Polynomial-Space

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1 Introduction

This paper outlines an inner product space with an infinite number of dimensions. Each of these dimensions represents a unique rational power of x, such that every possible rational power of x is represented by an axis in the space. Here, x does not refer to either a constant, a variable or a coordinate but instead refers to an abstract object. In such a space, each vector represents some polynomial function. Each component of the vector represents x raised to some power (this power is determined by the position of the component within the vector) multiplied by the value of this component. For example, if in this space, the 0^{th} component of a vector represented the coefficient of x^0, the 1^{st} component represented the coefficient of x^1, the 2^{nd} represented the coefficient of x^2 and so on, with the n^{th} component representing the coefficient of x^n, then, for example, the polynomial $3x^2 + 2x + 1$ would be represented by the vector:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

with an infinite number of zeroes filling up the components which aren't shown. For the rest of this paper, assume that any vector or matrix entries which are not shown have a value of zero. A vector space such as this in which all vectors represent polynomials would obey the intuitive rules of addition such that the vector which represents the sum of two polynomials would be the same as the sum of the vectors which represent the two polynomials. For example, if we take two polynomials, $x + 2$, and $5x^2 + 1$, the vectors which represent them would be

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 5 \\ \vdots \end{bmatrix} \text{ respectively}$$

The sum of these two vectors is

$$\begin{bmatrix} 3 \\ 1 \\ 5 \\ \vdots \end{bmatrix}$$

which represents the polynomial $5x^2 + x + 3$ which is the sum of the two original polynomials. Unfortunately, a space such as this wherein the n^{th} component of a vector represents the coefficient of x^n only leaves room for natural powers of x[1]. Instead, we want a vector which represents polynomials containing all rational powers of x. As such, we need to design a function $r(n)$ whose domain is the set of natural numbers, and which returns every rational number such that every rational number has a unique, finite and calculable index n. With such a function defined, we can create a space wherein the n^{th} component of a vector represents the coefficient of $x^{r(n)}$.

2 Finding an $r(n)$ function

To recap, we need a function $r(n)$ which has the following properties:

- Every single rational number (including negative rational numbers and 0) has one and only one natural index n, for which $r(n)$ gives that rational number.
- The value of $r(n)$ for every natural number n gives a unique rational number.

[1] In this context, and for the rest of this paper, the natural numbers includes 0

This section describes one such function and a proof that it obeys these properties. The remainder of this paper works with this one specific $r(n)$ function defined within this section, however most of the formulae are written in terms of $r(n)$ and any function which obeys the two properties will work just as well. The $r(n)$ function which we're about to define is designed from the following table.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
2	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50
3	0.00	0.33	0.67	1.00	1.33	1.67	2.00	2.33	2.67	3.00
4	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
5	0.00	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80
6	0.00	0.17	0.33	0.50	0.67	0.83	1.00	1.17	1.33	1.50
7	0.00	0.14	0.29	0.43	0.57	0.71	0.86	1.00	1.14	1.29
8	0.00	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13
9	0.00	0.11	0.22	0.33	0.44	0.56	0.67	0.78	0.89	1.00
10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90

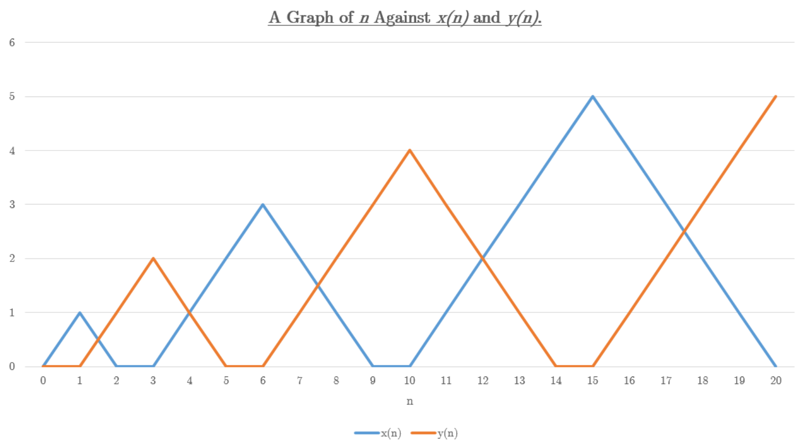
Along the top is the set of natural numbers (including 0), and along the left is the set of positive integers (excluding 0). In the cell directly underneath the number a and directly to the right of the number b, the value is $\frac{a}{b}$. As such, any rational number $\frac{a}{b}$ can be found in the cell with coordinates $(a, b-1)$ indexing from $(0,0)$. We can join these cells in such an order that although the table extends infinitely to the right and downwards, each cell is reached. This order is shown with these arrows.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
1	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
2	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50
3	0.00	0.33	0.67	1.00	1.33	1.67	2.00	2.33	2.67	3.00
4	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
5	0.00	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80
6	0.00	0.17	0.33	0.50	0.67	0.83	1.00	1.17	1.33	1.50
7	0.00	0.14	0.29	0.43	0.57	0.71	0.86	1.00	1.14	1.29
8	0.00	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13
9	0.00	0.11	0.22	0.33	0.44	0.56	0.67	0.78	0.89	1.00
10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90

This pattern of diagonal arrows continues infinitely, and will pass through every cell once. We need to define some functions which map from a distance along this diagonal curve to x and y coordinates. Below is a table of values which demonstrates exactly what these functions need to do.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
x(n)	0	1	0	0	1	2	3	2	1	0	0	1	2	3	4	5	4	3	2	1	0	...
y(n)	0	0	1	2	1	0	0	1	2	3	4	3	2	1	0	0	1	2	3	4	5	...

The following graph highlights the pattern.



