

# Discrete Maths Supervision 5

1.1 1a. RTP  $\forall$  sets A.  $A \subseteq A$

Equivalently,

$$\forall x. x \in A \Rightarrow x \in A$$

which is true by modus ponens

b. RTP  $\forall$  sets A, B, C.  $(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$

Assume  $A \subseteq B \wedge B \subseteq C$

Equivalently,  $(\forall x. x \in A \Rightarrow x \in B) \wedge (\forall y. y \in B \Rightarrow y \in C)$  (\*)

RTP  $(\forall z. z \in A \Rightarrow z \in C)$

By universal instantiation, let arbitrary  $z \in A$ .

By (\*),  $z \in B$

By (\*),  $z \in C$

□

c. RTP  $\forall$  sets A, B.  $(A \subseteq B \wedge B \subseteq A) \Leftrightarrow A = B$

Start with  $\Leftarrow$

Assume  $A = B$

Equivalently,  $\forall x. x \in A \Leftrightarrow x \in B$  (+)

RTP  $A \subseteq B \wedge B \subseteq A$

Equivalently,  $(\forall x. x \in A \Rightarrow x \in B) \wedge (\forall x. x \in B \Rightarrow x \in A)$  which is true by (+)

Next,  $\Rightarrow$

Assume  $A \subseteq B \wedge B \subseteq A$

Equivalently,  $(\forall x. x \in A \Rightarrow x \in B) \wedge (\forall x. x \in B \Rightarrow x \in A)$  (+)

RTP  $A = B$

Equivalently,

$\forall x. x \in A \Leftrightarrow x \in B$  which is true by (+)

□

2a. RTP  $\emptyset \subseteq S$

Equivently,  $\forall x. x \in \emptyset \Rightarrow x \in S$

which is true because there can be no such  $x \in \emptyset$

b. Start with  $\Rightarrow$

Assume  $\forall x. x \notin S \quad (+)$

RTP  $S = \emptyset$

Equivently,  $\nexists x \in S$  which we get from (+)

Next prove  $\Leftarrow$

Assume  $S = \emptyset$

Equivently  $\nexists x \in S$

$\therefore \forall x. x \notin S$

□  
3a. union:  $\{-1, 1, 2, 3, 4, 5, 7\}$

intersection:  $\{1, 3, 5\}$

b. union:  ~~$\{x \in \mathbb{R} \mid x = 6 \vee x > 7\}$~~   $\{x \in \mathbb{R} \mid x = 6 \vee x > 7\}$

intersection:  $\{x \in \mathbb{N} \mid x > 7\}$

4.  $\{(1, -1), (1, 1), (1, 3), (1, 5), (1, 7),$   
 $(2, -1), (2, 1), (2, 3), (2, 5), (2, 7),$   
 $(3, -1), (3, 1), (3, 3), (3, 5), (3, 7),$   
 $(4, -1), (4, 1), (4, 3), (4, 5), (4, 7),$   
 $(5, -1), (5, 1), (5, 3), (5, 5), (5, 7)\}$

5a.  $A_2 = \{2, 3, 1, 4\}$

$A_3 = \{3, 4, 2, 6\}$

$A_4 = \{4, 5, 3, 8\}$

$A_5 = \{5, 6, 4, 10\}$

b)  $\bigcup \{A_i \mid i \in I\} = \{1, 2, 3, 4, 5, 6, 8, 10\}$

$\bigcap \{A_i \mid i \in I\} = \{4\}$

$$6. \{ -1, 2, 4, 7 \}$$

7a. Start with  $\rightarrow$

$$\text{Assume } A^c = B$$

$$\text{Equivalently, } \forall x \in V. x \in A \iff x \notin B$$

$$A \cup B = \{x \in V \mid x \in A\} \cup \{x \in V \mid x \notin A\}$$

$$= \{x \in V \mid x \in A \vee x \notin A\}$$

$$= \{x \in V\}$$

$$= V$$

$$A \cap B = \{x \in V \mid x \in A\} \cap \{x \in V \mid x \notin A\}$$

$$= \{x \in V \mid x \in A \wedge x \in B\}$$

$$= \emptyset$$

Next prove  $\Leftarrow$

$$\text{Assume } A \cup B = V \stackrel{(*)}{\wedge} A \cap B = \emptyset$$

$$\text{By } (*) \quad \forall x \in V. x \in A \Rightarrow x \notin B$$

$$\begin{aligned} \text{By } (*) \quad & (\forall x \in V. x \in A \vee x \in B) \\ & \Rightarrow (\forall x \in V. x \notin B \Rightarrow x \in A) \end{aligned}$$

