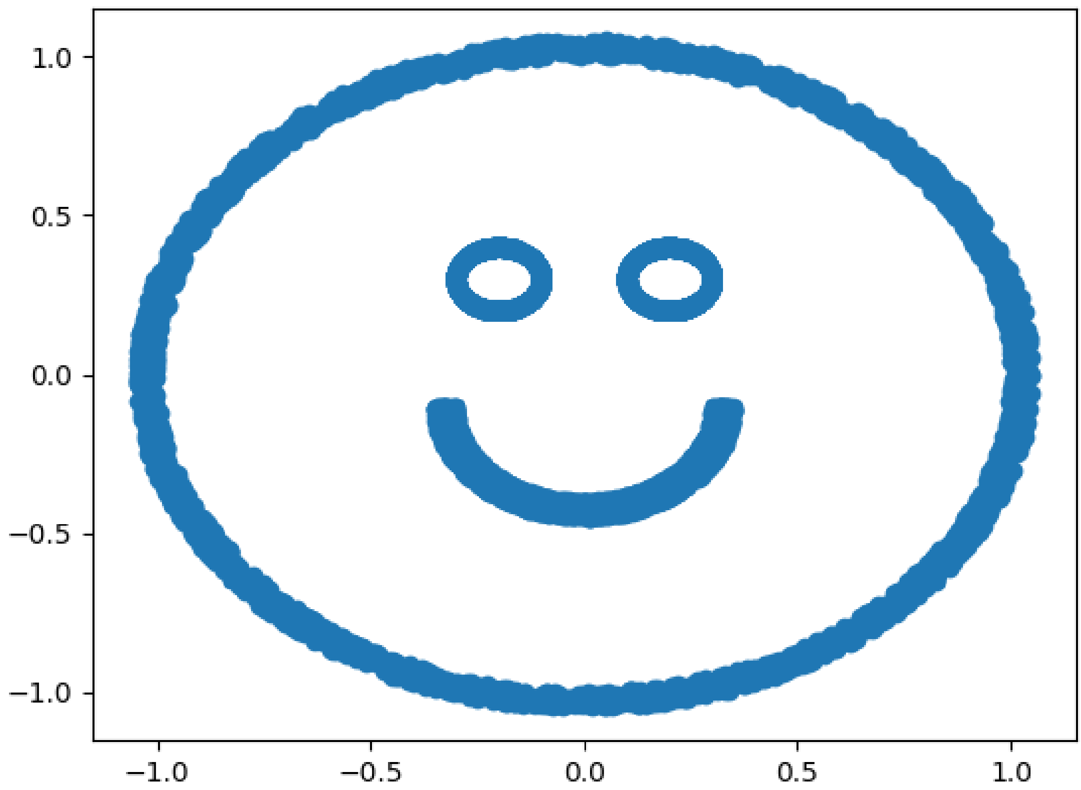
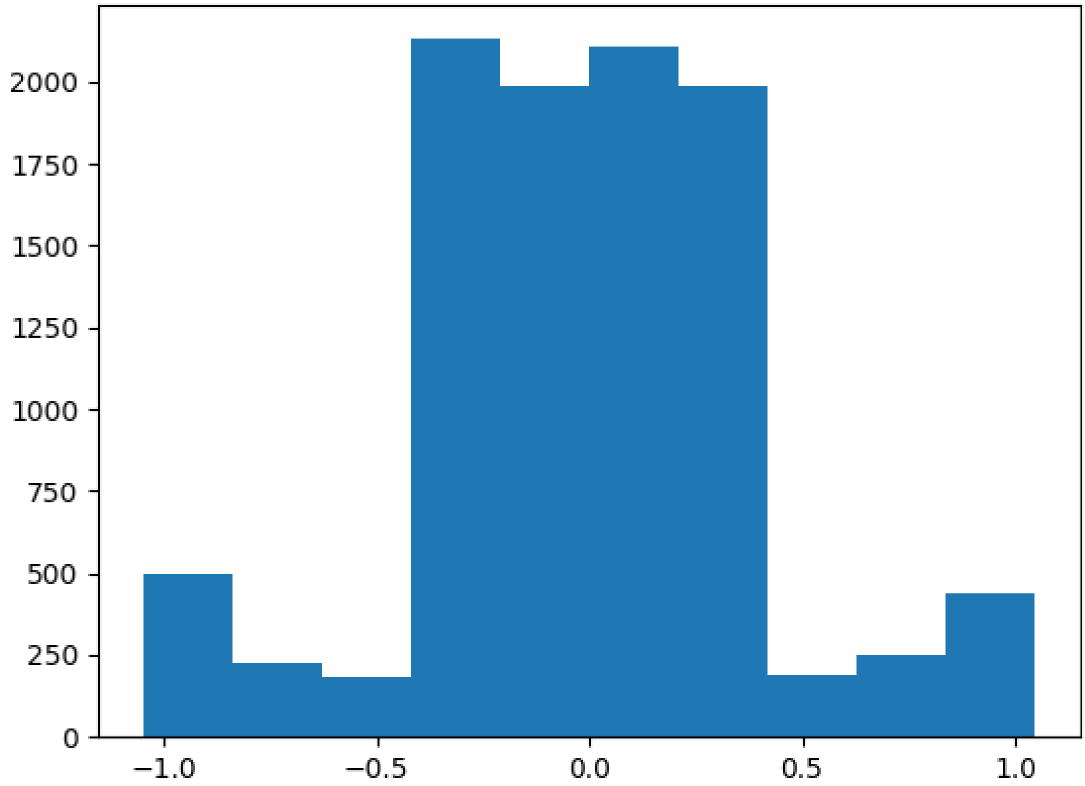
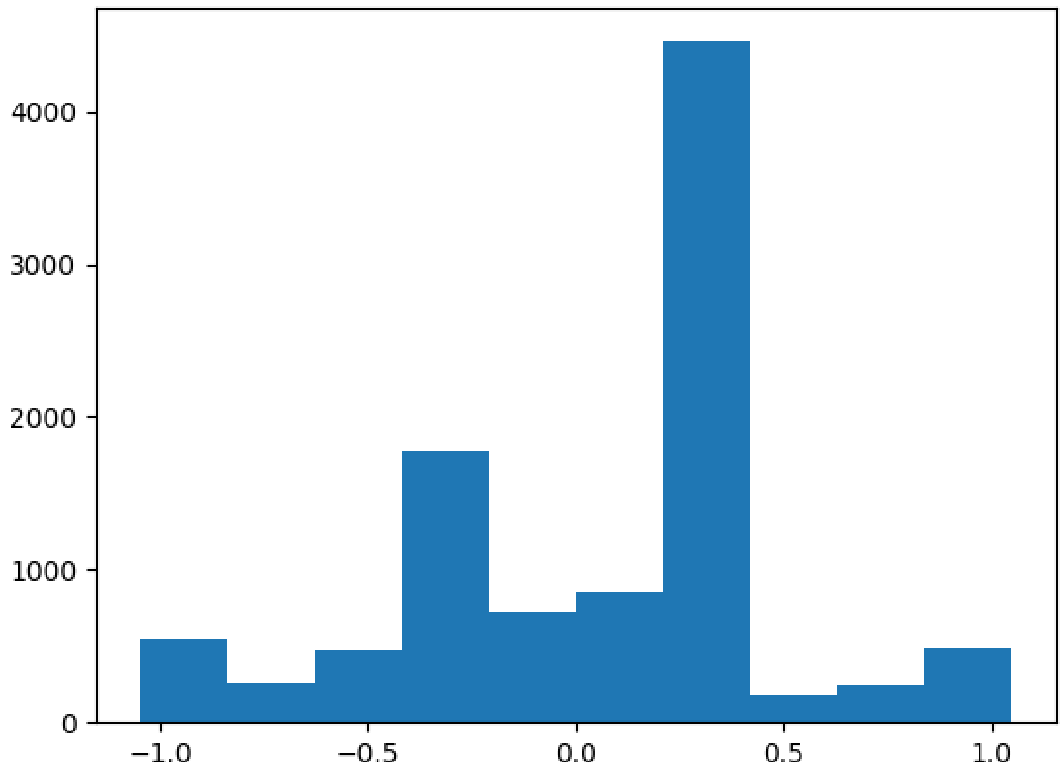
```python
def rxy():
    N = 10000
    face_radius = 1
    eye_radius = 0.1
    mouth_radius = 0.3
    mouth_height = -0.1
    line_thickness = 0.05
    eye_separation = 0.2
    eye_height = 0.3


    part = np.random.randint(low=0, high=4, size=(N))
    outer_ring = np.where(part == 0, 1, 0)
    left_eye = np.where(part == 1, 1, 0)
    right_eye = np.where(part == 2, 1, 0)
    mouth = np.where(part == 3, 1, 0)


    face_angle = np.random.uniform(low=0, high=2*np.pi, size=(N))
    face_r = np.random.uniform(low=face_radius, high=face_radius+line_thickness, size=(N))
    face_x = face_r * np.cos(face_angle)
    face_y = face_r * np.sin(face_angle)

    mouth_angle = face_angle/2.0
    mouth_r = np.random.uniform(low=mouth_radius, high=mouth_radius+line_thickness, size=(N))
    mouth_x = mouth_r * np.cos(mouth_angle)
    mouth_y = mouth_height - mouth_r * np.sin(mouth_angle)

    left_eye_x = eye_radius * np.cos(face_angle) + eye_separation
    right_eye_x = left_eye_x - (2 * eye_separation)
    eye_y = eye_radius * np.sin(face_angle) + eye_height

    x = (outer_ring * face_x) + (left_eye * left_eye_x) + (right_eye * right_eye_x) + (mouth * mouth_x)
    y = (outer_ring * face_y) + (left_eye * eye_y) + (right_eye * eye_y) + (mouth * mouth_y)

    return (x, y)
```