We can group the values of $x(n)$ and $y(n)$ into sections by seeing the points at which the function returns a value of 0. The groups look like this:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
x(n)	0	1	0	0	1	2	3	2	1	0	0	1	2	3	4	5	4	3	2	1	0
		4(0)+3					4(1)+3									4(2)+3					
y(n)	0	0	1	2	1	0	0	1	2	3	4	3	2	1	0	0	1	2	3	4	5
	4(0)+1			4(1)+1							4(2)+1										

As you can see, the k^{th} group of $x(n)$ is $4k+3$ values long, and the k^{th} group of $y(n)$ is $4k+1$ values long. In general, we can model both functions as having group lengths $4k+\Delta$, where for $x(n)$, $\Delta = 3$ and for $y(n)$, $\Delta = 1$ This means that in the first j groups combined, the number of values is equal to $\sum_{k=0}^{j} 4k+\Delta$. With some rearrangement, this becomes:

$$j\Delta + \Delta + 4\sum_{k=0}^{j} k \tag{1}$$

Since the sum of the first j natural numbers is $\frac{j^2+j}{2}$, this equation becomes the quadratic:

$$2j^2 + (2+\Delta)j + \Delta \tag{2}$$

If we set this equal to some number n, and solve for j, we get:

$$j = \frac{-2-\Delta+\sqrt{\Delta^2-4\Delta+4+8n}}{4} \tag{3}$$

And since we're only ever going to be using $\Delta = 1$ or $\Delta = 3$, we can simplify further to:

$$j = \frac{-2-\Delta+\sqrt{1+8n}}{4} \tag{4}$$

The significance of this equation is that given an index n, we can calculate how many groups of values came before it. Since this group number should always be an integer, we round j down to the next integer.

$$j = \left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor \tag{5}$$

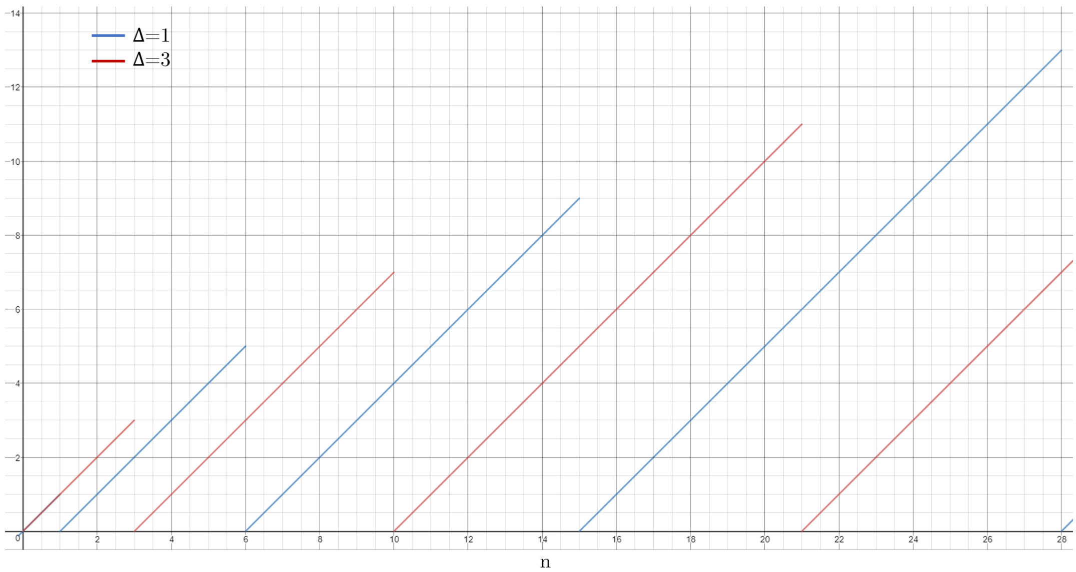
We can now plug this value of j back into the expression which finds the total number of values in the first j groups — expression (2) — giving us:

$$2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor^2 + (2+\Delta)\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + \Delta \tag{6}$$

This expression tells you the total number of values in all of the groups before the n^{th} value. If we subtract it from n, we get the distance through its group the n^{th} value appears. This expression looks like the following:

$$n - 2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor^2 - (2+\Delta)\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor - \Delta \tag{7}$$

The graph of this equation for $\Delta = 1$ and $\Delta = 3$ looks like this:



These are close to our $x(n)$ and $y(n)$ functions, except these continue increasing all the way through the group, whereas $x(n)$ and $y(n)$ increase until they reach the center of the group, and then start decreasing. To achieve this effect, we must first come up with an expression which determines the center of the group. This is half the length of the group. Since the current group number is:

$$\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + 1 \tag{8}$$

And the length of the k^{th} group is $4k+\Delta$, the length of the current group is:

$$4\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + 4 + \Delta \tag{9}$$

Therefore, the midpoint of the group is:

$$2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + 2 + \frac{\Delta}{2} \tag{10}$$

Rounding this down to the next integer gives the index which yields the highest value in the group.

$$\left\lfloor 2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + 2 + \frac{\Delta}{2} \right\rfloor \tag{11}$$

This can be rewritten as:

$$2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + 2 + \left\lfloor \frac{\Delta}{2} \right\rfloor \tag{12}$$

