Probability Supervision

L8-9 1. Let $X = X_1 + X_2 + X_3$

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1,X_2,X_3}(x-u-v, u, v)\,dv\,du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1}(x-u-v)\,f_{X_2}(u)\,f_{X_3}(v)\,dv\,du$$

$$= \int_0^1\int_0^1 f_{X_1}(x-u-v)\,dv\,du$$

Case 0: $x \leq 0$

$$\therefore f_X(x) = \int_0^1\int_0^1 0\,dv\,du = 0$$

Case 1: $0 \leq x \leq 1$

$$\therefore f_X(x) = \int_0^x\int_0^{x-u} dv\,du = \int_0^x (x-u)\,du = x^2 - \frac{x^2}{2} = \frac{x^2}{2}$$

Case 2: $1 \leq x \leq 2$

$$f_X(x) = \int_0^{x-1}\int_{x-u-1}^{1} dv\,du + \int_{x-1}^{1}\int_0^{x-u} dv\,du$$

$$= \int_0^{x-1}(2-x+u)\,du + \int_{x-1}^{1}(x-u)\,du$$

$$= 2x-2-x^2+x+\frac{x^2-2x+1}{2} + x - 1/2 - x^2 + x + \frac{x^2-2x+1}{2}$$

$$= -x^2 + 3x - 3/2$$

Case 3: $2 \leq x \leq 3$

$$f_X(x) = \int_{x-2}^{1}\int_{x-u-1}^{1} dv\,du$$

$$= \int_{x-2}^{1} (2-x+u)\, du$$

$$= 2 - x + \tfrac{1}{2} - 2x + 4 + x^2 - 2x - \frac{x^2 - 4x + 4}{2}$$

$$= \tfrac{x^2}{2} - 3x + \tfrac{9}{2}$$

Case 4: $x \geqslant 3$

$$f_X(x) = \int_0^1 \int_0^1 0 \, dv\, du = 0$$

$$\therefore f_X(x) = \begin{cases} x^2/2 & \text{if } 0 \leqslant x \leqslant 1 \\ -x^2 + 3x - 3/2 & \text{if } 1 \leqslant x \leqslant 2 \\ x^2/2 - 3x + 9/2 & \text{if } 2 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

L8-9. 5. Let $Z_1 = \frac{1}{\sigma\sqrt{n/2}}\left(\left(\sum_{i=1}^{n/2} X_i\right) - \tfrac{n}{2}\mu\right)$

$$= \sqrt{\frac{2}{\sigma^2 n}}\left(-\tfrac{n}{2}\mu + \sum_{i=1}^{n/2} X_i\right)$$

$$Z_2 = \sqrt{\frac{2}{\sigma^2 n}}\left(-\tfrac{n}{2}\mu + \sum_{i=n/2+1}^{n} X_i\right)$$

$$Z_3 = \sqrt{\frac{1}{\sigma^2 n}}\left(-n\mu + \sum_{i=1}^{n} X_i\right)$$

Let $A = \sum_{i=1}^{n/2} X_i$, $B = \sum_{i=n/2+1}^{n} X_i$, $C = \sum_{i=1}^{n} X_i$

