Axis Bending

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1 Introduction

A real-valued function $f(x)$ can be plotted on a graph as the set of all points which, for some real number x, lie a distance of $f(x)$ along the perpendicular to the point a distance of x along the x axis. For convention, positive values of $f(x)$ correspond to moving up the perpendicular, and negative values of $f(x)$ correspond to moving down the perpendicular. This set of points can be described as $y = f(x)$, or, in a parametrized form, $x(t) = t$, $y(t) = f(t)$. However, the choice to plot the function relative to the x axis is an arbitrary one, when in fact it is possible to do so relative to any other arbitrary curve, $g(x)$. Visually, this would look like plotting the set of points for which $y = f(x)$, and then bending the x axis such that it lies along the curve $y = g(x)$. Below are some examples of this, for $f(x) = x^2$ and $g(x) = kx^2$, for various values of k.

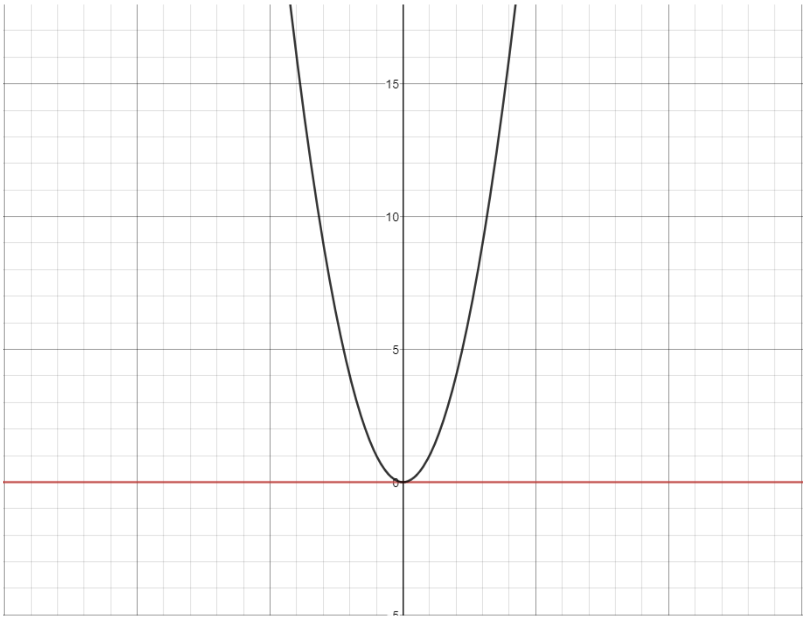


Figure 1: $k = 0$

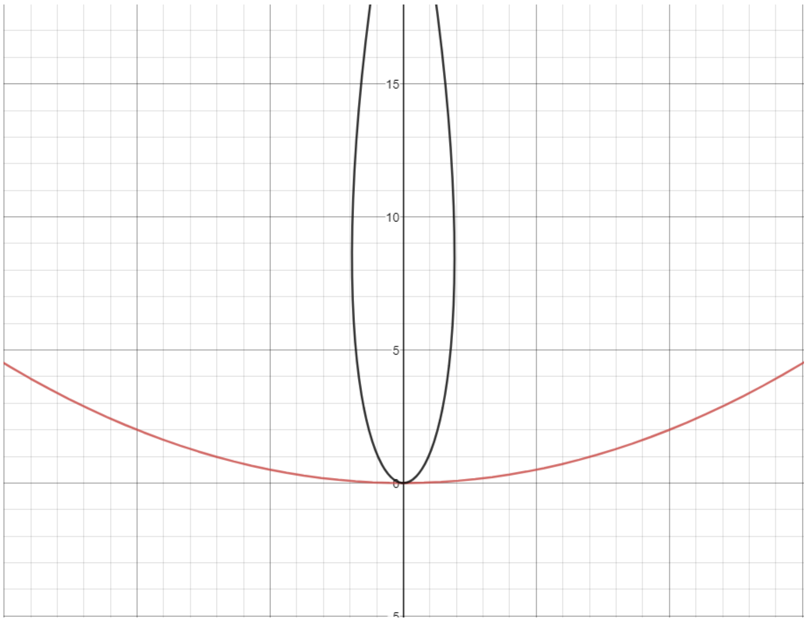


Figure 2: $k = 0.02$

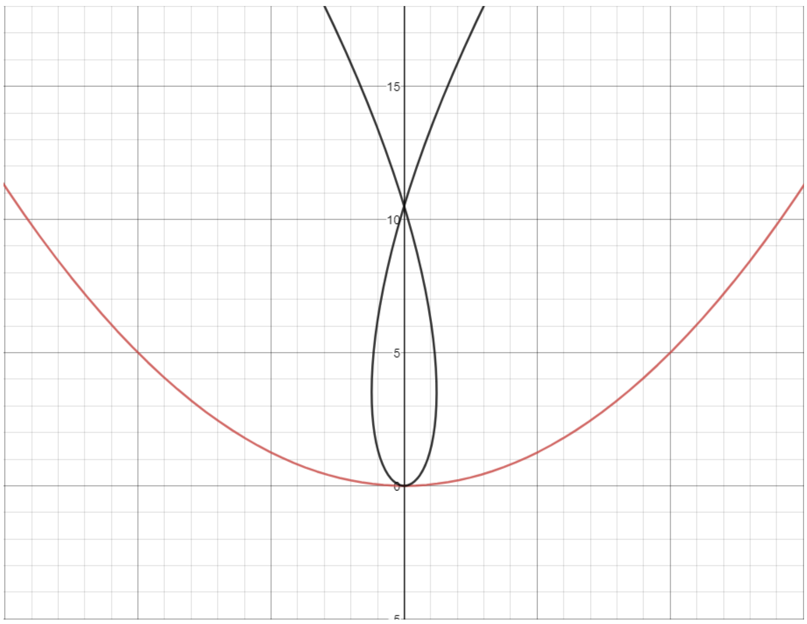


Figure 3: $k = 0.05$

Hopefully this should give some visual intuition for how bending the x axis would deform a plotted set of points. The aim of this paper is to fully describe the deformed set of points in parametrized form. In the remainder of the paper, it is assumed that $f(x)$ and $g(x)$ are continuous, and so the sets of points such that $y = f(x)$ and $y = g(x)$ are curves, and as such, so is the deformed set of points. However, many of the techniques generalize nicely to non-continuous functions. By the end of the paper we will have derived the following definition of the deformed curve:

$$x(t) = \frac{-g'(t)f(\int_0^t \sqrt{(g'(u))^2 + 1} \ \ du)}{\sqrt{(g'(t))^2 + 1}} + t$$

$$y(t) = \frac{f(\int_0^t \sqrt{(g'(u))^2 + 1} \ \ du)}{\sqrt{(g'(t))^2 + 1}} + g(t)$$

