Logic and Proof 1

1. ~~P → ¬P is unsatisfiable because there is no interpretation that satisfies it~~

$P \to \neg P$ is satisfied by $P \mapsto 0$

$\{P \to \neg P\}$ is not valid because the interpretation $P \mapsto 1$ does not satisfy $P \to \neg P$

2.

A	B	$\neg(A \wedge B)$	$\neg(A \vee B)$	$\neg A \vee \neg B$	$\neg A \wedge \neg B$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	1	0	1	0
1	1	0	0	0	0

A	B	C	$B \wedge C$	$A \vee B$	$A \vee C$	$A \vee (B \wedge C)$	$(A \vee B) \wedge (A \vee C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

A	B	C	$B \vee C$	$A \wedge B$	$A \wedge C$	$A \wedge (B \vee C)$	$(A \wedge B) \vee (A \wedge C)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	0	1	1	1
1	1	0	1	1	0	1	1
1	1	1	1	1	1	1	1

3.

Formula	Satisfying assignment	Verifying assignment
$P \rightarrow Q$	$P \mapsto 1,\ Q \mapsto 1$	$P \mapsto 1,\ Q \mapsto 0$
$P \vee Q \rightarrow P \wedge Q$	$P \mapsto 0,\ Q \mapsto 0$	$P \mapsto 1,\ Q \mapsto 0$
$\neg(P \vee Q \vee R)$	$P \mapsto 0,\ Q \mapsto 0,\ R \mapsto 0$	$P \mapsto 0,\ Q \mapsto 0,\ R \mapsto 1$
$\neg(P \wedge Q) \wedge \neg(Q \vee R) \wedge (P \vee R)$	$P \mapsto 1,\ Q \mapsto 0,\ R \mapsto 0$	$P \mapsto 0,\ Q \mapsto 0,\ R \mapsto 0$

4. $(P \rightarrow Q) \wedge (Q \rightarrow P)$

$\equiv (\neg P \vee Q) \wedge (\neg Q \vee P)$ CNF

$\equiv ((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)$

$\equiv (\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (P \wedge Q)$

$\equiv (\neg P \wedge \neg Q) \vee (P \wedge Q)$ DNF

Satisfiable $(P \mapsto 1,\ Q \mapsto 1)$

Not valid $(P \mapsto 1,\ Q \mapsto 0)$

ii. $((P \wedge Q) \vee R) \wedge \neg (P \vee R)$

$\cancel{\equiv ((P \wedge Q) \vee R) \wedge (\neg P \wedge \neg R)}$

$\cancel{\equiv (P \vee R) \wedge (Q \vee R) \wedge (\neg P \wedge \neg R)}$ CNF

$\cancel{\equiv P \wedge Q \wedge (\neg P \wedge \neg R}$

$((P \wedge Q) \vee R) \wedge \neg (P \vee R)$

$\equiv ((P \wedge Q) \vee R) \wedge \neg P \wedge \neg R$

$\equiv (P \vee R) \wedge (Q \vee R) \wedge \neg P \wedge \neg R$ CNF

$\cancel{\equiv (P \vee R) \wedge (Q \vee R) \wedge (\neg P \wedge (R \vee \neg R)) \wedge \neg R}$

$\equiv (((P \wedge Q) \wedge \neg P) \vee (R \wedge \neg P)) \wedge \neg R$

$\equiv (((P \wedge Q) \wedge \neg P) \cancel{\vee (R \wedge \neg P)} \wedge \neg R) \vee (R \wedge \neg P \wedge \neg R)$

$\equiv (P \wedge Q \wedge \neg P \wedge \neg R) \vee (R \wedge \neg P \wedge \neg R)$ DNF

$\equiv f$

Not satisfiable or valid because $\equiv f$

iii. $\neg (P \vee Q \vee R) \vee ((P \wedge Q) \vee R)$

$\equiv (\neg P \wedge \neg Q \wedge \neg R) \vee ((P \wedge Q) \vee R)$

$\equiv (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee R$ DNF

$\equiv ((\neg P \wedge \neg Q \wedge \neg R) \vee P) \wedge ((\neg P \wedge \neg Q \wedge \neg R) \vee Q) \vee R$

$\equiv ((\neg P \vee P) \wedge (\neg Q \vee P) \wedge (\neg R \vee P) \wedge (\neg P \vee Q) \wedge (\neg Q \vee Q) \wedge (\neg R \vee Q)) \vee R$

$\equiv (\neg P \vee P \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg R \vee R) \wedge (\neg P \vee Q \vee R) \wedge (\neg Q \vee Q \vee R) \wedge (Q \vee \neg R \vee R)$

$\equiv (P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R)$ CNF

Satisfiable $(P \mapsto 0, Q \mapsto 0, R \mapsto 1)$

Not valid $(P \mapsto 0, Q \mapsto 1, R \mapsto 0)$

5. See attached

6i.

