V4D2: Algebraic Topology II

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0 Preliminaries

0.1 Conventions

We begin with a series of conventions:

- We define a space from this point onwards as a compactly generated and weak Hausdorff topological space.
- $\underline{\mathrm{Top}}$ is the category of (NB: CGWH) spaces and continuous functions, which we call maps.
- $\underline{\mathrm{Top}}_*$ is the category of based spaces and based continuous functions.
- $[\cdot,\cdot]$ is the set of unbased homotopy classes of unbased functions.
- $[\cdot,\cdot]_*$ is the set of based homotopy classes of based functions.
- Groups will always be assumed to be topological groups. If a topology is not specified or canonical, then assume it is endowed with the discrete topology.
- To simplify notation, we will often denote the swapping of summands in a direct sum of vector spaces by φ_{swap}. The particular isometry is then to be assumed from context.

0.2 G-spaces

Definition 0.2.1. Let G be a group. A G-space is a space X with a continuous action by G, $G \times X \to X$. Similarly a based G-space is a based space X with a based continuous action by G, $G_+ \wedge X \to X$.

Definition 0.2.2. We define $\underline{\mathrm{Top}}^G$, the category of G-spaces and G-equivariant maps, and $\underline{\mathrm{Top}}^G_*$, the category of based G-spaces and based G-equivariant maps.

Definition 0.2.3. We define the balanced product $X \times_G Y$ of a G-space X and a G^{op}-space Y as the coequalizer of the two maps $m_X, m_Y : X \times G \times Y \to X \times Y$ defined by $m_X(x, g, y) = (xg, y)$ and $m_Y(x, g, y) = (x, gy)$. Similarly we can define the based balanced product $X \wedge_G Y$ as the coequalizer of the based analogue of m_X and m_Y.

Remark 0.2.4. The orbit space of a G-space X equals $\{*\} \times_G X$ and the orbit space of a based G-space equals $S^0 \wedge_G X$.

Remark 0.2.5. One may suppose that we have a concrete model for $X \times_G Y$ and $X \wedge_G Y$ as a suitable quotient of $X \times Y$ and $X \wedge Y$ respectively. However this is only true if G is compact. This is because taking quotients in the convenient category is in general not well behaved.

Definition 0.2.6. Suppose H is a closed subgroup of G. Let Y be a H-space. Then the induction of H, $\mathrm{Ind}_H^G(Y)$ equals $G \times_H Y$ is a G-space via left multiplication on G.

Dually we define $\mathrm{CoInd}_H^G(Y) = \mathrm{map}_H(G, Y)$, the subspace of $\mathrm{map}(G, Y)$ consisting of functions $f : G \to Y$ such that $f(hg) = hf(g)$ for all $h \in H$ and $g \in G$. This is a G-space via right multiplication on G.

Proposition 0.2.7. Ind_H^G and CoInd_H^G extend to functors from $\underline{\mathrm{Top}}^G$ to $\underline{\mathrm{Top}}^H$. Furthermore Ind_H^G is left adjoint to the restriction functor Res_H^G and CoInd_H^G is right adjoint to Res_H^G.

Proof. This is a exercise on the first exercise sheet. □

1 Orthogonal Spectra

We now begin the course proper. Recall that we have already seen a naive definition for the category of spectra in Algebraic Topology 1, namely the category of sequential spectra. However this is in some precise sense not a convenient category for conducting stable homotopy theory. The following section will mainly be concerned with the category of orthogonal spectra, and various constructions on it.

Definition 1.0.1. An inner product space V is a finite dimensional real vector space together with an inner product $\langle \cdot, \cdot \rangle$. We will write V for the inner product space $(V, \langle \cdot, \cdot \rangle_V)$

Definition 1.0.2. A linear isometric embedding (in short an isometry) is a linear map $T : V \to W$ between two inner product spaces V, W such that $\langle v, v' \rangle_V = \langle T(v), T(v') \rangle_W$.

Proposition 1.0.3. Every isometry is injective.

Definition 1.0.4. Given an inner product space V, we define $S^V = V \cup \{\infty\}$, the one point compactification of V. Given a map $\varphi : V \to W$, the functoriality of the one point compactification provides a map $S^V \to S^W$, which we denote S^φ.

Construction 1.0.5. We define $L(V, W)$ as the set of isometries from V to W. We define $O(V) = L(V, V)$. We turn $L(V, W)$ into a space as follows: First choose an orthonormal basis $b_1, \ldots, b_n$ of V and choose an isometry $\varphi : V \to W$. We then topologize $O(V)$ such that the clear bijection from $O(n)$, the group of orthogonal $n \times n$ matrices, to $O(V)$ given by the orthonormal basis $b_1, \ldots, b_n$ is a homeomorphism. We topologize $L(V, W)$ such that the map $O(W)/O(\mathrm{im}(\varphi)^\perp) \to L(V, W)$ given by $[f] \mapsto f \circ \varphi$ is a homeomorphism.

Lemma 1.0.6. The topology defined on $L(V, W)$ is independent of $b_1, \ldots, b_n$ and φ.

Example 1.0.7. S^V is an $O(V)$ space via the tautological action on V, with fixed points at 0 and ∞.

Construction 1.0.8. The composition $V \oplus W \longrightarrow V \times W \longrightarrow S^V \times S^W \longrightarrow S^V \wedge S^W$ extends to a continuous based map $S^{V \oplus W} \to S^V \wedge S^W$. We will call this the canonical maps and denote any map in this family by φ_{can}. The particular map will always be clear from context.

Lemma 1.0.9. The canonical map $S^{V \oplus W} \to S^V \wedge S^W$ is a homeomorphism for all inner product spaces V and W. Furthermore it is also a homeomorphism of $O(V) \times O(W)$ spaces.

Definition 1.0.10. An orthogonal spectrum X consists of the following data:

- A based space $X(V)$ for every inner product space V.
- For all inner product spaces V, W of equal dimension, based "action" maps $\rho_{V,W} : L_+(V,W) \wedge X(V) \to X(W)$ which satisfy the following two conditions:
  i) The $\rho_{V,W}$ are unital, by which we mean that $\rho_{V,V}(\mathrm{id}_V, \cdot) = \mathrm{id}_{X(V)}$.
  ii) The $\rho_{V,W}$ are associative, by which we mean that the following diagram commutes.

$$\begin{tikzcd} L(V,W)_+ \wedge L(U,V)_+ \wedge X(U) \arrow[r, "\mathrm{id}\wedge\rho_{U,V}"] \arrow[d, "\mathrm{comp}\wedge\mathrm{id}"] & L(V,W)_+ \wedge X(V) \arrow[d, "\rho_{V,W}"] \\ L(U,W)_+ \wedge X(U) \arrow[r, "\rho_{U,W}"] & X(W) \end{tikzcd}$$

- For every two arbitrary inner product spaces V, W, based "suspension" maps $\sigma_{V,W} : X(V) \wedge S^W \to X(V \oplus W)$, which satisfy the following three conditions:
  i) The $\sigma_{V,W}$ are unital, by which we mean that the following diagram commutes:

$$\begin{tikzcd} X(V) \wedge S^0 \arrow[r, "\cong"] \arrow[dr, "\sigma_{V,0}"] & X(V) \arrow[d, "\rho_{V,V\oplus 0}(\mathrm{inc},\cdot)"] \\ & X(V \oplus 0) \end{tikzcd}$$

  ii) The $\sigma_{V,W}$ are associative, by which we mean that the following diagram commutes:

$$\begin{tikzcd} X(U) \wedge S^V \wedge S^W \arrow[r, "\mathrm{id}\wedge\varphi_{\mathrm{can}}"] \arrow[d, "\sigma_{U,V}\wedge\mathrm{id}"] & X(U) \wedge S^{V\oplus W} \arrow[d, "\sigma_{U,V\oplus W}"] \\ X(U \oplus V) \wedge S^W \arrow[r, "\sigma_{U\oplus V,W}"] & X(U \oplus V \oplus W) \end{tikzcd}$$

  iii) The $\sigma_{V,W}$ and $\rho_{V,W}$ are compatible, by which we mean that the following diagram commutes:

$$\begin{tikzcd} L(V,V')_+ \wedge L(W,W')_+ \wedge X(V) \wedge S^W \arrow[r, "\oplus\wedge\sigma_{V,W}"] \arrow[d, "\rho_{V,V'}\wedge S^{L(W,W')}"] & L(V\oplus W, V'\oplus W')_+ \wedge X(V \oplus W) \arrow[d, "\rho_{V\oplus W,V'\oplus W'}"] \\ X(V') \wedge S^{W'} \arrow[r, "\sigma_{V',W'}"] & X(V' \oplus W') \end{tikzcd}$$

Remark 1.0.11. For a topological space X, X_+ denotes the space $X \cup \{*\}$, X union a disjoint basepoint. In the case of $L(X,Y)$, $L(X,Y)_+$ agrees with the one-point compactification of $L(X,Y)$, because $L(X,Y)$ is already compact.