Marginal distribution of X:



Marginal distribution of Y:



4000
3000
2000
1000
0
−1.0
−0.5
0.0
0.5
1.0

3. a. $Pr_{\Theta}(\theta) = \frac{d}{d\theta} P(\Theta \leq \theta)$

$$= \frac{d}{d\theta} \begin{cases} 1-(b_0/\theta)^{\alpha_0} & \text{if } \theta \geq b_0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{d}{d\theta} -b_0^{\alpha_0}\, \theta^{-\alpha_0} & \text{if } \theta \geq b_0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \alpha_0 b_0^{\alpha_0}\, \theta^{-\alpha_0 - 1} & \text{if } \theta \geq b_0 \\ 0 & \text{otherwise} \end{cases}$$

b. $\cancel{Pr_{\Theta}(\theta | x_1, \ldots x_n)}$

For a single observation x,

$$Pr_X(x | \Theta = \theta) = \frac{1}{\theta} \mathbb{1}_{0 \leq x \leq \theta}$$

Assuming independence,

$$Pr(x_1, \ldots x_n | \Theta = \theta) = \left(\frac{1}{\theta}\right)^n \prod_i \mathbb{1}_{0 \leq x_i \leq \theta}$$

$$= \frac{1}{\theta^n} \mathbb{1}_{\min(x) \geq 0} \mathbb{1}_{\max(x) \leq \theta}$$

$$Pr_\theta(\theta | x_1, \dots, x_n) = k\, Pr(x_1, \dots, x_n | \theta)\, Pr_\theta(\theta)$$

by Baye's rule, for some K

$$= K \alpha_0 \frac{b_0^{\alpha_0}}{\theta^{\alpha_0+1}} 1_{\theta \geq b_0} \frac{1}{\theta^n} 1_{\min(x) \geq 0} 1_{\max(x) \leq \theta}$$

$$= K \alpha_0 \frac{b_0^{\alpha_0}}{\theta^{\alpha_0+1+n}} 1_{\theta \geq b_0} 1_{\theta \geq \max(x)}$$

$$= \cancel{K' \alpha_0 b_0^{\alpha_0+n}}$$

$$= K' \frac{1}{\theta^{\alpha_0+n+1}} 1_{\theta \geq \max(b_0, \max(x))}$$

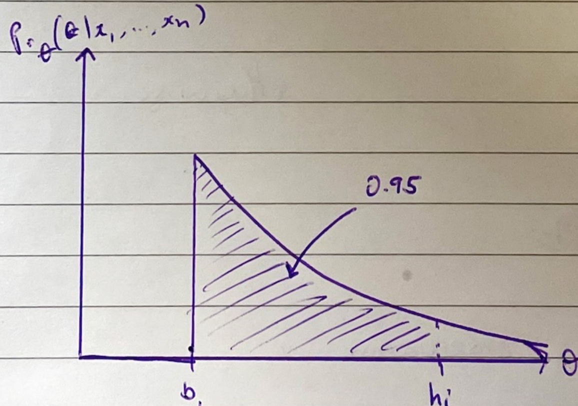
$\therefore (\theta | x_1, \dots, x_n) \sim \text{Pareto}(\alpha_0 + n, \max(b_0, \max x))$

