Inverse Problems — Example Sheet 2

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Question 1. *Let $\mathcal{U}$ be a Banach space and $J\colon \mathcal{U} \to \overline{\mathbb{R}}$ a functional. We define the* subdifferential *of J at any $v \in \mathcal{U}$ as*

$$\partial \mathcal{J}(v) \coloneqq \{p \in \mathcal{U}^* \mid J(u) \geq J(v) + \langle p, u - v\rangle \text{ for all } u \in \mathcal{U}\}.$$

Characterise the subdifferential for the

(a) absolute value function: $\mathcal{U} = \mathbb{R}$, $J(v) = |v|$,

(b) ℓ^1-norm: $\mathcal{U} = \ell^2$,

$$J(u) = \|u\|_{\ell^1} \coloneqq \begin{cases} \sum_{j=1}^\infty |u_j| & \text{if } u \in \ell^1; \\ \infty & \text{else.} \end{cases}$$

(c) characteristic function of the unit ball in $\mathbb{R}$: $\mathcal{U} = \mathbb{R}$, $J(u) = \chi_C(u)$, $C \coloneqq \{u \in \mathbb{R} : |u| \leq 1\}$.

(d) Total Variation $\mathrm{TV}\colon L^1(\Omega) \to \overline{\mathbb{R}}$, where $\Omega \subset \mathbb{R}^n$ is bounded Lipschitz

$$\mathrm{TV}(u) = \sup_{\varphi \in \mathcal{D}} \langle u, \boldsymbol{\nabla} \cdot \varphi\rangle, \quad \mathcal{D} = \{\varphi \in \mathcal{C}_0^\infty(\Omega; \mathbb{R}^n) : \|\varphi(x)\|_2 \leq 1 \ \forall x \in \Omega\}.$$

Solution. Note: the spaces $\mathcal{U}$ in parts (a) to (c) are Hilbert spaces, which means we can identify $\mathcal{U}^*$ with $\mathcal{U}$ (since any functional in $\mathcal{U}^*$ is of the form $\langle u, \cdot\rangle$ for some $u \in \mathcal{U}$).

(a) Let $v \in \mathbb{R}$. We know that $|\cdot|$ is differentiable at $v \neq 0$, so

$$v > 0 \implies \partial J(v) = \{1\} \quad \text{and} \quad v < 0 \implies \partial J(v) = \{-1\}.$$

For $v = 0$ we have

$$\begin{aligned} p \in \partial J(v) &\iff |u| \geq p \cdot u \text{ for all } u \in \mathbb{R} \\ &\iff p \in [-1, 1], \end{aligned}$$

so $\partial J(0) = [-1, 1]$.

(b) Let $v \in \ell^2$. Firstly, if $v \notin \ell^1 = \mathrm{dom}(J)$, then we have $\partial J(v) = \varnothing$. Assume now that $v \in \ell_1 \cap \ell_2$. Then we have, for $p \in \ell^2$, that

$$\begin{aligned} p \in \partial J(v) &\iff \|u\|_{\ell^1} \geq \|v\|_{\ell^1} + \langle p, u - v\rangle && \text{for all } u \in \ell^2 \\ &\iff \|u\|_{\ell^1} - \|v\|_{\ell^1} - \langle p, u - v\rangle \geq 0 && \text{for all } u \in \ell^2 \\ &\iff \sum_{j=1}^\infty |u_i| - |v_i| - p_i(u_i - v_i) \geq 0 && \text{for all } u \in \ell^2 \qquad (1) \\ &\overset{\star}{\iff} |x| - |v_i| - p_i(x - v_i) \geq 0 && \text{for all } i \in \mathbb{N} \text{ and } x \in \mathbb{R}. \qquad (2) \end{aligned}$$

We first prove the bi-implication $\star$. If (2) holds, it is clear that (1) holds. If (2) does not hold, then we can find x, i such that $|x| - |v_i| - p_i(x - v_i) < 0$. By now letting $u = xe_i$ in (1) we find that (1) does not hold.

However, if we define $H(x) \coloneqq |x|$, we see that eq. (2) is equivalent to $p_i \in \partial H(v_i)$ for all i. Therefore, by (a) we have

$$\partial J(v) = \left\{p \in \ell^2 \mid p_i = \operatorname{sign}(v_i) \text{ if } v_i \neq 0 \text{ and } p_i \in [-1, 1] \text{ for all } i\right\}.$$

(c) Clearly, if $|v| < 1$, then χ_C is differentiable with derivative 0 so $\partial J(v) = \{0\}$. If $|v| > 1$, then $v \notin \operatorname{dom}(J)$, and therefore $\partial J(v) = \varnothing$.