$$\therefore \forall x \in V. x \in A \iff x \notin B$$

$$\therefore A^c = B$$

$$b. (A^c)^c = \{x \in V \mid x \notin A^c\}$$

$$= \{x \in V \mid x \notin \{y \in V \mid y \in A\}\}$$

$$= \{x \in V \mid x \notin A = \text{false}\}$$

$$= \{x \in V \mid x \in A\}$$

$$= A$$

$$\begin{aligned}
 c. (A \cup B)^c &= \{x \in V \mid x \notin A \cup B\} \\
 &= \{x \in V \mid x \notin A \wedge x \notin B\} \\
 &= \{x \in V \mid x \notin A\} \cap \{x \in V \mid x \notin B\} \\
 &= A^c \cap B^c
 \end{aligned}$$

$$\begin{aligned}
 (A \cap B)^c &= \{x \in V \mid x \notin A \cap B\} \\
 &= \{x \in V \mid x \notin A \vee x \notin B\} \\
 &= \{x \in V \mid x \notin A\} \cup \{x \in V \mid x \notin B\} \\
 &= A^c \cup B^c
 \end{aligned}$$

1.2.1 a. Assume  $A \subseteq B$

~~$$\begin{aligned}
 &\text{E.g. } x \in A \Rightarrow x \in B \\
 &\text{P(A) } \subseteq \text{P(B)} \quad x \in A \Rightarrow x \in B
 \end{aligned}$$~~

$$\begin{aligned}
 P(B) &= \{x \mid x \in B\} \\
 &= \{x \mid x \in A\} \cup \{x \mid x \in B \wedge x \notin A\} \\
 &\supseteq P(A) \cup \{x \mid x \in B \wedge x \notin A\} \\
 \therefore P(A) &\subseteq P(B)
 \end{aligned}$$

b.  $A = \{\}, B = \{2\}$

$$P(A \cup B) = \{\{\}, \{\}, \{2\}, \{1, 2\}\}$$

$$P(A) \cup P(B) = \{\{\}, \{\}, \{2\}\} \not\supseteq P(A \cup B)$$

c.  $P(A) \subseteq P(A \cup B)$

$$P(B) \subseteq P(A \cup B)$$

$$\therefore P(A) \cup P(B) \subseteq P(A \cup B)$$

d.  $P(A) = \{x \mid x \in A\}$

$$P(B) = \{x \mid x \in B\}$$

$$\therefore P(A) \cap P(B) = \{x \mid x \in A \wedge x \in B\}$$

$$= \{x \mid x \in A \cap B\}$$

$$= P(A \cap B)$$

e. see d)

2a.  $A \cup B = B$

$$\Leftrightarrow \{x \in A \wedge x \in B\} \cup \{x \in B\} = \{x \in B\}$$

$$\Leftrightarrow (x \in A \wedge x \in B) \Leftrightarrow x \in B$$

$$\Leftrightarrow A \subseteq B$$

$$\Leftrightarrow B = A \cup \{x \mid x \in B \wedge x \notin A\}$$

$$\Leftrightarrow A \cap B = A$$

$$\Leftrightarrow x \notin B \Rightarrow x \notin A$$

$$d. \Leftrightarrow B^c \subseteq A^c$$



3a. Assume  $A \subseteq B \wedge C \subseteq D$

$$\begin{aligned}\therefore A \times C &= \{\forall i \in A, j \in C. (i, j)\} \\ &= \{\forall i \in B, j \in D. (i, j) \mid i \in A \wedge j \in C\} \\ &\subseteq \{\forall i \in B, j \in D. (i, j)\} \\ &= B \times C\end{aligned}$$

b. Let  $A = \{0\}$ ,  $B = \{1\}$ ,  $C = \{2\}$ ,  $D = \{3\}$

$$(A \times C) \times (B \cup D) = \{(0, 1), (0, 3), (2, 1), (2, 3)\}$$

$$(A \times B) \cup (C \times D) = \{(0, 1)\}$$

$$(A \times C) \times (B \cup D) \not\subseteq \{(0, 1)\}$$

Let  $A = \{a\}$ ,  $B = \{b\}$ ,  $C = \{c\}$ ,  $D = \{d\}$

$$(A \cup C) \times (B \cup D) = \{(a, b), (a, d), (c, b), (c, d)\}$$

$$(A \times B) \cup (C \times D) = \{(a, b), (c, d)\}$$

$$\therefore (A \cup C) \times (B \cup D) \not\subseteq (A \times B) \cup (C \times D)$$

c.  $A \subseteq A \cup C$ ,  $B \subseteq B \cup D$

$$\therefore \text{By a), } A \times B \subseteq (A \cup C) \times (B \cup D)$$

$$\text{By symmetry, } C \times D \subseteq (A \cup C) \times (B \cup D)$$

$$\therefore (A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$$

d. Let  $A = \{a\}$ ,  $B = \{b\}$ ,  $D = \{d\}$

$$A \times (B \cup D) = \{(a, b), (a, d)\}$$
$$(A \times B) \cup (A \times D) =$$

$$A \times (B \cup D) = \{i : i \in A, j \in B \cup D, (i, j)\}$$
$$= \{i : i \in A, j \in B, (i, j)\} \cup \{i : i \in A, j \in D, (i, j)\}$$
$$= (A \times B) \cup (A \times D)$$

e. See d)