Definition 1.0.12. A morphism of orthogonal spectra $f : X \to Y$ is a collection of based maps $f(V) : X(V) \to Y(V)$ such that the following two diagrams commute:

$$\begin{tikzcd} L(V,W)_+ \wedge X(V) \arrow[r, "\rho_{V,W}"] \arrow[d, "\mathrm{id}\wedge f(V)"] & X(W) \arrow[d, "f(W)"] \\ L(V,W)_+ \wedge Y(V) \arrow[r, "\rho_{V,W}"] & Y(W) \end{tikzcd} \qquad \begin{tikzcd} X(V) \wedge S^W \arrow[r, "\sigma_{V,W}"] \arrow[d, "f(V)\wedge\mathrm{id}"] & X(V \oplus W) \arrow[d, "f(V\oplus W)"] \\ Y(V) \wedge S^W \arrow[r, "\sigma_{V,W}"] & Y(V \oplus W) \end{tikzcd}$$

Definition 1.0.13. We write $\underline{\mathrm{Sp}}$ for the category consisting of orthogonal spectra and the morphisms of orthogonal spectra.

Remark 1.0.14.

- Part of the definition for $X \in \underline{\mathrm{Sp}}$ is a functor $X(\cdot)$ from the category inner product spaces and isometric isomorphisms to $\underline{\mathrm{Top}}_*$. However this is not sufficient for out considerations, because this functor has to respect the topology on $L(V,W)$ and $\mathrm{map}_*(X(V), X(W))$.
- Each $X \in \underline{\mathrm{Sp}}$ has an underlying sequential spectrum. This is given by $X_n = X(\mathbb{R}^n)$ and $\sigma_{n,1} = \sigma_{\mathbb{R}^n,\mathbb{R}}$.
- The $X(V)$ are $O(V)$ spaces via the action map $\rho_{V,V}$.
- Given a $\varphi \in L(V,W)$, we may denote $\rho_{V,W}(\varphi, \cdot)$ by $X(\varphi)$.

Lemma 1.0.15. Let $O(W)$ act on $L(V,W)_+ \wedge_{O(V)} X(V)$ via post composition on $L(V,W)$. Then the $\rho_{V,W}$ descend to a homeomorphism $L(V,W)_+ \wedge_{O(V)} X(V) \to X(W)$ of $O(W)$ spaces.

Proof. We will show that $\rho_{V,W} : L(V,W)_+ \wedge X(V) \to X(W)$ satisfies the universal property of the balanced product $L(V,W)_+ \wedge_{O(V)} X(V)$. That both compositions agree in the relevant diagram follows immediately from the associativity of $\rho_{V,W}$.

Now we have to show that $X(W)$ is universal with respect to this diagram. To do this we construct a section of $\rho_{V,W}$. Pick a $\varphi_0 \in L(V,W)$. We first identify $X(W) \cong S^0 \wedge X(W)$. Then we define s to be the following map $(0 \mapsto \varphi_0 \wedge \varphi_0^{-1}) \wedge \mathrm{id} : S^0 \wedge X(W) \to L(V,W)_+ \wedge L(V,W)_+ \wedge X(W)$ composed with the map $\mathrm{id} \wedge \rho_{W,V} : L(V,W)_+ \wedge L(V,W)_+ \wedge X(W) \to L(V,W)_+ \wedge X(V)$. That $\rho_{V,W} \circ s = \mathrm{id}_{X(W)}$ again follows from the associativity of $\rho_{V,W}$.

Now consider the following diagram, where the solid diagram commutes:

$$\begin{tikzcd}
L(V,W)_+ \wedge O(V) \wedge X(V) \arrow[r, shift left] \arrow[r, shift right] & L(V,W)_+ \wedge X(V) \arrow[r, "\rho_{V,W}"'] \arrow[dr, "f"'] & X(W) \arrow[l, bend right, "s"'] \arrow[d, dotted, "\phi"] \\
& & T
\end{tikzcd}$$

It should be clear that if a ϕ exists which makes the diagram commute, it must be equal to $f \circ s$. Finally one can check that $f \circ s$ truly does make the following diagram commute, and therefore that $X(W)$ is universal. $\square$

1.1 Homotopy Groups of Spectra

Definition 1.1.1. Let $i \in \mathbb{Z}$. We define the i -th homotopy group of $X \in \underline{\mathrm{Sp}}$ as the colimit $\mathrm{colim}_{n>>0}(\pi_{i+n}(X^n))$, where $X_n = X(\mathbb{R}^n)$, taken along the maps:

$$\pi_{i+n}(X^n) \xrightarrow{\cdot \wedge S^1} \pi_{i+n+1}(X^n \wedge S^1) \xrightarrow{\sigma_{n,1}^*} \pi_{i+n+1}(X_{n+1})$$

Remark 1.1.2.

- Because $\pi_{i+n}(X_n)$ is an abelian group for $i + n \geq 2$, we can take the colimit in the category of abelian groups.

- Because $\pi_i(\cdot)$ and taking colimits is functorial in spaces, $\pi_i(\cdot)$ is functorial in spectra.
- Note that the definition of homotopy groups only depends on the underlying sequential spectrum of an orthogonal spectrum. This is simply for simplicity. Taking the colimit over all $X(V)$ would result in the same groups, because a simple check shows that the sequential spectrum is cofinal in the resulting colimit.

Construction 1.1.3. Let $i \in \mathbb{Z}$, $n \in \mathbb{N}$ and V an inner product space. Let the dimension of V equal m. Then any map $f : S^{\mathbb{R}^{i+n} \oplus V} \to X(\mathbb{R}^n \oplus V)$ gives rise to a class $[f] \in \pi_i(X)$ by identifying $V \cong \mathbb{R}^m$ by an isometric isomorphism $\varphi : V \to \mathbb{R}^m$.

Lemma 1.1.4.

i) $[f]$ is independent of the choice of isometric isomorphism φ.

ii) $[f] = [f \diamond V]$, where $f \diamond V$ is the following composition

$$S^{\mathbb{R}^{i+n} \oplus V \oplus V} \cong S^{\mathbb{R}^{i+n} \oplus V} \wedge S^V \xrightarrow{f \wedge \mathrm{id}} X(\mathbb{R}^n \oplus V) \wedge S^V \xrightarrow{\sigma_{\mathbb{R}^n \oplus V, V}} X(\mathbb{R}^n \oplus V \oplus V)$$

Remark 1.1.5. The triangle operation should be considered a generalization of the stabilization maps in the colimit system in the definition of the homotopy groups of a spectrum. Therefore this lemma should come as no surprise.

Definition 1.1.6. Let $f : X \to Y$ be a map of orthogonal spectra.

i) f is a level equivilance if and only if all the maps $f(V) : X(V) \to Y(W)$ are weak equivalences.

ii) f is a stable equivilance if and only if all the induced maps $\pi_i(f)$ are isomorphisms.

Remark 1.1.7. Clearly a level equivilance is also a stable equivilance. The alternate direction is however not true. Unsurprisingly, in the area of stable homotopy theory the notion of stable equivilance is more natural.

1.2 Basic Examples

Construction 1.2.1. The suspension spectrum $\sum^\infty$ of a space $X \in \underline{\mathrm{Top}}_*$ is determined by the following data:

- $(\sum^\infty X)(V) = S^V \wedge X$,
- $\rho_{V,W} : L(V,W)_+ \wedge S^V \wedge X \to S^W \wedge X$ is given by $\varphi \wedge v \wedge x \mapsto \varphi(v) \wedge x$,
- $\sigma_{V,W} : S^V \wedge X \wedge S^W \to S^{V \oplus W} \wedge X$ is given by first swapping the last two coordinates, and then applying the canonical map φ_{can}.

Definition 1.2.2. We call the spectrum $\mathbb{S} = \sum^\infty S^0$ the sphere spectrum.