$$F_{Z_1}(x) = F_A\left(x\sigma\sqrt{n/2} + n/2\,\mu\right)$$

$$= F_B\left(x\sigma\sqrt{n/2} + n/2\,\mu\right) \quad \text{as } F_A = F_B$$

$$= F_{Z_2}(x)$$

$$f_{Z_1}(x) = \frac{d}{dx} F_A\left(x\sigma\sqrt{n/2} + n/2\,\mu\right)$$

$$= \sigma\sqrt{\frac{n}{2}}\, f_A\left(x\sigma\sqrt{n/2} + n/2\,\mu\right)$$

$$= f_{Z_2}$$

$$f_{Z_3} = \frac{d}{dx} F_C\left(x\sigma\sqrt{n} + n\mu\right)$$

$$= \sigma\sqrt{n}\, f_C\left(x\sigma\sqrt{n} + n\mu\right)$$

incomplete

6. Let $X_i = 1$ if the i^{th} throw is 6 else 0

Let $\mu = E[X_i] = 1/6$

$\sigma^2 = Var[X_i] = 5/36 \quad \therefore \sigma = \frac{\sqrt{5}}{6}$

Let $n = 1000$

$$Y = \sum_{i=1}^{n} X_i$$

Let $Z = \frac{1}{\sigma\sqrt{n}}(Y - n\mu)$ which is approximately distributed by $N(0,1)$

$$P(100 \leq Y \leq 200) = P\left(\frac{100-n\mu}{\sigma\sqrt{n}} \leq Z \leq \frac{200-n\mu}{\sigma\sqrt{n}}\right)$$

$$\cancel{= P(-80\sqrt{30} \leq Z \leq 40\sqrt{30})}$$

$$= P(-4\sqrt{2} \leq Z \leq 2\sqrt{2})$$

$$\simeq \int_{-4\sqrt{2}}^{2\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$$

$\therefore a = -4\sqrt{2}, \quad b = 2\sqrt{2}$

L10. 1. $X \sim Exp(\lambda)$

Let $Y = \frac{X_1 + X_2 + X_3 + X_4}{4}$

$E(X) = \frac{1}{4} \cdot 4E(X_i) = 1/\lambda \quad \therefore Y$ is unbiased

$\therefore 1/\lambda \simeq \frac{1}{4}(2+5+4+4) = \frac{15}{4}$

210.5. $P(Z=1) = 1/N$

$$P(Z=2) = \frac{N-1}{N} \cdot \frac{1}{N}$$

$$P(Z=k) = \left(\frac{N-1}{N}\right)^{k-1} \cdot \frac{1}{N} = \frac{(N-1)^{k-1}}{N^k} \quad \text{when } 1 \le k \le N$$

$$\cancel{E[Z] = \sum_{k=1}^{N} \frac{k(N-1)^{k-1}}{N^k}}$$

$$\cancel{\frac{1}{N}\sum_{k=1}^{N} k\left(\frac{N-1}{N}\right)^{k-1}} = \frac{1}{N-1}\sum_{k=1}^{N} k u^k$$

where $u = 1 - \frac{1}{N}$

$$E[Z] = \cancel{\frac{1}{N}\sum_{k=1}^{N} \frac{d}{du} u^k}$$

$$= \frac{1}{N-1}\left(\frac{u(Nu^{N+1} - (N+1)u^N + 1)}{(u-1)^2}\right)$$

$$= \frac{1}{N-1}(N^2-N)\left(N\cdot\left(\frac{N-1}{N}\right)^{N+1} - (N+1)\left(\frac{N-1}{N}\right)^N + 1\right)$$

$$= N\left(\frac{(N-1)^{N+1}}{N^N} - (N+1)\frac{(N-1)^N}{N^N} + 1\right)$$

$$= N\left(((N-1)-(N+1))\frac{(N-1)^N}{N^N} + 1\right)$$

$$= N\left(-2\frac{(N-1)^N}{N^N} + 1\right)$$

$$= -2\frac{(N-1)^N}{N^{N-1}} + N$$

$$E[Z] = \sum_{k=1}^{\infty} \frac{k(N-1)^{k-1}}{N^k}$$

$$= \frac{1}{N-1} \sum_{k=1}^{\infty} k \left(\frac{N-1}{N}\right)^k$$

$$= \frac{1}{N-1} \frac{\left(\frac{N-1}{N}\right)}{\left(\frac{N-1}{N} - 1\right)^2} \qquad *$$

$$= \frac{1}{N-1} \cdot \frac{N-1}{N\left(-\frac{1}{N}\right)^2}$$

$$= N$$

Proof of $*$:

Let $S = \sum_{k=1}^{\infty} k x^k = x \sum_{k=1}^{\infty} k x^{k-1} = x \sum_{k=1}^{\infty} \frac{d}{dx} x^k$

$$= x \frac{d}{dx} \sum_{k=1}^{\infty} x^k$$

$$= x \frac{d}{dx}\left(\frac{1}{1-x} - 1\right)$$

$$= x \cdot \frac{1}{(x-1)^2}$$

$$= \frac{x}{(x-1)^2} \quad \square$$

$$E\left[(Z-N)^2\right] = \sum_{k=1}^{\infty} \frac{(k-N)^2 (N-1)^{k-1}}{N^k}$$

$$= \sum_{k=1}^{\infty} k^2 \frac{(N-1)^{k-1}}{N^k} - 2\sum_{k=1}^{\infty} k \frac{(N-1)^{k-1}}{N^{k-1}} + \sum_{k=1}^{\infty} \frac{(N-1)^{k-1}}{N^{k-2}}$$

Let $u = \frac{N-1}{N}$

Let $E = E\left[(Z-N)^2\right]$

$$E = \frac{1}{N-1} \sum_{k=1}^{\infty} k^2 u^k - \frac{2N}{(N-1)} \sum_{k=1}^{\infty} k u^k + \frac{N^2}{(N-1)} \sum_{k=1}^{\infty} u^k$$

$$\sum_{k=1}^{\infty} k^2 u^k = u \sum_{k=1}^{\infty} k^2 u^{k-1} = u \sum_{k=1}^{\infty} \frac{d}{du} k u^k = u \frac{d}{du} \sum_{k=1}^{\infty} k u^k$$

$$= u \frac{d}{du} \cdot \frac{u}{(u-1)^2} = u \cdot \frac{(u-1)^2 - 2u(u-1)}{(u-1)^4}$$

$$= \frac{u(u-1-2u)}{(u-1)^3}$$

$$= \frac{u(-u-1)}{(u-1)^3}$$

$$= \frac{u(u+1)}{(1-u)^3}$$

$$\sum_{k=1}^{\infty} k u^k = \frac{u}{(1-u)^2}$$

$$\sum_{k=1}^{\infty} u^k = \frac{1}{1-u}$$

$$\therefore E = \frac{1}{N-1} \cdot \frac{u(u+1)}{(1-u)^3} - \frac{2N}{(N-1)} \cdot \frac{u}{(1-u)^2} + \frac{N^2}{(N-1)} \left(\frac{1}{1-u} - 1\right)$$

$$\cancel{= \frac{(u+1)}{(1-u)^2} - 2 + \frac{1}{N(N-1)}}$$

$$\cancel{= \left(\frac{N-1}{N} + 1\right) N^2 - 2 + \frac{1}{N(N-1)}}$$

$$\cancel{= \frac{N(N-1)}{N^2} \cdot 2 + \frac{1}{N(N-1)}}$$

$$= \frac{(1+u)}{(1-u)^2} - \frac{2}{(1-u)^2} + \frac{N}{1-u}$$

$$= \frac{N}{1-u} - \frac{1}{1-u}$$

$$= \frac{N-1}{\left(\frac{1}{N}\right)}$$

$$= N(N-1)$$

$$\therefore E[(Z-N)^2] = N(N-1)$$

An unbiased estimator for N can be obtained by taking the mean of many samples of Z.

L11-12.5. Let $X_i = 1$ if x_i is a local maximum, otherwise 0

Let $Y = \sum_{i=1}^{n} X_i$ = the number of local maxima with n candidates

~~$E[X_i] = \frac{\text{number of permutations in which } x_i > \text{everything before}}{\text{total number of permutations}}$~~

~~$= $~~

The first i values can be arranged in $i!$ ways, only $1/i$ th of which have the last (i^{th}) value being the largest

$$\therefore E[X_i] = \frac{1}{i} \qquad \therefore E[Y] = \sum_{i=1}^{n} \frac{1}{i}$$