Candidate number: 2031B
Paper 1
Question 5

a.i. $f(x) \simeq f(x_n) + \frac{f'(x_n)}{1!}(x - x_n)$

$\therefore$ If we want to solve $f(x) = 0$,

we have

$f(x_n) + f'(x_n)(x - x_n) \simeq 0$

~~$\therefore x_n = x \simeq$~~

$x - x_n \simeq -\frac{f(x_n)}{f'(x_n)}$

$\therefore x \simeq x_n - \frac{f(x_n)}{f'(x_n)}$

$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is a better guess for the root

ii. $$f(\alpha) = f(x_n) + \frac{f'(x_n)}{1!}(\alpha - x_n) + \frac{f''(x_n)}{2!}(\alpha - x_n)^2 + \cdots$$

$$= f(x_n) + f'(x_n)(\alpha - x_n) + O((\alpha - x_n)^2)$$

$$= 0$$

$$\therefore 0 = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) + O((\alpha - x_n)^2)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore e_{n+1} = \alpha - x_{n+1} = \alpha - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$$

$$= \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n)$$

$$= O((\alpha - x_n)^2)$$

$$= O(e_n^2)$$

For this to hold, $f'(x) \neq 0 \ \forall x \in [\alpha - |\alpha - x_0|, \alpha + |\alpha - x_0|]$

$f''(x)$ must be continuous in the same range as above

x_0 must be sufficiently close to α such that the higher order terms of the Taylor series can be ignored, as $(\alpha - x_0)^n$ is small

iii. If x_0 is very far away from α, the algorithm will not always converge

~~It is not always possible to know the derivative of the function you are trying to solve, at arbitrary inputs.~~

If for some estimate x_n, the derivative of the function is 0, the algorithm must terminate and does not find the root.

b. Let $x_k = 1 + \varepsilon$ where $\varepsilon > 0$, equivalently, $x_k > 1$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = 1 + \varepsilon - \frac{(1+\varepsilon)^2 - 1}{2(1+\varepsilon)}$$

$$= 1 + \frac{2\varepsilon(\varepsilon+1) - \varepsilon(\varepsilon+2)}{2(\varepsilon+1)}$$

$$= 1 + \frac{\varepsilon(2\varepsilon + 2 - \varepsilon - 2)}{2(\varepsilon+1)}$$

$$= 1 + \frac{\varepsilon^2}{2(\varepsilon+1)}$$

$\varepsilon > 0 \quad \therefore \quad \varepsilon^2 > 0 \wedge 2(\varepsilon+1) > 0$

$\therefore \dfrac{\varepsilon^2}{2(\varepsilon+1)} > 0 \quad \therefore \quad 1 + \dfrac{\varepsilon^2}{2(\varepsilon+1)} > 1$

$\therefore x_k > 1$

~~Assume $x_k = 1 + \delta$ where $\delta > 0$~~

~~Equivalently $x_n > 1$~~

~~$x_{k+1} = x_k - \frac{f(x_k)}{}$~~

$x_0 > 1$

$x_k > 1 \implies x_{k+1} > 1$

$\therefore x_n > 1 \quad \forall n \geq 1$

ii. $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1}{2x_n}$$

$x_n > 1 \quad \therefore x_n^2 > 1 \quad \therefore x_n^2 - 1 > 0$

$2x_n > 2 \quad \therefore \frac{x_n^2 - 1}{2x_n} > 0$

$\lim_{x_n \to \infty} \frac{x_n^2 - 1}{2x_n} = \frac{x_n}{2}$ and $\frac{x_n^2 - 1}{2x_n}$ is strictly increasing for $x_n > 1$

$$\therefore 0 < \frac{x_n^2 - 1}{2x_n} < \frac{x_n}{2}$$

$$\therefore -\frac{x_n}{2} < -\frac{x_n^2 - 1}{2x_n} < 0$$

$$\therefore \cancel{x_{n+1}} \; x_n - \frac{x_n}{2} < x_{n+1} < x_n$$

$$\therefore \frac{x_n}{2} < x_{n+1} < x_n$$