Cupcake Space

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1 Introduction

In this paper we will define and describe a 2D vector space called Cupcake space, denoted by the symbol $\acute{\mathbf{c}}$ (read as "c cherry"). There will be a unique continuous mapping from every point in $\acute{\mathbf{c}}$ to every point in $\mathbb{R}^2$. That is to say, for each point in an infinite Cartesian 2D plane, there is a single corresponding point in $\acute{\mathbf{c}}$, and if two points are arbitrarily close together in that Cartesian plane, they will also be arbitrarily close together in $\acute{\mathbf{c}}$. The space also has another property: the distance between every point in $\acute{\mathbf{c}}$ and the origin is less than (or equal to in some cases) 1. This is to say that $\acute{\mathbf{c}}$ is a bounded space, and can be modelled as the region inside a unit circle. One could think of this space as being an infinite Cartesian plane, but compressed into a finite region, bounded by a unit circle. To get a sense for the geometry of $\acute{\mathbf{c}}$, in this paper we will need to visualise the transformations between the infinite Cartesian plane (which we will refer to as $\mathbb{R}^2$) and $\acute{\mathbf{c}}$. For example, below is a graph of a region of $\mathbb{R}^2$. The horizontal gridlines (i.e. the curves with a constant integer y value) are shown in purple, and the vertical gridlines (the curves with a constant integer x value) are shown in green.

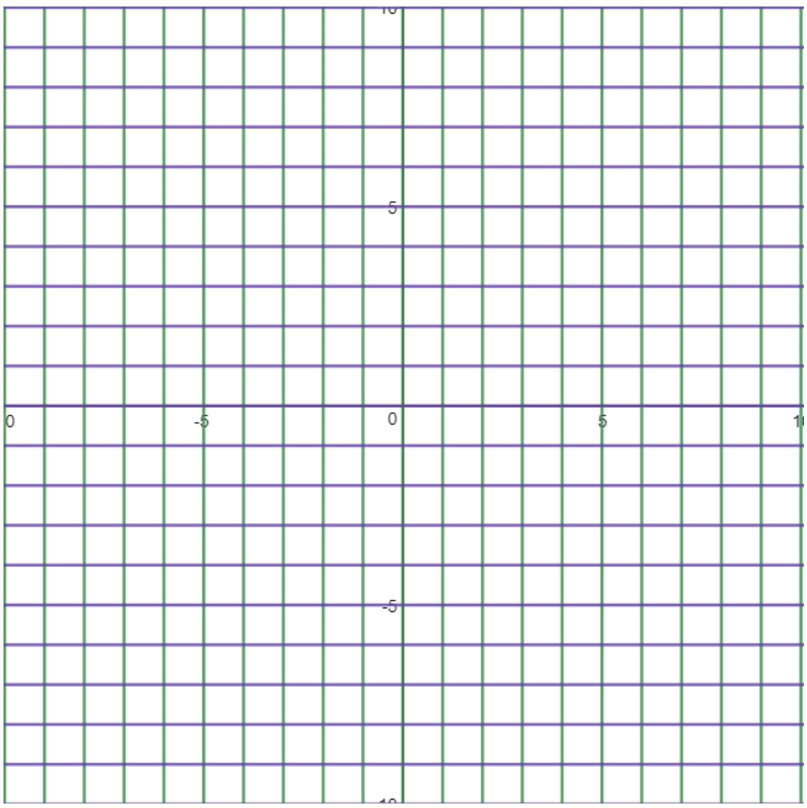

Figure 1: A region of $\mathbb{R}^2$ with the gridlines shown

Although only a finite region of $\mathbb{R}^2$ is shown in the graph, it is important to note that this space extends infinitely in all directions, and so do the curves. Now if we were to map every point in $\mathbb{R}^2$ (again, not just points in the finite region shown above, but every point in the entire infinite space) to its corresponding point in $\acute{\mathbf{c}}$, the gridlines would be transformed as follows:

