BGN: 2031B

Paper 2

Question 7

a. "$\Rightarrow$" first

Let $a, b, c \in \mathbb{Z}^+$ such that $\gcd(c, ab) = 1$

$\therefore \exists\, k, l \in \mathbb{Z}$ such that $kc + lab = 1$

$\therefore kc + (la)b = 1$

$la \in \mathbb{Z}$

$\therefore \gcd(c, b) = 1$

and $kc + (lb)a = 1$

$lb \in \mathbb{Z}$

$\therefore \gcd(c, a) = 1$

$\square$

"$\Leftarrow$" next

Let $a, b, c \in \mathbb{Z}^+$ such that $\gcd(c, a) = 1 \wedge \gcd(c, b) = 1$

$\therefore \exists\, k, l, m, n, \in \mathbb{Z}$ such that

$kc + la = 1 \quad \wedge \quad mc + nb = 1$

$\therefore (kc + la)(mc + nb) = 1$

$\therefore (kmc + mla + knb)c + (ln)ab = 1$

$kmc + mla + knb \in \mathbb{Z}$

$ln \in \mathbb{Z} \quad \therefore \gcd(c, ab) = 1 \quad \square$

bi. ~~Let~~ Assume $\forall m \in \mathbb{N}^+ .\ P(m+1) \Rightarrow P(m)$ *

Let n arbitrary $\in \mathbb{N}^+$

Assume $P(n+1)$ **

Let k arbitrary $\in \mathbb{N}^+$

Assume $k \leq n+1$

RTP: $P(k)$

Case 0: $k = n+1$

$\therefore P(k)$ by **

Case 1: $k = n$

$\therefore P(k)$ by *

Case 2: $k < n$

Assume $P(j)$ for $j = k+1$

$\therefore P(j-1)$ by *

$\therefore P(k)$

$\therefore$ By induction $P(k)$ is true

ii Assume $P(2)^{(\dagger)} \wedge (\forall m \in \mathbb{N}^+ . P(m) \Rightarrow P(2m))^{(*)}$
$\wedge (\forall m \in \mathbb{N}^+ . P(m+1) \Rightarrow (P m))^{(\ddagger)}$

Let n arbitrary $\in \mathbb{N}^+$

Let $a, b \in \mathbb{N}$ such that $n = 2^a - b$
where $b < 2^a$ and $n > 2^{a-1}$

Case 0: $b = 0$
Case 0.0: $a = 0$
$\therefore n = 1$
$P(n+1)$ is true by $(\dagger)$
$\therefore P(n)$ is true by $(\ddagger)$
Case 0.1: $a = 1$
$\therefore n = 2$
$\therefore P(n)$ by $(\dagger)$
Case 0.2: $a > 1$
Assume $P(2^{a-1})$
$\therefore P(2^a)$ by $(*)$
$\therefore P(n)$
By induction, $P(n)$ holds for all n where $b = 0$

Case 1: $b > 0$
Assume $P(2^a - (b-1))$
$\therefore P(2^a - b)$ by $(\ddagger)$
$\therefore P(n)$
$\therefore$ By induction $P(n)$ holds $\forall n \in \mathbb{N}^+$

c. i. $f(x) = x/2$

f is injective because for a given $y = f(x)$, the only value of x for which $y = f(x)$ is $2y$.

f is not surjective because e.g. $\nexists x \in I.\ f(x) = 0.6$
$\therefore f$ is not bijective

ii. $g(x) = \begin{cases} 2x & \text{if } x \leq 1/2 \\ 2x-1 & \text{otherwise} \end{cases}$

~~g(x)~~ g is surjective because $\forall y$ ~~f(x)~~ $\in I$ there exists a value of x (e.g. $y/2$) $\in I$ such that $y = g(x)$

g is not injective because $g(0.1) = g(0.6)$
$\therefore g$ is not bijective

iii. $h(x) = 1 - x$

h is ~~injective~~ surjective because $\forall y \in I.\ \exists x \in I$ (e.g. $x = 1-y$) s.t. $y = h(x)$

h is injective because $\forall y \in I.\ y = f(x)$ for some x, $\exists$ a unique value of $x \in I$ ($x = 1-y$) s.t. $y = h(x)$

$\therefore h$ is bijective
$h(1) = 0 \neq 1 \therefore h$ is not the identity