Maths Supervision Work 10

7. $xyz + x^3 + y^4 + z^5 = 0$

$$\therefore \frac{\partial}{\partial y}\Big|_z (xyz + x^3 + y^4 + z^5) = 0$$

$$\therefore z\frac{\partial}{\partial y}\Big|_z (xy) + 3x^2\frac{\partial x}{\partial y}\Big|_z + 4y^3 = 0$$

$$\therefore z\left(\frac{\partial x}{\partial y}\Big|_z y + x\right) + 3x^2\frac{\partial x}{\partial y}\Big|_z + 4y^3 = 0$$

$$(3z^2 + yz)\frac{\partial x}{\partial y}\Big|_z + xz + 4y^3 = 0$$

$$\therefore \frac{\partial x}{\partial y}\Big|_z = \frac{-xz - 4y^3}{3x^2 + yz}$$

$$\frac{\partial}{\partial z}\Big|_x (xyz + x^3 + y^4 + z^5) = 0$$

$$\therefore x\frac{\partial}{\partial z}\Big|_x (yz) + 4y^3\frac{\partial y}{\partial z}\Big|_x + 5z^4 = 0$$

$$\therefore x\left(\frac{\partial y}{\partial z}\Big|_x \cdot z + y\right) + 4y^3\frac{\partial y}{\partial z}\Big|_x + 5z^4 = 0$$

$$\therefore (xz + 4y^3)\frac{\partial y}{\partial z}\Big|_x + xy + 5z^4 = 0$$

$$\therefore \frac{\partial y}{\partial z}\Big|_x = \frac{-xy - 5z^4}{xz + 4y^3}$$

$$\frac{\partial}{\partial x}\Big|_y (xyz + x^3 + y^4 + z^5) = 0$$

$$\therefore y\frac{\partial}{\partial x}\Big|_y (xz) + 3x^2 + 5z^4\frac{\partial z}{\partial x}\Big|_y = 0$$

$$\therefore y\left(z + x\frac{\partial z}{\partial x}\Big|_y\right) + 3x^2 + 5z^4\frac{\partial z}{\partial x}\Big|_y = 0$$

$$\therefore (\cancel{yz}\ xy + 5z^4)\frac{\partial z}{\partial x}\Big|_y + 3x^2 + yz = 0$$

$$\therefore \frac{\partial z}{\partial x}\Big|_y = \frac{-3x^2 - yz}{xy + 5z^4}$$

$$\therefore \left.\frac{\partial x}{\partial y}\right|_z \cdot \left.\frac{\partial y}{\partial z}\right|_x \cdot \left.\frac{\partial z}{\partial x}\right|_y$$

$$= \left(\frac{-xz-4y^3}{3x^2+yz}\right)\left(\frac{-xy-5z^4}{xz+4y^3}\right)\left(\frac{-3x^2-yz}{xy+5z^4}\right)$$

$$= -\frac{(xz+4y^3)(xy+5z^4)(3x^2+yz)}{(xz+4y^3)(xy+5z^4)(3x^2+yz)}$$

$$= -1$$

8. $\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are basis vectors in x-y space

$\hat{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\hat{v} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ are basis vectors in u-v space

Consider a point $\vec{p} = x\hat{i} + y\hat{j} = u\hat{u} + v\hat{v}$

$$\cancel{u = \vec{p}\cdot\hat{u} = (x\hat{i}+y\hat{j})\cdot\hat{u} = \begin{pmatrix} x \\ y \end{pmatrix}\cdot\begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = x\cos\theta - y\sin\theta}$$

$$\cancel{v = \vec{p}\cdot\hat{v} = (x\hat{i}+y\hat{j})\cdot\hat{v} = \begin{pmatrix} x \\ y \end{pmatrix}\cdot\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}}$$

$$x = \vec{p}\cdot\hat{i} = (u\hat{u}+v\hat{v})\cdot\hat{i} = \begin{pmatrix} u\cos\theta - v\sin\theta \\ u\sin\theta + v\cos\theta \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= u\cos\theta - v\sin\theta$$

$$y = \vec{p}\cdot\hat{j} = \begin{pmatrix} u\cos\theta - v\sin\theta \\ u\sin\theta + v\cos\theta \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= u\sin\theta + v\cos\theta$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial^2 f}{\partial u^2} = \cos\theta\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial y}{\partial u}\right) + \sin\theta\left(\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial u}\right)$$

$$= \cos^2\theta\,\frac{\partial^2 f}{\partial x^2} + \sin^2\frac{\partial^2 f}{\partial y^2} + 2\cos\theta\sin\theta\,\frac{\partial^2 f}{\partial x\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}(-\sin\theta) + \frac{\partial f}{\partial y}\cdot\cos\theta$$

$$\frac{\partial^2 f}{\partial v^2} = -\sin\theta\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial y}{\partial v}\right) + \cos\theta\left(\frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial x\partial y}\,\frac{\partial x}{\partial v}\right)$$

$$= \sin^2\frac{\partial^2 f}{\partial x^2} + \cos^2\frac{\partial^2 f}{\partial y^2} - 2\cos\theta\sin\theta\,\frac{\partial^2 f}{\partial x\partial y}$$

$$\therefore \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial x^2} + (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial y^2} + (2\cos\theta\sin\theta - 2\cos\theta\sin\theta)\frac{\partial^2 f}{\partial x\partial y}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$