Reparameterizing the Birkhoff Polytope for Variational Permutation Inference: Supplementary Material

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A Alternative methods of discrete variational inference

Recently there have been a number of proposals for extending the reparameterization trick [Rezende et al., 2014, Kingma and Welling, 2014] to high dimensional discrete problems[1] by relaxing them to analogous continuous problems [Maddison et al., 2016, Jang et al., 2016, Kusner and Hernández-Lobato, 2016]. These approaches are based on the following observation: if $x \in \{0,1\}^N$ is a one-hot vector drawn from a categorical distribution, then the support of $p(x)$ is the set of vertices of the $N-1$ dimensional simplex. We can represent the distribution of x as an atomic density on the simplex.

A.1 The Gumbel-softmax method

Viewing x as a vertex of the simplex motivates a natural relaxation: rather than restricting ourselves to atomic measures, consider continuous densities on the simplex. To be concrete, suppose the density of x is defined by the transformation,

$$\begin{aligned}\xi_n &\overset{\text{iid}}{\sim} \text{Gumbel}(0,1)\\ \psi_n &= \log\theta_n + \xi_n\\ x &= \mathsf{softmax}(\psi/\tau)\\ &= \left(\frac{e^{\psi_1/\tau}}{\sum_{n=1}^N e^{\psi_n/\tau}}, \ldots, \frac{e^{\psi_N/\tau}}{\sum_{n=1}^N e^{\psi_n/\tau}}\right).\end{aligned}$$

The output x is now a point on the simplex, and the parameter $\theta = (\theta_1, \ldots, \theta_N)$ can be optimized via stochastic gradient ascent with the reparameterization trick.

The Gumbel distribution leads to a nicely interpretable model: when θ is a probability mass function, adding Gumbel noise and taking the argmax yields an exact sample from θ; the softmax is a natural relaxation. As the temperature τ goes to zero, the softmax converges to the argmax function. Ultimately, however, this is just a continuous relaxation of an atomic density to a continuous density.

Stick-breaking and rounding offer two alternative ways of conceiving a relaxed version of a discrete random variable, and both are amenable to reparameterization. However, unlike the Gumbel-Softmax, these relaxations enable extensions to more complex combinatorial objects, notably, permutations.

A.2 Stick-breaking

The stick-breaking transformation to the Birkhoff polytope presented in the main text contains a recipe for stick-breaking on the simplex. In particular, as we filled in the first row of the doubly-stochastic matrix, we were transforming a real-valued vector $\psi \in \mathbb{R}^{N-1}$ to a point in the simplex. We present this procedure for discrete variational inference again here in simplified form. Start with a reparameterization of a Gaussian vector,

$$\begin{aligned}\xi_n &\overset{\text{iid}}{\sim} \mathcal{N}(0,1),\\ \psi_n &= \mu_n + \eta_n\xi_n, \qquad 1 \le n \le N-1,\end{aligned}$$

parameterized by $\theta = (\mu_n, \eta_n)_{n=1}^{N-1}$. Then map this to the unit hypercube in a temperature-controlled manner with the logistic function,

$$z_n = \sigma(\psi_n/\tau),$$

where $\sigma(u) = (1+e^{-u})^{-1}$ is the logistic function. Finally, transform the unit hypercube to a point in the simplex:

$$\begin{aligned}x_1 &= z_1,\\ x_n &= z_n\left(1 - \sum_{m=1}^{n-1} x_m\right), \qquad 2 \le n \le N-1,\\ x_N &= 1 - \sum_{m=1}^{N-1} x_m,\end{aligned}$$

[1] Discrete inference is only problematic in the high dimensional case, since in low dimensional problems we can enumerate the possible values of x and compute the normalizing constant $p(y) = \sum_x p(y,x)$.

Here, z_n is the fraction of the remaining "stick" of probability mass assigned to x_n. This transformation is invertible, the Jacobian is lower-triangular, and the determinant of the Jacobian is easy to compute. Linderman et al. [2015] compute the density of x implied by a Gaussian density on ψ.