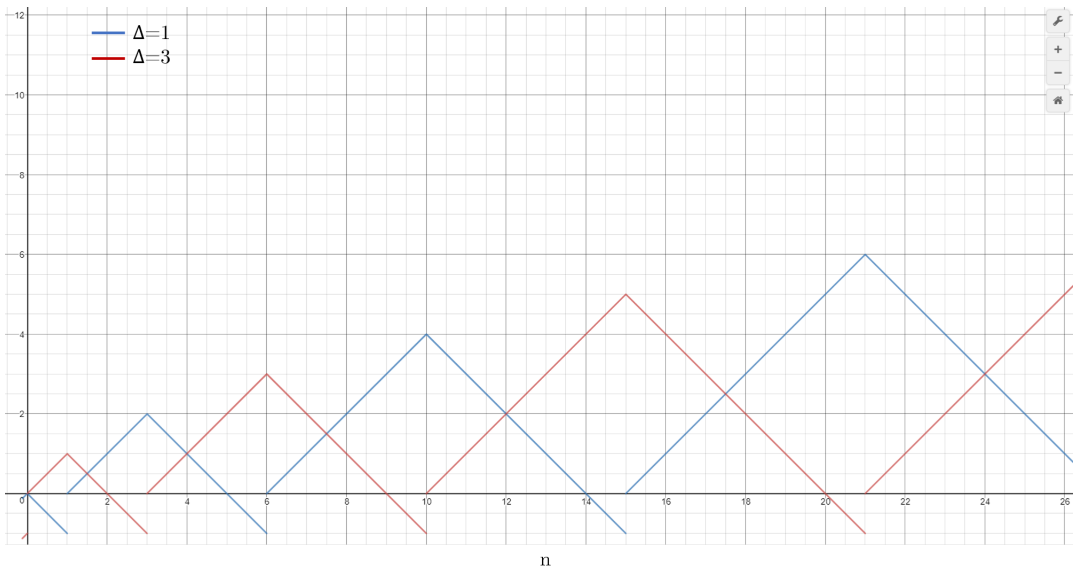
We now need an expression equal to expression (7) except it reverses direction when it reaches the value of expression (12). If we call expression (7) a and expression (12) b, the expression we want can be written as:

$$b - |a - b| \tag{13}$$

Substituting a and b, we get

$$2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor + 2 + \left\lfloor \frac{\Delta}{2} \right\rfloor - \left| n - 2\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor^2 - (4+\Delta)\left\lfloor \frac{-2-\Delta+\sqrt{1+8n}}{4} \right\rfloor - 2 - \Delta - \left\lfloor \frac{\Delta}{2} \right\rfloor \right| \tag{14}$$

Below is a graphical representation of this expression with $\Delta = 1$ and $\Delta = 3$.



For integer values of n, this yields exactly the same results as we want our $x(n)$ and $y(n)$ functions to yield. As such, we can define these functions as this expression with $\Delta = 3$ for $x(n)$ and $\Delta = 1$ for $y(n)$.

$$x(n) = 2\left\lfloor \frac{-5+\sqrt{1+8n}}{4} \right\rfloor + 3 - \left| n - 2\left\lfloor \frac{-5+\sqrt{1+8n}}{4} \right\rfloor^2 - 7\left\lfloor \frac{-5+\sqrt{1+8n}}{4} \right\rfloor - 6 \right| \tag{15}$$

$$y(n) = 2\left\lfloor \frac{-3+\sqrt{1+8n}}{4} \right\rfloor + 2 - \left| n - 2\left\lfloor \frac{-3+\sqrt{1+8n}}{4} \right\rfloor^2 - 5\left\lfloor \frac{-3+\sqrt{1+8n}}{4} \right\rfloor - 3 \right| \tag{16}$$

These equations give the grid coordinates of the number n units down the diagonal line in the grid of rational numbers. However, the grid coordinates are not as useful as the numerator and denominator of that n^{th} number. The numerator is the x coordinate, but the denominator is the y coordinate plus 1, because the numbers down the left hand side start from 1, not from 0. As such, we can change the definition of $y(n)$ such that its value is 1 greater as such:

$$y(n) = 2\left\lfloor \frac{-3+\sqrt{1+8n}}{4} \right\rfloor + 3 - \left| n - 2\left\lfloor \frac{-3+\sqrt{1+8n}}{4} \right\rfloor^2 - 5\left\lfloor \frac{-3+\sqrt{1+8n}}{4} \right\rfloor - 3 \right| \tag{17}$$

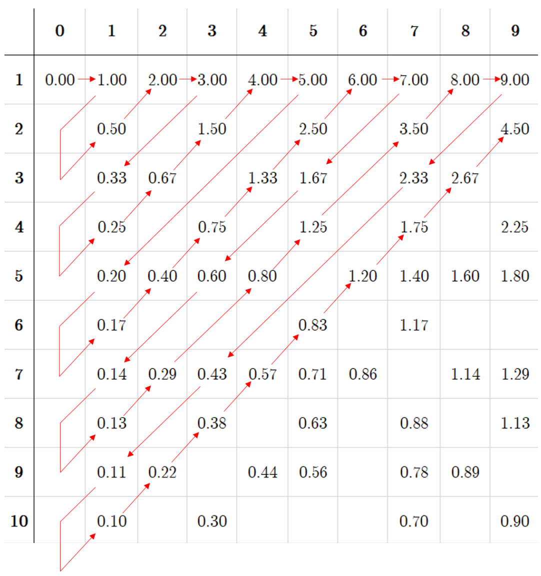
This means that the n^{th} number along the diagonal line is equal to $\frac{x(n)}{y(n)}$. There is one major problem with this function and it is that rational numbers appear more than once. In graphical terms, this means that instead of using this grid:

	0	1	2	3	4	5	6	7	8	9
1	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
2	0.00	0.50	1.00	1.50	2.00	2.50	3.00	3.50	4.00	4.50
3	0.00	0.33	0.67	1.00	1.33	1.67	2.00	2.33	2.67	3.00
4	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25
5	0.00	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80
6	0.00	0.17	0.33	0.50	0.67	0.83	1.00	1.17	1.33	1.50
7	0.00	0.14	0.29	0.43	0.57	0.71	0.86	1.00	1.14	1.29
8	0.00	0.13	0.25	0.38	0.50	0.63	0.75	0.88	1.00	1.13
9	0.00	0.11	0.22	0.33	0.44	0.56	0.67	0.78	0.89	1.00
10	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90

