Distribution Theory — Example Sheet 2

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We will write $\mathcal{R}$ and $\mathcal{F}$ for the reflection and Fourier transform operators.

Question 1. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, one of which has compact support. Show that the convolution $u * v$, defined as in your notes, is uniquely defined and gives rise to an element of $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. The convolution between $u, v \in \mathcal{D}'(\mathbb{R}^n)$ is defined by the formula

$$(u * v) * \varphi = u * (v * \varphi) \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1}$$

Recall that for all $u \in \mathcal{D}'(\mathbb{R}^n)\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have $\langle u, \varphi\rangle = (u * \check{\varphi})(0)$, and therefore $u * v$ should satisfy

$$\langle u * v, \varphi\rangle = ((u * v) * \check{\varphi})(0) = (u * (v * \check{\varphi}))(0) = \langle u, \mathcal{R}(v * \check{\varphi})\rangle.$$

We can now show existence and uniqueness:

1. Existence: define $w \colon \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ by

$$\langle w, \varphi\rangle = \langle u, \mathcal{R}(v * \check{\varphi})\rangle,$$

   then we will show that w satisfies $w * \varphi = u * (v * \varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

$$\begin{aligned}(w * \varphi)(x) &= \langle w, \tau_x \check{\varphi}\rangle = \langle u, \mathcal{R}(v * \mathcal{R}(\tau_x \check{\varphi}))\rangle = \langle u, \mathcal{R}(v * \tau_{-x}\varphi)\rangle \\ &= \langle u, \mathcal{R}\tau_{-x}(v * \varphi)\rangle = \langle u, \tau_x \mathcal{R}(v * \varphi)\rangle = (u * (v * \varphi))(x).\end{aligned}$$

2. Uniqueness: we have shown in the lectures that if $w * \varphi = w' * \varphi$ for all φ, then $w = w'$. This shows that eq. (1) uniquely defines $u * v$.

Now we prove that $u * v \in \mathcal{D}'(\mathbb{R}^n)$: by the previous equation we have

$$\langle u * v, \varphi\rangle = (u * (v * \check{\varphi}))(0) = \langle u, \widecheck{v * \check{\varphi}}\rangle.$$

Suppose u is compactly supported. Since $\widecheck{v * \check{\varphi}} \in \mathcal{E}(\mathbb{R}^n)$, there exists a compact $K \subseteq X$ and nonnegative C, N such that

$$\begin{aligned}\left|\langle u, \widecheck{v * \check{\varphi}}\rangle\right| &\leqslant C \sum_{\alpha \leqslant N} \sup_{x \in K} \left|\partial^\alpha(\widecheck{v * \check{\varphi}})\right| = C \sum_{\alpha \leqslant N} \sup_{x \in -K} |\partial^\alpha (v * \check{\varphi})| = C \sum_{\alpha \leqslant N} \sup_{x \in -K} |v * \partial^\alpha \check{\varphi}| \\ &= C \sum_{|\alpha| \leqslant N} \sup_{x \in -K} \left|\langle v, \tau_x \widecheck{\partial^\alpha \check{\varphi}}\rangle\right|.\end{aligned}$$

Note that if $\operatorname{supp} \varphi \subseteq K'$, then $\operatorname{supp} \check{\varphi} \subseteq -K'$, and for $x \in -K$ we find $\operatorname{supp} \tau_x \widecheck{\partial^\alpha \check{\varphi}} \subseteq -K' - K$. Then by the previous equation we find that there exists C', M with

$$|\langle u * v, \varphi\rangle| \leqslant C' \sum_{|\alpha| \leqslant N} \sum_{|\beta| \leqslant M} \sup_{x \in -K' - K} \partial^\beta(\tau_x \widecheck{\partial^\alpha \check{\varphi}}) \leqslant C' \sum_{|\alpha| \leqslant N} \sum_{|\beta| \leqslant M} \sup_x \left|\partial^{\alpha+\beta}\varphi\right| \leqslant C'' \sum_{|\alpha| \leqslant M+N} \sup_x |\partial^\alpha \varphi|,$$

which shows that $u * v \in \mathcal{D}'(\mathbb{R}^n)$. An analogous argument holds if v is compactly supported. $\square$

Question 19. *Compute the Fourier transforms of the functions*

(a) $\operatorname{sign}(x)$*;*

(b) $\arctan(x)$*;*

(c) $x \log|x| - x$*;*

(d) $\exp(i\omega x^2)$

in $\mathcal{S}'(\mathbb{R})$*, where* $\omega \in \mathbb{R}$*.*

Proof. (a) We have for $\varphi \in \mathcal{S}(\mathbb{R})$ that

$$\begin{aligned}
\langle \widehat{\operatorname{sign}}, \varphi\rangle &= \langle \operatorname{sign}, \hat{\varphi}\rangle = \int_{\mathbb{R}} \operatorname{sign}(\lambda)\hat{\varphi}(\lambda)\,\mathrm{d}\lambda = \int_{\mathbb{R}} \operatorname{sign}(\lambda)\int_{\mathbb{R}} e^{-i\lambda x}\varphi(x)\,\mathrm{d}x\,\mathrm{d}\lambda \\
&\overset{\star}{=} \int_{\mathbb{R}} \varphi(x)\int_{\mathbb{R}} \operatorname{sign}(\lambda)e^{-i\lambda x}\,\mathrm{d}\lambda\,\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\lim_{R\to\infty}\left(\int_0^R e^{-i\lambda x}\,\mathrm{d}\lambda - \int_{-R}^0 e^{-i\lambda x}\,\mathrm{d}\lambda\right)\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{|x|>\varepsilon}\varphi(x)\left(\frac{e^{ixR}+e^{-ixR}}{ix} - \frac{2}{ix}\right)\mathrm{d}x \\
&= \lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{\mathbb{R}}\frac{\varphi(x)\mathbb{1}_{|x|>\varepsilon}}{ix}\cdot e^{ixR}\,\mathrm{d}x + \lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{\mathbb{R}}\frac{\varphi(x)\mathbb{1}_{|x|>\varepsilon}}{ix}\cdot e^{-ixR}\,\mathrm{d}x + 2i\,\mathrm{P.V.}\left(\frac{1}{x}\right).
\end{aligned}$$