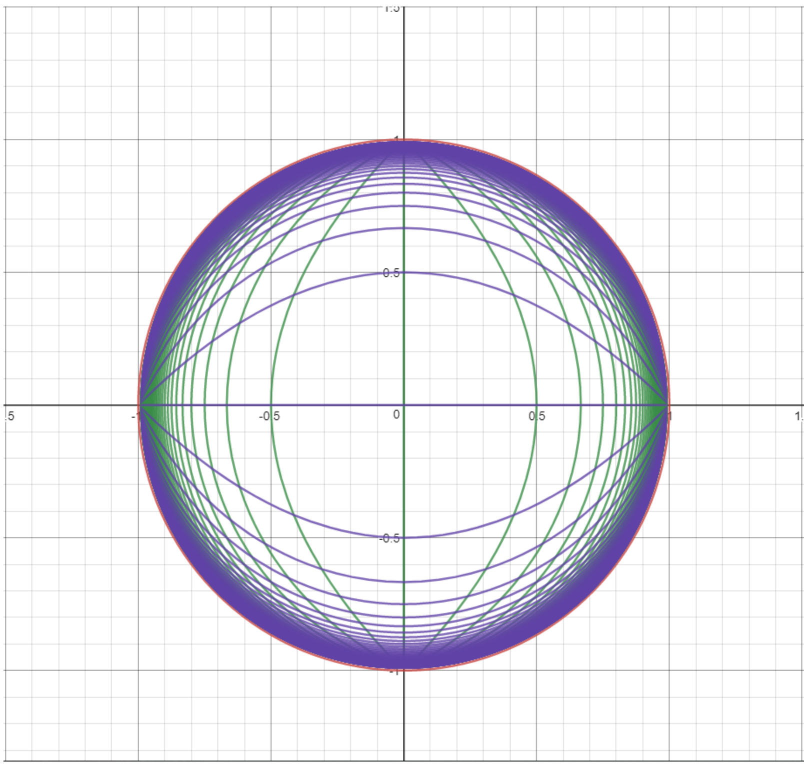

Figure 2: $\acute{\mathbf{c}}$ with transformed gridlines

As you can see, it is as though the infinite space $\mathbb{R}^2$ is compressed into the finite region of $\acute{\mathbf{c}}$. The circumference of the unit circle (the boundary of $\acute{\mathbf{c}}$) represents the points at infinity of $\mathbb{R}^2$. This means that if you observed the motion of a point a in $\mathbb{R}^2$, and the corresponding motion of its corresponding point $\acute{a}$ in $\acute{\mathbf{c}}$, as a gets further away from the origin of $\mathbb{R}^2$ (i.e. $|a|$ approaches infinity), you will observe $\acute{a}$ approach the boundary of $\acute{\mathbf{c}}$. The aim of this paper is to explore how points and curves are transformed when moving between $\acute{\mathbf{c}}$ and $\mathbb{R}^2$.

2 Rigorous definition of $\acute{\mathbf{c}}$

In the introduction we described $\acute{\mathbf{c}}$ qualitatively and gave a brief visual intuition for its geometry. However, in this section we will set out some mathematical properties which uniquely define the space. The first property is simple to describe. The origin of $\acute{\mathbf{c}}$ (i.e. the point at $(0, 0)$) maps to the origin of $\mathbb{R}^2$. Since these points are equivalent, we can refer to them both as $\mathbf{0}$.

2.1 Origin-Angle Preservation

The second property that $\acute{\mathbf{c}}$ has is that angles from the origin are preserved when mapping from $\acute{\mathbf{c}}$ to $\mathbb{R}^2$ and vice versa. To give a visual intuition of this property, imagine a diameter of $\acute{\mathbf{c}}$ with a certain gradient. If you were to map all the points on this line to their corresponding points in $\mathbb{R}^2$, the line would now extend infinitely in both directions, but it would have the same gradient. An example is shown below.

