Topic Models and Coordinate Ascent Variational Inference

STATS 305C: Applied Statistics

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Mixed Membership Models

Mixed membership models are designed for *grouped data*.

Each “data point” is itself a collection of observations. For example,

- in text analysis, a document is a collection of observed words.
- in social science, a survey is a collection of observed answers.
- in genetic sequencing, a genome is a collection of observed genes.

Mixed membership models look for patterns like the components of a mixture model, but allowing each data point to involve multiple components.

Notation for a mixed membership model

- Let $\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,D})$ denote n-th **data point**. It contains a **collection of observations**. To simplify notation, assume all data points are length D.
- Each data point reflects a **combination of** K **mixture components** with parameters $\{\boldsymbol{\theta}_k\}_{k=1}^K$. These are shared by all documents.
- Each data point has its own **mixture proportions**, $\pi_n \in \Delta_K$.
- Each **observation is assigned to one of the components** by $z_{n,d} \in \{1, \ldots, K\}$.

Topic models

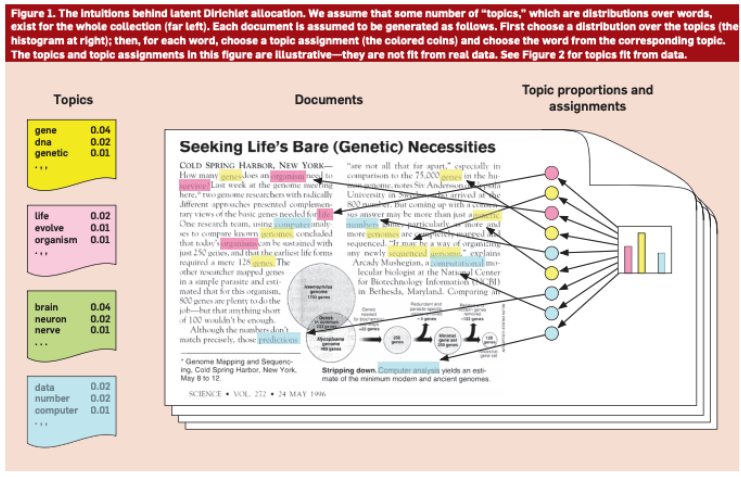
The most common mixed-membership model is the **topic model**, a generative model for documents.

Topic models are so common, they have their own nomenclature:

data set	*corpus*	collection of documents
data point	*document*	collection of words
observation	*word*	one element of a document
mixture component	*topic*	distribution over words
mixture proportions	*topic proportions*	distribution over topics
mixture assignment	*topic assignment*	which topic produced a word

Table: Rosetta stone for translating between mixed membership model and topic model notation.

Topic modeling intuition

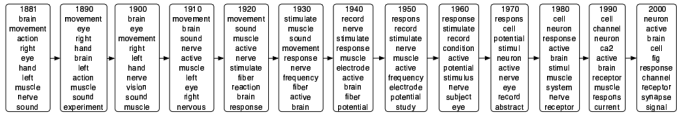


From Blei [2012].

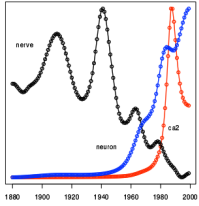
Topic modeling of New York Times articles

Figure 6.1: Posterior topics from the *The New York Times*.

Dynamic topic modeling of Science articles



"Neuroscience"



1887 Mental Science
1900 Hemianopsia in Migraine
1912 A Defence of the ``New Phrenology''
1921 The Synchronal Flashing of Fireflies
1932 Myoesthesis and Imageless Thought
1943 Acetylcholine and the Physiology of the Nervous System
1952 Brain Waves and Unit Discharge in Cerebral Cortex
1963 Errorless Discrimination Learning in the Pigeon
1974 Temporal Summation of Light by a Vertebrate Visual Receptor
1983 Hysteresis in the Force-Calcium Relation in Muscle
1993 GABA-Activated Chloride Channels in Secretory Nerve Endings

Figure 4. Examples from the posterior analysis of a 20-topic dynamic model estimated from the *Science* corpus. For two topics, we illustrate: (a) the top ten words from the inferred posterior distribution at ten year lags (b) the posterior estimate of the frequency as a function of year of several words from the same two topics (c) example articles throughout the collection which exhibit these topics. Note that the plots are scaled to give an idea of the shape of the trajectory of the words' posterior probability (i.e., comparisons across words are not meaningful).

From Blei and Lafferty [2006].

Topic modeling of congressional voting records

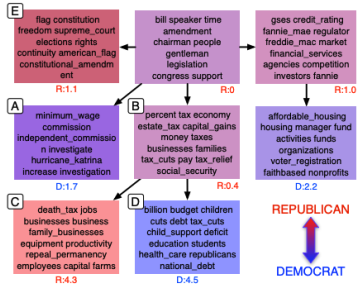


Figure 7.2: Topics discovered from Congressional floor debates using a downstream model to capture speaker's ideology. Many first-level topics are bipartisan (purple), while lower level topics are associated with specific ideologies (Democrats blue, Republicans red). For example, the "tax" topic (B) is bipartisan, but its Democratic-leaning child (D) focuses on social goals supported by taxes ("children", "education", "health care"), while its Republican-leaning child (C) focuses on business implications ("death tax", "jobs", "businesses"). The number below each topic denotes the magnitude of a learned regression parameter associated with that topic. Colors and the numbers beneath each topic show the regression parameter η associated with the topic. From Nguyen et al. [2013].

