Linear Dynamical Systems and State Space Models

STATS 305C: Applied Statistics

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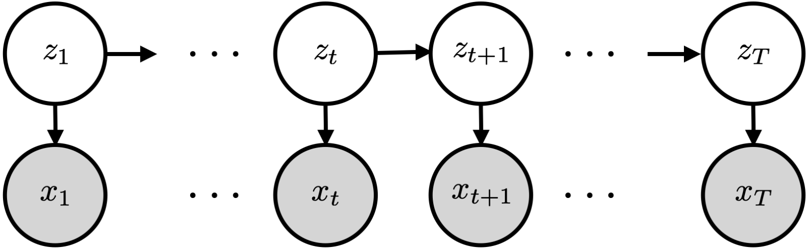
May 11, 2022

Hidden Markov Models

Hidden Markov Models (HMMs) assume a particular factorization of the joint distribution on latent states (z_t) and observations ($\boldsymbol{x}_t$). The graphical model looks like this:



This graphical model says that the joint distribution factors as,

$$p(z_{1:T}, \boldsymbol{x}_{1:T}) = p(z_1) \prod_{t=2}^{T} p(z_t \mid z_{t-1}) \prod_{t=1}^{T} p(\boldsymbol{x}_t \mid z_t). \tag{1}$$

We call this an HMM because $p(z_1) \prod_{t=2}^{T} p(z_t \mid z_{t-1})$ is a Markov chain.

Hidden Markov Models II

We are interested in questions like:

- What are the *predictive distributions* of $p(z_{t+1} \mid \boldsymbol{x}_{1:t})$?
- What is the *posterior marginal* distribution $p(z_t \mid \boldsymbol{x}_{1:T})$?
- What is the *posterior pairwise marginal* distribution $p(z_t, z_{t+1} \mid \boldsymbol{x}_{1:T})$?
- What is the *posterior mode* $z^\star_{1:T} = \arg\max p(z_{1:T} \mid \boldsymbol{x}_{1:T})$?
- How can we *sample the posterior* $p(\boldsymbol{z}_{1:T} \mid \boldsymbol{x}_{1:T})$ of an HMM?
- What is the *marginal likelihood* $p(\boldsymbol{x}_{1:T})$?
- How can we *learn the parameters* of an HMM?

Question: Why might these sound like hard problems?

State space models

Note that nothing above assumes that z_t is a discrete random variable!

HMM's are a special case of more general **state space models** with discrete states.

State space models assume the same graphical model but allow for arbitrary types of latent states.

For example, suppose that $\boldsymbol{z}_t \in \mathbb{R}^P$ are continuous valued latent states and that,

$$
\begin{aligned}
p(\boldsymbol{z}_{1:T}) &= p(\boldsymbol{z}_1) \prod_{t=2}^{T} p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}) && (2) \\
&= \mathcal{N}(\boldsymbol{z}_1 \mid \boldsymbol{b}_1, \boldsymbol{Q}_1) \prod_{t=2}^{T} \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{b}, \boldsymbol{Q}) && (3)
\end{aligned}
$$

This is called a Gaussian **linear dynamical system** (LDS).

Stability of Gaussian linear dynamical systems

Question: What is the asymptotic mean of a Gaussian LDS, $\lim_{t \to \infty} \mathbb{E}[\boldsymbol{z}_t]$?

Question: When is a Gaussian LDS stable? I.e. when is the asymptotic mean finite?

Message passing in HMMs

In the HMM with discrete states, we showed how to compute posterior marginal distributions using message passing,

$$p(z_t \mid \boldsymbol{x}_{1:T}) \propto \sum_{z_1} \cdots \sum_{z_{t-1}} \sum_{z_{t+1}} \cdots \sum_{z_T} p(z_{1:T}, \boldsymbol{x}_{1:T}) \tag{4}$$

$$= \alpha_t(z_t)\, p(\boldsymbol{x}_t \mid z_t)\, \beta_t(z_t) \tag{5}$$

where the *forward and backward messages* are defined recursively

$$\alpha_t(z_t) = \sum_{z_{t-1}} p(z_t \mid z_{t-1})\, p(\boldsymbol{x}_{t-1} \mid z_{t-1})\, \alpha_{t-1}(z_{t-1}) \tag{6}$$

$$\beta_t(z_t) = \sum_{z_{t+1}} p(z_{t+1} \mid z_t)\, p(\boldsymbol{x}_{t+1} \mid z_{t+1})\, \beta_{t+1}(z_{t+1}) \tag{7}$$

The initial conditions are $\alpha_1(z_1) = p(z_1)$ and $\beta_T(z_T) = 1$.