The temperature τ controls how concentrated $p(x)$ is at the vertices of the simplex, and with appropriate choices of parameters, in the limit $\tau \to 0$ we can recover any categorial distribution. In the other limit, as $\tau \to \infty$, the density concentrates on a point in the interior of the simplex determined by the parameters, and for intermediate values, the density is continuous on the simplex.

Finally, note that the logistic-normal construction is only one possible choice. We could instead let $z_n \sim \text{Beta}(\frac{a_n}{\tau}, \frac{b_n}{\tau})$. This would lead to the Dirichlet distribution on the simplex. The beta distribution is slightly harder to reparameterize since it is typically simulated with a rejection sampling procedure, but Naesseth et al. [2017] have shown how this can be handled with a mix of reparameterization and score-function gradients. Alternatively, the beta distribution could be replaced with the Kumaraswamy distribution, which is quite similar to the beta distribution but is easily reparameterizable.

A.3 Rounding

Rounding transformations also have a natural analog for discrete variational inference. Define the rounding operator,

$$\mathsf{round}(\psi) = \arg\min_{e_n} \|e_n - \psi\|^2,$$

which maps $\psi \in \mathbb{R}^N$ to the one-hot vectors e_n; i.e. the vectors in $\{0,1\}^N$ with n-th entry equal to one and all other entries equal zero. This is equivalent to defining $\mathsf{round}(\psi) = e_{n^*}$ where

$$\begin{aligned} n^* &= \arg\min_n \|e_n - \psi\|^2 \\ &= \arg\min_n \sum_{m \neq n} \psi_m^2 + (1-\psi_n)^2 \\ &= \arg\min_n \sum_{m \neq n} \psi_m^2 + \psi_n^2 - 2\psi_n + 1 \\ &= \arg\min_n \|\psi\|^2 - 2\psi_n + 1 \\ &= \arg\max_n \psi_n. \end{aligned}$$

In the case of a tie, let n^* be the smallest index n such that $\psi_n > \psi_m$ for all $m < n$. Rounding effectively partitions the space into N disjoint "Voronoi" cells,

$$V_n = \left\{\psi \in \mathbb{R}^N : \psi_n \geq \psi_m \, \forall m \ \wedge \ \psi_n > \psi_m \, \forall m < n\right\}.$$

By definition, $\mathsf{round}(\psi) = e_{n^*}$ for all $\psi \in V_{n^*}$

We define a map that pulls points toward their rounded values,

$$x = \tau\psi + (1-\tau)\mathsf{round}(\psi). \tag{1}$$

Proposition 1. *For $\tau \in [0,1]$, the map defined by (1) moves points strictly closer to their rounded values so that $\mathsf{round}(\psi) = \mathsf{round}(x)$.*

Proof. Note that the Voronoi cells are intersections of halfspaces and, as such, are convex sets. Since x is a convex combination of ψ and e_{n^*}, both of which belong to the convex set V_{n^*}, x must belong to V_{n^*} as well. □

Similarly, x will be a point on the simplex if an only if ψ is on the simplex as well. By analogy to the rounding transformations for permutation inference, in categorical inference we use a Gaussian distribution $\psi \sim \mathcal{N}(\mathsf{proj}(m), H)$, where $\mathsf{proj}(m)$ is the projection of $m \in \mathbb{R}_+^N$ onto the simplex. Still, the simplex has zero measure under the Gaussian distribution. It follows that the rounded points x will almost surely not be on the simplex either. The supposition of this approach is that this is not a problem: relaxing to the simplex is nice but not required.

