Morgan Saville

Maths Supervision Work 5

14. $I = \int_0^x e^{at} \cos(bt)\, dt$

$J = \int_0^x e^{at} \sin(bt)\, dt$

a. By integration by parts on I:

	D	I
+	e^{at}	$\cos(bt)$
−	ae^{at}	$\frac{1}{b}\sin(bt)$

$$I = \left[\frac{1}{b} e^{at} \sin(bt)\right]_0^x - \frac{a}{b}\int_0^x e^{at} \sin(bt)\, dt$$

$$= \frac{1}{b} e^{ax} \sin(bx) - \frac{a}{b} J$$

b. ~~By integration by parts on J:~~

	D	I
+	~~$\sin(bt)$~~	~~e^{at}~~
−	~~$b\cos(bt)$~~	~~$\frac{1}{a} e^{at}$~~

$$\cancel{J = \left[\frac{1}{a} e^{at} \sin(bt)\right]_0^x - \frac{b}{a}\int_0^x e^{at} \cos(bt)\, dt}$$

$$\cancel{= \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a}}$$

By integration by parts on I:

	D	I
+	$\cos(bt)$	e^{at}
−	$-b\sin(bt)$	$\frac{1}{a} e^{at}$

$$I = \left[\tfrac{1}{a}\cos(bt)e^{at}\right]_0^x + \tfrac{b}{a}\int_0^x e^{at}\sin(bt)\,dt$$

$$= \tfrac{1}{a}\left(e^{ax}\cos(bx) - 1\right) + \tfrac{b}{a}J$$

c. $\therefore \tfrac{1}{a}\left(e^{ax}\cos(bx) - 1\right) + \tfrac{b}{a}J = \tfrac{1}{b}e^{ax}\sin(bx) - \tfrac{a}{b}J$

$$b\left(e^{ax}\cos(bx) - 1\right) + b^2 J = ae^{ax}\sin(bx) - a^2 J$$

$$\therefore (a^2+b^2)J = e^{ax}\left(a\sin(bx) - b\cos(bx)\right) + b$$

$$\therefore J = \frac{e^{ax}\left(a\sin(bx) - b\cos(bx)\right) + b}{a^2+b^2}$$

$$I = \tfrac{1}{b}e^{ax}\sin(bx) - \tfrac{a}{b}J$$

$$= \tfrac{1}{b}e^{ax}\sin(bx) - \frac{ae^{ax}\left(a\sin(bx) - b\cos(bx)\right) + ab}{b(a^2+b^2)}$$

$$= \tfrac{1}{b}\left(\frac{e^{ax}(a^2+b^2)\sin(bx) - e^{ax}\left(a^2\sin(bx) - ab\cos(bx)\right) + ab}{a^2+b^2}\right)$$

$$= \frac{e^{ax}\left(b\sin(bx) + a\cos(bx)\right) + a}{a^2+b^2}$$

d. $I + iJ = \int_0^x e^{at}\cos(bt)\,dt + i\int_0^x e^{at}\sin(bt)\,dt$

$$= \int_0^x e^{at}\left(\cos(bt) + i\sin(bt)\right)dt$$

$$= \int_0^x e^{at}e^{ibt}\,dt$$

$$= \int_0^x e^{(a+bi)t}\,dt$$

$$= \left[\frac{1}{a+bi}e^{(a+bi)t}\right]_0^x$$

$$\therefore I + iJ = \frac{1}{a+bi}\left(e^{(a+bi)x} - 1\right)$$

$$= \frac{(a-bi)}{a^2+b^2}\left(e^{ax}(\cos(bx) + i\sin(bx)) - 1\right)$$

$$= \frac{(a-bi)(e^{ax}\cos(bx) + ie^{ax}\sin(bx) - 1)}{a^2+b^2}$$

$$= \frac{ae^{ax}\cos(bx) + aie^{ax}\sin(bx) - a - bie^{ax}\cos(bx) + be^{ax}\sin(bx) + bi}{a^2+b^2}$$

$$= \frac{e^{ax}(b\sin(bx) + a\cos(bx)) + a}{a^2+b^2} + i\,\frac{e^{ax}(a\sin(bx) - b\cos(bx)) + b}{a^2+b^2}$$

$$\therefore I = \frac{e^{ax}(b\sin(bx) + a\cos(bx)) + a}{a^2+b^2}$$

$$J = \frac{e^{ax}(a\sin(bx) - b\cos(bx)) + b}{a^2+b^2}$$

15a. Let $I_n = \int_0^1 (1-x^{n-1})^n \, dx$

By integration by parts on I_n:

	D	I
+	$(1-x^{n-1})^n$	1
-	$-n(n-1)x^{n-2}(1-x^{n-1})^{n-1}$	x

$$I_n = \left[x(1-x^{n-1})^n\right]_0^1 + n(n-1)\int_0^1 x^{n-1}(1-x^{n-1})^{n-1} \, dx$$

$$= n(n-1)\int_0^1 x^{n-1}(1-x^{n-1})^{n-1} \, dx$$

$$= n(1-n)\int_0^1 -x^{n-1}(1-x^{n-1})^{n-1} \, dx$$

$$= n(1-n)\int_0^1 \left((1-x^{n-1})(1-x^{n-1})^{n-1} - (1-x^{n-1})^{n-1}\right) dx$$

$$\therefore \cancel{I_n = n(1-n)\left(\int_0^1 (1-x^{n-1})^n dx - \int_0^1 (1-x^{n-1})^{n-1} dx\right)}$$

Let $I_n = \int_0^1 (1-x^3)^n \, dx$

By integration by parts on I_n:

	D	I
+	$(1-x^3)^n$	1
−	$-3n(1-x^3)^{n-1}x^2$	x

$$I_n = \left[x(1-x^3)^n\right]_0^1 + 3n\int_0^1 x^3(1-x^3)^{n-1}\,dx$$

$$= 3n\int_0^1 x^3(1-x^3)^{n-1}\,dx$$

$$= -3n\int_0^1 -x^3(1-x^3)^{n-1}\,dx$$

$$= -3n\int_0^1 \left((1-x^3)(1-x^3)^{n-1} - (1-x^3)^{n-1}\right)dx$$

$$= -3n\int_0^1 \left((1-x^3)^n - (1-x^3)^{n-1}\right)dx$$

$$= -3n\left(\int_0^1 (1-x^3)^n\,dx - \int_0^1 (1-x^3)^{n-1}\,dx\right)$$

$$= -3n(I_n - I_{n-1})$$

$$= 3nI_{n-1} - 3nI_n$$

$$\therefore (1+3n)I_n = 3nI_{n-1}$$

$$\therefore I_n = \frac{3n}{1+3n}I_{n-1}$$

$$= \left(1 - \frac{1}{3n+1}\right)I_{n-1}$$

$$I_0 = \int_0^1 1\,dx = 1$$

$$I_1 = \tfrac{3}{4} I_0 = \tfrac{3}{4}$$

$$I_2 = \tfrac{6}{7} I_1 = \tfrac{9}{14}$$

$$I_3 = \frac{9}{10} I_2 = \frac{81}{140}$$

$$I_4 = \frac{12}{13} I_3 = \frac{243}{455}$$

b. let $I_n = \int_0^{\pi/2} x^n \sin x \, dx$

By integration by parts on I_n:

	D	I
$+$	x^n	$\sin x$
$-$	nx^{n-1}	$-\cos x$
$+$	$n(n-1)x^{n-2}$	$-\sin x$

$$I_n = \left[-x^n \cos x + n x^{n-1} \sin x\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-1} \sin x \, dx$$

$$= \left(n\left(\tfrac{\pi}{2}\right)^{n-1} + \cancel{\left(\tfrac{\pi}{2}\right)}\right) - n(n-1)\, I_{n-2}$$

$$I_0 = \int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2} = 1$$

$$I_2 = \pi - 2I_0 = \pi - 2$$

$$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 12 I_2$$

$$= \frac{\pi^3}{2} - 12\pi + 24$$

$$I_6 = 6\left(\frac{\pi}{2}\right)^5 - 30 I_4$$

$$= \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720$$

c. Let $I_n = \int_0^1 x^n e^x \, dx$

By integration by parts on I_n

	D	I
+	x^n	e^x
−	nx^{n-1}	e^x

$$I_n = \left[x^n e^x\right]_0^1 - n\int_0^1 x^{n-1} e^x$$

$$= e - n I_{n-1}$$

$$I_0 = \int_0^1 e^x \, dx = \left[e^x\right]_0^1 = e - 1$$

$$I_1 = e - I_0 = 1$$

$$I_2 = e - 2I_1 = e - 2$$

$$I_3 = e - 3 I_2 = 6 - 2e$$

$$I_4 = e - 4 I_3 = 9e - 24$$

$$I_5 = e - 5 I_4 = 120 - 44e$$

16a. i $J(0,0) = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$

ii $J(0,1) = \int_0^{\pi/2} \sin\theta \, d\theta = 1$

iii $J(1,0) = \int_0^{\pi/2} \cos\theta \, d\theta = 1$

iv $J(1,1) = \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$

$= \frac{1}{2} \int_0^{\pi/2} 2\cos\theta \sin\theta \, d\theta$

$= \frac{1}{2} \int_0^{\pi/2} \sin 2\theta \, d\theta$

$= \frac{1}{2} \left[-\frac{1}{2} \cos 2\theta \right]_0^{\pi/2}$

$= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \right)$

$= \frac{1}{2}$

v. $J(m,1) = \int_0^{\pi/2} \cos^m\theta \sin\theta \, d\theta$

$= -\int_0^{\pi/2} -\sin\theta \cos^m\theta$

$= -\left[\frac{1}{m+1} \cos^{m+1}\theta \right]_0^{\pi/2}$

$= -\left(0 - \frac{1}{m+1} \right)$

$= \frac{1}{m+1}$

vi $J(1,n) = \int_0^{\pi/2} \cos\theta \sin^n\theta \, d\theta$

$= \left[\frac{1}{n+1} \sin^{n+1}\theta \right]_0^{\pi/2}$

$= \frac{1}{n+1}$

b. By integration by parts on $J(m,n)$:

