4.

a. $Pr(x_1, x_2, \dots x_n; \theta) = \prod_{i=1}^{n} P_r(x_i; \theta)$ assuming independence

$$= \prod_{i=1}^{n} (\theta 1_{x_i=1} + (1-\theta) 1_{x_i \neq 1})$$

note that $x_i = 1_{x_i=1}$, $1 - x_i = 1_{x_i \neq 1}$

$$= \prod_{i=1}^{n} (x_i\theta + (1-\theta)(1-x_i))$$

$$\cancel{= \prod_{i=1}^{n} (x_i\theta + 1 - x_i - \theta + \theta x_i)}$$

$$\cancel{= \prod_{i=1}^{n} (2x_i\theta - x_i - \theta + 1)}$$

$$= \theta^{\alpha} (1-\theta)^{n-\alpha}$$

where α = number of x_i such that $x_i = 1$

$$= \sum_{i=1}^{n} 1_{x_i=1}$$

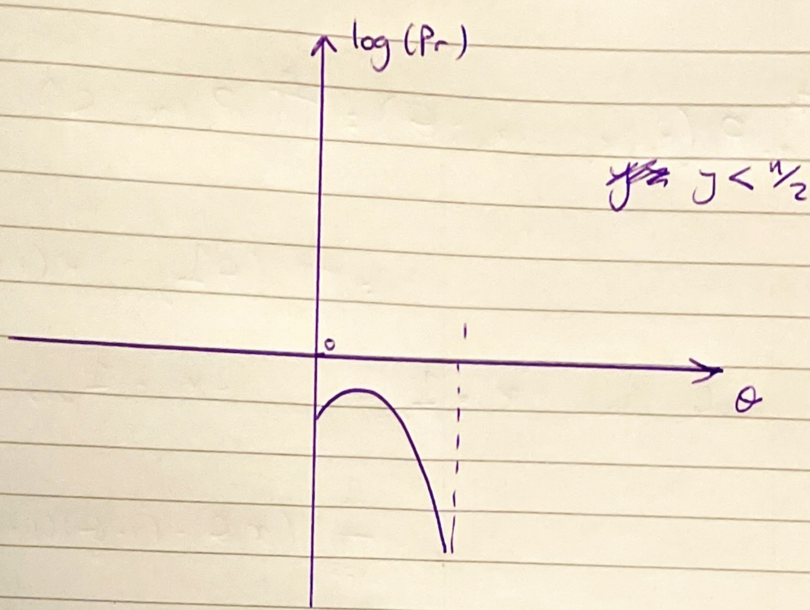
$$= \sum_{i=1}^{n} x_i$$

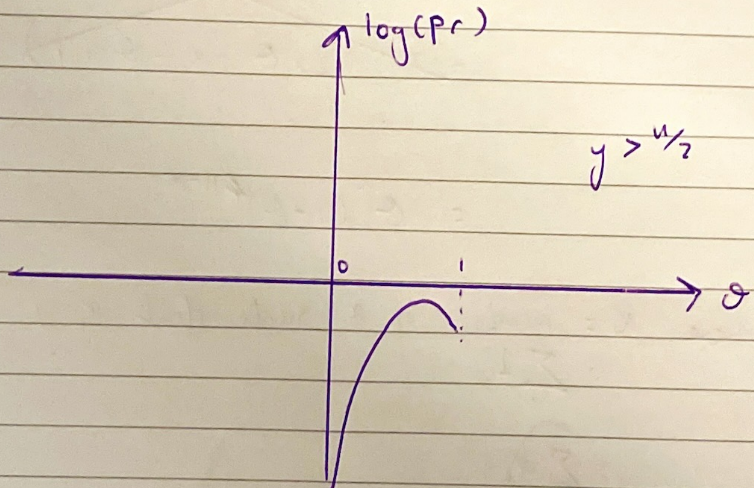
$$= y$$

$\therefore Pr(x_1, \dots, x_n; \theta) = \theta^{y} (1-\theta)^{n-y}$

b. $\log(Pr \dots) = \log(\theta^{y} (1-\theta)^{n-y})$

$$= y \log\theta + (n-y)\log(1-\theta)$$





c. $\frac{d}{d\theta} \log(\Pr \ldots) = y/\theta + \frac{y-n}{1-\theta} \doteq 0$

$\therefore (1-\theta)y + \theta(y-n) = 0$

$\therefore y - \theta y + \theta y - \theta n = 0$

$\therefore \theta = y/n$

$\hat{\theta} = \max(1/2,\ y/n)$

5. Let X be the length of an unbroken chain of failures

$X \sim Geo(1-\theta)$

$Pr(X=k) = (1-\theta)^k \theta$

$$\therefore Pr(X \geq k) = \sum_{k=n}^{\infty} Pr(X=k)$$

$$= \sum_{k=n}^{\infty} (1-\theta)^k \theta$$

$$= 1 - \sum_{k=0}^{n-1} (1-\theta)^k \theta$$

$$= 1 - \theta^n \frac{1-(1-\theta)^n}{1-(1-\theta)}$$

$$= 1 - \theta^{n-1} (1-(1-\theta)^n)$$

Looking for some n

such that $Pr(X \geq n) \leq 0.05$

$\therefore 1 - \theta^{n-1}(1-(1-\theta)^n) \leq 0.05$

$\therefore \theta^{n-1}(1-(1-\theta)^n) \geq 0.95$

$\therefore$ ~~[illegible]~~ not sure how to solve

1. $\mathbb{P}(X_3 = r \mid X_0 = g) = \sum_{x_2} \mathbb{P}(X_3 = r \mid X_2 = x_2)\, \mathbb{P}(X_2 = x_2 \mid X_0 = g)$

