BGN: 2031B
Paper 1
Question 13 Y

i. $\frac{d^2y}{dx^2} + y = 0$

$\cancel{\therefore -\frac{1}{y}\frac{d^2y}{dx^2} = 1}$

$\cancel{\therefore \int -\frac{1}{y}\frac{d^2y}{dx^2}\,dx = \int 1\,dx}$

$\cancel{\therefore -\ln|y|\frac{dy}{dx} = x + C}$

Assuming $y = e^{\lambda x}$ for some λ

$\lambda^2 + 1 = 0$

$\therefore \lambda = \pm i$

$\therefore y = \alpha e^{-ix} + \beta e^{ix} = A\cos x + B\sin x$ for some A, B

ii. Trial particular integral: $y = C\cos 2x + D$

$\frac{dy}{dx} = -2C\sin 2x$

$\frac{d^2y}{dx^2} = -4C\cos 2x$

$\frac{d^2y}{dx^2} + y = (C - 4C)\cos 2x + D$

$= -3C\cos 2x + D$

$\overset{!}{=} -\frac{1}{2}\cos 2x + \frac{1}{2}$

$\therefore D = \frac{1}{2}, \quad C = \frac{1}{6}$

$\therefore y = \frac{1}{6}\cos 2x + \frac{1}{2}$

$\therefore$ Particular solution:

$$y = A\cos x + B\sin x + \tfrac{1}{6}\cos 2x + \tfrac{1}{2}$$

iii. $y(0) = A + \tfrac{1}{6} + \tfrac{1}{2} = A + \tfrac{2}{3} \overset{!}{=} 0$

$$\therefore A = -\tfrac{2}{3}$$

$$y(\pi/2) = B + \tfrac{1}{2} \overset{!}{=} 0$$

$$\therefore B = -\tfrac{1}{2}$$

$$\therefore y = -\tfrac{2}{3}\cos x - \tfrac{1}{2}\sin x + \tfrac{1}{6}\cos 2x + \tfrac{1}{2}$$

b. $\frac{dy}{dx} = 2y + 4y^5$

$$\therefore \frac{1}{2y+4y^5}\frac{dy}{dx} = 1$$

$$\therefore \int \frac{1}{2y+4y^5}\frac{dy}{dx}\,dx = \int 1\,dx$$

$$\therefore \int \frac{1}{2y+4y^5}\,dy = x$$

(constant term is lumped into that of the left-hand integral)

$$\therefore x = \frac{1}{2}\int \frac{1}{y(1+2y^4)}\,dy$$

$\cancel{\text{Let } u = \ln y}$

$\cancel{\therefore \cancel{dx}\ du = \frac{1}{y}\,dy}$

$\cancel{y = e^u}$

$\cancel{\therefore x = \tfrac{1}{2}\int \frac{1}{1+2e^{4u}}\,du}$

$$\frac{1}{y(1+2y^4)} = \frac{1}{y} - \frac{2y^3}{1+2y^4}$$

$$\therefore x = \frac{1}{2}\left(\int \frac{1}{y}\,dy - 2\int \frac{y^3}{1+2y^4}\,dy\right)$$

$$= \tfrac{1}{2}\left(\ln|y| - \tfrac{1}{4}\ln|1+2y^4|\right) + c$$

$$\therefore x - c = \ln\left|\frac{y^{1/2}}{(1+2y^4)^{1/8}}\right|$$

$$\therefore Ae^{x} = \frac{y^{1/2}}{(1+2y^4)^{1/8}} \quad \text{for some } A$$

$$\therefore Be^{8x} = \frac{y^4}{1+2y^4} = \frac{1+2y^4-1}{1+2y^4}\cdot\frac{1}{2} \quad \text{for some } B$$

$$= \tfrac{1}{2} - \frac{1}{2+4y^4}$$

$$\therefore 2+4y^4 = \frac{1}{\tfrac{1}{2} - Be^{8x}} = \frac{2}{1-De^{8x}} \quad \text{for some } D$$

$$\therefore y^4 = \frac{1}{2-Ee^{8x}} - \tfrac{1}{2} \quad \text{for some } E$$

$$\therefore y = \pm\left(\frac{1}{2-Ee^{8x}} - \tfrac{1}{2}\right)^{1/4}$$