Maths Supervision

14.

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -M \qquad \therefore M^4 = -M^2 , \quad M^5 = M$$

$$\therefore (\theta M)^n = \begin{cases} -\theta^n M^2 & \text{if } n \equiv 0 \mod 4 \\ \theta^n M & \text{if } n \equiv 1 \mod 4 \\ \theta^n M^2 & \text{if } n \equiv 2 \mod 4 \\ -\theta^n M & \text{if } n \equiv 3 \mod 4 \end{cases}$$

$$\therefore \exp \theta M = I + \sum_{n=1}^{\infty} \frac{(\theta M)^n}{n!}$$

$$= I + \sum_{n=0}^{\infty} \frac{(\theta M)^{4n+1}}{(4n+1)!} + \frac{(\theta M)^{4n+2}}{(4n+2)!} + \frac{(\theta M)^{4n+3}}{(4n+3)!} + \frac{(\theta M)^{4n+4}}{(4n+4)!}$$

$$= I + \sum_{n=0}^{\infty} \frac{\theta^{4n+1}}{(4n+1)!} M + \frac{\theta^{4n+2}}{(4n+2)!} M^2 - \frac{\theta^{4n+3}}{(4n+3)!} M - \frac{\theta^{4n+4}}{(4n+4)!} M^2$$

$$\cancel{= I + M \left(\sum_{n=0}^{\infty} \theta^{4n+1} - \theta^{4n+3} \right.}$$

$$= I + M \left(\sum_{n=0}^{\infty} \frac{\theta^{4n+1}}{(4n+1)!} - \frac{\theta^{4n+3}}{(4n+3)!} \right) + M^2 \left(\sum_{n=0}^{\infty} \frac{\theta^{4n+2}}{(4n+2)!} - \frac{\theta^{4n+4}}{(4n+4)!} \right)$$

$$= I + M \left(\sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} \right) + M^2 \left(1 - \sum_{n=0}^{\infty} \frac{\theta^{4n}}{(4n)!} - \frac{\theta^{4n+2}}{(4n+2)!} \right)$$

$$= I + M \sin \theta + M^2 \left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} \right)$$

$$= I + M \sin \theta + M^2 (1 - \cos \theta)$$

$$\therefore \exp\theta_1 M \exp\theta_2 M$$

$$= \left(I + M\sin\theta_1 + M^2(1-\cos\theta_1)\right)\left(I + M\sin\theta_2 + M^2(1-\cos\theta_2)\right)$$

$$= \cancel{I^2 + IM\sin\theta_2 + IM^2(1-\cos\theta_2) + M^2\sin\theta_1\sin\theta_2 + M^3\sin\theta_1(1-\cos\theta_2)}$$
$$\cancel{+ M^4(1-\cos\theta_1)(1-\cos\theta_2)}$$

$$= I^2 + IM\sin\theta_2 + IM^2(1-\cos\theta_2) + IM\sin\theta_1 + M^2\sin\theta_1\sin\theta_2$$
$$+ M^3\sin\theta_1(1-\cos\theta_2) + IM^2(1-\cos\theta_1) + M^3(1-\cos\theta_1)\sin\theta_2$$
$$+ M^4(1-\cos\theta_1)(1-\cos\theta_2)$$

$$= I + M(\sin\theta_2 + \sin\theta_1) + M^2(1-\cos\theta_2 + \sin\theta_1\sin\theta_2 + 1 - \cos\theta_1)$$
$$+ M^3(\sin\theta_1(1-\cos\theta_2) + \sin\theta_2(1-\cos\theta_1)) + M^4(1-\cos\theta_1)(1-\cos\theta_2)$$

$$= I + M(\sin\theta_1 + \sin\theta_2 - \sin\theta_1 + \sin\theta_1\cos\theta_2 - \sin\theta_2 + \cos\theta_1\sin\theta_2)$$
$$+ M^2(2 - \cos\theta_1 - \cos\theta_2 + \sin\theta_1\sin\theta_2 - 1 - \cos\theta_1\cos\theta_2 + \cos\theta_1 + \cos\theta_2)$$

$$= I + M(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) + M^2(1 - (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2))$$

$$= I + M\sin(\theta_1+\theta_2) + M^2(1-\cos(\theta_1+\theta_2))$$

$$= \exp(\theta_1+\theta_2)M$$

$\square$

$$M^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -M$$

$$(M^2)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = M^2$$

$$\therefore (\exp\theta M)^T = I^T + M^T \sin\theta + (M^2)^T (1-\cos\theta)$$
$$= I - M\sin\theta + M^2(1-\cos\theta)$$

$$\therefore (\exp\theta M)(\exp\theta M)^T = (I + M\sin\theta + M^2(1-\cos\theta))(I - M\sin\theta + M^2(1-\cos\theta))$$

$$= I^2 + IM\sin\theta + IM^2(1-\cos\theta) - IM\sin\theta - M^2\sin^2\theta - M^3\sin\theta(1-\cos\theta)$$
$$+ IM^2(1-\cos\theta) + M^3\sin\theta(1-\cos\theta) + M^4(1-\cos\theta)^2$$

$$= I + 2M^2(1-\cos\theta) - M^2\sin^2\theta - M^2(1-\cos\theta)^2$$
$$= I + M^2(2 - 2\cos\theta - \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta)$$
$$= I + M^2(1 - \sin^2\theta - \cos^2\theta)$$
$$= I$$

This does not hold for all matrices, as that would imply that $A^T = A^{-1}$ for all matrices A, which is not the case.

16. $\det A = 4 - 6 = -2$

$\det B = 4 - 4 = 0$

$AB = \begin{pmatrix} 4 & 8 \\ 10 & 20 \end{pmatrix}$

$\det AB = 80 - 80 = 0$

$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix}$

$\det A^{-1} = \frac{1}{2}^2 (4 - 6) = \frac{1}{2}^2 (-2) = -\frac{1}{2}$

$\det AB = 0 = -2 \cdot 0 = \det A \det B$

$\det (A^{-1}) = -\frac{1}{2} = \frac{1}{-2} = \frac{1}{\det A}$