Distribution Theory and Applications — Example Sheet 1

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Question 1. *Construct a non-zero element of $\mathcal{D}(\mathbb{R})$ that vanishes outside $(0,1)$. Construct a non-zero of $\mathcal{D}(\mathbb{R}^n)$ that vanishes outside the ball $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.*

Proof. It is well-known that the function

$$\varphi\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto \begin{cases} 0 & \text{if } x \leqslant 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside $(0,\infty)$. The function $\psi(x) := \varphi(x)\varphi(1-x)$ is therefore also smooth and vanishes outside $(0,1)$.

Since ψ vanishes outside $(0,1)$, the function $\psi(x/\varepsilon)$ vanishes outside $(0,\varepsilon)$, and therefore the function $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$ vanishes outside B_ε. □

Question 2. *Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives*

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h).$$

Prove that $\operatorname{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in x for each multi-index α, i.e. prove

$$\lim_{|h| \to 0} \frac{\sup_x |\partial^\alpha R_N(x,h)|}{|h|^N} = 0$$

for each multi-index α.

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x,h) = \sum_{|\alpha| = N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th)\,\mathrm{d}t\,,$$

and note that $(N+1)! \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \cdots + h_n)^{N+1}$.

Proof. Since $\varphi \in \mathcal{D}(X)$, we know that $\operatorname{supp}\varphi \subseteq \overline{B_N}$ for some $N \in \mathbb{N}$. Now, suppose $\|h\| < 1$, then

$$\varphi(x+h) \neq 0 \implies \|x+h\| \leqslant N \implies \|x\| \leqslant \|x+h\| + \|h\| \leqslant N+1,$$

so if we define $\psi_h(x) = \varphi(x+h)$ then we know that $\operatorname{supp}\varphi_h \subseteq \overline{B_{N+1}}$.

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha| \leqslant N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x,h),$$

and since $\sum_{|\alpha|\leqslant N} \frac{h^\alpha}{\alpha!}\partial^\alpha\varphi(x)$ vanishes for $x \notin \overline{B_N}$, it is clear that $\operatorname{supp}(R_N(\cdot,h))$ must also be contained in $\overline{B_{N+1}}$ (again, for $\|h\| \leqslant 1$). This shows that $\operatorname{supp}(R_N)$ is contained in $\overline{B_{N+1}}$ for $|h|$ sufficiently small.

Now let β be a multi-index and define $C := \frac{1}{N!}\max_{|\alpha|=N+1, x\in\mathbb{R}^n}\left|(\partial^{\alpha+\beta})\varphi(x)\right|$ (note that C exists and is finite since all partial derivatives of φ have compact support), then we have

$$
\begin{aligned}
\left|\partial^\beta R_N(x,h)\right| &= \left|\partial^\beta \sum_{|\alpha|=N+1}\frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^\alpha\varphi)(x+th)\,\mathrm{d}t\right| \\
&\overset{\star}{=} \left|\sum_{|\alpha|=N+1}\frac{h^\alpha}{\alpha!}(N+1)\int_0^1 (1-t)^N(\partial^{\alpha+\beta}\varphi)(x+th)\,\mathrm{d}t\right| \\
&\leqslant \sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!}(N+1)\int_0^1 (1-t)^N\left|(\partial^{\alpha+\beta}\varphi)(x+th)\right|\mathrm{d}t \\
&\leqslant \left[\max_{|\alpha|=N+1, x\in\mathbb{R}^n}\left|(\partial^{\alpha+\beta})\varphi(x)\right|\right]\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!}(N+1) \\
&\leqslant C(N+1)!\sum_{|\alpha|=N+1}\frac{|h^\alpha|}{\alpha!} = C(|h_1|+\cdots+|h_n|)^{N+1}.
\end{aligned}
$$

Since this upper bound does not depend on x, we also have

$$
\sup_x \left|\partial^\beta R_N(x,h)\right| \leqslant C(|h_1|+\cdots+|h_n|)^{N+1},
$$

and we conclude that

$$
\frac{\sup_x\left|\partial^\beta R_N(x,h)\right|}{\|h\|^N} \leqslant \frac{C(|h_1|+\cdots+|h_n|)^{N+1}}{\|h\|^N} \leqslant \frac{CN^{N+1}\|h\|^{N+1}}{\|h\|^N} = CN^{N+1}\|h\| \to 0,
$$

and therefore that $\partial^\beta R_N(x,h) = o(\|h\|^n)$ for all multi-indices β. □

Question 3. *Which elements of $\mathcal{D}(X)$ can be represented as a power series on X?*

Solution. It is known that if two power series agree on an open set, they agree on the entire space. Since every $\varphi \in \mathcal{D}(X)$ is identically zero on some open set (outside its support), the only element of $\mathcal{D}(X)$ with a power series representation is the zero function.