4a. Let  $A = \{1\}$ ,  $B = \{1, 2\}$ ,  $C = \{2\}$ ,  $D = \{1, 2\}$

$$A \subseteq B \wedge C \subseteq D$$

$$A \uplus C = \{1, 2\}$$

$$B \uplus D = \emptyset$$

$$\{1, 2\} \neq \emptyset$$

b.  $(A \cup B) \uplus C = \{x | (x \in A \cup B \wedge x \notin C) \vee (x \notin A \cup B \wedge x \in C)\}$

$$= \{x | (x \in A \wedge x \notin C) \vee (x \in B \wedge x \in C) \vee (x \notin A \wedge x \in B \wedge x \in C)\}$$
$$\supseteq \{x | ((x \in A \wedge x \notin C) \vee (x \in A \wedge x \in C)) \vee ((x \in B \wedge x \in C) \vee (x \in B \wedge x \notin C))\}$$
$$= (A \uplus C) \cup (B \uplus C)$$

c. Let  $A = \{1\}$ ,  $B = \{2\}$ ,  $C = \{1, 2\}$

$$(A \uplus C) \cup (B \uplus C) = \{1, 2\}$$

$$(A \cup B) \uplus C = \emptyset$$

$$\{1, 2\} \neq \emptyset$$

d. Let  $A = \{1\}$ ,  $B = \{1, 2\}$ ,  $C = \{2\}$

$$(A \cap B) \uplus C = \{1, 2\}$$

$$(A \uplus C) \cap (B \uplus C) = \{1\}$$

$$\{1, 2\} \not\subseteq \{1\}$$

$$\begin{aligned}
 e. (A \oplus B) \cap (B \oplus C) &= \{x | (x \in A \wedge x \notin C) \vee (x \notin A \wedge x \in C)\} \cap \{x | (x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C)\} \\
 &= \{x | ((x \in A \wedge x \notin C) \vee (x \in A \wedge x \in C)) \wedge ((x \in B \wedge x \notin C) \vee (x \notin B \wedge x \in C))\} \\
 &= \{x | ((x \in A \wedge x \in C) \wedge (x \in B \wedge x \notin C)) \vee \\
 &\quad ((x \in A \wedge x \notin C) \wedge (x \in B \wedge x \in C)) \vee \\
 &\quad ((x \in A \wedge x \in C) \wedge (x \notin B \wedge x \in C))\} \\
 &= \{x | (x \in A \wedge x \in B \wedge x \notin C) \vee (x \in A \wedge x \in B \wedge x \in C)\} \\
 &\subseteq \{x | (x \in (A \cap B) \wedge x \notin C) \vee (x \in (A \cap B) \wedge x \in C)\} \\
 &= (A \cap B) \oplus C
 \end{aligned}$$

5.  $VF = \{x \in A | \exists S \in F, x \in S\} = A$

$$\begin{aligned}
 \cap U &= \{x \in A | \forall X \in U, x \in X\} \\
 &= \{x \in A | \forall X \in P(A), (\forall S \in F, S \subseteq X) \Leftrightarrow x \in X\} \\
 &= \{x \in A | \forall X \in F, (X = A) \Leftrightarrow x \in X\} \\
 &= \{x \in A | x \in A\} \\
 &= A \\
 &= VF
 \end{aligned}$$

~~VS CAUSES PROBLEMS~~

$$I = \{x \in A | \forall S \in F, x \subseteq S\} = \emptyset \vdash \cap F$$

6. Start with  $\Rightarrow$

Assume  $VF \subseteq U$

Equivalently  $\{x | \exists S \in F, x \in S\} \subseteq U$

~~Assume~~  $\forall X \in F, x \in X \Rightarrow x \in VF$   
 $\Rightarrow x \in U$

$\therefore \forall X \in F, x \in V$

Next prove  $\Leftarrow$

Assume  $\forall x \in F, x \subseteq V$

Equivalently,

$\forall X \in F, x \in X \Rightarrow x \in V$

$$\therefore \{\forall X \in F \mid x \in X\} = \bigcup F$$

$$= \{x \mid \exists X \in F, x \in X\}$$

$$\subseteq \{x \mid x \in V\}$$

$$= V$$

$$\therefore \bigcup F \subseteq V$$