \leqslant m \implies \partial^\alpha \varphi_m \to 0 \text{ uniformly for all } \alpha.$$

Since each φ_m also satisfies $\operatorname{supp} \varphi_m \subseteq K$, by definition 1.2 we have that $\varphi_m \to 0$, but also $\langle u, \varphi_m \rangle \to 1$, a contradiction. ☐

1.2 Limits in $\mathcal{D}'(X)$

Definition 1.7. We say that a sequence $(u_m) \subseteq \mathcal{D}'(X)$ converges to $u \in \mathcal{D}'(X)$ and write $u_m \to u$ if

$$\langle u_m, \varphi \rangle \to \langle u, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(X).$$

The following theorem is non-examinable but interesting:

Theorem 1.8. *Let (u_m) be a sequence in $\mathcal{D}'(X)$ such that $\lim_{m\to\infty} \langle u_m, \varphi \rangle$ exists for all $\varphi \in \mathcal{D}(X)$. Then the map $\langle u, \varphi \rangle := \lim_{m\to\infty} \langle u_m, \varphi \rangle$ is a distribution in X.*

Proof. This is a direct application of the uniform boundedness principle. ☐

Example 1.9. Let $X = \mathbb{R}$ and consider the sequence of functions $u_m \in L^1_{\mathrm{loc}}(\mathbb{R})$ defined by $u_m(x) = \sin(mx)$. Then, for all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle u_m, \varphi \rangle = \int_{\mathbb{R}} \sin(mx)\varphi(x) \, \mathrm{d}x = \frac{1}{m} \int_{\mathbb{R}} \cos(mx)\varphi'(x) \, \mathrm{d}x \leqslant \frac{1}{m} \int \left|\varphi'(x)\right| \mathrm{d}x \to 0.$$

Therefore, it holds that $u_m \to 0$ in $\mathcal{D}'(\mathbb{R})$. With our abuse of notation we write this as $\lim_{m\to\infty} \sin(mx) = 0$ in $\mathcal{D}'(\mathbb{R})$.

1.3 Basic operations

1.3.1 Differentiation and multiplication by smooth functions

For $u \in C^\infty(X)$ and $\varphi \in \mathcal{D}(X)$, we have noted that

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \partial^\alpha u \cdot \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_X u \cdot \partial^\alpha \varphi \,\mathrm{d}x = \langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle.$$

Since the RHS makes sense for any distribution u, we define

Definition 1.10. For $f \in C^\infty(X)$, $u \in \mathcal{D}'(X)$, we define $\partial^\alpha(fu)$ by

$$\langle \partial^\alpha(fu), \varphi \rangle := \langle u, (-1)^{|\alpha|} f \cdot \partial^\alpha \varphi \rangle$$

Remark. This definition outlines a more general pattern when working with distributions: first we take some well-defined operator on the collection of smooth maps, then we rewrite it to a form that is sensible for any distribution, and then we *define* that new form as the operator on distributions. This process is called *extending the definition by duality.*

Example 1.11. Let $u = \delta_x$, then we have

$$\langle \partial^\alpha \delta_x, \varphi \rangle = \langle \delta_x, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x)$$

Furthermore, consider the Heaviside function $H(x) = \mathbb{1}_{x \geqslant 0}$. We have

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{\mathbb{R}} H(x) \varphi'(x) \,\mathrm{d}x = -\int_0^\infty \varphi'(x) \,\mathrm{d}x = \varphi(0) = \langle \delta_0, \varphi \rangle,$$

so we write $H' = \delta_0$ *in the distributional sense.*

Lemma 1.12. *Suppose $u' \in \mathcal{D}'(\mathbb{R})$ satisfies $u' = 0$. Then u is constant (i.e., $\langle u, \varphi \rangle = \langle c, \varphi \rangle = c \int_{\mathbb{R}} \varphi \,\mathrm{d}x$ for some c).*

Proof. Fix any $\vartheta \in \mathcal{D}(\mathbb{R})$ with $\langle 1, \vartheta \rangle = 1$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and define

$$\varphi_A := \varphi - \langle 1, \varphi \rangle \vartheta, \quad \varphi_B := \langle 1, \varphi \rangle \vartheta \quad \text{such that } \varphi = \varphi_A + \varphi_B.$$

Note that $\langle 1, \varphi_A \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle \langle 1, \vartheta \rangle = 0$.