In the zero-temperature limit we obtain a discrete distribution on the vertices of the simplex. For $\tau \in (0,1]$ we have a distribution on $\mathcal{X}_\tau \subseteq \mathbb{R}^N$, the subset of the reals to which the rounding operation maps. (For $0 \leq \tau < 1$ this is a strict subset of $\mathbb{R}^N$.) To derive the density $q(x)$, we need the inverse transformation and the determinant of its Jacobian. From Proposition 1, it follows that the inverse transformation is given by,

$$\psi = \frac{1}{\tau}x - \frac{1-\tau}{\tau}\mathsf{round}(x).$$

As long as ψ is in the interior of its Voronoi cell, the round function is piecewise constant and the Jacobian is $\frac{\partial\psi}{\partial x} = \frac{1}{\tau}I$, and its determinant is τ^{-N}. Taken together, we have,

$$\begin{aligned} q(x; m, H) = \\ \tau^{-N}\mathcal{N}\left(\frac{1}{\tau}x - \frac{1-\tau}{\tau}\mathsf{round}(x);\, \mathsf{proj}(m), H\right) \\ \times \mathbb{I}[x \in \mathcal{X}_\tau]. \end{aligned}$$

Compare this to the density of the rounded random variables for permutation inference.

B Limit analysis for stick-breaking

We show that stick-breaking for discrete variational inference can converge to any categorical distribution in the zero-temperature limit. We do so with a sequence of propositions: first we show that in the zero-temperature limit, the distribution of $\sigma(\psi_n/\tau)$ converges to a Bernoulli distribution. The we show that when $\sigma(\psi_n/\tau)$ is Bernoulli (rather than a continuous density on the unit interval), the distribution on x obtained by applying the stick-breaking transformation to ψ is categorical.

Proposition 2. *Let $z = \sigma(\psi/\tau)$ with $\psi \sim \mathcal{N}(\mu, \eta^2)$. In the limit $\tau \to 0$ we have $z \sim \mathrm{Bern}(\Phi(-\frac{\mu}{\eta}))$, where $\Phi(\cdot)$ denotes the Gaussian cumulative distribution function (cdf).*

Proof. Let F_z be the cdf of the random variable z. Since z is a random variable on the unit interval, F_z is a non-decreasing function on $[0, 1]$ with $F_z(0) = 0$ and $F_z(1) = 1$. Reparameterize $\psi = \mu + \eta\xi$ where $\xi \sim \mathcal{N}(0, 1)$. Then we have,

$$\begin{aligned} F_z(u) &= \Pr(\sigma(\psi/\tau) < u) \\ &= \Pr(\psi < \tau\sigma^{-1}(u)) \\ &= \Pr(\xi < \tfrac{\tau}{\eta}\sigma^{-1}(u) - \tfrac{\mu}{\eta})) \\ &= \Phi(-\tfrac{\tau}{\eta}\sigma^{-1}(u) - \tfrac{\mu}{\eta}). \end{aligned}$$

By the continuity of Φ we have,

$$\lim_{\tau \to 0} F_z(u) = \Phi(-\tfrac{\mu}{\eta}) \qquad \text{for } u \in (0, 1).$$

This is the cdf of a Bernoulli random with probability $\rho = \Phi(-\frac{\mu}{\eta})$. □

Proposition 3. *As above, let $z_n = \sigma(\psi_n/\tau)$. When $z_n \sim \mathrm{Bern}(\rho_n)$ with $\rho_n \in [0, 1]$ for $n = 1, \ldots, N$, the random variable x obtained from applying the stick-breaking transformation to z will have an atomic distribution with atoms in the vertices of Δ_N; i.e, $x \sim \mathrm{Cat}(\pi)$ where*

$$\begin{aligned} \pi_1 &= \rho_1 \\ \pi_n &= \rho_n \prod_{m=1}^{n-1} (1 - \rho_m) \qquad n = 2, \ldots, N-1, \\ \pi_N &= \prod_{m=1}^{N-1} (1 - \rho_m). \end{aligned}$$