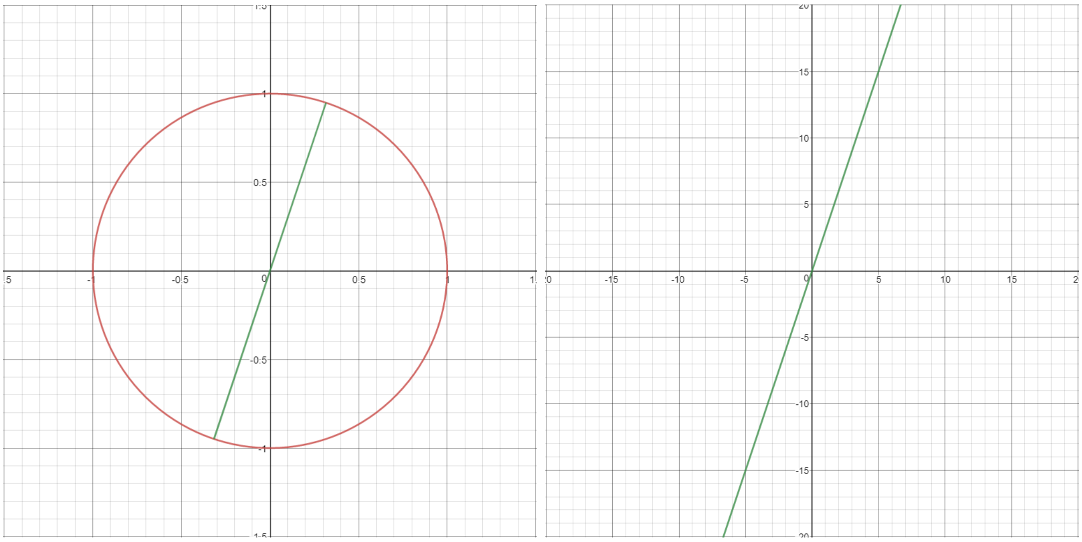

Figure 3: A diameter of $\acute{\mathbf{c}}$ (left) and its corresponding line in $\mathbb{R}^2$

To formalise this property, we can define a point on the boundary of $\acute{\mathbf{c}}$, $\acute{i}$. We will choose $\acute{i}$ to be the point with coordinates $(1, 0)$ in $\acute{\mathbf{c}}$. The corresponding point i in $\mathbb{R}^2$ has no precise coordinates because $\acute{i}$ is on the boundary of $\acute{\mathbf{c}}$ and so corresponds to a point on $\mathbb{R}^2$ which is an infinite distance away from the origin. However for our purposes, i does not need to have coordinates to be useful, only a direction. As such we need only to define i as the point at infinity along the positive x axis of $\mathbb{R}^2$. There is a non-rigorous sense in which $i = (\infty, 0)$. Now we can rigorously define the Origin-Angle preserving property as follows:

Let $\acute{a}$ be a point in $\acute{\mathbf{c}}$

Let a be the corresponding point to $\acute{a}$ in $\mathbb{R}^2$

$$\angle \acute{a}\mathbf{0}\acute{i} = a\mathbf{0}i$$

Below is a graphical example of this property.

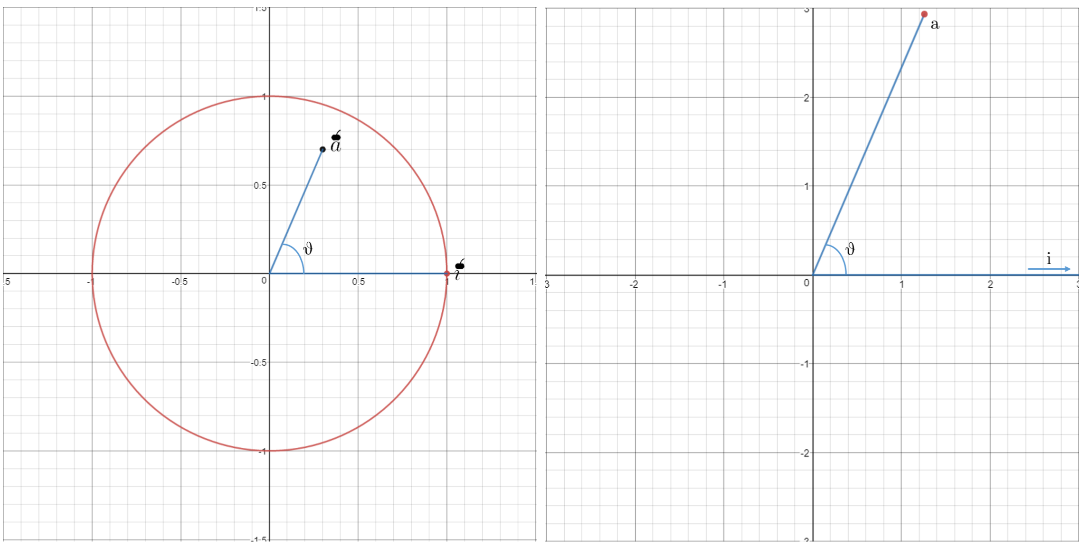

Figure 4: Demonstration of the preservation of angles from the origin when mapping from $\acute{\mathbf{c}}$ (left) to $\mathbb{R}^2$ (right)

This choice of $\acute{i}$ was arbitrary, as any other point on the boundary of $\acute{\mathbf{c}}$ would have worked equally well. However, this particular point was chosen so as to fit with the convention of measuring angles from the positive horizontal axis.