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}}{\Rightarrow \neg A, A}\,(\neg r)}{\neg\neg A \Rightarrow A}\,(\neg l)$$

ii.

$$\dfrac{\dfrac{\dfrac{}{A,B \Rightarrow B} \qquad \dfrac{}{A,B \Rightarrow A}}{A,B \Rightarrow B \wedge A}\,(\wedge r)}{A \wedge B \Rightarrow B \wedge A}\,(\wedge l)$$

iii.

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, A}}{A \Rightarrow B \vee A}\,(\vee r) \qquad \dfrac{\dfrac{}{B \Rightarrow B, A}}{B \Rightarrow B \vee A}\,(\vee r)}{A \vee B \Rightarrow B \vee A}\,(\vee l)$$

7:

$$
\dfrac{
  \dfrac{
    A, B, C \Rightarrow A
    \qquad
    \dfrac{
      \dfrac{}{A, B, C \Rightarrow B}
      \qquad
      \dfrac{}{A, B, C \Rightarrow C}
    }{A, B, C \Rightarrow B \wedge C}\ (\wedge r)
  }{
    \dfrac{A, B, C \Rightarrow A \wedge (B \wedge C)}{(A \wedge B), C \Rightarrow A \wedge (B \wedge C)}\ (\wedge l)
  }\ (\wedge r)
}{(A \wedge B) \wedge C \Rightarrow A \wedge (B \wedge C)}\ (\wedge l)
$$

$$
\cancel{
\dfrac{
  \dfrac{
    \dfrac{A, B, A \wedge C \Rightarrow}{A, B, A \wedge C \Rightarrow A \vee (B \wedge C)}
  }{A \wedge B, A \wedge C \Rightarrow A \vee (B \wedge C)}\ (\wedge l)
}{(A \vee B) \wedge (A \vee C) \Rightarrow A \vee (B \wedge C)}\ (\wedge l)
}
$$

ii.

$$
\begin{prooftree}
\AxiomC{$A, A \vee C \Rightarrow A, B \wedge C$}
\AxiomC{$B, A \vee C \Rightarrow A, B$}
\AxiomC{$B, A \Rightarrow A, C$}
\AxiomC{$B, C \Rightarrow A, C$}
\RightLabel{$(\vee l)$}
\BinaryInfC{$B, A \vee C \Rightarrow \cancel{B} A, C$}
\RightLabel{$(\wedge r)$}
\BinaryInfC{$B, A \vee C \Rightarrow A, B \wedge C$}
\RightLabel{$(\vee l)$}
\BinaryInfC{$A \vee B, A \vee C \Rightarrow A, B \wedge C$}
\RightLabel{$(\vee r)$}
\UnaryInfC{$A \vee B, A \vee C \Rightarrow A \vee (B \wedge C)$}
\RightLabel{$(\wedge l)$}
\UnaryInfC{$(A \vee B) \wedge (A \vee C) \Rightarrow A \vee (B \wedge C)$}
\end{prooftree}
$$

iii.

$$
\begin{prooftree}
\AxiomC{$A \Rightarrow A, B$}
\RightLabel{$(\neg r)$}
\UnaryInfC{$\Rightarrow \neg A, A, B$}
\AxiomC{$B \Rightarrow A, B$}
\RightLabel{$(\neg r)$}
\UnaryInfC{$\Rightarrow \neg B, A, B$}
\RightLabel{$(\wedge r)$}
\BinaryInfC{$\Rightarrow \neg A \wedge \neg B, A, B$}
\RightLabel{$(\vee r)$}
\UnaryInfC{$\Rightarrow \neg A \wedge \neg B, A \vee B$}
\RightLabel{$(\neg l)$}
\UnaryInfC{$\neg (A \vee B) \Rightarrow \neg A \wedge \neg B$}
\end{prooftree}
$$