Proof. From the stick-breaking definition, $x_1 = z_1$, $x_n = z_n(1 - \sum_{m<n} x_m)$, and $x_N = 1 - \sum_{m<N} x_m$. When $z_n \in \{0, 1\}$ for all $n = 1, \ldots, N-1$, we have the following equivalencies. For the first element,

$$x_1 = 1 \iff z_1 = 1;$$

for $1 < n < N - 1$:

$$x_n = 1 \iff (z_n = 1) \bigwedge_{m=1}^{n-1} (z_m = 0);$$

and for the last element,

$$x_N = 1 \iff \bigwedge_{m=1}^{N-1} (z_m = 0).$$

These events are mutually exclusive, implying that x will necessarily be a one-hot vector, i.e. a categorical random variable. Since $z_1, \ldots, z_{N-1}$ are independent Bernoulli random variables, the probabilities of these events are given by the $\pi, \ldots, \pi_N$ stated in the proposition. □

These two propositions, combined with the invertibility of the stick-breaking procedure, lead to our main result.

Lemma 1. *In the zero-temperature limit, stick-breaking of logistic-normal random variables can realize any categorical distribution on x.*

Proof. There is a one-to-one correspondence between $\pi \in \Delta_N$ and $\rho \in [0, 1]^{N-1}$. Specifically,

$$\begin{aligned} \rho_1 &= \pi_1 \\ \rho_n &= \frac{\pi_n}{\prod_{m=1}^{n=1} 1 - \rho_m} \quad \text{for } n = 2, \ldots, N-1. \end{aligned}$$

Since these are recursively defined, we can substitute the definition of ρ_m to obtain an expression for ρ_n in terms of π only. Thus, by Proposition 3, any desired categorical distribution π implies a set of Bernoulli parameters ρ. From Proposition 2, in the zero temperature limit, any desired ρ_n can be obtained with appropriate choice of Gaussian mean μ_n and variance η_n^2. Thus, stick-breaking can realize any categorical distribution when $\tau \to 0$. □

C Variational Autoencoders (VAE) with categorical latent variables

We considered the density estimation task on MNIST digits, as in Maddison et al. [2016], Jang et al. [2016], where observed digits are reconstructed from a latent discrete code. We used the continuous ELBO for training, and evaluated performance based on the marginal likelihood, estimated through the multi-sample variational objective of the discretized model. We compared against the methods of Jang et al. [2016], Maddison et al. [2016] and obtained the results in Table 1. While stick-breaking and rounding fare slightly worse than

the Gumbel-softmax method, they are readily extensible to more complex discrete objects, as shown in the main paper.

Table 1: Summary of results in VAE

Method	$-\log p(x)$
Gumbel-Softmax	106.7
Concrete	111.5
Rounding	121.1
Stick-breaking	119. 8

Figure 1 shows MNIST reconstructions using Gumbel-Softmax, stick-breaking and rounding reparameterizations. In all the three cases reconstructions are reasonably accurate, and there is diversity in reconstructions.

D Variational permutation inference details

Here we discuss more of the subtleties of variational permutation inference and present the mathematical derivations in more detail.

D.1 Continuous prior distributions.

Continuous relaxations require re-thinking the objective. As in Maddison et al. [2016], we maximize a relaxed ELBO, for which we need to specify a new continuous prior $p(X)$ over the relaxed discrete latent variables, here, over relaxations of permutation matrices. Moreover, it is critical to design sensible priors for relaxed permutations. Ideally, this prior should penalize values of X that are far from permutation matrices.

For our categorical experiment on MNIST we use a mixture of Gaussians around each vertex, $p(x) = \frac{1}{N}\sum_{n=1}^{N}\mathcal{N}(x \,|\, e_k, \eta^2)$. This can be extended to permutations, where use a mixture of Gaussians for each dimension,

$$p(X) = \prod_{m=1}^{N}\prod_{n=1}^{N}\frac{1}{2}\left(\mathcal{N}(x_{mn} \,|\, 0, \eta^2) + \mathcal{N}(x_{mn} \,|\, 1, \eta^2\right). \tag{2}$$

Although this prior puts significant mass around invalid points (e.g. $\mathbf{1}$), it penalizes X that far from $\mathcal{B}_N$.