2.2 Distance from the Origin

The third and final property which is needed to uniquely define $\acute{\mathbf{c}}$ is a notion of how the distance between the origin and some point $\acute{a}$ in $\acute{\mathbf{c}}$ (we can call this distance $\acute{r}$) is related to the distance between the origin and the corresponding point

a in $\mathbb{R}^2$ (we can call this distance r). We want this relationship to have the following properties:

$$r = 0 \implies \mathring{r} = 0$$

$$0 \leq r < \infty \implies 0 \leq \mathring{r} < 1$$

$$\lim_{r\to\infty} \mathring{r} = 1$$

$$\frac{d\mathring{r}}{dr} \text{ exists for all } 0 \leq r < \infty$$

This may seem conceptually tricky to understand. How can one map from an infinite interval to a finite interval? Mathematically this is quite common. Consider the function $f(x) = \frac{1}{x}$. As x varies from 1 to infinity, the value of $f(x)$ ranges from 1 to 0. As such, this function maps an infinite interval to a finite one. Here, x is analogous to r and $f(x)$ is analogous to $\mathring{r}$. Therefore, we want to construct a function where $f(x)$ ranges from 0 to 1, rather than from 1 to 0. We can modify $f(x)$ to be 1 subtract its original value, becoming $f(x) = 1 - \frac{1}{x}$. For this function, $f(x)$ ranges from 0 to 1 as x ranges from 1 to infinity. However, we want this to occur as x ranges from 0 to infinity. Therefore we can modify $f(x)$ to be its original value evaluated at $x+1$, becoming $f(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$. This does exactly what we need it to do: $f(x)$ ranges from 0 to 1 as x ranges from 0 to infinity. As such we can make this our relation between r and $\mathring{r}$.

$$\mathring{r} = \frac{r}{1+r}$$

We can also solve for r as follows:

$$r = \frac{\mathring{r}}{1-\mathring{r}}$$

This relationship does indeed satisfy the conditions outlined above, and is the relationship we will use. However, this is not the only relationship which satisfies them. For example, we could have also used the relationship $\mathring{r} = \frac{2}{\pi}\tan^{-1}(r)$ and $r = \tan(\frac{\pi}{2}\mathring{r})$ or any of an infinite number of other relationships. To give a visual intuition for how these distances are scaled, below is a graph of a set of curves. In $\mathbb{R}^2$, these curves are circles with integer radii (i.e. a circle with radius 1, radius 2, radius 3 etc.). In $\mathring{\mathbf{c}}$ the corresponding curves are shown.

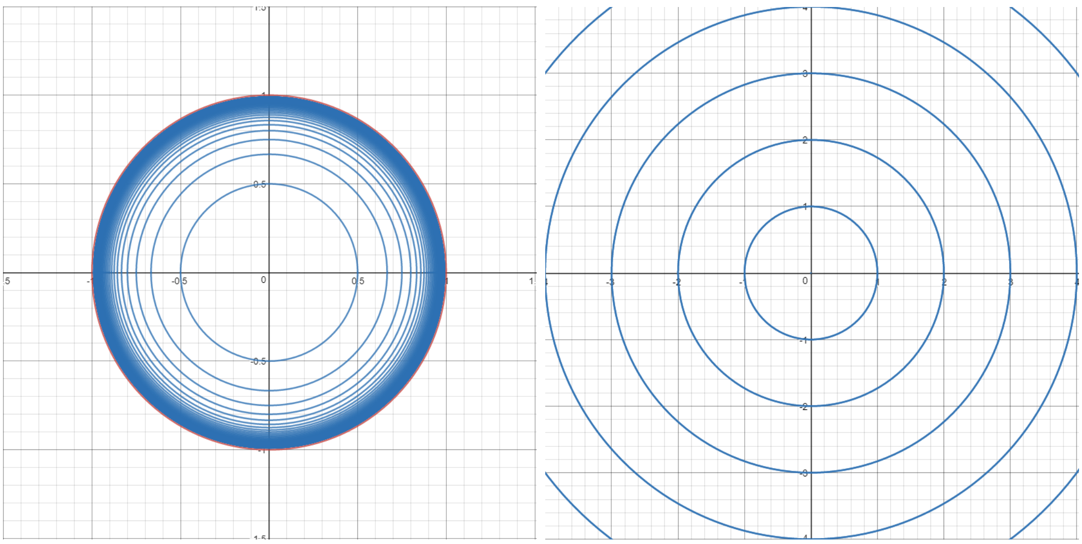

Figure 5: Circles with integer radii in $\mathbb{R}^2$ (right) and their corresponding curves in $\mathring{\mathbf{c}}$ (left)

In fact, the second and third properties alone (the Origin-Angle Preservation property and the $r/\mathring{r}$ relation) are enough to uniquely define the Cupcake space, because the third property ensures the first (i.e. when $r = 0$ then $\mathring{r} = 0$).

