BGN 2031B

Paper 1

Question 12R

a. $dV = r^2 \sin\varphi \cos\theta \, dr \, d\varphi \, d\theta$

$$M = \cancel{\int_0^R \int_0^{2\pi} \int_0^{\pi} Ar^{-2}\left(1-e^{-r/r_0}\right) r^2 \, d\varphi \, d\theta \, dr}$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} Ar^{-2}\left(1-e^{-r/r_0}\right) r^2 \, dr \, d\theta \, d\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} A\left(1-e^{-r/r_0}\right) dr \, d\theta \, d\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} \left[A\left(r+r_0 e^{-\frac{r}{r_0}}\right)\right]_0^R d\theta \, d\varphi$$

$$= \int_0^{\pi} \int_0^{2\pi} A\left(R+r_0 e^{-\frac{R}{r_0}} - r_0\right) d\theta \, d\varphi$$

$$= \int_0^{\pi} 2\pi A\left(R+r_0\left(e^{-\frac{R}{r_0}}-1\right)\right) d\varphi$$

$$= 2\pi^2 A\left(R+r_0\left(e^{-\frac{R}{r_0}}-1\right)\right)$$

$$= 2\pi^2 AR\left(1+\frac{r_0}{R}\left(e^{-\frac{R}{r_0}}-1\right)\right)$$

$$= 4\pi AR\, f\left(\frac{r_0}{R}\right) \text{ where } f\left(\frac{r_0}{R}\right) = \frac{\pi}{2}\left(1+\frac{r_0}{R}\left(e^{-\left(\frac{r_0}{R}\right)^{-1}}-1\right)\right)$$

ii. $\lim_{r_0/R \to \infty} f\left(\frac{r_0}{R}\right) = \lim_{x \to \infty} \frac{\pi}{2}\left(1 + x\left(e^{-1/x} - 1\right)\right)$

$\lim_{x \to \infty} \frac{e^{-1/x} - 1}{(1/x)} = \lim_{x \to \infty} \frac{-\ln x \, e^{-1/x}}{\ln x}$ by L'höpital's rule

$= \lim_{x \to \infty} -e^{-1/x} = 1$

$\therefore \lim_{x \to \infty} \frac{\pi}{2}\left(1 + x\left(e^{-1/x} - 1\right)\right) = \frac{\pi}{2}(1 + 1) = \pi$

$\lim_{r_0/R \to 0} f\left(\frac{r_0}{R}\right) = \lim_{x \to 0^+} \frac{\pi}{2}\left(1 + x\left(e^{-1/x} - 1\right)\right)$

$\lim_{x \to 0^+} e^{-1/x} - 1 = -1$

$\therefore \lim_{x \to 0^+} x\left(e^{-1/x} - 1\right) = 0$

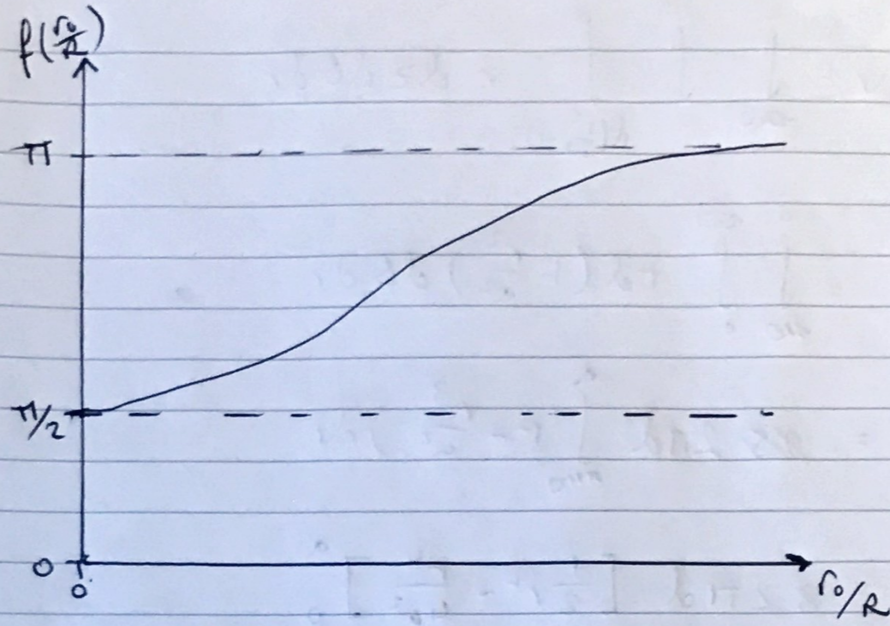
$\therefore \lim_{x \to 0^+} \frac{\pi}{2}\left(1 + x\left(e^{-1/x} - 1\right)\right)$

$= \frac{\pi}{2}$

$\therefore \lim_{r_0/R \to \infty} f\left(\frac{r_0}{R}\right) = \pi$

$\lim_{r_0/R \to 0} f\left(\frac{r_0}{R}\right) = \pi/2$

iii. $\cancel{f(0)}$



b. $dV = r\,dr\,d\theta\,dz$

$\frac{x^2+y^2}{a^2} = \frac{r^2}{a^2}$ $\therefore$ $\cancel{\frac{x^2}{a^2}+\frac{y^2}{a^2} < \frac{z}{d}+1 \Rightarrow r^2 < a^2\left(\frac{z}{d}+1\right)}$

$\cancel{\therefore -a\sqrt{\frac{z}{d}+1} < r < a\sqrt{\frac{z}{d}+1}}$

$\therefore \cancel{V = \int_{-\infty}^{0}\int_{0}^{2\pi}\int_{-a\sqrt{\frac{z}{d}+1}}^{a\sqrt{\frac{z}{d}+1}} r\,dr\,d\theta\,dz}$

$\cancel{= \int_{-\infty}^{0}\int_{0}^{2\pi}\left[\frac{r^2}{2}\right]_{-a\sqrt{\frac{z}{d}+1}}^{a\sqrt{\frac{z}{d}+1}} d\theta\,dz}$

$\therefore \frac{r^2}{a^2} < \frac{z}{d}+1$

$\therefore d\left(\frac{r^2}{a^2}-1\right) < z < 0$

$\therefore \frac{r^2}{a^2} < 1$

$\therefore r^2 < a^2$

$\therefore 0 < r < a$

$$\therefore V = \int_0^a \int_0^{2\pi} \int_{d\left(\frac{r^2}{a^2}-1\right)}^0 r \, dz \, d\theta \, dr$$

$$= \int_0^a \int_0^{2\pi} r d\left(1-\frac{r^2}{a^2}\right) d\theta \, dr$$

$$= 2\pi d \int_0^a \left(r - \frac{r^3}{a^2}\right) dr$$

$$= 2\pi d \left[\frac{1}{2}r^2 - \frac{r^4}{4a^2}\right]_0^a$$

$$= 2\pi d \left(\frac{a^2}{2} - \frac{a^2}{4}\right)$$

$$= \frac{1}{2}\pi a^2 d$$