What do the forward messages tell us?

The forward messages are equivalent to,

$$\alpha_t(z_t) = \sum_{z_1} \cdots \sum_{z_{t-1}} p(z_{1:t}, \boldsymbol{x}_{1:t-1}) \tag{8}$$

$$p(z_t, \boldsymbol{x}_{1:t-1}). \tag{9}$$

The normalized message is the *predictive distribution*,

$$\frac{\alpha_t(z_t)}{\sum_{z_t'} \alpha_t(z_t')} = \frac{p(z_t, \boldsymbol{x}_{1:t-1})}{\sum_{z_t'} p(z_t', \boldsymbol{x}_{1:t-1})} = \frac{p(z_t, \boldsymbol{x}_{1:t-1})}{p(\boldsymbol{x}_{1:t-1})} = p(z_t \mid \boldsymbol{x}_{1:t-1}), \tag{10}$$

The final normalizing constant yields the marginal likelihood, $\sum_{z_T} \alpha_T(z_T) = p(\boldsymbol{x}_{1:T})$.

Message passing in state space models

The same recursive algorithm applies (in theory) to any state space model with the same factorization, but the sums are replaced with integrals,

$$
\begin{aligned}
p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:T}) &\propto \int \mathrm{d}\boldsymbol{z}_1 \cdots \int \mathrm{d}\boldsymbol{z}_{t-1} \int \mathrm{d}\boldsymbol{z}_{t+1} \cdots \int \mathrm{d}\boldsymbol{z}_T \, p(\boldsymbol{z}_{1:T}, \boldsymbol{x}_{1:T}) && (11) \\
&= \alpha_t(\boldsymbol{z}_t) \, p(\boldsymbol{x}_t \mid \boldsymbol{z}_t) \, \beta_t(\boldsymbol{z}_t) && (12)
\end{aligned}
$$

where the *forward and backward messages* are defined recursively

$$
\begin{aligned}
\alpha_t(\boldsymbol{z}_t) &= \int p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}) \, p(\boldsymbol{x}_{t-1} \mid \boldsymbol{z}_{t-1}) \, \alpha_{t-1}(\boldsymbol{z}_{t-1}) \, \mathrm{d}\boldsymbol{z}_{t-1} && (13) \\
\beta_t(\boldsymbol{z}_t) &= \int p(\boldsymbol{z}_{t+1} \mid \boldsymbol{z}_t) \, p(\boldsymbol{x}_{t+1} \mid \boldsymbol{z}_{t+1}) \, \beta_{t+1}(\boldsymbol{z}_{t+1}) \, \mathrm{d}\boldsymbol{z}_{t+1} && (14)
\end{aligned}
$$

The initial conditions are $\alpha_1(\boldsymbol{z}_1) = p(\boldsymbol{z}_1)$ and $\beta_T(\boldsymbol{z}_T) \propto 1$.

Forward pass in a linear dynamical system

Consider an linear dynamical system (LDS) with Gaussian emissions,

$$p(\boldsymbol{x}_{1:T}, \boldsymbol{z}_{1:T}) = p(\boldsymbol{z}_1) \prod_{t=2}^{T} p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}) \tag{15}$$

$$= \mathcal{N}(\boldsymbol{z}_1 \mid \boldsymbol{b}_1, \boldsymbol{Q}_1) \prod_{t=2}^{T} \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{b}, \boldsymbol{Q}) \prod_{t=1}^{\mathcal{N}} (\boldsymbol{x}_t \mid \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{d}, \boldsymbol{R}) \tag{16}$$

Let's derive the forward message $\alpha_{t+1}(\boldsymbol{z}_{t+1})$. Assume $\alpha_t(\boldsymbol{z}_t) \propto \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t-1}, \Sigma_{t|t-1})$.

$$\alpha_{t+1}(\boldsymbol{z}_{t+1}) = \int p(z_{t+1} \mid z_t)\, p(\boldsymbol{x}_t \mid z_t)\, \alpha_t(z_t)\, \mathrm{d}z_t \tag{17}$$

$$= \int \mathcal{N}(\boldsymbol{z}_{t+1} \mid \boldsymbol{A}\boldsymbol{z}_t + \boldsymbol{b}, \boldsymbol{Q})\, \mathcal{N}(\boldsymbol{x}_t \mid \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{d}, \boldsymbol{R})\, \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t-1}, \Sigma_{t|t-1})\, \mathrm{d}\boldsymbol{z}_t \tag{18}$$

