Discrete Maths 8

9.1.1 $\overrightarrow{R_2}(\mathbb{N}) = \mathbb{Z}$

$\overleftarrow{R_2}(\mathbb{N}) = \{m \in \mathbb{N} \mid \forall n \in \mathbb{Z}.\ m = n^2 \Rightarrow n \in \mathbb{N}\}$

$= \mathbb{N} \setminus \{n^2 \mid n \in \mathbb{N} \wedge n > 0\}$

9.3a. $\overrightarrow{R}(\bigcup F) = \{b \in B \mid \exists X \in F.\ \exists a \in X.\ a R b\}$

$= \{b \in B \mid \exists X \in F.\ b \in \overrightarrow{R}(X)\}$

$= \bigcup \{\overrightarrow{R}(X) \mid X \in F\}$

b. $\overleftarrow{R}(\bigcap G) = \{a \in A \mid \forall b \in B.\ (a R b \Rightarrow b \in \bigcap G)\}$

$= \{a \in A \mid \forall b \in B.\ (a R b \Rightarrow \forall Y \in G.\ b \in Y)\}$

$= \{a \in A \mid \forall Y \in G.\ \forall b \in B.\ (a R b \Rightarrow b \in Y)\}$

$= \bigcap \{\overleftarrow{R}(Y) \mid \cancel{\ } Y \in G\}$

10.1a Let S be an arbitrary finite set with cardinality $n \in \mathbb{N}$

- $\# S = n = \# [n]$
- $[n] \subseteq \mathbb{N}$

$\therefore S \overset{n}{\cong} [n] \subseteq \mathbb{N}$

$\therefore S$ is countable

b. $\mathbb{N} \overset{\sim}{=} \mathbb{N} \subseteq \mathbb{N}$

$\therefore \mathbb{N}$ is countable

Consider $f : \mathbb{N} \Rightarrow \mathbb{Z} : n \mapsto \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \text{ is even} \\ -\lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \end{cases}$

Proof of injectivity:

RTP: ~~$\forall a \in \mathbb{N}, b, b' \in \mathbb{Z}$~~

$\forall a, a' \in \mathbb{N}, b \in \mathbb{Z}. \ a f b \wedge a' f b \Rightarrow a = a'$

Let $a, a' \in \mathbb{N}$, $b \in \mathbb{Z}$

Assume $a f b \wedge a' f b$

Case 0: a is even:

$\therefore \lceil \frac{a}{2} \rceil = b = \lceil \frac{a'}{2} \rceil \quad \therefore a'$ is even

$\therefore a = a' = 2b$

Case 1: a is odd

$\therefore -\lceil \frac{a}{2} \rceil = b = -\lceil \frac{a'}{2} \rceil \quad \therefore a'$ is odd

$\therefore -(a+1) = 2b = -(a'+1)$

$\therefore a = a'$

$\square$

Proof of surjectivity.

RTP: $\forall b \in \mathbb{Z}. \ \exists a \in \mathbb{N}. \ a f b$

Let $b \in \mathbb{Z}$.

Case 0: ~~[illegible]~~ $b \geq 0$

$\therefore$ Consider $a = 2b$

$\therefore \lceil \frac{a}{2} \rceil = \lceil \frac{2b}{2} \rceil = \lceil b \rceil = b$

$\therefore a f b$

Case 1: $b < 0$

Consider $a = -(2b+1)$

$-\lceil \frac{a}{2} \rceil = -\lceil \frac{-2b+1}{2} \rceil = -\lceil -b - \frac{1}{2} \rceil = b$

$\therefore a f b$

$\square$

$\therefore f$ is a bijection

$\therefore \mathbb{Z} \cong \mathbb{N} \subseteq \mathbb{N}$

$\therefore \mathbb{Z}$ is countable

Consider $g: \mathbb{Q} \Rightarrow \mathbb{Z} \times \mathbb{N} : \frac{p}{q} \longmapsto (p, q)$

g is clearly injective $\therefore$ there exists a bijection
$\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ (by 10.3)

$\therefore \mathbb{Z}$ is countable $\wedge$ $\mathbb{N}$ is countable

$\Rightarrow \mathbb{Z} \times \mathbb{N}$ is countable (by 10.2b)

