

Distribution Theory and Applications — Summary

Lucas Riedstra

November 7, 2020

Contents

1	Distributions	2
1.1	Test functions and distributions	2
1.2	Limits in the distribution space	3
1.3	Basic operations	4
1.3.1	Differentiation and multiplication by smooth functions	4
1.3.2	Reflection and translation	4
1.3.3	Convolution	5
1.4	Density of test functions in distribution space	6
2	Distributions with compact support	8
2.1	Test functions and distributions	8
2.2	Convolution between distributions	9
3	Tempered distributions and Fourier analysis	10
3.1	Functions of rapid decay	10
3.2	The Fourier transform on Schwartz functions	10
3.3	The Fourier transform on tempered distributions	12
3.4	Sobolev spaces	13
4	Applications of Fourier transform	14
4.1	Elliptic regularity	14
4.2	Fundamental solutions	16

1 Distributions

1.1 Test functions and distributions

Definition 1.1. Let $X \subseteq \mathbb{R}^n$ be open, then we define the set of *test functions* on X as

$$\mathcal{D}(X) := C_0^\infty(X) = \{f: X \rightarrow \mathbb{C} \mid f \text{ is smooth with compact support}\}.$$

Definition 1.2. Let $(\varphi_m) \subseteq \mathcal{D}(X)$. We say that $(\varphi_m) \rightarrow 0$ in $\mathcal{D}(X)$ if

1. there exists a compact $K \subseteq X$ such that $\text{supp } \varphi_m \subseteq K$ for all m ;
2. $\partial^\alpha \varphi_m \rightarrow 0$ uniformly for each multi-index α .

Note that, for any $\varphi, \psi \in \mathcal{D}(X)$ and any multi-index α we have

$$\int_X \varphi \cdot \partial^\alpha \psi \, dx = (-1)^{|\alpha|} \int_X \psi \cdot \partial^\alpha \varphi \, dx,$$

which follows from partial integration and the fact that all boundary terms vanish since φ and ψ have compact support.

Also, by Taylor's theorem, for any $\varphi \in \mathcal{D}(X)$, $x, h \in X$ and $N \in \mathbb{N}$ we have

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h) \quad \text{where } R_N(x, h) = o(|h|^N) \text{ uniformly in } x.$$

Definition 1.3. A *distribution* on X is a linear map $u: \mathcal{D}(X) \rightarrow \mathbb{C}$ if for every compact set $K \subseteq X$ there exist constants C, N such that for all $\varphi \in \mathcal{D}(X)$ with $\text{supp } \varphi \subseteq K$ we have

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi|. \quad (1)$$

Condition 1 is called the *seminorm condition*. If, in the seminorm condition, the same N can be used for every compact set $K \subseteq X$, then the least such N is called the *order* of u , written $\text{ord}(u)$.

The set of all distributions in X is denoted $\mathcal{D}'(X)$.

If $u \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}(X)$, then instead of $u(\varphi)$ we usually write $\langle u, \varphi \rangle$.

Recap 1.4. A function $f: X \rightarrow \mathbb{C}$ is called *locally integrable* if $\int_K |f| \, dx < \infty$ for all compact $K \subseteq X$.

The set of locally integrable functions on X is denoted $L_{\text{loc}}^1(X)$.

Example 1.5. Let $M \in \mathbb{N}$ and let $f_\alpha \in L_{\text{loc}}^1(X)$ for all $|\alpha| \leq M$. Define the linear map $T: \mathcal{D}(X) \rightarrow \mathbb{C}$ by

$$\langle T, \varphi \rangle := \sum_{|\alpha| \leq M} \int_X f_\alpha \cdot \partial^\alpha \varphi \, dx.$$

It is trivial that T is linear, and we verify that T is a distribution as follows: take $\varphi \in \mathcal{D}(X)$ with $\text{supp } \varphi \subseteq K$. Then we have

$$\begin{aligned} |\langle T, \varphi \rangle| &\leq \sum_{|\alpha| \leq M} \int_K |f_\alpha| \cdot |\partial^\alpha \varphi| \, dx \\ &\leq \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi| \cdot \int_K |f_\alpha| \, dx \\ &\leq \left(\max_\alpha \int_K |f_\alpha| \, dx \right) \sum_{|\alpha| \leq M} \sup |\partial^\alpha \varphi|. \end{aligned}$$

Therefore, the seminorm condition is satisfied with $N = M$. From this, it also follows that $\text{ord}(T) \leq M$.

A special case of the previous example is the case $M = 0$: in this case the distribution simply becomes

$$\langle \tau_f, \varphi \rangle = \int_X f \varphi \, dx.$$

Henceforth we will abuse notation: if $f \in L^1_{\text{loc}}(X)$, then we will write f instead of τ_f , i.e., $\langle f, \varphi \rangle = \int_X f \varphi \, dx$.

Lemma 1.6 (Sequential continuity). *Let $u: \mathcal{D}(X) \rightarrow \mathbb{C}$ be a linear map. Then u is a distribution if and only if, for every sequence $(\varphi_m) \subseteq \mathcal{D}(X)$ with $\varphi_m \rightarrow 0$ as in definition 1.2, we have $\langle u, \varphi_m \rangle \rightarrow 0$.*

Proof. ‘ \implies ’ If u is a distribution and $(\varphi_m) \rightarrow 0$, then $\text{supp } \varphi_m \subseteq K$ for some compact K , and by eq. (1) there exist C, N such that

$$|\langle u, \varphi_m \rangle| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \varphi_m| \rightarrow 0.$$

‘ \impliedby ’ Suppose there is a compact set K such that eq. (1) is not valid for any C, N . Let $m \in \mathbb{N}$ and $C = N = m$, then there is some φ_m with $\text{supp } (\varphi_m) \subseteq K$, and

$$|\langle u, \varphi_m \rangle| > m \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m|.$$

By dividing φ_m by $\langle u, \varphi_m \rangle \neq 0$, we may assume w.l.o.g. that $\langle u, \varphi_m \rangle = 1$. We now have a sequence (φ_m) such that

$$\frac{1}{m} > \sum_{|\alpha| \leq m} \sup |\partial^\alpha \varphi_m| \implies |\partial^\alpha \varphi_m| < \frac{1}{m} \quad \text{for } |\alpha| \leq m \implies \partial^\alpha \varphi_m \rightarrow 0 \text{ uniformly for all } \alpha.$$

Since each φ_m also satisfies $\text{supp } \varphi_m \subseteq K$, by definition 1.2 we have that $\varphi_m \rightarrow 0$, but also $\langle u, \varphi_m \rangle \rightarrow 1$, a contradiction. \square

