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Discrete Maths Supervision 7

6.1.1 a. RTP: $(\forall a \in A.\ aRa) \iff id_A \subseteq R$

First "$\Longrightarrow$"

Assume $\forall a \in A.\ aRa$

$\therefore \{(a,a) \mid \forall a \in A\} \subseteq R$

$\therefore id_A \subseteq R$

"$\Longleftarrow$" is trivially true

b. RTP: $(\forall a,b \in A.\ aRb \Rightarrow bRa) \iff R \subset R^{op}$

First "$\Longrightarrow$"

Assume $\forall a,b \in A.\ aRb \Rightarrow bRa$

~~$\forall a,b \in A$~~

$\therefore \forall a,b \in A.\ aRb \Rightarrow aR^{op}b$

$\therefore R \subseteq R^{op}$

Next "$\Longleftarrow$"

Assume $R \subseteq R^{op}$

$\therefore \forall a,b \in A:\ aRb \Rightarrow aR^{op}b$

$\Rightarrow bRa$

$\square$

c. RTP. $(\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc) \iff R \circ R \subseteq R$

First "$\Longrightarrow$"

Assume $\forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc$

$\therefore \forall a,b,c \in A.\ a(R \circ R)c \Rightarrow aRc$

$\therefore R \circ R \subseteq R$

Next "$\Longleftarrow$"

Assume $R \circ R \subseteq R$

~~aRoR~~ $\therefore \forall a,c \in A.\ a(R \circ R)c \Rightarrow aRc$

$\therefore \forall a,b,c \in A.\ aRb \wedge bRc \Rightarrow aRc$

$\square$

2. Reflexivity:

Let A a set.

$\#A = \#A$

$\therefore A \cong A$

Symmetry:

Let A, B sets

$A \cong B \iff B \cong A$ by definition of bijections

Transitivity:

Let A, B, C sets.

Assume $A \cong B \wedge B \cong C$

$\#A = \#B \wedge \#B = \#C$

$\therefore \#A = \#C$

$\therefore A \cong C$

3. Reflexivity is true by definition

Symmetry:

Assume $(a,b) \in id_A$

$\therefore a = b$

$\therefore (b,a) \in id_A$

Transitivity

Assume $(a,b) \in id_A \wedge (b,c) \in id_A$

$\therefore b = a \wedge c = b = a$

$\therefore (a,c) \in id_A$

Consider $f: id_A \to A : (a,a) \longmapsto a$ is a bijection

4. Reflexivity:

$\forall x \in \mathbb{Z}. \quad x \equiv x \pmod{m}$

$\therefore x \equiv_m x$

Symmetry:

$\forall x, y \in \mathbb{Z}. \quad x \equiv y \pmod{m} \iff y \equiv x \pmod{m}$

$\therefore x \equiv_m y \iff y \equiv_m x$

Transitivity:

$\forall x, y, z \in \mathbb{Z}. \quad x \equiv y \pmod{m} \land y \equiv z \pmod{m} \Rightarrow x \equiv z \pmod{m}$

$\therefore x \equiv_m y \land y \equiv_m z \Rightarrow x \equiv_m z$

5. Reflexivity

Let $(a,b) \in \mathbb{Z} \times \mathbb{N}^+$

$a \cdot b = a \cdot b \quad \therefore (a,b) \equiv (a,b)$

Symmetry

Let $(a,b), (x,y) \in \mathbb{Z} \times \mathbb{N}^+$

$a \cdot y = b \cdot x \iff x \cdot b = y \cdot a$

$\therefore (a,b) \equiv (x,y) \iff (x,y) \equiv (a,b)$

Transitivity

Let $(a,b), (x,y), (u,v) \in \mathbb{Z} \times \mathbb{N}^+$

$\equiv$ is transitive $\iff (ay = xb \land xv = uy \Rightarrow av = ub)$

Assume $ay = xb \land xv = uy$

RTP: $av = ub$

$av = \frac{xb}{y} \cdot \frac{uy}{x} = ub$

$\square$

6. Let $X, Y, Z \subseteq A$

Reflexivity:

$(X, X) \in E \Longleftarrow X \cap B = X \wedge B$

which is trivially true

Symmetry

$(X \cap B = Y \cap B) \Longleftrightarrow (Y \cap B = X \cap B)$

$\therefore (X, Y) \in E \Longleftrightarrow (Y, X) \in E$

Transitivity

Assume $(X, Y) \in E$ equivalently, $X \cap B = Y \cap B$

Assume $(Y, Z) \in E$ equivalently, $Y \cap B = Z \cap B$

RTP $(X, Z) \in E$ equivalently, $X \cap B = Z \cap B$

$X \cap B = Y \cap B = Z \cap B$ $\square$

6.2.1 a. Reflexivity:

$\forall a \in A.\ (a,a) \in E_1 \quad \therefore (a,a) \in E_1 \cup E_2$

Symmetry.

Assume $(a, b) \in E_1 \cup E_2$

$\therefore$ Either $(a, b) \in E_1$

$\therefore (b, a) \in E_1$

$\therefore (b, a) \in E_1 \cup E_2$

or $(a, b) \in E_2$

$\therefore (b, a) \in E_2$

$\therefore (b, a) \in E_1 \cup E_2$

Transitivity does not hold:

~~Assume $(a,b) \in E_1 \cup E_2 \wedge (b,c) \in E_1 \cup E_2$~~

Let $A = \{1, 2, 3\}$

Let $E_1 = id_A \cup \{(1,2), (2,1)\}$

Let $E_2 = id_A \cup \{(2,3), (3,2)\}$

$\therefore E_1$ and E_2 are equivalence relations on A.

~~E_2~~ $(1,2) \in E_1 \cup E_2 \wedge (2,3) \in E_1 \cup E_2$

but $(1,3) \notin E_1 \cup E_2$

$\therefore E_1 \cup E_2$ is not transitive and $\therefore$ not an equivalence relation.

b. Reflexivity:

$id_A \subseteq E_1 \wedge id_A \subseteq E_2 \Rightarrow id_A \subseteq E_1 \cap E_2$

Symmetry:

Assume $(a,b) \in E_1 \cap E_2$

$\therefore (a,b) \in E_1 \wedge (a,b) \in E_2$

$\therefore (b,a) \in E_1 \wedge (b,a) \in E_2$

$\therefore (b,a) \in E_1 \cap E_2$

Transitivity:

Assume $(a,b) \in E_1 \cap E_2 \wedge (b,c) \in E_1 \cap E_2$

$\therefore (a,b) \in E_1 \wedge (b,c) \in E_1 \wedge (a,b) \in E_2 \wedge (b,c) \in E_2$

$\therefore (a,c) \in E_1$ ~~E_2~~ $\wedge (a,c) \in E_2$

$\therefore (a,c) \in E_1 \cap E_2$

$\square$

2. Let $a_1, a_2 \in A$

RTP $\{x \mid x \in A \wedge x E a_1\} = \{x \mid x \in A \wedge x E a_2\} \Leftrightarrow a_1 E a_2$

" $\Leftarrow$ " :

Assume $a_1 E a_2$

$\therefore \forall x \in A.\ x E a_1 \Leftrightarrow x E a_2$ as E is transitive

$\therefore \{x \mid x \in A \wedge x E a_1\} = \{x \mid x \in A \wedge x E a_2\}$ as required

" $\Rightarrow$ "

Assume $\{x \mid x \in A \wedge x E a_1\} = \{x \mid x \in A \wedge x E a_2\}$

$\therefore x E a_1 \Leftrightarrow x E a_2$

$\therefore a_1 E a_1 \Rightarrow a_1 E a_2$

$\therefore a_1 E a_2$ as E is reflexive

$\square$

3a. Reflexivity:

$f(a) = f(a)$

$\therefore a \equiv_f a$

Symmetry

$(f(a) = f(b)) \Leftrightarrow (f(b) = f(a))$

$\therefore (a \equiv_f b) \Leftrightarrow (b \equiv_f a)$

Transitivity:

Assume $a \equiv_f b$ equivalently $f(a) = f(b)$

Assume $b \equiv_f c$ equivalently $f(b) = f(c)$

RTP: $a \equiv_f c$ equivalently $f(a) = f(c)$

$f(a) = f(b) = f(c)$

$\square$

b. RTP: E is an equivalence relation $\Rightarrow E = (\equiv_g)$
Equivalently: $id_A \subseteq E \wedge E \subseteq E^{op} \wedge E \circ E \subseteq E \Rightarrow (\forall a, b \in A.\ aEb \Leftrightarrow [a]_E = [b]_E$

Assume $id_A \subseteq E \wedge E \subseteq E^{op} \wedge E \circ E \subseteq E$
~~RTP: $\forall a, b$~~ Let $a, b \in A$
RTP: $aEb \Leftrightarrow [a]_E = [b]_E$
" $\Rightarrow$ ": Assume aEb
RTP: $\{x \mid x \in A \wedge xEa\} = \{x \mid x \in A \wedge xEb\}$
equivalently: $xEa \Leftrightarrow xEb$
which is evidently true as $aEb \wedge E$ is transitive

" $\Leftarrow$ " Assume $[a]_E = [b]_E$
Equivalently: $xEa \Leftrightarrow xEb$
$\therefore aEa \Rightarrow aEb$
$\therefore aEb$ as E is reflexive

$\square$

c. $A/_{\equiv_f} = \{[a]_{\equiv_f} \mid \forall a \in A\}$

~~$\{\{a' \mid f(a) = f(a')\} \mid \forall a \in A\}$~~

~~Consider $g: B \to A/_{\equiv_f}$: $b \mapsto$~~

$\forall b \in B.\ \exists a \in A.\ f(a) = b$
$\therefore$ Consider $g: B \twoheadrightarrow A/_{\equiv_f} : b \longmapsto [a]_{\equiv_f}$ where $b = f(a)$

~~g is~~ $b \in g(b)$ as $b \in \{a' \mid f(a') = f(a)\}$
and so ~~there~~ $g(b)$ is bijective

$\therefore B \cong A/_{\equiv_f}$

7.1.1. Surjective | Not surjective

$\mathrm{id}_{\mathbb{Z}}$ | $[n] \longrightarrow [n+1] : x \longmapsto x+1$

$\mathbb{Z} \twoheadrightarrow \mathbb{N} : x \mapsto |x|$ | $\mathbb{N} \longrightarrow \mathbb{N} : x \longmapsto 0$

$\mathbb{Z} \twoheadrightarrow \mathbb{Z} : x \mapsto -x$ | $\mathbb{N} \longrightarrow \mathbb{N} : x \longmapsto x^2$

2. i) ~~RTP~~ Let A a set
RTP: id_A is a surjection.
Equivalently: $\forall a \in A\ \exists b \in A .\ (b, a) \in \mathrm{id}_A$
$b = a$ satisfies this $\square$

~~Let R, S surjections on A~~
ii) Let A, B, C sets
Let $R : A \twoheadrightarrow B$, $S : B \twoheadrightarrow C$
RTP: $\forall c \in C\ \exists a \in A, b \in B .\ aRb \wedge bSc$
Let $c \in C$
$\therefore \exists b \in B .\ bSc$
Let b s.t. bSc
$\therefore \exists a \in A .\ aRb$
Let a s.t. aRb

$\therefore aRb \wedge bSc$

$\square$

7.2 Let $R: A \twoheadrightarrow B$, $S: X \twoheadrightarrow Y$

Let $Q = \{((a,x),(b,y)) \mid aRb \wedge xSy\}$

Let $(b,y) \in B \times Y$

RTP $\exists (a,x) \in A \times X$ s.t. $(a,x) Q (b,y)$

Equivalently, $\exists (a,x) \in A \times X$ s.t. $aRb \wedge xSy$

Equivalently, $\exists a \in A, x \in X$ s.t. $aRb \wedge xSy$

which is true as R and S are surjective.

