

Discrete Maths Supervision 6

$$2.1.1 S \circ R = \{(2, z), (3, x), (3, z)\}$$

2. RTP: \forall sets $A, B \subseteq R, S, T \subseteq A \times B, R \circ (S \circ T) = (R \circ S) \circ T$

Let A, B arbitrary sets

Let $R, S, T \subseteq A \times B$

$$S \circ T = \{($$

$$(T \circ S) \circ R : T \circ (S \circ R)$$

RTP: \forall sets $A, B, C, D, \forall R \subseteq A \times B, S \subseteq B \times C, T \subseteq C \times D, R \circ (S \circ T) = (R \circ S) \circ T$

Let A, B, C, D arbitrary sets

Let $R \subseteq A \times B, S \subseteq B \times C, T \subseteq C \times D$

$$S \circ T = \{(b \in B, d \in D) \mid \exists c \in C. (b, c) \in S \wedge (c, d) \in T\}$$

$$(T \circ S) \circ R$$

$$\therefore R \circ (S \circ T) = \{(a \in A, d \in D) \mid \exists b \in B. (a, b) \in R \wedge (\exists c \in C. (b, c) \in S \wedge (c, d) \in T)\}$$

$$= \{(a \in A, d \in D) \mid \exists c \in C. (\exists b \in B. (a, b) \in R \wedge (b, c) \in S) \wedge (c, d) \in T\}$$

$$= (R \circ S) \circ T = (S \circ R) \circ T$$

RTP: \forall sets $A, B \forall R \subseteq A \times B, R \circ R = R$

Let A, B arbitrary sets

Let $R \subseteq A \times B$

$$R \circ R = \{(a \in A, b \in B) \mid \exists b' \in B. (a, b') \in R \wedge b = b'\}$$

$$= \{(a \in A, b \in B) \mid b = b \text{ & } (a, b) \in R\}$$

$$= R$$

$$\begin{aligned} \exists_{\alpha} R \subseteq S &\Rightarrow ((a, b) \in R \Rightarrow (a, b) \in S) \\ &\Rightarrow ((b, a) \in R^{\text{op}} \Rightarrow (a, b) \in S \Rightarrow (b, a) \in S^{\text{op}}) \\ &\Rightarrow R^{\text{op}} \subseteq S^{\text{op}} \end{aligned}$$

$$\begin{aligned} b. (R \cap S) &= \{(a, b) \mid (a, b) \in R \wedge (a, b) \in S\} \\ \therefore (R \cap S)^{\text{op}} &= \{(b, a) \mid (a, b) \in R \wedge (a, b) \in S\} \\ &= \{(b, a) \mid (b, a) \in R^{\text{op}} \wedge (b, a) \in S^{\text{op}}\} \\ &= R^{\text{op}} \cap S^{\text{op}} \end{aligned}$$

$$\begin{aligned} c. (R \cup S) &= \{(a, b) \mid (a, b) \in R \vee (a, b) \in S\} \\ \therefore (R \cup S)^{\text{op}} &= \{(b, a) \mid (a, b) \in R \vee (a, b) \in S\} \\ &= \{(b, a) \mid (b, a) \in R^{\text{op}} \vee (b, a) \in S^{\text{op}}\} \\ &= R^{\text{op}} \cup S^{\text{op}} \end{aligned}$$

4. RTP: $\forall \text{sets } A, \forall R \subseteq A \times A, (\forall a \neq b \in A. (a, b) \in R \Leftrightarrow (b, a) \notin R) \Leftrightarrow R \cap R^{\text{op}} \subseteq \text{id}_A$

Let A arbitrary set

Let $R \subseteq A \times A$

Start with " \Rightarrow "

Assume $\forall a \neq b \in A. (a, b) \in R \Leftrightarrow (b, a) \notin R$

RTP $R \cap R^{\text{op}} \subseteq \text{id}_A$

$$\begin{aligned} R \cap R^{\text{op}} &= \{(a, b) \mid (a, b) \in R \wedge (a, b) \in R^{\text{op}}\} \\ &= \{(a, b) \mid (a, b) \in R \wedge (b, a) \in R\} \\ &\not\subseteq \{(a, b) \mid a = b\} \\ &\subseteq \text{id}_A \end{aligned}$$

Next prove " \Leftarrow "

Assume $R \cap R^{\text{op}} \subseteq \text{id}_A$

RTP $\forall a \neq b \in A. (a, b) \in R \Leftrightarrow (b, a) \notin R$

Equivently, $\forall a \neq b \in A. (a, b) \in R \Rightarrow (b, a) \notin R$
by contrapositive.