We would use this grid:

	0	1	2	3	4	5	6	7	8	9
1	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
2		0.50		1.50		2.50		3.50		4.50
3		0.33	0.67		1.33	1.67		2.33	2.67	
4		0.25		0.75		1.25		1.75		2.25
5		0.20	0.40	0.60	0.80		1.20	1.40	1.60	1.80
6		0.17				0.83		1.17		
7		0.14	0.29	0.43	0.57	0.71	0.86		1.14	1.29
8		0.13		0.38		0.63		0.88		1.13
9		0.11	0.22		0.44	0.56		0.78	0.89	
10		0.10		0.30				0.70		0.90

In which all cells in which the number directly above and the number directly to the left share a factor greater than one is left blank. This means that all non-negative rational numbers appear only once. This means that the diagonal line passing through all the cells would look like the following:



This means that we need to design some function $d(n)$ which translates from an absolute distance n through the line (which includes blank spaces) to a relative distance $d(n)$ (which skips over them). This table of values shows how this function should behave.

n	0	1	2	3	4	5	6	7	8	9
d(n)	0	1	4	5	6	8	11	12	13	14

In order to create a function which yields these values, we first need to create an expression which equals 1 if cell j is blank and 0 otherwise. The blankness of a cell is determined by the greatest common factor of the numerator and the denominator. If and only if this value is anything other than 1, then the cell is blank. As such, the reciprocal of the greatest common factor is 1 if the cell is not blank, and will be between 0 and 1 (non-inclusive) if the cell is blank. Therefore it follows that rounding this value down to the next integer yields 0 if the cell is blank and 1 otherwise. Subtracting this from 1 yields 1 if cell j is blank and 0 otherwise.

$$1 - \left\lfloor \frac{1}{\gcd(x(j), y(j))} \right\rfloor \tag{18}$$

If you take the relative index of the previous non-blank cell, add 1, and then add the number of consecutive blank cells after it, you'll get the index of the next non-blank cell. This function is based the following:

$$d(n) = d(n-1) + 1 + \sum_{k=d(n-1)+1}^{\infty} (1 - \left\lfloor \frac{1}{\gcd(x(k), y(k))} \right\rfloor) \tag{19}$$

However, this value is always infinite because it considers every single following blank cell, not just the consecutive ones. As such, we can take the product of the value of expression (18) for every string of cells of increasing length. As such, if any string of cells contains even a single non-blank cell, the value will be 0, otherwise it will be 1. As such, the sum will be equal to the number of sequences of consecutive blank cells. It therefore follows that it will equal the number of consecutive blank cells since the previous non-blank cell. This means that the function $d(n)$ is defined as such:

$$d(n) = d(n-1) + 1 + \sum_{k=d(n-1)+1}^{\infty} \prod_{j=d(n-1)+1}^{k} (1 - \left\lfloor \frac{1}{\gcd(x(j), y(j))} \right\rfloor) \tag{20}$$

However, since this function is defined recursively, we need to define a starting value. Since the 0^{th} cell isn't blank, our starting value is $d(0) = 0$. At first glance, it may appear that the infinite sum may cause some problems for computation, however it is not necessary to add an infinite number of terms. This is due to the fact that the sum is for adding together consecutive strings of blank cells, and as soon as a single non-blank cell is reached, the current and all future terms will be 0, and so none of the remaining terms make any difference to the total, and you can stop the calculation. Now that we have a function which gives the index of the n^{th} number in the grid, and two for calculating the numerator and denominator from a given index, we can create a function $r\prime(n)$ which produces all non-negative rational numbers.

$$r\prime(n) = \frac{x(d(n))}{y(d(n))} \tag{21}$$

Below is a table of values for $r\prime(n)$.

n	0	1	2	3	4	5	6	7	8	9
r′(n)	0	1	$\frac{1}{2}$	2	3	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{2}$	4

If we want to include negatives, we can spread out this function such that each value is followed by its negative as such:

n	0	1	2	3	4	5	6	7	8	9
r(n)	0	1	-1	$\frac{1}{2}$	$-\frac{1}{2}$	2	-2	3	-3	$\frac{1}{3}$

As such we can define $r(n)$ in terms of $r\prime(n)$ as such:

$$r(n) = r\prime(\left\lceil \frac{n}{2} \right\rceil)(-1)^{n+1} \tag{22}$$

The $\left\lceil \frac{n}{2} \right\rceil$ spreads out the inputs and the $(-1)^{n+1}$ term makes sure that every even numbered term is negative and every odd term is positive. We can then substitute $r\prime(n)$, giving us:

$$r(n) = \frac{x(d(\left\lceil \frac{n}{2} \right\rceil))}{y(d(\left\lceil \frac{n}{2} \right\rceil))}(-1)^{n+1} \tag{23}$$

