Maths Supervision Work 12

16c. $\psi = x^4 - 6x^2y^2 + y^4$

$$\frac{\partial \psi}{\partial x} = 4x^3 - 12xy^2$$

$$\frac{\partial \psi}{\partial y} = -12x^2y + 4y^3$$

$$\frac{\partial^2 \psi}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial^2 \psi}{\partial y^2} = -12x^2 + 12y^2$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 12x^2 - 12y^2 - 12x^2 + 12y^2$$

$$= 0$$

$$y = 0 \implies \frac{\partial \psi}{\partial y} = -12x^2y + 4y^3 = 0$$

$$y = x \implies \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x} = -12x^2y + 4y^3 - 4x^3 + 12xy^2$$

$$= -12x^3 + 4x^3 - 4x^3 + 12x^3$$

$$= 0$$

ii. $\psi = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y$

$$\frac{\partial \psi}{\partial x} = 2\sin x \cos x \cosh^2 y + 2\sin x \cos x \sinh^2 y$$

$$= \sin(2x)(\cosh^2 y + \sinh^2 y)$$

$$= \sin(2x)\cosh(2y)$$

$$\frac{\partial \psi}{\partial y} = 2\sin^2 x \cosh y \sinh y - 2\cos^2 x \sinh y \cosh y$$

$$= 2\cosh y \sinh y\,(\sin^2 x - \cos^2 x)$$

$$= -\cos(2x)\sinh(2y)$$

$$\frac{\partial^2 \psi}{\partial x^2} = 2\cos(2x)\cosh(2y)$$

$$\frac{\partial^2 \psi}{\partial y^2} = -2\cos(2x)\cosh(2y)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2\cos(2x)\cosh(2x) - 2\cos(2x)\cosh(2x)$$

$$= 0$$

iii. $\psi = \ln \frac{x^2+(y-1)^2}{x^2+(y+1)^2} = \ln(x^2+(y-1)^2) - \ln(x^2+(y+1)^2)$

$$\frac{\partial \psi}{\partial x} = \frac{2x}{x^2+(y-1)^2} - \frac{2x}{x^2+(y+1)^2}$$

$$\frac{\partial \psi}{\partial y} = \frac{2(y-1)}{x^2+(y-1)^2} - \frac{2(y+1)}{x^2+(y+1)^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2x^2+2(y-1)^2-4x^2}{(x^2+(y-1)^2)^2} - \frac{2x^2+2(y+1)^2-4x^2}{(x^2+(y+1)^2)^2}$$

$$= \frac{-2x^2+2(y-1)^2}{(x^2+(y-1)^2)^2} - \frac{-2x^2+2(y+1)^2}{(x^2+(y+1)^2)^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{2x^2+2(y-1)^2-4(y-1)^2}{(x^2+(y-1)^2)^2} - \frac{2x^2+2(y+1)^2-4(y+1)^2}{(x^2+(y+1)^2)^2}$$

$$= \frac{2x^2-2(y-1)^2}{(x^2+(y-1)^2)^2} - \frac{2x^2-2(y+1)^2}{(x^2+(y+1)^2)^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{-2x^2+2(y-1)^2}{(x^2+(y-1)^2)^2} + \frac{2x^2-2(y-1)^2}{(x^2+(y-1)^2)^2} - \frac{-2x^2+2(y+1)^2}{(x^2+(y+1)^2)^2} - \frac{2x^2-2(y+1)^2}{(x^2+(y+1)^2)^2}$$

$$= 0$$

$$y=0 \Rightarrow \psi = \ln \frac{x^2+(y-1)^2}{x^2+(y+1)^2} = \ln \frac{x^2+1}{x^2+1} = \ln 1 = 0$$

17. $\Psi(x,y) = \sin\frac{\pi y}{a} e^{-\pi x/a} + 2\sin\frac{2\pi y}{a} e^{-2\pi\frac{x}{a}}$

$$\frac{\partial\Psi}{\partial x} = -\frac{\pi}{a}\sin\frac{\pi y}{a} e^{-\frac{\pi x}{a}} - \frac{4\pi}{a}\sin\frac{2\pi y}{a} e^{-2\pi\frac{x}{a}}$$

$$\frac{\partial\Psi}{\partial y} = \frac{\pi}{a}\cos\frac{\pi y}{a} e^{-\frac{\pi x}{a}} + \frac{4\pi}{a}\cos\frac{2\pi y}{a} e^{-2\pi\frac{x}{a}}$$

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{\pi^2}{a^2}\sin\frac{\pi y}{a} e^{-\frac{\pi x}{a}} + \frac{8\pi^2}{a^2}\sin\frac{2\pi y}{a} e^{-\frac{2\pi x}{a}}$$

$$\frac{\partial^2\Psi}{\partial y^2} = -\frac{\pi^2}{a^2}\sin\frac{\pi y}{a} e^{-\frac{\pi x}{a}} - \frac{8\pi^2}{a^2}\sin\frac{2\pi y}{a} e^{-2\pi\frac{x}{a}}$$

$$= -\frac{\partial^2\Psi}{\partial x^2}$$

$\therefore \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0 \quad \therefore \Psi$ satisfies Laplace's eqn.

$\Psi(x,0) =$ $\sin(0) e^{-\frac{\pi x}{a}} + 2\sin(0) e^{-\frac{2\pi x}{a}} = 0$

$\Psi(x,a) = \sin(\pi) e^{-\frac{\pi x}{a}} + 2\sin(2\pi) e^{-\frac{2\pi x}{a}} = 0 + 0 = 0$

$\Psi(0,y) = \sin\left(\frac{\pi y}{a}\right)e^0 + 2\sin\left(\frac{2\pi y}{a}\right)e^0$

$= \sin\frac{\pi y}{a} + 2\sin\frac{2\pi y}{a}$

$\therefore \Psi$ satisfies both boundary conditions.