1.2 Limits in the distribution space

Definition 1.7. We say that a sequence $(u_m) \subseteq \mathcal{D}'(X)$ converges to $u \in \mathcal{D}'(X)$ and write $u_m \rightarrow u$ if

$$\langle u_m, \varphi \rangle \rightarrow \langle u, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(X).$$

The following theorem is non-examinable but interesting:

Theorem 1.8. *Let (u_m) be a sequence in $\mathcal{D}'(X)$ such that $\lim_{m \rightarrow \infty} \langle u_m, \varphi \rangle$ exists for all $\varphi \in \mathcal{D}(X)$. Then the map $\langle u, \varphi \rangle := \lim_{m \rightarrow \infty} \langle u_m, \varphi \rangle$ is a distribution in X .*

Proof. This is a direct application of the uniform boundedness principle. \square

Example 1.9. Let $X = \mathbb{R}$ and consider the sequence of functions $u_m \in L^1_{\text{loc}}(\mathbb{R})$ defined by $u_m(x) = \sin(mx)$. Then, for all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle u_m, \varphi \rangle = \int_{\mathbb{R}} \sin(mx) \varphi(x) \, dx = \frac{1}{m} \int_{\mathbb{R}} \cos(mx) \varphi'(x) \, dx \leq \frac{1}{m} \int_{\mathbb{R}} |\varphi'(x)| \, dx \rightarrow 0.$$

Therefore, it holds that $u_m \rightarrow 0$ in $\mathcal{D}'(\mathbb{R})$. With our abuse of notation we write this as $\lim_{m \rightarrow \infty} \sin(mx) = 0$ in $\mathcal{D}'(\mathbb{R})$.

1.3 Basic operations

1.3.1 Differentiation and multiplication by smooth functions

For $u \in C^\infty(X)$ and $\varphi \in \mathcal{D}(X)$, we have noted that

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \partial^\alpha u \cdot \varphi \, dx = (-1)^{|\alpha|} \int_X u \cdot \partial^\alpha \varphi \, dx = \langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle.$$

Since the RHS makes sense for any distribution u , we define

Definition 1.10. For $f \in C^\infty(X)$, $u \in \mathcal{D}'(X)$, we define $\partial^\alpha(fu)$ by

$$\langle \partial^\alpha(fu), \varphi \rangle := \langle u, (-1)^{|\alpha|} f \cdot \partial^\alpha \varphi \rangle$$

Remark. This definition outlines a more general pattern when working with distributions: first we take some well-defined operator on the collection of smooth maps, then we rewrite it to a form that is sensible for any distribution, and then we *define* that new form as the operator on distributions. This process is called *extending the definition by duality*.

Example 1.11. Let $u = \delta_x$, then we have

$$\langle \partial^\alpha \delta_x, \varphi \rangle = \langle \delta_x, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x)$$

Furthermore, consider the Heaviside function $H(x) = \mathbb{1}_{x \geq 0}$. We have

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{\mathbb{R}} H(x) \varphi'(x) \, dx = -\int_0^\infty \varphi'(x) \, dx = \varphi(0) = \langle \delta_0, \varphi \rangle,$$

so we write $H' = \delta_0$ in the distributional sense.

Lemma 1.12. Suppose $u' \in \mathcal{D}'(\mathbb{R})$ satisfies $u' = 0$. Then u is constant (i.e., $\langle u, \varphi \rangle = \langle c, \varphi \rangle = c \int_{\mathbb{R}} \varphi \, dx$ for some c).

Proof. Fix any $\vartheta \in \mathcal{D}(\mathbb{R})$ with $\langle 1, \vartheta \rangle = 1$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and define

$$\varphi_A := \varphi - \langle 1, \varphi \rangle \vartheta, \quad \varphi_B := \langle 1, \varphi \rangle \vartheta \quad \text{such that } \varphi = \varphi_A + \varphi_B.$$

Note that $\langle 1, \varphi_A \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle \langle 1, \vartheta \rangle = 0$.

We claim that the function $\Phi_A(x) := \int_{-\infty}^x \varphi_A(y) \, dy$ has compact support: since $\text{supp } \varphi_A \subseteq [a, b]$ for some $a, b \in \mathbb{R}$, clearly $\Phi_A(x) = 0$ for $x < a$, while for $x > b$ we have $\Phi_A(x) = \langle 1, \varphi_A \rangle = 0$. Obviously, it holds that $\Phi'_A = \varphi_A$. Now we compute

$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle = \langle u, \Phi'_A \rangle + \langle 1, \varphi \rangle \langle u, \vartheta \rangle = -\langle u', \Phi_A \rangle + c \langle 1, \varphi \rangle = \langle c, \varphi \rangle.$$

Since φ was chosen arbitrarily this shows that u is constant. □

1.3.2 Reflection and translation

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, define the *translation of φ by h* by

$$(\tau_h \varphi)(x) := \varphi(x - h),$$

and the *reflection of φ* by $\check{\varphi}(x) := \varphi(-x)$.

Extending the definitions of translation and reflection by duality yields the following:

Definition 1.13. For $u \in \mathcal{D}'(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, define

$$\langle \tau_h u, \varphi \rangle := \langle u, \tau_{-h} \varphi \rangle \quad \text{and} \quad \langle \check{u}, \varphi \rangle := \langle u, \check{\varphi} \rangle.$$

Lemma 1.14. For $u \in \mathcal{D}'(\mathbb{R}^n)$, define $V_h \in \mathcal{D}'(\mathbb{R}^n)$ for $0 \neq h \in \mathbb{R}^n$ by

$$V_h := \frac{\tau_{-h}u - u}{\|h\|}$$

If $(h_j) \subseteq \mathbb{R}^n$ is a sequence for which $\lim_{j \rightarrow \infty} \frac{h_j}{\|h_j\|} = m \in S^{n-1}$, then $V_{h_j} \rightarrow \sum_i m_i \frac{\partial u}{\partial x_i}$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. By definition, we can write $\langle V_h, \varphi \rangle = \frac{1}{\|h\|} \langle u, \tau_h \varphi - \varphi \rangle$. Now Taylor's theorem tells us that

$$(\tau_h \varphi - \varphi)(x) = \varphi(x - h) - \varphi(x) = - \sum_i h_i \frac{\partial \varphi}{\partial x_i} + R(x, h),$$

where $R(x, h) = o(\|h\|)$ in $D(\mathbb{R}^n)$ (exercise sheet 1, question 2).

