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Discrete Maths Supervision 6

2.1.1 $S \circ R = \{(2, z), (3, x), (3, z)\}$

2. ~~RTP: $\forall$ sets A, B $\forall R, S, T \in A \times B$ $R \circ (S \circ T) = (R \circ S) \circ T$~~
~~Let A, B arbitrary sets~~
~~Let $R, S, T \in A \times B$~~

~~$S \circ T = \{($~~

$(T \circ S) \circ R = T \circ (S \circ R)$

RTP: $\forall$ sets A, B, C, D, $\forall R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$, $\cancel{R \circ (S \circ T) = (R \circ S) \circ T}$

Let A, B, C, D arbitrary sets
Let $R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$

$S \circ T = \{(b \in B, d \in D) \mid \exists c \in C.\ (b, c) \in S \wedge (c, d) \in T\}$

$(T \circ S) \circ R$
$\cancel{R \circ (S \circ T)} = \{(a \in A, d \in D) \mid \exists b \in B.\ (a, b) \in R \wedge (\exists c \in C.\ (b, c) \in S \wedge (c, d) \in T)\}$

$= \{(a \in A, d \in D) \mid \exists c \in C.\ (\exists b \in B.\ (a, b) \in R \wedge (b, c) \in S) \wedge (c, d) \in T\}$

$= \cancel{(R \circ S) \circ T}\ T \circ (S \circ R)$

RTP: $\forall$ sets A, B $\forall R \subseteq A \times B$, $id_B \circ R = R$

Let A, B arbitrary sets
Let $R \in A \times B$

$id_B \circ R = \{(a \in A, b \in B) \mid \exists b' \in B.\ (a, b') \in R \wedge b = b'\}$
$= \{(a \in A, b \in B) \mid (a, b) \in R\}$
$= R$

As composition is not commutative, you still need to prove that id_A is a 'right identity', that is
R o id_A = R
for whichever R the composition is defined.

Better psentation: Let (b,a)∈R^{op}, then (a,b)∈R, so ....

3a. $R \subseteq S \Rightarrow ((a,b) \in R \Rightarrow (a,b) \in S)$
$\Rightarrow ((b,a) \in R^{op} \Rightarrow (a,b) \in S \Rightarrow (b,a) \in S^{op})$
$\Rightarrow R^{op} \subseteq S^{op}$

b. $(R \cap S) = \{(a,b) \mid (a,b) \in R \wedge (a,b) \in S\}$
$\therefore (R \cap S)^{op} = \{(b,a) \mid (a,b) \in R \wedge (a,b) \in S\}$
$= \{(b,a) \mid (b,a) \in R^{op} \wedge (b,a) \in S^{op}\}$
$= R^{op} \cap S^{op}$

c. $(R \cup S) = \{(a,b) \mid (a,b) \in R \vee (a,b) \in S\}$
$\therefore (R \cup S)^{op} = \{(b,a) \mid (a,b) \in R \vee (a,b) \in S\}$
$= \{(b,a) \mid (b,a) \in R^{op} \vee (b,a) \in S^{op}\}$
$= R^{op} \cup S^{op}$

4. RTP: $\forall$ sets A, $\forall R \subseteq A \times A$. $(\underline{\forall a \neq b \in A.\ (a,b) \in R \Leftrightarrow (b,a) \notin R}) \Leftrightarrow R \cap R^{op} \subseteq id_A$

antisymmetry means that at most one of (a,b) or (b,a) is in R when a is not b. What you write says exactly one of them is in R. Thinking about partial orders, that means that there might be elements that are incomparable.

Let A arbitrary set
Let $R \subseteq A \times A$
Start with "$\Rightarrow$"
Assume $\forall a \neq b \in A.\ (a,b) \in R \Leftrightarrow (b,a) \notin R$
RTP $R \cap R^{op} \subseteq id_A$

$R \cap R^{op} = \{(a,b) \mid (a,b) \in R \wedge (a,b) \in R^{op}\}$
$= \{(a,b) \mid (a,b) \in R \wedge (b,a) \in R\}$
$\subseteq \{(a,b) \mid a = b\}$
$\subseteq id_A$

Next prove "$\Leftarrow$"
Assume $R \cap R^{op} \subseteq id_A$
RTP $\forall a \neq b \in R.\ (a,b) \in R \Leftrightarrow (b,a) \notin R$
Equivalently, $\forall a \neq b \in R.\ (a,b) \in R \Rightarrow (b,a) \notin R$
by contrapositive.