Let $a, b, c \in A$

~~RSP~~ ~~(a, b) ∈ R~~ ~~(b, a) ∈ R~~ Assume $(a, b) \in R$

RTP: $(b, a) \notin R$

$$\{(a, b) \mid (a, b) \in R \wedge (b, a) \in R\} \subseteq id_A$$

~~(a, b) ∈ R~~ ~~(b, a) ∈ R~~

$$(a, b) \in R \wedge (b, a) \in R \Rightarrow a = b \\ \Rightarrow \text{false}$$

$\therefore (b, a) \in R \Rightarrow \text{false}$

$\therefore (b, a) \notin R$

2.2. 1.a) ~~(VF) ∘ R~~ ~~(A × B) ∘ R~~

$$\begin{aligned}(VF) \circ R &= \{(x, b) \mid \exists a. (x, a) \in R \wedge (a, b) \in VF\} \\ &= \{(x, b) \mid \exists a. (x, a) \in R \wedge (\exists s \in F. (a, b) \in s)\} \\ &= \{(x, b) \mid \exists a \in A, s \in F. (x, a) \in R \wedge (a, b) \in s\} \\ &= V(\{\{(x, b) \mid \exists a \in A, (x, a) \in R \wedge (a, b) \in s\} \mid s \in F\}) \\ &= V(\{R \circ s \mid s \in F\})\end{aligned}$$

$$\begin{aligned}b) R \circ (VF) &= \{(a, y) \mid \exists b. (a, b) \in VF \wedge (b, y) \in R\} \\ &= \{(a, y) \mid \exists b. (\exists s \in F. (a, b) \in s) \wedge (b, y) \in R\} \\ &= \{(a, y) \mid \exists b \in B, s \in F. (a, b) \in s \wedge (b, y) \in R\} \\ &= V(\{\{(a, y) \mid \exists b. (a, b) \in s \wedge (b, y) \in R\} \mid s \in F\}) \\ &= V(\{R \circ s \mid s \in F\})\end{aligned}$$

$$2. R^{0+} = \text{rel} \left(\sum_{n \in \mathbb{N}} \text{mat}(R^{0n}) \right)$$

$$= \text{rel} \left(\sum_{n \in \mathbb{N}} \text{mat}(R)^n \right)$$

$$R \circ R^{0+} = \text{rel} \left(\text{mat}(R) \cdot \sum_{n \in \mathbb{N}} \text{mat}(R)^n \right)$$

$$= \text{rel} \left(\sum_{n \in \mathbb{N}} \text{mat}(R)^{n+1} \right)$$

$\therefore R^{0+} = \{(a, b) \mid \exists \text{ a path of length } \geq 1 \text{ from } a \text{ to } b \text{ in } R\}$

i. RTP: ~~$R \subseteq R^{0+}$~~ & $R \subseteq R^{0+} \cap R^{0+}$ is transitive

First, prove $R \subseteq R^{0+}$

$R = \{(a, b) \mid \exists \text{ a path of length } 1 \text{ from } a \text{ to } b \text{ in } R\}$

$\therefore R \subseteq R^{0+}$ by definition

Next prove R^{0+} is transitive

Assume $(a, b) \in R^{0+} \cap (b, c) \in R^{0+}$

RTP $(a, c) \in R^{0+}$

Let α = the path length from a to b in R^{0+}

Let β = the path length from b to c in R^{0+}

$$\alpha + \beta > 1$$

\therefore there exists a path of length ≥ 1 from a to c in R^{0+}

□

0.1.1 $A_2 \Rightarrow A_2 :$

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{ {},  
  { (1, 1) },  
  { (1, 2) },  
  { (2, 1) },  
  { (2, 2) },  
  { (1, 1), (2, 1) },  
  { (1, 1), (2, 2) },  
  { (1, 2), (2, 1) },  
  { (1, 2), (2, 2) } }
```

$A_2 \Rightarrow A_3 :$