8i. $A \leftrightarrow B \ \hat{=}\ (\neg A \vee B) \wedge (\neg B \vee A)$

$$
\dfrac{
  \dfrac{
    \dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, \neg A, B}\,(\neg r)
  }{\Gamma \Rightarrow \Delta, \neg A \vee B}\,(\vee r)
  \qquad
  \dfrac{
    \dfrac{B, \Gamma \Rightarrow \Delta, A}{\vdots}
  }{\Gamma \Rightarrow \Delta, \neg B \vee A}\,\text{similar}
}{
  \dfrac{\Gamma \Rightarrow \Delta, (\neg A \vee B) \wedge (\neg B \vee A)}{\Gamma \Rightarrow \Delta, A \leftrightarrow B}
}\,(\wedge r)
$$

$$
\dfrac{A, \Gamma \Rightarrow \Delta, B \qquad B, \Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \leftrightarrow B}\,(\leftrightarrow r)
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma, \neg A \Rightarrow \Delta, B}\,(\neg l)
      }{\Gamma, \neg A, \neg B \Rightarrow \Delta}\,(\neg l)
      \qquad
      \dfrac{
        \dfrac{}{\Gamma, A \Rightarrow \Delta, A}
      }{\Gamma, \neg A, A \Rightarrow \Delta}\,(\neg l)
    }{\Gamma, \neg A, \neg B \vee A \Rightarrow \Delta}\,(\vee l)
    \qquad
    \dfrac{
      \dfrac{
        \dfrac{}{\Gamma, B \Rightarrow \Delta, B}
      }{\Gamma, B, \neg B \Rightarrow \Delta}\,(\neg l)
      \qquad
      \Gamma, B, A \Rightarrow \Delta
    }{\Gamma, B, \neg B \vee A \Rightarrow \Delta}\,(\vee l)
  }{\Gamma, \neg A \vee B, \neg B \vee A \Rightarrow \Delta}\,(\vee l)
}{
  \dfrac{\Gamma, (\neg A \vee B) \wedge (\neg B \vee A) \Rightarrow \Delta}{\Gamma, A \leftrightarrow B \Rightarrow \Delta}
}\,(\wedge l)
$$

$$\frac{\Gamma \Rightarrow A, B, \Delta \qquad \Gamma, A, B \Rightarrow \Delta}{\Gamma, A \leftrightarrow B \Rightarrow \Delta} \; (\leftrightarrow l)$$

ii. $A \oplus B \triangleq \cancel{\neg(A \leftrightarrow B)} \; A \leftrightarrow \neg B$

$$\dfrac{\dfrac{\dfrac{\Gamma, B \Rightarrow A, \Delta}{\Gamma \Rightarrow A, \neg B, \Delta}\,(\neg r) \qquad \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma, A, \neg B \Rightarrow \Delta}\,(\neg l)}{\Gamma, A \leftrightarrow \neg B \Rightarrow \Delta}\,(\leftrightarrow l)}{\Gamma, A \oplus B \Rightarrow \Delta}$$

$$\frac{\Gamma, A \Rightarrow B, \Delta \qquad \Gamma, B \Rightarrow A, \Delta}{\Gamma, A \oplus B \Rightarrow \Delta} \; (\oplus l)$$

$$\dfrac{\dfrac{(\neg r)\,\dfrac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \Rightarrow \neg B, \Delta} \qquad \dfrac{\Gamma \Rightarrow A, B, \Delta}{\Gamma, \neg B \Rightarrow A, \Delta}\,(\neg l)}{\Gamma \Rightarrow \Delta, A \leftrightarrow \neg B}\,(\leftrightarrow r)}{\Gamma \Rightarrow \Delta, A \oplus B}$$

$$\frac{\Gamma, A, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow \Delta, A \oplus B} \; (\oplus r)$$

9 i.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A, B}}{A, \neg A \Rightarrow B}\ (\neg l)}{(A \wedge \neg A) \Rightarrow B}\ (\wedge l)}{\Rightarrow (A \wedge \neg A) \rightarrow B}\ (\rightarrow r)$$

ii.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A, B}}{\Rightarrow A, A \rightarrow B}\ (\rightarrow r) \qquad \dfrac{}{A \Rightarrow A}}{(A \rightarrow B) \rightarrow A \Rightarrow A}\ (\rightarrow l)}{\Rightarrow ((A \rightarrow B) \rightarrow A) \rightarrow A}\ (\rightarrow r)$$

10. $(1) \not\Rightarrow (2)$

$(2) \Rightarrow (1)$

11. ~~$\exists x_1 \exists x_2 \exists x_3 \ldots \exists x_m ($~~

$\exists x_1, x_2, x_3, \ldots x_m \ (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_1 \neq x_m \wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge \ldots \wedge x_2 \neq x_m$
$\wedge \ldots \wedge x_{m-1} \neq x_m)$

(There exist m elements of the domain such that none of them are equal to any other)

$\forall x_1, x_2, x_3, \ldots x_m \ (x_1 = x_2 \vee x_1 = x_3 \vee \ldots \vee x_1 = x_m \vee x_2 = x_3 \vee x_2 = x_4 \vee \ldots x_2 = x_m$
$\vee \ldots \vee x_{m-1} = x_m)$

(For all octuples from the domain, at least one pair will be equal)