Definition 1.2.3. For $X \in \underline{\mathrm{Top}}_*$, we call $\pi_i(\sum^\infty X)$ the i-th stable homotopy group of X.

Lemma 1.2.4. For $i \leq 0$, $\pi_i(\sum^\infty X) = 0$.

Example 1.2.5. We know from the calculations of algebraic topology 1 that the stable homotopy group of the sphere spectrum is zero in negative degrees, $\mathbb{Z}$ for $i = 0$, and finite in all positive degrees. For example, $\pi_1(\mathbb{S}) = \mathbb{Z}/2\mathbb{Z}$ is generated by the Hopf map $\eta : S^3 \to S^2$.

Construction 1.2.6. Let A be an abelian group, and let X be a based space with base point x_0. Then the A-linearization $A[X]$ of X is the space of finite formal linear combinations of elements of A and K, after quotienting by the equivilance relation $ax_0 = 0$. This is a based set, with base point 0. We topologize $A[X]$ with the quotient topology induced by the following surjection $\coprod_{n\in\mathbb{N}} A^n \times X^n \to \sum_{i\in\mathbb{N}} A^n \times X^n$, $(a_1, \dots, a_n, x_1, \dots, x_n) \mapsto \sum a_i x_i$.

Remark 1.2.7. $A[X]$ is naturally a topological abelian group, via addition of formal sums. Furthermore the functor $\mathbb{Z}[\cdot]$ is left adjoint to the forgetful functor from topological abelian groups to $\underline{\mathrm{Top}}_*$.

Theorem 1.2.8. The functor $A[\cdot]$ sends cofiber sequences to fiber sequences. Furthermore the functors $h_*(\cdot) = \pi_*(A[\cdot])$ is a reduced homology theory on finite CW complexes whose coefficient groups are A for $i = 0$ and zero otherwise.

Remark 1.2.9. The proof of this theorem can be found in the original article from Dold-Thom, and is extremely similar to the Dold-Thom Theorem from Algebraic Topology 1. In fact it is in some sense even easier, because the map $A[X] \to A[X/A]$ is in fact a fibration.

Corollary 1.2.10. $A[S^n] \cong K(A, n)$.

Construction 1.2.11. We construct the Eilenberg-Maclane spectrum HA of an abelian group A. $HA(V) = A[S^V]$, $\rho_{V,W}$ is defined by $\varphi \wedge \sum a_v v \mapsto \sum a_v \varphi(v)$, $\sigma_{V,W}$ is defined by $(\sum a_v v) \wedge w \mapsto \sum a_v (v, w)$.

Definition 1.2.12. $X \in \underline{\mathrm{Sp}}$ is called an Ω-spectrum if all the $\tilde{\sigma}_{V,W} : X(V) \to \Omega^W X(V \oplus W)$ adjoint to $\sigma_{V,W}$ are weak equivalences.

Lemma 1.2.13. If X is an Ω-spectrum, then all the maps in the colimit system definition $\pi_i(X)$ are isomorphisms. In particular $\pi_i(X) = \pi_i(X_0)$ for $i \geq 0$ and $\pi_0(X_{-i})$ for $i \leq 0$.

Proposition 1.2.14. $\pi_i(HA) = A$ if $i = 0$ and equals $\{0\}$ otherwise.

Proof. Considering the following commutative diagram:

$$\begin{tikzcd}
{[S^n, X_n]_*} \arrow[r, "- \wedge S^1"] \arrow[dr, "\mathrm{counit}^*"'] & {[S^{n+1}, X^n \wedge S^1]_*} \arrow[r, "\sigma^*_{n,1}"] \arrow[d, "\cong"] & {[S^{n+1}, X_{n+1}]_*} \arrow[d, "\cong"] \\
& {[S^n, \Omega(X^n \wedge S^1)]_*} \arrow[r] & {[S^n, \Omega X_{n+1}]_*}
\end{tikzcd}$$

The bottom composition equals $[S^n, \tilde{\sigma}_{n,1}]$, hence is an isomorphism by assumption. Therefore the top composition is an isomorphism. □

Lemma 1.2.15. X is an Ω-spectrum if and only if the $\tilde{\sigma}_{n,1}$ are weak equivalences.

Proof. The important identity is the following: $\sigma_{n,m} = \sigma_{n+m-1,1} \circ \sigma_{n+m-2,1} \circ \cdots \circ \sigma_{n+1,1} \circ \sigma_{n,1}$, which comes from the associativity of the sigmas applied inductively. One then applies the adjoint relation to this, to show that $\sigma_{\tilde{n,m}}$ is a weak equivilance, assuming each $\tilde{\sigma}_{n,1}$ is. One can then clearly generalize this to all $\sigma_{V,W}$. □

Proposition 1.2.16. *HA is an Ω-spectrum.*

Proof. This follows from the Dold-Thom theorem quoted earlier. In particular the fact that $A[\cdot]$ sends cofiber sequences to fiber sequences. Consider the following diagram:

$$\begin{tikzcd}
A[S^n] \arrow[r] \arrow[d, "\mathrm{adj}"] & A[CS^n] \arrow[r] \arrow[d, "\cong"] & A[\sum S^n] \arrow[d, "\mathrm{id}"] \\
\Omega(A[\sum S^n]) \arrow[r] & P(A[\sum S^n]) \arrow[r] & A[\sum S^n]
\end{tikzcd}$$

Both the top and bottom row are fiber sequences, and the left two maps are homotopy equivalences. The middle map exists because both CS^n and $P(A[\sum S^n])$ are contractible. Then the five lemma implies that $\pi_n(\tilde{\sigma}_{n,1})$ is an isomorphism, proving the lemma. □

1.3 Suspension, Loops, and Shift

We now describe three important functors from $\underline{\mathrm{Sp}}$ to itself which shift the homotopy groups of spectra up or down.

Construction 1.3.1. Let $A \in \underline{\mathrm{Top}}_*$, $(X, \rho^X, \sigma^X) \in \underline{\mathrm{Sp}}$.

i) We then define $X \wedge A \in \underline{\mathrm{Sp}}$ with the following data:
   - $(X \wedge A)(V) = X(V) \wedge A$,
   - $\rho_{V,W} = \rho^X_{V,W} \wedge \mathrm{id}_A$,
   - $\sigma_{V,W} = (\sigma^X_{V,W} \wedge \mathrm{id}_A) \circ (\mathrm{twist})$.

ii) Dually we define $\mathrm{map}_*(A, X) \in \underline{\mathrm{Sp}}$ with the following data:
   - $(\mathrm{map}_*(A, X))(V) = \mathrm{map}_*(A, X(V))$,
   - $\rho_{V,W}$ sends $\varphi \wedge f$ to $\rho^X_{V,W}(\varphi, \cdot) \circ f$,
   - $\sigma_{V,W}$ sends $f \wedge w$ to $\sigma^X_{V,W}(\cdot, w) \circ f$.

Remark 1.3.2. Given $X \in \underline{\mathrm{Sp}}$ we write ΩX for the spectrum $\mathrm{map}_*(S^1, X)$.

The importance of these constructions comes from the following:

Proposition 1.3.3. *$\cdot \wedge A : \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$ is left adjoint to $\mathrm{map}_*(A, \cdot) : \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$.*

Proof. This follows from the classical adjunction between suspension and mapping spaces in $\underline{\mathrm{Top}}$, which we can apply level-wise. □

Proposition 1.3.4.

i) The adjunction isomorphisms combined with a coordinate swap:

$$\mathrm{map}_*(S^{i+n}, \Omega^m S) \cong \mathrm{map}_*(S^{i+n+m}, X) \cong \mathrm{map}_*(S^{i+m+n}, X)$$

give rise to an isomorphism $\pi_i(\Omega^m X) \to \pi_{i+m}(X)$.

ii) The maps $[S^{i+n}, X_n]_* \to [S^{i+n} \wedge S^m, X_n \wedge S^m]_* \cong [S^{i+m+n}, X_n \wedge S_m]_*$ give rise to an isomorphism $\pi_i(X) \to \pi_i(X) \to \pi_{i+m}(X \wedge S^m)$.

Proof. i) Because each map is an isomorphism, it is enough to confirm that the adjunction maps are compatible with the stabilization maps in the colimit system defining the π_i. Given $f : S^{i+n} \to \Omega^m X_n$ we write $f^\flat : S^{i+m+n} \to X_n$ for the adjoint of f.