by the law of total probability

$$= \sum_{x_2} P_{x_2 r}\, \mathbb{P}(X_2 = x_2 \mid X_0 = g)$$

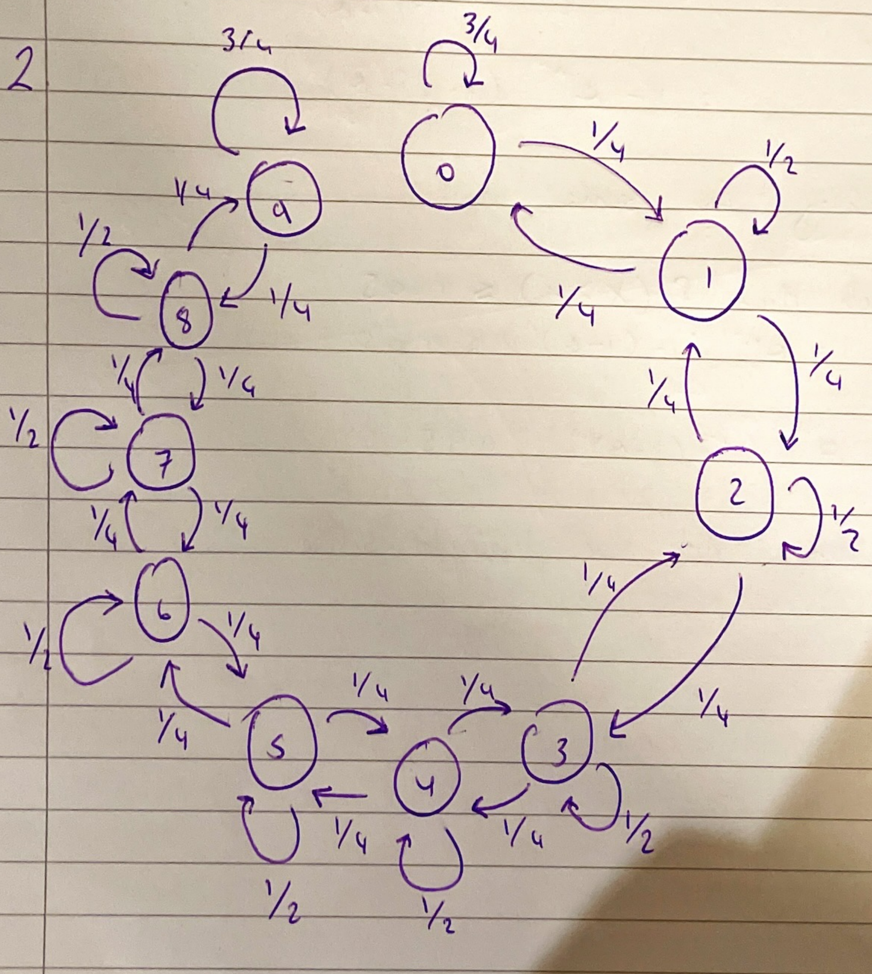
$$= \sum_{x_2} P_{x_2 r} \sum_{x_1} \mathbb{P}(X_2 = x_2 \mid X_1 = x_1)\, \mathbb{P}(X_1 = x_1 \mid X_0 = g)$$

by the law of total probability

$$= \sum_{x_2} P_{x_2 r} \sum_{x_1} P_{x_1 x_2}\, P_{g x_1}$$

$$= \sum_{x_1, x_2} P_{g x_1}\, P_{x_1 x_2}\, P_{x_2 r}$$

2.



3. Let π be a stationary distribution of the Markov chain.

$$\therefore \mathbb{P}(X_0 = a) = \pi_a$$
$$\mathbb{P}(X_0 = b) = \pi_b = 1 - \pi_a$$

$$\mathbb{P}(X_1 = a) = \sum_{x_0} \mathbb{P}(X_1 = a | X_0 = x_0)\, \mathbb{P}(X_0 = x_0)$$

$$= \mathbb{P}(X_1 = a | X_0 = a)\pi_a + \mathbb{P}(X_1 = a | X_0 = b)\pi_b$$

$$= (1-\alpha)\pi_a + \beta(1-\pi_a)$$

$$= \pi_a - \alpha\pi_a + \beta - \beta\pi_a$$

$$\overset{!}{=} \pi_a$$

$$\therefore \beta - \pi_a(\alpha+\beta) = 0$$

$$\therefore \pi_a = \frac{\beta}{\alpha+\beta}, \quad \pi_b = \frac{\alpha}{\alpha+\beta}$$

This is a unique solution.

5a.

7. ~~[illegible]~~

Let Π = a stationary distribution for X such that

$Pr_{X_0}(k) = \Pi_k$

$Pr_{X_1}(k) = \mu + \lambda(Pr_{X_0}(k) - \mu)$ by the shifting properties of a normal distribution

$= \mu + \lambda(\Pi_k - \mu)$

$\stackrel{!}{=} \Pi_k$

$\therefore \mu + \lambda\Pi_k - \Pi_k - \lambda\mu = 0$

$\therefore \Pi_k(\lambda - 1) = \mu(\lambda - 1)$

$\therefore \Pi_k = \mu$

$\therefore \Pi \sim N(\mu, 0)$?? doesn't seem right