Discrete Maths Supervision 5

1.1 1a. RTP $\forall$ sets A. $A \subseteq A$

Equivalently,

$\forall x.\ x \in A \Rightarrow x \in A$

which is true by modus ponens

b. RTP $\forall$ sets A, B, C. $(A \subseteq B \land B \subseteq C) \Rightarrow A \subseteq C$

Assume $A \subseteq B \land B \subseteq C$

Equivalently, $(\forall x.\ x \in A \Rightarrow x \in B) \land (\forall y.\ y \in B \Rightarrow y \in C)$ $(\dagger)$ $(\ddagger)$

RTP $(\forall z.\ z \in A \Rightarrow z \in C)$

By universal instantiation, let arbitrary $z \in A$.

By $(\dagger)$, $z \in B$

By $(\ddagger)$, $z \in C$

$\square$

c. RTP $\forall$ sets A, B. $(A \subseteq B \land B \subseteq A) \Longleftrightarrow A = B$

Start with $\Leftarrow$

Assume $A = B$

Equivalently, $\forall x.\ x \in A \Longleftrightarrow x \in B$ $(\dagger)$

RTP $A \subseteq B \land B \subseteq A$

Equivalently, $(\forall x.\ x \in A \Rightarrow x \in B) \land (\forall x.\ x \in B \Rightarrow x \in A)$ which is true by $(\dagger)$.

Next, $\Rightarrow$

Assume $A \subseteq B \land B \subseteq A$

Equivalently, $(\forall x.\ x \in A \Rightarrow x \in B) \land (\forall x.\ x \in B \Rightarrow x \in A)$ $(\ddagger)$

RTP $A = B$

Equivalently,

$\forall x.\ x \in A \Longleftrightarrow x \in B$ which is true by $(\ddagger)$

$\square$

2a. RTP $\emptyset \subseteq S$

Equivalently, $\forall x.\ x \in \emptyset \Rightarrow x \in S$

which is true because there can be no such $x \in \emptyset$

b. Start with $\Rightarrow$

Assume $\forall x.\ x \notin S$ (†)

RTP $S = \emptyset$

Equivalently, $\nexists x \in S$ which we get from (†)

Next prove $\Leftarrow$

Assume $S = \emptyset$

Equivalently $\nexists x \in S$

$\therefore \forall x.\ x \notin S$

$\square$

3a. union: $\{-1, 1, 2, 3, 4, 5, 7\}$

intersection: $\{1, 3, 5\}$

b. union: $\{x \in \mathbb{R} \mid x = 6 \vee x > 7\}$

intersection: $\{x \in \mathbb{N} \mid x > 7\}$

4. $\{(1,-1), (1,1), (1,3), (1,5), (1,7),$
$(2,-1), (2,1), (2,3), (2,5), (2,7),$
$(3,-1), (3,1), (3,3), (3,5), (3,7),$
$(4,-1), (4,1), (4,3), (4,5), (4,7),$
$(5,-1), (5,1), (5,3), (5,5), (5,7)\}$

5a. $A_2 = \{2, 3, 1, 4\}$

$A_3 = \{3, 4, 2, 6\}$

$A_4 = \{4, 5, 3, 8\}$

$A_5 = \{5, 6, 4, 10\}$

b. $\bigcup\{A_i \mid i \in I\} = \{1, 2, 3, 4, 5, 6, 8, 10\}$

$\bigcap\{A_i \mid i \in I\} = \{4\}$

$A \dot\cup B = (A \times \{1\}) \cup (B \times \{2\})$

6. $\{-1, 2, 4, 7\}$ $= \{(1,1), (2,1), \dots, (-1,2), \dots\}$

7a. Start with $\Rightarrow$

Assume $A^c = B$

Equivalently, $\forall x \in U.\ x \in A \iff x \notin B$

$A \cup B = \{x \in U \mid x \in A\} \cup \{x \in U \mid x \notin A\}$

$= \{x \in U \mid x \in A \vee x \notin A\}$

$= \{x \in U\}$

$= U$

$A \cap B = \{x \in U \mid x \in A\} \cap \{x \in U \mid x \notin A\}$

$= \{x \in U \mid x \in A \wedge x \notin A\}$

$= \emptyset$

Next prove $\Leftarrow$

Assume $A \cup B = U^{(\dagger)} \wedge A \cap B = \emptyset^{(\ddagger)}$

$\ddagger$ says $\forall x\ (x \notin A \cap B)$, then ....

