Modern Statistical Methods — Example Sheet 4

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Question 1. *content...*

Proof. content... $\square$

Question 2. *Consider the matrix* $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1.01 \end{bmatrix}$ *and its perturbation* $\hat{Q} = Q + \begin{bmatrix} 0 & 0.01 \\ 0.01 & 0 \end{bmatrix}$. *Show that the eigenvalues are stable to perturbation, but the top eigenvector is not.*

Proof. Clearly Q has eigenvalues $\lambda_1 = 1.01, \lambda_2 = 1$, with eigenvectors $\mathbf{x}_1 = \mathbf{e}_2, \mathbf{x}_2 = \mathbf{e}_1$.

Now, we have

$$\hat{Q} = \begin{bmatrix} 1 & 0.01 \\ 0.01 & 1.01 \end{bmatrix} \implies \det\left(\lambda I - \hat{Q}\right) = (\lambda - 1)(\lambda - 1.01) - 0.01^2 = \lambda^2 - 2.01\lambda + 1.0099,$$

which gives eigenvectors

$$\lambda = \frac{2.01 \pm \sqrt{4.0401 - 4.0396}}{2} = 1.005 \pm \frac{\sqrt{5}}{2} \cdot 0.01 \approx \{0.9938, 1.0162\}$$

which is indeed very close to the eigenvalues 1 and 1.01.

Letting $\tilde{\lambda}_1 = 1.005 + 0.01\frac{\sqrt{5}}{2}$, we have by the top row that

$$\hat{Q}x = \lambda x \iff x + 0.01y = \lambda x \implies x = \frac{0.01}{\lambda - 1} y = \frac{1}{2}(\sqrt{5} - 1)y,$$

so the top eigenvector has been perturbed from $(0, 1)^\top$ to $(\frac{1}{2}(\sqrt{5} - 1), 1)^\top$, which upon normalisation is a change of more than $\pi/6$ radians. $\square$