BGN: 2031B
Paper 2
Section A

1. $$\begin{vmatrix} \cosh u - \lambda & \sinh u \\ \sinh u & \cosh u - \lambda \end{vmatrix} = 0$$

$\therefore (\cosh u - \lambda)^2 - \sinh^2 u = 0$

$\therefore \cosh u - \lambda = \pm \sinh u$

$\therefore \lambda = \cosh u \pm \sinh u$

$= \dfrac{(e^u + e^{-u}) \pm (e^u - e^{-u})}{2}$

$= e^u$ or e^{-u}

$\therefore \lambda_1 = e^u \qquad \lambda_2 = e^{-u}$

For $\vec{x}_1$, $\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix} = e^u \begin{pmatrix} 1 \\ y \end{pmatrix}$

$\therefore \cosh u + y \sinh u = e^u$

$\therefore y = \frac{1}{\sinh u}(e^u - \cosh u) = \dfrac{2e^u - e^u - e^{-u}}{e^u - e^{-u}}$

$= \dfrac{e^u - e^{-u}}{e^u - e^{-u}} = 1$

To check: $\sinh u + \cosh u = e^u = e^u \cdot 1$

$\therefore \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\vec{x}_2$

$$\begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = e^{-u}\begin{pmatrix} 1 \\ y \end{pmatrix}$$

$\therefore \cosh u + y \sinh u = e^{-u}$

$$\therefore y = \frac{2e^{-u} - e^{u} - e^{-u}}{e^{u} - e^{-u}} = \frac{e^{-u} - e^{u}}{e^{u} - e^{-u}} = -1$$

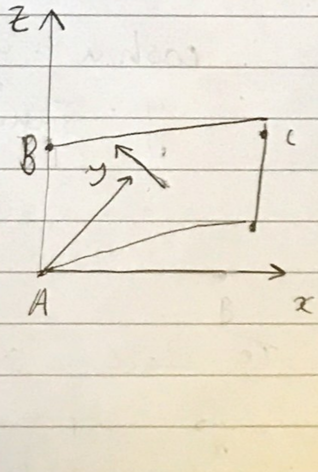
To check: $\sinh u - \cosh u = -e^{-u} = e^{-u} \cdot -1$

$$\therefore \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2. $\vec{a} = \vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$$= \hat{i}(-1) - \hat{j}(-1) + \hat{k}(0)$$

$$= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$



3. $\frac{dy}{dx} + y = e^{2x}$

$IF = e^{\int dx} = e^{x}$

$\therefore e^{x}\frac{dy}{dx} + e^{x}y = e^{3x}$

$\therefore \frac{d}{dx}(e^{x}y) = e^{3x}$

$\therefore e^{x}y = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

$\therefore y = \frac{1}{3}e^{2x} + Ce^{-x}$

4. For $n = 0$

$I = \int_{-\infty}^{\infty} x^{0}e^{-ax^2}\,dx = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$

Let $u = \sqrt{a}\,x$ $\qquad \lim_{x\to-\infty} u = -\infty$

$du = \sqrt{a}\,dx$ $\qquad \lim_{x\to\infty} u = \infty$

$\therefore I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{a}}e^{-u^2}\,du = \frac{1}{\sqrt{a}}\cdot\sqrt{\pi} = \sqrt{\frac{\pi}{a}}$

For $n = 1$

$J = \int_{-\infty}^{\infty} xe^{-ax^2}\,dx$

~~Let~~ $\cancel{u = e^{-ax^2}}$

$\cancel{du = -2axe^{-ax^2}}$

$\cancel{\lim_{x\to-\infty} u = \lim_{x\to-\infty} \frac{-2ax}{e^{ax^2}} = \lim_{x\to-\infty} \frac{-2a}{2axe^{ax^2}}}$ ~~by L'Hôpital's rule~~