By sequential continuity, we have

$$\lim_{j \rightarrow \infty} \langle V_{h_j}, \varphi \rangle = \langle u, - \sum_i m_i \frac{\partial \varphi}{\partial x_i} \rangle = \langle \sum_i m_i \frac{\partial u}{\partial x_i}, \varphi \rangle,$$

which shows that $V_{h_j} \rightarrow \sum_i m_i \frac{\partial u}{\partial x_i}$ in $\mathcal{D}'(\mathbb{R}^n)$. □

1.3.3 Convolution

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, note that $(\tau_x \check{\varphi})(y) = \check{\varphi}(y - x) = \varphi(x - y)$.

Definition 1.15. For $u \in C^\infty(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the *convolution* $u * \varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ as

$$(u * \varphi)(x) := \int_{\mathbb{R}^n} u(x - y) \varphi(y) dy = \int_{\mathbb{R}^n} u(y) \varphi(x - y) dy = \langle u, \tau_x \check{\varphi} \rangle.$$

Since the RHS makes sense for any $u \in \mathcal{D}'(\mathbb{R}^n)$, we extend the definition this way: for $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the convolution $u * \varphi$ as

$$(u * \varphi)(x) := \langle u, \tau_x \check{\varphi} \rangle.$$

Lemma 1.16. Let $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and define $\Phi_x(y) := \varphi(x, y)$. Suppose for any $x \in \mathbb{R}^n$ there exists a neighbourhood $N(x)$ and a compact $K \subseteq \mathbb{R}^m$ such that $\varphi(x, y)$ for all $x \in N(x)$, $y \notin K$.

Then $x \mapsto \langle u, \Phi_x \rangle$ is differentiable with

$$\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi_x \rangle$$

for any $u \in \mathcal{D}'(\mathbb{R}^m)$.

Proof. Fix $x \in \mathbb{R}^n$, then by Taylor's formula we have

$$\Phi_{x+h}(y) = \Phi_x(y) + \sum_{i=1}^n h_i \frac{\partial \varphi}{\partial x_i} + R(x, y, h),$$

where $\partial_y^\alpha R(x, y, h) = o(\|h\|)$, uniformly in y , for any multi-index α . Furthermore, by assumption there exists a compact K such that for h small enough, $\text{supp } R(x, \cdot, h) \subseteq K$. Therefore, $R(x, \cdot, h)$ is a test function for h small enough.

Combining the previous two facts shows that $R(x, \cdot, h) = o(\|h\|)$ in $\mathcal{D}(\mathbb{R}^m)$ as $h \rightarrow 0$.

Let $u \in \mathcal{D}'(\mathbb{R}^m)$, then we find by sequential continuity that $\langle u, R(x, \cdot, h) \rangle$ is also $o(\|h\|)$, and therefore

$$\langle u, \Phi_{x+h} \rangle = \langle u, \Phi_x \rangle + \sum_{i=1}^n h_i \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle + o(\|h\|).$$

This shows that $x \mapsto \langle u, \Phi_x \rangle$ is differentiable with

$$\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle.$$

From this the result follows. □

Corollary 1.17. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $u * \varphi$ is differentiable with $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi$.*

Proof. Apply the previous lemma with $\Phi_x(y) := \varphi(x - y)$. \square

Due to the previous corollary, we often call $u * \varphi$ a *regularisation* of u .

Convention. If $u \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}(X)$, then instead of $\langle u, \varphi \rangle$ we also write $\langle u(t), \varphi(t) \rangle$ (or with any other dummy variable) when the variable used for φ is not directly clear.

1.4 Density of test functions in distribution space

Lemma 1.18. *If $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, then*

$$(u * \varphi) * \psi = u * (\varphi * \psi).$$

Proof. Fix $x \in \mathbb{R}^n$. Now we write

$$\begin{aligned} ((u * \varphi) * \psi)(x) &= \int_{\mathbb{R}^n} \langle u(z), (\tau_y \check{\varphi})(z) \rangle \cdot \psi(x - y) \, dy \\ &= \int_{\mathbb{R}^n} \langle u(z), (\tau_{x-y} \check{\varphi})(z) \rangle \cdot \psi(y) \, dy \\ &= \int_{\mathbb{R}^n} \langle u(z), \psi(y) (\tau_{x-y} \check{\varphi})(z) \rangle \, dy. \end{aligned}$$

We would like to interchange integral and application of u , and we will have to justify this using Riemann sums:

$$\begin{aligned} \int_{\mathbb{R}^n} \langle u(z), \psi(y) (\tau_{x-y} \check{\varphi})(z) \rangle \, dy &= \lim_{\varepsilon \downarrow 0} \sum_{m \in \mathbb{Z}^n} \langle u(z), \psi(\varepsilon m) \varphi(x - z - \varepsilon m) \rangle \varepsilon^n \\ &\stackrel{*}{=} \lim_{\varepsilon \downarrow 0} \langle u(z), \sum_{m \in \mathbb{Z}^n} \psi(\varepsilon m) \varphi(x - z - \varepsilon m) \varepsilon^n \rangle \\ &\stackrel{**}{=} \left\langle u(z), \int_{\mathbb{R}^n} \psi(y) \varphi(x - z - y) \, dy \right\rangle \\ &= \langle u(z), (\varphi * \psi)(x - z) \rangle = \langle u(z), \widetilde{(\tau_x \varphi * \psi)}(z) \rangle = (u * (\varphi * \psi))(x). \end{aligned}$$

Here, $*$ is by the fact that the sum is finite since ψ has compact support, while $**$ is by sequential continuity of u and the fact that the Riemann sum converges to the convolution integral *in the space of test functions* (non-examinable fact). \square

We will use the following trick many times:

Proposition 1.19. *For any $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $\langle u, \varphi \rangle = (u * \check{\varphi})(0)$.*

Proof. We have $(u * \check{\varphi})(0) = \langle u, \tau_0 \varphi \rangle = \langle u, \varphi \rangle$. \square

For example, from this trick it follows that if $u * \varphi = 0$ for all φ , then $u = 0$.

Theorem 1.20. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, there exists a sequence $(\varphi_k) \subseteq \mathcal{D}(\mathbb{R}^n)$ such that $\varphi_k \rightarrow u$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi \, dx = 1$, and set $\psi_k(x) := k^n \psi(kx)$. Note that $\int_{\mathbb{R}^n} \psi_k \, dx = 1$.

Now, fix any $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi \equiv 1$ on $\{\|x\| < 1\}$ and $\chi \equiv 0$ on $\{\|x\| < 2\}$. Define $\chi_k(x) := \chi(x/k)$, so that $\lim_{k \rightarrow \infty} \chi_k(x) = 1$ for all x . We will set

$$\varphi_k(x) := \chi_k(x)(u * \psi_k)(x).$$

Clearly we have $\varphi_k \in \mathcal{D}(\mathbb{R}^n)$ since each χ_k has compact support.