We claim that the function $\Phi_A(x) := \int_{-\infty}^x \varphi_A(y) \,\mathrm{d}y$ has compact support: since $\operatorname{supp} \varphi_A \subseteq [a, b]$ for some $a, b \in \mathbb{R}$, clearly $\Phi_A(x) = 0$ for $x < a$, while for $x > b$ we have $\Phi_A(x) = \langle 1, \varphi_A \rangle = 0$. Obviously, it holds that $\Phi_A' = \varphi_a$. Now we compute

$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle = \langle u, \Phi_A' \rangle + \langle 1, \varphi \rangle \langle u, \vartheta \rangle = -\langle u', \Phi_A \rangle + c \langle 1, \varphi \rangle = \langle c, \varphi \rangle.$$

Since φ was chosen arbitrarily this shows that u is constant. □

1.3.2 Reflection and translation

For $\varphi \in \mathcal{D}(\mathbb{R}^n), h \in \mathbb{R}^n$, define the *translation of φ by h* by

$$(\tau_h \varphi)(x) := \varphi(x - h),$$

and the *reflection of φ* by $\check{\varphi}(x) := \varphi(-x)$.

Extending the definitions of translation and reflection by duality yields the following:

Definition 1.13. For $u \in \mathcal{D}'(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, define

$$\langle \tau_h u, \varphi \rangle := \langle u, \tau_{-h} \varphi \rangle \quad \text{and} \quad \langle \check{u}, \varphi \rangle := \langle u, \check{\varphi} \rangle.$$

Lemma 1.14. *For $u \in \mathcal{D}'(\mathbb{R}^n)$, define $V_h \in \mathcal{D}'(\mathbb{R}^n)$ for $0 \neq h \in \mathbb{R}^n$ by*

$$V_h := \frac{\tau_{-h}u - u}{\|h\|}$$

If $(h_j) \subseteq \mathbb{R}^n$ is a sequence for which $\lim_{j\to\infty} \frac{h_j}{\|h_j\|} = m \in S^{n-1}$, then $V_{h_j} \to \sum_i m_i \frac{\partial u}{\partial x_i}$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. By definition, we can write $\langle V_h, \varphi\rangle = \frac{1}{\|h\|}\langle u, \tau_h\varphi - \varphi\rangle$. Now Taylor's theorem tells us that

$$(\tau_h\varphi - \varphi)(x) = \varphi(x-h) - \varphi(x) = -\sum_i h_i \frac{\partial \varphi}{\partial x_i} + R(x,h),$$

where $R(x,h) = o(\|h\|)$ in $D(\mathbb{R}^n)$ (exercise sheet 1, question 2).

By sequential continuity, we have

$$\lim_{j\to\infty}\langle V_{h_j}, \varphi\rangle = \langle u, -\sum_i m_i \frac{\partial\varphi}{\partial x_i}\rangle = \langle \sum_i m_i \frac{\partial u}{\partial x_i}, \varphi\rangle,$$

which shows that $V_{h_j} \to \sum_i m_i \frac{\partial u}{\partial x_i}$ in $\mathcal{D}'(\mathbb{R}^n)$. $\square$

1.3.3 Convolution

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, note that $(\tau_x\check{\varphi})(y) = \check{\varphi}(y-x) = \varphi(x-y)$.

Definition 1.15. For $u \in C^\infty(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the *convolution* $u * \varphi\colon \mathbb{R}^n \to \mathbb{C}$ as

$$(u*\varphi)(x) := \int_{\mathbb{R}^n} u(x-y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} u(y)\varphi(x-y)\,\mathrm{d}y = \langle u, \tau_x\check{\varphi}\rangle.$$

Since the RHS makes sense for any $u \in \mathcal{D}'(\mathbb{R}^n)$, we extend the definition this way: for $u \in \mathcal{D}'(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the convolution $u * \varphi$ as

$$(u*\varphi)(x) := \langle u, \tau_x\check{\varphi}\rangle.$$

Lemma 1.16. *Let $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and define $\Phi_x(y) := \varphi(x,y)$. Suppose for any $x \in \mathbb{R}^n$ there exists a neighbourhood $N(x)$ and a compact $K \subseteq \mathbb{R}^m$ such that $\varphi(x,y)$ for all $x \in N(x), y \notin K$.*

Then $x \mapsto \langle u, \Phi_x\rangle$ is differentiable with

$$\partial_x^\alpha\langle u, \Phi_x\rangle = \langle u, \partial_x^\alpha \Phi_x\rangle$$

for any $u \in \mathcal{D}'(\mathbb{R}^m)$.

