

Discrete Maths 8

9.1.1 $\overrightarrow{R}_2(N) = \mathbb{Z}$

$\leftarrow R_2(N) = \{m \in N \mid \forall n \in \mathbb{Z}, m = n^2 \Rightarrow n \in N\}$
 $= N \setminus \{n^2 \mid n \in N \wedge n > 0\}$

9.3a. $\overrightarrow{R}(VF) = \{b \in B \mid \exists x \in F, \exists a \in X, a R b\}$

$$= \{b \in B \mid \exists x \in F, b \in \overrightarrow{R}(x)\}$$

$$= V \{\overrightarrow{R}(x) \mid x \in F\}$$

b. $\overleftarrow{R}(AG) = \{a \in A \mid \forall b \in B, (a R b \Rightarrow b \in AG)\}$

$$= \{a \in A \mid \forall b \in B, (a R b \Rightarrow \forall Y \in G, b \in Y)\}$$

$$= \{a \in A \mid \forall Y \in G, \forall b \in B, (a R b \Rightarrow b \in Y)\}$$

$$= \cap \{\overleftarrow{R}(Y) \mid Y \in G\}$$

10.1a Let S be an arbitrary finite set with cardinality $n \in \mathbb{N}$

- $\#S = n = \#[n]$

- $[n] \subseteq \mathbb{N}$

- $\therefore S \cong [n] \subseteq \mathbb{N}$

- $\therefore S$ is countable

Give a surjection $\mathbb{N} \rightarrow [n] = S$ to complete the proof.

- $\mathbb{N} \cong \mathbb{N} \subseteq \mathbb{N}$

- \mathbb{N} is countable

Consider $f: \mathbb{N} \rightarrow \mathbb{Z}: n \mapsto$

$$\begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \text{ is even} \\ -\lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \end{cases}$$

Proof of injectivity:

RTP: ~~$\forall a, a' \in \mathbb{N}, b \in \mathbb{Z}. afb \wedge a'fb \Rightarrow a = a'$~~

$$\forall a, a' \in \mathbb{N}, b \in \mathbb{Z}. afb \wedge a'fb \Rightarrow a = a'$$

Let $a, a' \in \mathbb{N}, b \in \mathbb{Z}$

Assume $afb \wedge a'fb$

Case 0: a is even

$$\therefore \lceil \frac{a}{2} \rceil = b = \lceil \frac{a'}{2} \rceil \quad \therefore a' \text{ is even}$$

$$\therefore a = a' = 2b$$

Case 1: a is odd

$$\therefore -\lceil \frac{a}{2} \rceil = b = -\lceil \frac{a'}{2} \rceil \quad \therefore a' \text{ is odd}$$

$$\therefore -(a+1) = 2b = -(a'+1)$$

$$\therefore a = a'$$

□

Proof of surjectivity.

RTP: $\forall b \in \mathbb{Z}. \exists a \in \mathbb{N}. afb$

Let $b \in \mathbb{Z}$.

Case 0: ~~$b \geq 0$~~ $b \geq 0$

∴ Consider $a = 2b$

$$\therefore \lceil \frac{a}{2} \rceil = \lceil \frac{2b}{2} \rceil = \lceil b \rceil = b$$

∴ $a \neq b$

Case 1: $b < 0$

Consider $a = -(2b+1)$

$$-\lceil \frac{a}{2} \rceil = -\lceil \frac{-2b-1}{2} \rceil = -\lceil b - \frac{1}{2} \rceil = b$$

∴ $a \neq b$

□

∴ f is a bijection

∴ $\mathbb{Z} \cong \mathbb{N} \subseteq \mathbb{N}$

∴ \mathbb{Z} is countable

Consider $g: \mathbb{Q} \Rightarrow \mathbb{Z} \times \mathbb{N}: \frac{p}{q} \mapsto (p, q)$

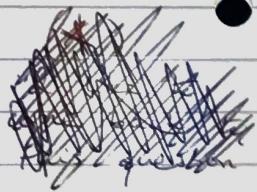
g is clearly injective ∴ There exists a bijection
 ~~$\mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{N}$~~ $\mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$ (by 10.3)

∴ \mathbb{Z} is countable $\sim \mathbb{N}$ is countable

⇒ $\mathbb{Z} \times \mathbb{N}$ is countable (by 10.2b)

⇒ ~~\mathbb{Q}~~ is countable

Great!