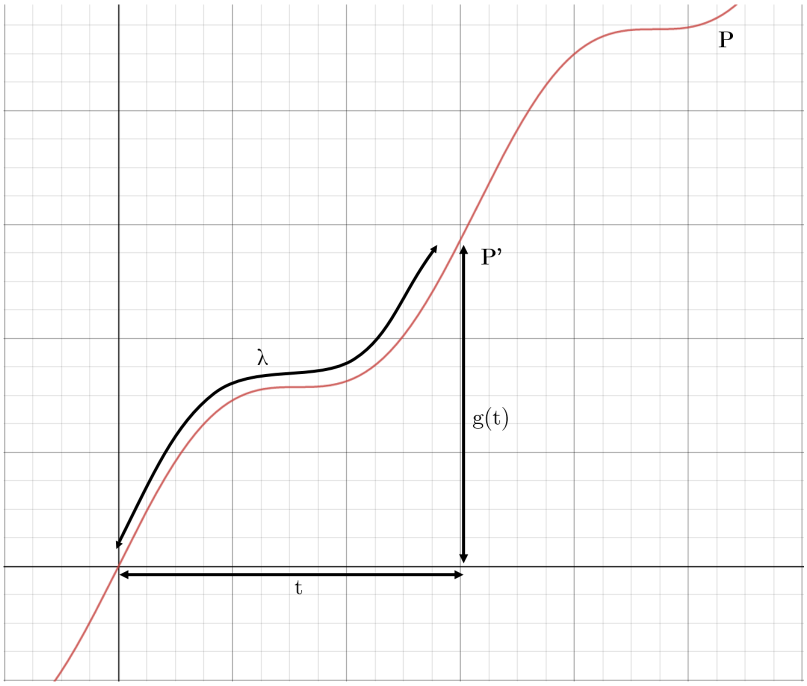
2 Precise Definition of the Curve

This notion of "bending" the x axis in order to deform a curve is an intuitively helpful, but ultimately ambiguous one. As such we will now set out a precise definition of the deformed curve L. First, imagine the curve $y = g(x)$. We will call this curve P. Before, we referred to this curve as the "bent x axis", however we will no longer call it this, as it is important to note that all of the geometry done in this paper uses the regular Cartesian notion of x and y coordinates. In other words, do not think of this transformation as the x axis itself changing, but rather simply choosing to plot the function $f(x)$ from a curve other than the x axis (specifically, curve P). Now we can define the curve L as the set of all points which, for some point P' on P which is a is a distance λ along P, lie a distance of $f(\lambda)$ along the line which is normal to P at P'.

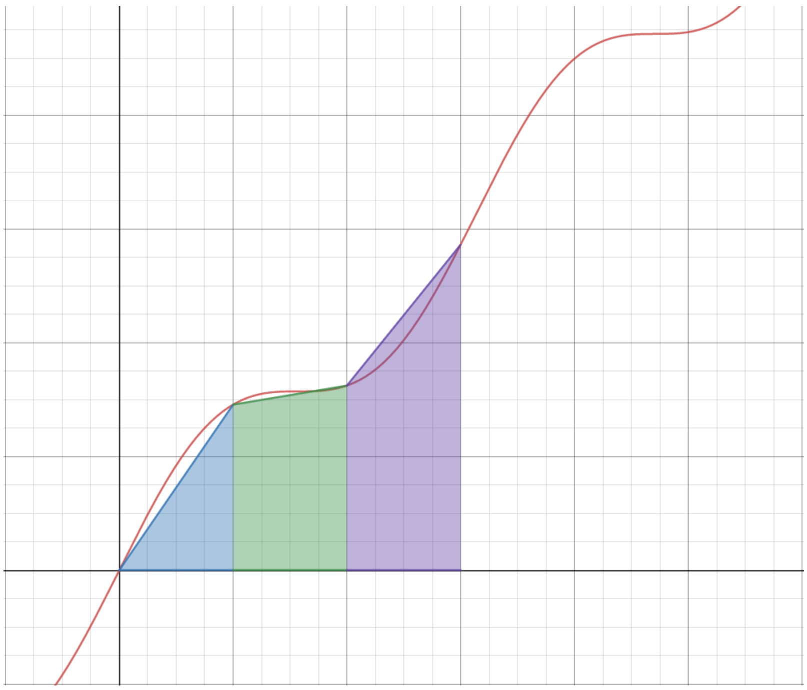
3 Deriving the Curve

3.1 Calculating Arclength

The first step in deriving the parametrization of L is calculating the arclength of P at a given x coordinate. However, for reasons which will come apparent later, we will refer to this x coordinate as t, and it will be the parameter of our curve L. If we define P' to be the point $(t, g(t))$, then the arclength of P between 0 and t will be the value λ as defined in Section 2.



In order to calculate λ, we can split the curve up into columns of width Δu and take the sum of their arclengths. We can approximate these arclengths by modelling them as straight lines.