The update step

The first step is the **update step**, where we **condition on** the emission $\boldsymbol{x}_t$,

Exercise: Expand the densities, collect terms, and complete the square to compute the mean $\boldsymbol{\mu}_{t|t}$ and covariance $\boldsymbol{\Sigma}_{t|t}$ after the update step,

$$\mathcal{N}(\boldsymbol{x}_t \mid \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{d}, \boldsymbol{R})\, \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t-1}, \boldsymbol{\Sigma}_{t|t-1}) \propto \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t}, \boldsymbol{\Sigma}_{t|t}). \tag{19}$$

The update step II

Write the joint distribution,

$$p(\boldsymbol{z}_t, \boldsymbol{x}_t \mid \boldsymbol{x}_{1:t-1}) = \mathcal{N}(\boldsymbol{x}_t \mid \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{d}, \boldsymbol{R})\, \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t-1}, \Sigma_{t|t-1}) \tag{20}$$

$$= \mathcal{N}\left(\begin{bmatrix} \boldsymbol{z}_t \\ \boldsymbol{x}_t \end{bmatrix} \,\middle|\, \begin{bmatrix} \boldsymbol{\mu}_{t|t-1} \\ \boldsymbol{C}\boldsymbol{\mu}_{t|t-1} + \boldsymbol{d} \end{bmatrix}, \begin{bmatrix} \Sigma_{t|t-1} & \Sigma_{t|t-1}\boldsymbol{C}^\top \\ \boldsymbol{C}\Sigma_{t|t-1} & \boldsymbol{C}\Sigma_{t|t-1}\boldsymbol{C}^\top + \boldsymbol{R} \end{bmatrix} \right) \tag{21}$$

Since $(\boldsymbol{z}_t, \boldsymbol{x}_t)$ are jointly Gaussian, $\boldsymbol{z}_t$ must be conditionally Gaussian as well,

$$p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t}) = \mathcal{N}(\boldsymbol{\mu}_{t|t}, \Sigma_{t|t}). \tag{22}$$

Exercise: Now use the **Schur complement** from Week 1 to solve for $\boldsymbol{\mu}_{t|t}$ and $\Sigma_{t|t}$

The update step III

Exercise: Write $\mu_{t|t}$ and $\Sigma_{t|t}$ in terms of the **Kalman gain**,

$$\boldsymbol{K}_t = \Sigma_{t|t-1}\boldsymbol{C}^\top(\boldsymbol{C}\Sigma_{t|t-1}\boldsymbol{C}^\top + \boldsymbol{R})^{-1} \tag{23}$$

What is the Kalman gain doing?

The predict step

The predict step yields $p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t}) = \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t}, \Sigma_{t|t})$. To complete the forward pass, we need the **predict step**,

$$\begin{aligned}
\alpha_{t+1}(\boldsymbol{z}_{t+1}) &= \int p(\boldsymbol{z}_{t+1} \mid \boldsymbol{z}_t)\, p(\boldsymbol{x}_t \mid \boldsymbol{z}_t)\, \alpha_t(\boldsymbol{z}_t)\, \mathrm{d}\boldsymbol{z}_t && (24) \\
&= \int \mathcal{N}(\boldsymbol{z}_{t+1} \mid \boldsymbol{A}\boldsymbol{z}_t + \boldsymbol{b}, \boldsymbol{Q})\, \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t}, \Sigma_{t|t})\, \mathrm{d}\boldsymbol{z}_t && (25) \\
&= \mathcal{N}(\boldsymbol{z}_{t+1} \mid \boldsymbol{\mu}_{t+1|t}, \Sigma_{t+1|t}) && (26)
\end{aligned}$$

Exercise: Solve for the mean $\boldsymbol{\mu}_{t+1|t}$ and covariance $\Sigma_{t+1|t}$ after the predict step.

