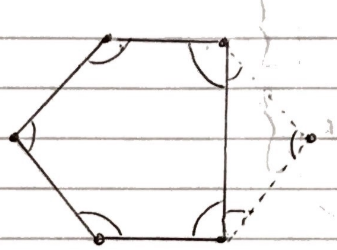
Morgan Saville

Discrete Maths 4

4.1.1 Proof of ~~[illegible]~~ $\forall$ ~~[illegible]~~ $n \geq 3 \in \mathbb{N}$, (*) the interior angles in an n-gon sum to $180(n-2)°$

Base case:
Take $n = 3$, an n-gon is a triangle $\therefore \sum_i \theta_i = 180°$
$\therefore$ (*) holds for $n = 3$

Assume (*) for $n = k$.
When $n = k+1$, consider ~~[illegible]~~ a k-gon with one extra point outside it

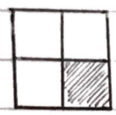


As shown above, the sum of internal angles in the resulting $(k+1)$-gon = that of the k-gon + $180°$
$= 180°(k-2) + 180°$
$= 180°((k+1)-2)$
$\therefore$ (*) holds for $n = k+1$

$\therefore$ (*) holds $\forall n \geq 3 \in \mathbb{N}$

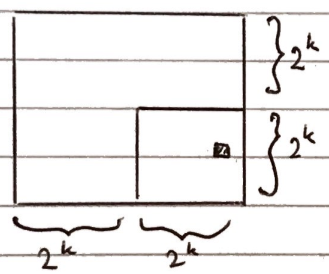
2. Proof of $\forall n \in \mathbb{Z}^+$, (*) a $2^n \times 2^n$ square grid with one square removed can be tiled with L-shaped pieces consisting of 3 squares.

Base case:
Take $n=1$



By symmetry, it does not matter which tile is removed, the resulting grid can be tiled with one such L piece.

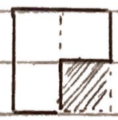
Assume (*) holds for $n=k$



The ~~small~~ square region of side length 2^k is tileable.
RTP: The large L-shaped region is tileable.

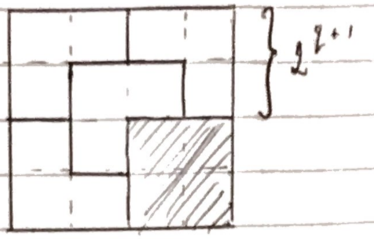
Proof of (**) that an L-shaped region of short-side length 2^p can be tiled by 3-square L-shapes $\forall p \in \mathbb{Z}_0^+$

Base case:
Take $p=0$



$\therefore$ (**) holds for $p=0$

Assume (**) holds for $p=q$
Take $p=q+1$



Where the inner L-pieces shown have short side length 2^q and are therefore tileable.

$\therefore$ $(**)$ holds for $p = q+1$

$\therefore$ $(**)$ holds $\forall\, p \in \mathbb{Z}_0$

$\therefore$ $(**)$ holds for $p = k$

$\therefore$ $(*)$ holds for $n = k+1$

$\therefore$ $(*)$ holds $\forall\, n \in \mathbb{Z}^+$

4.2.1 a. $(2^n-1)\cdot\sum_{i=0}^{m-1} 2^{in} = (2^n-1)\sum_{i=0}^{m-1}(2^n)^i$ which is a geometric seri

$$= (2^n-1)\cdot\frac{1-(2^n)^m}{1-2^n}$$

$$= 2^{mn}-1$$

b. Let $k = mn$ where $m, n \neq 1 \in \mathbb{Z}$

$2^k-1 = 2^{mn}-1 = (2^n-1)\cdot\sum_{i=0}^{m-1}2^{in}$ which is composite, as $2^n-1 \in \mathbb{Z} \neq 1$

Proof that $\sum_{i=0}^{k-1} r^i = \frac{1-r^k}{1-r}$ $(*)$ $\forall\, k \in \mathbb{Z}^+$ $\forall\, r \in \mathbb{R}$

Take $k=1$,

$\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{1-r^1}{1-r}$

$\therefore$ $(*)$ holds for $k=1$

Assume (*) holds for ~~all~~ $k = n$

Take $k = n+1$

$$\sum_{i=0}^{n} r^i = \sum_{i=0}^{n-1} r^i + r^n = \frac{1-r^n}{1-r} + r^n$$

$$= \frac{1-r^n}{1-r} + \frac{(1-r)r^n}{1-r}$$

$$= \frac{1-r^n+r^n-r^{n+1}}{1-r}$$

$$= \frac{1-r^{n+1}}{1-r}$$

$\therefore$ (*) holds for $k = n+1$

$\therefore$ (*) holds $\forall k \in \mathbb{Z}^+$

2. Proof that $\forall n \in \mathbb{N}$ (*) $\forall x \in \mathbb{R}.\ x \geq -1 \Rightarrow (1+x)^n \geq 1+nx$

Take $n = 0$

~~Assume x≥-1~~

$(1+x)^n = 1 = 1+nx$

~~1+x≥x~~

$\therefore$ (*) holds for $n = 0$

Assume (*) holds for $n = k$

Take $n = k+1$

Assume $x \geq -1$

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \quad \text{because } 1+x \geq 0 \\ &= 1+x+kx+kx^2 \\ &= 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x \quad \text{because } kx^2 \geq 0\end{aligned}$$

$\therefore$ (*) holds for $n = k+1$

$\therefore$ (*) holds $\forall n \in \mathbb{N}$

3a. Let (*) = Cassini's identity.