So we have to show that $[(f \diamond \mathbb{R}^k)^\flat]$ and $[f^\flat \diamond \mathbb{R}^k]$ are represented by the same map. Such a map is the following: $S^{i+m+n+k} \to X_{n+k}$ given by $u, w, v, z \to \sigma_{n,k}(f(u, v)(w), z)$. We leave it to the reader to confirm this fact.

ii) A similar computation shows that the suspension homomorphisms are compatible with stabilization. This implies that we get a well-defined homomorphism $\cdot \wedge S^m : \pi_i(X) \to \pi_{i+m}(X \wedge S^m)$. However in contrast to previously, the maps are not isomorphisms level-wise and so we are not done.

To show $\cdot \wedge S^m$ is an isomorphism we construct an inverse $\Phi : \pi_{i+m}(X \wedge S^m) \to \pi_i(X)$. This is given by $[f] \mapsto \Phi([f])$, which by an abuse of notation we set equal to $[\Phi(f)]$. If $[f]$ is a function from S^{i+m+n} to $X_n \wedge S^m$, we define $\Phi(f)$ to be the following composition:

$$S^{i+m+n} \xrightarrow{f} X_n \wedge S^m \xrightarrow{\sigma_{n,m}} X(\mathbb{R}^n \oplus \mathbb{R}^m) \xrightarrow{X(\varphi_{\mathrm{swap}})} X(\mathbb{R}^m \oplus \mathbb{R}^n)$$

One should once again confirm that Φ commutes with stabilization, and therefore induces a map on homotopy groups.

Then $(\Phi \circ (\cdot \wedge S^m))[f] = [X(\varphi_{\mathrm{swap}}) \circ (f \diamond \mathbb{R}^m) \circ S^{\varphi^{-1}_{\mathrm{swap}}}]$, which equals $[f \diamond \mathbb{R}^m]$. This is because both representations become homotopic after stabilizing by $\mathbb{R}^{n+m}$. We have seen arguments to this effect before.

Conversely, $(\cdot \wedge S^m) \circ \Phi)[f] = [X(\varphi_{\mathrm{swap}}) \circ (f \diamond \mathbb{R}^m) \circ S^{\varphi^{-1}_{\mathrm{swap}}} = [f \diamond \mathbb{R}^m] = [f]$.

□

Corollary 1.3.5. The adjunction unit $\eta : X \to \Omega^m(X \wedge S^m)$ and counit $\epsilon : (\Omega^m X) \wedge S^m \to X$ are stable equivalences. In particular $\cdot \wedge S^m$ and $\Omega^m(\cdot)$ are inverse functors up to stable equivilance.

Proof. This follows immediately from applying the functors π_i to the triangles defining the counit and unit. □

Construction 1.3.6. Let V be an inner product space. The V-th shift of $(X, \rho^X, \sigma^X) \in \underline{\mathrm{Sp}}$ is the spectrum $\mathrm{Sh}^V(X)$ given by the following data:

- $(\mathrm{Sh}^V(X))(U) = X(U \oplus V)$,
- $\rho_{U,W} : L(U,W)_+ \wedge X(U \oplus V) \to X(U \oplus W)$ is given by $\varphi \wedge x \mapsto X(\varphi \oplus \mathrm{id}_V)(x)$,
- $\sigma_{U,W} : X(U \oplus V) \wedge S^W \to X(U \oplus W \oplus V)$ is given by $X(\mathrm{id}_U \oplus \varphi_{\mathrm{swap}}) \circ \sigma^X_{U \oplus V,W}$.

The shift functor enjoys many useful properties:

Lemma 1.3.7. The functors $\cdot \wedge A$, $\mathrm{map}_*(A, \cdot)$ and Sh^V satisfy:

i) $(\mathrm{Sh}^V X) \wedge A = \mathrm{Sh}^V(X \wedge A)$,

ii) $\mathrm{map}_*(A, \mathrm{Sh}^V X) = \mathrm{Sh}^V(\mathrm{map}_*(A, X))$,

iii) $\mathrm{Sh}^V \circ \mathrm{Sh}^W$ is naturally equivalent to $\mathrm{Sh}^{V \oplus W}$.

Construction 1.3.8. We define a map $\lambda^V_X : X \wedge S^V \to \mathrm{Sh}^V X$ by $\lambda^V_X(U) = \sigma^X_{U,V}$. This induces a natural transformation from $\cdot \wedge V$ to Sh^V.

Remark 1.3.9. We note that λ^X_V is only well-defined because of the application of $X(\varphi_{\mathrm{swap}})$ in the definition of the shift functor. This in particular means that λ^V_X would not exist on the level of sequential spectrum. This is one example where the extra structure of orthogonal spectra works in our favour.

Construction 1.3.10. We define the shift homomorphism $\mathrm{sh} : \pi_i X \to \pi_{i+m}(\mathrm{Sh}^m X)$ as follows: Given a map $f : S^{i+n} \to X_n$, we define $\mathrm{sh}([f])$ as the equivilance class of the following composition in $\pi_{i+m}(\mathrm{Sh}^m X)$:

$$S^{i+m+n} \xrightarrow{\varphi^{-1}_{\mathrm{can}}} S^{i+n} \wedge S^m \xrightarrow{f \wedge \mathrm{id}} X_n \wedge S^m \xrightarrow{\sigma_{n,m}} X(\mathbb{R}^n \oplus \mathbb{R}^m)$$

This is well-defined by similar arguments as before. In particular, it is clear that two different maps $f, g : S^{i+n} \to X_n$ such that $[f] \cong [g]$ are mapped into the same equivilance class in $\pi_{i+m}(\mathrm{Sh}^m X)$. One must then also confirm that the map defined level-wise commutes with stabilization, up to taking equivilance classes in $\pi_{i+m}(\mathrm{Sh}^m X)$.

Proposition 1.3.11.

i) For all $m \geq 0 : \mathrm{sh}^m$ is an isomorphism.

ii) The map $\lambda^m_X : X \wedge S^m \to \mathrm{Sh}^m X$ and its adjoint $\tilde{\lambda}^m_X : X \to \Omega^m(\mathrm{sh}^m X)$ are natural stable equivalences.

Proof.

i) We define an inverse $\Phi : \pi_{i+m}(\mathrm{Sh}^m X) \to \pi_i$ as follows: given a map $g : S^{i+m+n} \to X(\mathbb{R}^m \oplus \mathbb{R}^n)$, we define $\Phi([g])$ as $[g]$. This is clearly well-defined, and an inverse for the shift map.

ii) We can unravel the definition of sh^m to see that it agrees with the following composition:

$$\pi_i(X) \xrightarrow{\tilde{\lambda}_*} \pi_i(\Omega^m(\mathrm{Sh}^m X)) \xrightarrow{\cong} \pi_{i+m}(\mathrm{Sh}^m X)$$

Since sh^m is an isomorphism the two out of three rule implies that $\tilde{\lambda}_*$ is an isomorphism. Then λ is an isomorphism, again by the two out of three rule and the commutativity of the following diagram:

$$\begin{tikzcd} X \wedge S^m \arrow[rr, "\lambda_*"] \arrow[dr, "\tilde{\lambda}_* \wedge \mathrm{id}"'] & & \mathrm{sh}^m X \\ & \Omega^m(\mathrm{Sh}^m X) \wedge S^m \arrow[ur, "\epsilon_*"'] & \end{tikzcd}$$

This shows that both λ and $\tilde{\lambda}$ are stable equivalences.

□

1.4 (Co)fiber Sequences

All spaces, maps and homotopies in the following section are assumed to be based.

We will show in the following section how suitably defined cofiber and fiber sequences of spectra give right to long exact sequences in homotopy groups. To begin we recall the relevant definitions from Algebraic Topology 1. This will also allow us to set conventions. In the following, and throughout the remainder of the notes, the unit interval I will have its basepoint at 0. We will also frequently identify $I^n/\partial I^n$ with $S^n = \mathbb{R}^n \cup \{\infty\}$ by the maps t^n given by $t^n([x_1, \ldots, x_n]) = (\frac{2x_1-1}{x_1(1-x_1)}, \ldots, \frac{2x_1-1}{x_n(1-x_n)}$.

Definition 1.4.1.

- We define the cone CA of a space A to be $A \wedge I$. Then the mapping cone C_f of a map $f : A \to B$ equals $CA \cup_f B$. We have the following maps:

  $$B \overset{i}{\hookrightarrow} C_f \xrightarrow{p} A \wedge S^1$$

  i is the inclusion of B into C_f and p is the map which collapses B and sends $a \wedge x$ to $a \wedge t^1(x)$.