Question 4. *Prove the C^∞ Urysohn lemma: if K is a compact subset of $X \subseteq \mathbb{R}^n$, show that one can find a $\varphi \in \mathcal{D}(X)$ such that $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ on a neighborhood of K.*

Solution. Let $K \subseteq U_1$. Define $U_2 := U_1 + B(0,1)$ and let $\chi = \mathbb{1}_{U_2}$. Now let $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n}\psi\,\mathrm{d}x = 1$ and $\operatorname{supp}\psi \subseteq B(0,1)$. The we compute

$$
(\chi * \psi)(x) = \int_{\mathbb{R}^n}\chi(y)\psi(x-y)\,\mathrm{d}y = \int_{U_2}\psi(x-y)\,\mathrm{d}y\,.
$$

Clearly, $\chi * \psi \in \mathcal{D}(X)$, and furthermore, we have for $x \in U_1$ that

$$
\int_{U_2}\psi(x-y)\,\mathrm{d}y = \int_{U_2-x}\psi(z)\,\mathrm{d}z \overset{\star}{=} 1,
$$

since $B(0,1) \subseteq U_2 - x$. This proves the claim.

Question 5. *Given $T \in \mathcal{D}'(X)$, the derivative $\partial^\alpha T$ is defined by*

$$\langle \partial^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha \varphi\rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Show that $\partial^\alpha T \in \mathcal{D}'(X)$. If $\operatorname{ord}(T) = m$ what can you say about $\operatorname{ord}(\partial^\alpha T)$?

Proof. Let $K \subseteq X$ be compact and $\varphi \in \mathcal{D}(X)$. Since T is a distribution, we know that there exists constants C, N such that

$$|\langle T, \varphi\rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left|\partial^\beta \varphi\right|.$$

Letting $M := |\alpha|$, we find

$$|\langle \partial^\alpha T, \varphi\rangle| = |\langle T, \partial^\alpha \varphi\rangle| \leqslant C \sum_{|\beta| \leqslant N} \sup \left|\partial^{\alpha+\beta}\varphi\right| \leqslant C \sum_{|\beta| \leqslant M+N} \sup \left|\partial^\beta \varphi\right|.$$

We conclude that $\partial^\alpha T$ is a distribution, and that if $\operatorname{ord}(T) = m$, $\operatorname{ord}(\partial^\alpha T) \leqslant m + |\alpha|$. □

Question 6. *Given $T \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$, prove that for each multi-index α*

$$\partial^\alpha(Tf) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

in $\mathcal{D}'(X)$.

Proof. Let $\varphi \in \mathcal{D}(X)$, then by definition we have $\langle \partial^\alpha(Tf), \varphi\rangle = \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi\rangle$. Approximate T by a sequence $(\psi_n) \subseteq \mathcal{D}'(X)$, then we find

$$\begin{aligned}
\langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi\rangle &= \lim_{n\to\infty} \langle \psi_n, (-1)^{|\alpha|} f \partial^\alpha \varphi\rangle = \lim_{n\to\infty} (-1)^{|\alpha|} \int_X \psi_n(x) f(x) \partial^\alpha \varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \partial^\alpha(\psi_n(x) f(x)) \varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_X \left(\sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \cdot \partial^{\alpha-\beta}\psi_n(x)\right) \varphi(x)\,\mathrm{d}x \\
&= \lim_{n\to\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta}\psi_n, \varphi\rangle = \lim_{n\to\infty} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle \psi_n, \partial^\beta f \partial^{\alpha-\beta}\varphi\rangle \\
&= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle T, \partial^\beta f \partial^{\alpha-\beta}\varphi\rangle = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} T, \varphi\rangle \\
&= \left\langle \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T, \varphi \right\rangle.
\end{aligned}$$

□

Question 7. *Let (x_k) be a sequence in X with no limit point in X. Consider the family of linear maps $u_\alpha \colon \mathcal{D}(X) \to \mathbb{C}$ defined by*

$$\langle u_\alpha, \varphi\rangle = \sum_{k=1}^\infty \partial^\alpha \varphi(x_k)$$

for each multi-index α. For what α is $u_\alpha \in \mathcal{D}'(X)$? What is $\operatorname{ord}(u_\alpha)$?