```
{ {},  
  { (1, a) },  
  { (1, b) },  
  { (1, c) },  
  { (2, a) },  
  { (2, b) },  
  { (2, c) },  
  { (1, a), (2, a) },  
  { (1, a), (2, b) },  
  { (1, a), (2, c) },  
  { (1, b), (2, a) },  
  { (1, b), (2, b) },  
  { (1, b), (2, c) },  
  { (1, c), (2, a) },  
  { (1, c), (2, b) },  
  { (1, c), (2, c) } }
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$A_2 \Rightarrow A_4 :$

```
{ {},  
  { (a, 1) },  
  { (a, 2) },  
  { (b, 1) },  
  { (b, 2) },  
  { (c, 1) },  
  { (c, 2) },  
  { (a, 1), (b, 1) },  
  { (a, 1), (b, 2) },  
  { (a, 2), (b, 1) },  
  { (a, 2), (b, 2) },  
  { (a, 1), (c, 1) },  
  { (a, 1), (c, 2) },  
  { (a, 2), (c, 1) },  
  { (a, 2), (c, 2) },  
  { (b, 1), (c, 1) },  
  { (b, 1), (c, 2) },  
  { (b, 2), (c, 1) },  
  { (b, 2), (c, 2) },  
  { (a, 1), (b, 1), (c, 1) },  
  { (a, 1), (b, 1), (c, 2) },  
  { (a, 1), (b, 2), (c, 1) },  
  { (a, 1), (b, 2), (c, 2) },  
  { (a, 2), (b, 1), (c, 1) },  
  { (a, 2), (b, 1), (c, 2) },  
  { (a, 2), (b, 2), (c, 1) },  
  { (a, 2), (b, 2), (c, 2) } }
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$A_3 \Rightarrow A_3 :$

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{ {},
```

) 3.1.1 A₂ A₂

{(a, a)},
{(a, b)},
{(a, c)},
{(b, a)},
{(b, b)},
{(b, c)},
{(c, a)},
{(c, b)},
{(c, c)},
{(a, a), (b, a)},
{(a, a), (b, b)},
{(a, a), (b, c)},
{(a, b), (b, a)},
{(a, b), (b, b)},
{(a, b), (b, c)},
{(a, c), (b, a)},
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{(a, a), (c, a)},
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{(a, b), (b, b), (c, c)},
{(a, b), (b, c), (c, a)},
{(a, b), (b, c), (c, b)},
{(a, b), (b, c), (c, c)},
{(a, c), (b, a), (c, a)},
{(a, c), (b, a), (c, b)},
{(a, c), (b, a), (c, c)},
{(a, c), (b, b), (c, a)},
{(a, c), (b, b), (c, b)},
{(a, c), (b, b), (c, c)},
{(a, c), (b, c), (c, a)},
{(a, c), (b, c), (c, b)},
{(a, c), (b, c), (c, c)}}

~~3.1.1 $A_2 \Rightarrow A_2$~~ : $\{\{(1,1), (2,1)\}, \{(1,1), (2,2)\}, \{(1,2), (2,1)\}, \{(1,2), (2,2)\}\}$
 ~~$A_2 \Rightarrow A_3$~~ : $\{\{\{(1,a), (2,a)\}, \{(1,a), (2,b)\}, \{(1,a), (2,c)\},$
 $\{(1,b), (2,a)\}, \{(1,b), (2,b)\}, \{(1,b), (2,c)\},$
 $\{(1,c), (2,a)\}, \{(1,c), (2,b)\}, \{(1,c), (2,c)\}\}$
 ~~$A_3 \Rightarrow A_2$~~ : $\{\{\{(a,1), (b,1), (c,1)\}, \{(a,1), (b,1), (c,2)\}, \{(a,1), (b,2), (c,1)\},$
 $\{(a,1), (b,2), (c,2)\}, \{(a,2), (b,1), (c,1)\}, \{(a,2), (b,1), (c,2)\},$
 $\{(a,2), (b,2), (c,1)\}, \{(a,2), (b,2), (c,2)\}\}$
 ~~$A_3 \Rightarrow A_3$~~ : $\{\{\{(a,a), (b,a), (c,a)\}, \{(a,a), (b,a), (c,b)\}, \{(a,a), (b,a), (c,c)\},$
 $\{(a,a), (b,b), (c,a)\}, \{(a,a), (b,b), (c,b)\}, \{(a,a), (b,b), (c,c)\},$
 $\{(a,a), (b,c), (c,a)\}, \{(a,a), (b,c), (c,b)\}, \{(a,a), (b,c), (c,c)\},$
 $\{(a,a), (b,a), (c,a)\}, \{(a,b), (b,a), (c,a)\}, \{(a,b), (b,a), (c,b)\},$
 $\{(a,b), (b,b), (c,a)\}, \{(a,b), (b,b), (c,b)\}, \{(a,b), (b,b), (c,c)\},$
 $\{(a,b), (b,c), (c,a)\}, \{(a,b), (b,c), (c,b)\}, \{(a,b), (b,c), (c,c)\},$
 $\{(a,c), (b,a), (c,a)\}, \{(a,c), (b,a), (c,b)\}, \{(a,c), (b,a), (c,c)\},$
 $\{(a,c), (b,b), (c,a)\}, \{(a,c), (b,b), (c,b)\}, \{(a,c), (b,b), (c,c)\},$
 $\{(a,c), (b,c), (c,a)\}, \{(a,c), (b,c), (c,b)\}, \{(a,c), (b,c), (c,c)\}\}$

