BGN: 2031B
Paper 2
Question 12 X

a. A stationary point is a point on the surface of $f$ in $\mathbb{R}^3$ at which the gradient in every direction is 0. i.e. the surface is locally flat (parallell to the $x$-$y$ plane)

Since the gradient in the $x$ direction is 0, $\frac{\partial f}{\partial x} = 0$ at a stationary point and likewise $\frac{\partial f}{\partial y} = 0$

$\therefore \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \nabla f = 0$ at a stationary point.

b. $\frac{\partial f}{\partial x} = 2 \cdot \frac{(1+x^2+y^2)(-2x) - (y-x^2)(2x)}{(1+x^2+y^2)^2}$

$\doteq 0$

$\therefore$ ~~Either~~ Either $x = 0$

or $1 + x^2 + y^2 - y + x^2 = 0$

$\therefore 2x^2 = y - y^2 - 1$

$\therefore x = \pm \sqrt{\frac{y - y^2 - 1}{2}}$

$$\frac{\partial f}{\partial y} = 2\,\frac{(1+x^2+y^2)(1) - (y-x^2)(2y)}{(1+x^2+y^2)^2}$$

$$\stackrel{!}{=} 0$$

$$\therefore\ 1 + x^2 + y^2 - 2y^2 + 2x^2y = 0$$

$$\therefore\ 1 + x^2 - y^2 + 2x^2y = 0$$

Case 1: $x = 0$

$$\therefore\ 1 - y^2 = 0$$

$$\therefore\ y = \pm 1$$

Case 2: $x^2 = \frac{y - y^2 - 1}{2}$

$$\therefore\ 1 + \frac{y-y^2-1}{2} - y^2 + y^2 - y^3 - y = 0$$

$$\therefore\ 2 - y^2 - 1 - 2y^3 - y = 0$$

$$\therefore\ 2y^3 + y^2 + y - 1 = 0$$

$$\cancel{y(2y^2 + y + 1) = 1}$$

Which has no solutions such that $\frac{y-y^2-1}{2} \geq 0$

$\therefore$ Stationary points are at $(0,1)$, $(0,-1)$

$$f(0,1) = \frac{2}{2} = 1$$

$$f(0,-1) = \frac{-2}{2} = -1$$

$\therefore$ Stationary points at $(0,1,1)$ and $(0,-1,-1)$