Consider the point $v = 1$, then we have

$$p \in \partial J(\chi_C) \iff \chi_C(u) \geq p \cdot (u - 1) \ \forall u.$$

For $u > 1$, this equation is satisfied regardless of p. Therefore, the above equation is equivalent to

$$p \cdot (u - 1) \leq 0 \ \forall u \leq 1,$$

which is satisfied for all $p \geq 0$, so we conclude $\partial J(1) = [0, \infty)$. Analogously, we find $\partial J(-1) = (-\infty, 0]$. We conclude that

$$\partial J(v) = \begin{cases} \varnothing & \text{if } |v| > 1; \\ (-\infty, 0] & \text{if } v = -1; \\ \{0\} & \text{if } v \in (-1, 1); \\ [0, \infty) & \text{if } v = 1. \end{cases}$$

(d) Let $f \in L^1(\Omega) \setminus \mathrm{BV}(\Omega)$, then clearly $\partial \operatorname{TV}(f) = \varnothing$. Now suppose $f \in \mathrm{BV}(\Omega)$. It is known that the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$. Therefore, we have for $p \in L^\infty(\Omega)$ that

$$p \in \partial \operatorname{TV}(f) \iff \operatorname{TV}(g) \geq \operatorname{TV}(f) + \int_\Omega p(x)(g - f)(x) \, \mathrm{d}x \ \ \forall g \in L^1(\Omega)$$

I do not know how to continue from here.

Question 2. *Recall that a function $J \colon \mathcal{U} \to \overline{\mathbb{R}}$ is called absolutely one-homogeneous if $J(\lambda u) = |\lambda| J(u)$ for all $\lambda \in \mathbb{R}, u \in \mathcal{U}$. Let J be convex, proper, l.s.c. and absolutely one-homogeneous.*

(a) Show that $p \in \partial J(v)$ if and only if $p \in \partial J(0)$ and $J(v) = \langle p, v\rangle$. Therefore,

$$D_J^p(u, v) = J(u) - \langle p, u\rangle.$$

Show that

$$\partial J(0) = \bigcup_{u \in \mathcal{U}} \partial J(u).$$

(b) Show that the Bregman distances associated with absolutely one-homogeneous functionals fulfill the triangle inequality in the first argument, i.e., for all $u, v, w \in \mathcal{U}$ and $p \in \partial J(w)$ there is

$$D_J^p(u + v, w) \leq D_J^p(u, w) + D_J^p(v, w).$$

(c) Show that the convex conjugate $J^*(\cdot)$ is the characteristic function of the convex set $\partial J(0)$. Compare this to the results of Exercise 2(a)(i).

Proof. It is clear that $J(0) = 0$.

(a) Suppose $p \in \partial J(v)$. Then we have $J(u) \geq J(v) + \langle p, u - v\rangle$ for all u, which we can rewrite as $J(u) - \langle p, u\rangle \geq J(v) - \langle p, v\rangle$. Plugging in $u = 0$ we obtain $J(v) - \langle p, v\rangle \leq 0$, but plugging in $u = 2v$ we obtain

$$2(J(v) - \langle p, v\rangle) = J(2v) - \langle p, 2v\rangle \geq J(v) - \langle p, v\rangle \implies J(v) - \langle p, v\rangle \geq 0,$$

so we conclude $J(v) - \langle p, v\rangle = 0$ or $J(v) = \langle p, v\rangle$. This also implies that

$$J(u) \geq \langle p, u\rangle \text{ for all } u \implies p \in \partial J(0).$$

Conversely, if $p \in \partial J(0)$ and $J(v) = \langle p, v\rangle$, then for all u we have

$$J(u) \geq \langle p, u\rangle + (J(v) - \langle p, v\rangle) \implies p \in \partial J(v).$$

This concludes the first claim.

From this claim, it follows that $\partial J(u) \subseteq \partial J(0)$ for all $u \in \mathcal{U}$, and therefore trivially $\partial J(0) = \cup_u \partial J(u)$.

(b) Note that we have

$$J(u+v) = 2J\left(\frac{1}{2}u + \frac{1}{2}v\right) \leq 2\left(\frac{1}{2}J(u) + \frac{1}{2}J(v)\right) = J(u) + J(v),$$

and therefore

$$D_J^p(u+v, w) = J(u+v) - \langle p, u+v\rangle \leq J(u) + J(v) - \langle p, u\rangle - \langle p, v\rangle = D_J^p(u, w) + D_J^p(v, w).$$

(c) We can reason analogously to 2(a)(i): we have

$$J^*(v) = \sup_{u \in U}(\langle v, u\rangle - J(u)).$$

Suppose that $v \notin \partial J(0)$, i.e., $\langle v, u^*\rangle - J(u) = \xi > 0$ for some u^*. Then we have for all $\lambda > 0$ that

$$\langle v, \lambda u^*\rangle - J(\lambda u^*) = \lambda \xi,$$

and letting $\lambda \to \infty$ shows $J^*(v) = \infty$.

On the other hand, suppose that $v \in \partial J(0)$, i.e., $\langle v, u\rangle - J(u) \leq 0$ for all u. Then the supremum is attained in $u = 0$ and therefore we have $J^*(v) = 0$.

It follows that $J^*(v) = \partial J(0)$, which is indeed also what we saw in 2(a)(i), since the subdifferential of the norm at 0 is exactly $\{v \in \mathcal{U}^* : \|v\|_{\mathcal{U}^*} \leq 1\}$.

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