- We define the based path space $\mathrm{Path}_*(B)$ of a space B to equal $\mathrm{map}_*(I, B)$. This is naturally based by taking the constant path. Then we define the homotopy fiber, $F_f = \mathrm{Path}_*(B) \times_B A = \{(\gamma, a) \in \mathrm{Path}_*(B) \times A \mid \gamma(0) = *_B, \gamma(1) = f(a)\}$. This is again naturally based by taking the constant path. We then have the following maps:

  $$\Omega B \xrightarrow{j} F_f \xrightarrow{q} A$$

  where i is given by $\omega \mapsto (\omega \circ t^1, *_A)$ and q is the projection onto the second coordinate.

Proposition 1.4.2. Both compositions $p \circ i$ and $q \circ j$ are null. Furthermore the following triangle commutes up to homotopy:

$$
\begin{tikzcd}
CA \cup_f CB \arrow[d, "p_a \cup *"'] \arrow[dr, "* \cup p_b"] & \\
A \wedge S^1 \arrow[r, "f \wedge S^{-\mathrm{id}}"'] & B \wedge S^1
\end{tikzcd}
$$

Proof. This should be familiar from Algebraic Topology 1. $\square$

Lemma 1.4.3. Consider the following based diagram:

$$
\begin{tikzcd}
& Z \arrow[d, "\beta"] & \\
A \arrow[r, "f"] & B \arrow[r, "i"] & C_f
\end{tikzcd}
$$

where $i \circ \beta \cong *$. Then there exists a based map $h : Z \wedge S^1 \to A \wedge S^1$ such that the following triangle commutes up to homotopy:

$$
\begin{tikzcd}
Z \wedge S^1 \arrow[r, "h"] \arrow[dr, "\beta \wedge \mathrm{id}"'] & A \wedge S^1 \arrow[d, "f \wedge \mathrm{id}"] \\
& B \wedge S^1
\end{tikzcd}
$$

Proof. Pick a null homotopy $H : i\beta \cong *$. Then consider the following diagram:

$$
\begin{tikzcd}
Z \wedge I \arrow[r, "H"] \arrow[d, two heads] & C_f \arrow[r, "p"] & A \wedge S^1 \\
Z \wedge (I/\partial I) \cong Z \wedge S \arrow[urr, dotted, "\exists! h"'] & &
\end{tikzcd}
$$

The map $h : Z \wedge S^1 \to A \wedge S^1$ exists because of the universal property of the quotient map. In particular the top composition is constant on the top copy of Z because H is a null-homotopy, and is constant on the bottom copy of Z because $p \circ i$ is null.

We claim that the diagram below completes the proof. Note first that $p_2 \cup *$, which collapses one copy of CZ, is a homotopy equivilance. Then following the diagram on the outside gives exactly the triangle we hope to show commutes. But every inside diagram commutes and therefore the outside diagram commutes.

$$
\begin{tikzcd}
CZ \cup_{Z \times 1} CZ \arrow[rr, "="] \arrow[d, "\mathrm{id} \cup *"'] \arrow[dr, "H \cup C(\beta)"] & & CZ \cup_{Z \times 1} CZ \arrow[d] \arrow[dr, bend left, "p_2 \cup *"] & \\
Z \wedge S^1 \arrow[dr, bend right, "h"'] & CA \cup_f CB \arrow[d, "p_A \cup *"] \arrow[dr, "* \cup p_B"] & Z \wedge S^1 \arrow[r, "Z \wedge S^{-\mathrm{id}}"] \arrow[d, "\beta \wedge S^1"] & Z \wedge S^1 \arrow[d, "\beta \wedge \mathrm{id}"] \\
& A \wedge S^1 \arrow[r, "f \wedge \mathrm{id}"'] & B \wedge S^1 \arrow[r, "\mathrm{id} \wedge S^{-\mathrm{id}}"] & B \wedge S^1
\end{tikzcd}
$$

$\square$

Construction 1.4.4. Let $f : X \to Y$ be a map of spectra. Then the mapping cone C_f of f is the spectrum determined by the following data:

- $(C_f)(V) = C_{f(V)}$
- $\rho_{V,W} : L(V,W)_+ \wedge C_{f(V)} \to C_{f(W)}$ is defined to be the push-out of the maps $\rho^Y_{V,W}$ and $\rho^Y(V,W)$.
- $\sigma_{V,W} : C_{f(V)} \wedge S^W \to C_{f(V \oplus W)}$ is defined to be the push-out of the maps $\sigma^X_{V,W}$ and $\sigma_{V,W}$.

Similarly to the case of spaces, we then have maps: $Y \xrightarrow{i} C_f \xrightarrow{p} X \wedge S^1$ defined level wise by i and p. From this we can define the connecting homomorphism $\delta : \pi_{i+1}(C_f) \xrightarrow{p_*} \pi_{i+1}(X \wedge S^1) \xleftarrow{\cong} \pi_i(X)$.

Construction 1.4.5. Let $f : X \to Y$ be a map of spectra. Then the homotopy fibre F_f of f is the spectrum determined by the following data:

- $(F_f)(V) = F_{f(V)}$
- $\rho_{V,W} : L(V,W)_+ \wedge F_{f(V)} \to F_{f(W)}$ is defined to be the pull-back of the maps $\rho^Y_{V,W}$ and $\rho^Y_{V,W}$.
- $\sigma_{V,W} : C_{f(V)} \wedge S^W \to C_{f(V \oplus W)}$ is defined to be the pull-back of the maps $\sigma^X_{V,W}$ and $\sigma^Y_{V,W}$.

Similarly to the case of spaces, we then have maps: $\Omega Y \xrightarrow{j} F_f \xrightarrow{q} X$ defined level wise by j and q. From this we can define the connecting homomorphism $\delta : \pi_{i+1}(Y) \xleftarrow{\cong} \pi_i(\Omega Y) \xrightarrow{i_*} \pi_i(F_f)$.

Theorem 1.4.6. For every map of spectra $f : X \to Y$ the following long exact sequences of abelian groups is exact:

$$\ldots \longrightarrow \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \xrightarrow{i_*} \pi_i(C_f) \xrightarrow{\delta} \pi_{i-1}(X) \longrightarrow \ldots$$

$$\ldots \longrightarrow \pi_i(X) \xrightarrow{f_*} \pi_i(Y) \xrightarrow{\delta} \pi_{i-1}(F_f) \xrightarrow{p_*} \pi_{i-1}(X) \longrightarrow \ldots$$

Proof. The exactness of the fiber sequence of spaces implies that the sequence is exact level-wise. Now taking sequential colimits is exact, and so we immediately obtain the exactness of the second long exact sequence of spectra.

It is more difficult to prove the exactness of the long exact sequence. This is of course to be expected, because there is no analogue of this long exact sequence for spaces.

We first prove exactness at $\pi_i(Y)$. First note that $i_* f_*$ is zero, because it is induced by map level-wise constant maps. Noe consider $\beta : S^{i+n} \to Y_n$ such that $[\beta] \in \ker(i_*)$. After stabilizing we can assume that $i \circ \beta$ is null-homotopic. So we can apply 1.4.3 to obtain a map $h : S^{i+n} \wedge S^1 \to S_n \wedge S^1$ such that the following triangle commutes:

$$\begin{tikzcd} S^{i+n} \wedge S^1 \arrow[r, "h"] \arrow[dr, "\beta\wedge\mathrm{id}"'] & X_n \wedge S^1 \arrow[d, "f_n\wedge\mathrm{id}"] \\ & Y_n \wedge S^1 \end{tikzcd}$$

Then $f_*[\sigma_{n,1} \circ h] = [f_{n+1} \circ \sigma_{n,1} \circ h] = [\sigma_{n,1} \circ (f_n \wedge S^1) \circ h] = [\sigma_{n,1} \circ \beta \wedge S^1] = [\beta \diamond \mathbb{R}] = [\beta]$.

