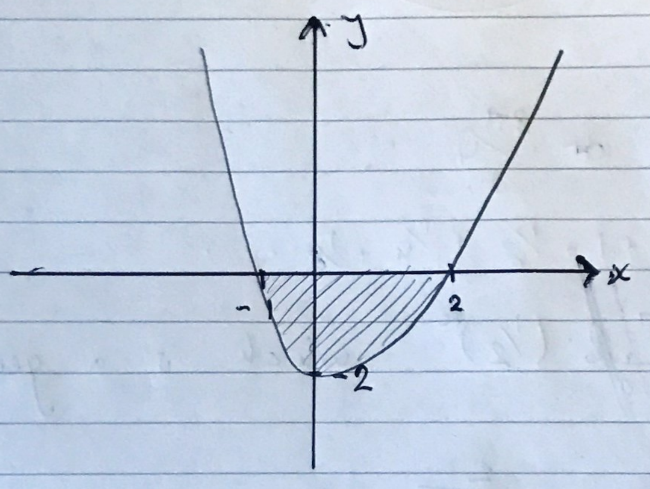
BGN: 2031B
Paper 1
Section A

1. $y^2 = (\sin x^3 + \cos 2x)^2$

$\therefore \frac{d}{dx} y^2 = 2(\sin x^3 + \cos x)(3x^2 \cos x^3 - 2\sin 2x)$

2. $(x^2 - x - 2) = (x+1)(x-2)$



$\therefore x^2 - x - 2 < 0 \iff$ ~~$x < -1 \cup x > 2$~~
$-1 < x < 2$

3. $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1 \Rightarrow \cos^2\theta = \frac{\cos 2\theta + 1}{2}$
$= 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$\therefore \sin^4\theta + \cos^4\theta = \left(\frac{1-\cos 2\theta}{2}\right)^2 + \left(\frac{1+\cos 2\theta}{2}\right)^2$

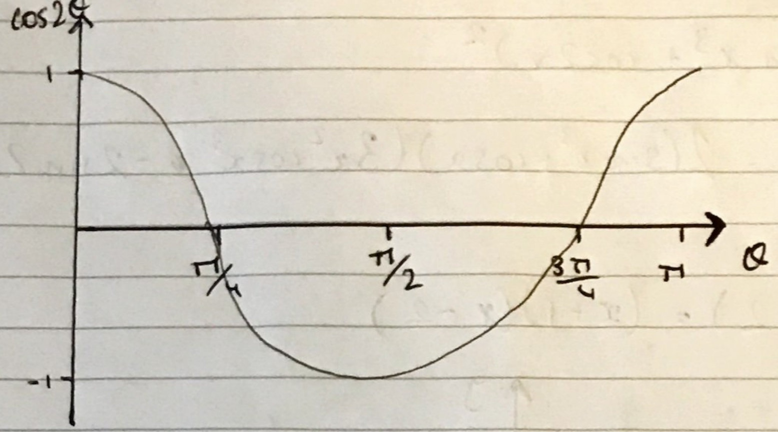
$= \frac{1 + \cos^2 2\theta - 2\cos 2\theta + 1 + \cos^2 2\theta + 2\cos 2\theta}{4}$

$= \frac{2 + 2\cos^2 2\theta}{4} = \frac{1 + \cos^2 2\theta}{2} \doteq \frac{1}{2}$

$\therefore \ 1 + \cos^2 2\theta = 1$

$\therefore \ \cos^2 2\theta = 0$

$\therefore \ \cos 2\theta = 0$



$\therefore \ \theta = \pi/4$ or $3\pi/4$

4. Let $S = 2 + 1 + 1/2 + 1/4 + 1/8 \dots$

$= \sum_{k=0}^{\infty} 2 \cdot (1/2)^k$ which is a geometric series

$= \frac{2}{1 - 1/2} = \frac{2}{(1/2)} = 4$

Let $R = 2 - 1 + 1/2 - 1/4 + 1/8 \dots$

$= \sum_{k=0}^{\infty} 2 (-1/2)^k$ which is a geometric series

$= \frac{2}{1 - (-1/2)} = \frac{2}{1 + 1/2} = \frac{2}{(3/2)} = 4/3$

5. $f(x) = xe^{-2x^2}$

$$f'(x) = e^{-2x^2} + x(-4x)e^{-2x^2}$$

$$= (1-4x^2)e^{-2x^2}$$

$$\stackrel{!}{=} 0$$

$\therefore 1-4x^2 = 0$

$\therefore 4x^2 = 1$

$\therefore x^2 = 1/4$

$\therefore x = \pm 1/2 \qquad \therefore x = 1/2$ given $0 \leq x \leq 1$

$f(1/2) = 1/2\, e^{-2/4} = 1/2\, e^{-1/2}$

$\cancel{f(-1/2) = -1/2\, e^{-1/2}}$

$f(0) = 0$

$f(1) = e^{-2}$

$\therefore$ The minimum value occurs at $x = 0$ and is $f(0) = 0$

The maximum value occurs at $x = 1/2$ and is $f(1/2) = \dfrac{1}{2\sqrt{e}}$

6. Let $\vec{a} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\vec{b} = \begin{pmatrix}2\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$

$\hat{a} = \frac{1}{|\vec{a}|}\vec{a} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$\hat{b} = \frac{1}{|\vec{b}|}\vec{b} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