$\therefore Q$ is surjective

Let $T = \{((a,0),(b,0)) \mid \forall a \in A, b \in B.\ aRb\}$
$\cup \{((x,1),(y,1)) \mid \forall x \in X, y \in Y.\ xRy\}$

~~Let $b \in B$:~~

Let $\beta \in B \uplus Y$. RTP: $\exists \alpha \in A \times X$ s.t. $\alpha T \beta$

Case 1: $\beta = (b,0)$ for some $b \in B$

$\therefore \exists a \in A$ s.t. aRb

Let a s.t. aRb

$\therefore (a,0) \in A \times X$

$(a,0) T \beta$

Case 2: $\beta = (y,1)$ for some $y \in Y$

$\therefore \exists x \in X$ s.t. xSy

Let x s.t. xSy

$\therefore (x,1) \in A \times X$

$(x,1) T \beta$

$\therefore T$ is surjective.

8.1.1 Injective

$id_{\mathbb{Z}}$

$\mathbb{Z} \rightarrowtail \mathbb{Z} : x \mapsto -x$

$\{0\} \rightarrowtail \mathbb{N} : x \mapsto 5$

Not injective

$\mathbb{Z} \to \mathbb{N} : x \mapsto |x|$

$\mathbb{N} \to \{0\} : x \mapsto 0$

$\mathbb{Z} \to \mathbb{N} : x \mapsto x^2$

2. ~~RTP~~ Let A a set

RTP id_A is an injection

Equivalently, $(b, a) \in id_A \wedge (c, a) \in id_A \implies b = c$

Assume $(b, a) \in id_A$

$\therefore b = a$

Assume $(c, a) \in id_A$

$\therefore c = a$

$\therefore b = c$

$\square$

Let A, B, C sets

Let $R : A \rightarrowtail B$, $S : B \rightarrowtail C$

RTP: $S \circ R$ is an injection

~~Equivalently $a(S \circ R)d \wedge a(S \circ R)c \implies c = d$~~

~~Assume $a(S \circ R)d$~~

~~$\therefore$ Let $b \in B$ s.t. $aRb \wedge bSd$~~

~~Assume $a(S \circ R)c$~~

~~Let $b' \in B$ s.t. $aRb' \wedge b'Sc$~~

~~$aRb \wedge aRb' \implies b = b'$~~

~~$\therefore bSd \wedge bSc \implies c = d$~~

$\square$ Equivalently, $a(S \circ R)c \wedge a'(S \circ R)c \implies a = a'$

Assume $a(S \circ R)c$

Let $b \in B$ s.t. $aRb \wedge bSc$

Assume $a'(S \circ R)c$

Let $b' \in B$ s.t. $a'Rb' \wedge b'Sc$

$bSc \wedge b'Sc \Rightarrow b = b'$

$\therefore a R b \wedge a' R b \Rightarrow a = a'$

$\square$

8.2 Let $R : A \rightarrowtail B$, $S : X \rightarrowtail Y$

Let $Q = \{((a, x), (b, y)) \mid aRb \wedge xRy\}$

Let $a, a' \in A$, $b \in B$, $x, x' \in X$, $y \in Y$

Assume $(a, x) \, Q \, (b, y)$

$\therefore aRb,\ xSy$

Assume $(a', x') \, Q \, (b, y)$

$\therefore a'Rb,\ x'Sy$

$aRb \wedge a'Rb \Rightarrow a = a'$

$xSy \wedge x'Sy \Rightarrow x = x'$

$\therefore (a, x) = (a', x')$

$\therefore Q$ is injective

Let $T = \{((a,0),(b,0)) \mid aRb\} \cup \{((x,1),(y,1)) \mid xSy\}$

Let $\alpha, \alpha' \in A \times X$, $\beta \in B \times Y$

Assume $\alpha T \beta \wedge \alpha' T \beta$

Case 0: $\alpha = (a, 0)$ for some $a \in A$