Next we can apply the level-wise homotopy equivilance: $C_i = CY \cup_f CX \cong X \wedge S^1$. This induces an equivilance of the specta C_i and $X \wedge S^1$. We then obtain the following diagram:

$$\begin{tikzcd}
C_f \arrow[r, "i_i"] \arrow[d, "="] & C_i \arrow[r, "p_i"] \arrow[d, "\cong"] & Y \wedge S^1 \arrow[d, "\mathrm{id} \wedge S^{-\mathrm{id}}"] \\
C_f \arrow[r, "p"'] & X \wedge S^1 \arrow[r, "f \wedge \mathrm{id}"'] & Y \wedge S^1
\end{tikzcd}$$

The left square commutes on the nose, and the right square commutes up to homotopy by 1.4.2. Applying π_i we obtain a section of the long exact sequence. We know the top row is exact, and therefore we know the bottom row is exact. This shows exactness at $X \wedge S^1$. Continuing in this way we can show exactness everywhere in the long exact sequence. □

Corollary 1.4.7.

i) For every family $(X^j)_{j \in J}$ and $i \in \mathbb{Z}$, the canonical map $\bigoplus_{j \in J} X^j \to \pi_i(\bigvee_{j \in J} X^j)$ is an isomorphism.

ii) For every finite family $(X_j)_{j \in J}$ the canonical map $\pi_i(\prod_{j \in J} X^j) \to \prod_{j \in J} X^j$ is an isomorphism.

iii) For every finite family $(X^j)_{j \in J}$, the canonical map $\bigvee_{j \in J} X_j \to \prod_{j \in J} X_j$ is a stable equivilance.

Remark 1.4.8. The final claim of the corollary can loosely be interpreted as saying that $\underline{\mathrm{Sp}}$ is additive up to stable equivilance.

Proof.

i) the inclusion $i_X : X \to X \vee Y$ has a retract, because it does level-wise. Then the long exact sequence associated to i_X splits into short exact sequences which split. So $\pi_i(X) \oplus \pi_i(Y) \to \pi_i(X \vee Y)$ is an isomorphism. We can then inductively show an isomorphism for finite wedges. Next consider the case $J = \mathbb{N}$.

$$\begin{aligned}
\bigoplus_{j \in \mathbb{N}} \pi_i(X^j) &\cong \varinjlim_N \big(\bigoplus_{j \leq N} \pi(X^j) \big) \\
&\cong \varinjlim_N (\pi_i(\bigvee_{j \in N} X^j)) \\
&\cong \varinjlim_N \varinjlim_{n>>0} (\pi_{i+n}(\bigvee_{j \leq N} X^j)) \\
&\cong \varinjlim_{n>>0} (\pi_{i+n}(\varinjlim (\bigvee_{j \leq N} X^j_n) \\
&\cong \varinjlim_{n>>0} (\pi_{i+n}(\bigvee_{j \in \mathbb{N}} X^j_n)) \\
&\cong \pi_i(\bigvee_{j \in J} X^j)
\end{aligned}$$

The fourth isomorphism must be argued for. It follows from the fact that colimits commute, and that π commutes with sequential colimits along closed inclusions.

ii) This is immediate when one unwraps the definitions. Important is to recall that finite products of abelian groups are direct sums, hence commute with colimit.

iii) Consider the following sequence of maps:

$$\begin{tikzcd} \bigoplus_{j\in J} \pi_i(X^j) \arrow[r, "\cong"] \arrow[rrr, bend right, "\cong"] & \pi_i(\bigvee X_j) \arrow[r] & \pi_i(\prod X^j) \arrow[r, "\cong"] & \prod \pi_i(X_j) \end{tikzcd}$$

All the maps but the middle are isomorphisms by the previous statements of this proposition, and the existence of a canonical isomorphism from finite products of abelian groups to direct sums. So the middle map is an isomorphism.

□

1.5 Categorical Constructions

Proposition 1.5.1. The category of orthogonal spectra admit all small limits and colimits, and they can be computed level-wise.

Proof. This will be proven in a homework sheet. □

Construction 1.5.2. We write $\underline{\mathrm{Sp}}(X, Y)$ for the set of maps of spectra $f : X \to Y$. We consider $\underline{\mathrm{Sp}}(X, Y)$ as a subset of $\prod \mathrm{map}_*(X_n, Y_n)$ and endow it with the subspace topology. We choose the level-wise constant map as the canonical base-point.

Lemma 1.5.3. The functor $X \wedge \cdot : \underline{\mathrm{Top}}_* \to \underline{\mathrm{Sp}}$ is left adjoint to the functor $\underline{\mathrm{Sp}}(X, \cdot) : \underline{\mathrm{Sp}} \to \underline{\mathrm{Top}}_*$.

Proof. This follows from the level-wise adjunction. □

Construction 1.5.4. We define the function spectrum, or internal hom of maps of spectra as the spectrum $\mathrm{Hom}(X, Y)$ given by the following data:

- $\mathrm{Hom}(X, Y)(V) = \underline{\mathrm{Sp}}(X, \mathrm{Sh}^V(Y))$,
- $\rho_{V,W} : L(V, W)_+ \wedge \underline{\mathrm{Sp}}(X, \mathrm{Sh}^V Y) \to \underline{\mathrm{Sp}}(X, \mathrm{Sh}^W Y)$ is given by $\varphi \wedge f \mapsto \rho_{V,W}(\varphi, f)$, where $\rho_{V,W}(\varphi, f)(U)$ is given by the following composition: $X(U) \xrightarrow{f(V)} Y(U \oplus V) \xrightarrow{Y(\mathrm{id}\oplus\varphi)} Y(U \oplus W)$,
- $\sigma_{V,W} : \underline{\mathrm{Sp}}(X, \mathrm{Sh}^V Y) \wedge S^W \to \underline{\mathrm{Sp}}(X, \mathrm{Sh}^{V\oplus W} Y)$ is given by $f \wedge w \mapsto \sigma_{V,W}(f, w)$, where $\sigma_{V,W}(f, w)(U)$ is given by the following composition: $X(U) \xrightarrow{f(U)} Y(U \oplus V) \xrightarrow{\sigma_{U\oplus V,W}(\cdot,w)} Y(U \oplus V \oplus W)$.

Lemma 1.5.5. Given $X, Y \in \underline{\mathrm{Sp}}$, $\mathrm{Hom}(X, \mathrm{Sh}^V Y) \cong \mathrm{Sh}^V(\mathrm{Hom}(X, Y))$.

Proposition 1.5.6. The evaluation functor $\mathrm{ev}_V : \underline{\mathrm{Sp}} \to \underline{\mathrm{Top}}_*$ given by $X \mapsto X(V)$ admits a left topological adjoint: $F_V : \underline{\mathrm{Top}}_* \to \underline{\mathrm{Sp}}$.

Remark 1.5.7. Recall that the morphism sets in both $\underline{\mathrm{Top}_*}$ and $\underline{\mathrm{Sp}}$ are in fact enriched as topological spaces. A topological adjunction is then a adjunction where the adjunction isomorphism is in fact continuous as a map of spaces.

Before we begin the rigorous proof, we discuss what form we should expect F_V to take.

We first note that the uniqueness of adjoints implies that it is enough to construct $F_V(S^0)$. If F_V were to exist, then $F_V K \cong (F_V S^0) \wedge K$. This is because of the following sequence of natural isomorphisms and the uniqueness of adjoints:

$$\begin{aligned}
\underline{\mathrm{Sp}}((F_V S^0) \wedge K, Y) &\cong \underline{\mathrm{Sp}}(F_V S^0, \mathrm{map}_*(K, Y)) \\
&\cong \mathrm{map}_*(S^0, \mathrm{map}_*(K, Y(V))) \\
&\cong \mathrm{map}_*(K, Y(V)) \\
&\cong \underline{\mathrm{Sp}}(F_V K, Y)
\end{aligned}$$

Now suppose F_V exists, then by Yoneda's lemma and the adjunction between ev and F_V:

$$\begin{aligned}
F_V(S^0)(W) &\cong \mathrm{map}_*(S^0, F_V(S^0)(W)) \\
&\cong \underline{\mathrm{Sp}}(F_W(S^0), F_V(S^0)) \\
&\cong \mathrm{Nat}(\underline{\mathrm{Sp}}(F_W(S^0), \cdot), \underline{\mathrm{Sp}}(F_V(S^0), \cdot)) \\
&\cong \mathrm{Nat}(\mathrm{ev}_W, \mathrm{ev}_V)
\end{aligned}$$

So we should think of $F_V(S^0)(W)$ as the "space" of all transformations $X(W) \to X(V)$ natural in $X \in \underline{\mathrm{Sp}}$.