3 Mapping points

In this section we will create a method for mapping a point (x, y) in $\mathbb{R}^2$ to a point $(\mathring{x}, \mathring{y})$ in $\mathring{\mathbf{c}}$ and vice versa.

3.1 $\mathbb{R}^2$ to $\mathring{\mathbf{c}}$

First we will consider a point $a = (x, y)$ in $\mathbb{R}^2$. The corresponding point $\mathring{a} = (\mathring{x}, \mathring{y})$ must make the same angle from the positive horizontal axis. Therefore the tangent of the angles must be the same. That is to say that the ratio of the vertical to the horizontal distance must be the same for a and $\acute{a}$. We can write and rearrange this relation as follows:

$$\frac{y}{x} = \frac{\acute{y}}{\acute{x}}$$

$$\frac{\acute{x}}{x} = \frac{\acute{y}}{y} = k$$

$$\therefore \; \acute{x} = kx$$

$$\acute{y} = ky$$

Now we need to find the appropriate value of k. We will do this by finding the distance r of a from the origin and also the distance $\acute{r}$ of $\acute{a}$ from the origin in terms of k. We can then solve for k using the relation defined in the previous section.

$$r = \sqrt{x^2 + y^2}$$

$$\begin{aligned} \acute{r} &= \sqrt{(kx)^2 + (ky)^2} \\ &= k\sqrt{x^2 + y^2} \\ &= kr \end{aligned}$$

$$\therefore \; k = \frac{\acute{r}}{r}$$

We can recall the relationship between r and $\acute{r}$:

$$\begin{aligned} k &= \frac{\acute{r}}{r} \\ &= \frac{\left(\frac{r}{1+r}\right)}{r} \\ &= \frac{1}{1+r} \\ &= \frac{1}{1+\sqrt{x^2+y^2}} \end{aligned}$$

We can plug this into the formulae for $\acute{x}$ and $\acute{y}$:

$$\acute{x} = \frac{x}{1+\sqrt{x^2+y^2}}$$

$$\acute{y} = \frac{y}{1+\sqrt{x^2+y^2}}$$

And these are the coordinates of $\acute{a}$.

3.2 $\acute{\mathbf{c}}$ to $\mathbb{R}^2$

We will now carry out that same process in reverse in order to map from a pair of known coordinates $(\acute{x}, \acute{y})$ of a point $\acute{a}$ in $\acute{\mathbf{c}}$ to the coordinates (x, y) of its corresponding point a in $\mathbb{R}^2$. We can simply use the same equations as before:

$$\begin{aligned} \acute{x} = kx &\implies x = \frac{1}{k}\acute{x} \\ \acute{y} = ky &\implies y = \frac{1}{k}\acute{y} \\ k = \frac{\acute{r}}{r} &\implies \frac{1}{k} = \frac{r}{\acute{r}} \end{aligned}$$

$$\therefore \quad \frac{1}{k} = \frac{\left(\frac{\acute{r}}{1-\acute{r}}\right)}{\acute{r}}$$

$$= \frac{1}{1-\acute{r}}$$

$$= \frac{1}{1-\sqrt{\acute{x}^2+\acute{y}^2}}$$

$$\therefore \quad x = \frac{\acute{x}}{1-\sqrt{\acute{x}^2+\acute{y}^2}}$$

$$y = \frac{\acute{y}}{1-\sqrt{\acute{x}^2+\acute{y}^2}}$$

And these are the coordinates of a. We can now map from a point in $\acute{\mathbf{c}}$ to a point in $\mathbb{R}^2$ and vice versa.

4 Mapping curves

Now that we can map a single point, it is an easy step to map entire curves. This is because if we write our curve in parametric form (that is to say, the coordinates are functions not of each other, but of a shared parameter, often thought of as "time"), then for a given time, the curve yields an individual point. First we will consider mapping a curve in $\mathbb{R}^2$ to the corresponding curve in $\acute{\mathbf{c}}$. When written in parametric form, this should look familiar, as we know $(x(t), y(t))$ from $\alpha \leq t \leq \beta$, and wish to find $(\acute{x}(t), \acute{y}(t))$ for t in the same range. For the same reasons as in the previous section, we can simply introduce the parameter into the exact same formulae:

$$\acute{x}(t) = \frac{x(t)}{1+\sqrt{x^2(t)+y^2(t)}}$$

$$\acute{y}(t) = \frac{y(t)}{1+\sqrt{x^2(t)+y^2(t)}}$$

And for mapping from $\acute{\mathbf{c}}$ to $\mathbb{R}^2$:

$$x(t) = \frac{\acute{x}(t)}{1+\sqrt{\acute{x}^2(t)+\acute{y}^2(t)}}$$

$$x(t) = \frac{\acute{y}(t)}{1+\sqrt{\acute{x}^2(t)+\acute{y}^2(t)}}$$

Below are some examples:

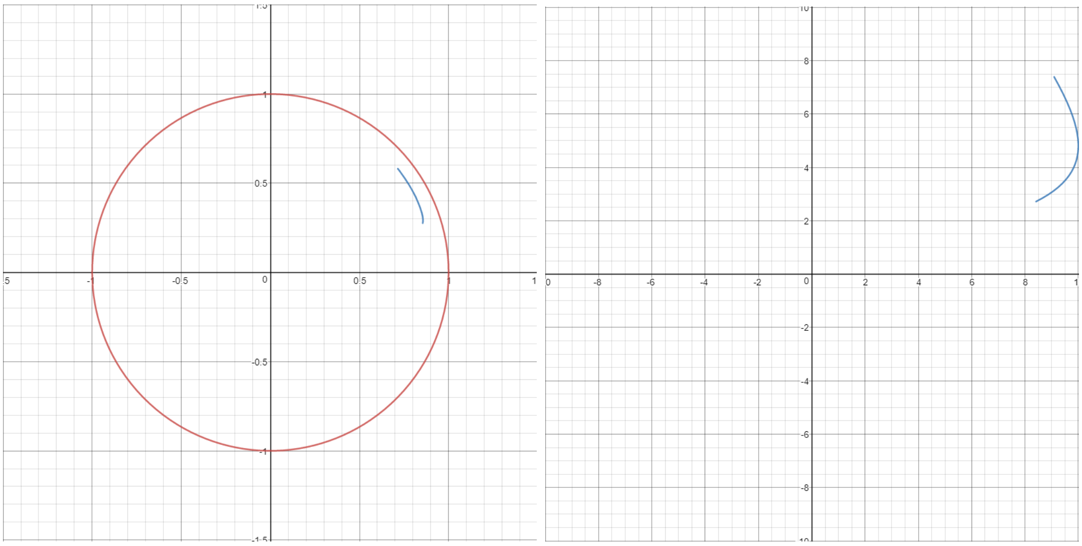

Figure 6: Curve $(x(t), y(t))$ where $x(t) = 10\sin(t)$, $y(t) = e^t$ from $1 \leq t \leq 2$ in $\mathbb{R}^2$ (right) and corresponding curve $(\acute{x}(t), \acute{y}(t))$ from $1 \leq t \leq 2$ in $\acute{\mathbf{c}}$ (left)

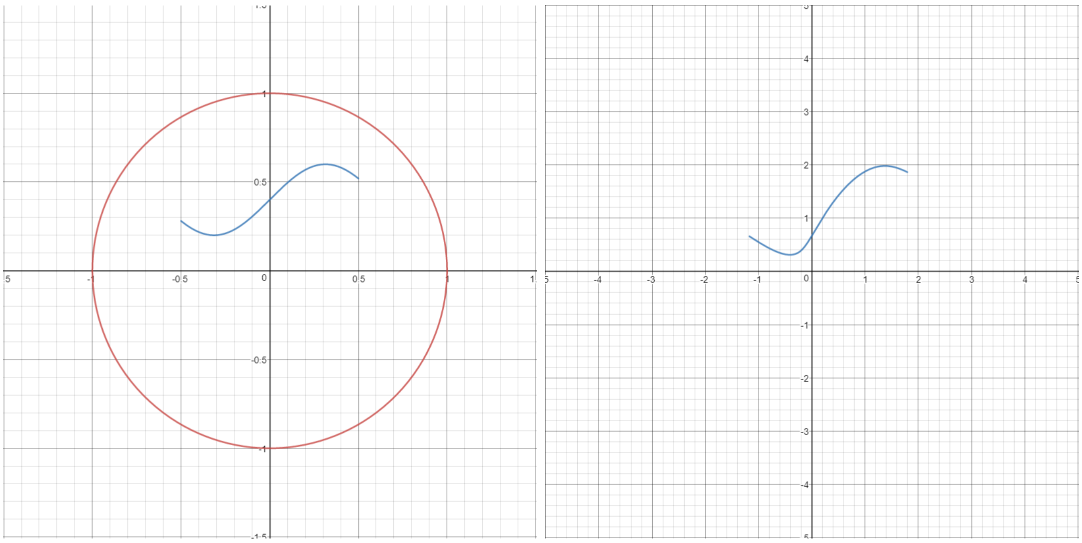

Figure 7: Curve $(\acute{x}(t), \acute{y}(t))$ where $\acute{x}(t) = t$, $\acute{y}(t) = \frac{\sin(5t)}{5} + 0.4$ from $-0.5 \leq t \leq 0.5$ in $\acute{\mathbf{c}}$ (left) and corresponding curve $(x(t), y(t))$ from $-0.5 \leq t \leq 0.5$ in $\mathbb{R}^2$ (right)