Proof. Fix $x \in \mathbb{R}^n$, then by Taylor's formula we have

$$\Phi_{x+h}(y) = \Phi_x(y) + \sum_{i=1}^n h_i \frac{\partial\varphi}{\partial x_i} + R(x,y,h),$$

where $\partial_y^\alpha R(x,y,h) = o(\|h\|)$, uniformly in y, for any multi-index α. Furthermore, by assumption there exists a compact K such that for h small enough, $\operatorname{supp} R(x,\cdot,h) \subseteq K$. Therefore, $R(x,\cdot,h)$ is a test function for h small enough.

Combining the previous two facts shows that $R(x,\cdot,h) = o(\|h\|)$ in $\mathcal{D}(\mathbb{R}^m)$ as $h \to 0$.

Let $u \in \mathcal{D}'(\mathbb{R}^m)$, then we find by sequential continuity that $\langle u, R(x,\cdot,h)\rangle$ is also $o(\|h\|)$, and therefore

$$\langle u, \Phi_{x+h}\rangle = \langle u, \Phi_x\rangle + \sum_{i=1}^n h_i\langle u, \frac{\partial \Phi_x}{\partial x_i}\rangle + o(\|h\|).$$

This shows that $x \mapsto \langle u, \Phi_x\rangle$ is differentiable with

$$\frac{\partial}{\partial x_i}\langle u, \Phi_x\rangle = \langle u, \frac{\partial\Phi_x}{\partial x_i}\rangle.$$

From this the result follows. $\square$

Corollary 1.17. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $u * \varphi$ is differentiable with $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi$.*

Proof. Apply the previous lemma with $\Phi_x(y) := \varphi(x - y)$. ☐

Due to the previous corollary, we often call $u * \varphi$ a *regularisation* of u.

Convention. If $u \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}(X)$, then instead of $\langle u, \varphi \rangle$ we also write $\langle u(t), \varphi(t) \rangle$ (or with any other dummy variable) when the variable used for φ is not directly clear.

1.4 Density of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{D}'(\mathbb{R}^n)$

Lemma 1.18. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, then*

$$(u * \varphi) * \psi = u * (\varphi * \psi).$$

Proof. Fix $x \in \mathbb{R}^n$. Now we write

$$\begin{aligned} ((u * \varphi) * \psi)(x) &= \int_{\mathbb{R}^n} \langle u(z), (\tau_y \check{\varphi})(z) \rangle \cdot \psi(x - y) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \langle u(z), (\tau_{x-y} \check{\varphi})(z) \rangle \cdot \psi(y) \,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \langle u(z), \psi(y) (\tau_{x-y} \check{\varphi})(z) \rangle \,\mathrm{d}y \,. \end{aligned}$$

We would like to interchange integral and application of u, and we will have to justify this using Riemann sums:

$$\begin{aligned} \int_{\mathbb{R}^n} \langle u(z), \psi(y)(\tau_{x-y}\check{\varphi})(z) \rangle \,\mathrm{d}y &= \lim_{\varepsilon \downarrow 0} \sum_{m \in \mathbb{Z}^n} \langle u(z), \psi(\varepsilon m) \varphi(x - z - \varepsilon m) \rangle \varepsilon^n \\ &\overset{*}{=} \lim_{\varepsilon \downarrow 0} \langle u(z), \sum_{m \in \mathbb{Z}^n} \psi(\varepsilon m) \varphi(x - z - \varepsilon m) \varepsilon^n \rangle \\ &\overset{**}{=} \left\langle u(z), \int_{\mathbb{R}^n} \psi(y) \varphi(x - z - y) \,\mathrm{d}y \right\rangle \\ &= \langle u(z), (\varphi * \psi)(x - z) \rangle = \langle u(z), (\tau_x \widetilde{\varphi * \psi})(z) \rangle = (u * (\varphi * \psi))(x). \end{aligned}$$

Here, $*$ is by the fact that the sum is finite since ψ has compact support, while $**$ is by sequential continuity of u and the fact that the Riemann sum converges to the convolution integral *in the space of test functions* (non-examinable fact). ☐

We will use the following trick many times:

Proposition 1.19. *For any $u \in \mathcal{D}'(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $\langle u, \varphi \rangle = (u * \check{\varphi})(0)$.*

Proof. We have $(u * \check{\varphi})(0) = \langle u, \tau_0 \varphi \rangle = \langle u, \varphi \rangle$. ☐

For example, from this trick it follows that if $u * \varphi = 0$ for all φ, then $u = 0$.