Here is the 'mistake'. The two statements are not equivalent, as the latter does not imply the former (<= may not hold)

Let $a \neq b \in A$

~~RTP (a,b) ∈ R ⇒ (b,a) ∉ R~~ Assume $(a,b) \in R$

RTP: $(b,a) \notin R$

$\{(a,b) \mid (a,b) \in R \wedge (b,a) \in R\} \subseteq id_A$

$\therefore (a,b) \in R \wedge (b,a) \in R \Rightarrow a = b$

$\Rightarrow false$

The proof is correct, other than the small mistake above.

$\therefore (b,a) \in R \Rightarrow false$

$\therefore (b,a) \notin R$

2.2.1 a) ~~$(\bigcup F)\circ R = (A \times B)\circ R$~~

$$
\begin{aligned}
(\bigcup F)\circ R &= \{(x,b) \mid \exists a.\ (x,a) \in R \wedge (a,b) \in \bigcup F\} \\
&= \{(x,b) \mid \exists a.\ (x,a) \in R \wedge (\exists S \in F.\ (a,b) \in S)\} \\
&= \{(x,b) \mid \exists a \in A, S \in F.\ (x,a) \in R \wedge (a,b) \in S\} \\
&= \bigcup(\{\{(x,b) \mid \exists a.\ (x,a) \in R \wedge (a,b) \in S\} \mid S \in F\}) \\
&= \bigcup(\{S \circ R \mid S \in F\})
\end{aligned}
$$

b)
$$
\begin{aligned}
R \circ (\bigcup F) &= \{(a,y) \mid \exists b.\ (a,b) \in \bigcup F \wedge (b,y) \in R\} \\
&= \{(a,y) \mid \exists b.\ (\exists S \in F.\ (a,b) \in S) \wedge (b,y) \in R\} \\
&= \{(a,y) \mid \exists b \in B, S \in F.\ (a,b) \in S \wedge (b,y) \in R\} \\
&= \bigcup(\{\{(a,y) \mid \exists b.\ (a,b) \in S \wedge (b,y) \in R\} \mid S \in F\}) \\
&= \bigcup(\{R \circ S \mid S \in F\})
\end{aligned}
$$

2. $R^{o+} = rel\left(\sum_{n \in \mathbb{N}}^{1} mat(R^{on})\right)$

$= rel\left(\sum_{n \in \mathbb{N}}^{1} mat(R)^n\right)$

$\therefore R \circ R^{o*} = rel\left(mat(R) \cdot \sum_{n \in \mathbb{N}}^{1} mat(R)^n\right)$

$= rel\left(\sum_{n \in \mathbb{N}}^{1} mat(R)^{n+1}\right)$

$\therefore R^{o+} = \{(a,b) \mid \exists$ a path of length ≥ 1 from a to b in $R\}$

i. RTP: ~~R⊆R∘R~~ $R \subseteq R^{o+} \wedge R^{o+}$ is transitive

First, prove $R \subseteq R^{o+}$

$R = \{(a,b) \mid \exists$ a path of length 1 from a to ~~b~~ b in $R\}$

$\therefore R \subseteq R^{o+}$ by definition

Next prove R^{o+} is transitive

Assume $(a,b) \in R^{o+} \wedge (b,c) \in R^{o+}$

RTP $(a,c) \in R^{o+}$

some (there might be plenty)

Let α = ~~the~~ path length from a to b in R^{o+}

Let β = ~~the~~ path length from b to c in R^{o+}

$\alpha + \beta > 1$

$\therefore$ there exists a path of length ≥ 1 from a to c in R^{o+}

$\therefore (a,c) \in R^{o+}$

$\square$

What about part (ii)? It is a fun proof by induction!

