Topics — Example Sheet 2

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Question 2. *For $\beta \in (0,1]$ and $L > 0$, let $\mathcal{F}_2(\beta, L)$ denote the class of densities on $\mathbb{R}^2$ that satisfy*

$$|f(x,y) - f(x_0,y_0)| \le L(|x-x_0|^\beta + |y-y_0|^\beta)$$

for all $(x,y),(x_0,y_0) \in \mathbb{R}^2$. Let K be a non-negative kernel on $\mathbb{R}$ with $\mu_\beta(K)$ and $R(K)$ finite. Given i.i.d. pairs $(X_1,Y_1),\dots,(X_n,Y_n)$, consider the kernel density estimator $\hat{f}_n$ obtained using a product kernel, i.e.,

$$\hat{f}_n(x_0,y_0) \coloneqq \frac{1}{nh^2}\sum_{i=1}^n K\left(\frac{x_0-X_i}{h}\right)K\left(\frac{y_0-Y_i}{h}\right).$$

Find a bound on $\operatorname{MSE}\left\{\hat{f}_n(x_0,y_0)\right\}$ *that holds uniformly for all $f \in \mathcal{F}_2(\beta,L)$ and $(x_0,y_0) \in \mathbb{R}^2$.*

Proof. First we compute a variance bound following the proof of proposition 19: noting that $\hat{f}_n(x,y) = \frac{1}{n}\sum_{i=1}^n K_h(x-X_i)K_h(y-Y_i)$, we have

$$\begin{aligned}
\operatorname{Var}\hat{f}_n(x,y) &= \frac{1}{n}\operatorname{Var}(K_h(x-X_i)K_h(y-Y_i)) \le \frac{1}{n}\mathbb{E}[K_h^2(x-X_i)K_h^2(y-Y_i)] \\
&= \frac{1}{nh^4}\iint_{\mathbb{R}^2} K^2(\frac{x-w}{h})K^2(\frac{y-z}{h})f(w,z)\,\mathrm{d}w\,\mathrm{d}z \\
&\le \frac{\|f\|_\infty}{nh^2}\iint_{\mathbb{R}^2} K^2(s)K^2(t)\,\mathrm{d}s\,\mathrm{d}t = \frac{\|f\|_\infty R(K)^2}{nh^2}.
\end{aligned}$$

Next, we compute a bias bound following the proof of proposition 22. Note that we have

$$\begin{aligned}
\operatorname{Bias}\hat{f}_n(x,y) &= \frac{1}{h^2}\iint_{\mathbb{R}^2} K(\frac{x-w}{h})K(\frac{y-z}{h})f(w,z)\,\mathrm{d}w\,\mathrm{d}z - f(x,y) \\
&= \iint_{\mathbb{R}^2} K(s)K(t)\{f(x-sh,y-th)-f(x,y)\}\,\mathrm{d}s\,\mathrm{d}t\,,
\end{aligned}$$

and taking absolute values gives

$$\begin{aligned}
\left|\operatorname{Bias}\hat{f}_n(x,y)\right| &\le \iint_{\mathbb{R}^2} K(s)K(t)|f(x-sh,y-th)-f(x,y)|\,\mathrm{d}s\,\mathrm{d}t \\
&\le L\iint_{\mathbb{R}^2} K(s)K(t)\left(|sh|^\beta + |th|^\beta\right)\mathrm{d}s\,\mathrm{d}t \\
&= 2Lh^\beta \int_{\mathbb{R}} K(s)\int_{\mathbb{R}} |t|K(t)\,\mathrm{d}t\,\mathrm{d}s \\
&= 2Lh^\beta\mu_\beta(K)\int_{\mathbb{R}} K(s)\,\mathrm{d}s = 2Lh^\beta\mu_\beta(K).
\end{aligned}$$

We therefore have

$$\operatorname{MSE}\hat{f}_n(x,y) \le \frac{1}{nh^2}\|f\|_\infty R(K)^2 + 4L^2\mu_\beta(K)^2h^{2\beta}.$$

Completely analogous to the proof of theorem 23, we can show that $\|f\|_\infty$ is bounded uniformly over $\mathcal{F}_2(\beta, L)$, and the minimiser of MSE is of order $n^{-1/(2\beta+2)}$. Plugging this into the expression gives

$$\sup_{(x,y)} \sup_{f \in \mathcal{F}} \operatorname{MSE} \hat{f}_n(x,y) \leq C n^{-2\beta/(2\beta+2)},$$

for some C depending only on β, L, K. $\square$

Question 10. *Let $n \geq 3$, let $a \leq x_1 < \cdots < x_n \leq b$, and let $\boldsymbol{g} \in \mathbb{R}^n$. Prove that the natural cubic spline interpolant to $\boldsymbol{g}$ at $x_1, \ldots, x_n$ is the unique minimiser of $R(\tilde{g}) = \int_a^b \tilde{g}''(x)^2 \, \mathrm{d}x$ over all $\tilde{g} \in \mathcal{S}_2[a,b]$ that interpolate $\boldsymbol{g}$ at $x_1, \ldots, x_n$.*

Hint: *Let g be the natural cubic spline interpolant, $h \coloneqq \tilde{g} - g$, and consider $\int_a^b g''(x)h''(x) \, \mathrm{d}x$.*

Solution. As in the hint we let g be the natural cubic spline interpolant, $\tilde{g} \in \mathcal{S}_2[a,b]$, and $h \coloneqq \tilde{g} - g$. We know that g is a cubic polynomial p_i on each interval $[x_i, x_{i+1}]$ (denote its leading coefficient by c_i), that g'' is continuous, and that $g'' = 0$ on $[a, x_1]$ and on $[x_n, b]$. Furthermore, we know that $h(x_i) = 0$ for all i. Using this, we can write

$$\begin{aligned}
\int_a^b g''(x)h''(x) \, \mathrm{d}x &= \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} p_i''(x)h''(x) \, \mathrm{d}x \\
&= \sum_{i=1}^{n-1} \left([p_i''(x)h'(x)]_{x=x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} p_i'''(x)h'(x) \, \mathrm{d}x \right) \\
&= -\sum_{i=1}^{n-1} c_i \int_{x_i}^{x_{i+1}} h'(x) \, \mathrm{d}x \\
&= -\sum_{i=1}^{n-1} c_i (h(x_{i+1}) - h(x_i)) = 0.
\end{aligned}$$

Now we find

$$R(g) = \int_a^b g''(x)^2 \, \mathrm{d}x = \int_a^b g''(x)(\tilde{g}''(x) - h''(x)) \, \mathrm{d}x = \int_a^b g''(x)\tilde{g}''(x) \overset{\text{CS}}{\leq} \sqrt{R(g)}\sqrt{R(\tilde{g})},$$

with equality if and only if $g'' = \tilde{g}''$ (by Cauchy-Schwarz). Rearranging gives $\sqrt{R(g)} \leq \sqrt{R(\tilde{g})} \implies R(g) \leq R(\tilde{g})$. Since $\tilde{g} \in \mathcal{S}_2[a,b]$ was arbitrary, we deduce that g is a minimiser of R over all function in $\mathcal{S}_2$.

For uniqueness: we already know that any other minimiser $h \in \mathcal{S}_2$ must satisfy $g'' = h''$ a.e., so $g - h$ is a polynomial of degree 1 a.e., and by continuity we know that $g - h$ is a polynomial of degree 1. However, since g and h must agree on $n \geq 3$ points it follows that $g = h$.