Now, take any $\vartheta \in \mathcal{D}(\mathbb{R}^n)$, then we have

$$\begin{aligned}\langle \varphi_k, \vartheta \rangle &= \langle \chi_k(u * \psi_k), \vartheta \rangle = \langle u * \psi_k, \chi_k \vartheta \rangle = \left[(u * \psi_k) * \widetilde{\chi_k \vartheta} \right](0) \\ &= \left[u * (\psi_k * \widetilde{\chi_k \vartheta}) \right](0).\end{aligned}$$

Now we compute $\psi_k * \widetilde{\chi_k \vartheta}$: note that

$$\begin{aligned}(\psi_k * \widetilde{\chi_k \vartheta})(x) &= \int_{\mathbb{R}^n} k^n \psi(k(x-y)) \chi\left(-\frac{y}{k}\right) \vartheta(-y) \, dy \\ &= \int_{\mathbb{R}^n} \psi(y) \chi\left(\frac{y}{k^2} - \frac{x}{k}\right) \vartheta\left(\frac{y}{k} - x\right) \, dy \\ &= \vartheta(-x) + R_k(-x) = (\vartheta + \widetilde{R_k})(x)\end{aligned}$$

where $R_k(x) = \int_{\mathbb{R}^n} \psi(y) \left[\chi\left(\frac{y}{k^2} + \frac{x}{k}\right) \vartheta\left(\frac{y}{k} + x\right) - \vartheta(x) \right] \, dy$.

So

$$\langle \varphi_k, \vartheta \rangle = (u * (\vartheta + \widetilde{R_k}))(0) = (u * \check{\vartheta})(0) + (u * \widetilde{R_k})(0) = \langle u, \vartheta \rangle + \langle u, R_k \rangle.$$

We must now only prove that $R_k \rightarrow 0$ in $\mathcal{D}(\mathbb{R}^n)$, and then by sequential continuity it follows that $\varphi_k \rightarrow u$ in $\mathcal{D}'(\mathbb{R}^n)$. \square

2 Distributions with compact support

Definition 2.1. Let $Y \subseteq X$ be open and $u \in \mathcal{D}'(X)$. We say that u *vanishes* on Y if $\langle u, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(Y)$.

Definition 2.2. For $u \in \mathcal{D}'(X)$, we define the *support* of u as

$$\text{supp } u := X \setminus \bigcup \{Y \subseteq X \mid Y \text{ open, } u \text{ vanishes on } Y\}.$$

For example, the support of δ_x is simply $\{x\}$.

2.1 Test functions and distributions

We will now consider a bigger space of test functions, which yields a smaller space of distributions.

Definition 2.3. We define $\mathcal{E}(X)$ as the space of smooth functions $\varphi: X \rightarrow \mathbb{C}$. We say that a sequence $(\varphi_m) \subseteq \mathcal{E}(X)$ converges to 0 if $\partial^\alpha \varphi \rightarrow 0$ uniformly on compact subsets of X for every multi-index α .

Definition 2.4. We define $\mathcal{E}'(X)$ as the space of linear maps $u: \mathcal{E}(X) \rightarrow \mathbb{C}$ for which there exists a compact $K \subseteq X$ and nonnegative constants C, N such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \quad (2)$$

for all $\varphi \in \mathcal{E}(X)$.

Lemma 2.5 (Sequential continuity). *Let $u: \mathcal{E}(X) \rightarrow \mathbb{C}$ be a linear map. Then $u \in \mathcal{E}'(X)$ if and only if, for every sequence $(\varphi_m) \subseteq \mathcal{E}(X)$ with $\varphi_m \rightarrow 0$, we have $\langle u, \varphi_m \rangle \rightarrow 0$.*

Proof. **TODO:** □

Lemma 2.6. *If $u \in \mathcal{E}'(X)$, then $u|_{\mathcal{D}(X)}$ defines an element of $\mathcal{D}'(X)$ with compact support and finite order.*

Conversely, for each $u \in \mathcal{D}'(X)$ with compact support there exists a unique extension $\tilde{u} \in \mathcal{E}'(X)$ with $\text{supp}(\tilde{u}) = \text{supp}(u)$ and $\tilde{u}|_{\mathcal{D}(X)} = u$.

Proof. Let $u \in \mathcal{E}'(X)$, so that there exists a compact $K \subseteq X$ with $|\langle u, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$. Now, for any compact $K' \subseteq X$ and any φ with $\text{supp } \varphi \subseteq K'$, eq. (1) is clearly satisfied, and we can use the same N for all compact K' , so clearly $u|_{\mathcal{D}(X)}$ is an element of $\mathcal{D}'(X)$ with finite order. Finally, suppose φ is supported in $X \setminus K$, then it is clear that $\langle u, \varphi \rangle = 0$, which proves that $\text{supp } u \subseteq K$ and therefore that u has compact support.

Now suppose $u \in \mathcal{D}'(X)$ has compact support, let $\rho \in \mathcal{D}(X)$ be 1 in a neighbourhood of $\text{supp } u$, and define $\tilde{u} \in \mathcal{E}'(X)$ by

$$\langle \tilde{u}, \varphi \rangle := \langle u, \rho \varphi \rangle \quad \forall \varphi \in \mathcal{E}(X).$$

Clearly \tilde{u} is an element of $\mathcal{E}'(X)$ since $\text{supp}(\rho \varphi) \subseteq \text{supp } \rho$ and

$$|\langle \tilde{u}, \varphi \rangle| = |\langle u, \rho \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{\text{supp}(\rho)} |\partial^\alpha (\rho \varphi)| \stackrel{*}{\leq} C' \sum_{|\alpha| \leq N} \sup_{\text{supp } \rho} |\partial^\alpha \varphi|,$$

where \star follows from the Leibniz rule. It is also clear that $\text{supp } \tilde{u} = \text{supp } u$.

Finally we will show uniqueness: suppose \tilde{v} is an extension of u with $\text{supp } \tilde{v} = \text{supp } u$, and write any $\varphi \in \mathcal{E}(X)$ as $\varphi = \rho \varphi + (1 - \rho) \varphi = \varphi_0 + \varphi_1$. Then since $\varphi_0 \in \mathcal{D}(X)$ and φ_1 vanishes on a neighbourhood of $\text{supp } u$, we find

$$\langle \tilde{v}, \varphi \rangle = \langle \tilde{v}, \varphi_0 + \varphi_1 \rangle = \langle \tilde{v}, \varphi_0 \rangle = \langle u, \varphi_0 \rangle,$$

which is independent of choice of extension. □

2.2 Convolution between distributions

Definition 2.7. Define for $u \in \mathcal{E}'(\mathbb{R}^n)$, $\varphi \in \mathcal{E}(\mathbb{R}^n)$ the *convolution*

$$(u * \varphi)(x) = \langle u, \tau_x \check{\varphi} \rangle.$$

This convolution satisfies the same properties as the convolution between $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$. Also, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $u * \varphi \in \mathcal{D}(\mathbb{R}^n)$.

