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# Discrete Maths Supervision 6

2.1.1 $S \circ R = \{(2,z), (3,x), (3,z)\}$

2. ~~RTP: $\forall$ sets $A, B$ $\forall R, S, T \in A \times B$ $R \circ (S \circ T) = (R \circ S) \circ T$~~
~~Let $A, B$ arbitrary sets~~
~~Let $R, S, T \in A \times B$~~

~~$S \circ T = \{($~~

$(T \circ S) \circ R = T \circ (S \circ R)$

RTP: $\forall$ sets $A, B, C, D$, $\forall R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$, $\cancel{R \circ (S \circ T) = (R \circ S) \circ T}$

Let $A, B, C, D$ arbitrary sets
Let $R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$

$S \circ T = \{(b \in B, d \in D) \mid \exists c \in C. (b,c) \in S \land (c,d) \in T\}$

$(T \circ S) \circ R$

$\cancel{R \circ (S \circ T)} = \{(a \in A, d \in D) \mid \exists b \in B. (a,b) \in R \land (\exists c \in C. (b,c) \in S \land (c,d) \in T)\}$

$= \{(a \in A, d \in D) \mid \exists c \in C. (\exists b \in B. (a,b) \in R \land (b,c) \in S) \land (c,d) \in T\}$

$= \cancel{(R \circ S) \circ T}$ $T \circ (S \circ R)$

RTP: $\forall$ sets $A, B$ $\forall R \in A \times B$, $\mathrm{id}_B \circ R$ $\cancel{R \circ \mathrm{id}_B} = R$

Let $A, B$ arbitrary sets
Let $R \in A \times B$

$\cancel{R \circ \mathrm{id}_B} = \{(a \in A, b \in B) \mid \exists b' \in B. (a,b') \in R \land b = b'\}$

$\mathrm{id}_B \circ R$

$= \{(a \in A, b \in B) \mid \cancel{\exists b'} (a,b) \in R\}$

$= R$

3a. $R \subseteq S \Rightarrow \big((a,b) \in R \Rightarrow (a,b) \in S\big)$

$\Rightarrow \big((b,a) \in R^{op} \Rightarrow (a,b) \in S \Rightarrow (b,a) \in S^{op}\big)$

$\Rightarrow R^{op} \subseteq S^{op}$

b. $(R \wedge S) = \{(a,b) \mid (a,b) \in R \wedge (a,b) \in S\}$

$\therefore (R \wedge S)^{op} = \{(b,a) \mid (a,b) \in R \wedge (a,b) \in S\}$

$= \{(b,a) \mid (b,a) \in R^{op} \wedge (b,a) \in S^{op}\}$

$= R^{op} \cap S^{op}$

c. $(R \cup S) = \{(a,b) \mid (a,b) \in R \vee (a,b) \in S\}$

$\therefore (R \cup S)^{op} = \{(b,a) \mid (a,b) \in R \vee (a,b) \in S\}$

$= \{(b,a) \mid (b,a) \in R^{op} \vee (b,a) \in S^{op}\}$

$= R^{op} \cup S^{op}$

4. RTP: $\forall$ sets $A$, $\forall R \subseteq A \times A$. $(\forall a \neq b \in A.\ (a,b) \in R \Leftrightarrow (b,a) \notin R) \Leftrightarrow R \cap R^{op} \subseteq id_A$

Let $A$ arbitrary set

Let $R \subseteq A \times A$

Start with "$\Rightarrow$"

Assume $\forall a \neq b \in A.\ (a,b) \in R \Leftrightarrow (b,a) \notin R$

RTP $R \cap R^{op} \subseteq id_A$

$R \cap R^{op} = \{(a,b) \mid (a,b) \in R \wedge (a,b) \in R^{op}\}$

$= \{(a,b) \mid (a,b) \in R \wedge (b,a) \in R\}$

$\subseteq \{(a,b) \mid a = b\}$

$\subseteq id_A$

Next prove "$\Leftarrow$"

Assume $R \cap R^{op} \subseteq id_A$

RTP $\forall a \neq b \in R.\ (a,b) \in R \Leftrightarrow (b,a) \notin R$

Equivalently, $\forall a \neq b \in R.\ (a,b) \in R \Rightarrow (b,a) \notin R$ by contrapositive.