Theorem 1.20. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, there exists a sequence $(\varphi_k) \subseteq \mathcal{D}(\mathbb{R}^n)$ such that $\varphi_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi \,\mathrm{d}x = 1$, and set $\psi_k(x) := k^n \psi(kx)$. Note that $\int_{\mathbb{R}^n} \psi_k \,\mathrm{d}x = 1$.

Now, fix any $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi \equiv 1$ on $\{\|x\| < 1\}$ and $\chi \equiv 0$ on $\{\|x\| < 2\}$. Define $\chi_k(x) := \chi(x/k)$, so that $\lim_{k \to \infty} \chi_k(x) = 1$ for all x. We will set

$$\varphi_k(x) := \chi_k(x)(u * \psi_k)(x).$$

Clearly we have $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ since each χ_k has compact support.

Now, take any $\vartheta \in \mathcal{D}(\mathbb{R}^n)$, then we have

$$\begin{aligned}\langle \varphi_k, \vartheta \rangle = \langle \chi_k(u * \psi_k), \vartheta \rangle = \langle u * \psi_k, \chi_k \vartheta \rangle &= \left[(u * \psi_k) * \widecheck{\chi_k \vartheta} \right](0) \\ &= \left[u * \left(\psi_k * \widecheck{\chi_k \vartheta} \right) \right](0).\end{aligned}$$

Now we compute $\psi_k * \widecheck{\chi_k \vartheta}$: note that

$$\begin{aligned}(\psi_k * \widecheck{\chi_k \vartheta})(x) &= \int_{\mathbb{R}^n} k^n \psi(k(x-y)) \chi\Big(-\frac{y}{k}\Big) \vartheta(-y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \psi(y) \chi\Big(\frac{y}{k^2} - \frac{x}{k}\Big) \vartheta(\frac{y}{k} - x) \, \mathrm{d}y \\ &= \vartheta(-x) + R_k(-x) = (\widecheck{\vartheta + R_k})(x)\end{aligned}$$

where $R_k(x) = \int_{\mathbb{R}^n} \psi(y) \big[\chi\big(\frac{y}{k^2} + \frac{x}{k}\big) \vartheta\big(\frac{y}{k} + x\big) - \vartheta(x) \big] \, \mathrm{d}y$.

So

$$\langle \varphi_k, \vartheta \rangle = (u * (\widecheck{\vartheta + R_k}))(0) = (u * \check{\vartheta})(0) + (u * \widecheck{R_k})(0) = \langle u, \vartheta \rangle + \langle u, R_k \rangle.$$

We must now only prove that $R_k \to 0$ in $\mathcal{D}(\mathbb{R}^n)$, and then by sequential continuity it follows that $\varphi_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$. □

2 Distributions with compact support

Definition 2.1. Let $Y \subseteq X$ be open and $u \in \mathcal{D}'(X)$. We say that u *vanishes* on Y if $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(Y)$.

Definition 2.2. For $u \in \mathcal{D}'(X)$, we define the *support* of u as

$$\operatorname{supp} u := X \setminus \bigcup \{Y \subseteq X \mid Y \text{ open}, u \text{ vanishes on } Y\}.$$

For example, the support of δ_x is simply $\{x\}$.

2.1 Test functions and distributions

We will now consider a bigger space of test functions, which yields a smaller space of distributions.

Definition 2.3. We define $\mathcal{E}(X)$ as the space of smooth functions $\varphi \colon X \to \mathbb{C}$. We say that a sequence $(\varphi_m) \subseteq \mathcal{E}(X)$ converges to 0 if $\partial^\alpha \varphi \to 0$ uniformly on compact subsets of X for every multi-index α.

Definition 2.4. We define $\mathcal{E}'(X)$ as the space of linear maps $u \colon \mathcal{E}(X) \to \mathbb{C}$ for which there exists a compact $K \subseteq X$ and nonnegative constants C, N such that

$$|\langle u, \varphi \rangle| \leqslant C \sum_{\alpha \leqslant N} \sup_K |\partial^\alpha \varphi| \tag{2}$$

for all $\varphi \in \mathcal{E}(X)$.

