BGN: 2031B
Paper 1
Question 18Z

a.i. $Tr(M) = a^2 + 1 - 1 = a^2 \doteq 3$

$\therefore a = \pm\sqrt{3}$, b can be anything

ii. M is invertible $\iff$ det M $\neq 0$

$\therefore a^2(-1) - b(b) \cancel{+} -3(-b-1) \neq 0$

$\therefore$ ~~b~~

$Tr(M) \Rightarrow a = \pm\sqrt{3} \Rightarrow a^2 = 3$

$\therefore -3 - b^2 + 3b + 1 \neq 0$

$\therefore b^2 - 3b + 2 \neq 0$

$\therefore (b-2)(b-1) \neq 0$

$\therefore b \neq 2 \wedge b \neq 1$

$\therefore Tr(M) = 3$ and M is invertible

$\Rightarrow a = \pm\sqrt{3}$ and $b \neq 2$ and $b \neq 1$

b.i. C is a square, diagonal matrix

~~$\therefore$ BC is a square diagonal matrix~~

Let $C = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\therefore BC = \begin{pmatrix} \alpha a & \beta b \\ \alpha c & \beta d \end{pmatrix}$

$AB = \begin{pmatrix} -5a-4c & -5b-4d \\ 4a+5c & 4b+5d \end{pmatrix} \stackrel{!}{=} BC = \begin{pmatrix} \alpha a & \beta b \\ \alpha c & \beta d \end{pmatrix}$

~~$\therefore$ The solution is Let $\alpha = -5$, $\beta = -5$,~~

$\therefore \cancel{\begin{pmatrix} -5a-4c & -5b-4d \\ 4a+5c & 4b+5d \end{pmatrix} = \begin{pmatrix} -5a & -5b \\ -5c & -5d \end{pmatrix}}$

$\cancel{4c = 0}$

$\therefore \cancel{c = 0}$

$\therefore \cancel{4a = 0}$

$\therefore \cancel{a = 0}$

$\cancel{\text{Let } \alpha = 1}$

$\therefore \cancel{-5a - 4c = a}$

$\therefore \cancel{-6a = 4c}$

$\therefore \cancel{a = -\tfrac{2}{3}c}$

$\therefore \cancel{4\left(-\tfrac{2}{3}c\right) + 5c = 0}$

$\therefore \cancel{\left(12 - \tfrac{8}{3}\right)c = 0}$

$\therefore \cancel{c = 0 \quad \therefore a = 0}$

$-5a - 4c = \alpha a$

$\therefore (5+\alpha)a = -4c$

$\therefore a = \dfrac{-4c}{5+\alpha}$

$4a + 5c = \alpha c$

$\therefore \dfrac{-16c}{5+\alpha} + 5c = \alpha c$

$\therefore$ Either $c = 0$ or

$-16 + 5(5+\alpha) = \alpha(5+\alpha)$

$\therefore -16 + 25 + 5\alpha = 5\alpha + \alpha^2$

$\therefore \alpha^2 = 9$

$\therefore \alpha = \pm 3 \quad \therefore$ Let $\alpha = 3 \quad \therefore a = -\tfrac{1}{2}c$

$-5b - 4d = \beta d$

$\therefore b = \frac{-4d}{5+\beta}$ by symmetry

$4b + 5d = \beta d$

$\therefore \beta = \pm 3$ Let $\beta = -3$

$\therefore b =$ ~~by symmetry~~ $-2d$

~~wlog. Let $c = d = 2$~~

$\cancel{\therefore B = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}}$ $\quad \cancel{C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}}$

ii. $\cancel{B^{-1} = \frac{1}{|B|}\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}}$

$= \frac{1}{-2+2}$

wlog. Let $c = 2$, $d = 1$

$\therefore a = -1$, $b = -2$

$B = \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

ii. $|B| = -1 + 4 = 3$

$B^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$

c. Let $X = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \cdots & x_{0n} \\ -x_{01} & x_{11} & x_{12} & \cdots & x_{1n} \\ -x_{02} & -x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{0n} & -x_{1n} & -x_{2n} & \cdots & x_{nn} \end{pmatrix}$

Let $Y = \begin{pmatrix} y_{00} & y_{01} & y_{02} & \cdots & y_{0n} \\ y_{10} & y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{n0} & y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$

$$Z = Y X Y^{-1}$$

$$\mathrm{Tr}(Z) = \sum_{k=0}^{n} Z_{kk}$$

$$= \sum_{k=0}^{n} Y_{ki} (\cancel{X} Y^{-1})_{ik}$$

$$= \sum_{k=0}^{n} Y_{ki} \cancel{X}_{ij} Y^{-1}_{jk}$$

$$= -\sum_{k=0}^{n} Y_{ki} X_{ji} Y^{-1}_{jk}$$

$$= -\sum_{k=0}^{n}$$

$$= -\sum_{k=0}^{n} (Y^{-1})^{T}_{kj} X_{ji} (Y^{T})_{ik}$$

$$= -\sum_{k=0}^{n} (Y^{-1})^{T}_{kj} (X Y^{T})_{jk}$$

$$= -\sum_{k=0}^{n} \left((Y^{-1})^T X Y^T \right)_{kk}$$

$$= -\sum_{k=0}^{n} Z^T_{kk}$$

$$= - \mathrm{Tr}(Z^T)$$

$$= - \mathrm{Tr}(Z)$$

$$\therefore \mathrm{Tr}(Z) = 0$$

d. i. $\begin{pmatrix}-1\\c\end{pmatrix}\cdot\begin{pmatrix}b\\d\end{pmatrix}=0$

$\therefore -b + cd = 0$

$cd = b$

ii. $cd = b$ †

$d \neq 1$ ‡

$c \neq b$ which is given by † and ‡

$\therefore cd = b \wedge d \neq 1$

e. $\begin{pmatrix}a & b\\c & 3\end{pmatrix} = \begin{pmatrix}a & \frac{b+c}{2}\\ \frac{b+c}{2} & 3\end{pmatrix} + \begin{pmatrix}0 & \frac{b-c}{2}\\ \frac{c-b}{2} & 0\end{pmatrix}$