Definition 2.8. Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of which has compact support, define $u * v: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{E}(\mathbb{R}^n)$ by

$$(u * v) * \varphi = u * (v * \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We will show (example sheet 2, question 1) that $u * v$ is uniquely defined and gives rise to an element of $\mathcal{D}'(\mathbb{R}^n)$ via $\langle u * v, \varphi \rangle := [(u * v) * \check{\varphi}](0)$.

Lemma 2.9. *Given $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of which has compact support, we have $u * v = v * u$.*

Proof. First we note that $(u * \varphi) * \psi = u * (\varphi * \psi)$ holds if u has compact support and at least one of φ, ψ has compact support.

Fix $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, we see from our earlier shown properties that

$$(u * v) * (\varphi * \psi) = u * (v * (\varphi * \psi)) = u * ((v * \varphi) * \psi) = u * (\psi * (v * \varphi)) = (u * \psi) * (v * \varphi).$$

If we interchange u and v in the above, that is equivalent to interchanging φ and ψ , which we know must yield the same result. This shows $u * v$ and $v * u$ agree on $\varphi * \psi$ for all $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$. Defining $E = u * v - v * u$, we find that $0 = E * (\varphi * \psi) = (E * \varphi) * \psi$ for all $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, so $E * \varphi = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, so $E = 0$. \square

3 Tempered distributions and Fourier analysis

3.1 Functions of rapid decay

Definition 3.1. For any $f: \mathbb{R}^n \rightarrow \mathbb{C}$ and multi-indices α, β we define $\|f\|_{\alpha, \beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f|$.

We define the *Schwartz space*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha, \beta} < \infty \text{ for all } \alpha, \beta \right\}.$$

We say that a sequence $(\varphi_n) \subseteq \mathcal{S}(\mathbb{R}^n)$ converges to 0 if $\|\varphi_n\|_{\alpha, \beta} \rightarrow 0$ for every α, β .

Example 3.2. The function $x \mapsto \exp(-\|x\|^2)$ lies in $\mathcal{S}(\mathbb{R}^n)$.

Proposition 3.3. For all n we have that $\mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then for all $N \in \mathbb{N}$ we have

$$\int_{\mathbb{R}^n} |\varphi(x)| dx = \int_{\mathbb{R}^n} (1 + \|x\|)^{-N} (1 + \|x\|)^N |\varphi(x)| dx \stackrel{?}{\leq} C \sum_{|\alpha| \leq N} \|\varphi\|_{\alpha, 0} \int_{\mathbb{R}^n} (1 + \|x\|)^{-N} dx.$$

Since $\int_{\mathbb{R}^n} (1 + \|x\|)^{-N} dx$ is finite for N large enough (??), this proves the claim. \square

Definition 3.4. A linear map $u: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$ is called a *tempered distribution* if there exists constants C, N such that

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|, |\beta| \leq N} \|\varphi\|_{\alpha, \beta} \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

The space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

This definition is equivalent to sequential continuity.

3.2 The Fourier transform on Schwartz functions

Convention. We write $D := -i\partial$ and $D^\alpha = (-i)^{|\alpha|} \partial^\alpha$.

Definition 3.5. For $f \in L^1(\mathbb{R}^n)$, define the *Fourier transform* of f by

$$[\mathcal{F}(f)](\lambda) = \hat{f}(\lambda) := \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} f(x) dx \quad \text{where } \lambda \in \mathbb{R}^n.$$

Lemma 3.6. If $f \in L^1(\mathbb{R}^n)$, then \hat{f} is continuous.

Proof. If $\lambda_m \rightarrow \lambda \in \mathbb{R}^n$, then by the dominated convergence theorem we have

$$\hat{f}(\lambda_m) = \int_{\mathbb{R}^n} e^{-i\lambda_m \cdot x} f(x) dx \stackrel{\text{DCT}}{=} \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} f(x) dx = \hat{f}(\lambda),$$

where we were able to use the dominated convergence theorem since the integrand is absolutely bounded by $|f|$ and $f \in L^1$. \square

It turns out that this idea generalises: the faster the function f decays, the smoother the Fourier transform \hat{f} is.

Lemma 3.7. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have $\mathcal{F}[D_x^\alpha \varphi](\lambda) = \lambda^\alpha \hat{\varphi}(\lambda)$ and $\mathcal{F}[x^\beta \varphi](\lambda) = (-1)^{|\beta|} D_\lambda^\beta \hat{\varphi}(\lambda)$.

Proof. Since $|x^\alpha D^\beta \varphi| \rightarrow 0$ as $\|x\| \rightarrow \infty$, we have using integration by parts

$$\begin{aligned}\mathcal{F}[D_\lambda^\alpha \varphi](\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} D_x^\alpha \varphi(x) \, dx \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_x^\alpha (e^{-i\lambda \cdot x}) \varphi(x) \, dx \\ &= \lambda^\alpha \hat{\varphi}(\lambda).\end{aligned}$$

For the second part of the lemma, by differentiation under the integral sign we get

$$\begin{aligned}\mathcal{F}[x^\beta \varphi](\lambda) &= \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} x^\beta \varphi(x) \, dx \\ &= \int_{\mathbb{R}^n} ((-D_\lambda)^\beta e^{-i\lambda \cdot x}) \varphi(x) \, dx \\ &= (-1)^{|\beta|} D_\lambda^\beta \hat{\varphi}(\lambda).\end{aligned}$$

□

We define the *inverse Fourier transform* by

$$\mathcal{F}^{-1}[\hat{f}](x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \hat{f}(\lambda) \, d\lambda.$$

We will now show that on $\mathcal{S}(\mathbb{R}^n)$, the inverse Fourier transform is indeed an inverse:

Theorem 3.8. *The Fourier transform \mathcal{F} defines a continuous isomorphism from $\mathcal{S}(\mathbb{R}^n)$ to itself.*

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. First, we show that $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$: by the previous lemma we have for multi-indices α, β that

$$\begin{aligned}|\lambda^\alpha (-D_\lambda)^\beta \hat{\varphi}(\lambda)| &= |\lambda^\alpha \mathcal{F}[x^\beta \varphi](\lambda)| = |\mathcal{F}[D_x^\alpha (x^\beta \varphi)](\lambda)| = \left| \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} D_x^\alpha (x^\beta \varphi) \, dx \right| \\ &\leq \int_{\mathbb{R}^n} |D^\alpha (x^\beta \varphi)| \, dx,\end{aligned}\tag{3}$$

which is finite since $D^\alpha (x^\beta \varphi)$ is also a Schwartz function and therefore integrable.

From the previous lemma we also infer that $\hat{\varphi}$ is smooth, so indeed we have $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. From eq. (3) it is also easily seen that if $\varphi_m \rightarrow 0$ in $\mathcal{S}(\mathbb{R}^n)$, then $\hat{\varphi}_m \rightarrow 0$ in $\mathcal{S}(\mathbb{R}^n)$ also, which shows that \mathcal{F} is continuous.

To prove surjectivity and injectivity, we will show that $\mathcal{F}^{-1}[\mathcal{F}[f]](x) = f(x)$ (???). Indeed we have

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y)} \varphi(y) \, dy \, d\lambda \\ &= \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y) - \varepsilon \|\lambda\|^2} \varphi(y) \, dy \, d\lambda \\ &\stackrel{\star}{=} \frac{1}{(2\pi)^n} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y) - \varepsilon \|\lambda\|^2} \, d\lambda \, dy,\end{aligned}$$

where \star follows from Fubini's theorem.

Now we note that

$$\int_{\mathbb{R}^n} e^{i\lambda \cdot (x-y) - \varepsilon \|\lambda\|^2} \, d\lambda = \prod_{j=1}^n \int_{\mathbb{R}} e^{i\lambda_j (x_j - y_j) - \varepsilon \lambda_j^2} \, d\lambda_j \stackrel{\star\star}{=} \prod_{j=1}^n \left(\frac{\pi}{\varepsilon} \right)^{1/2} e^{-\frac{(x_j - y_j)^2}{4\varepsilon}} = \left(\frac{\pi}{\varepsilon} \right)^{n/2} e^{-\frac{\|x-y\|^2}{4\varepsilon}}.$$

To explain **, **TODO:** .

and plugging that into the above yields

$$\begin{aligned}\mathcal{F}^{-1}[\mathcal{F}[\varphi]](x) &= \lim_{\varepsilon \downarrow 0} 2^{-n} (\pi\varepsilon)^{-n/2} \int_{\mathbb{R}^n} \varphi(y) e^{-\|x-y\|^2/(4\varepsilon)} dy \\ &\stackrel{***}{=} \pi^{-n/2} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^n} \varphi(x - 2\sqrt{\varepsilon}y') e^{-\|y'\|^2} dy \\ &\stackrel{\text{DCT}}{=} \pi^{-n/2} \varphi(x) \int_{\mathbb{R}^n} e^{-\|y'\|^2} dy = \varphi(x),\end{aligned}$$

where *** follows from the substitution $x - y = 2\sqrt{\varepsilon}y'$.

Finally, continuity of \mathcal{F}^{-1} is easily shown with an argument analogous to that for continuity of \mathcal{F} (???). \square

3.3 The Fourier transform on tempered distributions

Lemma 3.9. For $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^n} \varphi(x) \hat{\psi}(x) dx = \int_{\mathbb{R}^n} \hat{\varphi}(x) \psi(x) dx$.

Proof. This follows from Fubini's theorem:

$$\int_{\mathbb{R}^n} \varphi(x) \hat{\psi}(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(x) \psi(\lambda) e^{-i\lambda \cdot x} d\lambda dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi(\lambda) \varphi(x) e^{-i\lambda \cdot x} dx d\lambda = \int_{\mathbb{R}^n} \psi(\lambda) \hat{\varphi}(\lambda) d\lambda.$$

\square

The above result can be rewritten as $\langle \hat{\varphi}, \psi \rangle = \langle \varphi, \hat{\psi} \rangle$, which motivates the definition of the Fourier transform for tempered distributions:

Definition 3.10. For $u \in \mathcal{S}'(\mathbb{R}^n)$, we define its Fourier transform by

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Using sequential continuity and theorem 3.8, it is easily seen that \hat{u} is indeed an element of $\mathcal{S}'(\mathbb{R}^n)$.

Example 3.11. It is easily checked that $\delta_0 \in \mathcal{S}'(\mathbb{R}^n)$, and we can compute

$$\langle \hat{\delta}_0, \varphi \rangle = \langle \delta_0, \hat{\varphi} \rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x) dx = \langle 1, \varphi \rangle,$$

so we can write $\hat{\delta}_0 = 1$. Analogously, by the Fourier inversion theorem we have

$$\langle \hat{1}, \varphi \rangle = \langle 1, \hat{\varphi} \rangle = \int_{\mathbb{R}^n} \hat{\varphi}(\lambda) d\lambda = (2\pi)^n \varphi(0) = \langle (2\pi)^n \delta_0, \varphi \rangle,$$

so we write $\hat{1} = (2\pi)^n \delta_0$.

We can easily generalise lemma 3.7 to the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, so

$$\mathcal{F}[D^\alpha u] = \lambda^\alpha \hat{u}, \quad \mathcal{F}[x^\beta u] = (-D^\beta) \hat{u}.$$

Theorem 3.12. The Fourier transform \mathcal{F} extends to a continuous isomorphism on $\mathcal{S}'(\mathbb{R}^n)$.

Proof. We claim that $\check{u} = (2\pi)^{-n} \mathcal{F}[\mathcal{F}[u]]$. To check this, note that by the Fourier inversion theorem we have for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ that

$$\check{\varphi}(x) = \varphi(-x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\lambda \cdot x} \hat{\varphi}(\lambda) d\lambda = (2\pi)^{-n} \mathcal{F}[\mathcal{F}[\varphi]](x),$$

and therefore

$$\langle \check{u}, \varphi \rangle = \langle u, \check{\varphi} \rangle = \langle u, (2\pi)^{-n} \mathcal{F}[\mathcal{F}[\varphi]] \rangle = \langle (2\pi)^{-n} \mathcal{F}[\mathcal{F}[u]], \varphi \rangle.$$

This shows that \mathcal{F} is bijective (since $\mathcal{F} \circ \mathcal{F}$ is bijective). For continuity of \mathcal{F} and its inverse: using theorem 3.8, we find

$$\begin{aligned} u_m &\rightarrow 0 \text{ in } \mathcal{S}'(\mathbb{R}^n) \\ \iff \langle u_m, \varphi \rangle &\rightarrow 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \\ \iff \langle u_m, \hat{\varphi} \rangle &\rightarrow 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \\ \iff \langle \hat{u}_m, \varphi \rangle &\rightarrow 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \\ \iff \hat{u}_m &\rightarrow 0 \text{ in } \mathcal{S}'(\mathbb{R}^n). \end{aligned}$$

□

3.4 Sobolev spaces

Convention. We write $\langle \lambda \rangle := (1 + \|\lambda\|^2)^{1/2}$ for $\lambda \in \mathbb{R}^n$. Note that $\langle \lambda \rangle \sim 1$ as $\|\lambda\| \rightarrow 0$ and $\langle \lambda \rangle \rightarrow \|\lambda\|$ as $\|\lambda\| \rightarrow \infty$.

Definition 3.13. For $s \in \mathbb{R}$, define the *Sobolev space* $H^s(\mathbb{R}^n)$ to be the set of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ for which \hat{u} can be identified with a measurable function $\hat{u}: \mathbb{R}^n \rightarrow \mathbb{C}$ such that $\langle \lambda \rangle^s \hat{u} \in L^2(\mathbb{R}^n)$.

For $X \subseteq \mathbb{R}^n$ open, we define the *localised Sobolev space* $H_{\text{loc}}^s(X)$ by setting

$$u \in H_{\text{loc}}^s(X) \iff \varphi u \in H^s(\mathbb{R}^n) \text{ for all } \varphi \in \mathcal{D}(X).$$

Lemma 3.14 (Sobolev lemma). *If $u \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$, then u is continuous.*

Proof. We will show that \hat{u} is integrable. By Cauchy-Schwarz, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{u}(\lambda)| \, d\lambda &= \int_{\mathbb{R}^n} \langle \lambda \rangle^{-s} \langle \lambda \rangle^s |\hat{u}(\lambda)| \, d\lambda \\ &\leq \left(\int_{\mathbb{R}^n} \langle \lambda \rangle^{-2s} \, d\lambda \right)^{1/2} \|u\|_{H^s} \\ &= C \|u\|_{H^s} \left(\int_0^\infty r^{n-1} (1+r^2)^{-s} \, dr \right)^{1/2}, \end{aligned}$$

where the last line follows from using polar coordinates and C is the area of the $(n-1)$ -sphere.

Writing $s = \frac{n}{2} + \varepsilon$, we find that the integrand $r^{n-1} (1+r^2)^{-s}$ is of order $O(r^{-1-2\varepsilon})$ as $r \rightarrow \infty$, and therefore the integral is finite, so indeed we have $\hat{u} \in L^1(\mathbb{R}^n)$.

By applying theorem 3.8 to a test function, we can show that $u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\lambda \cdot x} \hat{u}(\lambda) \, d\lambda$, which is continuous by the dominated convergence theorem. □

Corollary 3.15. *If $u \in H^s(\mathbb{R}^n)$ for every $s > n/2$, then $u \in C^\infty(\mathbb{R}^n)$.*

4 Applications of Fourier transform

4.1 Elliptic regularity

Recall that $D = -i\partial$. If p is an N -th order polynomial, then $p(D)$ is called an N -th order differential operator.

Definition 4.1. For an N -th order differential operator $p(D) = \sum_{|\alpha| \leq N} c_\alpha D^\alpha$, define its *principal symbol* $\sigma_p(\lambda)$ by

$$\sigma_p(\lambda) := \sum_{|\alpha|=N} c_\alpha \lambda^\alpha \quad (\lambda \in \mathbb{R}^n).$$

The operator $p(D)$ is called *elliptic* if $\sigma_p(\lambda) \neq 0$ for $\lambda \neq 0$.

Lemma 4.2. If $p(D)$ is an N -th order elliptic partial differential operator, then there exist $R > 0$ such that, $C > 0$ such that

$$|p(\lambda)| \geq C \langle \lambda \rangle^N \quad \text{if } \|\lambda\| > R.$$

Proof. Let $C_0 > 0$ be the minimum of $|\sigma_p|$ on S^{n-1} , then for $\lambda \neq 0$ we have

$$|\sigma_p(\lambda)| = \left| \sum_{|\alpha|=N} c_\alpha \lambda^\alpha \right| = \|\lambda\|^N |\sigma_p(\lambda/\|\lambda\|)| \geq \|\lambda\|^N C_0.$$

By the triangle inequality we find

$$|p(\lambda)| \geq |\sigma_p(\lambda)| - |p(\lambda) - \sigma_p(\lambda)| \geq \left[C_0 - \left| \frac{p(\lambda) - \sigma_p(\lambda)}{\|\lambda\|^N} \right| \right] \|\lambda\|^N$$

Since $p - \sigma_p$ is a polynomial of order $N - 1$, we can choose R sufficiently large s.t. $\left| \frac{p(\lambda) - \sigma_p(\lambda)}{\|\lambda\|^N} \right| < C_0/2$. Since $\langle \lambda \rangle \sim \|\lambda\|$ for λ large enough, we find that there exists C such that

$$|p(\lambda)| \geq \frac{C_0}{2} \|\lambda\|^N \geq C \langle \lambda \rangle^N$$

for $\|\lambda\| > R$. □

We will try to prove the *elliptic regularity theorem*:

Theorem 4.3 (Elliptic regularity). Suppose $p(D)$ is an N -th order elliptic partial differential operator and $u \in \mathcal{D}'(X)$ satisfies $p(D)u \in H_{\text{loc}}^s(X)$, then $u \in H_{\text{loc}}^{s+N}(X)$.

Corollary 4.4. If $p(D)$ is N -th order elliptic and $p(D)u \in C^\infty(X)$, then $u \in C^\infty(X)$.

We will first prove an “easy version” of theorem 4.3 using a *parametrix*:

Definition 4.5. If $p(D)$ is an N -th order differential operator, then $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a *parametrix* for $p(D)$ if

$$p(D)E = \delta_0 + \omega \quad \text{for some } \omega \in \mathcal{E}(\mathbb{R}^n).$$

Lemma 4.6. Every elliptic partial differential operator $p(D)$ has a parametrix which is smooth on $\mathbb{R}^n \setminus \{0\}$.

Proof. Since $p(D)$ is elliptic, we can choose $R > 0$, $C > 0$ such that $|p(\lambda)| \geq C \langle \lambda \rangle^N$ for $\|\lambda\| > R$.

Fix some $\chi_R \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi_R = 1$ on $\|\lambda\| \leq R$ and $\chi_R = 0$ on $\|\lambda\| > R + 1$, and define

$$\hat{E}(\lambda) := \frac{1 - \chi_R(\lambda)}{p(\lambda)}.$$

Then \tilde{E} is smooth and for λ sufficiently large we have $|\hat{E}(\lambda)| \lesssim \langle \lambda \rangle^{-N}$ since χ_R vanishes for large λ , which implies $\hat{E} \in \mathcal{S}'(\mathbb{R}^n)$. Therefore, $p(\lambda)\hat{E} = 1 - \chi_R(\lambda)$ is also a tempered distribution and we can take its inverse Fourier transform $p(D)E = \delta_0 + \omega$ for some $\omega \in \mathcal{S}(\mathbb{R}^n)$, which shows that E is a parametrix.

