The Lemon Theorem

Consider a function defined on the integers ≥ 2:

$$f(x) = \frac{\sum_{k=0}^{x-2}(k+1)x^k}{\sum_{k=0}^{x-2}(x-1-k)x^k}$$

E.g. $f(10) = \dfrac{987654321}{123456789}$

The lemon theorem: $f(x) \simeq x-2$ for large x

Proof:

$$a(x) = \sum_{k=0}^{x-2}(k+1)x^k$$

$$b(x) = \sum_{k=0}^{x-2}(x-k-1)x^k$$

Note that $f(x) = \dfrac{a(x)}{b(x)}$

Consider the expression $a(x) - b(x)(x-2)$

$$a(x)-b(x)(x-2) = \left(\sum_{k=0}^{x-2}(k+1)x^k\right) - x\left(\sum_{k=0}^{x-2}(x-k-1)x^k\right) + 2\left(\sum_{k=0}^{x-2}(x-k-1)x^k\right)$$

$$= \left(\sum_{k=0}^{x-2}(k+1)x^k\right) - \left(\sum_{k=0}^{x-2}(x-2)(x-k-1)x^k\right)$$

$$= \left(\sum_{k=0}^{x-2}(k+1)x^k\right) - \left(\sum_{k=0}^{x-2}(x^2-(k+3)x+2(k+1))x^k\right)$$

$$= \left(\sum_{k=0}^{x-2}(k+1)x^k\right) - \left(\sum_{k=0}^{x-2}x^{k+2} - (k+3)x^{k+1} + 2(k+1)x^k\right)$$

$$= \left(\sum_{k=0}^{x-2}(k+1)x^k\right) - \left(\sum_{k=0}^{x-2}x^{k+2}\right) + \left(\sum_{k=0}^{x-2}(k+3)x^{k+1}\right) - \left(\sum_{k=0}^{x-2}2(k+1)x^k\right)$$

$$= -\left(\sum_{k=0}^{x-2} x^{k+2}\right) + \left(\sum_{k=0}^{x-2} (k+3)x^{k+1}\right) - \left(\sum_{k=0}^{x-2} (k+1)x^{k}\right)$$

$$= -\left(\sum_{k=2}^{x} x^{k}\right) + \left(\sum_{k=1}^{x-1} (k+2)x^{k}\right) - \left(\sum_{k=0}^{x-2} (k+1)x^{k}\right)$$

$$= -\left(x^{x-1} + x^{x} + \sum_{k=2}^{x-2} x^{k}\right) + \left(3x + (x+1)x^{x-1} + \sum_{k=2}^{x-2} (k+2)x^{k}\right) - \left(1 + 2x + \sum_{k=2}^{x-2} (k+1)x^{k}\right)$$

$$= -x^{x-1} - x^{x} + 3x + (x+1)x^{x-1} - 1 - 2x + \left(\sum_{k=2}^{x-2} -x^{k}\right) + \left(\sum_{k=2}^{x-2} (k+2)x^{k}\right) + \left(\sum_{k=2}^{x-2} -(k+1)x^{k}\right)$$

$$= -x^{x-1} - x^{x} + x + x^{x} + x^{x-1} - 1 + \sum_{k=2}^{x-2}\left(-x^{k} + (k+2)x^{k} - (k+1)x^{k}\right)$$

$$= x - 1 + \sum_{k=2}^{x-2} x^{k}(k+2-k-1-1)$$

$$= x - 1 + \sum_{k=2}^{x-2} x^{k}(0)$$

$$= x - 1 + \sum_{k=2}^{x-2} 0$$

$$= x - 1$$

$\therefore a(x) - b(x)(x-2) = x - 1$

$\therefore a(x) = x - 1 + b(x)(x-2)$

$\therefore \dfrac{a(x)}{b(x)} = \dfrac{x-1}{b(x)} + x - 2$

$\therefore f(x) = \dfrac{x-1}{b(x)} + x - 2$

Consider the term $\frac{x-1}{b(x)}$

$x-1$, when written in base-x positional notation, is a single digit number

$b(x)$, when written in base-x positional notation, is an $x-1$ digit number, with a leading digit of 1.

These two facts can be written as follows:

$$x-1 < 10_x$$

$$b(x) \geq \overbrace{1000\ldots0}^{x-1}{}_x$$

$$\therefore \frac{x-1}{b(x)} \leq \frac{10_x}{\overbrace{1000\ldots0}^{x-1}{}_x}$$

$$\therefore \frac{x-1}{b(x)} \leq \frac{1}{\overbrace{1000\ldots0}^{x-2}{}_x}$$

This shows that $\frac{x-1}{b(x)}$ must get smaller as x gets larger, and $\frac{x-1}{b(x)}$ gets arbitrarily close to 0 as x approaches infinity.

$$\therefore \frac{x-1}{b(x)} \simeq 0 \text{ for large } x$$

$$\therefore \frac{x-1}{b(x)} + x - 2 \simeq x-2 \text{ for large } x$$

$$\therefore f(x) \simeq x-2 \text{ for large } x$$

QED.