Inverse Problems — Example Sheet 1

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Note: when writing a norm of a vector $v \in V$, I will simply write $\|v\|$ and not $\|v\|_V$, unless it is unclear in which space v lives. The same holds for inner products.

Question 1. *For $\Omega = [0,1]^2$ and $\mathcal{X} \in L^2(\Omega)$, we consider the integral operator $A\colon \mathcal{X} \to \mathcal{X}$ with*

$$(Au)(y) \coloneqq \int_\Omega k(x,y)u(x)\,\mathrm{d}x\,,$$

for $k \in L^2(\Omega \times \Omega)$. Show that

(a) A is linear with respect to u,

(b) A is a bounded linear operator, i.e. $\|Au\|_{\mathcal{X}} \leq \|A\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}\|u\|_{\mathcal{X}}$. Give also an estimate for $\|A\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}$,

(c) the adjoint A^* is given via

$$(A^*v)(y) = \int_\Omega k(y,x)v(x)\,\mathrm{d}x\,.$$

(d) A is a compact operator, i.e. $A \in \mathcal{K}(\mathcal{X},\mathcal{X})$.

Hint: *you may use the fact that if an operator A can be written as a limit (in the operator norm) of finite-rank operators then A is compact. An operator B is called finite-rank if* $\dim(B) < \infty$.

Solution. (a) Let $\alpha, \beta \in \mathbb{R}$, $u, v \in L^2(\Omega)$ and $y \in \Omega$. Then we have

$$\begin{aligned}
(A(\alpha u + \beta v))(y) &= \int_\Omega k(x,y)(\alpha u + \beta v)(x)\,\mathrm{d}x \\
&= \int_\Omega k(x,y)(\alpha u(x) + \beta v(x))\,\mathrm{d}x \\
&= \alpha \int_\Omega k(x,y)u(x)\,\mathrm{d}x + \beta \int_\Omega k(x,y)v(x)\,\mathrm{d}x \\
&= (\alpha Au)(y) + (\beta Av)(y) = (\alpha Au + \beta Av)(y).
\end{aligned}$$

Since equality holds for all $y \in \Omega$ we find $A(\alpha u + \beta v) = \alpha Au + \beta Av$, which proves that A is linear.

(b) Let $u \in L^2(\Omega)$, then we have

$$\|Au\|^2 = \int_\Omega ((Au)(y))^2\,\mathrm{d}y = \int_\Omega \left(\int_\Omega k(x,y)u(x)\,\mathrm{d}x\right)^2 \mathrm{d}y = \int_\Omega \langle k(\cdot,y), u(\cdot)\rangle^2\,\mathrm{d}y\,.$$

Now we apply Cauchy-Schwarz and find

$$\int_\Omega \langle k(\cdot,y),u(\cdot)\rangle^2 \,\mathrm{d}y \le \int_\Omega \|k(\cdot,y)\|^2 \|u\|^2 \,\mathrm{d}y \overset{\star}{=} \|u\|^2 \iint_{\Omega^2} k^2(x,y)\,\mathrm{d}x\,\mathrm{d}y = \|u\|^2\|k\|^2,$$

where $\star$ follows from Fubini's theorem since the integrand is nonnegative. Taking square roots on both sides we find that $\|Au\| \le \|k\|\|u\|$, so A is bounded with $\|A\| \le \|k\|$.

(c) We know that the adjoint is the unique operator that satisfies $\langle Au, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in \mathcal{X}$. Let $u, v \in \mathcal{X}$, then we compute

$$\begin{aligned}\langle Au, v\rangle &= \int_\Omega (Au)(y)\cdot v(y)\,\mathrm{d}y = \int_\Omega \left(\int_\Omega k(x,y)u(x)\,\mathrm{d}x\right) v(y)\,\mathrm{d}y \\ &= \int_\Omega\int_\Omega k(x,y)u(x)v(y)\,\mathrm{d}x\,\mathrm{d}y \overset{\star}{=} \int_\Omega\int_\Omega k(x,y)u(x)v(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_\Omega u(x)\left(\int_\Omega k(x,y)v(y)\,\mathrm{d}y\right)\mathrm{d}x = \langle u, A^*v\rangle\end{aligned}$$

where $(A^*v)(x) = \int_\Omega k(x,y)v(y)\,\mathrm{d}y$ as required. Here $\star$ follows from Fubini's theorem.

