13S. BGN: 2031B
Paper 2
Question 13S

a. $\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (-ay+e^{-r^2}x) & (ax+e^{-r^2}y) & (e^{-r^2}z) \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}(e^{-r^2}z) - \frac{\partial}{\partial z}(ax+e^{-r^2}y)\right)$$

$$-\hat{j}\left(\frac{\partial}{\partial x}(e^{-r^2}z) - \frac{\partial}{\partial z}(-ay+e^{-r^2}x)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(ax+e^{-r^2}y) - \frac{\partial}{\partial y}(-ay+e^{-r^2}x)\right)$$

$$= \hat{i}\left(-2yze^{-r^2} + 2yze^{-r^2}\right) + \hat{j}\left(-2xze^{-r^2} + 2xze^{-r^2}\right)$$

$$+ \hat{k}\left(a - 2xye^{-r^2} + a + 2xye^{-r^2}\right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2a \end{pmatrix}$$

b. $\nabla \cdot \nabla \times F = \nabla \cdot \begin{pmatrix} 0 \\ 0 \\ 2a \end{pmatrix} = \frac{\partial}{\partial z} 2a = 0$

c. $\nabla \times F = \vec{0} \iff a = 0$

d. $$\vec{dr} = \begin{pmatrix} -2\pi n t^{n-1} \sin(2\pi t^n) \\ 2\pi n t^{n-1} \cos(2\pi t^n) \\ 0 \end{pmatrix} dt$$

$$\vec{F} = \begin{pmatrix} -a\sin(2\pi t^n) + e^{-2}\cos(2\pi t^n) \\ a\cos(2\pi t^n) + e^{-2}\sin(2\pi t^n) \\ e^{-2} \end{pmatrix}$$

$$\therefore \vec{F} \cdot \vec{dr} = \Big(-2\pi n t^{n-1} \sin(2\pi t^n)\big(-a\sin(2\pi t^n) + e^{-2}\cos(2\pi t^n)\big) + 2\pi n t^{n-1}\cos(2\pi t^n)\big(a\cos(2\pi t^n) + e^{-2}\sin(2\pi t^n)\big) + 0\Big) dt$$

$$= \Big(2\pi n t^{n-1}\big(a(\cos^2(2\pi t^n) + \sin^2(2\pi t^n)) + e^{-2}(\cos(2\pi t^n)\sin(2\pi t^n) - \cos(2\pi t^n)\sin(2\pi t^n))\big)\Big) dt$$

$$= \big(2\pi n t^{n-1}(a \cdot 1 + e^{-2}(0))\big) dt$$

$$= 2\pi a n t^{n-1} \, dt$$

$$\therefore \int_{\Gamma} \vec{F} \cdot \vec{dr} = \int_0^1 2\pi a n t^{n-1} \, dt$$

$$= \left[2\pi a t^n\right]_0^1$$

$$= 2\pi a$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 2a \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2a$$

$$\vec{dS} = \vec{k}\, r\, dr\, d\theta$$

$$\therefore \int_D \vec{\nabla} \times \vec{F} \cdot dS = \int_0^{2\pi} \int_0^1 2a\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[a r^2 \right]_0^1 d\theta$$

$$= \int_0^{2\pi} a\, d\theta$$

$$= \left[a\theta \right]_0^{2\pi}$$

$$= 2\pi a$$

$$\therefore I = 2\pi a$$

f. $$\vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \phi \\ \frac{\partial}{\partial y} \phi \\ \frac{\partial}{\partial z} \phi \end{pmatrix}$$

$$\therefore e^{-r^2} z = \frac{\partial}{\partial z} \phi$$

$$\therefore \phi = \int e^{-r^2} z \, \partial z \quad \text{with } x, y \text{ constant}$$

$$= -\tfrac{1}{2} e^{-r^2} + C(x,y)$$

$$\therefore \frac{\partial}{\partial y} \phi = \frac{\partial}{\partial y} \left(-\tfrac{1}{2} e^{-r^2} + C(x,y) \right)$$

$$= \frac{\partial}{\partial y} C(x,y) + e^{-r^2} y \overset{!}{=} ax + e^{-r^2} y$$

$$\therefore \frac{\partial}{\partial y} C(x,y) = ax$$

$$\therefore C(x,y) = \int ax \, dy \quad \text{with } x \text{ constant}$$

$$= axy + D(x)$$

$$\therefore \phi = -\tfrac{1}{2} e^{-r^2} + axy + D(x)$$

$$\frac{\partial}{\partial x}\phi = e^{-r^2} x + ay + \frac{\partial}{\partial x} D \overset{!}{=} e^{-r^2} x - ay$$

$$\therefore ay + \frac{\partial}{\partial x} D = -ay$$

$$\therefore \frac{\partial}{\partial x} D = -2ay$$

$$\therefore D = \int -2ay \, dx = A - 2axy$$

$$\therefore \phi = -\tfrac{1}{2} e^{-r^2} - axy + A \quad \text{for some constant } A$$

One such field is $\phi(x,y,z) = -\tfrac{1}{2} e^{-r^2} - axy$