5 Geodesics and Distances

Now we need to define a notion of the "length" of a curve in $\acute{\mathbf{c}}$. Instead of simply using the arclength of the curve in $\acute{\mathbf{c}}$, we will instead define the length to be the arclength of the corresponding curve in $\mathbb{R}^2$. As such, a geodesic (path of shortest length) between two points $\acute{a}$ and $\acute{b}$ in $\acute{\mathbf{c}}$ is not the straight line between them in $\acute{\mathbf{c}}$, but rather the curve which corresponds to the straight line between the corresponding points a and b in $\mathbb{R}^2$. We will work backwards, starting with the straight line between a and b in $\mathbb{R}^2$. We can write this curve in parametric form as follows:

$$x(t) = a_x(1-t) + b_x t$$

$$y(t) = a_y(1-t) + b_y t$$

From $0 \leq t \leq 1$. We can rewrite a_x, a_y, b_x and b_y in terms of $\acute{a}_x$, $\acute{a}_y$, $\acute{b}_x$ and $\acute{b}_y$ as follows:

$$x(t) = \frac{\acute{a}_x(1-t)}{1-\sqrt{\acute{a}_x^2 + \acute{a}_y^2}} + \frac{\acute{b}_x t}{1-\sqrt{\acute{b}_x^2 + \acute{b}_y^2}}$$

$$y(t) = \frac{\acute{a}_y(1-t)}{1-\sqrt{\acute{a}_x^2 + \acute{a}_y^2}} + \frac{\acute{b}_y t}{1-\sqrt{\acute{b}_x^2 + \acute{b}_y^2}}$$

From the previous section, we know that:

$$\acute{x}(t) = \frac{x(t)}{1 + \sqrt{x^2(t) + y^2(t)}}$$

$$\acute{y}(t) = \frac{y(t)}{1 + \sqrt{x^2(t) + y^2(t)}}$$

And, although the result is quite inelegant, we can use this to find our geodesic:

$$\acute{x}(t) = \frac{\left(\frac{\acute{a}_x(1-t)}{1-\sqrt{\acute{a}_x^2+\acute{a}_y^2}} + \frac{\acute{b}_x t}{1-\sqrt{\acute{b}_x^2+\acute{b}_y^2}}\right)}{1 + \sqrt{\left(\frac{\acute{a}_x(1-t)}{1-\sqrt{\acute{a}_x^2+\acute{a}_y^2}} + \frac{\acute{b}_x t}{1-\sqrt{\acute{b}_x^2+\acute{b}_y^2}}\right)^2 + \left(\frac{\acute{a}_y(1-t)}{1-\sqrt{\acute{a}_x^2+\acute{a}_y^2}} + \frac{\acute{b}_y t}{1-\sqrt{\acute{b}_x^2+\acute{b}_y^2}}\right)^2}}$$

$$\acute{y}(t) = \frac{\left(\frac{\acute{a}_y(1-t)}{1-\sqrt{\acute{a}_x^2+\acute{a}_y^2}} + \frac{\acute{b}_y t}{1-\sqrt{\acute{b}_x^2+\acute{b}_y^2}}\right)}{1 + \sqrt{\left(\frac{\acute{a}_x(1-t)}{1-\sqrt{\acute{a}_x^2+\acute{a}_y^2}} + \frac{\acute{b}_x t}{1-\sqrt{\acute{b}_x^2+\acute{b}_y^2}}\right)^2 + \left(\frac{\acute{a}_y(1-t)}{1-\sqrt{\acute{a}_x^2+\acute{a}_y^2}} + \frac{\acute{b}_y t}{1-\sqrt{\acute{b}_x^2+\acute{b}_y^2}}\right)^2}}$$

This is the parametrization of the geodesic between $\acute{a} = (\acute{a}_x, \acute{a}_y)$ and $\acute{b} = (\acute{b}_x, \acute{b}_y)$ in $\acute{\mathbf{c}}$, when $0 \leq t \leq 1$. Below is an example of two points in $\acute{\mathbf{c}}$ and the geodesic between them described by the above parametrization.