To prove that E is smooth on $\mathbb{R}^n \setminus \{0\}$, consider for $\|\lambda\| > R + 1$

$$|\mathcal{F}[D^\beta(x^\alpha E)]| = |\lambda^\beta D^\alpha \hat{E}| = \left| \lambda^\beta D^\alpha \left(\frac{1}{p(\lambda)} \right) \right| \stackrel{\star}{\lesssim} \|\lambda\|^{|\beta| - |\alpha| - N},$$

where \star can be shown with an induction argument. For each β , we can simply choose $|\alpha|$ large enough such that $\mathcal{F}[D^\beta(x^\alpha E)] \in L^1(\mathbb{R}^n)$, and therefore $D^\beta(x^\alpha E)$ is continuous for $|\alpha|$ large enough. Since β was randomly chosen, E will be smooth outside the origin. \square

We will now consider the proof of theorem 4.3 in the special case that u and $f := p(D)u$ have compact support.

Proof. Let E be a parametrix for P , then we have

$$u = \delta_0 * u = [p(D)E - \omega] * u = p(D)E * u - \omega * u = E * f - \omega * u.$$

Since u has compact support, $\omega * u$ will be a Schwartz function, and it can be shown that

$$|\mathcal{F}[E * f](\lambda)| = |\hat{E}(\lambda)\hat{f}(\lambda)| \lesssim \langle \lambda \rangle^{-N} |\hat{f}(\lambda)|,$$

which shows that $f \in H^s(\mathbb{R}^n) \implies u \in H^{s+N}(\mathbb{R}^n)$. \square

To prove theorem 4.3 in general, we will need some facts which are proved on the second example sheet:

1. If $s > t$ then $H^s(\mathbb{R}^n) \subseteq H^t(\mathbb{R}^n)$;
2. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $u \in H^s(\mathbb{R}^n)$, then $\varphi u \in H^s(\mathbb{R}^n)$;
3. If $u \in \mathcal{E}'(\mathbb{R}^n)$, then $u \in H^t(\mathbb{R}^n)$ for some $t \in \mathbb{R}$;
4. If $u \in H^s(\mathbb{R}^n)$, then $D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$.

Now we prove the theorem:

Proof. Fix $\varphi \in \mathcal{D}(X)$, we wish to prove that $\varphi u \in H^{s+N}(\mathbb{R}^n)$ given that $p(D)u \in H_{\text{loc}}^s(X)$. Choosing $M \in \mathbb{N}$, we introduce a collection $\{\psi_0, \dots, \psi_M\} \subseteq \mathcal{D}(X)$ such that

$$\text{supp}(\varphi) \subseteq \text{supp}(\psi_M) \subseteq \dots \subseteq \text{supp}(\psi_0), \quad \psi_{i-1} = 1 \text{ on } \text{supp } \psi_i, \quad \psi_M = 1 \text{ on } \text{supp } \varphi.$$

Consider $\psi_0 u \in \mathcal{E}'(\mathbb{R}^n)$. Then there exists $t \in \mathbb{R}$ for which $\varphi_0 u \in H^t(\mathbb{R}^n)$. We compute

$$p(D)(\psi_1 u) = \psi_1 p(D)u + [p(D), \psi_1](u) = \psi_1 f + [p(D), \psi_1](\psi_0 u),$$

where the last equality follows from the fact that $\psi_0 u \equiv u$ on $\text{supp } \psi_1$. Now note that $[p(D), \psi_1]$ is an order $N - 1$ differential operator. So we have $\psi_1 f \in H^s(\mathbb{R}^n)$ and $[p(D), \psi_1](\psi_0 u) \in H^{t-N+1}(\mathbb{R}^n)$. Setting $\tilde{A}_1 := \min(s, t - N + 1)$ we find that $p(D)(\psi_1 u) \in H^{\tilde{A}_1}(\mathbb{R}^n)$.

Since $|p(\lambda)| \gtrsim \langle \lambda \rangle^N$, we find that

$$\begin{aligned} p(D)(\psi_1 u) \in H^{\tilde{A}_1}(\mathbb{R}^n) &\implies \int_{\mathbb{R}^n} \langle \lambda \rangle^{2\tilde{A}_1} |p(\lambda)\mathcal{F}[\psi_1 u](\lambda)|^2 d\lambda \\ &\implies \int_{\mathbb{R}^n} \langle \lambda \rangle^{2\tilde{A}_1 + 2N} |\mathcal{F}[\psi_1 u](\lambda)|^2 d\lambda \\ &\implies \psi_1 u \in H^{\tilde{A}_1 + N}(\mathbb{R}^n). \end{aligned}$$

Define $A_1 := \tilde{A}_1 + N = \min\{s + N, t + 1\}$, then we have shown that $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$. By carrying on inductively, we can show that $\psi_M u \in H^{A_M}(\mathbb{R}^n)$ where $A_M = \min\{s + N, t + M\}$. By choosing M large enough we conclude $\psi_M u \in H^{s+N}(\mathbb{R}^n)$, and since $\psi_M = 1$ on $\text{supp } \varphi$, this also shows that $\varphi u \in H^{s+N}(\mathbb{R}^n)$. Since φ was randomly chosen, it follows that $u \in H_{\text{loc}}^{s+N}(X)$. \square

4.2 Fundamental solutions

Definition 4.7. Let $p(D)$ be a partial differential operator, then $E \in \mathcal{D}'(\mathbb{R}^n)$ is called a *fundamental solution* for $p(D)$ if $p(D)E = \delta_0$.

Example 4.8. Let $z = x_1 + ix_2 \in \mathbb{C}$ and define the Cauchy-Riemann operator as $\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right)$. It can be shown that $E := \frac{1}{\pi z}$ is a fundamental solution of this equation.

Example 4.9. Let $p(D) = \frac{\partial}{\partial t} - \Delta x$ be the heat operator (where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$) with coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Then it can be shown that

$$E := \begin{cases} (4\pi t)^{-n/2} \exp\left(-\frac{\|x\|^2}{4t}\right), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a fundamental solution.

Furthermore, if f has compact support, then $u = E * f$ solves $p(D)u = f$, since in this case

$$p(D)(E * f) = (p(D)E * f) = \delta_0 * f = f.$$