Q. RTP: \forall sets $A, B \neq \emptyset \subset X \times Y$. R is a partial function $\Leftrightarrow R \circ R^{\text{op}} \subseteq \text{id}_B$

Let A, B arbitrary sets

Let $R \subseteq A \times B$

Start with " \Rightarrow "

Assume R is a partial function

Equivalently ~~$\forall a, b, c \in A \times B$ $(a, b) \in R \wedge (a, c) \in R \Rightarrow b = c$~~

~~$\forall a, b, c \in A \times B$ $(a, b) \in R \wedge (a, c) \in R \Rightarrow b = c$~~ (T)

RTP: $R \circ R^{\text{op}} \subseteq \text{id}_B$

$$\begin{aligned}
 R \circ R^{\text{op}} &= \{(b, b') \mid \exists a. (b, a) \in R^{\text{op}} \wedge (a, b') \in R\} \\
 &= \{(b, b') \mid \exists a. (a, b) \in R \wedge (a, b') \in R\} \\
 &\subseteq \{(b, b') \mid b = b'\} \quad \text{by (T)} \\
 &= \text{id}_B
 \end{aligned}$$

Next prove " \Leftarrow "

Assume $R \circ R^{\text{op}} \subseteq \text{id}_B$

~~equivalently: $\{(b, b') \mid \exists a. (a, b) \in R \wedge (a, b') \in R\} \subseteq \text{id}_B$~~

$\therefore \forall a, b, b'. (a, b) \in R \wedge (a, b') \in R \Rightarrow b = b'$

~~RTP: $\forall a \in A \Rightarrow b \in B$ s.t. $a \sim a$~~

~~Let $a \in A$~~

□

3a. RTP: ~~s.t.~~ \forall sets A , id_A is a partial function

~~Let A arbitrary set~~

~~RTP: $\forall a, b, c. (a, b) \in \text{id}_A \wedge (a, c) \in \text{id}_A \Rightarrow b = c$~~

~~Let $a, b, c \in A$~~

Assume $(a, b) \in \text{id}_A \Rightarrow b = a$

$(a, c) \in \text{id}_A \Rightarrow c = a$

$\therefore b = c$ □

b. Let A, B, C arbitrary sets

Let $S: A \rightarrow B$, $T: B \rightarrow C$

$RTP: T \circ S: A \rightarrow C$

Equivalently, $\forall a \in A, c \in C, c' \in C, (a, c) \in T \circ S \wedge (a, c') \in T \circ S$
 $\Rightarrow c = c'$

Let $a \in A, c \in C, c' \in C$

Assume $(a, c) \in T \circ S$

let ~~b~~ s.t. $(a, b) \in S \wedge (b, c) \in T$

~~Let b~~

Assume $(a, c') \in T \circ S$

Let b' s.t. $(a, b') \in S \wedge (b', c') \in T$

S is a partial function $\therefore b = b'$

$\therefore (b, c) \in T \wedge (b, c') \in T \Rightarrow T$ is a partial function

$\therefore c = c'$ □

3.2.1 Reflexivity:

RTP: $\forall R: A \rightarrow B, R \subseteq R$

Let R be an arbitrary partial function from A to B . RTP $R \subseteq R$ which is true by definition.

Transitivity:

~~RTP $\forall R: A \rightarrow B, R \subseteq R$~~

~~$\forall R: A \rightarrow B, R \subseteq R$~~

Let R, S, T arbitrary partial functions from A to B .

RTP $R \subseteq S \wedge S \subseteq T \Rightarrow R \subseteq T$

Assume $R \subseteq S$, equivalently, $\forall x. x \in R \Rightarrow x \in S$ (†)

Assume $S \subseteq T$, equivalently, $\forall x. x \in S \Rightarrow x \in T$ (‡)

RTP $\forall x. x \in R \Rightarrow x \in T$

Let $x \in R$

$\therefore x \in S$ by (†)

$\therefore x \in T$ by (‡)

□

Antisymmetry:

Let R, S arbitrary partial functions from A to B .

