Distribution Theory and Applications — Summary

Lucas Riedstra

October 17, 2020

Contents

1 Distributions

1.1 Test functions and distributions

Definition 1.1. Let $X \subseteq \mathbb{R}^n$ be open, then we define the set of *test functions* on X as

$$\mathcal{D}(X) := C_0^\infty(X) = \{f \colon X \to \mathbb{C} \mid f \text{ is smooth with compact support}\}.$$

Definition 1.2. Let $(\varphi_m) \subseteq \mathcal{D}(X)$. We say that $(\varphi_m) \to 0$ in $\mathcal{D}(X)$ if

1. there exists a compact $K \subseteq X$ such that $\operatorname{supp} \varphi_m \subseteq K$ for all m;
2. $\partial^\alpha \varphi_m \to 0$ uniformly for each multi-index α.

Note that, for any $\varphi, \psi \in \mathcal{D}(X)$ and any multi-index α we have

$$\int_X \varphi \cdot \partial^\alpha \psi \, \mathrm{d}x = (-1)^{|\alpha|} \int_X \psi \cdot \partial^\alpha \varphi \, \mathrm{d}x \,,$$

which follows from partial integration and the fact that all boundary terms vanish since φ and ψ have compact support.

Also, by Taylor's theorem, for any $\varphi \in \mathcal{D}(X)$, $x, h \in X$ and $N \in \mathbb{N}$ we have

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h) \quad \text{where } R_N(x,h) = o(|h|^N) \text{ uniformly in } x.$$

Definition 1.3. A *distribution* on X is a linear map $u \colon \mathcal{D}(X) \to \mathbb{C}$ if for every compact set $K \subseteq X$ there exist constants C, N such that for all $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp} \varphi \subseteq K$ we have

$$|u(\varphi)| \leqslant C \sum_{|\alpha| \leqslant N} \sup |\partial^\alpha \varphi|. \tag{1}$$

Condition 1 is called the *seminorm condition.* If, in the seminorm condition, the same N can be used for every compact set $K \subseteq X$, then the least such N is called the *order* of u, written $\operatorname{ord}(u)$.

The set of all distributions in X is denoted $\mathcal{D}'(X)$.

If $u \in \mathcal{D}'(X)$ and $\varphi \in \mathcal{D}(X)$, then instead of $u(\varphi)$ we usually write $\langle u, \varphi \rangle$.

Recap 1.4. A function $f \colon X \to \mathbb{C}$ is called *locally integrable* if $\int_K |f| \, \mathrm{d}x < \infty$ for all compact $K \subseteq X$.

The set of locally integrable functions on X is denoted $L^1_{\mathrm{loc}}(X)$.

Example 1.5. Let $M \in \mathbb{N}$ and let $f_\alpha \in L^1_{\mathrm{loc}}(X)$ for all $|\alpha| \leqslant M$. Define the linear map $T \colon \mathcal{D}(X) \to \mathbb{C}$ by

$$\langle T, \varphi \rangle := \sum_{|\alpha| \leqslant M} \int_X f_\alpha \cdot \partial^\alpha \varphi \, \mathrm{d}x \,.$$

It is trivial that T is linear, and we verify that T is a distribution as follows: take $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp} \varphi \subseteq K$. Then we have

$$\begin{aligned}
|\langle T, \varphi \rangle| &\leqslant \sum_{|\alpha| \leqslant M} \int_K |f_\alpha| \cdot |\partial^\alpha \varphi| \, \mathrm{d}x \\
&\leqslant \sum_{|\alpha| \leqslant M} \sup |\partial^\alpha \varphi| \cdot \int_K |f_\alpha| \, \mathrm{d}x \\
&\leqslant \left(\max_\alpha \int_K |f_\alpha| \, \mathrm{d}x \right) \sum_{|\alpha| \leqslant M} \sup |\partial^\alpha \varphi|.
\end{aligned}$$

Therefore, the seminorm condition is satisfied with $N = M$. From this, it also follows that $\operatorname{ord}(T) \leqslant M$.

A special case of the previous example is the case $M = 0$: in this case the distribution simply becomes

$$\langle T_f, \varphi \rangle = \int_X f\varphi \, \mathrm{d}x \,.$$

Henceforth we will abuse notation: if $f \in L^1_{\mathrm{loc}}(X)$, then we will write f instead of T_f, i.e., $\langle f, \varphi \rangle = \int_X f\varphi \, \mathrm{d}x$.