10.2.1 ~~Prop.~~ Let S an infinite set

Let $f: N \rightarrow S$ be a surjection

$$\therefore \forall s \in S \cdot \exists n \in N \cdot n f s \quad (\text{P})$$

~~Lemma g: N → S~~

Let $g: S \rightarrow N: s \mapsto \min \{n \in N \mid n f s\}$

g is a total function by (P)

g is ~~surjective~~^{injective} because f is a function

g is surjective because S is infinite

It is only surjective, if f is injective. Otherwise, if let's say $f(0)=f(2)$,

$\therefore g$ is a bijection

$\therefore g^{-1}$ is a bijection $N \rightarrow S$

2a. Let X be a countable set

Let $S \subseteq X$

~~Prop.~~ $\exists Y \subseteq N$ s.t. $X \cong Y$

$\therefore \exists Z \subseteq Y \subseteq N$ s.t. $S \cong Z$

$\therefore S$ is countable

b. Let S, T be countable sets

~~def~~

\therefore let $X \subseteq \mathbb{N}$, f a bijection $S \xrightarrow{\sim} X$, $N \xrightarrow{\sim} S$
Let $Y \subseteq \mathbb{N}$, g a bijection $T \xrightarrow{\sim} Y$, $N \xrightarrow{\sim} T$

let $\mathbb{Z} \subseteq \mathbb{N}$ s.t. $\# \mathbb{Z} = \# X + \# Y$

RSP: \exists a bijection $\mathbb{Z} \rightarrow S \# T$

Consider $h: \mathbb{Z} \rightarrow S \# T: n \mapsto$

$$\begin{cases} f\left(f\left(\frac{n}{2}\right), 0\right) & \text{if } n \text{ is even} \\ g\left(\frac{n-1}{2}\right), 1 & \text{if } n \text{ is odd} \end{cases}$$

h is a bijection

$\therefore S \# T$ is countable.

There is a reason that the suggested definition of enumerable and countable was suggested instead of the one you use here, which is classically equivalent.

I will try to give you an idea in person.

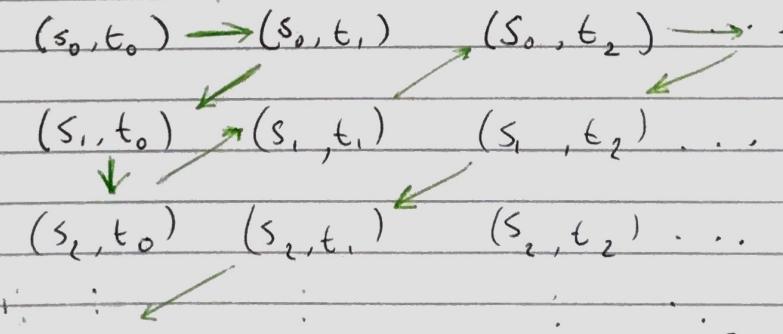
Product:

~~Set $S \times T$ is S.E.~~ Case 0: either S or T or both is infinite

$\forall n \in \mathbb{X}$. Let $s_n \in S$ s.t. $f(n) = s_n$

$\forall n \in \mathbb{Y}$ Let $t_n \in T$ s.t. $g(n) = t_n$

Consider the ^{infinite} table



Starting at (s_0, t_0) and following the diagonal path for n steps and reaching (s_u, t_v) .
 $j: \mathbb{N} \rightarrow S \# T: n \mapsto (s_n, t_n)$ is a bijection
 $\therefore S \# T$ is countable

Case 1: S and T are both finite.

$$\#(S \times T) = (\#S) \times (\#T) = \#[(\#S) \times (\#T)]$$

$$\therefore S \times T \cong [(\#S) \times (\#T)] \subseteq \mathbb{N}$$

$\therefore S \times T$ is countable

3. Let S an infinite set

~~exists a by 10.2.1~~

~~exists a~~

$$c \Rightarrow a \text{ by 10.2.1}$$

$$a \Rightarrow c \text{ by definition}$$

$$\therefore a \Leftrightarrow c$$

RTP. $b \Leftrightarrow a$

" \Leftarrow " first:

There is a bijection $\mathbb{N} \rightarrow S$

\Rightarrow There is a bijection $S \rightarrow \mathbb{N}$

\Rightarrow There is an injection $S \rightarrow \mathbb{N}$

□

" \Rightarrow " next.

Assume there is an injection $f: S \rightarrow \mathbb{N}$

~~exists a~~

$\therefore \forall x, x' \in S, y \in \mathbb{N}, x f y \wedge x' f y \Rightarrow x = x'$

$\therefore S$ is countable

\therefore There exists a bijection $S \rightarrow \mathbb{N}$

□

Let f an injection $P(X) \rightarrow X$
let g be the retraction of f

Consider $B = \{A \subseteq X \mid f(A) \notin A\}$

$$\therefore g(B) \in B \iff g(B) \notin B$$

which is a contradiction

$$\therefore f \text{ an injection } P(X) \rightarrow X$$

Great!