Lemma 2.5 (Sequential continuity)**.** *Let $u \colon \mathcal{E}(X) \to \mathbb{C}$ be a linear map. Then $u \in \mathcal{E}'(X)$ if and only if, for every sequence $(\varphi_m) \subseteq \mathcal{E}(X)$ with $\varphi_m \to 0$, we have $\langle u, \varphi_m \rangle \to 0$.*

Proof. TODO: □

Lemma 2.6. *If $u \in \mathcal{E}'(X)$, then $u{\restriction}_{\mathcal{D}(X)}$ defines an element of $\mathcal{D}'(X)$ with compact support and finite order.*

Conversely, for each $u \in \mathcal{D}'(X)$ with compact support there exists a unique extension $\tilde{u} \in \mathcal{E}'(X)$ with $\operatorname{supp}(\tilde{u}) = \operatorname{supp}(u)$ and $\tilde{u}{\restriction}_{\mathcal{D}(X)} = u$.

Proof. Let $u \in \mathcal{E}'(X)$, so that there exists a compact $K \subseteq X$ with $|\langle u, \varphi \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup_K |\partial^\alpha \varphi|$. Now, for any compact $K' \subseteq X$ and any φ with $\operatorname{supp} \varphi \subseteq K'$, eq. (1) is clearly satisfied, and we can use the same N for all compact K', so clearly $u{\restriction}_{\mathcal{D}(X)}$ is an element of $\mathcal{D}'(X)$ with finite order. Finally, suppose φ is supported in $X \setminus K$, then it is clear that $\langle u, \varphi \rangle = 0$, which proves that $\operatorname{supp} u \subseteq K$ and therefore that u has compact support.

Now suppose $u \in \mathcal{D}'(X)$ has compact support, let $\rho \in \mathcal{D}(X)$ be 1 in a neighbourhood of $\operatorname{supp} u$, and define $\tilde{u} \in \mathcal{E}'(X)$ by

$$\langle \tilde{u}, \varphi \rangle := \langle u, \rho\varphi \rangle \quad \forall \varphi \in \mathcal{E}(X).$$

Clearly $\tilde{u}$ is an element of $\mathcal{E}'(X)$ since $\operatorname{supp}(\rho\varphi) \subseteq \operatorname{supp} \rho$ and

$$|\langle \tilde{u}, \varphi \rangle| = |\langle u, \rho\varphi \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup_{\operatorname{supp}(\rho)} |\partial^\alpha(\rho\varphi)| \overset{\star}{\leqslant} C' \sum_{|\alpha| \leqslant N} \sup_{\operatorname{supp} \rho} |\partial^\alpha \varphi|,$$

where $\star$ follows from the Leibniz rule. It is also clear that $\operatorname{supp} \tilde{u} = \operatorname{supp} u$.

Finally we will show uniqueness: suppose $\tilde{v}$ is an extension of u with $\operatorname{supp} \tilde{v} = \operatorname{supp} u$, and write any $\varphi \in \mathcal{E}(X)$ as $\varphi = \rho\varphi + (1-\rho)\varphi = \varphi_0 + \varphi_1$. Then since $\varphi_0 \in \mathcal{D}(X)$ and φ_1 vanishes on a neighbourhood of $\operatorname{supp} u$, we find

$$\langle \tilde{v}, \varphi \rangle = \langle \tilde{v}, \varphi_0 + \varphi_1 \rangle = \langle \tilde{v}, \varphi_0 \rangle = \langle u, \varphi_0 \rangle,$$

which is independent of choice of extension. □

2.2 Convolution between $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$

Definition 2.7. Define for $u \in \mathcal{E}'(\mathbb{R}^n), \varphi \in \mathcal{E}(\mathbb{R}^n)$ the *convolution*

$$(u * \varphi)(x) = \langle u, \tau_x \check{\varphi} \rangle.$$

This convolution satisfies the same properties as the convolution between $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$. Also, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $u * \varphi \in \mathcal{D}(\mathbb{R}^n)$.

Definition 2.8. Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of which has compact support, define $u * v \colon \mathcal{D}(\mathbb{R}^n) \to \mathcal{E}(\mathbb{R}^n)$ by

$$(u * v) * \varphi = u * (v * \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We will show (example sheet 2, question 1) that $u * v$ is uniquely defined and gives rise to an element of $\mathcal{D}'(\mathbb{R}^n)$ via $\langle u * v, \varphi \rangle := [(u * v) * \check{\varphi}](0)$.