Solution. Let $K \subseteq X$ be compact. Since (x_k) does not have a limit point, only finitely many of the x_k lie in K (otherwise (x_k) would have a subsequence contained in K which would have a convergent subsequence). Without loss of generality, assume that $x_1, \dots, x_n \in K$, and $x_{n+1}, x_{n+2}, \dots, \notin K$. Now, for any $\varphi \in \mathcal{D}(X)$ with $\operatorname{supp}(\varphi) \subseteq K$ we find

$$|\langle u, \varphi\rangle| = \left|\sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)\right| = \left|\sum_{k=1}^{n} \partial^\alpha \varphi(x_k)\right| \leqslant \sum_{k=1}^{n} |\partial^\alpha \varphi(x_k)| \leqslant n \cdot \sup |\partial^\alpha \varphi| \leqslant n \cdot \sum_{|\beta| \leqslant |\alpha|} \sup \left|\partial^\beta \varphi\right|.$$

This shows that $u_\alpha \in \mathcal{D}'(X)$ for any α, with $\operatorname{ord}(u_\alpha) \leqslant |\alpha|$. We claim that this is an equality, i.e., $\operatorname{ord}(u_\alpha) = |\alpha|$. TODO: How to show??

Question 8. *Find the most general solution to the equations*

(a) $u' = 1$,

(b) $xu' = \delta_0$,

(c) $(e^{2\pi i x} - 1)u' = 0$

in $\mathcal{D}'(\mathbb{R})$.

Solution. Let $\varphi \in \mathcal{D}(X)$.

(a) If $u' = 1$ then we find

$$\int_{\mathbb{R}} \varphi(x)\, \mathrm{d}x = \langle 1, \varphi\rangle = \langle u', \varphi\rangle = -\langle u, \varphi'\rangle,$$

so for any $c \in \mathbb{R}$ we find by partial integration $\langle u, \varphi'\rangle = -\int_{\mathbb{R}} \varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}} (x + c)\varphi'(x)\, \mathrm{d}x$. From this we deduce that $u = x + c$ for some c.

(b) If $xu' = \delta_0$ then we have

$$\varphi(0) = \langle \delta_0, \varphi\rangle = \langle xu', \varphi\rangle = \langle u', x\varphi\rangle = \langle u, -(x\varphi)'\rangle = \langle u, -x\varphi' - \varphi\rangle$$

Note that the above is satisfied for $u = -\delta_0 + c$ for any constant c. TODO: is this the most general solution?

(c) Since $e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$, intuitively it must be the case that u' is 0, except "on $\mathbb{Z}$", whatever that may mean. Therefore, we guess that, for any sequence $(\alpha_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ and constant $c \in \mathbb{C}$, the map

$$u = c + \sum_{n\in\mathbb{Z}} \alpha_n \mathbb{1}_{x \geqslant n}.$$

We compute the derivative of u. It is easily seen that $u' = \sum_{n\in\mathbb{Z}} \alpha_n \delta_n$ (the infiniteness of the sum does not pose a problem since the test functions are compactly supported, so $\langle u, \varphi\rangle$ will always be a finite sum). From this, we see that

$$\langle (e^{2\pi i x} - 1)u', \varphi\rangle = \langle u', (e^{2\pi i x} - 1)\varphi\rangle = \sum_{n\in\mathbb{Z}} \alpha_n (e^{2\pi i n} - 1)\varphi(n) = 0,$$

so u satisfies the equation. TODO: Why is this the most general solution? Intuitively clear, but how to make this rigorous?

Question 9. *Define the distribution* $u \in \mathcal{D}'(\mathbb{R}^2)$ *by the locally integrable function* $u(x, y) = \mathbb{1}_{x \geqslant y}$. *Show that* $\partial_x^2 u - \partial_y^2 u = 0$ *in* $\mathcal{D}'(\mathbb{R}^2)$. *Can you give a physical interpretation of this result?*