D.2 Computing the ELBO

Here we show how to evaluate the ELBO. Note that the stick-breaking and rounding transformations are compositions of invertible functions, $g_\tau = h_\tau \circ f$ with $\Psi = f(\Xi; \theta)$ and $X = h_\tau(\Psi)$. In both cases, f takes in a matrix of independent standard Gaussians (Ξ) and transforms it with the means and variances in θ to output a matrix Ψ with entries $\psi_{mn} \sim \mathcal{N}(\mu_{mn}, \eta^2_{mn})$. Stick-breaking and rounding differ in the temperature-controlled transformations $h_\tau(\Psi)$ they use to map Ψ toward the Birkhoff polytope.

To evaluate the ELBO, we must compute the density of $q_\tau(X; \theta)$. Let $J_h(u) = \frac{\partial h(U)}{\partial U}\big|_{U=u}$ denote the Jacobian of a function h evaluated at value u. By the change of variables theorem and properties of the determinant,

$$\begin{aligned} q_\tau(X; \theta) &= p\big(h_\tau^{-1}(X); \theta\big) \times \left|J_{h_\tau^{-1}}(X)\right| \\ &= p\big(h_\tau^{-1}(X); \theta\big) \times \left|J_{h_\tau}(h_\tau^{-1}(X))\right|^{-1}. \end{aligned}$$

Now we appeal to the law of the unconcious statistician to compute the entropy of $q_\tau(X; \theta)$,

$$\begin{aligned} &\mathbb{E}_{q_\tau(X;\theta)}\Big[-\log q(X;\theta)\Big] \\ &\quad= \mathbb{E}_{p(\Psi;\theta)}\Big[-\log p(\Psi;\theta) + \log\left|J_{h_\tau}(\Psi)\right|\Big] \\ &\quad= \mathbb{H}(\Psi;\theta) + \mathbb{E}_{p(\Psi;\theta)}\Big[\left|J_{h_\tau}(\Psi)\right|\Big]. \end{aligned} \tag{3}$$

Since Ψ consists of independent Gaussians, the entropy is simply,

$$\mathbb{H}(\Psi;\theta) = \frac{1}{2}\sum_{m,n}\log(2\pi e\eta^2_{mn}).$$

We estimate the second term of equation (3) using Monte-Carlo samples. For both transformations, the Jacobian has a simple form.

Jacobian of the rounding transformation. The rounding transformation is given in matrix form in the main text, and we restate it here in coordinate-wise form for convenience,

$$x_{mn} = [h_\tau(\Psi)]_{mn} = \tau\psi_{mn} + (1-\tau)[\mathsf{round}(\Psi)]_{mn}.$$

This transformation is piecewise linear with jumps at the boundaries of the "Voronoi cells;" i.e., the points where $\mathsf{round}(X)$ changes. The set of discontinuities has Lebesgue measure zero so the change of variables theorem still applies. Within each Voronoi cell, the rounding operation is constant, and the Jacobian is,

$$\log\left|J_{h_\tau}(\Psi)\right| = \sum_{m,n}\log\tau = N^2\log\tau.$$

For the rounding transformation with given temperature, the Jacobian is constant.

Jacobian of the stick-breaking transformation. Here h_τ consists of two steps: map $\Psi \in \mathbb{R}^{N-1\times N-1}$