Now, in order to calculate the arclength of the column starting at $x = t'$, we need to know the change in x value, which is simply the width of the column Δu, and the change in y value, which is $g(t' + \Delta u) - g(t')$. However, we will advance this value by $\frac{\Delta u}{\Delta u}$ for reasons which will become apparent later. As such, the change in y value is $\frac{g(t'+\Delta u)-g(t')}{\Delta u} \cdot \Delta u$ According to the Pythagorean theorem, the arclength of the column starting at t' is $\sqrt{(\Delta u)^2 + \left(\frac{g(t'+\Delta u)-g(t')}{\Delta u} \cdot \Delta u\right)^2}$. We can simplify this as follows:

$$\sqrt{(\Delta u)^2 + \left(\frac{g(t' + \Delta u) - g(t')}{\Delta u} \cdot \Delta u\right)^2} = \sqrt{(\Delta u)^2 \left(1 + \left(\frac{g(t' + \Delta u) - g(t')}{\Delta u}\right)^2\right)}$$

$$= \sqrt{1 + \left(\frac{g(t' + \Delta u) - g(t')}{\Delta u}\right)^2} \sqrt{(\Delta u)^2}$$

$$= \sqrt{1 + \left(\frac{g(t' + \Delta u) - g(t')}{\Delta u}\right)^2} \quad \Delta u$$

As such, the sum of the arclengths of all of the columns can be calculated as follows:

$$\sum_{u=0}^{t} \sqrt{1 + \left(\frac{g(u + \Delta u) - g(u)}{\Delta u}\right)^2} \quad \Delta u$$

It is important to note that the summation variable u is increasing in increments of Δu. As previously noted, this is only an approximation of the arclength of the actual curve P between 0 and t. However, this approximation is arbitrarily good for sufficiently small Δu, and in the limiting case wherein Δu approaches 0, this approximation will be equal to the desired arclength λ.

$$\lambda = \lim_{\Delta u \to 0} \sum_{u=0}^{t} \sqrt{1 + \left(\frac{g(u + \Delta u) - g(u)}{\Delta u}\right)^2} \quad \Delta u$$

Luckily, we can simplify this formula by taking a look at the definition of the derivative and the definite integral. The definition of the derivative is as follows:

$$g'(u) = \lim_{\Delta u \to 0} \frac{g(u + \Delta u) - g(u)}{\Delta u}$$

The definition of a definite integral is as follows:

$$\int_a^b A(u) \ du = \lim_{\Delta u \to 0} \sum_{u=a}^{b} (A(u) \ \ \Delta u)$$

Where again, the summation variable u on the right hand side is increasing in increments of Δu. We can substitute these definitions into our formula for λ to simplify:

$$\lambda = \int_0^t \sqrt{(g'(u))^2 + 1} \ \ du$$

To recap, this value λ is the distance along the curve P to the point P', where P' is the point on P with an x coordinate of t. We chose t as the parameter of the curve L because, since P is the curve plotted by a real-valued function $g(x)$, there are no two points on P with the same x coordinate. In other words, there is a one-to-one continuous mapping between values of t and points on P.

3.2 Normal Vector to P'

A point is on the curve L if it lies a distance of $f(\lambda)$ along the line which is normal to P at P'. We will describe this line using vectors. First, we will calculate the gradient of the tangent line to P at P'. Since P' has an x coordinate of t, the gradient will be $g'(t)$. Since the normal line is perpendicular to the tangent, and the gradient of a perpendicular to a line is the negative reciprocal of the gradient of the line itself, the gradient of the normal line is $\frac{1}{-g'(t)}$. This means that for every k units along the normal line in the x direction, $\frac{k}{-g'(t)}$ units are moved in the y direction. In vector form, this means that the line is defined as the set of all vectors of the form:

$$\begin{bmatrix} t \\ g(t) \end{bmatrix} + k \begin{bmatrix} 1 \\ \frac{1}{-g'(t)} \end{bmatrix}$$