Proof. Let $f \in \mathcal{D}(\mathbb{R}^2)$, then we have

$$\begin{aligned}
\langle \partial_x^2 u - \partial_y^2 u, f\rangle &= \langle \partial_x^2 u, f\rangle - \langle \partial_y^2 u, f\rangle = \langle u, \partial_x^2 f\rangle - \langle u, \partial_y^2 f\rangle = \langle u, \partial_x^2 f - \partial_y^2 f\rangle \\
&\overset{\star}{=} \int_{-\infty}^{\infty}\int_{y}^{\infty} \partial_x^2 f(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_{-\infty}^{\infty}\int_{-\infty}^{x} \partial_y^2 f(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} \partial_x f(y,y)\,\mathrm{d}y - \int_{-\infty}^{\infty} \partial_y f(x,x)\,\mathrm{d}x \\
&= -\int_{-\infty}^{\infty} (\partial_x f + \partial_y f)(x,x)\,\mathrm{d}x\,.
\end{aligned}$$

Here, $\star$ follows from Fubini's theorem. Define $g(x) = f(x,x)$, then it is easily seen that $g'(x) = \partial_x f(x,x) + \partial_y f(x,x)$, so we find that

$$\langle \partial_x^2 u - \partial_y^2 u, f\rangle = -\int_{-\infty}^{\infty} g'(x)\,\mathrm{d}x = \lim_{x\to-\infty} g(x) - \lim_{x\to\infty} g(x) = 0 - 0 = 0.$$

This shows that $\partial_x u - \partial_y u = 0$, or equivalently, that u satisfies the wave equation. ☐

Question 10. *Compute $\Delta(\|x\|^{2-n})$ in $\mathcal{D}'(\mathbb{R}^n)$ for $n \geqslant 3$, i.e. compute*

$$\langle \Delta(\|x\|^{2-n}), \varphi\rangle = \langle \|x\|^{2-n}, \Delta\varphi\rangle = \int \frac{\Delta\varphi}{\|x\|^{n-2}}\,\mathrm{d}x$$

for arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Note that $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\Delta = \sum_i (\frac{\partial}{\partial x_i})^2$.

Hint: *use $\int \mathrm{d}x = \int_{\|x\|\leqslant\varepsilon} \mathrm{d}x + \int_{\|x\|>\varepsilon} \mathrm{d}x$ and treat each integral separately.*

Solution. We follow the hint: let $\varepsilon > 0$, then we first compute

$$\int_{\|x\|>\varepsilon} \frac{\Delta\varphi}{\|x\|^{n-2}}\,\mathrm{d}x = \sum_{i=1}^{n} \int_{\|x\|>\varepsilon} \frac{\frac{\partial^2}{\partial x_i^2}\varphi}{\|x\|^{n-2}}\,\mathrm{d}x = \sum_{i=1}^{n} \int_{\|x\|>\varepsilon} \varphi \cdot \frac{\partial^2}{\partial x_i^2}\|x\|^{2-n}\,\mathrm{d}x\,.$$

Now, it is easily computed that $\frac{\partial}{\partial x_i}\|x\| = \frac{x_i}{\|x\|}$, and therefore

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}\|x\|^{2-n} &= \frac{\partial}{\partial x_i}(2-n)x_i\|x\|^{-n} = (2-n)\Big(\|x\|^{-n} - n x_i^2 \|x\|^{-n-2}\Big) \\
&= (n-2)\|x\|^{-n}\left(n\left(\frac{x_i}{\|x\|}\right)^2 - 1\right).
\end{aligned}$$

TODO: finish

Question 11. *Let (f_k) be the sequence of smooth functions defined by $f_k(x) = \frac{1}{\pi}\frac{k}{(kx)^2+1}$. Prove that $f_k \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$. Compute the limits*

(a) $\lim_{k\to\infty} k^2 x e^{-k^2x^2}$,

(b) $\lim_{k\to\infty} k^3 e^{ikx}$,

(c) $\lim_{k\to\infty} \frac{\sin(kx)}{\pi x}$,

in $\mathcal{D}'(\mathbb{R})$.

Solution. It is easily seen that every f_k is locally integrable. Furthermore, noting that $\arctan(kx)' = \frac{k}{(kx)^2+1}$, we see by the dominated convergence theorem that

$$\begin{aligned}\lim_{k\to\infty}\langle f_k,\varphi\rangle &= \frac{1}{\pi}\lim_{k\to\infty}\int_{\mathbb{R}}\frac{k}{(kx)^2+1}\varphi(x)\,\mathrm{d}x = -\frac{1}{\pi}\lim_{k\to\infty}\int_{\mathbb{R}}\arctan(kx)\varphi'(x)\,\mathrm{d}x\\ &= -\frac{1}{\pi}\int_{\mathbb{R}}\lim_{k\to\infty}\arctan(kx)\varphi'(x)\,\mathrm{d}x = -\frac{1}{\pi}\left(2\frac{\pi}{2}\int_0^\infty\varphi'(x)\,\mathrm{d}x\right) = \varphi(0),\end{aligned}$$

which proves $f_k \to \delta_0$.