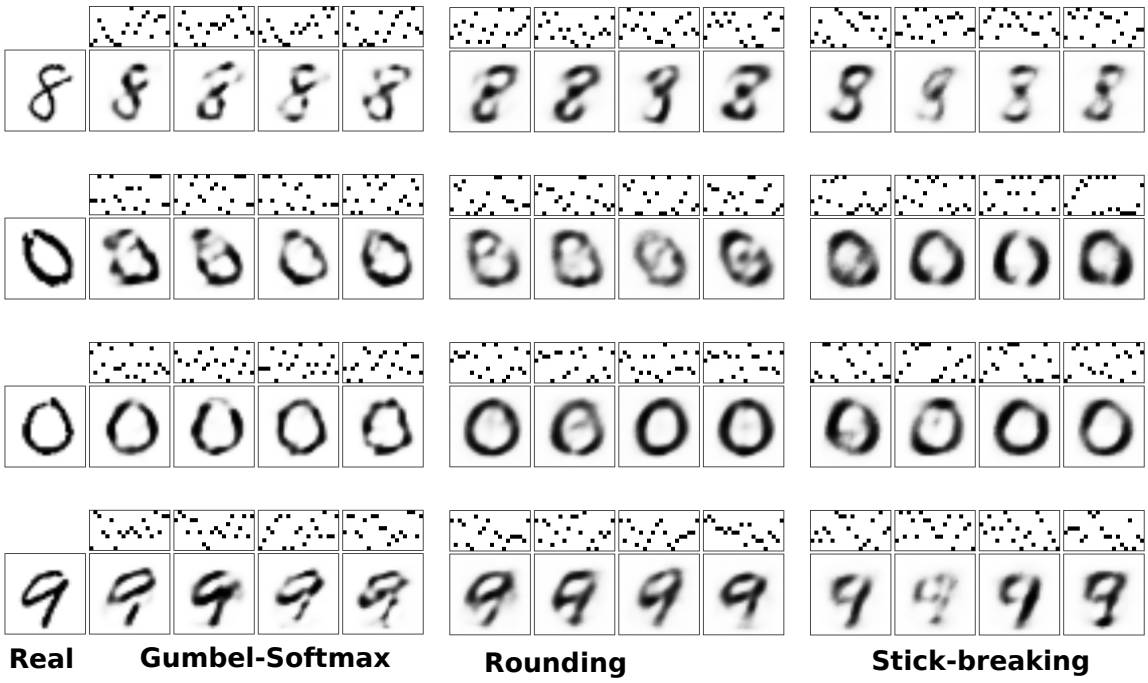

Figure 1: Examples of true and reconstructed digits from their corresponding random codes using with $K = 20$ categorical variables with $N = 10$ possible values.

to $Z \in [0,1]^{N-1 \times N-1}$ with a temperature-controlled, elementwise logistic function, then map Z to $X \in \mathcal{B}_N$ with the stick-breaking transformation.

As with the standard stick-breaking transformation to the simplex, our transformation to the Birkhoff polytope is feed-forward; i.e. to compute x_{mn} we only need to know the values of z up to and including the (m,n)-th entry. Consequently, the Jacobian of the transformation is triangular, and its determinant is simply the product of its diagonal.

We derive an explicit form in two steps. With a slight abuse of notation, note that the Jacobian of $h_\tau(\Psi)$ is given by the chain rule,

$$J_{h_\tau}(\Psi) = \frac{\partial X}{\partial \Psi} = \frac{\partial X}{\partial Z}\frac{\partial Z}{\partial \Psi}.$$

Since both transformations are bijective, the determinant is,

$$\left|J_{h_\tau}(\Psi)\right| = \left|\frac{\partial X}{\partial Z}\right| \left|\frac{\partial Z}{\partial \Psi}\right|.$$

the product of the individual determinants. The first determinant is,

$$\left|\frac{\partial X}{\partial Z}\right| = \prod_{m=1}^{N-1}\prod_{n=1}^{N-1}\frac{\partial x_{mn}}{\partial z_{mn}} = \prod_{m=1}^{N-1}\prod_{n=1}^{N-1}(u_{mn} - \ell_{mn}).$$

The second transformation, from Ψ to Z, is an elementwise, temperature-controlled logistic transformation such that,

$$\begin{aligned}\left|\frac{\partial Z}{\partial \Psi}\right| &= \prod_{m=1}^{N-1}\prod_{n=1}^{N-1}\frac{\partial z_{mn}}{\partial \psi_{mn}} \\ &= \prod_{m=1}^{N-1}\prod_{n=1}^{N-1}\frac{1}{\tau}\sigma\left(\psi_{mn}/\tau\right)\sigma\left(-\psi_{mn}/\tau\right).\end{aligned}$$

double check

It is important to note that the transformation that maps $Z \to X$ is only piecewise continuous: the function is not differentiable at the points where the bounds change; for example, when changing Z causes the active upper bound to switch from the row to the column constraint or vice versa.

conclude this section.