RTP. $R \subseteq S \wedge S \subseteq R \Rightarrow S = R$

Assume $R \subseteq S$, equivalently, $\forall x. x \in R \Rightarrow x \in S$

Assume $S \subseteq R$, equivalently, $\forall x. x \in S \Rightarrow x \in R$

$\forall x. x \in R \Leftrightarrow x \in S$

$\therefore R = S$

□

2. Let ~~$\exists F \subseteq \text{PFun}(A, B)$~~ $F \subseteq \text{PFun}(A, B)$

Assume $F \neq \emptyset$

$$\cap F = \{(a, b) / \forall R \in F, (a, b) \in R\}$$

RTP: $\forall a, b, c \quad (a, b) \in \cap F \wedge (a, c) \in \cap F \Rightarrow b = c$

Let ~~$\exists a \in A, b \in B, c \in B$~~ $a \in A, b \in B, c \in B$

Assume $(a, b) \in \cap F$

i. $\forall R \in F, (a, b) \in R$

Assume $(a, c) \in \cap F$

ii. $\forall R \in F, (a, c) \in R$

~~RTP: $b=c$~~

Let $R \in F$

$(a, b) \in R \wedge (a, c) \in R$

$\therefore b = c$

3. Let $f, g \in \text{PFun}(A, B)$

Assume $\exists h \in \text{PFun}(A, B)$

Let $h \in \text{PFun}(A, B)$

~~Assume $f \subseteq h \subseteq g$~~

RTP: $f \cup g \in \text{PFun}(A, B)$

Equivalently, $\forall a, b, c \quad (a, b) \in f \cup g \wedge (a, c) \in f \cup g \Rightarrow b = c$

Let $a \in A, b, c \in B$

Assume $(a, b) \in f \cup g$

$\therefore (a, b) \in f \vee (a, b) \in g$

i. $(a, b) \in h$ in either case

Assume $(a, c) \in f \cup g$

$\therefore (a, c) \in f \vee (a, c) \in g$

ii. $(a, c) \in h$ in either case

$\therefore b = c$

□

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$$\begin{aligned}
 4.11. \quad A_1 \Rightarrow A_2 &: \left\{ \left\{ (1,1), (2,1) \right\}, \left\{ (1,1), (2,2) \right\}, \left\{ (1,2), (2,1) \right\}, \left\{ (1,2), (2,2) \right\} \right\} \\
 A_2 \Rightarrow A_3 &: \left\{ \left\{ (1,a), (2,a) \right\}, \left\{ (1,a), (2,b) \right\}, \left\{ (1,a), (2,c) \right\}, \right. \\
 &\quad \left. \left\{ (1,b), (2,a) \right\}, \left\{ (1,b), (2,b) \right\}, \left\{ (1,b), (2,c) \right\}, \right. \\
 &\quad \left. \left\{ (1,c), (2,a) \right\}, \left\{ (1,c), (2,b) \right\}, \left\{ (1,c), (2,c) \right\} \right\} \\
 A_3 \Rightarrow A_2 &: \left\{ \left\{ (a,1), (b,1), (c,1) \right\}, \left\{ (a,1), (b,1), (c,2) \right\}, \left\{ (a,1), (b,2), (c,1) \right\}, \right. \\
 &\quad \left. \left\{ (a,1), (b,2), (c,2) \right\}, \left\{ (a,2), (b,1), (c,1) \right\}, \left\{ (a,2), (b,1), (c,2) \right\}, \right. \\
 &\quad \left. \left\{ (a,2), (b,2), (c,1) \right\}, \left\{ (a,2), (b,2), (c,2) \right\} \right\} \\
 A_3 \Rightarrow A_3 &: \left\{ \left\{ (a,a)(b,a), (c,a) \right\}, \left\{ (a,a)(b,a), (c,b) \right\}, \left\{ (a,a)(b,a), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,a)(b,b), (c,a) \right\}, \left\{ (a,a)(b,b), (c,b) \right\}, \left\{ (a,a)(b,b), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,a)(b,c), (c,a) \right\}, \left\{ (a,a)(b,c), (c,b) \right\}, \left\{ (a,a)(b,c), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,b), (b,a), (c,a) \right\}, \left\{ (a,b), (b,a), (c,b) \right\}, \left\{ (a,b), (b,a), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,b), (b,b), (c,a) \right\}, \left\{ (a,b), (b,b), (c,b) \right\}, \left\{ (a,b), (b,b), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,b), (b,c), (c,a) \right\}, \left\{ (a,b), (b,c), (c,b) \right\}, \left\{ (a,b), (b,c), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,c), (b,a), (c,a) \right\}, \left\{ (a,c), (b,a), (c,b) \right\}, \left\{ (a,c), (b,a), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,c), (b,b), (c,a) \right\}, \left\{ (a,c), (b,b), (c,b) \right\}, \left\{ (a,c), (b,b), (c,c) \right\}, \right. \\
 &\quad \left. \left\{ (a,c), (b,c), (c,a) \right\}, \left\{ (a,c), (b,c), (c,b) \right\}, \left\{ (a,c), (b,c), (c,c) \right\} \right\}
 \end{aligned}$$