By $(\ddagger)$ $\forall x \in U.\ x \in A \Rightarrow x \notin B$

By $(\dagger)$ $(\forall x \in U.\ x \in A \vee x \in B)$

$\Rightarrow (\forall x \in U.\ x \notin B \Rightarrow x \in A)$

$\therefore \forall x \in U.\ x \in A \iff x \notin B$

$\therefore A^c = B$

b. $(A^c)^c = \{x \in U \mid x \notin A^c\}$

$= \{x \in U \mid x \notin \{y \in U \mid y \notin A\}\}$

✓ $= \{x \in U \mid x \notin A = \text{false}\}$

$= \{x \in U \mid x \in A\}$

$= A$

c. $(A\cup B)^c = \{x\in U \mid x\notin A\cup B\}$ $\Rightarrow \{x \in A \mid \neg(x\in A \wedge x\in B)\}$

$= : \{x\in U \mid x\notin A \wedge x\notin B\}$

$= \{x\in U \mid x\notin A\} \cap \{x\in U \mid x\notin B\}$

$= A^c \cap B^c$

$(A\cap B)^c = \{x\in U \mid x\notin A\cap B\}$

$= \{x\in U \mid x\notin A \vee x\notin B\}$

$= \{x\in U \mid x\notin A\} \cup \{x\in U \mid x\notin B\}$

$= A^c \cup B^c$

✓

1.2.1 a. Assume $A\subseteq B$

~~Equivalently $\forall x, x\in A \Rightarrow x\in B$~~

~~$P(A) = \{x \mid x\subseteq A\}$~~ $x\subseteq A \Rightarrow x\subseteq B$

$\boxed{P(B) = \{x \mid x\subseteq B\} = \{x \mid x\subseteq A\}}$ $\cup \{x \mid x\subseteq B \wedge x\not\subseteq A\}$

$= P(A) \cup \{x \mid x\subseteq B \wedge x\not\subseteq A\}$

$\therefore P(A)\subseteq P(B)$

You already assume what you need to prove. Use the def. of subset instead.

"Let $x\in P(A)$, then .... so $x\in P(B)$"

b. $A=\{1\}$, $B=\{2\}$ ✓

$P(A\cup B) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$

$P(A)\cup P(B) = \{\{\}, \{1\}, \{2\}\}$ ~~[illegible]~~ $\not\supseteq P(A\cup B)$

c. $P(A)\subseteq P(A\cup B)$

$P(B)\subseteq P(A\cup B)$ ✓

$\therefore P(A)\cup P(B)\subseteq P(A\cup B)$

d. $P(A) = \{x \mid x \overset{\subseteq}{\in} A\}$ $\{x \mid x\in A\} = A$

$P(B) = \{x \mid x\in B\}$

$\therefore P(A)\cap P(B) = \{x \mid x\in A \wedge x\in B\}$

$= \{x \mid x\in A\cap B\}$

$= P(A\cap B)$

Try this again.

e. see d)

2a. $A \cup B = B$

$\Leftrightarrow$ ~~$\{x \mid x \in A\} \cup \{x \mid x \in B\} = \{x \mid x \in B\}$~~

$\Leftrightarrow \forall x\ (x \in A \vee x \in B) \Leftrightarrow x \in B)$

$\Leftrightarrow \forall x\ (x \in A \Rightarrow x \in B)$

b. $\Leftrightarrow A \subseteq B$

$\Leftrightarrow B = A \vee \{x \mid x \in B \wedge x \notin A\}$

c. $\Leftrightarrow A \cap B = A$

$\Leftrightarrow \forall x\ x \notin B \Rightarrow x \notin A$

d. $\Leftrightarrow B^c \subseteq A^c$

on the left elements | conditions on the right

$\{(i,j) : i \in A, j \in C\}$

3a. Assume $A \subseteq B \wedge C \subseteq D$

$\therefore A \times C = \{\forall i \in A, j \in C.\ (i,j)\}$

$= \{\forall i \in B, j \in D.\ (i,j) \mid i \in A \wedge j \in C\}$

$\subseteq \{\forall i \in B, j \in D.\ (i,j)\}$

$= B \times C$

b. ~~Let $A = \{0\}$, $B = \{1\}$, $C = \{2\}$, $D = \{3\}$~~

~~$(A \cup C) \times (B \cup D) = \{(0,1), (0,3), (2,1), (2,3)\}$~~

~~$(A \times B) \cup (C \times D) = \{(0,1)$~~

~~$(A \cup C) \times (B \cup D) = \{$~~

Let $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{d\}$

$(A \cup C) \times (B \cup D) = \{(a,b), (a,d), (c,b), (c,d)\}$

$(A \times B) \cup (C \times D) = \{(a,b), (c,d)\}$

$\therefore (A \cup C) \times (B \cup D) \not\subseteq (A \times B) \cup (C \times D)$

c. $A \subseteq A \cup C$, $B \subseteq B \cup D$

$\therefore$ By a), $A \times B \subseteq (A \cup C) \times (B \cup D)$

By symmetry, $C \times D \subseteq (A \cup C) \times (B \cup D)$

$\therefore (A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

d. ~~$A = \{a\}$, $B = \{b\}$, $D = \{d\}$~~

~~$A \times (B \cup D) = \{(a,b), (a,d)\}$~~
~~$(A \times B) \cup (A \times D) =$~~

$$A \times (B \cup D) = \{\forall i \in A, j \in B \cup D.\ (i,j)\}$$
$$= \{\forall i \in A, j \in B.\ (i,j)\} \cup \{\forall i \in A, j \in D.\ (i,j)\}$$
$$= (A \times B) \cup (A \times D)$$

e. See d)

Try again after rereading about disjoint unions!