Lemma 1.6. *Let $u \colon \mathcal{D}(X) \to \mathbb{C}$ be a linear map. Then u is a distribution if and only if, for every sequence $(\varphi_j) \subseteq \mathcal{D}(X)$ with $\varphi_j \to 0$ as in definition 1.2, we have $\langle u, \varphi_m \rangle \to 0$.*

Proof. '$\implies$' If u is a distribution and $(\varphi_m) \to 0$, then $\operatorname{supp} \varphi_m \subseteq K$ for some compact K, and by eq. (1) there exist C, N such that

$$|\langle u, \varphi_m \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup |\partial^\alpha \varphi_m| \to 0.$$

'$\impliedby$' Suppose there is a compact set K such that eq. (1) is not valid for any C, N. Let $m \in \mathbb{N}$ and $C = N = m$, then there is some φ_m with $\operatorname{supp}(\varphi_m) \subseteq K$, and

$$|\langle u, \varphi_m \rangle| > m \sum_{|\alpha| \leqslant m} \sup |\partial^\alpha \varphi_m|.$$

By dividing φ_m by $\langle u, \varphi_m \rangle \neq 0$, we may assume w.l.o.g. that $\langle u, \varphi_m \rangle = 1$. We now have a sequence (φ_m) such that

$$\frac{1}{m} > \sum_{|\alpha| \leqslant m} \sup |\partial^\alpha \varphi_m| \implies |\partial^\alpha \varphi_m| < \frac{1}{m} \quad \text{for } |\alpha| \leqslant m \implies \partial^\alpha \varphi_m \to 0 \text{ uniformly for all } \alpha.$$

Since each φ_m also satisfies $\operatorname{supp} \varphi_m \subseteq K$, by definition 1.2 we have that $\varphi_m \to 0$, but also $\langle u, \varphi_m \rangle \to 1$, a contradiction. □

1.2 Limits in $\mathcal{D}'(X)$

Definition 1.7. We say that a sequence $(u_m) \subseteq \mathcal{D}'(X)$ converges to $u \in \mathcal{D}'(X)$ and write $u_m \to u$ if

$$\langle u_m, \varphi \rangle \to \langle u, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{D}(X).$$

The following theorem is non-examinable but interesting:

Theorem 1.8. *Let (u_m) be a sequence in $\mathcal{D}'(X)$ such that $\lim\limits_{m \to \infty} \langle u_m, \varphi \rangle$ exists for all $\varphi \in \mathcal{D}(X)$. Then the map $\langle u, \varphi \rangle := \lim\limits_{m \to \infty} \langle u_m, \varphi \rangle$ is a distribution in X.*

Proof. This is a direct application of the uniform boundedness principle. □

Example 1.9. Let $X = \mathbb{R}$ and consider the sequence of functions $u_m \in L^1_{\mathrm{loc}}(\mathbb{R})$ defined by $u_m(x) = \sin(mx)$. Then, for all $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\langle u_m, \varphi \rangle = \int_{\mathbb{R}} \sin(mx) \varphi(x) \, \mathrm{d}x = \frac{1}{m} \int_{\mathbb{R}} \cos(mx) \varphi'(x) \, \mathrm{d}x \leqslant \frac{1}{m} \int \left| \varphi'(x) \right| \mathrm{d}x \to 0.$$

Therefore, it holds that $u_m \to 0$ in $\mathcal{D}'(\mathbb{R})$. With our abuse of notation we write this as $\lim\limits_{m \to \infty} \sin(mx) = 0$ in $\mathcal{D}'(\mathbb{R})$.