4.12. RTP: $\forall a \in A \exists b \in B \cdot a R b \iff id_A \subseteq R^{\text{op}} \circ R$

Start with " \Rightarrow "

Assume $\forall a \in A \exists b \in B \cdot a R b \quad (1)$

~~Let $a \in A$~~
~~Let $b \in B$~~
~~Let $c \in C$~~

RTP $id_A \subseteq R^{\text{op}} \circ R$

$$\begin{aligned}
 R^{\text{op}} \circ R &= \{ (a, a') \mid \exists b \in B \cdot (a, b) \in R \wedge (b, a') \in R^{\text{op}} \} \\
 &= \{ (a, a') \mid \exists b \in B \cdot (a, b) \in R \wedge (a', b) \in R \}
 \end{aligned}$$

~~Let a, a' . $a = a' \wedge \exists b \in B \cdot (a, b) \in R \Rightarrow (a', b) \in R$~~

By (1): ~~Let $a = a' \in A \Rightarrow \exists b \in B \cdot (a, b) \in R \wedge (a', b) \in R$~~

$$\therefore id_A \subseteq R^{\text{op}} \circ R$$

Next prove " \Leftarrow "

Assume $id_A \subseteq R^{\text{op}} \circ R$

Equivalently, $a = a' \in A \Rightarrow \exists b \in B \cdot (a, b) \in R \wedge (a', b) \in R$

Equivalently, $\forall a \in A \cdot \exists b \in B \cdot (a, b) \in R$

3a) RFP: $\forall a \in A \exists a' \in A. (a, a') \in id_A$
 Equivalently, $\forall a \in A \exists a' \in A. a = a'$
 which is satisfied by $a' = a$

b.) RTP: $\forall \text{sets } A, B \subset \text{if } f: A \rightarrow B, g: B \rightarrow C. g \circ f \text{ is a function}$
 Let A, B, C arbitrary sets
 Let $f: A \rightarrow B, g: B \rightarrow C$
 RTP: $\forall a \in A g \circ f$ is a function.

By 3.1.3, $g \circ f$ is a partial function

RTP: $\forall a \in A \exists c \in C \text{ s.t. } (a, c) \in g \circ f$

Let $a \in A$

Let $b \in B$ s.t. $(a, b) \in f$

Let $c \in C$ s.t. $(b, c) \in g$

$\therefore (a, c) \in g \circ f \quad \square$

$$4.21. \begin{aligned} f &= \{(a, a+1) \mid a \in A\} & \text{where } A = \mathbb{N} \\ g &= \{(a, 2a) \mid a \in A\} \end{aligned}$$

$$f \circ g = \{(a, 2a+1) \mid a \in A\}$$

$$g \circ f = \{(a, 2a+2) \mid a \in A\}$$

$$(0, 1) \in f \circ g$$

$$(0, 1) \notin g \circ f$$

$$\therefore f \circ g \neq g \circ f$$

$$\text{def. } X_{A \cup B}(x) = (x \in A \cup B) \cancel{=} (x \in A) \vee (x \in B)$$

$$= X_A \text{ or } X_B$$

$$X_A \quad X_B \quad X_{A \cup B} \quad \max(X_A, X_B)$$

0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

$$\therefore X_{A \cup B}(x) = \max(X_A(x), X_B(x))$$

$$\chi_{A \cap B}(z) = (z \in A \cap B) = (z \in A) \wedge (z \in B) = \chi_A(z) \text{ AND } \chi_B(z)$$

$$\chi_A \quad \chi_B \quad \chi_{A \cap B} \quad \min(\chi_A, \chi_B)$$

0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

$$\therefore \chi_{A \cap B}(z) = \min(\chi_A(z), \chi_B(z))$$

$$\chi_{A^c}(z) = (z \in A^c) = (\bar{z} \notin A) = \text{NOT } (z \in A) = \text{NOT } \chi_A(z)$$

$$\chi_A \quad \chi_{A^c} \quad 1 - \chi_A$$

0	1	1
1	0	0

b. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$$\chi_{A \Delta B} = (z \in A \Delta B) = (z \in A) \text{ XOR } (z \in B) = \chi_A \text{ XOR } \chi_B$$

$$\chi_A \quad \chi_B \quad \chi_{A \Delta B} \quad \chi_A + \chi_B$$

0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	0

- 5.1.1. ~~$f: \mathbb{Z} \rightarrow N$~~ $f(x) = x^2$ has no retraction
 ~~$f: \mathbb{Z} \rightarrow \mathbb{Z}$~~ $f(x) = 2x$ has one retraction
 ~~$f: N \rightarrow \{0\}$~~ $f(x) = 0$ has many retractions.
 ~~$f: \{0\} \rightarrow \{0, 1\}$~~ $f(x) = 0$ has two retractions.
 merely, if $g(x) = 0$ and $g(x) = 1$
- a. ~~$f: \mathbb{Z} \rightarrow N$~~ . $f(x) + (x)$ has
 $f: \{0\} \rightarrow \{0, 1\}$ if $f(x) = 0$
 has many retractions including
 $g(x) = 0$ and $g(x) = x$
- b. ~~$f: \mathbb{Z} \rightarrow N$~~ . $f(x) = |x|$ has no section
 ~~$f: \mathbb{Z} \rightarrow \mathbb{Z}$~~ . $f(x) = x+1$ has one section