4a) Let $A = \{1\}$, $B = \{1,2\}$, $C = \{2\}$, $D = \{1,2\}$

$$A \subseteq B \wedge C \subseteq D$$
$$A \uplus C = \{1,2\}$$
$$B \uplus D = \emptyset$$
$$\{1,2\} \not\subseteq \emptyset$$

b. $(A \cup B) \uplus C = \{x \mid (x \in A \cup B \wedge x \notin C) \vee (x \notin A \cup B \wedge x \in C)\}$

$$= \{x \mid (x \in A \wedge x \notin C) \vee (x \in B \wedge x \in C) \vee (x \notin A \wedge x \notin B \wedge x \in C)$$
$$= \{x \mid ((x \in A \wedge x \notin C) \vee (x \notin A \wedge x \in C)) \vee ((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C))$$
$$= (A \uplus C) \cup (B \uplus C)$$

c. Let $A = \{1\}$, $B = \{2\}$, $C = \{1,2\}$

$$(A \uplus C) \cup (B \uplus C) = \{1,2\}$$
$$(A \cup B) \uplus C = \emptyset$$
$$\{1,2\} \not\subseteq \emptyset$$

d. Let $A = \{1\}$, $B = \{1,2\}$, $C = \{2\}$

$$(A \cap B) \uplus C = \{1,2\}$$
$$(A \uplus C) \cap (B \uplus C) = \{1\}$$
$$\{1,2\} \not\subseteq \{1\}$$

e. $(A \uplus C) \cap (B \uplus C) = \{x \mid (x \in A \wedge x \notin C) \vee (x \notin A \wedge x \in C)\} \cap \{x \mid (x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C)\}$

$= \{x \mid ((x \in A \wedge x \notin C) \vee (x \notin A \wedge x \in C)) \wedge ((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C))\}$

$= \{x \mid ((x \in A \wedge x \notin C) \wedge (x \in B \wedge x \notin C)) \vee$

$((x \in A \wedge x \notin C) \wedge (x \notin B \wedge x \in C)) \vee$

$((x \notin A \wedge x \in C) \wedge (x \in B \wedge x \notin C)) \vee$

$((x \notin A \wedge x \in C) \wedge (x \notin B \wedge x \in C))\}$

$= \{x \mid (x \in A \wedge x \in B \wedge x \notin C) \vee (x \notin A \wedge x \notin B \wedge x \in C)\}$

$\subseteq \{x \mid (x \in (A \cap B) \wedge x \notin C) \vee (x \notin (A \cap B) \wedge x \in C)\}$

$= (A \cap B) \uplus C$

5. $\bigcup F = \{x \in A \mid \exists S \in F.\ x \in S\} = A$

Why also $\Leftarrow$?
The def. says

$\bigcap \mathcal{U} = \{x \in A \mid \forall X \in \mathcal{U}.\ x \in X\}$

$= \{x \in A \mid \forall X \in \mathcal{P}(A).\ (\forall S \in F.\ S \subseteq X) \Leftrightarrow x \in X\}$

$= \{x \in A \mid \forall X \in F.\ (X = A) \Leftrightarrow x \in X\}$ $\forall X\ (X \in \mathcal{U} \Rightarrow x \in X)$

$= \{x \in A \mid x \in A\}$

$= A$

$= \bigcup F$

We'll talk about these in person.

$\cancel{1 = \{X \subseteq A \mid \exists S \in F.\ S \subseteq X\}}$

$1 = \{X \subseteq A \mid \forall S \in F.\ X \subseteq S\} = \emptyset = \bigcap F$

6. Start with $\Rightarrow$

Assume $\bigcup F \subseteq U$

Equivalently $\{x \mid \exists S \in F.\ x \in S\} \subseteq U$

~~Assume Mix~~ $\forall X \in F\ \forall x \in X \Rightarrow x \in \bigcup F$

$\Rightarrow x \in U$

$\therefore \forall X \in F.\ X \subseteq U$

Next prove $\Leftarrow$

Assume $\forall X \in F.\ X \subseteq V$

Equivalently,

$\forall \cancel{X} \in F.\ x \in X \Rightarrow x \in V$

$\therefore \{\cancel{\forall} X \in F \mid x \in X\} = \bigcup F$

$= \{x \mid \exists X \in F.\ x \in X\}$

$\subseteq \{x \mid x \in V\}$

$= V$

$\therefore \bigcup F \subseteq V$