1.3 Basic operations

1.3.1 Differentiation and multiplication by smooth functions

For $u \in C^\infty(X)$ and $\varphi \in \mathcal{D}(X)$, we have noted that

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \partial^\alpha u \cdot \varphi \,\mathrm{d}x = (-1)^{|\alpha|} \int_X u \cdot \partial^\alpha \varphi \,\mathrm{d}x = \langle u, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle.$$

Since the RHS makes sense for any distribution u, we define

Definition 1.10. For $f \in C^\infty(X)$, $u \in \mathcal{D}'(X)$, we define $\partial^\alpha (fu)$ by

$$\langle \partial^\alpha (fu), \varphi \rangle := \langle u, (-1)^{|\alpha|} f \cdot \partial^\alpha \varphi \rangle$$

Remark. This definition outlines a more general pattern when working with distributions: first we take some well-defined operator on the collection of smooth maps, then we rewrite it to a form that is sensible for any distribution, and then we *define* that new form as the operator on distributions. This process is called *extending the definition by duality.*

Example 1.11. Let $u = \delta_x$, then we have

$$\langle \partial^\alpha \delta_x, \varphi \rangle = \langle \delta_x, (-1)^{|\alpha|} \partial^\alpha \varphi \rangle = (-1)^{|\alpha|} \partial^\alpha \varphi(x)$$

Furthermore, consider the Heaviside function $H(x) = \mathbb{1}_{x \geqslant 0}$. We have

$$\langle H', \varphi \rangle = -\langle H, \varphi' \rangle = -\int_{\mathbb{R}} H(x) \varphi'(x) \,\mathrm{d}x = -\int_0^\infty \varphi'(x) \,\mathrm{d}x = \varphi(0) = \langle \delta_0, \varphi \rangle,$$

so we write $H' = \delta_0$ *in the distributional sense.*

Lemma 1.12. *Suppose $u' \in \mathcal{D}'(\mathbb{R})$ satisfies $u' = 0$. Then u is constant (i.e., $\langle u, \varphi \rangle = \langle c, \varphi \rangle = c \int_{\mathbb{R}} \varphi \,\mathrm{d}x$ for some c).*

Proof. Fix any $\vartheta \in \mathcal{D}(\mathbb{R})$ with $\langle 1, \vartheta \rangle = 1$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and define

$$\varphi_A := \varphi - \langle 1, \varphi \rangle \vartheta, \quad \varphi_B := \langle 1, \varphi \rangle \vartheta \quad \text{such that } \varphi = \varphi_A + \varphi_B.$$

Note that $\langle 1, \varphi_A \rangle = \langle 1, \varphi \rangle - \langle 1, \varphi \rangle \langle 1, \vartheta \rangle = 0$.

We claim that the function $\Phi_A(x) := \int_{-\infty}^x \varphi_A(y) \,\mathrm{d}y$ has compact support: since $\operatorname{supp} \varphi_A \subseteq [a, b]$ for some $a, b \in \mathbb{R}$, clearly $\Phi_A(x) = 0$ for $x < a$, while for $x > b$ we have $\Phi_A(x) = \langle 1, \varphi_A \rangle = 0$. Obviously, it holds that $\Phi_A' = \varphi_a$. Now we compute

$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle = \langle u, \Phi_A' \rangle + \langle 1, \varphi \rangle \langle u, \vartheta \rangle = -\langle u', \Phi_A \rangle + c \langle 1, \varphi \rangle = \langle c, \varphi \rangle.$$

Since φ was chosen arbitrarily this shows that u is constant. ☐

1.3.2 Reflection and translation

For $\varphi \in \mathcal{D}(\mathbb{R}^n), h \in \mathbb{R}^n$, define the *translation of φ by h* by

$$(T_h \varphi)(x) := \varphi(x - h),$$

and the *reflection of* φ by $\check{\varphi}(x) := \varphi(-x)$.

Extending the definitions of translation and reflection by duality yields the following:

Definition 1.13. For $u \in \mathcal{D}'(\mathbb{R}^n)$, $h \in \mathbb{R}^n$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, define

$$\langle T_h u, \varphi \rangle := \langle u, T_{-h} \varphi \rangle \quad \text{and} \quad \langle \check{u}, \varphi \rangle := \langle u, \check{\varphi} \rangle.$$

Lemma 1.14. *For $u \in \mathcal{D}'(\mathbb{R}^n)$, define $V_h \in \mathcal{D}'(\mathbb{R}^n)$ for $0 \neq h \in \mathbb{R}^n$ by*

$$V_h := \frac{T_{-h}u - u}{\|h\|}$$

If $(h_j) \subseteq \mathbb{R}^n$ is a sequence for which $\lim_{j\to\infty} \frac{h_j}{\|h_j\|} = m \in S^{n-1}$, then $V_{h_j} \to \sum_i m_i \frac{\partial u}{\partial x_i}$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof. By definition, we can write $\langle V_h, \varphi\rangle = \frac{1}{\|h\|}\langle u, T_h\varphi - \varphi\rangle$. Now Taylor's theorem tells us that

$$(T_h\varphi - \varphi)(x) = \varphi(x-h) - \varphi(x) = -\sum_i h_i \frac{\partial \varphi}{\partial x_i} + R(x,h),$$

where $R(x,h) = o(\|h\|)$ in $D(\mathbb{R}^n)$ (exercise sheet 1, question 2).