	Retractions	Sections
0	$f: N \rightarrow N: f(x) = x^2$	$f: N \rightarrow N: f(x) = 2^x$
1	$f: \mathbb{Z} \rightarrow \mathbb{Z}: f(x) = x+1$	$f: \mathbb{Z} \rightarrow \mathbb{Z}: f(x) = 1-x$
Many	$f: \{0, 1\} \rightarrow \{0, 1\}$ $f(x) = 0$	$f: \mathbb{Z} \rightarrow N: f(x) = x$

2. a. 2 : $[0..n] \rightarrow [-n..n]$: $x \mapsto x$
 $[0..n] \rightarrow [-n..n]$: $x \mapsto -x$

b. retraction (x) = $\begin{cases} \log_2 x & \text{if } \log_2 x \text{ is an integer} \\ \text{anything} & \text{otherwise} \end{cases}$ (n cases)
 ($2^n + 1 - n$ cases)

\therefore there are $(1+n)(2^n + 1 - n)$ retractions

3i. $A = \mathbb{N}$

$B = \mathbb{N}$

$f(x) = 2x$

The retraction: $x \mapsto \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \text{anything} & \text{otherwise} \end{cases}$

There is no section, because ~~there is~~ on odd inputs there is no natural number which, when doubled, gives that input.

Ques. Let A be some arbitrary set

RTP: $\forall a' \in A \exists! a \in A \cdot (a, a') \in \text{id}_A$

Let $a' \in A \cdot (a', a') \in \text{id}_A \therefore \exists a \in A \cdot (a, a') \in \text{id}_A$

RTP: $\forall a_1, a_2 \in A \cdot ((a_1, a') \in \text{id}_A \wedge (a_2, a') \in \text{id}_A) \Rightarrow a_1 = a_2$

Let $a_1, a_2 \in A$

Assume $(a_1, a') \in \text{id}_A \therefore a_1 = a'$

Assume $(a_2, a') \in \text{id}_A \therefore a_2 = a'$

$\therefore a_1 = a_2 \quad \square$

b. RTP. \forall sets A, B, C . \forall bijections $f: A \rightarrow B, g: B \rightarrow C$. $g \circ f$ is a bijection

Let A, B, C arbitrary sets

Let $f: A \rightarrow B, g: B \rightarrow C$ be bijections

RTP: $\forall c \in C \exists! a \in A \cdot g \circ f(a) = c$

Let $c \in C$

$\therefore \exists b \in B \cdot g(b) = c$

$\therefore \exists a \in A \cdot f(a) = b$

$\therefore g \circ f(a) = c \wedge a$ is unique

\square

Let

5. By associativity,

$$g \circ f \circ h = g \circ (f \circ h) : g \circ \text{id}_B = g$$

$$= (g \circ f) \circ h = \text{id}_A \circ h = h$$

$$\therefore g = h$$

5.2.1a. Assume $r \circ s = \text{id}_A$

$$\text{RTP } (s \circ r) \circ (s \circ r) = s \circ r$$

$$(s \circ r) \circ (s \circ r) = s \circ r \circ s \circ r$$

$$= s \circ (r \circ s) \circ r$$

$$= s \circ \text{id}_A \circ r$$

$$= s \circ r$$

□

b. Let $e: B \rightarrow B$ s.t. $e \circ e = e$

RTP: \exists set A , $s: A \rightarrow B$, $r: B \rightarrow A$ s.t. $s \circ r = e \wedge r \circ s = \text{id}_A$
 Let $A = \{(e(x), \odot) \mid x \in B\}$

$$r: B \rightarrow A : r(x) = (e(x), \odot)$$

$$s: e \circ \text{Pr}_1$$

RTP: $s \circ r = e \quad (1) \wedge r \circ s = \text{id}_A \quad (2)$

(1) First:

$$\begin{aligned} (s \circ r)(x) &= (e \circ \text{Pr}_1)((e(x), \odot)) \\ &= e(e(x)) \\ &= e(x) \end{aligned}$$

$$\therefore s \circ r = e$$

Next (2)