~~D~~	~~I~~
~~$\cos^m\theta$~~	~~$\sin^n\theta$~~
~~$-m\sin\theta\cos^{m-1}\theta$~~	

	D	I
+	$\cos^{m-1}\theta$	$\sin^n\theta\cos\theta$
−	$-(m-1)\cos^{m-2}\theta\sin\theta$	$\frac{1}{n+1}\sin^{n+1}\theta$

$$\therefore J(m,n) = \left[\frac{1}{n+1}\cos^{m-1}\theta\sin^{n+1}\theta\right]_0^{\pi/2} + \frac{m-1}{n+1}\int_0^{\pi/2}\cos^{m-2}\theta\sin^{n+2}\theta\,d\theta$$

$$= \frac{m-1}{n+1}J(m-2,n+2)$$

$$\therefore J(m-2,n+2) = \frac{n+1}{m-1}J(m,n) \qquad (*)$$

Next, note:

$$J(m,n) = \int_0^{\pi/2}\cos^m\theta\sin^n\theta\,d\theta$$

$$= \int_0^{\pi/2}\cos^m\theta\sin^n\theta\left(\cos^2\theta+\sin^2\theta\right)d\theta$$

$$= \int_0^{\pi/2}\left(\cos^{m+2}\theta\sin^n\theta + \cos^m\theta\sin^{n+2}\theta\right)d\theta$$

$$= \int_0^{\pi/2}\cos^{m+2}\theta\sin^n\theta\,d\theta + \int_0^{\pi/2}\cos^m\theta\sin^{n+2}\theta\,d\theta$$

$$= J(m+2,n) + J(m,n+2) \qquad (**)$$

$$\therefore J(m-2,n) = J(m,n) + J(m-2,n+2)$$

Substituting $(*)$,

$$J(m-2,n) = J(m,n)\left(1+\frac{n+1}{m-1}\right)$$

$$\therefore J(m-2,n) = J(m,n)\left(\frac{m+n}{m-1}\right)$$

$$\therefore J(m,n) = \frac{m-1}{m+n} J(m-2,n)$$

~~Substituting n-2 into (* *),~~

$$\cancel{J(m, n-2) = J(m+2, n-2) + J(m-2, n)}$$

From (*),

$$J(m,n) = \frac{n-1}{m+1} J(m+2, n-2) \quad (***)$$

$$\therefore J(m+2,n-2) = \frac{m+1}{n-1} J(m,n)$$

From (**),

$$J(m, n-2) = J(m+2, n-2) + J(m,n)$$

Substituting (***),

$$J(m, n-2) = \left(1+\frac{m+1}{n-1}\right) J(m,n)$$

$$= \frac{m+n}{n-1} J(m,n)$$

$$\therefore J(m,n) = \frac{n-1}{m+n} J(m, n-2) = \frac{m-1}{m+n} J(m-2,n)$$

As required.

c. i. $J(5,3) = \frac{4}{8} J(3,3) = \frac{1}{2}\cdot\frac{2}{6} J(1,3)$

$= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{4} J(1,1)$

$= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$= \frac{1}{24}$

~~ii. J(6,5)~~

ii. $J(6,5) = \frac{5}{11} J(4,5)$

$= \frac{5}{11} \cdot \frac{3}{9} J(2,5)$

$= \frac{5}{11} \cdot \frac{1}{3} \cdot \frac{1}{7} J(0,5)$

$= \frac{5}{11} \cdot \frac{1}{3} \cdot \frac{1}{7} \cdot \frac{4}{5} J(0,3)$

$= \frac{5}{11} \cdot \frac{1}{3} \cdot \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} J(0,1)$

$= \frac{8}{693}$

iii. $J(4,8) = \frac{3}{12} J(2,8)$

$= \frac{1}{4} \cdot \frac{1}{10} J(0,8)$

$= \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{7}{8} J(0,6)$

$= \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot J(0,4)$

$= \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot J(0,2)$

$= \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{7}{8} \cdot 5/6 \cdot 3/4 \cdot \frac{1}{2} \cdot J(0,0)$

$= \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{7}{8} \cdot 5/6 \cdot 3/4 \cdot 1/2 \cdot \pi/2$

$\cancel{= \frac{105\pi}{3840}} = \frac{7\pi}{2048}$

$\cancel{= \frac{7\pi}{256}}$