Maths Supervision Work 13

2 i.

$$I = \int_{x=0}^{2} \int_{y=x/2}^{1} 2xy^2 \, dy \, dx$$

$$= \int_{x=0}^{2} \left[\frac{2}{3} x y^3 \right]_{x/2}^{1} dx$$

$$= \int_{0}^{2} \left(\left(\frac{2}{3} x \right) - \left(\frac{2}{3} \cdot \frac{x^4}{8} \right) \right) dx$$

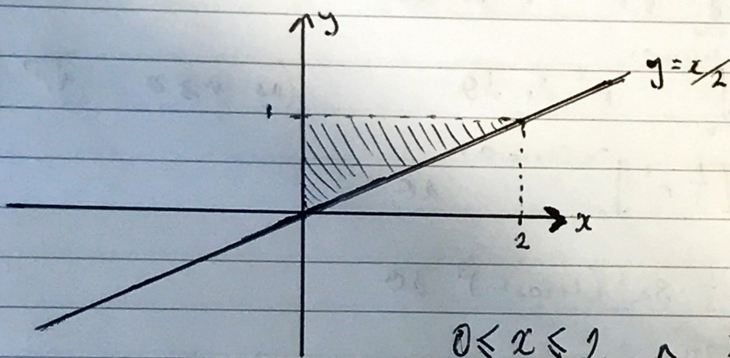
$$= \int_0^2 \left(\frac{2}{3} x - \frac{x^4}{12} \right) dx$$

$$= \left[\frac{1}{3} x^2 - \frac{1}{60} x^5 \right]_0^2$$

$$= \frac{4}{3} - \frac{32}{60}$$

$$= \frac{80-32}{60} = \frac{48}{60} = \frac{24}{30} = \frac{12}{15} = \frac{4}{5}$$

ii.



$0 \leq x \leq 2 \ \wedge \ \frac{x}{2} \leq y \leq 1$

is the same region as

$0 \leq y \leq 1 \ \wedge \ 0 \leq x \leq 2y$

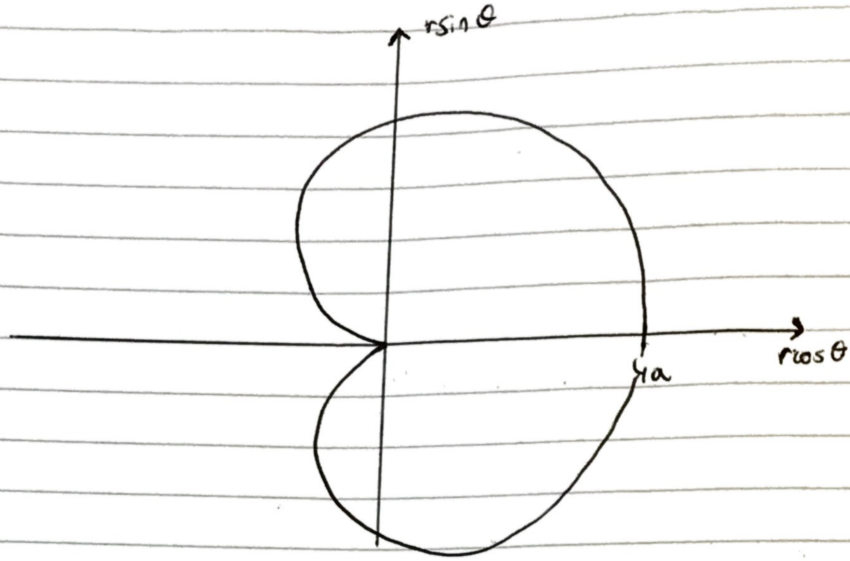
$$\therefore I = \int_{y=0}^{1} \int_{x=0}^{2y} 2xy^2 \, dx \, dy$$

$$= \int_{y=0}^{1} \left[x^2 y^2 \right]_0^{2y} dy$$

$$= \int_0^1 4y^4 \, dy$$

$$= \left[\frac{4}{5} y^5 \right]_0^1 = \frac{4}{5}$$

3.



D is described by $0 \leq \theta \leq 2\pi \wedge 0 \leq r \leq 2a(1+\cos\theta)$

$dxdy = r\,dr\,d\theta$

$$I = \int_D (x^2+y^2)^{1/2}\,dx\,dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2a(1+\cos\theta)} \sqrt{r^2}\; r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2a(1+\cos\theta)} r^2\,dr\,d\theta \qquad (\text{as } r \geq 0 \therefore \sqrt{r^2} = r \;\forall r \in D)$$

$$= \int_0^{2\pi} \left[\frac{1}{3} r^3\right]_0^{2a(1+\cos\theta)} d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} \cdot 8a^3 (1+\cos\theta)^3\,d\theta$$

$$= \frac{8a^3}{3} \int_0^{2\pi} (1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= \frac{8a^3}{3} \int_0^{2\pi} \left(1 + 3\cos\theta + \frac{3}{2}\cos 2\theta + \frac{3}{2} + \cos\theta(1-\sin^2\theta)\right) d\theta$$

$$= \frac{4a^3}{3} \int_0^{2\pi} (5 + 8\cos\theta + 3\cos 2\theta - 2\cos\theta\sin^2\theta)\,d\theta$$

$$= \frac{4a^3}{3} \left[5\theta + 8\sin\theta + \frac{3}{2}\sin 2\theta - \frac{2}{3}\sin^3\theta\right]_0^{2\pi}$$

$$= \frac{4a^3}{3}\left((10\pi) - (0)\right) = \frac{40a^3\pi}{3}$$

4 The given region can be described by

$0 \leq \theta \leq 2\pi \quad \wedge \quad 0 \leq r \leq 1$

$x^2 + y^2 = r^2$

$x^2 = r^2 \cos^2 \theta$

$dx\, dy = r\, dr\, d\theta$

$$\therefore I = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^2 \cos^2\theta \, (1-r^2)\, r \, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \cos^2\theta \, d\theta \int_{r=0}^{1} (r^3 - r^5)\, dr$$

$$= \frac{1}{2}\int_{0}^{2\pi} (\cos 2\theta + 1)\, d\theta \; \left[\frac{1}{4} r^4 - \frac{1}{6} r^6\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{2\pi} \left(\frac{1}{4} - \frac{1}{6}\right)$$

$$= \frac{1}{24}(2\pi - 0)$$

$$= \frac{\pi}{12}$$