Completing the recursions

That wraps up the recursions! All that's left is the base case, which comes from the initial state distribution,

$$\boldsymbol{\mu}_{1|0} = \boldsymbol{b}_1 \qquad \text{and} \quad \boldsymbol{\Sigma}_{1|0} = \boldsymbol{Q}_1. \tag{27}$$

Computing the marginal likelihood

Like in the discrete HMM, we can compute the marginal likelihood along the forward pass.

$$
\begin{aligned}
p(\boldsymbol{x}_{1:T}) &= \prod_{t=1}^{T} p(\boldsymbol{x}_t \mid \boldsymbol{x}_{1:t-1}) && (28)\\
&= \prod_{t=1}^{T} \int p(\boldsymbol{x}_t \mid \boldsymbol{z}_t)\, p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t-1})\, \mathrm{d}\boldsymbol{z}_t && (29)\\
&= \prod_{t=1}^{T} \int \mathcal{N}(\boldsymbol{x}_t \mid \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{d}, \boldsymbol{R})\, \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{\mu}_{t|t-1}, \Sigma_{t|t-1})\, \mathrm{d}\boldsymbol{z}_t && (30)
\end{aligned}
$$

Exercise: Obtain a closed form expression for the integrals.

Computing the smoothing distributions

- The forward pass gives us the filtering distributions $p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t})$. How can we compute the smoothing distributions, $p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:T})$?
- In the discrete HMM we essentially ran the *same algorithm in reverse* to get the backward messages, starting from $\beta_T(\boldsymbol{z}_T) \propto 1$.
- We can do the same sort of thing here, but it's a bit funky because we need to start with an improper Gaussian distribution $\beta_T(\boldsymbol{z}_T) \propto \mathcal{N}(\boldsymbol{0}, \infty \boldsymbol{I})$.
- Instead, we'll derive an alternative way of computing the smoothing distributions.

Bayesian Smoothing

Note: $\boldsymbol{z}_t$ is conditionally independent of $\boldsymbol{x}_{t+1:T}$ given $\boldsymbol{z}_{t+1}$, so

$$p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t+1}, \boldsymbol{x}_{1:T}) = p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t+1}, \boldsymbol{x}_{1:t}) \tag{31}$$

$$= \frac{p(\boldsymbol{z}_t, \boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:t})}{p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:t})} \tag{32}$$

$$= \frac{p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t})\, p(\boldsymbol{z}_{t+1} \mid \boldsymbol{z}_t)}{p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:t})} \tag{33}$$

Question: what rules did we apply in each of these steps?

Bayesian Smoothing II

Now we can write the joint distribution as,

$$p(\boldsymbol{z}_t, \boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:T}) = p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:T})p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:T}) \tag{34}$$

$$= \frac{p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t})\, p(\boldsymbol{z}_{t+1} \mid \boldsymbol{z}_t)\, p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:T})}{p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:t})}. \tag{35}$$

Marginalizing over $\boldsymbol{z}_{t+1}$ gives us,

$$p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:T}) = p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t}) \int \frac{p(\boldsymbol{z}_{t+1} \mid \boldsymbol{z}_t)\, p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:T})}{p(\boldsymbol{z}_{t+1} \mid \boldsymbol{x}_{1:t})}\, \mathrm{d}\boldsymbol{z}_{t+1} \tag{36}$$

Question: Can we compute each of these terms?

The Rauch-Tung-Striebel Smoother, aka Kalman Smoother

These recursions apply to any Markovian state space model. For the special case of a Gaussian linear dynamical system, the smoothing distributions are again Gaussians,

$$p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:T}) = \mathcal{N}(\boldsymbol{z}_t \mid \mu_{t|T}, \Sigma_{t|T}) \tag{37}$$

where

$$\mu_{t|T} = \mu_{t|t} + \boldsymbol{G}_t(\mu_{t+1|T} - \mu_{t+1|t}) \tag{38}$$

$$\Sigma_{t|T} = \Sigma_{t|t} + \boldsymbol{G}_t(\Sigma_{t+1|T} - \Sigma_{t+1|t})\boldsymbol{G}_t^\top \tag{39}$$

$$\boldsymbol{G}_t \triangleq \Sigma_{t|t}\boldsymbol{A}^\top \Sigma_{t+1|t}^{-1}. \tag{40}$$

This is called the **Rauch-Tung-Striebel (RTS) smoother** or the **Kalman smoother**.

Kalman smoothing in information form

So far we've worked with the *mean parameters* $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, but working with *natural parameters* $\boldsymbol{J}$ and $\boldsymbol{h}$ offers another perspective.

