20/10/20

Morgan Saville

Supervision Work 1

A10) a) $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

b) $\vec{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\left(\begin{pmatrix}3\\1\\4\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}\right)$

$= \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\4\end{pmatrix}$

Suppose $\vec{r} = \begin{pmatrix}-2\\0\\4\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\4\end{pmatrix}$

$\therefore \cancel{-2 = 1 + 2\lambda}$ $\quad 0 = 1 + 0\lambda = 1$

$\therefore \cancel{\lambda = -\tfrac{3}{2}}$ $\quad$ which is a contradiction

Suppose $\vec{r} = \begin{pmatrix}5\\1\\8\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\4\end{pmatrix}$

$\therefore 5 = 1 + 2\lambda$ $\quad 8 = 0 + 4\lambda$ $\quad 1 = 1 + 0\lambda$

$\therefore \lambda = 2$ $\quad \therefore \lambda = 2$

$\therefore$ no contradiction

Suppose $\vec{r} = \begin{pmatrix}1\\1\\-4\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\4\end{pmatrix}$

$\therefore 1 = 1 + 2\lambda$ $\quad -4 = 0 + 4\lambda$

$\therefore \lambda = 0$ $\quad \therefore \lambda = -1$

which is a contradiction

Suppose $\vec{r} = \begin{pmatrix}-1/2\\1/2\\2\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\0\\4\end{pmatrix}$

$\therefore 1/2 = 1 + 0\lambda = 1$

which is a contradiction

$\therefore$ only $\vec{r} = \cancel{d} = \begin{pmatrix}5\\1\\8\end{pmatrix}$ is on the line through $\vec{a}$ and $\vec{b}$

A11) a) For the line through $\vec{a}$ and $\vec{b}$

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda\left(\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

For the line through $\vec{c}$ and $\vec{d}$

$$\vec{r} = \vec{c} + \lambda(\vec{d} - \vec{c})$$

$$= \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \lambda\left(\begin{pmatrix} 2 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

b) For the line through $\vec{a}$ and $\vec{b}$

$$\frac{x-1}{3} = \frac{y-1}{1} = \frac{z-1}{4}$$

For the line through $\vec{c}$ and $\vec{d}$

$$\frac{x-3}{-1} = \frac{y-1}{-1} = \frac{z-6}{1}$$

$$\vec{r}_1 = \vec{r}_2 \iff \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}3\\1\\6\end{pmatrix} + \mu\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$\therefore 1 + 3\lambda = 3 - \mu$

$\therefore \mu = 2 - 3\lambda$

$1 + \lambda = 1 - \mu = 1 - (2 - 3\lambda)$

$= 3\lambda - 1$

$\therefore 2\lambda = 2$

$\therefore \lambda = 1$, $\mu = -1$

To check that the lines do intersect:

$1 + 1\cdot 4 = 5 = 6 + (-1)\cdot 1$ ✓

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}4\\2\\5\end{pmatrix}$$

$\therefore$ The lines intersect at the point with position vector $\begin{pmatrix}4\\2\\5\end{pmatrix}$

Alternatively,

$\frac{x-1}{3} = y - 1 = \frac{z-1}{4}$ (*) $\Longrightarrow 4x - 4 = 12y - 12 = 3z - 3$

and $3 - x = 1 - y = z - 6$ (**)

$\therefore (4x-4) + 4(3-x) = (12y-12) + 4(1-y)$

$\therefore 8 = 8y - 8$

$\therefore y = 2$

B4)

$\therefore 3 - x = 1 - 2 = -1$

$\therefore x = 4$

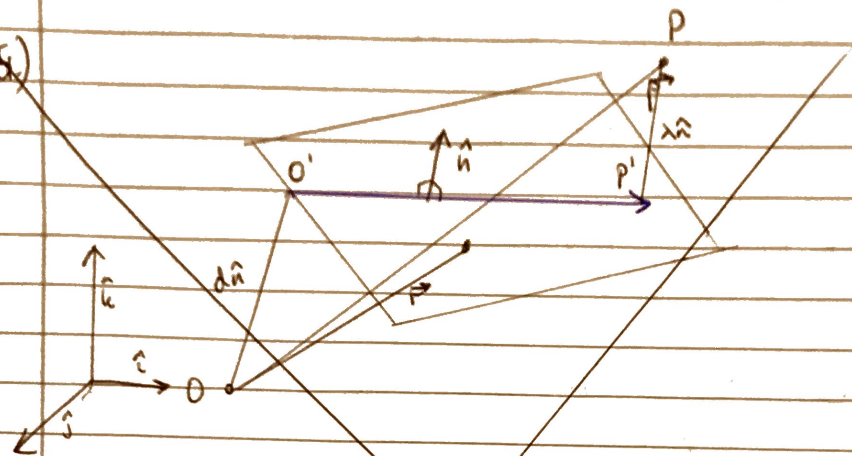
$z - 6 = 1 - 2 = -1$

$\therefore z = \cancel{7}\ 5$

$(x, y, z) = (4, 2, 5)$ satisfies both $(*)$ and $(**)$

$\therefore$ the lines intersect at the point with position vector $\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$

B5)



Let O' = the closest point on the plane to O
Let P' = the closest point on the plane to P
such that $\overrightarrow{O'P'}$ is in the plane

$$\vec{p} = \overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P'} + \overrightarrow{P'P}$$
$$= d\hat{n} + \overrightarrow{O'P'} + \lambda\hat{n}$$
$$= (d + \lambda)\hat{n} + \overrightarrow{O'P'}$$

$$\therefore \vec{p} \cdot \hat{n} = (d+\lambda)\hat{n} \cdot \hat{n} + \overrightarrow{O'P'} \cdot \hat{n}$$
$$= d + \lambda + 0$$

$$\therefore \lambda = \vec{p} \cdot \hat{n} - d$$

B4) a) $n = \overrightarrow{AB} \times \overrightarrow{AC}$

$$= \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \times \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\hat{n} = \frac{1}{|n|} n = \frac{1}{\sqrt{1+4+0}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

b) $\left(\vec{r} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = 0$

c) $\cancel{\left(\vec{r} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\right) \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = 0}$ ~~where~~

Let $\vec{r}$ = the position vector of the point in the plane closest to the origin

$\therefore \vec{r} = \lambda \hat{n}$ where λ = the perpendicular distance to the plane from the origin

$(\vec{r} - \vec{a}) \cdot \hat{n} = 0$ for some point with position vector $\vec{a}$ on the plane

$\therefore \vec{r} \cdot \hat{n} - \vec{a} \cdot \hat{n} = 0$

$\therefore \lambda \hat{n} \cdot \hat{n} - \vec{a} \cdot \hat{n} = 0$

$\therefore \lambda = \vec{a} \cdot \hat{n}$ $\forall$ point on the plane with position vector $\vec{a}$

$\therefore$ w.l.o.g. take $\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$$\therefore \lambda = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}}(-3 + 4 + 0)$$

$$= \frac{1}{\sqrt{5}}$$