Take $n=0$,

$F_n \cdot F_{n+2} = F_0 \cdot F_2 = 0 \cdot 1 = 0$

$F_{n+1}^2 + (-1)^{n+1} = F_1^2 - 1 = 1^2 - 1 = 0$

$\therefore$ (*) holds for $n=0$

~~Assume (*) holds for $n=k$~~

~~Take $n=k+1$~~

~~$F_{k+1} \cdot F_{k+3}$~~

Take $n=1$,

$F_1 \cdot F_3 = 1 \cdot 2 = 2$

$F_2^2 + (-1)^2 = 1^2 + 1 = 2$

$\therefore$ (*) holds for $n=1$

Assume (*) holds for $n=k$

Take $n=k+2$

$$F_{k+3}^2 + (-1)^{k+3} = (F_{k+1} + F_{k+2})^2 + (-1)^{k+1}$$

$$= F_{k+1}^2 + F_{k+2}^2 + 2F_{k+1}F_{k+2} + (-1)^{k+1}$$

$$= (F_{k+1}^2 + (-1)^{k+1}) + F_{k+2}^2 + 2F_{k+1}F_{k+2}$$

$$= F_k F_{k+2} + F_{k+2}^2 + 2F_{k+1}F_{k+2}$$

$$= F_{k+2}(F_k + 2F_{k+1} + F_{k+2})$$

$$= F_{k+2}(F_{k+1} + 2F_{k+2})$$

$$= F_{k+2}(F_{k+2} + F_{k+3})$$

$$= F_{k+2} F_{k+4}$$

$\therefore$ (*) holds for $n=k+1$ $\therefore$ (*) holds $\forall n \in \mathbb{N}$

b. Let $(*) := \left(F_{n+k+1} = F_{k+1} \cdot F_{n+1} + F_k \cdot F_n \quad \forall n \in \mathbb{N} \right)$

Take $k=0$

$F_{n+1} = F_1 \cdot F_{n+1} + F_0 \cdot F_n$

$= \cancel{F_{n+1}}$

$\therefore (*)$ holds for $k=0$

Take $k=1$

$F_{n+2} = \cancel{F_2 F_{n+1}} \; F_{n+1} + F_n$

$= F_2 F_{n+1} + F_1 F_n$

$\therefore (*)$ holds for $k=1$

Assume $(*)$ holds for both $k=p$ and $k=p+1$

$\cancel{F_{p+2} F_{n+1}} \; F_{n+p+3} = F_{n+p+2} + F_{n+p+1}$

$= F_{p+2} F_{n+1} + F_{p+1} F_n + F_{p+1} F_{n+1} + F_p F_n$

$= (F_{p+2} + F_{p+1}) F_{n+1} + (F_{p+1} + F_p) F_n$

$= F_{p+3} F_{n+1} + F_{p+2} F_n$

$\therefore (*)$ holds for $k=p+2$

$\therefore (*)$ holds $\forall k \in \mathbb{N}$

c. Let $(*) = (F_n \mid F_{ln})$ $\forall n \in \mathbb{N}$

Take $l=0$

$F_n \mid F_0$

$\therefore (*)$ holds for $l=0$

Assume $(*)$ holds for $l=p$

Take $l=p+1$

$F_{(p+1)n} = F_{pn+n} = F_{pn} F_{n+1} + F_{pn-1} F_n$ by part b)

$= kF_n F_{n+1} + F_{pn-1} F_n$ for some $k \in \mathbb{Z}$

$= F_n (kF_{n+1} + F_{pn-1})$

$kF_{n+1} + F_{pn-1} \in \mathbb{Z}$ $\therefore F_n \mid F_{(p+1)n}$

$\therefore (*)$ holds for $l=p+1$

$\therefore (*)$ holds $\forall l \in \mathbb{N}$

d. Let $(*) = \gcd(F_{n+2}, F_{n+1}) = 1$ and terminates in n steps.

Take $n=0$

$\gcd(F_2, F_1) = \gcd(1, 1) = 1$ in 0 steps

$\therefore (*)$ holds for $n=0$

Assume $(*)$ for $n=k$

Take $n=k+1$

$\gcd(F_{k+3}, F_{k+2})$

$F_{k+3} = 1 \cdot F_{k+2} + F_{k+1}$ } 1 step

$\gcd(F_{k+2}, F_{k+1}) = 1$ } k steps

$\therefore \gcd(F_{k+3}, F_{k+2}) = 1$ in $k+1$ steps

$\therefore (*)$ holds for $n = k+1$

$\therefore (*)$ holds $\forall$ $n \in \mathbb{N}$

e.i.