To recap, We can now define $r(n)$ in terms of $x(n)$, $y(n)$ and $d(n)$ as follows:

$$x(n) = 2\left\lfloor\frac{-5+\sqrt{1+8n}}{4}\right\rfloor + 3 - \left|n - 2\left\lfloor\frac{-5+\sqrt{1+8n}}{4}\right\rfloor^2 - 7\left\lfloor\frac{-5+\sqrt{1+8n}}{4}\right\rfloor - 6\right| \tag{24}$$

$$y(n) = 2\left\lfloor\frac{-3+\sqrt{1+8n}}{4}\right\rfloor + 3 - \left|n - 2\left\lfloor\frac{-3+\sqrt{1+8n}}{4}\right\rfloor^2 - 5\left\lfloor\frac{-3+\sqrt{1+8n}}{4}\right\rfloor - 3\right| \tag{25}$$

$$d(0) = 0 \tag{26}$$

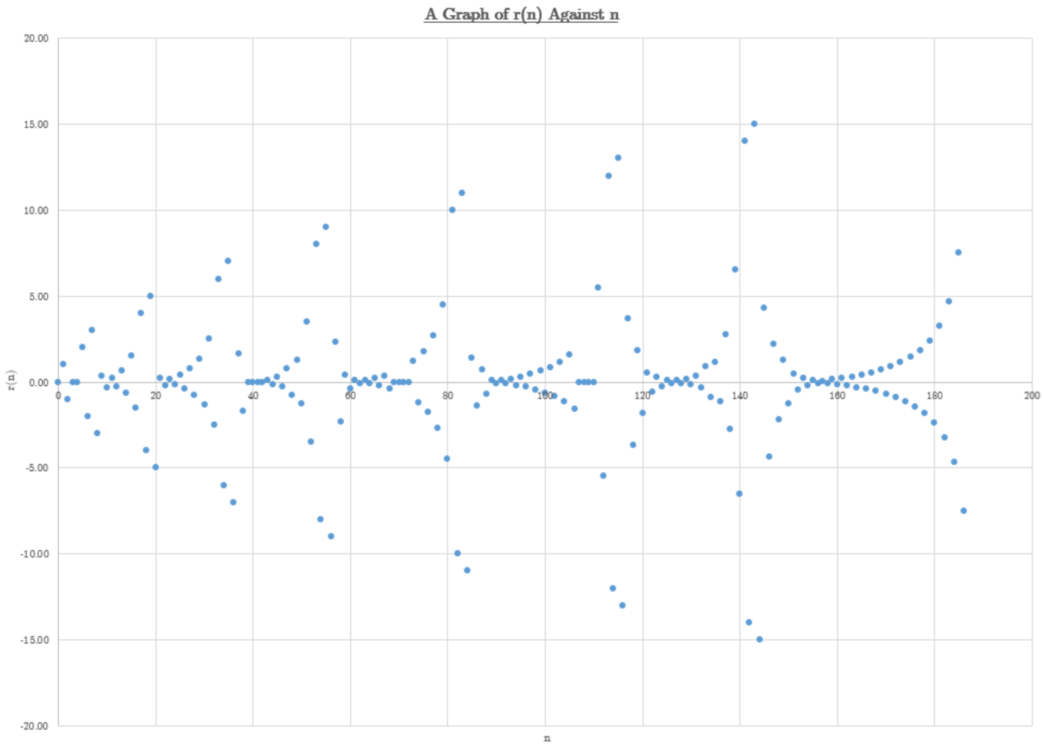
$$d(n) = d(n-1) + 1 + \sum_{k=d(n-1)+1}^{\infty} \prod_{j=d(n-1)+1}^{k} (1 - \left\lfloor\frac{1}{\gcd(x(j), y(j))}\right\rfloor) \tag{27}$$

$$r(n) = \frac{x(d(\lceil\frac{n}{2}\rceil))}{y(d(\lceil\frac{n}{2}\rceil))}(-1)^{n+1} \tag{28}$$

The above definition is the $r(n)$ function which we will use in the remainder of this paper. Rather than using the function itself, we can refer to the table of the first 40 values below:

n	r(n)	
0	0.00	0
1	1.00	1
2	-1.00	-1
3	0.50	1/2
4	-0.50	-1/2
5	2.00	2
6	-2.00	-2
7	3.00	3
8	-3.00	-3
9	0.33	1/3
10	-0.33	- 1/3
11	0.25	1/4
12	-0.25	- 1/4
13	0.67	2/3
14	-0.67	- 2/3
15	1.50	3/2
16	-1.50	- 3/2
17	4.00	4
18	-4.00	-4
19	5.00	5
20	-5.00	-5
21	0.20	1/5
22	-0.20	- 1/5
23	0.17	1/6
24	-0.17	- 1/6
25	0.40	2/5
26	-0.40	- 2/5
27	0.75	3/4
28	-0.75	- 3/4
29	1.33	4/3
30	-1.33	- 4/3
31	2.50	5/2
32	-2.50	- 5/2
33	6.00	6
34	-6.00	-6
35	7.00	7
36	-7.00	-7
37	1.67	5/3
38	-1.67	- 5/3
39	0.60	3/5

Plotting these values on a scatter graph gives the following:



Below is a python script which generates a given r value. Although this may not be the fastest algorithm for generating the r values, it was designed this way so as to bear a resemblance to the mathematical functions themselves. However there have been some adjustments to handle the computation of the infinite sum and the recursion in the $d(n)$ function.

```
from math import floor, ceil, sqrt
from fractions import gcd

def x(n):
    return 2 * floor((-5+sqrt(1+8*n))/4.0) + 3 - abs(n - 2 * pow(floor((-5+sqrt(1+8*n))/4.0), 2) - 7*floor((-5+sqrt(1+8*n))/4.0) - 6)

def y(n):
    return 2 * floor((-3+sqrt(1+8*n))/4.0) + 3 - abs(n - 2 * pow(floor((-3+sqrt(1+8*n))/4.0), 2) - 5*floor((-3+sqrt(1+8*n))/4.0) - 3)

def d(n):
    if n == 0:
        return 0
    _sum = 0
    k = d(n-1) + 1
    while True:
        prod = 1
        for j in range(d(n-1) + 1, k + 1):
            prod *= 1-floor(1/float(gcd(x(j), y(j))))
        if prod == 0:
            break
        _sum += prod
        k += 1
    return d(n-1) + 1 + _sum

def r(n):
    return x(d(ceil(n/2.0)))/y(d(ceil(n/2.0))) * pow(-1, n+1)

for i in range(40):
    print(r(i))
```