E Experiment details

We used Tensorflow [Abadi et al., 2016] for the VAE experiments, slightly changing the code made available from Jang et al. [2016]. For experiments on synthetic matching and the C. elegans example we used Autograd [Maclaurin et al., 2015], explicitly avoiding propagating gradients through the non-differentiable round operation, which requires solving a matching problem.

We used the ADAM with learning rate 0.1 for opti-

mization. For rounding, the parameter vector H defined in **??**(iii) was constrained to lie in the interval $[0.1, 0.5]$. Also, for rounding, we used ten iterations of the Sinkhorn-Knopp algorithm, to obtain points in the Birkhoff polytope. For stick-breaking the variances ν defined in **??** were constrained between $1e-8$ and 1.0. In either case, the temperature, along with maximum values for the noise variances were calibrated using a grid search.

In the C. elegans example we considered the symmetrized version of the adjacency matrix described in [Varshney et al., 2011]; i.e. we used $A' = (A + A^\top)/2$, and the matrix W was chosen antisymmetric, with entries sampled randomly with the sparsity pattern dictated by A'. To avoid divergence, the matrix W was then re-scaled by 1.1 times its spectral radius. This choice, although not essential, induced a reasonably well behaved linear dynamical system, rich in non-damped oscillations. We used a time window of $T = 1000$ time samples, and added spherical standard noise at each time.

References

M. Abadi, A. Agarwal, P. Barham, E. Brevdo, Z. Chen, C. Citro, G. S. Corrado, A. Davis, J. Dean, M. Devin, et al. Tensorflow: Large-scale machine learning on heterogeneous distributed systems. *arXiv preprint arXiv:1603.04467*, 2016.

E. Jang, S. Gu, and B. Poole. Categorical reparameterization with Gumbel-softmax. *arXiv preprint arXiv:1611.01144*, 2016.

D. P. Kingma and M. Welling. Auto-encoding variational Bayes. In *International Conference on Learning Representations*, 2014.

M. J. Kusner and J. M. Hernández-Lobato. GANs for sequences of discrete elements with the Gumbel-softmax distribution. *arXiv preprint arXiv:1611.04051*, 2016.

S. Linderman, M. Johnson, and R. P. Adams. Dependent multinomial models made easy: Stick-breaking with the polya-gamma augmentation. In *Advances in Neural Information Processing Systems*, pages 3456–3464, 2015.

D. Maclaurin, D. Duvenaud, and R. P. Adams. Autograd: Effortless gradients in numpy. In *ICML 2015 AutoML Workshop*, 2015.

C. J. Maddison, A. Mnih, and Y. W. Teh. The concrete distribution: A continuous relaxation of discrete random variables. *arXiv preprint arXiv:1611.00712*, 2016.

C. Naesseth, F. Ruiz, S. Linderman, and D. Blei. Reparameterization gradients through acceptance-rejection sampling algorithms. In *Artificial Intelligence and Statistics*, pages 489–498, 2017.

D. J. Rezende, S. Mohamed, and D. Wierstra. Stochastic backpropagation and approximate inference in deep generative models. In *Proceedings of The 31st International Conference on Machine Learning*, pages 1278–1286, 2014.

L. R. Varshney, B. L. Chen, E. Paniagua, D. H. Hall, and D. B. Chklovskii. Structural properties of the caenorhabditis elegans neuronal network. *PLoS computational biology*, 7(2):e1001066, 2011.