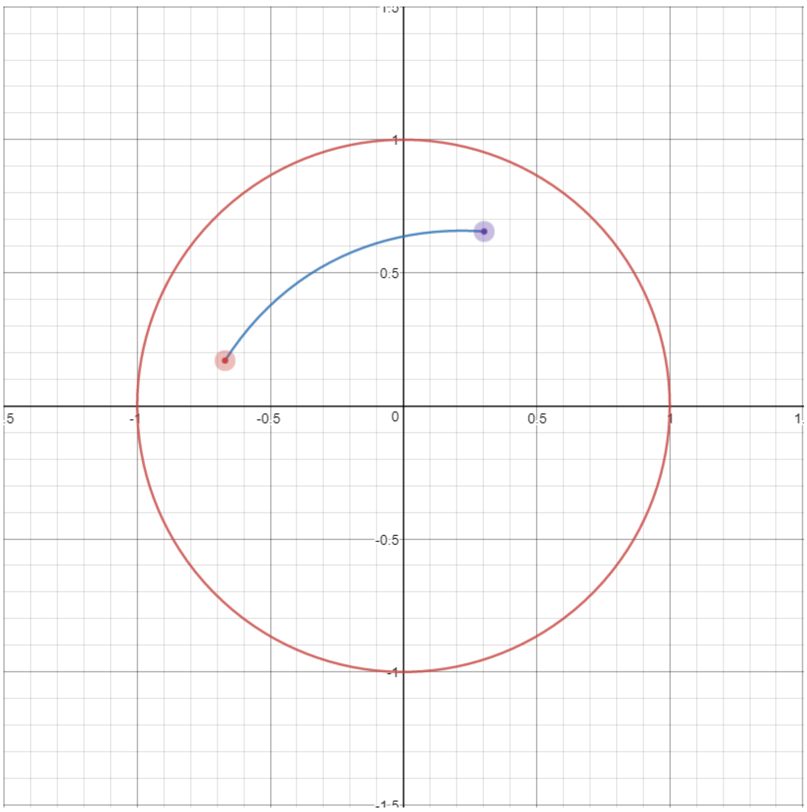

Figure 8: The geodesic between two points in $\acute{\mathbf{c}}$

Now we can define the distance between two points $\acute{a}$ and $\acute{b}$ in $\acute{\mathbf{c}}$ as the length of the straight line between their corresponding points a and b in $\mathbb{R}^2$. We can write this as follows:

$$d(\acute{a}, \acute{b}) = \sqrt{(b_x - a_x)^2 + (b_y - a_x)^2}$$

And now we can again rewrite a_x, a_y, b_x and b_y in terms of $\acute{a}_x$, $\acute{a}_y$, $\acute{b}_x$ and $\acute{b}_y$ as follows:

$$d(\acute{a}, \acute{b}) = \sqrt{\left(\frac{\acute{b}_x}{1 - \sqrt{\acute{b}_x^2 + \acute{b}_y^2}} - \frac{\acute{a}_x}{1 - \sqrt{\acute{a}_x^2 + \acute{a}_y^2}}\right)^2 + \left(\frac{\acute{b}_y}{1 - \sqrt{\acute{b}_x^2 + \acute{b}_y^2}} - \frac{\acute{a}_y}{1 - \sqrt{\acute{a}_x^2 + \acute{a}_y^2}}\right)^2}$$

6 Conclusion

Cupcake space is an example of how an infinite 2D space can be modelled and examined as a finite 2D region. In this paper we have demonstrated that such a space is possible, and also that phenomena such as curves, geodesics and distances can be observed in the Cupcake space, and these observations can be meaningfully interpreted to make conclusions about the infinite space. Furthermore, no information about the infinite space is lost when observing it in the form of a Cupcake space. This is useful because it allows us to concretely talk about the points at, and even beyond infinity. If you were to consider only the infinite space, these points can be difficult to pin down, however in the Cupcake space it is trivial to define a point which exists on or outside the circumference. In fact, there is no reason to limit the concept to two dimensions, as the logic generalises seamlessly to an N-dimensional Cupcake space, mapping an N-dimensional infinite hyperplane into a finite unit N-dimensional hypersphere. Finitely bounding an infinite space also allows us to join infinite spaces together, perhaps joining the borders of Cupcake spaces. This would take a difficult concept (that beyond the points at infinity of one infinite space, there is another infinite space), and making it as easy to visualise as two finite shapes which share a boundary. This, I believe, is the true utility of Cupcake space: To create a visual and instinctual intuition for transfinite concepts.