Now for $\dim(V) = \dim(W)$, the definition of maps of spectra implies that $L(V, W)_+$ parametrizes a whole family of such maps by $\phi \mapsto X(\phi)$.

Similarly for $W = V \oplus V'$ such that $\dim(V) \leq \dim(W)$, $S^{V'}$ parametrizes such a family of natural transformations via $v' \mapsto \sigma(\cdot, v')$.

These are somehow also the only maps we should expect, because they are the only constraints on the maps $f : X \to Y$ in $\underline{\mathrm{Sp}}$.

Now we have to generalize this to all inner product spaces V, W, including those which do nicely decompose as above.

To this end we consider the following construction:

Construction 1.5.8. Let V, W be inner product spaces. We define the orthogonal complement bundle $\xi(V, W) \to L(V, W)$ as the vector bundle with total space $\xi(V, W) = \{(\varphi, w) \in L(V, W) \times W \mid w \in \mathrm{im}(\varphi)^\perp\}$. The map $\xi(V, W) \to L(V, W)$ is then the projection to the first component. The fiber of a point φ is then clearly $\mathrm{im}(\varphi)^\perp$, the orthogonal complement of $\mathrm{im}(\varphi)$.

We then define $\mathcal{J}(V, W) = \mathrm{Thom}(\xi(V, W))$, the Thom space of $\xi(V, W)$. We do not include the definition of a Thom space here, because in our case it simply coincides with the one-point compactification of $\xi(V,W)$.

Example 1.5.9. Let V, W be inner product spaces.

- If $V = 0$, then $\xi(V,W) = W$ and so $\mathcal{J}(V,W) = S^W$.
- If $\dim(V) = \dim(W)$, every fibre is zero and so $\xi(V,W) \cong L(V,W)$. In particular $\mathcal{J}(V,W) \cong L(V,W)_+$, agreeing with the intuitive picture above.
- If $\dim(V) > \dim(W)$, then $\mathcal{J}(V,W) = \{*\}$.

Lemma 1.5.10. Let V, W be inner product spaces such that $\dim(W) \geq \dim(V)$, every choice of $\varphi \in L(V,W)$ gives rise to homeomorphisms:

$$\begin{aligned} O(W)/O(\mathrm{im}(\varphi)^\perp) &\to L(V,W) \\ [\psi] &\mapsto \psi \circ \varphi \\ O(W)_+ \wedge_{O(\mathrm{im}(\varphi)^\perp)} S^{\mathrm{im}}(\varphi)^\perp &\to \mathcal{J}(V,W) \\ [\psi, w] &\mapsto (\psi \circ \varphi, \psi(w)) \end{aligned}$$

Proof. Note that because all spaces involved are compact Hausdorff, it is enough to ensure that the defined maps are continuous bijections. We leave this as an exercise for the reader. □

Remark 1.5.11. The functor F_V is called the free spectrum functor, and we call $F_V K$ the spectrum freely generated by $K \in \underline{\mathrm{Top}}_*$ in level V.

Construction 1.5.12. We define the spectrum $F_V(S^0)$ as follows:

- $F_V(S^0)(W) = \mathcal{J}(V,W)$.
- $\rho_{W,W'} : L(V,W)_+ \wedge \mathcal{J}(V,W) \to \mathcal{J}(V,W')$ is defined by $\psi \wedge (\varphi, w) \mapsto (\psi \circ \varphi, \varphi(w))$.
- $\sigma_{W,W'} : \mathcal{J}(V,W) \wedge S^{W'} \to \mathcal{J}(V, W \oplus W')$ is defined by $(\varphi, w) \wedge w' \mapsto (\mathrm{incl}_1 \circ \varphi, (w, w'))$.

We are now in a position to prove 1.5.6.

Proof. As we have seen before, it is enough to show $F_V(S^0)$ satisfies the universal property of the adjunction. In other words we have to show that $\underline{\mathrm{Sp}}(F_V(S^0), Y)$ is homeomorphic tomap$_*(S^0, Y(V) \cong Y(V)$ via the map $f \mapsto f(V)(\mathrm{id}_V, 0)$.

To do this we construct an inverse $Y(V) \to \underline{\mathrm{Sp}}(F_V(S^0), Y)$ given by $y \mapsto \hat{y}$. $\hat{y}$ is the map of spectra given level-wise by $\hat{y}(W) : \mathcal{J}(V,W) \to Y(W)$ such that $\hat{y}(\varphi, w) = Y(\tilde{\varphi} \oplus \mathrm{im}(\varphi)^\perp)(\sigma_{V,\mathrm{im}(\varphi)^\perp}(y, w)$, where $\tilde{\varphi}$ is φ with its codomain restricted to its image.

That this an inverse for the initial map is a computation left to the reader. To show that this is continuous we first note that by the definition of the topology on $\underline{\mathrm{Sp}}(F_V(S^0), Y)$, it is enough to check that the induced maps $Y(V) \to \mathrm{map}_*((F_V(S^0))_n, Y_n)$. By applying adjunction twice it is enough to show that $(\varphi, w) \mapsto (y \mapsto (\hat{y})_N(\varphi, w)$ is continuous. Precomposing by a homeomorphism from lemma 1.5.10 we can rewrite this in a form where continuity becomes obvious. We leave this to the reader. $\square$

Example 1.5.13. For $V = 0$, $F_0(S^0) = \mathbb{S}$. So F_0 agrees with $\sum^\infty$. We therefore call ev_0 the infinite loop functor.

Example 1.5.14. We call $S^{-V} = F_V(S^0)$ the $(-V)$-th sphere. This is not an arbitrary assignment. Once we have constructed smash products of spectra, we will see that there exists a stable equivilance between $S^{-V} \wedge S^V$ and $\mathbb{S}$.

Remark 1.5.15. We briefly remark that the evaluation functor ev_V in fact also admits a left adjoint, G_V, which is given by $G_V K(W) = \mathcal{J}(V, W) \wedge_{O(V)} K$. We call this the semi-free spectrum of K in level V.

1.6 Alternative Definition

There are various different definitions of orthogonal spectra, all of which give rise to equivalent categories. The first example of this is the category of coordinatized orthogonal spectrum. This was considered in the first exercise sheet, and therefore we do not treat it here.

More interesting is the following definition:

Definition 1.6.1. Let $\mathcal{J}$ be the category whose objects are inner product spaces, and whose spaces of morphisms are the Thom spaces $\mathcal{J}(V, W)$. We define composition by $(\psi, w) \circ (\varphi, v) \mapsto (\psi \circ \varphi, \psi(v) + w)$. We then define the category of $\mathcal{J}$-spaces, or alternatively of Mandell-May spectra, as the category whose objects are functors $\bar{X} : J \to \underline{\mathrm{Top}}_*$ and whose morphisms are natural transformations of J-spaces.

Construction 1.6.2. We define a functor $\mathcal{F}$ from $\mathcal{J}$-spaces to spectra. Given a $\mathcal{J}$-space $\bar{X}$ we define the spectrum $\mathcal{F}(\bar{X})$ by:

- $\mathcal{F}(\bar{X})(V) = \bar{X}(V)$,
- $\rho_{V,W}(\varphi, \cdot) = \bar{X}((\varphi, 0))$,
- $\sigma_{V,W}(\cdot, w) = \bar{X}(((\mathrm{incl}_1 : V \to V \oplus W), w))$.

We also define a functor $\mathcal{G}$ from spectra to $\mathcal{J}$-spaces. Given a spectrum X, we define $\mathcal{G}(X)$ as follows:

- $\mathcal{G}(X)(V) = X(V)$.
- The map $\mathcal{J}(V, W) \to \mathrm{map}_*(X(V), X(W))$ induced by the functor is $\mathcal{G}(X)$ is given by the adjoint of the map $\mathcal{J}(V, W) \wedge X(V) \to X(W)$, where $(\varphi, w) \wedge x \mapsto \rho_{V \oplus \mathrm{im}(\varphi)^\perp, W}(\varphi \oplus \mathrm{id}_{\mathrm{im}\varphi^\perp}, \sigma_{V, \mathrm{im}\varphi^\perp}(x, w))$.

Theorem 1.6.3. $\mathcal{F}$ and $\mathcal{G}$ are inverse isomorphisms of categories.

1.7 Smash Product

In the following section we will define a smash product on the category of spectra, which will turn it into a symmetric monoidal category with strict unit.