$\Rightarrow \mathbb{Q}$ is countable

102.1 ~~[illegible]~~ Let S an infinite set
Let $f: \mathbb{N} \to S$ be a surjection

$\therefore \forall s \in S . \exists n \in \mathbb{N} . nfs$ $(\dagger)$

~~Let $g: \mathbb{N} \to S$~~

Let $g: S \longrightarrow \mathbb{N}: s \longmapsto \min\{n \in \mathbb{N} \mid nfs\}$

g is a total function by $(\dagger)$

g is ~~surjective~~ injective because f is a function

g is surjective because S is infinite

$\therefore g$ is a bijection

$\therefore g^{-1}$ is a bijection $\mathbb{N} \longmapsto S$

2a. Let X be a countable set
Let $S \subseteq X$

~~[illegible]~~ $\exists Y \subseteq \mathbb{N}$ s.t. $X \cong Y$
$\therefore \exists Z \subseteq Y \subseteq \mathbb{N}$ s.t. $S \cong Z$

$\therefore S$ is countable

6. Let S, T ~~sets~~ countable sets

$\therefore$ Let $X \subseteq \mathbb{N}$. f a bijection $X \to S$

Let $Y \subseteq \mathbb{N}$. g a bijection $Y \to T$

Let $Z \subseteq \mathbb{N}$ s.t. $\# Z = \# X + \# Y$

RTP: $\exists$ a bijection $Z \to S \uplus T$

Consider $h : Z \to S \uplus T : n \longmapsto \begin{cases} (f(\lfloor \frac{n}{2} \rfloor), 0) & \text{if } n \text{ is even} \\ (g(\lfloor \frac{n}{2} \rfloor), 1) & \text{if } n \text{ is odd} \end{cases}$

h is a bijection

$\therefore S \uplus T$ is countable.

<u>Product:</u>

Case 0: either S or T or both is infinite

$\forall n \in X$. Let $s_n \in S$ s.t. $f(n) = s_n$

$\forall n \in Y$ Let $t_n \in T$ s.t. $g(n) = t_n$

Consider the infinite table

$$\begin{array}{cccc}
(s_0, t_0) & (s_0, t_1) & (s_0, t_2) & \cdots \\
(s_1, t_0) & (s_1, t_1) & (s_1, t_2) & \cdots \\
(s_2, t_0) & (s_2, t_1) & (s_2, t_2) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

Starting at (s_0, t_0) and following the diagonal path for n steps and reaching (s_u, t_v).

$j : \mathbb{N} \to S \times T : n \mapsto (s_u, t_v)$ is a bijection

$\therefore S \times T$ is countable

Case 1: S and T are both finite.

$\#(S \times T) = (\# S) \times (\# T) = \# [(\# S) \times (\# T)]$

$\therefore S \times T \cong [(\# S) \times (\# T)] \subseteq \mathbb{N}$

$\therefore S \times T$ is countable

3. Let S an infinite set

~~c ⇒ a by 10.2.1~~

~~RTP: b ⇒ a~~

$c \Rightarrow a$ by 10.2.1

$a \Rightarrow c$ by definition

$\therefore a \Leftrightarrow c$

RTP. $b \Leftrightarrow a$

"$\Leftarrow$" first:

There is a bijection $\mathbb{N} \to S$

$\Rightarrow$ There is a bijection $S \to \mathbb{N}$

$\Rightarrow$ There is an injection $S \to \mathbb{N}$

$\square$

"$\Rightarrow$" next.

Assume there is an injection $f: S \to \mathbb{N}$

~~$\therefore \forall x, x' \in S,\ y \in \mathbb{N}$~~

$\therefore \forall x, x' \in S,\ y \in \mathbb{N}.\ x f y \wedge x' f y \Rightarrow x = x'$

$\therefore S$ is countable

$\therefore$ There exists a bijection $S \to \mathbb{N}$

$\square$

4. Let f an injection $P(X) \to X$

Let g be the retraction of f

Consider $B = \{A \subseteq X \mid f(A) \notin A\}$

$\therefore g(B) \in B \iff g(B) \notin B$

which is a contradiction

$\therefore \nexists$ an injection $P(X) \to X$