~~for all $x \in B$ $s(r(x)) = x$~~

Let $x \in A = (e(y), \odot)$

$$\begin{aligned}(r \circ s)(x) &= (e(e(p_r((e(y), \odot)))), \odot) \\&= (e(e(y)), \odot) \\&= (e(y), \odot) \\&= x\end{aligned}$$

$$\therefore r \circ s = id_A$$

□

c. $p \circ q \circ p = p$

$$p \circ q \circ p \circ q = p \circ q$$

$$(p \circ q) \circ (p \circ q) = p \circ q$$

$\therefore p \circ q$ is idempotent

$$p \circ q \circ p = p$$

$$q \circ p \circ q \circ p = q \circ p$$

$$(q \circ p) \circ (q \circ p) = q \circ p$$

$\therefore q \circ p$ is idempotent

2i. RTP: $A \cong A$

The bijection is id_A

ii. RTP: $A \cong B \Rightarrow B \cong A$; Assume $A \cong B$

Let f be a bijection from A to B

f^{-1} is a bijection from B to A

iii. RTP $(A \cong B \wedge B \cong C) \Rightarrow A \cong C$

Assume $A \cong B \wedge B \cong C$

Let f = a bijection from A to B

Let g = a bijection from B to C

$g \circ f$ is a bijection from A to C

Assume $A \cong X \wedge B \cong Y$

Let $f = \text{a bijection from } A \text{ to } X$. $g = \text{a bijection from } B \text{ to } Y$

iv). RTP $P(A) \cong P(X)$

Let $f(a) \in X$, ~~then $f(a)$ is a~~

~~$\{f(a), f(a)\} \subset X$~~

~~Let $F: P(A) \rightarrow P(X) = x \mapsto \{f(a) \mid a \in x\}$~~

is a bijection ~~from~~

v). RTP $A \times B \cong X \times Y$

Let $F: A \times B \rightarrow X \times Y = \{ (f(a), g(b)) \mid \forall (a, b) \in A \times B \}$

is a bijection

vi) RTP $A \cup B \cong X \uplus Y$

Let $F: A \cup B \rightarrow X \uplus Y = \{ f(a) \mid a \in A \} \uplus \{ g(b) \mid b \in B \}$

is a bijection

vii). RTP: $\text{Rel}(A, B) \cong \text{Rel}(X, Y)$

Equivalently, $P(A \times B) \cong P(X \times Y)$

$A \times B \cong X \times Y$ by v)

$P(A \times B) \cong P(X \times Y) \cup$ by iv)

viii) RTP: $(A \Rightarrow B) \cong (X \Rightarrow Y)$

Let $F: (A \Rightarrow B) \rightarrow (X \Rightarrow Y): x \mapsto \{ (f(a), g(b)) \mid \forall (a, b) \in (A \Rightarrow B) \}$

is a bijection

ix) RTP: $(A \Rightarrow B) \cong (X \Rightarrow Y)$

Let $F: (A \Rightarrow B) \rightarrow (X \Rightarrow Y): x \mapsto \{ (f(a), g(b)) \mid \forall (a, b) \in (A \Rightarrow B) \}$

is a bijection

x) RTP: $\text{Bij}(A, B) \cong \text{Bij}(X, Y)$

Let $F: \text{Bij}(A, B) \rightarrow \text{Bij}(X, Y): x \mapsto \{ (f(a), g(b)) \mid \forall (a, b) \in \text{Bij}(A, B) \}$

is a bijection

3a. $\# P([n]) = 2^n$ because for each $x \in [n]$, it is either in the subset or not.

$$\#[2^n] = 2^n = \# P([n]) \therefore \text{there is a bijection}$$

b. $\# ([m] \times [n]) = \# [m] \times \# [n]$
 $= m \times n$
 $= \# [m \times n]$

c. $\# ([m] \uplus [n]) = \# [m] + \# [n]$
 $= m + n$
 $= \# [m + n]$

d. $\# ([m] \Rightarrow [n]) = (n+1)^m$ because for each ^{of the m} elements in $[m]$, it has $n+1$ options for a mapping: one of the n elements of $[n]$, or to be undefined.
 $= \# [(n+1)^m]$

e. $\# ([m] \Rightarrow [n]) = n^m$ because for each of the m elements in $[m]$, it can be mapped to any of the n elements in $[n]$
 $= \# [n^m]$

f. $\# B_{ij}([n], [n]) = n!$ because for the first element of the first set, it has n possible images. The second has $n-1$ and so on until the n^{th} element has only 1 possible image $\therefore \# = \sum_{i=0}^{n-1} (n-i) = n!$
 $= \# [n!]$