BGN: 2191A
P4
Q8

a. Case 0: int (base case)

RTP: int $\leq$ int

$$\frac{}{\text{int} \leq \text{int}} \leq \text{int}$$

$\blacksquare$

Case 1: $A \to B$

RTP: $(A \to B) \leq (A \to B)$
Assume $A \leq A \wedge B \leq B$ (induction hypothesis)

$$\frac{A \leq A \qquad B \leq B}{(A \to B) \leq (A \to B)} \leq \to$$

$\blacksquare$

Case 2: $\{l_1 : A_1, \ldots l_n : A_n\}$

RTP: $\{l_1 : A_1, \ldots l_n : A_n\} \leq \{l_1 : A_1, \ldots, l_n : A_n\}$
Assume $\forall i \in [1, n].\ A_i \leq A_i$ (induction hypothesis)

$$\frac{n \leq n \qquad A_1 \leq A_1 \quad \cdots \quad A_n \leq A_n}{\{l_1 : A_1, \ldots, l_n : A_n\} \leq \{l_1 : A_1, \ldots, l_n : A_n\}} \leq \text{rec}$$

$\blacksquare$

QED

6. Let A', B', C'

Assume $A' \leq B' \wedge B' \leq C'$

RTP: $A' \leq C'$

Case 0: $C' = \text{int}$ (base case)

The only rule which can derive a relation of the form $\alpha \leq \text{int}$ is $(\leq \text{int})$. $B' \leq C' \Rightarrow B' = \text{int}$, $\therefore B' = \text{int}$

Likewise $\therefore$ $A' \leq B' \Rightarrow A' = \text{int}$, $\therefore A' = \text{int}$

$\text{int} \leq \text{int}$ by $(\leq \text{int})$, $\therefore A' \leq C'$ ■

Case 1: $C' = A_3 \to B_3$

The only rule which can derive a relation of the form $\alpha \leq A_3 \to B_3$ is $(\geq \to)$ $\therefore$ $B' \leq C' \Rightarrow B' = A_2 \to B_2$ where

$A_3 \leq A_2 \wedge B_2 \leq B_3$

Likewise since $A' \leq A_2 \to B_2$, A' must be of the form $A_1 \to B_1$ where $A_2 \leq A_1 \wedge B_1 \leq B_2$, by $(\geq \to)$

Assume $A_3 \leq A_2 \wedge A_2 \leq A_1 \Rightarrow A_3 \leq A_1$

$B_1 \leq B_2 \wedge B_2 \leq B_3 \Rightarrow B_1 \leq B_3$ (induction hypothesis)

$\therefore A_3 \leq A_1 \wedge B_1 \leq B_3$

$\therefore A_1 \to B_1 \leq A_3 \to B_3$ $(\leq \to)$

$\therefore A' \leq C'$ ■

Case 2: $C' = \{l_1 : C_1, \ldots, l_m : C_m\}$

The only rule which can derive a relation of the form $\alpha \leq \{l_1 : C_1, \ldots, l_m : C_m\}$ is ($\leq$rec)

$\therefore B' \leq C' \Rightarrow B' = \{l_1 : B_1, \ldots, l_n : B_n\}$

where $n \leq m$, $B_1 \leq C_1, \ldots, B_n \leq C_n$

Likewise, $A' \leq B' \Rightarrow A' = \{l_1 : A_1, \ldots, l_k : A_k\}$

where $k \leq n$, $A_1 \leq B_1, \ldots, A_k \leq B_k$

$\therefore k \leq m$

Assume $\forall i \in [1, k] .\ A_i \leq B_i \wedge B_i \leq C_i \Rightarrow A_i \leq C_i$ (induction hypothesis)

$$\frac{k \leq m \qquad A_1 \leq C_1 \ \ldots \ A_k \leq C_k}{\{l_1 : A_1, \ldots l_k : A_k\} \leq \{l_1 : C_1, \ldots l_m : C_m\}} \ \leq\text{rec}$$

$\therefore A' \leq C'$ ∎

QED