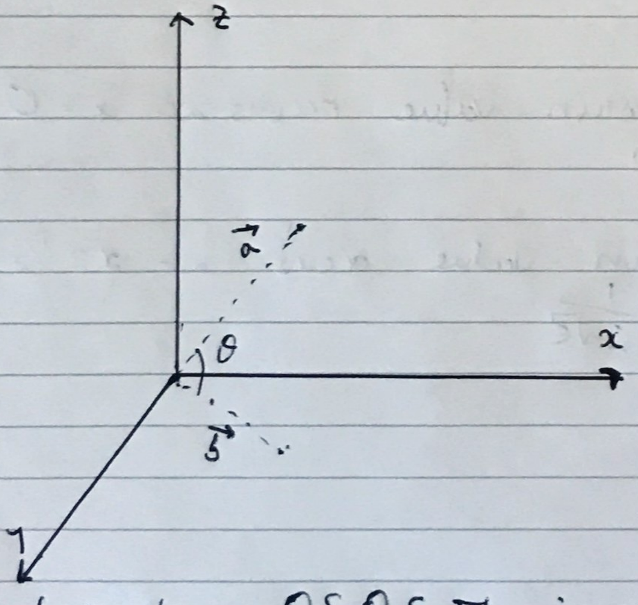
$\cos\theta = \hat{a}\cdot\hat{b} = \frac{1}{\sqrt{6}}(1+1+0) = \frac{2}{\sqrt{2}\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$

$= \sqrt{\frac{2}{3}}$

$\therefore \cos^2\theta = \frac{2}{3}$

$\therefore \sin^2\theta = 1 - \cos^2\theta = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore \sin\theta = \pm\frac{1}{\sqrt{3}}$



By observation, $0 \leq \theta \leq \pi$ $\therefore \sin\theta \geq 0$

$\therefore \sin\theta = \frac{1}{\sqrt{3}}$

7. $f(x) = x^3 - 5x - 1$

$f'(x) = 3x^2 - 5$

Stationary points at $f'(x) = 3x^2 - 5 = 0$

$\therefore 3x^2 = 5$

$\therefore x^2 = 5/3$

$\therefore x = \pm\sqrt{5/3}$ $= \pm\sqrt{1.\dot{6}}$

$\therefore 1 < \sqrt{5/3} < 2$

$-2 < -\sqrt{5/3} < -1$

$f(0) = -1$

$f(3) = 11$

$f(2) = -3$

$f(1) = -5$

$f(-1) = 3$

$f(-2) = 1$

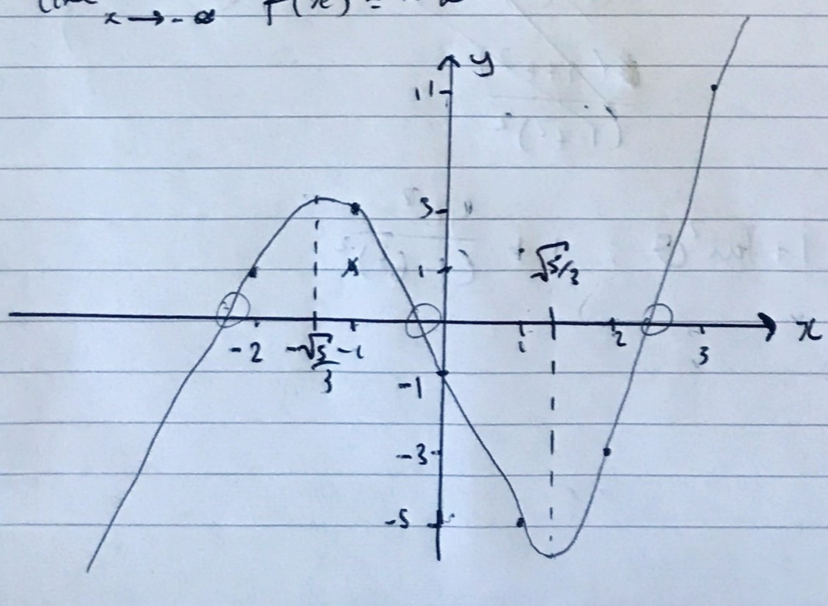
$f''(x) = 6x$

$\therefore f''(-\sqrt{5/3}) < 0 \quad \therefore$ Maximum at $x = -\sqrt{5/3}$

$f''(\sqrt{5/3}) > 0 \quad \therefore$ Minimum at $x = \sqrt{5/3}$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$



There are 3 real solutions

8. $I = \int x^3 e^{-x^4}\,dx$

$= -\frac{1}{4}\int -4x^3 e^{-x^4}\,dx$

$\frac{d}{dx} e^{-x^4} = -4x^3 e^{-x^4}$

$\therefore\ I = -\frac{1}{4} e^{-x^4} + C$

9. $\tan\theta = \dfrac{\tan\theta/2 + \tan\theta/2}{1 - \tan^2\theta/2}$ by the compound angle formula $\tan(A+B)$ $\cancel{\frac{\tan A + \tan B}{1-\tan A \tan B}}$

$= \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \dfrac{2t}{1-t^2}$

$\cancel{\sec^2\theta = \frac{d}{d\theta}\tan\theta = \frac{d}{d\theta}\frac{2t}{1-t^2}}$

$\cancel{= \frac{2(1-t^2) - 2t(-2t)}{(1-t^2)^2}\cdot\frac{dt}{d\theta}}$

$\cancel{= \frac{2 - 2t^2 + 4t^2}{(1-t^2)^2}\cdot \frac{1}{2}\sec^2\frac{\theta}{2}}$

$\cancel{= \frac{(1+t^2)}{(1-t^2)^2}}$

$\sec^2\theta = 1 + \tan^2\theta = 1 + \dfrac{4t^2}{(1-t^2)^2}$

10. $9x^2+16y^2=25$

$$\therefore \frac{d}{dx}(9x^2+16y^2)=\frac{d}{dx}25$$

$$\therefore 18x+32y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{18x}{32y}=-\frac{9x}{16y}$$

$$\therefore \left.\frac{dy}{dx}\right|_{(x=1,\ y=1)}=-\frac{9}{16}$$