(d) It is known that for any compact set $X \subseteq \mathbb{R}^n$, polynomials lie dense in $L^2(X)$. Therefore, there exists a sequence of polynomials p_n such that $p_n \to k$ in $L^2([0,1]^4)$. It is easily seen that for any polynomial p, the operator

$$(A_p u)(y) := \int_\Omega p(x,y)u(x)\,\mathrm{d}x$$

has finite rank: let $p(z) = \sum_{|\alpha|\le n} c_\alpha z^\alpha$ (where $z \in [0,1]^4$ and α is a multi-index), then we find

$$(A_p u)(y) = \sum_{|\alpha|\le n} c_\alpha \int_\Omega x_1^{\alpha_1}x_2^{\alpha_2}y_1^{\alpha_3}y_2^{\alpha_4}u(x)\,\mathrm{d}x = \sum_{|\alpha|\le n} c_\alpha \left(\int_\Omega x_1^{\alpha_1}x_2^{\alpha_2}u(x)\,\mathrm{d}x\right) y_1^{\alpha_3}y_2^{\alpha_4},$$

so $A_p u$ lies in the Span $\{y_1^{\alpha_1}y_2^{\alpha_2} \mid \alpha_1+\alpha_2 \le n\}$, and therefore has finite rank. By (b), we find that $\|A - A_n\| \le \|k - p_n\| \to 0$, which shows that $A_n \to A$ in operator norm. We conclude that A is compact.

Question 2. *We consider the problem of differentiation, formulated as the inverse problem of finding u from $Au = f$ with the integral operator $A\colon L^2([0,1]) \to L^2([0,1])$ defined as*

$$(Au)(y) := \int_0^y u(x)\,\mathrm{d}x\,.$$

(a) Let f be given by

$$f(x) := \begin{cases} 0 & x < \frac{1}{2}, \\ 1 & x > \frac{1}{2}. \end{cases}$$

Show that $f \in \overline{\mathcal{R}(A)}$.

(b) Let f be given as in a). Show that $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$. Hint: Consider the Picard criterion.

(c) Prove or falsify: "The Moore-Penrose inverse of A is continuous."

Solution. (a) We want to show that we can approximate f by a sequence (Au_n) for some $(u_n) \subseteq L^2[0,1]$. To this end, define for $n \geq 2$

$$u_n(x) = \begin{cases} 0 & \left|x - \frac{1}{2}\right| > \frac{1}{n}, \\ \frac{n}{2} & \left|x - \frac{1}{2}\right| \leq \frac{1}{n}. \end{cases}$$

Clearly $u \in L^2[0,1]$, and we have

$$f_n(y) := (Au_n)(y) = \int_0^y u_n(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } y < \frac{1}{2} - \frac{1}{n}, \\ \frac{n}{2}(y - \frac{1}{2} + \frac{1}{n}) & \text{if } \frac{1}{2} - \frac{1}{n} \leq y \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \text{if } y > \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Therefore we find

$$\begin{aligned} \|f_n - f\|^2 &= \int_0^1 (f_n - f)^2(x)\,\mathrm{d}x \\ &= 2\int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \frac{n^2}{4}(x - \frac{1}{2} - \frac{1}{n})^2\,\mathrm{d}x \\ &= \frac{n^2}{2}\int_0^{1/n} x^2\,\mathrm{d}x = \frac{1}{6n} \to 0, \end{aligned}$$

so $f_n \to f$ in $L^2[0,1]$. Since $f_n \in \mathcal{R}(A)$ this shows $f \in \overline{\mathcal{R}(A)}$.

(b) In example 2.2.12, it is shown that for this operator, the Picard criterion is

$$2\sum_{j=1}^{\infty} \sigma_j^{-2}\left(\int_0^1 f(s)\sin\!\left(\sigma_j^{-1}s\right)\mathrm{d}s\right)^2 < \infty, \tag{1}$$

where $\sigma_j = \frac{2}{(2j-1)\pi}$.

We compute

$$\int_0^1 f(s)\sin\!\left(\sigma_j^{-1}s\right)\mathrm{d}s = \int_{1/2}^1 \sin\!\left(\sigma_j^{-1}s\right)\mathrm{d}s = \sigma_j\left[\cos\!\left(\frac{1}{2}\sigma_j^{-1}\right) - \cos\!\left(\sigma_j^{-1}\right)\right].$$

We have

$$\cos\!\left(\sigma_j^{-1}\right) = \cos\!\left(\frac{(2j-1)\pi}{2}\right) = 0 \quad \text{and} \quad \cos\!\left(\frac{1}{2}\sigma_j^{-1}\right) = \cos\!\left(\frac{(2j-1)\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}.$$

Plugging this into eq. (1) gives that

$$2\sum_{j=1}^{\infty} \sigma_j^{-2}\left(\int_0^1 f(s)\sin\!\left(\sigma_j^{-1}s\right)\mathrm{d}s\right)^2 = 2\sum_{j=1}^{\infty} \sigma_j^{-2}(\sigma_j^2/2) = \sum_{j=1}^{\infty} 1 = \infty,$$

so f does not satisfy the Picard criterion and therefore $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$.

(c) The Moore-Penrose inverse of A is discontinuous. This can be seen by theorem 2.1.11: we have in (b) an element $f \in \overline{\mathcal{R}(A)} \setminus \mathcal{R}(A)$, so $\mathcal{R}(A)$ is not closed, so $A^\dagger$ is discontinuous.