From Boyd-Graber et al. [2017].

The generative process for a mixed membership model

The generative model process is:

- for each mixture component $k = 1, \ldots, K$, sample its parameter $\boldsymbol{\theta}_k \sim p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi})$
- for each data point $n = 1, \ldots, N$:
  - sample mixture proportions $\boldsymbol{\pi}_n \sim \mathrm{Dir}(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha})$
  - for each observation $d = 1, \ldots, D$:
    - sample mixture assignment $z_{n,d} \in \{1, \ldots, K\}$ from a cateogrical distribution $z_{n,d} \sim \mathrm{Cat}(\boldsymbol{\pi}_n)$
    - sample observation $x_{n,d} \sim p(x \mid \boldsymbol{\theta}_{z_{n,d}})$

The mixed membership model allows sharing at the dataset level (all data points share the same components) while allowing variability at the data point level (each data point has its own mixture proportions).

The generative process for a topic model

Slide 9 in the language of topic modeling:

- for each **topic** $k = 1, \ldots, K$, sample its parameter $\boldsymbol{\theta}_k \sim p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi})$
- for each **document** $n = 1, \ldots, N$:
  - sample **topic proportions** $\pi_n \sim \mathrm{Dir}(\pi_n \mid \boldsymbol{\alpha})$
  - for each **word** $d = 1, \ldots, D$:
    - sample **topic assignment** $z_{n,d} \in \{1, \ldots, K\}$ from a cateogrical distribution $z_{n,d} \sim \mathrm{Cat}(\pi_n)$
    - sample **word** $x_{n,d} \in \{1, \ldots, V\}$ from a cateogrical distribution, $x_{n,d} \sim \mathrm{Cat}(\boldsymbol{\theta}_{z_{n,d}})$

The topic model captures sharing at the corpus level (all documents share the same topics) while allowing variability at the data point level (each document weights topics differently).

The joint distribution

The joint probability for a general mixed membership model is,

$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \boldsymbol{z}_n, \boldsymbol{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) = \prod_{k=1}^K p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left[p(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^D p(z_{n,d} \mid \boldsymbol{\pi}_n)\, p(x_{n,d} \mid \boldsymbol{\theta}_{z_{n,d}}) \right] \tag{1}$$

As in mixture models, we can write this equivalently as

$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \boldsymbol{z}_n, \boldsymbol{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) = \prod_{k=1}^K p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left[p(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^D \prod_{k=1}^K \left[\pi_{n,k}\, p(x_{n,d} \mid \boldsymbol{\theta}_k) \right]^{\mathbb{I}[z_{n,d}=k]} \right] \tag{2}$$

Graphical Model

Exercise: Draw the graphical models for a mixture model and a mixed membership model.

Topic model data types

Picture: Draw $\boldsymbol{x}$, $\boldsymbol{z}$, π, and θ.

Latent Dirichlet allocation

Latent Dirichlet Allocation (LDA) [Blei et al., 2003] is the most widely used topic model.

It assumes conjugate Dirichlet-Categorical model for the topics $\boldsymbol{\theta}_k \in \Delta_V$ and words $x_{n,d} \in \{1, \ldots, V\}$,

$$\boldsymbol{\theta}_k \overset{\text{iid}}{\sim} \text{Dir}(\boldsymbol{\phi}), \tag{3}$$

$$\boldsymbol{\pi}_n \overset{\text{iid}}{\sim} \text{Dir}(\boldsymbol{\alpha}), \tag{4}$$

$$z_{n,d} \overset{\text{iid}}{\sim} \text{Cat}(\boldsymbol{\pi}_n) \tag{5}$$

$$x_{n,d} \overset{\text{iid}}{\sim} \text{Cat}(\boldsymbol{\theta}_{z_{n,d}}) \tag{6}$$

Latent Dirichlet allocation II

Plugging in these assumptions, the joint probability is,

$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \boldsymbol{z}_n, \boldsymbol{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) \tag{7}$$

$$= \prod_{k=1}^K \mathrm{Dir}(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left[\mathrm{Dir}(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^D \pi_{n,z_{n,d}} \theta_{z_{n,d},x_{n,d}} \right] \tag{8}$$

$$= \prod_{k=1}^K \mathrm{Dir}(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \left(\mathrm{Dir}(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{d=1}^D \left[\prod_{k=1}^K \pi_{n,k}^{\mathbb{I}[z_{n,d}=k]} \right] \left[\prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{\mathbb{I}[x_{n,d}=v]\mathbb{I}[z_{n,d}=k]} \right] \right)$$

$$= \left[\prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{\phi_v - 1} \right] \prod_{n=1}^N \left(\left[\prod_{k=1}^K \pi_{n,k}^{\alpha_k + N_{n,\cdot,k} - 1} \right] \left[\prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{N_{n,v,k}} \right] \right) \tag{9}$$

where

- $N_{n,v,k} = \sum_{d=1}^D \mathbb{I}[x_{n,d} = v]\mathbb{I}[z_{n,d} = k]$ is the number of instances of word v in document n assigned to topic k, and,
- $N_{n,\cdot,k} = \sum_{v=1}^V N_{n,v,k} = \sum_{d=1}^D \mathbb{I}[z_{n,d} = k]$ is the number of words in document n assigned to topic k.

Gibbs sampling for LDA

As usual, sample each variable from its conditional distribution, holding the rest fixed.

- **Topic assignments:** The assignments are conditionally independent given the topic parameters and proportions,

$$p(z_{n,d} = k \mid x_{n,d} = v, \{\boldsymbol{\theta}_k\}_{k=1}^K, \boldsymbol{\pi}_n) \propto \pi_{n,k}\theta_{k,v} \tag{10}$$

- **Topic proportions:** Let $N_{n,k} = \sum_{d=1}^D \mathbb{I}[z_{n,d} = k]$ denote the number of words in document n assigned to topic k. Then,

$$p(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}, \boldsymbol{z}_n) \propto \mathrm{Dir}(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) \prod_{k=1}^K \pi_{n,k}^{N_{n,\cdot,k}} = \mathrm{Dir}([\alpha_1 + N_{n,\cdot,1}, \ldots, \alpha_K + N_{n,\cdot,K}]). \tag{11}$$

- **Topic parameters:** Let $N_{k,v} = \sum_{n=1}^N \sum_{d=1}^D \mathbb{I}[z_{n,d} = k]\mathbb{I}[x_{n,d} = v]$ denote the number of times word v was assigned to topic k across all documents. Then,

$$p(\boldsymbol{\theta}_k \mid \{\boldsymbol{z}_n, \boldsymbol{x}_n\}, \boldsymbol{\phi}) \propto \mathrm{Dir}(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) \prod_{n=1}^N \prod_{v=1}^V \theta_{k,v}^{N_{n,v,k}} = \mathrm{Dir}([\phi_1 + \sum_{n=1}^N N_{n,v,k}, \ldots, \phi_V + \sum_{n=1}^N N_{n,V,k}]).$$

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Why does LDA produce sharp topics?

Consider the log posterior as a function of $\boldsymbol{\theta}_k$ and $\boldsymbol{\pi}_n$. From eq. 8, it is,

$$\sum_{k=1}^{K} \log p(\boldsymbol{\theta}_k \mid \boldsymbol{\phi}) + \sum_{n=1}^{N} \log p(\boldsymbol{\pi}_n \mid \boldsymbol{\alpha}) + \sum_{n=1}^{N} \sum_{d=1}^{D} \left(\log \pi_{n,z_{n,d}} + \log \theta_{z_{n,d},x_{n,d}} \right) \tag{13}$$

The double sum over *n* and *d* dominates this expression.

It encourages topic proportions and topic probabilities to both be large, but recall that both $\boldsymbol{\pi}_n$ and $\boldsymbol{\theta}_k$ are constrained to the simplex.

For $\log \pi_{n,z_{n,d}}$ to be large, the posterior should assign all words to as few topics as possible.

For $\log \theta_{z_{n,d},x_{n,d}}$ to be large, the topics should put high probability on as few words as possible.

These goals are at odds. The LDA posterior balances these goals to find topics with sharply co-occuring words.

Topic Models and Coordinate Ascent Variational Inference

- Model: Topic Models
- **Algorithm: Coordinate Ascent Variational Inference (CAVI)**

Taking stock

We've covered a number of posterior inference algorithms thus far:

- **Exact inference:** for simple models (e.g. conjugate exponential family models) where the posterior is available in closed form.
- **Gibbs sampling:** an MCMC algorithm that iteratively samples conditional distributions for one variable at a time. This works well for conditionally conjugate models with weak correlations.
- **Metropolis-Hastings:** a very general MCMC algorithm to sample the posterior, and the building block for many other MCMC techniques.
- **Hamiltonian Monte Carlo:** an MCMC algorithm to draw samples from the posterior by leveraging gradients of the log joint probability. This works well for more general posteriors over continuous variables.

Variational inference

MCMC methods are asymptotically unbiased (though for finite samples there is a transient bias that shrinks as $O(S^{-1})$. The real issue is variance: it only shrinks as $O(S^{-1/2})$.

Motivation: With finite computation, can we get better posterior estimates by trading asymptotic bias for smaller variance?

Idea: approximate the posterior by with a simple, parametric form (though not strictly a Gaussian on the mode!). Optimize to find the approximation that is as "close" as possible to the posterior.

Notation

This notation could be a bit confusing. Let,

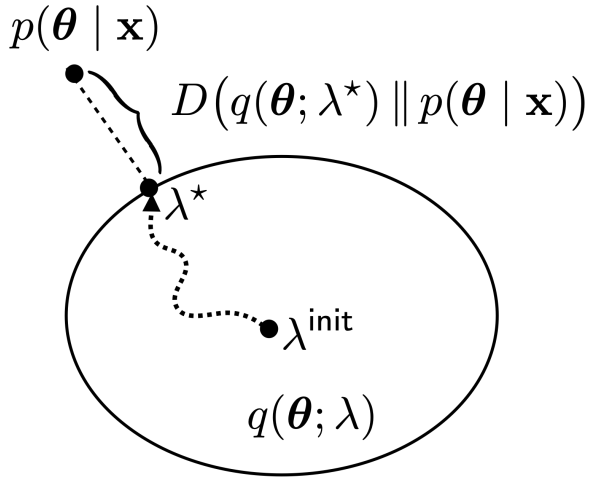
- $\boldsymbol{\vartheta} \in \mathbb{R}^J$ denote **all of latent variables and parameters** we wish to infer.
  In LDA, $\boldsymbol{\vartheta} = \{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \boldsymbol{z}_n\}_{n=1}^N$.

In contrast to last week, here we will obtain a **full posterior over parameters and latent variables**.

Likewise, let

- $p(\boldsymbol{\vartheta} \mid \boldsymbol{x})$ denote the true posterior distribution we want to approximate.
- $q(\boldsymbol{\vartheta}; \boldsymbol{\lambda})$ denote a parametric *variational approximation* to the posterior where...
- $\boldsymbol{\lambda}$ denotes the *variational parameters* that we will optimize.
- $D(q \,||\, p)$ denote a *divergence measure* that takes in two distributions q and p and returns a measure of how similar they are.

A view of variational inference



Key questions

- *What parametric family should we use?*
  - This lecture: the **mean-field family**.
- *How should we measure closeness?*
  - This lecture: the **Kullback-Leibler (KL)** divergence.
- *How do we find the closest distribution in that family?*
  - This lecture: **coordinate ascent.**

These choices are what Blei et al. [2017] call **coordinate ascent variational inference** CAVI).

The mean-field family

The *mean-field family* gets its name from statistical mechanics. It treats each latent variable and parameter as independent with its own variational parameter,

$$q(\boldsymbol{\vartheta}; \boldsymbol{\lambda}) = \prod_{j=1}^{J} q(\vartheta_j; \lambda_j). \tag{14}$$

For example, in LDA the mean field approximation treats each topic, topic proportion, and topic assignment as independent,

$$q(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \boldsymbol{z}_n\}_{n=1}^N; \boldsymbol{\lambda}) = \prod_{k=1}^{K} q(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}) \prod_{n=1}^{N} q(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) \prod_{n=1}^{N} \prod_{d=1}^{D} q(z_{n,d}; \boldsymbol{\lambda}_{n,d}^{(z)}) \tag{15}$$

Question: Is this a good approximation to the posterior?

The Kullback-Leibler (KL) divergence

The KL divergence is a measure of closeness between two distributions. It is defined as,

$$D_{\mathrm{KL}}\left(q(\boldsymbol{\vartheta};\boldsymbol{\lambda}) \,\|\, p(\boldsymbol{\vartheta} \mid \boldsymbol{x})\right) = \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log \frac{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}{p(\boldsymbol{\vartheta} \mid \boldsymbol{x})}\right] \tag{16}$$

$$= \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log q(\boldsymbol{\vartheta};\boldsymbol{\lambda})\right] - \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log p(\boldsymbol{\vartheta} \mid \boldsymbol{x})\right] \tag{17}$$

It has some nice properties:

- It is non-negative.
- It is zero iff $q(\boldsymbol{\vartheta};\boldsymbol{\lambda}) \equiv p(\boldsymbol{\vartheta} \mid \boldsymbol{x})$.
- It is defined in terms of expectations wrt q.

But it's also a bit weird...

- It's asymmetric ($D_{\mathrm{KL}}\left(q \,\|\, p\right) \neq D_{\mathrm{KL}}\left(p \,\|\, q\right)$).

The evidence lower bound (ELBO) from another angle

More concerning, the KL divergence involves the posterior $p(\boldsymbol{\vartheta} \mid \boldsymbol{x})$, which we cannot compute!

But notice that...

$$
\begin{aligned}
D_{\mathrm{KL}}\left(q(\boldsymbol{\vartheta};\boldsymbol{\lambda}) \,\|\, p(\boldsymbol{\vartheta} \mid \boldsymbol{x})\right) &= \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log q(\boldsymbol{\vartheta};\boldsymbol{\lambda})\right] - \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log p(\boldsymbol{\vartheta} \mid \boldsymbol{x})\right] && (18)\\
&= \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log q(\boldsymbol{\vartheta};\boldsymbol{\lambda})\right] - \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log p(\boldsymbol{\vartheta},\boldsymbol{x})\right] + \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log p(\boldsymbol{x})\right] && (19)\\
&= \underbrace{\mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log q(\boldsymbol{\vartheta};\boldsymbol{\lambda})\right] - \mathbb{E}_{q(\boldsymbol{\vartheta};\boldsymbol{\lambda})}\left[\log p(\boldsymbol{\vartheta},\boldsymbol{x})\right]}_{\text{negative ELBO, } -\mathscr{L}(\boldsymbol{\lambda})} + \underbrace{\log p(\boldsymbol{x})}_{\text{evidence}} && (20)
\end{aligned}
$$

The first term involves the log joint, which we can compute, and the last term is independent of the variational parameters!

Rearranging, we see that $\mathscr{L}(\boldsymbol{\lambda})$ is a lower bound on the marginal likelihood, aka the evidence,

$$
\mathscr{L}(\boldsymbol{\lambda}) = \log p(\boldsymbol{x}) - D_{\mathrm{KL}}\left(q(\boldsymbol{\vartheta};\boldsymbol{\lambda}) \,\|\, p(\boldsymbol{\vartheta} \mid \boldsymbol{x})\right) \leq \log p(\boldsymbol{x}). \tag{21}
$$

That's why we call it the **evidence lower bound (ELBO)**.

Viewer discretion advised...

https://www.youtube.com/watch?v=jugUBL4rEIM

Optimizing the ELBO with coordinate ascent

We want to find the variational parameters $\boldsymbol{\lambda}$ that minimize the KL divergence or, equivalently, maximize the ELBO.

For the mean-field family, we can typically do this via **coordinate ascent**.

Consider optimizing the parameters for one factor $q(\vartheta_j; \lambda_j)$. As a function of λ_j, the ELBO is,

$$\begin{aligned}
\mathscr{L}(\boldsymbol{\lambda}) &= \mathbb{E}_{q(\vartheta_j;\lambda_j)}\left[\mathbb{E}_{q(\boldsymbol{\vartheta}_{\neg j};\boldsymbol{\lambda}_{\neg j})}\left[\log p(\boldsymbol{\vartheta}, \boldsymbol{x})\right]\right] - \mathbb{E}_{q(\vartheta_j;\lambda_j)}[\log q(\vartheta_j; \lambda_j)] + c && (22)\\
&= \mathbb{E}_{q(\vartheta_j;\lambda_j)}\left[\mathbb{E}_{q(\boldsymbol{\vartheta}_{\neg j};\boldsymbol{\lambda}_{\neg j})}\left[\log p(\vartheta_j \mid \boldsymbol{\vartheta}_{\neg_j}, \boldsymbol{x})\right]\right] - \mathbb{E}_{q(\vartheta_j;\lambda_j)}[\log q(\vartheta_j; \lambda_j)] + c' && (23)\\
&= -D_{\mathrm{KL}}\left(q(\vartheta_j; \lambda_j) \parallel \tilde{p}(\vartheta_j)\right) + c'' && (24)
\end{aligned}$$

where

$$\tilde{p}(\vartheta_j) \propto \exp\left\{\mathbb{E}_{q(\boldsymbol{\vartheta}_{\neg j};\boldsymbol{\lambda}_{\neg j})}\left[\log p(\vartheta_j \mid \boldsymbol{\vartheta}_{\neg j}, \boldsymbol{x})\right]\right\} \tag{25}$$

The ELBO is maximized wrt λ_j when this KL is minimized; i.e. when $q(\vartheta_j; \lambda_j) = \tilde{p}(\vartheta_j)$, the exponentiated expected log conditional probability, holding all other factors fixed.

Lap 5: Mixed Membership Models and Variational Inference

- Model: Mixed Membership Models
- Algorithm: Coordinate Ascent Variational Inference (CAVI)
  - **CAVI for LDA**

Coordinate Ascent Variational Inference for LDA

Let's derive the CAVI updates for LDA.

Assume a mean field family, and assume each factor is of the same exponential family form as the corresponding prior:

$$q(z_{n,d}; \boldsymbol{\lambda}_{n,d}^{(z)}) = \mathrm{Cat}(z_{n,d}; \boldsymbol{\lambda}_{n,d}^{(z)}) \tag{26}$$

$$q(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) = \mathrm{Dir}(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) \tag{27}$$

$$q(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}) = \mathrm{Dir}(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}). \tag{28}$$

so $\boldsymbol{\lambda}_{n,d}^{(z)} \in \Delta_K$, $\boldsymbol{\lambda}_n^{(\pi)} \in \mathbb{R}_+^K$, and $\boldsymbol{\lambda}_k^{(\theta)} \in \mathbb{R}_+^V$ are the variational parameters.

(It turns out, for conjugate exponential family models, the optimal variational factors are of the same form as the prior anyway!)

CAVI updates for the topic assignments

Recall that the optimal CAVI updates are of the form in Eq. 25.

We already derived the conditional distributions for Gibbs sampling (Slide 16).

For the topic assignments, the CAVI update is,

$$\log q(z_{n,d} = k; \boldsymbol{\lambda}_{n,d}^{(z)}) = \mathbb{E}_{q(\pi_n)q(\theta_k)}\left[\log \pi_{n,k} + \log p(x_{n,d} \mid \theta_k)\right] + c \tag{29}$$

$$= \mathbb{E}_{q(\pi_n)}\left[\log \pi_{n,k}\right] + \mathbb{E}_{q(\theta_k)}\left[\log \theta_{k,x_{n,d}}\right] + c \tag{30}$$

$$= \log \mathrm{Cat}(z_{n,d} = k; \boldsymbol{\lambda}_{n,d}^{(z)}) \tag{31}$$

$$\Rightarrow \log \lambda_{n,d,k}^{(z)} = \mathbb{E}_{q(\pi_n)}\left[\log \pi_{n,k}\right] + \mathbb{E}_{q(\theta_k)}\left[\log \theta_{k,x_{n,d}}\right] + c \tag{32}$$

Since $\boldsymbol{\lambda}_{n,d}^{(z)}$ must sum to one,

$$\lambda_{n,d,k}^{(z)} = \frac{\exp\left\{\mathbb{E}_{q(\pi_n)}\left[\log \pi_{n,k}\right] + \mathbb{E}_{q(\theta_k)}\left[\log \theta_{k,x_{n,d}}\right]\right\}}{\sum_{j=1}^{K} \exp\left\{\mathbb{E}_{q(\pi_n)}\left[\log \pi_{n,j}\right] + \mathbb{E}_{q(\theta_j)}\left[\log \theta_{j,x_{n,d}}\right]\right\}} \tag{33}$$

Expectations under Dirichlet distributions

The variational factors for π_n and θ_k are Dirichlet distributions.

The necessary expectations have closed form expressions:

$$\mathbb{E}_{\mathrm{Dir}(\pi;\boldsymbol{\alpha})}[\log \pi_k] = \psi(\alpha_k) - \psi\Big(\sum_{j=1}^{K} \alpha_j\Big) \tag{34}$$

where $\psi(\cdot)$ is the *digamma function*, the logarithmic derivative of the gamma function.

CAVI updates for the topic proportions

Referring back to Slide 16, we see the CAVI update is,

$$\log q(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}) = \mathbb{E}_{q(\boldsymbol{z}_n)}\left[\log \mathrm{Dir}(\boldsymbol{\pi}_n; \boldsymbol{\alpha}) + \sum_{k=1}^{K} N_{n,k} \log \pi_{n,k}\right] + c \tag{35}$$

$$= \sum_{k=1}^{K} (\alpha_k - 1 + \mathbb{E}_{q(\boldsymbol{z}_n)}[N_{n,k}]) \log \pi_{n,k} \tag{36}$$

$$= \log \mathrm{Dir}\left(\boldsymbol{\pi}_n; \boldsymbol{\lambda}_n^{(\pi)}\right) \tag{37}$$

$$\Rightarrow \boldsymbol{\lambda}_n^{(\pi)} = \left[\alpha_1 + \mathbb{E}_{q(\boldsymbol{z}_n)}[N_{n,1}], \ldots, \alpha_K + \mathbb{E}_{q(\boldsymbol{z}_n)}[N_{n,K}]\right], \tag{38}$$

where

$$\mathbb{E}_{q(\boldsymbol{z}_n)}[N_{n,k}] = \sum_{d=1}^{D} \mathbb{E}_{q(z_{n,d})}[\mathbb{I}[z_{n,d} = k]] = \sum_{d=1}^{D} \lambda_{n,d,k}^{(z)}. \tag{39}$$

CAVI updates for the topic parameters

The topic parameter updates are similar

$$\log q(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}) = \mathbb{E}_{q(\boldsymbol{z})}\left[\log \mathrm{Dir}(\boldsymbol{\theta}_k; \boldsymbol{\phi}) + \sum_{v=1}^{V} N_{k,v} \log \theta_{k,v}\right] + c \tag{40}$$

$$= \sum_{v=1}^{V} (\phi_v - 1 + \mathbb{E}_{q(\boldsymbol{z})}[N_{k,v}]) \log \theta_{k,v} \tag{41}$$

$$= \log \mathrm{Dir}\left(\boldsymbol{\theta}_k; \boldsymbol{\lambda}_k^{(\theta)}\right) \tag{42}$$

$$\Rightarrow \boldsymbol{\lambda}_k^{(\theta)} = \left[\phi_1 + \mathbb{E}_{q(\boldsymbol{z})}[N_{k,1}], \ldots, \phi_V + \mathbb{E}_{q(\boldsymbol{z})}[N_{k,V}]\right], \tag{43}$$

where

$$\mathbb{E}_{q(\boldsymbol{z})}[N_{k,v}] = \sum_{n=1}^{N}\sum_{d=1}^{D} \mathbb{E}_{q(z_{n,d})}[\mathbb{I}[z_{n,d} = k]]\mathbb{I}[x_{n,d} = v] = \sum_{n=1}^{N}\sum_{d=1}^{D} \lambda_{n,d,k}^{(z)} \mathbb{I}[x_{n,d} = v]. \tag{44}$$

Calculating the ELBO

Dropping hyperparameters and variational parameters, the ELBO is,

$$\mathscr{L}(\boldsymbol{\lambda}) = \mathbb{E}_q[\log p(\{\boldsymbol{x}_n, \boldsymbol{z}_n, \boldsymbol{\pi}_n\}_{n=1}^N, \{\boldsymbol{\theta}_k\}_{k=1}^K)] - \mathbb{E}_q[\log q(\{\boldsymbol{z}_n, \boldsymbol{\pi}_n\}_{n=1}^N, \{\boldsymbol{\theta}_k\}_{k=1}^K)] \tag{45}$$

Thanks to the factorization of the joint distribution and the variational posterior, this simplifies,

$$\begin{aligned}\mathscr{L}(\boldsymbol{\lambda}) = &\sum_{n=1}^N \mathbb{E}_q[\log p(\boldsymbol{\pi}_n)] + \sum_{k=1}^K \mathbb{E}_q[\log p(\boldsymbol{\theta}_k)] \\ &+ \sum_{n=1}^N \sum_{d=1}^D \mathbb{E}_q[\log p(z_{n,d} \mid \boldsymbol{\pi}_n)] + \mathbb{E}_q[\log p(x_{n,d} \mid z_{n,d}, \boldsymbol{\theta})] \\ &- \sum_{k=1}^K \mathbb{E}_q[\log q(\boldsymbol{\theta}_k)] - \sum_{n=1}^N \mathbb{E}_q[\log q(\boldsymbol{\pi}_n)] - \sum_{n=1}^N \sum_{d=1}^D \mathbb{E}_q[\log q(z_{n,d})]\end{aligned} \tag{46}$$

The first two terms are Dirichlet cross entropies and the last three terms are entropies. Tensorflow probability has already implemented these functions for you, so this is easier to calculate than you might think!

Calculating the ELBO II

The middle terms are,

$$\mathbb{E}_q[\log p(z_{n,d} \mid \boldsymbol{\pi}_n)] = \sum_{k=1}^{K} \mathbb{E}_{q(z_{n,d})}[\mathbb{I}[z_{n,d} = k]] \cdot \mathbb{E}_{q(\boldsymbol{\pi}_n)}[\log \pi_{n,k}] \tag{47}$$

$$= \sum_{k=1}^{K} \lambda^{(z)}_{n,d,k} \cdot \mathbb{E}_{q(\boldsymbol{\pi}_n)}[\log \pi_{n,k}] \tag{48}$$

and

$$\mathbb{E}_q[\log p(x_{n,d} \mid z_{n,d}, \boldsymbol{\theta})] = \sum_{k=1}^{K} \mathbb{E}_{q(z_{n,d})}[\mathbb{I}[z_{n,d} = k]] \cdot \mathbb{E}_{q(\boldsymbol{\theta}_k)}[\log \theta_{k,x_{n,d}}] \tag{49}$$

$$= \sum_{k=1}^{K} \lambda^{(z)}_{n,d,k} \cdot \mathbb{E}_{q(\boldsymbol{\theta}_k)}[\log \theta_{k,x_{n,d}}] \tag{50}$$

Again, these involve only simple expectations of Dirichlet distributions.

Bag of word counts assumption and exchangeability

LDA models the words as **exchangeable** random variables. That is, the joint distribution is invariant to permutations:

$$p(\boldsymbol{x}_n) = p(x_{n,1}, \ldots, x_{n,D}) \equiv p(x_{n,\sigma(1)}, \ldots, p(x_{n,\sigma(D)})) \tag{51}$$

where $\sigma(\cdot)$ is any permutation of the indices $\{1, \ldots, D\}$.

In text modeling, this is called the **bag of words** assumption.

In LDA, this manifests in the joint probability (eq. 9, reproduced below) only depending on **word counts**,

$$p(\{\boldsymbol{\theta}_k\}_{k=1}^K, \{\boldsymbol{\pi}_n, \boldsymbol{z}_n, \boldsymbol{x}_n\}_{n=1}^N \mid \boldsymbol{\phi}, \boldsymbol{\alpha}) \propto \left[\prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{\phi_v - 1}\right] \prod_{n=1}^N \left(\left[\prod_{k=1}^K \pi_{n,k}^{\alpha_k + N_{n,\cdot,k} - 1}\right] \left[\prod_{k=1}^K \prod_{v=1}^V \theta_{k,v}^{N_{n,v,k}}\right] \right) \tag{52}$$

Working with word counts

The fact that the joint probability depends only on word counts suggests an alternative representation of the data.

Let, $\boldsymbol{y}_n \in \mathbb{N}^V$ denote the n-th document represented as a vector of word counts; i.e.,

$$y_{n,v} = \sum_{d=1}^{D} \mathbb{I}[x_{n,d} = v]. \tag{53}$$

Typically these vector will be **sparse**.

Likewise, let $\boldsymbol{c}_{n,v} \in \mathbb{N}^K$ be a vector of **latent counts** denoting how many times word v in document n was attributed to each of the K topics; i.e.,

$$c_{n,v,k} = \sum_{d=1}^{D} \mathbb{I}[x_{n,d} = v]\, \mathbb{I}[z_{n,d} = k] \tag{54}$$

We must have that $\sum_{k=1}^{K} c_{n,v,k} = y_{n,v}$.

LDA with word counts and the corresponding data types

Picture: Draw the data types for $\boldsymbol{y}$, $\boldsymbol{c}$, π, and θ.

Gibbs and CAVI updates with word counts

In terms of word counts, the conditional distribution of $\boldsymbol{c}_{n,v} \in \mathbb{N}^K$ is,

$$p(\boldsymbol{c}_{n,v} \mid y_n, \{\boldsymbol{\theta}_k\}_{k=1}^K, \boldsymbol{\pi}_n) = \text{Mult}\left(\boldsymbol{c}_{n,v} \,\middle|\, y_n, \left[\frac{\pi_{n,1}\theta_{1,v}}{\sum_{k=1}^K \pi_{n,k}\theta_{k,v}}, \ldots, \frac{\pi_{n,K}\theta_{K,v}}{\sum_{k=1}^K \pi_{n,k}\theta_{k,v}}\right]\right) \tag{55}$$

Instead of sampling each $z_{n,d}$ in the Gibbs sampler, we can directly sample count vectors $\boldsymbol{c}_{n,v}$.

Likewise, for CAVI, instead of having parameters for each $q(z_{n,d} \mid \boldsymbol{\lambda}_{n,d}^{(z)})$, only store one for each term v,

$$q(\boldsymbol{c}_{n,v}; \boldsymbol{\lambda}_{n,v}^{(c)}) = \text{Mult}(\boldsymbol{c}_{n,v} \mid y_{n,v}, \boldsymbol{\lambda}_{n,v}^{(c)}) \tag{56}$$

$$\lambda_{n,v,k}^{(c)} = \frac{\exp\left\{\mathbb{E}_{q(\pi_n)}\left[\log \pi_{n,k}\right] + \mathbb{E}_{q(\boldsymbol{\theta}_k)}\left[\log \theta_{k,v}\right]\right\}}{\sum_{j=1}^K \exp\left\{\mathbb{E}_{q(\pi_n)}\left[\log \pi_{n,j}\right] + \mathbb{E}_{q(\boldsymbol{\theta}_j)}\left[\log \theta_{j,v}\right]\right\}} \tag{57}$$

From these variational parameters, it's easy to compute the expected summary counts,

$$\mathbb{E}_q[N_{n,\cdot,k}] = \sum_{v=1}^V y_{n,v}\lambda_{n,v,k}^{(c)} \qquad \mathbb{E}_q[N_{\cdot,k,v}] = \sum_{n=1}^N y_{n,v}\lambda_{n,v,k}^{(c)} \tag{58}$$

Scaling up to very large datasets

There are a few tricks to make LDA much more scalable.

First, to save memory, you only need to track, $\boldsymbol{\lambda}_{n,v}^{(c)} = [\lambda_{n,v,1}^{(c)}, \ldots, \lambda_{n,v,K}^{(c)}]$ if $y_{n,v} > 0$.

Likewise, you can process documents in rolling fashion, discarding $\boldsymbol{\lambda}_{n,v}^{(c)}$ once you've updated $\mathbb{E}_q[N_{k,v}]$ and $\boldsymbol{\lambda}_n^{(\pi)}$.

Finally, you can use **stochastic variational inference** [Hoffman et al., 2013] to work with mini-batches of documents to get Monte Carlo estimates of $\mathbb{E}_q[N_{k,v}]$.

SVI can be seen as **stochastic gradient ascent** on the ELBO using **natural gradients** Amari [1998]; i.e., gradient descent preconditioned with the Fisher information matrix.

Evaluating topic models

The key hyperparameter is K, the number of topics. By now, we've seen a few different ways of setting these "complexity knobs."

Question: what approaches could we take?

Blei recommends another method that differs slightly from what we've seen thus far. He suggests evaluating,

$$p(\boldsymbol{x}_{n'}^{\text{out}} \mid \boldsymbol{x}_{n'}^{\text{in}}, \{\boldsymbol{x}_n\}_{n=1}^N) = \int p(\boldsymbol{x}_{n'}^{\text{out}} \mid \pi_{n'}, \{\boldsymbol{\theta}_k\}_{k=1}^K)\, p(\pi_{n'} \mid \boldsymbol{x}_{n'}^{\text{in}}, \{\boldsymbol{\theta}_k\}_{k=1}^K)\, p(\{\boldsymbol{\theta}_k\}_{k=1}^K \mid \{\boldsymbol{x}_n\}_{n=1}^N)\, \mathrm{d}\pi_{n'} \tag{59}$$

where

- $\boldsymbol{x}_{n'}^{\text{out}}$ consists of a subset of words in a held-out document n'
- $\boldsymbol{x}_{n'}^{\text{in}}$ are the remaining words in that document, which are used to estimate the topic proportions, and
- $\{\boldsymbol{x}_n\}_{n=1}^N$ are the training documents used to estimate the topics $\{\boldsymbol{\theta}_k\}_{k=1}^K$.

Evaluating topic models II

Question: why not simply compare the ELBO for different values of K? It's related to the marginal likelihood, after all.

Other mixed membership models

- LDA evolved from a long line of work on topic modeling. Deerwester et al. [1990] proposed **latent semantic analysis** and Hofmann [1999] proposed a probabilistic version called the **aspect model**.
- Pritchard et al. [2000] developed MM models in **population genetics**.
- Erosheva et al. [2007] used MM models for **survey data**.
- Airoldi et al. [2008] developed MM models for **community detection in networks**. Gopalan and Blei [2013] developed a stochastic variational inference algorithm for this model.
- Gopalan et al. [2013] proposed **Poisson matrix factorization**, which is closely related to LDA.

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