3 Partial Vector Notation

Using this $r(n)$ function, we can say that the 0^{th} component of a vector in polynomial-space represents the coefficient of x^0, the 1^{st} component represents the coefficient of x^1, the 2^{nd} component represents the coefficient of x^{-1} and in general the n^{th} component represents the coefficient of $x^{r(n)}$. For example, the polynomial $3x^2 + 2x + 1$ would be represented by the vector:

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Similarly, the expression $5\sqrt{x} + 9x$ would be represented by the vector:

$$\begin{bmatrix} 0 \\ 9 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

It is becoming evident that these vectors, due to their infinite number of terms, are becoming very unwieldy. As such we will define a new notation which cuts out all of the zeroes of a vector. Below is an example of how to translate an infinite vector into a compressed vector:

$$\begin{bmatrix} 0 \\ 9 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \equiv \begin{bmatrix} 9 \\ 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$$

The square-bracketed vector on the right hand side shows all of the non-zero values of the vector, and the adjacent component in the curly brackets shows the position in which that value appeared in the original infinite vector. This compressed vector notation will be called Partial Vector Notation and will be used widely throughout the remainder of this paper. This means that for a general partial vector like the following:

$$\begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \begin{Bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{Bmatrix}$$

The value k_a represents the coefficient of $x^{r(p_a)}$. For example, the following partial vector:

$$\begin{bmatrix} 1 \\ 7 \\ 2 \\ 9 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 4 \\ 10 \end{Bmatrix}$$

Represents the polynomial function $9x^{-\frac{1}{3}} + 2x^{-\frac{1}{2}} + 7x + 1$. When calculating the sum of partial vectors, the result is the same as if normal vector addition was performed on the infinite vectors. For example:

$$\begin{bmatrix} 1 \\ 7 \\ 2 \\ 9 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 4 \\ 10 \end{Bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \begin{Bmatrix} 0 \\ 4 \\ 15 \end{Bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 6 \\ 9 \\ 8 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 4 \\ 10 \\ 15 \end{Bmatrix}$$

4 Polynomial Expansions

There are some very common functions in mathematics such as the trigonometric functions (sine, cosine, secant, cosecant, their inverses, their hyperbolic counterparts, etc.) and the exponential functions and logarithms etc., which, although they are not considered polynomial functions, can be expressed as infinite polynomials whose coefficients follow a well-defined pattern. As such, a function with the property that it becomes arbitrarily close to its Taylor function (or other form of polynomial expansion) as more terms are added can be considered an infinite polynomial and as such can be expressed as a vector in our polynomial-space. For example, the function $\sin(x)$ can be written as:

$$\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \dots \tag{29}$$

In explicit form, this is:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \tag{30}$$

To frame this in terms of coefficients of powers of x, the sin function has odd integer powers of x, and the coefficient of x^k is $\frac{(-1)^{\frac{k-1}{2}}}{k!}$. As such, it can be written as the partial vector:

$$\begin{bmatrix} 1 \\ -\frac{1}{6} \\ \frac{1}{120} \\ -\frac{1}{5040} \\ \vdots \end{bmatrix} \begin{Bmatrix} 1 \\ 7 \\ 19 \\ 35 \\ \vdots \end{Bmatrix}$$

The indices 1, 7, 19 and 35 are used because the needed powers of x are 1, 3, 5 and 7, and $r(1) = 1$, $r(3) = 7$, $r(5) = 19$ and $r(7) = 35$. Since this vector has an infinite number of terms, even in partial vector form, we will instead refer to the vector as $\overrightarrow{\sin}$, because it is the vector corresponding to the sin function. In a similar way, using the polynomial expansions of other functions we can create a list of vectors of various functions:

Vector	**Polynomial Expansion**	**Partial Vector**	**Pattern**
$\overrightarrow{\sin}$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$	$\begin{bmatrix} 1 \\ -\frac{1}{6} \\ \frac{1}{120} \\ -\frac{1}{5040} \\ \vdots \end{bmatrix} \begin{Bmatrix} 1 \\ 7 \\ 19 \\ 35 \\ \vdots \end{Bmatrix}$	$\overrightarrow{\sin}_k = \left\lfloor \frac{1}{1+\lvert r(k)-\lvert r(k)\rvert\rvert} \right\rfloor \left\lfloor \frac{1}{1+\left\lvert r(k)+1-2\left\lfloor \frac{r(k)+1}{2} \right\rfloor\right\rvert} \right\rfloor \frac{(-1)^{\frac{r(k)-1}{2}}}{r(k)!}$
$\overrightarrow{\cos}$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$	$\begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{24} \\ -\frac{1}{720} \\ \vdots \end{bmatrix} \begin{Bmatrix} 0 \\ 5 \\ 17 \\ 33 \\ \vdots \end{Bmatrix}$	$\overrightarrow{\cos}_k = \left\lfloor \frac{1}{1+\lvert r(k)-\lvert r(k)\rvert\rvert} \right\rfloor \left\lfloor \frac{1}{1+\left\lvert r(k)-2\left\lfloor \frac{r(k)}{2} \right\rfloor\right\rvert} \right\rfloor \frac{(-1)^{\frac{r(k)-1}{2}}}{r(k)!}$
$\overrightarrow{\exp}$	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	$\begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \\ \frac{1}{6} \\ \vdots \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 5 \\ 7 \\ \vdots \end{Bmatrix}$	$\overrightarrow{\exp}_k = \left\lfloor \frac{1}{1+\lvert r(k)-\lvert r(k)\rvert\rvert} \right\rfloor \left\lfloor \frac{1}{1+\lvert r(k)-\lfloor r(k)\rfloor\rvert} \right\rfloor \frac{1}{r(k)!}$

For now, we will use only these three, because most other trigonometry/exponential functions are only equal to their Taylor series within a certain range of x values.