By sequential continuity, we have

$$\lim_{j\to\infty} \langle V_{h_j}, \varphi\rangle = \langle u, -\sum_i m_i \frac{\partial \varphi}{\partial x_i}\rangle = \langle \sum_i m_i \frac{\partial u}{\partial x_i}, \varphi\rangle,$$

which shows that $V_{h_j} \to \sum_i m_i \frac{\partial u}{\partial x_i}$ in $\mathcal{D}'(\mathbb{R}^n)$. $\square$

1.3.3 Convolution

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, note that $(T_x\check{\varphi})(y) = \check{\varphi}(y-x) = \varphi(x-y)$.

Definition 1.15. For $u \in C^\infty(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the *convolution* $u * \varphi \colon \mathbb{R}^n \to \mathbb{C}$ as

$$(u * \varphi)(x) := \int_{\mathbb{R}^n} u(x-y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} u(y)\varphi(x-y)\,\mathrm{d}y = \langle u, T_x\check{\varphi}\rangle.$$

Since the RHS makes sense for any $u \in \mathcal{D}'(\mathbb{R}^n)$, we extend the definition this way: for $u \in \mathcal{D}'(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n)$, we define the convolution $u * \varphi$ as

$$(u * \varphi)(x) := \langle u, T_x\check{\varphi}\rangle.$$

Lemma 1.16. *Let $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and define $\Phi_x(y) := \varphi(x,y)$. Suppose for any $x \in \mathbb{R}^n$ there exists a neighbourhood $N(x)$ and a compact $K \subseteq \mathbb{R}^m$ such that $\varphi(x,y)$ for all $x \in N(x), y \notin K$.*

Then $x \mapsto \langle u, \Phi_x\rangle$ is differentiable with

$$\partial_x^\alpha \langle u, \Phi_x\rangle = \langle u, \partial_x^\alpha \Phi_x\rangle$$

for any $u \in \mathcal{D}'(\mathbb{R}^m)$.

Proof. Fix $x \in \mathbb{R}^n$, then by Taylor's formula we have

$$\Phi_{x+h}(y) = \Phi_x(y) + \sum_{i=1}^n h_i \frac{\partial \varphi}{\partial x_i}(x,y) + R(x,y,h),$$

where $\partial_y^\alpha R(x,y,h) = o(\|h\|)$, uniformly in y, for any multi-index α. Furthermore, by assumption there exists a compact K such that for h small enough, $\operatorname{supp} R(x,\cdot,h) \subseteq K$. Therefore, $R(x,\cdot,h)$ is a test function for h small enough.

Combining the previous two facts shows that $R(x,\cdot,h) = o(\|h\|)$ in $\mathcal{D}(\mathbb{R}^m)$ as $h \to 0$.

Let $u \in \mathcal{D}'(\mathbb{R}^m)$, then we find by sequential continuity that $\langle u, R(x,\cdot,h)\rangle$ is also $o(\|h\|)$, and therefore

$$\langle u, \Phi_{x+h}\rangle = \langle u, \Phi_x\rangle + \sum_{i=1}^n h_i \langle u, \frac{\partial \Phi_x}{\partial x_i}\rangle + o(\|h\|).$$

This shows that $x \mapsto \langle u, \Phi_x \rangle$ is differentiable with

$$\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial \Phi_x}{\partial x_i} \rangle.$$

From this the result follows. $\square$

Corollary 1.17. *If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then $u * \varphi$ is differentiable with $\partial^\alpha (u * \varphi) = u * \partial^\alpha \varphi$.*

Proof. Apply the previous lemma with $\Phi_x(y) := \varphi(x - y)$. $\square$