(a) We have $-(\frac{1}{2}e^{-k^2x^2})' = k^2xe^{-k^2x^2}$ and therefore

$$\lim_{k\to\infty}\int_{\mathbb{R}}k^2xe^{-k^2x^2}\varphi(x)\,\mathrm{d}x = \frac{1}{2}\lim_{k\to\infty}\int_{\mathbb{R}}e^{-k^2x^2}\varphi'(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\lim_{k\to\infty}e^{-k^2x^2}\varphi'(x)\,\mathrm{d}x = 0,$$

so the sequence converges to 0 in $\mathcal{D}'(\mathbb{R})$.

(b) We have

$$\lim_{k\to\infty}k^3\int_{\mathbb{R}}e^{ikx}\varphi(x)\,\mathrm{d}x = \lim_{k\to\infty}2\pi k^3\mathcal{F}^{-1}[\varphi](k).$$

Since φ is a Schwarz function, its inverse Fourier transform is also a Schwarz function, and in particular $k^3\mathcal{F}^{-1}[\varphi](k) \to 0$ as $k \to \infty$.

(c) This sequence converges to δ_0, although I have no idea how to prove it.

Question 12. *Compute the limit*

$$\lim_{k\to\infty}\frac{1}{2}+\sum_{m=1}^{k}\cos(\pi mx)$$

in $\mathcal{D}'(-1,1)$.

Solution. Note that

$$f_k(x) := \frac{1}{2}+\sum_{m=1}^{k}\cos(\pi mx) = \frac{1}{2}\big(1+e^{i\pi x}+e^{-i\pi x}+\cdots+e^{i\pi mx}+e^{-i\pi mx}\big) = \frac{1}{2}\sum_{m=-k}^{k}(e^{i\pi x})^m.$$

By viewing the last term as a geometric series $\frac{1}{2}e^{-ik\pi x}\sum_{m=0}^{2k+1}(e^{ix})^m$ we can compute that

$$f_k(x) = \frac{\sin\big((n+\frac{1}{2})\pi x\big)}{2\sin\big(\frac{1}{2}\pi x\big)}.$$

It can be shown (???) that $f_k \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$.

Question 13. *We define the* principal value *of* $1/x$*, written* p.v.$(1/x)$*, by*

$$\langle \text{p.v.}(1/x),\varphi\rangle = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x \quad \textit{for all } \varphi\in\mathcal{D}(\mathbb{R}).$$

Prove that p.v.$(1/x) \in \mathcal{D}'(\mathbb{R})$ *and that* $\operatorname{ord}(\text{p.v.}(1/x)) = 1$*. Show that*

$$\lim_{\varepsilon\to 0}\frac{1}{x-i\varepsilon} = \text{p.v.}\left(\frac{1}{x}\right)+i\pi\delta_0 \quad \textit{in } \mathcal{D}'(\mathbb{R}).$$

Proof. First we must show that p.v.$(1/x)$ is well-defined. ?? $\square$

Question 14. *??*

Proof. ?? $\square$

Question 15. *??*

Proof. ?? $\square$

Question 16. *Define the distribution $u \in \mathcal{E}'(\mathbb{R}^3)$ by the locally integrable function $u(x) = \mathbb{1}_{|x|\leqslant 1}$. Prove that $-\sum_i x_i(\frac{\partial u}{\partial x_i}) = \mathrm{d}\sigma_2$ in $\mathcal{E}'(\mathbb{R}^3)$, where $\mathrm{d}\sigma_2$ is the surface element on the sphere $S^2 \subseteq \mathbb{R}^3$.*

Proof. We have

$$\begin{aligned}\left\langle -\sum_i x_i \frac{\partial u}{\partial x_i}, \varphi \right\rangle &= -\sum_i \left\langle \frac{\partial u}{\partial x_i}, x_i\varphi \right\rangle = \sum_i \left\langle u, \frac{\partial}{\partial x_i}(x_i\varphi) \right\rangle = \left\langle u, \sum_i \frac{\partial}{\partial x_i}(x_i\varphi) \right\rangle \\ &= \int_{\|x\|\leqslant 1} \sum_i \frac{\partial}{\partial x_i}(x_i\varphi)\,\mathrm{d}x\end{aligned}$$

$\square$