$\therefore \beta = (b, 0)$ for some $b \in B$ s.t. aRb

$\therefore \alpha' = (a', 0)$ for some $a' \in A$ s.t. $a'Rb$

$aRb \wedge a'Rb \Rightarrow a = a'$

$\therefore \alpha = \alpha'$

Case 1: $\alpha = (x, 0)$ for some $x \in X$

$\therefore \beta = (y, 0)$ for some $y \in Y$ s.t. xSy

$\therefore \alpha' = (x', 0)$ for some $x' \in X$ s.t. $x'Sy$

$xSy \wedge x'Sy \Rightarrow x = x'$

$\therefore \alpha = a'$

$\therefore T$ is injective.

9.9.1 Direct image $\{n \mid n, n^2 \in \mathbb{Z}\} \subseteq \mathbb{Z}$
Inverse image: $\{n^2 \mid n \in \mathbb{Z}\} \subseteq \mathbb{N}$

2a. Let $X \subseteq A$

$$\bigcup_{x\in X} \vec{R}(\{x\}) = \bigcup_{x\in X} \{y \mid xRy\}$$

$$= \{y \mid \exists x \in X.\ xRy\}$$

$$= \vec{R}X$$

b. Let $Y \subseteq B$

$\{a \in A \mid \vec{R}(\{a\}) \subseteq Y\}$
$= \{a \mid a \in A \wedge \{y \mid \exists a' \in \{a\}.\ a'Ry\} \subseteq Y\}$
$= \{a \mid a \in A \wedge \{y \mid aRy\} \subseteq Y\}$
$= \{a \mid a \in A \wedge (\exists y \in Y.\ aRy)\}$
$= \overleftarrow{R}(Y)$

2.1 $\#(\vec{f}(X)) = \#X$, as for each element x of X, there exists exactly one element $b \in B$ s.t. xfb

$\therefore \vec{f}(X) \cong X$

2. ~~$X \subseteq A$, $Y \subseteq B$~~

Let $Y \subseteq B$
RTP $\exists X \subseteq A$ s.t. $X \vec{f} Y$
Equivalently, $\exists X \subseteq A$ s.t. $\{b \mid \exists x \in X.\ xfb\} = Y$
$X = \{a \mid \forall y \in Y.\ xfy\}$ is well defined as f is surjective and clearly satisfies the above property
□

9.3a. $\overleftarrow{f}(\emptyset) = \{a \mid \exists b \in \emptyset .\ afb\} = \emptyset$

b. $\overleftarrow{f}(X \cup Y) = \{a \mid \exists b \in X \cup Y .\ afb\}$

$= \{a \mid (\exists b \in X .\ afb) \vee (\exists b \in Y .\ afb)\}$

$= \{a \mid \exists b \in X .\ afb\} \cup \{a \mid \exists b \in Y .\ afb\}$

$= \overleftarrow{f}(X) \cup \overleftarrow{f}(Y)$

c. $\overleftarrow{f}(B) = \{a \mid \exists b \in B .\ afb\} = A$ as f is a function

d. $\overleftarrow{f}(X \cap Y) = \{a \mid \exists b \in X \cap Y .\ afb\}$

$= \{a \mid \exists b .\ b \in X \wedge b \in Y .\ afb\}$

$= \{a \mid (\exists b \in X .\ afb) \wedge (\exists b \in Y .\ afb)\}$ as f is a function

$= \{a \mid \exists b \in X .\ afb\} \cap \{a \mid \exists b \in Y .\ afb\}$

$= \overleftarrow{f}(X) \cap \overleftarrow{f}(Y)$

e. $\overleftarrow{f}(X^c) = \{a \mid \exists b \in X^c .\ afb\}$

$= \{a \mid \nexists b \in X .\ afb\}$

$= \{a \mid \exists b \in X .\ afb\}^c$

$= (\overleftarrow{f}(X))^c$