Lemma 2.9. *Given $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of which has compact support, we have $u * v = v * u$.*

Proof. First we note that $(u * \varphi) * \psi = u * (\varphi * \psi)$ holds if u has compact support and at least one of φ, ψ has compact support.

Fix $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, we see from our earlier shown properties that

$$(u * v) * (\varphi * \psi) = u * (v * (\varphi * \psi)) = u * ((v * \varphi) * \psi) = u * (\psi * (v * \varphi)) = (u * \psi) * (v * \varphi).$$

If we interchange u and v in the above, that is equivalent to interchanging φ and ψ, which we know must yield the same result. This shows $u * v$ and $v * u$ agree on $\varphi * \psi$ for all $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$. Defining $E = u * v - v * u$, we find that $0 = E * (\varphi * \psi) = (E * \varphi) * \psi$ for all $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, so $E * \varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, so $E = 0$. □

3 Tempered distributions and Fourier analysis

3.1 Functions of rapid decay

Definition 3.1. For any $f \colon \mathbb{R}^n \to \mathbb{C}$ and multi-indices α, β we define $\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \varphi \right|$.

We define the *Schwarz space*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty \text{ for all } \alpha, \beta \right\}.$$

We say that a sequence $(\varphi_n) \subseteq \mathcal{S}(\mathbb{R}^n)$ converges to 0 if $\|\varphi_m\|_{\alpha,\beta} \to 0$ for every α, β.

Example 3.2. The function $x \mapsto \exp\left(-\|x\|^2\right)$ lies in $\mathcal{S}(\mathbb{R}^n)$.

Proposition 3.3. *For all n we have that $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then for all $N \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^n} |\varphi(x)| \,\mathrm{d}x = \int_{\mathbb{R}^n} (1 + \|x\|)^{-N} (1 + \|x\|)^N |\varphi(x)| \,\mathrm{d}x \overset{?}{\leqslant} C \sum_{|\alpha| \leqslant N} \|\varphi\|_{\alpha,0} \int_{\mathbb{R}^n} (1 + \|x\|)^{-N} \,\mathrm{d}x \,.$$

Since $\int_{\mathbb{R}^n} (1 + \|x\|)^{-N} \,\mathrm{d}x$ is finite for N large enough (??), this proves the claim. □

Definition 3.4. A linear map $u \colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called a *tempered distribtion* if there exists constants C, N such that

$$|\langle u, \varphi \rangle| \leqslant C \sum_{|\alpha|, |\beta| \leqslant N} \|\varphi\|_{\alpha,\beta} \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This definition is equivalent to sequential continuity.

3.2 The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$

Convention. We write $D := -i\partial$ and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$.

Definition 3.5. For $f \in L^1(\mathbb{R}^n)$, define the *Fourier transform* of f by

$$[\mathcal{F}(f)](\lambda) = \hat{f}(\lambda) := \int_{\mathbb{R}^n} e^{-i\lambda x} f(x)\,\mathrm{d}x \quad \text{where } \lambda \in \mathbb{R}^n.$$

Lemma 3.6. *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ is continuous.*

Proof. If $\lambda_m \to \lambda \in \mathbb{R}^n$, then by the dominated convergence theorem we have

$$\hat{f}(\lambda_m) = \int_{\mathbb{R}^n} e^{-i\lambda_m x} f(x)\,\mathrm{d}x \overset{\text{DCT}}{=} \int_{\mathbb{R}^n} e^{-i\lambda x} f(x)\,\mathrm{d}x = \hat{f}(\lambda),$$

where we were able to use the dominated convergence theorem since the integrand is absolutely bounded by $|f|$ and $f \in L^1$. □

It turns out that this idea generalises: the faster the function f decays, the smoother the Fourier transform $\hat{f}$ is.

Lemma 3.7. *For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}[D_x^\alpha \varphi](\lambda) = \lambda^\alpha \hat{\varphi}(\lambda)$ and $\mathcal{F}[x^\beta \varphi](\lambda) = (-1)^{|\beta|} D_\lambda^\beta \hat{\varphi}(\lambda)$.*

Proof. Since $\left|x^\alpha D^\beta \varphi\right| \to 0$ as $\|x\| \to \infty$, we have using integration by parts

$$\begin{aligned}
\mathcal{F}[D_\lambda^\alpha \varphi](\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda x} D_x^\alpha \varphi(x)\,\mathrm{d}x \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_x^\alpha (e^{-i\lambda x})\varphi(x)\,\mathrm{d}x \\
&= \lambda^\alpha \hat{\varphi}(\lambda).
\end{aligned}$$

