Morgan Saville

Maths Supervision 3

F13. Let $Z_N = \sum_{n=1}^{N} (\cos n\theta + i\sin n\theta)$

such that $S_N = \sum_{n=1}^{N} \sin\theta = Im(Z_N)$

and $C_N = \sum_{n=1}^{N} \cos\theta = Re(Z_N)$

$Z_N = \sum_{n=1}^{N} e^{in\theta} = \sum_{n=1}^{N} (e^{i\theta})^n$ which is a geometric series with first term $e^{i\theta}$ and common ratio $e^{i\theta}$

$$\therefore Z_N = \frac{e^{i\theta}(1-e^{iN\theta})}{1-e^{i\theta}}$$

$$= \frac{e^{i\theta}(1-e^{iN\theta})(1-e^{-i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$

$$= \frac{(e^{i\theta}-1)(1-e^{iN\theta})}{2-e^{i\theta}-e^{-i\theta}}$$

$$= \frac{(e^{i\theta}-1)(1-e^{iN\theta})}{2-2\cos\theta}$$

$$= \frac{e^{i\theta}-e^{i(N+1)\theta}-1+e^{iN\theta}}{2-2\cos\theta}$$

$$= \frac{(-1+\cos\theta+\cos N\theta-\cos(N+1)\theta)+i(\sin\theta+\sin N\theta-\sin(N+1)\theta)}{2-2\cos\theta}$$

$$\therefore C_N = Re(Z_N) = \frac{-1+\cos\theta+\cos N\theta-\cos(N+1)\theta}{2-2\cos\theta}$$

and $S_N = Im(Z_N) = \dfrac{\sin\theta+\sin N\theta-\sin(N+1)\theta}{2-2\cos\theta}$

a. $\sum_{n=1}^{5} \sin n\theta = S_5 = \frac{\sin\theta + \sin 5\theta - \sin 6\theta}{2 - 2\cos\theta}$

b. $\sum_{n=1}^{N} \cos n\theta = C_N = \frac{-1 + \cos\theta + \cos N\theta - \cos(N+1)\theta}{2 - 2\cos\theta}$

F11 ~~F11~~ a. $e^{i\theta} = \cos\theta + i\sin\theta$

b. $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

$= e^{in\theta}$

$= \cos n\theta + i\sin n\theta$ QED

c. Let $s = \cos 2\theta + i\sin 2\theta$

$= (\cos\theta + i\sin\theta)^2$ by De Moivre's theorem

$= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$

$= \cos^2\theta - (1 - \cos^2\theta) + 2i\cos\theta\sin\theta$

$= 2\cos^2\theta - 1 + 2i\cos\theta\sin\theta$

Note that $\cos 2\theta = Re(s)$

$= 2\cos^2\theta - 1$

d. $\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$= \cos^2\theta - \sin^2\theta$

$= \cos^2\theta - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

F14. ~~x = 7 + 24 cos 3t + 7 sin 3t~~

The particle is oscilating around $x = 7$, so the displacement from the centre of oscilation

$d = 24\cos 3t + 7\sin 3t$

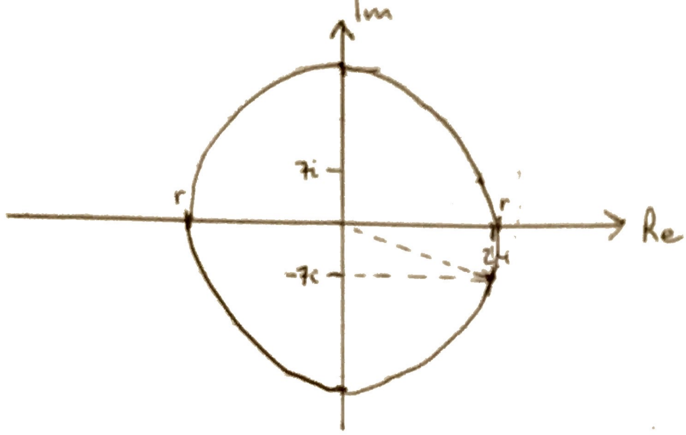
$X = (24 - 7i)e^{3it} = (24 - 7i)(\cos 3t + i\sin 3t)$

$= 24\cos 3t + 7\sin 3t + 24i\sin 3t - 7i\cos 3t$

$Re(X) = 24\cos 3t + 7\sin 3t = d$ QED

b. Amplitude = max $|d|$ as t varies
= max $|Re(X)|$ as t varies

The locus of X as t varies is:



a circle centred on the origin with radius r, where the point $24-7i$ lies on the circle

$\therefore r = |24-7i| = \sqrt{24^2+7^2} = 25$

$\therefore Re(X)$ varies from -25 to 25

$\therefore$ The maximum value of $|Re(X)| = 25$

$\therefore$ Amplitude $= 25$

c. $\arg(24-7i) = -\arctan\left(\frac{7}{24}\right)$

$|24-7i| = 25$

$\therefore 24-7i = 25e^{-i\arctan\frac{7}{24}}$

$\therefore X = 25e^{-i\arctan\frac{7}{24}} e^{i3t} = 25e^{i(3t-\arctan\frac{7}{24})}$

$\therefore$ setting $\dot{x}(t) = 0$, we get

$$\mathrm{Re}\left(25e^{i(3t - \arctan\frac{7}{24})}\right) = 0$$

$$\therefore 25\cos\left(3t - \arctan\tfrac{7}{24}\right) = 0$$

$$\therefore \cos\left(3t - \arctan\tfrac{7}{24}\right) = 0$$

$$\therefore 3t - \arctan\tfrac{7}{24} = (2n + \tfrac{1}{2})\pi \quad \text{for some } n \in \mathbb{Z}$$

$$3t = \arctan\tfrac{7}{24} + (2n + \tfrac{1}{2})\pi$$

$$\therefore t = \tfrac{1}{3}\left(\arctan\tfrac{7}{24} + (2n + \tfrac{1}{2})\pi\right)$$

taking the two smallest values of n — n_1 and n_2 — such that the corresponding values for t — t_1 and t_2 — satisfy $t_1, t_2 > 0$

$$n_1 = 0, \quad n_2 = 1$$

$$t_1 = \tfrac{1}{3}\left(\arctan\tfrac{7}{24} + \tfrac{\pi}{2}\right) \simeq 0.618$$
$$t_2 = \tfrac{1}{3}\left(\arctan\tfrac{7}{24} + \tfrac{5\pi}{2}\right) \simeq 2.713$$

d. the stationary points occur $\Rightarrow d = \pm$ Amplitude
$= \pm 25$

$$\therefore x = 7 \pm 25$$
$$= -18 \text{ or } 32$$

$\therefore$ the stationary points occur at distances 18 and 32 from the origin.