Let $a \neq b \in A$

~~RTP $(a,b) \in R \Rightarrow (b,a) \notin R$~~ Assume $(a,b) \in R$

RTP: $(b,a) \notin R$

$\{(a,b) \mid (a,b) \in R \wedge (b,a) \in R\} \subseteq id_A$

$\therefore (a,b) \in R \wedge (b,a) \in R \Rightarrow a = b$

$\Rightarrow \text{false}$

$\therefore (b,a) \in R \Rightarrow \text{false}$

$\therefore (b,a) \notin R$

2.2.1 a)

$$
\begin{aligned}
(\bigcup F) \circ R &= \{(x,b) \mid \exists a.\ (x,a) \in R \wedge (a,b) \in \bigcup F\} \\
&= \{(x,b) \mid \exists a.\ (x,a) \in R \wedge (\exists S \in F.\ (a,b) \in S)\} \\
&= \{(x,b) \mid \exists a \in A, S \in F.\ (x,a) \in R \wedge (a,b) \in S\} \\
&= \bigcup(\{\{(x,b) \mid \exists a.\ (x,a) \in R \wedge (a,b) \in S\} \mid S \in F\}) \\
&= \bigcup(\{S \circ R \mid S \in F\})
\end{aligned}
$$

b)

$$
\begin{aligned}
R \circ (\bigcup F) &= \{(a,y) \mid \exists b.\ (a,b) \in \bigcup F \wedge (b,y) \in R\} \\
&= \{(a,y) \mid \exists b.\ (\exists S \in F.\ (a,b) \in S) \wedge (b,y) \in R\} \\
&= \{(a,y) \mid \exists b \in B, S \in F.\ (a,b) \in S \wedge (b,y) \in R\} \\
&= \bigcup(\{\{(a,y) \mid \exists b, (a,b) \in S \wedge (b,y) \in R\} \mid S \in F\}) \\
&= \bigcup(\{R \circ S \mid S \in F\})
\end{aligned}
$$

2. $R^{0+} = \mathrm{rel}\left(\sum_{n \in \mathbb{N}} \mathrm{mat}(R^{0n})\right)$

$= \mathrm{rel}\left(\sum_{n \in \mathbb{N}} \mathrm{mat}(R)^n\right)$

$R \circ R^{0*} = \mathrm{rel}\left(\mathrm{mat}(R) \cdot \sum_{n \in \mathbb{N}} \mathrm{mat}(R)^n\right)$

$= \mathrm{rel}\left(\sum_{n \in \mathbb{N}} \mathrm{mat}(R)^{n+1}\right)$

$\therefore R^{0+} = \{(a,b) \mid \exists$ a path of length $\geq 1$ from a to b in $R\}$

i. RTP: $R \subseteq R^{0+} \wedge R^{0+}$ is transitive

First, prove $R \subseteq R^{0+}$

$R = \{(a,b) \mid \exists$ a path of length 1 from a to b in $R\}$

$\therefore R \subseteq R^{0+}$ by definition

Next prove $R^{0+}$ is transitive

Assume $(a,b) \in R^{0+} \wedge (b,c) \in R^{0+}$

RTP $(a,c) \in R^{0+}$

Let $\alpha$ = the path length from a to b in $R^{0+}$

Let $\beta$ = the path length from b to c in $R^{0+}$

$\alpha + \beta > 1$

$\therefore$ there exists a path of length $\geq 1$ from a to c in $R^{0+}$

$\therefore (a,c) \in R^{0+}$

$\square$