Let's go back to the basics,

$$
\begin{aligned}
p(\boldsymbol{z}_{1:T} \mid \boldsymbol{x}_{1:T}) &\propto p(\boldsymbol{z}_{1:T}, \boldsymbol{x}_{1:T}) && (41) \\
&= p(\boldsymbol{z}_1) \prod_{t=2}^{T} p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}) \prod_{t=1}^{T} p(\boldsymbol{x}_t \mid \boldsymbol{z}_t) && (42) \\
&= \mathcal{N}(\boldsymbol{z}_1 \mid \boldsymbol{b}_1, \boldsymbol{Q}_1) \prod_{t=2}^{T} \mathcal{N}(\boldsymbol{z}_t \mid \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{b}, \boldsymbol{Q}) \prod_{t=1}^{T} \mathcal{N}(\boldsymbol{x}_t \mid \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{d}, \boldsymbol{R}) && (43)
\end{aligned}
$$

Kalman smoothing in information form II

Expand the Gaussian densities,

$$p(\boldsymbol{z}_{1:T} \mid \boldsymbol{x}_{1:T}) \propto \exp\Big\{ -\frac{1}{2}(\boldsymbol{z}_1 - \boldsymbol{b}_1)^\top \boldsymbol{Q}_1^{-1}(\boldsymbol{z}_1 - \boldsymbol{b}_1) \tag{44}$$

$$-\frac{1}{2}\sum_{t=2}^{T}(\boldsymbol{z}_t - \boldsymbol{A}\boldsymbol{z}_{t-1} - \boldsymbol{b})^\top \boldsymbol{Q}^{-1}(\boldsymbol{z}_t - \boldsymbol{A}\boldsymbol{z}_{t-1} - \boldsymbol{b}) \tag{45}$$

$$-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{x}_t - \boldsymbol{C}\boldsymbol{z}_t - \boldsymbol{d})^\top \boldsymbol{R}^{-1}(\boldsymbol{x}_t - \boldsymbol{C}\boldsymbol{z}_t - \boldsymbol{d})\Big\} \tag{46}$$

This is a giant quadratic expression in $\boldsymbol{z}_{1:T}$; i.e. a multivariate normal distribution on $\mathbb{R}^{TD}$.

We can write it in terms of its natural parameters $\boldsymbol{J} \in \mathbb{R}^{TD \times TD}$ and $\boldsymbol{h} \in \mathbb{R}^{TD}$

Kalman smoothing in information form III

Question: Which entries in $\boldsymbol{J}$ are nonzero?

Duality between message passing and sparse linear algebra

Recall that to get mean from the natural parameters we have,

$$p(\boldsymbol{z}_{1:T} \mid \boldsymbol{x}_{1:T}) = \mathcal{N}(\boldsymbol{z}_{1:T} \mid \boldsymbol{J}^{-1}\boldsymbol{h}, \boldsymbol{J}^{-1}). \tag{47}$$

In other words, the posterior mean is the solution of a linear system $\boldsymbol{J}^{-1}\boldsymbol{h}$.

Typically, this would cost $O((TD)^3)$, but since $\boldsymbol{J}$ is block-tridiagonal (or more generally, banded), we can compute it in only $O(TD^3)$ time.

The algorithm for solving this sparse linear system is essentially the same as the message passing algorithm we derived today.

Message passing in nonlinear dynamical systems

Question: What would you do if you were given a nonlinear model, $p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}) = \mathcal{N}(z_t \mid f(\boldsymbol{z}_{t-1}), \boldsymbol{Q})$?

Sequential Monte Carlo

Recall that the forward messages are proportional to the predictive distributions $p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t-1})$. We can view the forward recursions as an expectation,

$$\alpha_t(\boldsymbol{z}_t) = \int p(\boldsymbol{z}_t \mid z_{t-1})\, p(\boldsymbol{x}_{t-1} \mid \boldsymbol{z}_{t-1})\, \alpha_{t-1}(\boldsymbol{z}_{t-1})\, \mathrm{d}\boldsymbol{z}_{t-1} \tag{48}$$

$$\propto \mathbb{E}_{\boldsymbol{z}_{t-1} \sim p(\boldsymbol{z}_{t-1} \mid \boldsymbol{x}_{1:t-2})} \left[p(\boldsymbol{z}_t \mid z_{t-1})\, p(\boldsymbol{x}_{t-1} \mid \boldsymbol{z}_{t-1}) \right] \tag{49}$$

One natural idea is to approximate this expectation with Monte Carlo,

$$\hat{\alpha}_t(\boldsymbol{z}_t) \approx \frac{1}{S} \sum_{s=1}^{S} \left[w_{t-1}^{(s)}\, p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}^{(s)}) \right] \tag{50}$$

where we have defined the **weights** $w_{t-1}^{(s)} \triangleq p(\boldsymbol{x}_{t-1} \mid \boldsymbol{z}_{t-1}^{(s)})$.

