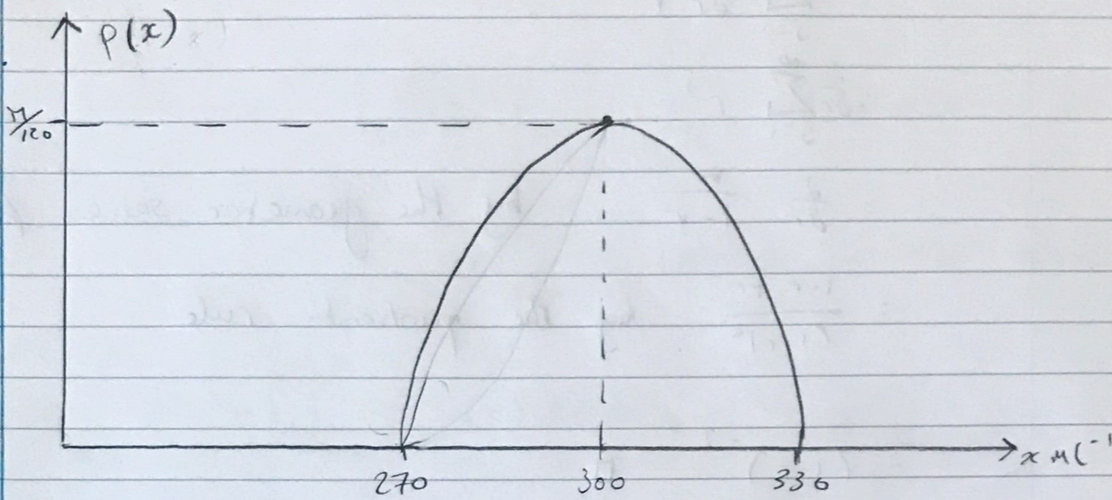
BGN: 2032 B
Paper 2
Question 14 V

a i:



ii: $P(290 \leq v \leq 310) = \int_{290}^{310} P(v)\,dv$

$= \int_{290\,ml}^{310\,ml} \frac{\pi}{120} \cos\left(\frac{\pi(v-300\,ml)}{60\,ml}\right) (ml)^{-1}\,dv$

$= \frac{\pi}{120} \int_{290}^{310} \cos\left(\frac{\pi}{60}v - 5\right) dv$

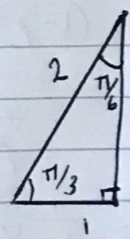
$= \frac{\pi}{120} \left[\frac{60}{\pi} \sin\left(\frac{\pi}{60}v - 5\right)\right]_{290}^{310}$

$= \left[\frac{1}{2} \sin\left(\frac{\pi(v-300)}{60}\right)\right]_{290}^{310}$

$= \frac{1}{2}\left(\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\right)$

$= \sin(\pi/6)$

$= 1/2$



iii. Let $I = \sum_{j=1}^{\infty} j r^{j-1}$

$$= \sum_{j=1}^{\infty} \frac{d}{dr} r^j$$

$$= \frac{d}{dr} \sum_{j=1}^{\infty} r^j$$

$$= \frac{d}{dr} \frac{r}{1-r}$$ by the geometric series formula

$$= \frac{1-r+r}{(1-r)^2}$$ by the quotient rule

$$= (1-r)^{-2} \quad \square$$

iv. X = number of test up to and including first failure

Let p = probability of passing the test $= 1 - 1/2 = 1/2$

$$P(X=k) = p^{k-1} \cancel{p} (1-p)$$

$$\therefore E[X] = \sum_{j=1}^{\infty} j P(X=j) = \sum_{j=1}^{\infty} j p^{j-1} (1-p)$$

$$= (1-p) \sum_{j=1}^{\infty} j p^{j-1}$$

$$= (1-p)(1-p)^{-2}$$

$$= \frac{1}{1-p}$$

$$= \frac{1}{1-1/2}$$

$$= 1/(1/2) = 2$$

b. Let v_1 = volume of first coffee
v_2 = volume of second coffee

$V = V_1 + V_2$

$$P(V \geq 630) = \int_{-\infty}^{\infty} \rho_{V_1}(t)\, P(V_2 \geq 630-t)\, dt$$

$$= \int_{270}^{330} \rho_{V_1}(t) \int_{630-t}^{330} \rho_{V_2}(s)\, ds\, dt \quad \text{if } 630-t \leq 330$$

$$= \int_{300}^{330} \rho_{V_1}(t) \int_{630-t}^{330} \rho_{V_2}(s)\, ds\, dt$$

$$= \int_{300}^{330} \int_{630-t}^{330} \frac{\pi^2}{120^2} \cos\left(\frac{\pi}{60}t - 5\right) \cos\left(\frac{\pi}{60}s - 5\right) ds\, dt$$

$$= \int_{300}^{330} \frac{\pi}{120} \cos\left(\frac{\pi}{60}t - 5\right) \left[\frac{1}{2} \sin\left(\frac{\pi(s-300)}{60}\right)\right]_{630-t}^{330} dt$$

$$= \int_{300}^{330} \frac{\pi}{120} \cos\left(\frac{\pi}{60}t - 5\right) \left(\frac{1}{2} \cdot \sin\frac{\pi}{2} - \frac{1}{2} \sin\left(\frac{\pi(330-t)}{60}\right)\right) dt$$

$$= \int_{300}^{330} \frac{\pi}{240} \cos\left(\frac{\pi(t-300)}{60}\right) dt - \int_{300}^{330} \frac{\pi}{240} \cos\left(\frac{\pi(t-300)}{60}\right) \sin\left(\frac{\pi(330-t)}{60}\right) dt$$

$$= \left[\frac{1}{4} \sin\left(\frac{\pi(t-300)}{60}\right)\right]_{300}^{330} + \int_{300}^{330} \frac{\pi}{240} \cos\left(\frac{\pi(t-300)}{60}\right) \sin\left(\frac{\pi(t-330)}{60}\right) dt$$

$$= \frac{1}{4} + \int_{300}^{330} \frac{\pi}{240} \cos\left(\frac{\pi(t-300)}{60}\right) \sin\left(\frac{\pi(t-300) - 30\pi}{60}\right) dt$$

Let $u = \frac{t-300}{30} \Rightarrow du = \frac{1}{30}\, dt$

$\therefore P(V \geq 630) =$

$$\frac{1}{4} + \int_0^1 \frac{\pi}{8} \cos\left(\frac{\pi}{2}u\right) \sin\left(\frac{\pi}{2}u - \pi\right) du$$

$$\sin\left(\frac{\pi}{2}u - \pi\right) = \sin\left(\frac{\pi}{2}u\right)\cos(\pi) - \cos\left(\frac{\pi}{2}u\right)\sin(\pi)$$

$$= -\sin\left(\frac{\pi}{2}u\right)$$

$$\therefore P(V \geq 630) = \frac{1}{4} + \int_0^1 -\frac{\pi}{8} \cos\left(\frac{\pi}{2}u\right) \sin\left(\frac{\pi}{2}u\right)$$

$$= \frac{1}{4} + \left[\frac{1}{8}\cos^2\left(\frac{\pi}{2}u\right)\right]_0^1$$

$$= \frac{1}{4} + \left(0 - \frac{1}{8}\right)$$

$$= \frac{1}{8}$$