5 Inner Product

In order to make this vector space more useful when talking about polynomials, we want to define an inner product function such that the inner product of two vectors f and g which represent two polynomials, is the vector which represents the product of those two polynomials. The k^{th} component of this inner product can be defined as follows:

$$(f, g)_k = \sum_{j=0}^{\infty} f_j g_{r^{-1}(k-r(j))} \tag{31}$$

This mirrors the fact that when you multiply two polynomials, the coefficient of x^k is the sum of the products of all the pairs of coefficients of powers of x which sum to k. Although this definition contains an infinite sum, for all polynomials with a finite number of terms, there is only a finite number of non-zero terms in the sum. An example of the inner product function can be seen by the fact that the product of the following polyomials:

$$(3x^2 + 2x + 1)(6x^2 + 5x + 4) = 18x^4 + 27x^3 + 28x^2 + 13x + 4 \tag{32}$$

Can be represented by the inner product of vectors representing them:

$$\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 5 \end{Bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 5 \end{Bmatrix} \right) = \begin{bmatrix} 4 \\ 13 \\ 27 \\ 28 \\ 18 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 5 \\ 7 \\ 17 \end{Bmatrix} \tag{33}$$

Using this, and the Taylor-expanded vectors defined above, we can create vectors which represent functions using any combination of composition by multiplication and addition. For example, we could have a vector representing $2x\sin(x)+5$ using the following expression:

$$[2]\ \{1\} \cdot \overrightarrow{sin} + [5]\ \{0\} \tag{34}$$

And this creates a valid vector.

6 Probabilistic Uses

One task which is very well encapsulated by this polynomial-space is outcomes of probabilistic events such as the rolling of dice. For example, with two standard fair dice with six faces each valued 1 to 6, you may wish to know the probability of the sum of the values of each die being 4. In order to do this with vectors, you would first need to consider each die to be a polynomial function like the following:

$$D_0(x) = 1x^1 + 1x^2 + 1x^3 + 1x^4 + 1x^5 + 1x^6 \tag{35}$$

In this function, the powers of x represent the values of the faces, and the coefficients represent how many sides with that face exist. The die D_0 therefore has 1 face with a value of 1, 1 face with a value of 2, 1 face with a value of 3 and so on. This also corresponds to the fact that if you were to roll the dice, the values 1 to 6 would occur in the ratio $1:1:1:1:1:1$. If we want to see the possible sums of the outcomes if we roll two of this die, we multiply the function by itself:

$$D_0(x)^2 = 1x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + 1x^{12} \tag{36}$$

As with before, the powers of x represent the possible outcomes (in this case, 2 to 12) and the coefficients show the relative frequency of getting these results. It also shows that rolling the die D_0 twice and adding the results is equivalent to rolling one die with 36 faces: 1 face with a value of 2, 2 faces with a value of 3, 3 faces with a value of 4, 4 faces with a value of 5, 5 faces with a value of 6, 6 faces with a value of 7, 5 faces with a value of 8, 4 faces with a value of 9, 3 faces with a value of 10, 2 faces with a value of 11 and 1 face with a value of 12. As another example, imagine a die D_1 with 10 faces. 3 of these faces have a value of 1, 5 of them have a value of 2, and the other 2 faces have a value of $\frac{1}{2}$. This would be written in polynomial form as:

$$D_1(x) = 3x^1 + 5x^2 + 2x^{\frac{1}{2}} \tag{37}$$

If you rolled the die D_0 and then rolled the die D_1 and then added up the result, the possible outcomes could be represented by the product of the two functions:

$$D_0(x) \times D_1(x) = 2x^{\frac{3}{2}} + 3x^2 + 2x^{\frac{5}{2}} + 8x^3 + 2x^{\frac{7}{2}} + 8x^4 + 2x^{\frac{9}{2}} + 8x^5 + 2x^{\frac{11}{2}} + 8x^6 + 2x^{\frac{13}{2}} + 8x^7 + 5x^8 \tag{38}$$

And, like before, the powers of x represent the possible outcomes, and the coefficient represents the relative frequency of that outcome. Similarly, it shows that rolling D_0 followed by D_1 and summing the results is equivalent to rolling a 60-sided die whose sides have values as defined above. This example demonstrates the fact that the values of the sides do not have to be integers, or even positive. Modelling the dice as polynomial functions works for any real values. However, if we are to interpret those polynomial functions as vectors in polynomial-space, only rational values can be used. Below is the partial vector notation for the polynomial function D_0:

$$\overrightarrow{D_0} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 5 \\ 7 \\ 17 \\ 19 \\ 33 \end{Bmatrix} \tag{39}$$

Equation 36 can be rephrased as the inner product between this vector and itself:

$$(\overrightarrow{D_0}, \overrightarrow{D_0}) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \begin{Bmatrix} 5 \\ 7 \\ 17 \\ 19 \\ 33 \\ 35 \\ 53 \\ 55 \\ 81 \\ 83 \\ 113 \end{Bmatrix} \tag{40}$$

7 Complex and Imaginary Vector Properties

Complex numbers come up very frequently and naturally in infinite dimensional vector spaces. For example, several vectors in the space will have imaginary magnitudes. For example the following vector:

$$\overrightarrow{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ 2 \\ 2\sqrt{2} \\ 4 \\ 4\sqrt{2} \\ 8 \\ \vdots \end{bmatrix} \tag{41}$$

such that the k^{th} component

$$\overrightarrow{v}_k = \sqrt{2^k} \tag{42}$$

Since the magnitude of a vector is the square root of the sum of the squares of its components as per the Pythagorean theorem, the magnitude of the vector this vector can be defined as follows:

$$|\overrightarrow{v}| = \sqrt{1 + 2 + 4 + 8 + 16 + \ldots} \tag{43}$$

In order to solve this, we must first calculate the sum of all of the powers of 2. We can solve this with simple algebra. We first set the series equal to some variable s.