For all real values of k. We now want to normalize the vector on the right (i.e. choose a value of k such that the magnitude of the vector is 1). We can achieve this by first scaling the vector by $-g'(t)$ in order to remove the fraction from the vector. As such, we will choose a new sliding component j.

$$\overrightarrow{n} = j \begin{bmatrix} -g'(t) \\ 1 \end{bmatrix}$$

We can calculate the magnitude of this vector as follows:

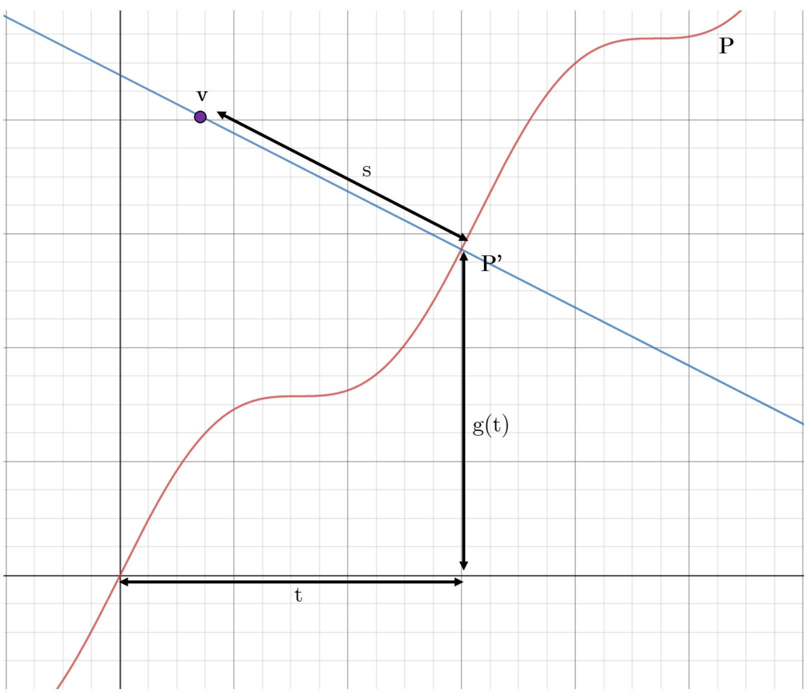
$$|\overrightarrow{n}| = |j|\sqrt{(g'(t))^2 + 1}$$

We can assert that this is equal to 1 and then solve for j:

$$\begin{aligned} 1 &\overset{!}{=} |j|\sqrt{(g'(t))^2 + 1} \\ \therefore \quad |j| &= \frac{1}{\sqrt{(g'(t))^2 + 1}} \\ \therefore \quad j &= \pm\frac{1}{\sqrt{(g'(t))^2 + 1}} \end{aligned}$$