Definition 1.7.1. Let $X, Y, Z \in \underline{\mathrm{Sp}}$. A bimorphism $b : (X, Y) \to Z$ is a collection of based maps $b_{V,W} : X(V) \wedge Y(W) \to Z(V \oplus W)$ such that the following two diagrams commute:

$$\begin{tikzcd} L(V,V')_+ \wedge X(V) \wedge L(W,W')_+ \wedge Y(W) \arrow[r, "\oplus\wedge b"] \arrow[d, "\rho\wedge\rho"] & L(V\oplus W, V'\oplus W') \wedge Z(V \oplus W) \arrow[d, "\rho"] \\ X(V') \wedge X(W') \arrow[r, "b"] & Z(V' \oplus W') \end{tikzcd}$$

$$\begin{tikzcd} X(V) \wedge S^{V'} \wedge Y(W) \wedge S^{W'} \arrow[r, "b\wedge\varphi_{\mathrm{can}}"] \arrow[d, "\sigma\wedge\sigma"] & Z(V \oplus W) \wedge S^{V'\oplus W'} \arrow[d, "\sigma"] \\ X(V \oplus V') \wedge X(W \oplus W') \arrow[r, "b"] & Z(V \oplus W \oplus V' \oplus W') \end{tikzcd}$$

We will write $\mathrm{Bimor}((X,Y),Z)$ for the set of all bimorphisms.

Remark 1.7.2. First note that b is a bimorphism if and only if the maps $b_{V,W}$ define maps of spectra $X \wedge Y(W) \to \mathrm{sh}^W(Z)$ and $Y \wedge X(V) \to \mathrm{sh}^V(Z)$ for fixed W, V respectively.

Definition 1.7.3. Let $X, Y \in \underline{\mathrm{Sp}}$. Then a smash product of X and Y is a pair $(X \wedge Y, \imath_{X,Y})$, where $X \wedge Y \in \underline{\mathrm{Sp}}$ and $i_{X,Y}$ is a bimorphism such that for all other spectra Z $i^*_{X,Y} : \underline{\mathrm{Sp}}((X \wedge Y), Z) \to \mathrm{Bimor}((X,Y),Z)$ given by $f \mapsto f \circ i_{X,Y}$.

Construction 1.7.4. We construct $X \wedge Y$ as coordinatized spectrum. This is simply a matter of convenience, and allows us to avoid the issue of taking large colimits. We have seen before how colimits over all inner product spaces are determined by the $\{\mathbb{R}^n\}_{n\in N}$, and so we should expect the result of our colimit to not depend on this choice.

In particular we define $X \wedge Y$ level-wise by setting $(X \wedge Y)_n$ to equal the coequalizer of the following pair of maps:

$$\bigvee_{i+j+k=n} O(n)_+ \wedge_{O(i)\times O(j)\times O(k)} X_i \wedge S^j \wedge Y_k \underset{2}{\overset{1}{\rightrightarrows}} \bigvee_{p+q} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q$$

where the first map sends the (i,j,k)-th summand to the (i,j+k)-th summand via the map $\mathrm{id}_{O(n)} \wedge \sigma^X_{i,j} \wedge \mathrm{id}_Y$ and the second map sends the (i,j,k)-th summand to the (i,j+k)-th summand via the map $Y(\varphi_{\mathrm{swap}}) \circ (\mathrm{id}_{O(n)} \wedge \mathrm{id}_X \wedge \sigma^Y_{k,j}) \circ (\mathrm{swap})$.

Note that the definition of the coequalizer gives the following relation in $(X \wedge Y)_n$:

$$[\varphi, \sigma^X(x,u), y] = [\varphi, x, Y(\varphi_{\mathrm{swap}}) \circ \sigma^Y(y,u)]$$

We turn $(X \wedge Y)_n$ into an $O(n)$ space via postcomposing on the direct summand of $O(n)_+$ in each summand. Now there are two ways to define the suspension maps, namely by either applying σ^X

or σ^Y and then taking the induced map on the equalizer. However the relation above, and the construction of the balanced product guarantees that they agree.

Finally we define the biholomorphism $i_{X,Y}$ as the collection of maps given by:

$$\begin{aligned} X_p \wedge X_q &\to O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \\ x \wedge y &\mapsto [\mathrm{id}, x, y] \end{aligned}$$

, after postcomposing by the inclusion into the wedge product.

Theorem 1.7.5. $(X \wedge Y), i_{X,Y})$ is a smash product.

Proof. It is enough to check the universal property for coordinatized spectra and bimorphisms of coordinate spectra.

We sketch the bijection. First note that bimorphisms from $(X, Y) \to Z$ are by definition in bijection to collections of based $O(p) \times O(q)$ equivariant maps $X_p \wedge Y_q \to Z_{p+q}$ compatible with suspension in each variable. This is, by the induction adjunction in bijection with the collection of based $O(n)$ maps $O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge X_q \to Z_{p+q}$ compatible with suspensions in either variable. Now one can confirm that precomposing with $i_{X,Y}$ gives an bijection of this set to maps of coordinate spectra $X \wedge Y \to Z$. The specific bijection is in face given by the universal property of the coequalizer, where commuting with suspensions is equivalent to coequalizing the relevant diagram. $\square$

Lemma 1.7.6. The assignment $\mathrm{Bimor}((X,Y),Z) \to \underline{\mathrm{Sp}}(X, \mathrm{Hom}(X,Y))$ given by currying is a well-defined bijection natural in X, Y and Z.

Corollary 1.7.7. $\cdot \wedge Y : \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$ is a functor and is left adjoint to $\mathrm{Hom}(Y, \cdot) : \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$. Similarly $\cdot \wedge \cdot : \underline{\mathrm{Sp}} \times \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$ is functorial in both variables.

Example 1.7.8. Recall that $F_0 \cong \sum^\infty$ as functors. Therefore:

$$\underline{\mathrm{Sp}}(\mathbb{S} \wedge X, Z) \cong \underline{\mathrm{Sp}}(F_0(S^0), \mathrm{Hom}(X,Z)) \cong \mathrm{Hom}(X,Z)(0) \cong \underline{\mathrm{Sp}}(X,Z)$$

Then uniqueness of adjoints implies that there exists a canonical isomorphism $\mathbb{S} \wedge X \cong X$. Similarly there is a canonical isomorphism $X \wedge \mathbb{S}$.

Remark 1.7.9. Because of the freedom in choosing a smash of two spectra, given by the fact that taking coequalizers is only well-defined up to canonical isomorphism, we can in fact set $\mathbb{S} \wedge X = X \wedge \mathbb{S} = X$ with $i_{X,\mathbb{S}} : (X, \mathbb{S}) \to X$ given level-wise by $\sigma^X_{V,W}$. This is the convention we will take from now on.

Construction 1.7.10. Using the uniqueness of adjoints we can construct a unique isomorphism $\alpha_{X,Y,Z} : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$, called the associativity isomorphism. Similarly because there is a clear bijection $\mathrm{Bimor}((X,Y), \cdot) \cong \mathrm{Bimor}((Y,X), \cdot$, the uniqueness of adjoints gives a unique isomorphism $\tau : X \wedge Y \to Y \wedge X$, called the symmetry isomorphism. Both of these maps also commute with the bimorphism involved.

Theorem 1.7.11. The functor $\cdot \wedge \cdot : \underline{\mathrm{Sp}} \times \underline{\mathrm{Sp}} \to \underline{\mathrm{Sp}}$ and the maps $\alpha_{X,Y,Z}$ and $\tau_{X,Y}$ for all $X, Y, Z \in \underline{\mathrm{Sp}}$ endows $\underline{\mathrm{Sp}}$ with the structure of a symmetric monoidal category with a strict unit given by $\mathbb{S}$.

Proposition 1.7.12. Let $K, K' \in \underline{\mathrm{Top}}_*$ and let $X \in \underline{\mathrm{Sp}}$.

i) $X \wedge K$ equals $X \wedge \sum^\infty K$.

ii) $\mathrm{map}_*(K, \cdot)$ and $\mathrm{Hom}_*(\sum^\infty K, \cdot)$ agree up to a natural isomorphism.

iii) $F_{V \oplus V'}(KK') = F_V(K) F_{V'} K'$

iv) $\sum^\infty$ sends smash products of based spaces to smash products of spectra.

Proof.

i) We claim that the bimorphism $K : (X, \sum^\infty K) \to X \wedge K$ given level-wise by $\sigma^X \wedge \mathrm{id}_K$ is a universal biproduct. **TODO!:**

□

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