$$s = 1 + 2 + 4 + 8 + 16 + \ldots \tag{44}$$

We then double both sides:

$$2s = 2 + 4 + 8 + 16 + 32 + \ldots \tag{45}$$

We can then add 1 to both sides:

$$2s + 1 = 1 + 2 + 4 + 8 + 16 + \ldots \tag{46}$$

We can see that this is the same as the original series, so we can now say that:

$$2s + 1 = s \tag{47}$$

Subtracting $s + 1$ from both sides leaves:

$$s = -1 \tag{48}$$

And as such:

$$1 + 2 + 4 + 8 + 16 + \ldots = -1 \tag{49}$$

This allows us to work out the magnitude of that vector.

$$|\overrightarrow{v}| = \sqrt{-1} = i \tag{50}$$

As you can see, the magnitude of this vector is the imaginary constant i. This shows that infinite-dimensional vector spaces lend themselves readily to complex numbers. Another example of this is the following vector:

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 2 \\ \sqrt{5} \\ \sqrt{6} \\ \vdots \end{bmatrix} \tag{51}$$

such that the k^{th} component

$$\overrightarrow{u}_k = \sqrt{k+1} \tag{52}$$

The magnitude of this vector is therefore the following:

$$|\overrightarrow{u}| = \sqrt{1+2+3+4+5+\ldots} \tag{53}$$

Since the sum of all the natural numbers equals $-\frac{1}{12}$, then the magnitude is equal to:

$$|\overrightarrow{u}| = \frac{\sqrt{12}}{12}i \tag{54}$$

Since complex numbers come up so naturally in infinite vector spaces, it seems natural to allow the components of the vectors themselves to have complex values. This allows for vectors representing polynomials with imaginary and complex coefficients.

8 Derivative Matrix

In this section we will design an infinite dimensional square matrix[2] which represents a linear transformation. When this transformation is applied to a vector representing some polynomial function, the result will be the vector which represents that same polynomial function's derivative. Since each column of the matrix represents where it takes each of the basis vectors of the polynomial-space, each column will have only one non-zero value. We want the matrix to map the basis vector representing x^k to be scaled by a factor of k and rotated parallel to the basis vector representing x^{k-1}. As such, for the k^{th} column of the matrix, the non-zero value will be equal to $r(k)$ to handle the scaling, and will be in the $r^{-1}(r(k)-1)^{th}$ row, so as to decrease the exponent by 1. To generalise this rule, we need to check, for a given column k, whether the r function, evaluated on a given row j is equal to $r(k)-1$, and so whether a non-zero value will be put in the cell. To do this, observe that the following expression:

$$|r(j) - (r(k)-1)| \tag{55}$$

will equal zero when the this condition is met, and some positive value when it is not. We can add 1 and reciprocate to make this value easier to manipulate.

$$\frac{1}{1+|r(j)-(r(k)-1)|} \tag{56}$$

This expression will equal 1 when the condition is met and will be between 0 and 1 (non-inclusive) when the condition is not met. We can round this expression down and multiply by $r(k)$.

$$r(k)\left\lfloor \frac{1}{1+|r(j)-(r(k)-1)|} \right\rfloor \tag{57}$$

This expression yields $r(k)$ when the condition is met and 0 otherwise. We can use this expression to fill each cell of the matrix, $\mathbf{D}$.

$$\mathbf{D}_{jk} = r(k)\left\lfloor \frac{1}{1+|r(j)-(r(k)-1)|} \right\rfloor \tag{58}$$

Below are the first few elements of the matrix.

$$\mathbf{D} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & \ldots \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{59}$$

[2] A matrix with an infinite number of rows and columns

The values which aren't shown also follow the rule defined in Equation 58. Below is an example of how this matrix could be applied to a vector.

$$\mathbf{D}\begin{bmatrix}1\\2\\3\end{bmatrix}\begin{Bmatrix}0\\1\\5\end{Bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix} \tag{60}$$

This corresponds to the fact that the derivative of $3x^2 + 2x + 1$ is $6x + 2$.

9 Antiderivative Matrix

In a similar way, we can define a matrix which maps each vector which represents a polynomial to the vector which represents the antiderivative of that polynomial. Instead of simply finding $\mathbf{D}^{-1}$, we will define the antiderivative matrix $\mathbf{A}$ using a similar method to before. We want each column k of the matrix to be zero for all rows j except when $r(j)$ is equal to $r(k)+1$, in which case we want it to be equal to $\frac{1}{r(k)+1}$. Using a very similar process to before, we can define a rule for each cell of the matrix.

$$\mathbf{A}_{jk} = \frac{1}{r(k)+1}\left\lfloor \frac{1}{1+|r(j)-(r(k)+1)|}\right\rfloor \tag{61}$$

Below are the first few values of the matrix.

$$\mathbf{A} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \cdots \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{62}$$

It is however important to note that when $r(k) = -1$, then this will result in division by 0, and so we have replaced it with a 0 in this case. It is the equivalent problem of trying to antidifferentiate x^{-1}, which yields $\ln x$, a function which is not expressable as a polynomial. As such, the antiderivative matrix will indeed map all vectors which represent a polynomial to the vector which represents the antiderivative of that polynomial, unless that polynomial contains a non-zero coefficient for x^{-1}, or, equivalently, the vector contains a non-zero value in the 2^{nd} component, in which case this term will be ignored.