Let $a_1 = \alpha_0 + n$

$b_1 = \max(b_0, \max x)$

$\therefore (\theta | x_1, \dots, x_n) \sim \text{Pareto}(a_1, b_1)$

c.



$P(b_1 \leq h_i | \text{data}) = 0.95$

$\therefore 1 - \left(\frac{b_1}{h_i}\right)^{\alpha_1} = 0.95$

$\therefore h_i = b_1 (0.05)^{-1/\alpha_1}$

5. ~~Let X = number of failures~~
~~X ~ Bin(n, θ)~~

~~Pdx~~

$$P(n \text{ failures} \mid \theta) = \theta^n$$

$$Pr_\theta(\theta \mid n \text{ failures}) = K \cancel{P} \, P(n \text{ failures} \mid \theta) \, Pr_\theta(\theta) \text{ by Bayes}$$

for some K

$$= K \theta^n \left(\varepsilon \mathbb{1}_{\theta \le 1/2} + (2-\varepsilon) \mathbb{1}_{\theta > 1/2} \right)$$

$$\int_0^1 K \theta^n \left(\varepsilon \mathbb{1}_{\theta \le 1/2} + (2-\varepsilon) \mathbb{1}_{\theta > 1/2} \right) d\theta = 1$$

$$\therefore K \left(\varepsilon \int_0^{1/2} \theta^n \, d\theta + (2-\varepsilon) \int_{1/2}^1 \theta^n \, d\theta \right) = 1$$

$$\therefore K \left(\varepsilon \left[\frac{\theta^{n+1}}{n+1} \right]_0^{1/2} + (2-\varepsilon) \left[\frac{\theta^{n+1}}{n+1} \right]_{1/2}^1 \right) = 1$$

$$\therefore K \left(\frac{\varepsilon}{2^{n+1}(n+1)} + \frac{2-\varepsilon}{n+1} - \frac{2-\varepsilon}{2^{n+1}(n+1)} \right) = 1$$

$$\therefore K \left(\frac{\varepsilon + 2^{n+1}(2-\varepsilon) - 2 + \varepsilon}{(n+1)2^{n+1}} \right) = 1$$

$$\therefore K = \frac{(n+1)2^{n+1}}{\varepsilon + 2^{n+1}(2-\varepsilon) - 2 + \varepsilon} = \frac{(n+1)2^{n+1}}{2\varepsilon + 2^{n+1}(2-\varepsilon) - 2}$$

$$\therefore Pr_\theta(\theta \mid n \text{ failures}) = \frac{(n+1)2^{n+1}}{2\varepsilon + 2^{n+1}(2-\varepsilon) - 2} \theta^n \left(\varepsilon \mathbb{1}_{\theta \le 1/2} + (2-\varepsilon) \mathbb{1}_{\theta > 1/2} \right)$$

$$\therefore P(\theta \le 1/2) = \int_0^{1/2} \frac{(n+1)2^{n+1}}{2\varepsilon + 2^{n+1}(2-\varepsilon) - 2} \theta^n \varepsilon \, d\theta$$

$$= \frac{\varepsilon(n+1)2^{n+1}}{2\varepsilon + 2^{n+1}(2-\varepsilon) - 2} \left[\frac{\theta^{n+1}}{n+1} \right]_0^{1/2}$$

$$= \text{''} \quad \frac{1}{(n+1)2^{n+1}} = \frac{\varepsilon}{2\varepsilon + 2^{n+1}(2-\varepsilon) - 2}$$

$$= 0.5$$

$$\therefore \varepsilon + 2^n (2-\varepsilon) - 1 = \varepsilon$$

$$\therefore 2^n = \frac{1}{2-\varepsilon}$$

$$\therefore n = \log_2 \frac{1}{2-\varepsilon}$$

8a. $Pr_{M,H}(\mu, h) = Pr_M(\mu)\, Pr_H(h)$ assuming independence

$$= \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\mu-5}{3}\right)^2} \times \left(0.99 \times 1_{h=\text{healthy}} + 0.01 \times 1_{h=\text{sick}}\right)$$

b. $Pr(x_1, \ldots, x_{30}, y \mid \mu, h) = \left(\prod_i Pr(x_i \mid \mu)\right) Pr(y \mid h, \mu)$ assuming independence

$$= \left(\prod_i \frac{1}{3.2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{3.2}\right)^2}\right) \cdot \left(\frac{1}{2.1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{2.1}\right)^2} 1_{h=\text{healthy}} + \frac{1}{3.2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{3.2}\right)^2} 1_{h=\text{sick}}\right)$$

$$= \left(\frac{1}{3.2\sqrt{2\pi}}\right)^{30} e^{-\frac{1}{2(3.2)^2}\sum_i (x_i-\mu)^2}$$

c. $Pr_{M,H}(\mu, h \mid x_1, \ldots, x_{30}, y) = k\, Pr(x_1, \ldots x_{30}, y \mid \mu, h)\, Pr_{M,H}(\mu, h)$

by Bayes' rule for some k

$$= k' e^{-\frac{1}{2(3.2)^2}\sum_i (x_i-\mu)^2} e^{-\frac{1}{2}\left(\frac{\mu-5}{3}\right)^2} \left(0.99 \cdot 1_{h=\text{healthy}} + 0.01 \cdot 1_{h=\text{sick}}\right)$$

$$\cdot \left(\frac{1}{2.1} e^{-\frac{1}{2}\left(\frac{y}{2.1}\right)^2} 1_{h=\text{healthy}} + \frac{1}{3.2} e^{-\frac{1}{2}\left(\frac{y-\mu}{3.2}\right)^2} 1_{h=\text{sick}}\right)$$

// TODO

$temp \sim N(\alpha + \beta_1 \sin(2\pi t) + \beta_2 \cos(2\pi t) + \gamma(t - 2000), \sigma^2)$

//TODO

1a. $\log \Pr(x_1, \dots x_n) = \log\left(\prod_i \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2}\right)$ assuming independence

$$= \text{constant} + \sum_i -\tfrac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2 + \log(\sigma^{-n})$$

$$= \text{constant} - \frac{1}{2\sigma^2} \sum_i (x_i^2 + \mu^2 - 2x_i\mu) - n\log\sigma$$

$$= \text{constant} - \frac{1}{2\sigma^2}(n\overline{x^2} + n\mu^2 - 2n\bar{x}\mu) - n\log\sigma$$

$$\frac{\partial \log \Pr(\dots)}{\partial \mu} = \frac{1}{2\sigma}(2n\mu - 2n\bar{x}) \overset{!}{=} 0$$

$$\therefore \mu = \bar{x}$$

$$\frac{\partial \log \Pr(\dots)}{\partial \sigma} = (n\overline{x^2} + n\mu^2 - 2n\bar{x}\mu) \cdot \frac{1}{4\sigma^3} - \frac{n}{\sigma} \overset{!}{=} 0$$

$$\therefore \overline{x^2} + \mu^2 - 2\bar{x}\mu = 4\sigma^2 = \bar{x}^2 - \overline{x^2}$$

$$\therefore \sigma = \frac{\sqrt{\bar{x}^2 - \overline{x^2}}}{2}$$

```python
import numpy as np

def sigma(x):
    mu_mle = np.mean(mu)
    sigma_mle = np.sqrt(mu_mle * mu_mle - np.mean(x*x)) * 0.5
    return sigma_mle


def parametric_sample(x):
    n = x.size[0]
    mu_mle = np.mean(mu)
    sigma_mle = sigma(x)
    new_x = np.random.normal(mu_mle, sigma_mle, size=(n, 100000))
    new_sigma = sigma(new_x)
    plt.hist(new_sigma) # Mark off confidence interval from this histogram
```