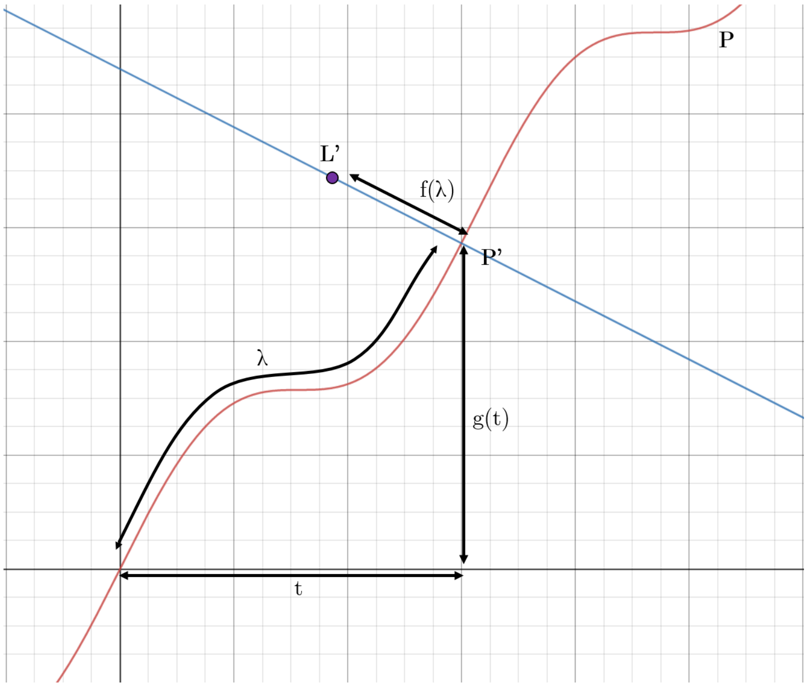
The choice of whether to use the positive value of j or the negative value of j determines which way the normal vector is to be facing. In this case, we want the positive value of j in order to preserve the sense that points on the curve L are being plotted "above" the curve P when $f(\lambda)$ is positive, and "below" the curve P when $f(\lambda)$ is negative. If we were to scale this vector by a value of s, then the magnitude would be s. As such, the vector from the origin to the point v on the normal line to P at P' which is a distance s away from P' is the following:

$$\overrightarrow{0v} = \begin{bmatrix} t \\ g(t) \end{bmatrix} + \frac{s}{\sqrt{(g'(t))^2 + 1}} \begin{bmatrix} -g'(t) \\ 1 \end{bmatrix}$$



3.3 Plotting $f(\lambda)$

For a given value of t, we now know the distance λ along the curve P until the point P', and we know the vector between the origin and the point which is an arbitrary distance s away from P' along the normal line of P at P'. We can use this information to calculate the coordinates of the point L' which is the point on the curve L for a given value of t. By definition, L' should be the point which is a distance of $f(\lambda)$ away from P' along the normal line to P at P'. In other words, it is the point v as defined above, evaluated at $s = f(\lambda)$.



As such, we know the formula for the vector between the origin and L':

$$\overrightarrow{0L'} \begin{bmatrix} t \\ g(t) \end{bmatrix} + \frac{f(\lambda)}{\sqrt{(g'(t))^2+1}} \begin{bmatrix} -g'(t) \\ 1 \end{bmatrix}$$

Therefore we can write the x and y values of L' as follows:

$$x = \frac{-g'(t)f(\lambda)}{\sqrt{(g'(t))^2+1}} + t$$

$$y = \frac{f(\lambda)}{\sqrt{(g'(t))^2+1}} + g(t)$$

Since L' is dependent on t, we can rewrite these coordinates as functions of t:

$$x(t) = \frac{-g'(t)f(\lambda)}{\sqrt{(g'(t))^2+1}} + t$$

$$y(t) = \frac{f(\lambda)}{\sqrt{(g'(t))^2+1}} + g(t)$$

Finally, we can substitute in our definition of $\lambda = \int_0^t \sqrt{(g'(u))^2+1}\ du$:

$$x(t) = \frac{-g'(t)f(\int_0^t \sqrt{(g'(u))^2+1}\ du)}{\sqrt{(g'(t))^2+1}} + t$$

$$y(t) = \frac{f(\int_0^t \sqrt{(g'(u))^2+1}\ du)}{\sqrt{(g'(t))^2+1}} + g(t)$$

This is our parametrization of the curve L as defined in the introduction. We will now explore an example.

4 Example

This section gives an example of the parametrization of L for $f(x) = x^2$ and $g(x) = \sin(x)$:

$$x(t) = \frac{-\cos(t)(\int_0^t \sqrt{\cos^2(u)+1}\ du)^2}{\sqrt{\cos^2(t)+1}} + t$$

$$y(t) = \frac{(\int_0^t \sqrt{\cos^2(u)+1}\ du)^2}{\sqrt{\cos^2(t)+1}} + \sin(t)$$

Below is a graph of this curve, where the x axis is not shown, but is instead replaced by the curve $y = \sin(x)$: