

Distribution Theory and Applications — Example Sheet 1

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Question 1. Construct a non-zero element of $\mathcal{D}(\mathbb{R})$ that vanishes outside $(0, 1)$. Construct a non-zero element of $\mathcal{D}(\mathbb{R}^n)$ that vanishes outside the ball $B_\varepsilon = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$.

Proof. It is well-known that the function

$$\varphi: \mathbb{R} \rightarrow \mathbb{R}: x \mapsto \begin{cases} 0 & \text{if } x \leq 0; \\ e^{-1/x} & \text{if } x > 0, \end{cases}$$

is smooth and vanishes outside $(0, \infty)$. The function $\psi(x) := \varphi(x)\varphi(1-x)$ is therefore also smooth and vanishes outside $(0, 1)$.

Since ψ vanishes outside $(0, 1)$, the function $\psi(x/\varepsilon)$ vanishes outside $(0, \varepsilon)$, and therefore the function $\mathbf{x} \mapsto \psi(\|\mathbf{x}\|/\varepsilon)$ vanishes outside B_ε . \square

Question 2. Given $\varphi \in \mathcal{D}(X)$, Taylor's theorem gives

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h).$$

Prove that $\text{supp}(R_N)$ is contained in some fixed compact $K \subseteq X$ for $|h|$ sufficiently small. Show also that $\partial^\alpha R_N = o(|h|^N)$ uniformly in x for each multi-index α , i.e. prove

$$\lim_{|h| \rightarrow 0} \frac{\sup_x |\partial^\alpha R_N(x, h)|}{|h|^N} = 0$$

for each multi-index α .

Hint: you may find it convenient to use the following form of the remainder

$$R_N(x, h) = \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th) dt,$$

and note that $(N+1)! \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} = (h_1 + \dots + h_n)^{N+1}$.

Proof. Since $\varphi \in \mathcal{D}(X)$, we know that $\text{supp } \varphi \subseteq \overline{B_N}$ for some $N \in \mathbb{N}$. Now, suppose $\|h\| < 1$, then

$$\varphi(x+h) \neq 0 \implies \|x+h\| \leq N \implies \|x\| \leq \|x+h\| + \|h\| \leq N+1,$$

so if we define $\psi_h(x) = \varphi(x+h)$ then we know that $\text{supp } \psi_h \subseteq \overline{B_{N+1}}$.

By Taylor's theorem, we have

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h),$$

and since $\sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x)$ vanishes for $x \notin \overline{B_N}$, it is clear that $\text{supp}(R_N(\cdot, h))$ must also be contained in $\overline{B_{N+1}}$ (again, for $\|h\| \leq 1$). This shows that $\text{supp}(R_N)$ is contained in $\overline{B_{N+1}}$ for $|h|$ sufficiently small.

Now let β be a multi-index and define $C := \frac{1}{N!} \max_{|\alpha|=N+1, x \in \mathbb{R}^n} |(\partial^{\alpha+\beta})\varphi(x)|$ (note that C exists and is finite since all partial derivatives of φ have compact support), then we have

$$\begin{aligned} |\partial^\beta R_N(x, h)| &= \left| \partial^\beta \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th) dt \right| \\ &\stackrel{*}{=} \left| \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^{\alpha+\beta} \varphi)(x+th) dt \right| \\ &\leq \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} (N+1) \int_0^1 (1-t)^N |(\partial^{\alpha+\beta} \varphi)(x+th)| dt \\ &\leq \left[\max_{|\alpha|=N+1, x \in \mathbb{R}^n} |(\partial^{\alpha+\beta})\varphi(x)| \right] \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} (N+1) \\ &\leq C(N+1)! \sum_{|\alpha|=N+1} \frac{|h^\alpha|}{\alpha!} = C(|h_1| + \dots + |h_n|)^{N+1}. \end{aligned}$$

Since this upper bound does not depend on x , we also have

$$\sup_x |\partial^\beta R_N(x, h)| \leq C(|h_1| + \dots + |h_n|)^{N+1},$$

and we conclude that

$$\frac{\sup_x |\partial^\beta R_N(x, h)|}{\|h\|^N} \leq \frac{C(|h_1| + \dots + |h_n|)^{N+1}}{\|h\|^N} \leq \frac{CN^{N+1} \|h\|^{N+1}}{\|h\|^N} = CN^{N+1} \|h\| \rightarrow 0,$$

and therefore that $\partial^\beta R_N(x, h) = o(\|h\|^n)$ for all multi-indices β . \square

Question 3. Which elements of $\mathcal{D}(X)$ can be represented as a power series on X ?

Solution. It is known that if two power series agree on an open set, they agree on the entire space. Since every $\varphi \in \mathcal{D}(X)$ is identically zero on some open set (outside its support), the only element of $\mathcal{D}(X)$ with a power series representation is the zero function.

Question 4. Prove the C^∞ Urysohn lemma: if K is a compact subset of $X \subseteq \mathbb{R}^n$, show that one can find a $\varphi \in \mathcal{D}(X)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on a neighborhood of K .

Solution. Let $K \subseteq U_1$. Define $U_2 := U_1 + B(0, 1)$ and let $\chi = \mathbb{1}_{U_2}$. Now let $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \psi dx = 1$ and $\text{supp } \psi \subseteq B(0, 1)$. Then we compute

$$(\chi * \psi)(x) = \int_{\mathbb{R}^n} \chi(y) \psi(x-y) dy = \int_{U_2} \psi(x-y) dy.$$

Clearly, $\chi * \psi \in \mathcal{D}(X)$, and furthermore, we have for $x \in U_1$ that

$$\int_{U_2} \psi(x-y) dy = \int_{U_2-x} \psi(z) dz \stackrel{*}{=} 1,$$

since $B(0, 1) \subseteq U_2 - x$. This proves the claim.

Question 5. Given $T \in \mathcal{D}'(X)$, the derivative $\partial^\alpha T$ is defined by

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Show that $\partial^\alpha T \in \mathcal{D}'(X)$. If $\text{ord}(T) = m$ what can you say about $\text{ord}(\partial^\alpha T)$?

Proof. Let $K \subseteq X$ be compact and $\varphi \in \mathcal{D}(X)$. Since T is a distribution, we know that there exists constants C, N such that

$$|\langle T, \varphi \rangle| \leq C \sum_{|\beta| \leq N} \sup |\partial^\beta \varphi|.$$

Letting $M := |\alpha|$, we find

$$|\langle \partial^\alpha T, \varphi \rangle| = |\langle T, \partial^\alpha \varphi \rangle| \leq C \sum_{|\beta| \leq N} \sup |\partial^{\alpha+\beta} \varphi| \leq C \sum_{|\beta| \leq M+N} \sup |\partial^\beta \varphi|.$$

We conclude that $\partial^\alpha T$ is a distribution, and that if $\text{ord}(T) = m$, $\text{ord}(\partial^\alpha T) \leq m + |\alpha|$. \square

Question 6. Given $T \in \mathcal{D}'(X)$ and $f \in C^\infty(X)$, prove that for each multi-index α

$$\partial^\alpha(Tf) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

in $\mathcal{D}'(X)$.

Proof. Let $\varphi \in \mathcal{D}(X)$, then by definition we have $\langle \partial^\alpha(Tf), \varphi \rangle = \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle$. Approximate T by a sequence $(\psi_n) \subseteq \mathcal{D}'(X)$, then we find

$$\begin{aligned} \langle T, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle &= \lim_{n \rightarrow \infty} \langle \psi_n, (-1)^{|\alpha|} f \partial^\alpha \varphi \rangle = \lim_{n \rightarrow \infty} (-1)^{|\alpha|} \int_X \psi_n(x) f(x) \partial^\alpha \varphi(x) dx \\ &= \lim_{n \rightarrow \infty} \int_X \partial^\alpha (\psi_n(x) f(x)) \varphi(x) dx \\ &= \lim_{n \rightarrow \infty} \int_X \left(\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f(x) \cdot \partial^{\alpha-\beta} \psi_n(x) \right) \varphi(x) dx \\ &= \lim_{n \rightarrow \infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} \psi_n, \varphi \rangle = \lim_{n \rightarrow \infty} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle \psi_n, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle \\ &= \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (-1)^{|\alpha-\beta|} \langle T, \partial^\beta f \partial^{\alpha-\beta} \varphi \rangle = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \langle \partial^\beta f \partial^{\alpha-\beta} T, \varphi \rangle \\ &= \left\langle \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T, \varphi \right\rangle. \end{aligned}$$

\square

Question 7. Let (x_k) be a sequence in X with no limit point in X . Consider the family of linear maps $u_\alpha: \mathcal{D}(X) \rightarrow \mathbb{C}$ defined by

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)$$

for each multi-index α . For what α is $u_\alpha \in \mathcal{D}'(X)$? What is $\text{ord}(u_\alpha)$?

Solution. Let $K \subseteq X$ be compact. Since (x_k) does not have a limit point, only finitely many of the x_k lie in K (otherwise (x_k) would have a subsequence contained in K which would have a convergent subsequence). Without loss of generality, assume that $x_1, \dots, x_n \in K$, and $x_{n+1}, x_{n+2}, \dots \notin K$. Now, for any $\varphi \in \mathcal{D}(X)$ with $\text{supp}(\varphi) \subseteq K$ we find

$$|\langle u, \varphi \rangle| = \left| \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k) \right| = \left| \sum_{k=1}^n \partial^\alpha \varphi(x_k) \right| \leq \sum_{k=1}^n |\partial^\alpha \varphi(x_k)| \leq n \cdot \sup |\partial^\alpha \varphi| \leq n \cdot \sum_{|\beta| \leq |\alpha|} \sup |\partial^\beta \varphi|.$$

This shows that $u_\alpha \in \mathcal{D}'(X)$ for any α , with $\text{ord}(u_\alpha) \leq |\alpha|$. We claim that this is an equality, i.e., $\text{ord}(u_\alpha) = |\alpha|$. **TODO:** How to show??

Question 8. Find the most general solution to the equations

(a) $u' = 1$,

(b) $xu' = \delta_0$,

(c) $(e^{2\pi i x} - 1)u' = 0$

in $\mathcal{D}'(\mathbb{R})$.

Solution. Let $\varphi \in \mathcal{D}(X)$.

(a) If $u' = 1$ then we find

$$\int_{\mathbb{R}} \varphi(x) dx = \langle 1, \varphi \rangle = \langle u', \varphi \rangle = -\langle u, \varphi' \rangle,$$

so for any $c \in \mathbb{R}$ we find by partial integration $\langle u, \varphi' \rangle = -\int_{\mathbb{R}} \varphi(x) dx = \int_{\mathbb{R}} (x + c)\varphi'(x) dx$. From this we deduce that $u = x + c$ for some c .

(b) If $xu' = \delta_0$ then we have

$$\varphi(0) = \langle \delta_0, \varphi \rangle = \langle xu', \varphi \rangle = \langle u', x\varphi \rangle = \langle u, -(x\varphi)' \rangle = \langle u, -x\varphi' - \varphi \rangle$$

Note that the above is satisfied for $u = -\delta_0 + c$ for any constant c . **TODO:** is this the most general solution?

(c) Since $e^{2\pi i x} = 1 \iff x \in \mathbb{Z}$, intuitively it must be the case that u' is 0, except “on \mathbb{Z} ”, whatever that may mean. Therefore, we guess that, for any sequence $(\alpha_n)_{n \in \mathbb{Z}} \subseteq \mathbb{C}$ and constant $c \in \mathbb{C}$, the map

$$u = c + \sum_{n \in \mathbb{Z}} \alpha_n \mathbb{1}_{x \geq n}.$$

We compute the derivative of u . It is easily seen that $u' = \sum_{n \in \mathbb{Z}} \alpha_n \delta_n$ (the infiniteness of the sum does not pose a problem since the test functions are compactly supported, so $\langle u, \varphi \rangle$ will always be a finite sum). From this, we see that

$$\langle (e^{2\pi i x} - 1)u', \varphi \rangle = \langle u', (e^{2\pi i x} - 1)\varphi \rangle = \sum_{n \in \mathbb{Z}} \alpha_n (e^{2\pi i n} - 1)\varphi(n) = 0,$$

so u satisfies the equation. **TODO:** Why is this the most general solution? Intuitively clear, but how to make this rigorous?

Question 9. Define the distribution $u \in \mathcal{D}'(\mathbb{R}^2)$ by the locally integrable function $u(x, y) = \mathbb{1}_{x \geq y}$. Show that $\partial_x^2 u - \partial_y^2 u = 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Can you give a physical interpretation of this result?

Proof. Let $f \in \mathcal{D}(\mathbb{R}^2)$, then we have

$$\begin{aligned}
\langle \partial_x^2 u - \partial_y^2 u, f \rangle &= \langle \partial_x^2 u, f \rangle - \langle \partial_y^2 u, f \rangle = \langle u, \partial_x^2 f \rangle - \langle u, \partial_y^2 f \rangle = \langle u, \partial_x^2 f - \partial_y^2 f \rangle \\
&\stackrel{*}{=} \int_{-\infty}^{\infty} \int_y^{\infty} \partial_x^2 f(x, y) \, dx \, dy - \int_{-\infty}^{\infty} \int_{-\infty}^x \partial_y^2 f(x, y) \, dy \, dx \\
&= - \int_{-\infty}^{\infty} \partial_x f(y, y) \, dy - \int_{-\infty}^{\infty} \partial_y f(x, x) \, dx \\
&= - \int_{-\infty}^{\infty} (\partial_x f + \partial_y f)(x, x) \, dx.
\end{aligned}$$

Here, \star follows from Fubini's theorem. Define $g(x) = f(x, x)$, then it is easily seen that $g'(x) = \partial_x f(x, x) + \partial_y f(x, x)$, so we find that

$$\langle \partial_x^2 u - \partial_y^2 u, f \rangle = - \int_{-\infty}^{\infty} g'(x) \, dx = \lim_{x \rightarrow -\infty} g(x) - \lim_{x \rightarrow \infty} g(x) = 0 - 0 = 0.$$

This shows that $\partial_x u - \partial_y u = 0$, or equivalently, that u satisfies the wave equation. \square

Question 10. Compute $\Delta(\|x\|^{2-n})$ in $\mathcal{D}'(\mathbb{R}^n)$ for $n \geq 3$, i.e. compute

$$\langle \Delta(\|x\|^{2-n}), \varphi \rangle = \langle \|x\|^{2-n}, \Delta \varphi \rangle = \int \frac{\Delta \varphi}{\|x\|^{n-2}} \, dx$$

for arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Note that $\|x\| = (x_1^2 + \dots + x_n^2)^{1/2}$ and $\Delta = \sum_i (\frac{\partial}{\partial x_i})^2$.

Hint: use $\int dx = \int_{\|x\| \leq \varepsilon} dx + \int_{\|x\| > \varepsilon} dx$ and treat each integral separately.

Solution. We follow the hint: let $\varepsilon > 0$, then we first compute

$$\int_{\|x\| > \varepsilon} \frac{\Delta \varphi}{\|x\|^{n-2}} \, dx = \sum_{i=1}^n \int_{\|x\| > \varepsilon} \frac{\frac{\partial^2}{\partial x_i^2} \varphi}{\|x\|^{n-2}} \, dx = \sum_{i=1}^n \int_{\|x\| > \varepsilon} \varphi \cdot \frac{\partial^2}{\partial x_i^2} \|x\|^{2-n} \, dx.$$

Now, it is easily computed that $\frac{\partial}{\partial x_i} \|x\| = \frac{x_i}{\|x\|}$, and therefore

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2} \|x\|^{2-n} &= \frac{\partial}{\partial x_i} (2-n)x_i \|x\|^{-n} = (2-n) \left(\|x\|^{-n} - nx_i^2 \|x\|^{-n-2} \right) \\
&= (n-2) \|x\|^{-n} \left(n \left(\frac{x_i}{\|x\|} \right)^2 - 1 \right).
\end{aligned}$$