$= 0$

The integrand is odd and the interval is symmetric

$\therefore \int = 0$

More precisely:

$$\forall k \in \mathbb{R}^+, \quad \int_{-k}^{k} x e^{-ax^2} \, dx = 0$$

$$\therefore \lim_{k \to \infty} \int_{-k}^{k} x e^{-ax^2} \, dx = 0$$

5. $H = pv - L(x, v)$

$$= pv - \tfrac{1}{2}mv^2 + V(x)$$

$$\left.\frac{\partial H}{\partial v}\right|_{p,x} = p - mv \stackrel{!}{=} 0$$

$$\therefore \quad v = p/m$$

$$\frac{\partial^2 H}{\partial v^2} = -m < 0 \quad \therefore \quad v = p/m \text{ is a local maximum}$$

$$\left. H \right|_{v = p/m} = \frac{p^2}{m} - \frac{1}{2}\frac{p^2}{m} + V(x)$$

$$= \frac{p^2}{2m} + V(x)$$

6. $F \times G = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ -y & x & 0 \end{vmatrix}$

$= \hat{i}(-xz) - \hat{j}(yz) + \hat{k}(x^2+y^2)$

$= \begin{pmatrix} -xz \\ -yz \\ x^2+y^2 \end{pmatrix}$

$\nabla \cdot (F \times G) = \frac{\partial}{\partial x}(-xz) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}(x^2+y^2)$

$= -z - z + 0$

$= -2z$

7. $\sum_{n=0}^{\infty} P(X=n) \stackrel{!}{=} 1$

$\therefore \sum_{n=0}^{\infty} ce^{-n} = c\sum_{n=0}^{\infty} (1/e)^n$ (which is a geometric series)

$= \frac{c}{1-1/e} = 1$

$\therefore c = 1 - 1/e$

$P(X \geq N) = 1 - P(X < N)$

$= 1 - \sum_{n=0}^{n-1} ce^{-n}$

$= 1 - \frac{c(1-(1/e)^n)}{1-1/e} = 1 - \frac{(1-1/e)(1-e^{-n})}{1-1/e}$

$$= 1 - (1 - e^{-n})$$

$$= e^{-n}$$

8. Consider the polynomial $p(x) = 1 - x^2 + x^4 - x^6 + \ldots$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$p(x)(1+x^2) = (1+x^2) \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k} + \sum_{k=0}^{\infty} (-1)^k x^{2(k+1)}$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k} - \sum_{k=1}^{\infty} (-1)^k x^{2k}$$

$$= (-1)^0 x^{2 \cdot 0}$$

$$= 1$$

$\therefore p(x) = \frac{1}{1+x^2}$ $\therefore p(x)$ is the Taylor series of $\frac{1}{1+x^2}$

$\therefore$ the n^{th} term of the Taylor series (not including 0 terms) is $(-1)^{n-1} x^{2(n-1)}$

~~And the n^{th} term including 0 terms is coef~~

~~$(-1)^{n/2 - 1} x^{n-2}$ if n is even~~

The coefficient of x^n is $\begin{cases} (-1)^{n/2} & \text{if } n \text{ even} \\ 0 & \text{otherwise} \end{cases}$

4. $\frac{\partial u}{\partial t} = a \cdot \frac{\partial}{\partial t} f(x+at)$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial u} f(x+at)$

$$\cancel{\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial}{\partial x} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} a \frac{\partial}{\partial t} f(x+at)}$$

$$\cancel{= a \frac{\partial^2}{\partial x \partial t} f(x+at)}$$

$$\cancel{= a \frac{\partial}{\partial t} f(x+at) + \frac{\partial}{\partial x} f(x+at) = 0}$$

10. $dS = r\,dr\,d\theta$

$$\vec{n}\,dS = \begin{pmatrix} 0 \\ 0 \\ r\,dr\,d\theta \end{pmatrix}$$

$$\therefore F \cdot \vec{n}\,dS = 0 + 0 + \frac{e^z}{1+z^2}\,r\,dr\,d\theta$$

$\forall$ points on D, $z = 0$

$$\therefore F \cdot \vec{n}\,dS = 0 \quad \frac{e^0}{1+0}\,r\,dr\,d\theta = r\,dr\,d\theta$$

$$\therefore \cancel{\int_D F \cdot \vec{n}\,dS = 0}$$

$$\therefore \int_D F \cdot \vec{n}\,dS = \int_0^1 \int_0^{2\pi} r\,d\theta\,dr$$

$$= \int_0^1 \left[r\theta \right]_0^{2\pi} dr$$

$$= \int_0^1 2\pi r\,dr$$

$$= \left[\pi r^2 \right]_0^1 dr$$

$$= \pi$$