Maths Supervision

2b.

$Ax = y$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ -1 & 1 & 0 & b \\ -1 & -1 & 1 & c \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 2 & 1 & b+a \\ 0 & 0 & 2 & c+a \end{array}\right)$$

$2z = c+a \Rightarrow z = \frac{c+a}{2}$

$2y + z = b + a$

$\therefore 2y + \frac{c+a}{2} = b+a$

$\therefore 2y = b + a - \frac{c+a}{2} = \frac{2b+a-c}{2}$

$\therefore y = \frac{2b+a-c}{4}$

$x + y + z = a$

$\therefore x + \frac{2b+a-c}{4} + \frac{2c+2a}{4} = a$

$\therefore x + \frac{2b+3a+c}{4} = a$

$\therefore x = \frac{4a-2b-3a-c}{4} = \frac{a-2b-c}{4}$

$$e_1 = \begin{pmatrix} \frac{1-2\cdot 0-0}{4} \\ \frac{2\cdot 0+1-0}{4} \\ \frac{0+1}{2} \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/2 \end{pmatrix}$$

$$e_2 = \begin{pmatrix} \frac{0 - 2\cdot 1 - 0}{4} \\ \frac{2\cdot 1 + 0 - 0}{4} \\ \frac{0+0}{2} \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} \frac{0\cdot 0 - 2\cdot 0 - 1}{4} \\ \frac{2\cdot 0 + 0 - 1}{4} \\ \frac{1+0}{2} \end{pmatrix} = \begin{pmatrix} -1/4 \\ -1/4 \\ 1/2 \end{pmatrix}$$

let $M = (e_1 | e_2 | e_3) = \begin{pmatrix} 1/4 & -1/2 & -1/4 \\ 1/4 & 1/2 & -1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

$$\det(A) = (1-0) - (-1-0) + (1-(-1))$$
$$= 1 + 1 + 2$$
$$= 4$$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 2 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} A^T = \frac{1}{4} \begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix} = M$$

$\therefore$ The matrix whose columns are e_1, e_2, e_3 is the inverse of A

29. Let $\vec{x}$ be an eigenvector such that

$$A\vec{x} = \lambda\vec{x} \quad \text{for some scalar } \lambda, \quad \vec{x} \neq \vec{0}$$

$$\therefore (A - \lambda I)\vec{x} = 0$$

$$\therefore \det(A - \lambda I) = 0$$

$$\therefore \det\begin{pmatrix} 4-\lambda & -2 & 0 \\ -2 & 3-\lambda & -2 \\ 0 & -2 & 2-\lambda \end{pmatrix} = 0$$

$$\therefore (4-\lambda)((3-\lambda)(2-\lambda)-4) + 2(-2(2-\lambda)-0) + 0 = 0$$

$$\therefore (4-\lambda)(6+\lambda^2-5\lambda-4) - 8 + 4\lambda = 0$$

$$(4-\lambda)(2 - 5\lambda + \lambda^2) - 8 + 4\lambda = 0$$

$$\therefore 8 - 20\lambda + 4\lambda^2 - 2\lambda + 5\lambda^2 - \lambda^3 - 8 + 4\lambda = 0$$

$$\therefore -\lambda^3 + 9\lambda^2 - 18\lambda = 0$$

$$\therefore \lambda^3 - 9\lambda^2 + 18\lambda = 0$$

$$\therefore \lambda(\lambda^2 - 9\lambda + 18) = 0$$

$$\therefore \lambda(\lambda-3)(\lambda-6) = 0$$

$$\therefore \lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = 6$$

For λ_1, let $\vec{x}_1 = \begin{pmatrix} 1 \\ b \\ c \end{pmatrix}$ be a corresponding eigenvector

$$\left(\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ -2 & 3 & -2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right) \quad \text{with } x = 1$$

$\therefore$

$$\cancel{\left(\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -2 & a & 0 \end{array}\right)}$$

$\therefore 4 - 2y = 0$

$\therefore y = 2$

$-2 + 3y - 2z = 0$

$\therefore 4 - 2z = 0$

$\therefore z = 2$

$0 - 2y + 2z = -4 + 4 = 0$

$\therefore \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad |\vec{x}_1| = 3 \quad \therefore \vec{e}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

For λ_2 let $\vec{x}_2 = \begin{pmatrix} 1 \\ b \\ c \end{pmatrix}$ be ~~a~~ corresponding eigenvector

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right) \quad \text{with } x = 1$$

$1 - 2y = 0 \quad \therefore y = \frac{1}{2}$

$-2 - 2z = 0$

$\therefore z = -1$

$0 - 2y - z = 0 - 2(\frac{1}{2}) + 1 = 0$

$\therefore \vec{x}_2 = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix} \quad |\vec{x}_2| = \frac{3}{2} \quad \therefore \vec{e}_2 = \frac{2}{3}\begin{pmatrix} 1 \\ \frac{1}{2} \\ -1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

For λ_3, let $\vec{x}_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be ~~the~~ corresponding eigenvector

$$\left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -3 & -2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right) \quad \text{with } x = 1$$

$\therefore -2 - 2y = 0$

$\therefore y = -1$

$-2 - 3y - 2z = 0$

$\therefore -2 + 3 - 2z = 0$

$\therefore z = 1/2$

$0 - 2y - 4z = 2 - \frac{4}{2} = 0$

$\therefore \vec{x}_3 = \begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix} \quad |\vec{x}_3| = 3/2 \quad \therefore \vec{e}_3 = 2/3 \begin{pmatrix} 1 \\ -1 \\ 1/2 \end{pmatrix}$

$= 1/3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\vec{y} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$

$\vec{y} \cdot \vec{e}_1 = 1/3 (6 - 6 + 0) = 0$

$\vec{y} \cdot \vec{e}_2 = 1/3 (12 - 3) = 9/3 = 3$

$\vec{y} \cdot \vec{e}_3 = 1/3 (12 + 6) = 18/3 = 6$

$$\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{e}_3 \cdot \vec{e}_3 = 1$$

$$\vec{e}_1 \cdot \vec{e}_2 = \tfrac{1}{4}(2+2-4) = 0$$

$$\vec{e}_1 \cdot \vec{e}_3 = \tfrac{1}{9}(2-4+2) = 0$$

$$\vec{e}_2 \cdot \vec{e}_3 = \tfrac{1}{9}(4-2-2) = 0$$

$$\vec{y} \cdot \vec{e}_1 = p_1 = 0$$

$$\vec{y} \cdot \vec{e}_2 = p_2 = 3$$

$$\vec{y} \cdot \vec{e}_3 = p_3 = 6$$

$$\therefore \vec{y} = 0\vec{e}_1 + 3\vec{e}_2 + 6\vec{e}_3$$

$$\vec{x} = q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3$$

$$A\vec{x} = q_1 \cdot 0\vec{e}_1 + q_2 \cdot 3\vec{e}_2 + q_3 \cdot 6\vec{e}_3$$

$$= 0\vec{e}_1 + 3q_2\vec{e}_2 + 6q_3\vec{e}_3$$

$$= \vec{y}$$

$$= 0\vec{e}_1 + 3\vec{e}_2 + 6\vec{e}_3$$

$\therefore q_2 = q_3 = 1$, while q_1 can vary

$$\therefore \vec{x} = \frac{1}{3}\begin{pmatrix} q_1 + 2 + 2 \\ 2q_1 + 1 - 2 \\ 2q_1 - 2 + 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} q_1 + 4 \\ 2q_1 - 1 \\ 2q_1 - 1 \end{pmatrix}$$

Eg. with $q_1 = 2$, $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ but this is not unique as q_1 can be any real number