How do we sample $\boldsymbol{z}_{t-1}^{(s)} \overset{\text{iid}}{\sim} p(\boldsymbol{z}_{t-1} \mid \boldsymbol{x}_{1:t-2})$? Let's sample the normalized $\hat{\alpha}_{t-1}(\boldsymbol{z}_{t-1})$ instead!

Sequential Monte Carlo II

The normalizing constant is,

$$\int \hat{\alpha}_{t-1}(\boldsymbol{z}_{t-1}) \, \mathrm{d}\boldsymbol{z}_{t-1} = \frac{1}{S} \sum_{s=1}^{S} w_{t-2}^{(s)} \int p(\boldsymbol{z}_{t-1} \mid \boldsymbol{z}_{t-2}^{(s)}) \, \mathrm{d}\boldsymbol{z}_{t-1} = \frac{1}{S} \sum_{s=1}^{S} w_{t-2}^{(s)}. \tag{51}$$

Use this to define the *normalized forward message* (i.e. the Monte Carlo estimate of the predictive distribution) is,

$$\bar{\alpha}_{t-1}(\boldsymbol{z}_{t-1}) \triangleq \frac{\hat{\alpha}_{t-1}(\boldsymbol{z}_{t-1})}{\int \hat{\alpha}_{t-1}(\boldsymbol{z}'_{t-1}) \, \mathrm{d}\boldsymbol{z}'_{t-1}} = \sum_{s=1}^{S} \bar{w}_{t-2}^{(s)} \, p(\boldsymbol{z}_{t-1} \mid \boldsymbol{z}_{t-2}^{(s)}) \tag{52}$$

where $\bar{w}_{t-2}^{(s)} = \frac{w_{t-2}^{(s)}}{\sum_{s'} w_{t-2}^{(s')}}$ is the normalized weight of sample $\boldsymbol{z}_{t-2}^{(s)}$.

The normalized forward message is just a mixture distribution with weights $\bar{w}_{t-2}^{(s)}$!

Putting it all together

Combining the above, we have the following algorithm for the forward pass:

1. Let $\bar{\alpha}_1(\boldsymbol{z}_1) = p(z_1)$
2. For $t = 1, \ldots, T$:
   a. Sample $\boldsymbol{z}_t^{(s)} \overset{\text{iid}}{\sim} \bar{\alpha}_t(\boldsymbol{z}_t)$ for $s = 1, \ldots, S$
   b. Compute weights $w_t^{(s)} = p(\boldsymbol{x}_t \mid \boldsymbol{z}_t^{(s)})$ and normalize $\bar{w}_t^{(s)} = w_t^{(s)} / \sum_{s'} w_t^{(s')}$.
   c. Compute normalized forward message $\bar{\alpha}_{t+1}(\boldsymbol{z}_{t+1}) = \sum_{s=1}^{S} \bar{w}_t^{(s)} p(\boldsymbol{z}_{t+1} \mid \boldsymbol{z}_t^{(s)})$.

This is called **sequential Monte Carlo** (SMC) using the model dynamics as the proposal.

Note that Step 2a can **resample** the same $\boldsymbol{z}_{t-1}^{(s)}$ multiple times according to its weight.

Question: How can you approximate the marginal likelihood $p(\boldsymbol{x}_{1:T})$ using the weights? *Hint: look back to Slide 7.*

Generalizations

- Instead of sampling $\bar{\alpha}_t(\boldsymbol{z}_t)$, we could have sampled with a **proposal distribution** $r(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}^{(s)})$ instead and corrected for it by defining the weights to be,

$$w_t^{(s)} = \frac{p(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}^{(s)})\, p(\boldsymbol{x}_t \mid \boldsymbol{z}_t)}{r(\boldsymbol{z}_t \mid \boldsymbol{z}_{t-1}^{(s)})} \tag{53}$$

  Moreover, the proposal distribution can "look ahead" to future data $\boldsymbol{x}_t$.

References I