We claim the first two terms go to 0: this is because the term in the integral is the Fourier transform of $\frac{\varphi(x)\mathbb{1}_{|x|>\varepsilon}}{ix}$ evaluated at $\pm R$, and since the function is in L^1, its Fourier transform decays to 0 as $|R| \to \infty$.

We conclude $\widehat{\operatorname{sign}} = 2i\mathrm{P.V.}(\frac{1}{x})$.

(b) We know that $\arctan'(x) = \frac{1}{1+x^2} =: f(x)$, then we have $\widehat{\arctan}(\lambda) = \frac{1}{i\lambda}\hat{f}(\lambda)$ (in the distributional sense).

We have, using Fubini and the fact that $\langle \hat{f}, \varphi\rangle$ is finite, that

$$\langle \hat{f}, \varphi\rangle = \int_{\mathbb{R}} \frac{\hat{\varphi}(\lambda)}{1+\lambda^2}\,\mathrm{d}\lambda = \int_{\mathbb{R}} \varphi(x)\lim_{R\to\infty}\int_{-R}^R \frac{e^{-i\lambda x}}{1+\lambda^2}\,\mathrm{d}\lambda\,\mathrm{d}x \overset{\star}{=} \int_{\mathbb{R}} \varphi(x)\left(\pi e^{-|x|}\right)\mathrm{d}x\,,$$

from which it follows that the Fourier transform of $\frac{1}{1+x^2}$ is given by $\pi e^{-|\lambda|}$, and therefore the Fourier transform of arctan is given by $\frac{\pi}{i\lambda}e^{-|\lambda|}$.

(c) The derivative of this function is $\log(|x|)$, and the derivative of $\log(|x|)$ is $\mathrm{P.V.}(1/x)$ which we saw in the previous example sheet. By (a) we know that the Fourier transform of sign is $2i\mathrm{P.V.}(1/x)$, so we have

$$\mathcal{F}[2i\mathrm{P.V.}(1/x)] = \mathcal{F}[\mathcal{F}[\operatorname{sign}]] = (2\pi)^n\widetilde{\operatorname{sign}} \implies \mathcal{F}[\mathrm{P.V.}(1/x)] = \frac{(2\pi)^n}{2i}\widetilde{\operatorname{sign}}.$$

Note that $\widetilde{\operatorname{sign}} = -\operatorname{sign}$ we conclude

$$\mathcal{F}[x\log|x| - x](\lambda) = \frac{(2\pi)^n}{2i\lambda^2}\operatorname{sign}(\lambda).$$

(d) Clearly, if $\omega = 0$, the function is 1 and its Fourier transform is $2\pi\delta_0$, so assume $\omega \neq 0$. We have analogously to (b), with $f(x) = \exp(i\omega x^2)$, that

$$\langle \hat{f}, \varphi\rangle = \int_{\mathbb{R}} \hat{\varphi}(\lambda)e^{i\omega\lambda^2}\,\mathrm{d}\lambda = \int_{\mathbb{R}} \varphi(x)\lim_{R\to\infty}\int_{-R}^R e^{i\omega\lambda^2 - i\lambda x}\,\mathrm{d}\lambda\,\mathrm{d}x\,.$$

Now, by completing the square we have

$$i(\omega\lambda^2 - x\lambda) = i\left(\sqrt{\omega}\lambda - \frac{x}{2\sqrt{\omega}}\right)^2 - \frac{ix^2}{4\omega},$$

and therefore

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^{R} e^{i\omega\lambda^2 - i\lambda x}\,\mathrm{d}\lambda = e^{-ix^2/(4\omega)}\lim_{R\to\infty}\int_{-R}^{R} e^{i(\sqrt{\omega}\lambda - \frac{x}{2\sqrt{\omega}})^2}\,\mathrm{d}\lambda = \frac{e^{-ix^2/(4\omega)}}{\sqrt{\omega}}\lim_{R\to\infty}\int_{-R}^{R} e^{i\lambda^2}\,\mathrm{d}\lambda \\ = \frac{e^{-ix^2/(4\omega)}}{\sqrt{\omega}}(1+i)\sqrt{\frac{\pi}{2}},\end{aligned}$$

where we use that the *Fresnel integral* $\int_{-\infty}^{\infty} e^{ix^2}\,\mathrm{d}x$ is known.
Plugging this back into our original equation yields

$$\langle \hat{f}, \varphi\rangle = \int_{\mathbb{R}} \varphi(x)\frac{e^{-ix^2/(4\omega)}}{\sqrt{\omega}}(1+i)\sqrt{\frac{\pi}{2}}\,\mathrm{d}x\,,$$

which shows that

$$\hat{f}(\lambda) = (1+i)e^{-i\lambda^2/(4\omega)}\sqrt{\frac{\pi}{2\omega}}.$$

in the distributional sense.

$\square$