For the second part of the lemma, by differentiation under the integral sign we get

$$\begin{aligned}
\mathcal{F}[x^\beta \varphi](\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda x} x^\beta \varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n} ((-D_\lambda)^\beta e^{-i\lambda x})\varphi(x)\,\mathrm{d}x \\
&= (-1)^{|\beta|} D_\lambda^\beta \hat{\varphi}(\lambda).
\end{aligned}$$

□

We define the *inverse Fourier transform* by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\lambda x} \hat{f}(\lambda)\,\mathrm{d}\lambda\,.$$

We will now show that on $\mathcal{S}(\mathbb{R}^n)$, the inverse Fourier transform is indeed an inverse:

Theorem 3.8. *The Fourier transform $\mathcal{F}$ defines a continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to itself.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. First, we show that $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$: by the previous lemma we have for multi-indices α, β that

$$\begin{aligned}
\left|\lambda^\alpha (-D_\lambda)^\beta \hat{\varphi}(\lambda)\right| = \left|\lambda^\alpha \mathcal{F}[x^\beta \varphi](\lambda)\right| = \left|\mathcal{F}[D_x^\alpha (x^\beta \varphi)](\lambda)\right| &= \left|\int_{\mathbb{R}^n} e^{-i\lambda x} D^\alpha (x^\beta \varphi)\,\mathrm{d}x\right| \\
&\leqslant \int_{\mathbb{R}^n} \left|D^\alpha (x^\beta \varphi)\right| \mathrm{d}x\,, \qquad (3)
\end{aligned}$$

which is finite since $D^\alpha(x^\beta\varphi)$ is also a Schwarz function and therefore integrable.

From the previous lemma we also infer that $\hat{\varphi}$ is smooth, so indeed we have $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. From eq. (3) it is also easily seen that if $\varphi_m \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, then $\hat{\varphi}_m \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ also, which shows that $\mathcal{F}$ is continuous.

To prove surjectivity and injectivity, we will show that $\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x)$ (???). Indeed we have

$$\begin{aligned}
\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\lambda(x-y)} \varphi(y) \,\mathrm{d}y \,\mathrm{d}\lambda \\
&= \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\lambda(x-y) - \varepsilon\|\lambda\|^2} \varphi(y) \,\mathrm{d}y \,\mathrm{d}\lambda \\
&\overset{\star}{=} \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} e^{i\lambda(x-y) - \varepsilon\|\lambda\|^2} \,\mathrm{d}\lambda \,\mathrm{d}y \,,
\end{aligned}$$

where $\star$ follows from Fubini's theorem.

Now we note that

$$\int_{\mathbb{R}^n} e^{i\lambda(x-y) - \varepsilon\|\lambda\|^2} \,\mathrm{d}\lambda = \prod_{i=1}^n \int_{\mathbb{R}} e^{i\lambda_j(x_j - y_j) - \varepsilon\lambda_j^2} \,\mathrm{d}\lambda \overset{\star\star}{=} \prod_{i=1}^n \left(\frac{\pi}{e}\right)^{1/2} e^{-\frac{(x_i - y_i)^2}{4\varepsilon}} = \left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-\frac{\|x-y\|^2}{4\varepsilon}}.$$

To explain $\star\star$, TODO: .

and plugging that into the above yields

$$\begin{aligned}
\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \lim_{\varepsilon \downarrow 0} 2^{-n} (\pi\varepsilon)^{-n/2} \int_{\mathbb{R}^n} \varphi(y) e^{-\|x-y\|^2/(4\varepsilon)} \,\mathrm{d}y \\
&\overset{\star\star\star}{=} \pi^{-n/2} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \varphi(x - 2\sqrt{\varepsilon} y') e^{-\|y'\|^2} \,\mathrm{d}y \\
&\overset{\text{DCT}}{=} \pi^{-n/2} \varphi(x) \int_{\mathbb{R}^n} e^{-\|y'\|^2} \,\mathrm{d}y = \varphi(x),
\end{aligned}$$

where $\star\star\star$ follows from the substitution $x - y = 2\sqrt{\varepsilon} y'$.

Finally, continuity of $\mathcal{F}^{-1}$ is easily shown with an argument analogous to that for continuity of $\mathcal{F}$ (????). $\square$