3.1.1 | $A_2 \Rightarrow A_2$:

```
{{},
{(1, 1)},
{(1, 2)},
{(2, 1)},
{(2, 2)},
{(1, 1), (2, 1)},
{(1, 1), (2, 2)},
{(1, 2), (2, 1)},
{(1, 2), (2, 2)}}
```

$A_2 \Rightarrow A_3$:

```
{{},
{(1, a)},
{(1, b)},
{(1, c)},
{(2, a)},
{(2, b)},
{(2, c)},
{(1, a), (2, a)},
{(1, a), (2, b)},
{(1, a), (2, c)},
{(1, b), (2, a)},
{(1, b), (2, b)},
{(1, b), (2, c)},
{(1, c), (2, a)},
{(1, c), (2, b)},
{(1, c), (2, c)}}
```

$A_3 \Rightarrow A_2$:

```
{{},
{(a, 1)},
{(a, 2)},
{(b, 1)},
{(b, 2)},
{(c, 1)},
{(c, 2)},
{(a, 1), (b, 1)},
{(a, 1), (b, 2)},
{(a, 2), (b, 1)},
{(a, 2), (b, 2)},
{(a, 1), (c, 1)},
{(a, 1), (c, 2)},
{(a, 2), (c, 1)},
{(a, 2), (c, 2)},
{(b, 1), (c, 1)},
{(b, 1), (c, 2)},
{(b, 2), (c, 1)},
{(b, 2), (c, 2)},
{(a, 1), (b, 1), (c, 1)},
{(a, 1), (b, 1), (c, 2)},
{(a, 1), (b, 2), (c, 1)},
{(a, 1), (b, 2), (c, 2)},
{(a, 2), (b, 1), (c, 1)},
{(a, 2), (b, 1), (c, 2)},
{(a, 2), (b, 2), (c, 1)},
{(a, 2), (b, 2), (c, 2)}}
```

$A_3 \Rightarrow A_3$:

```
{{},
```

```
{(a, a)},
{(a, b)},
{(a, c)},
{(b, a)},
{(b, b)},
{(b, c)},
{(c, a)},
{(c, b)},
{(c, c)},
{(a, a), (b, a)},
{(a, a), (b, b)},
{(a, a), (b, c)},
{(a, b), (b, a)},
{(a, b), (b, b)},
{(a, b), (b, c)},
{(a, c), (b, a)},
{(a, c), (b, b)},
{(a, c), (b, c)},
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{(b, c), (c, c)},
{(a, a), (b, a), (c, a)},
{(a, a), (b, a), (c, b)},
{(a, a), (b, a), (c, c)},
{(a, a), (b, b), (c, a)},
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{(a, a), (b, b), (c, c)},
{(a, a), (b, c), (c, a)},
{(a, a), (b, c), (c, b)},
{(a, a), (b, c), (c, c)},
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{(a, b), (b, a), (c, b)},
{(a, b), (b, a), (c, c)},
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{(a, b), (b, b), (c, c)},
{(a, b), (b, c), (c, a)},
{(a, b), (b, c), (c, b)},
{(a, b), (b, c), (c, c)},
{(a, c), (b, a), (c, a)},
{(a, c), (b, a), (c, b)},
{(a, c), (b, a), (c, c)},
{(a, c), (b, b), (c, a)},
{(a, c), (b, b), (c, b)},
{(a, c), (b, b), (c, c)},
{(a, c), (b, c), (c, a)},
{(a, c), (b, c), (c, b)},
{(a, c), (b, c), (c, c)}}
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Out of curiosity did you type them or write a program to give them?

(there is a extra bracket on the empty partial function)

3.1.1 $A_2 \Rightarrow A_2$: $\{\{(1,1),(2,1)\}, \{(1,1),(2,2)\}, \{(1,2),(2,1)\}, \{(1,2),(2,2)\}\}$

$A_2 \Rightarrow A_3$: $\{\{(1,a),(2,a)\}, \{(1,a),(2,b)\}, \{(1,a),(2,c)\},$
$\{(1,b),(2,a)\}, \{(1,b),(2,b)\}, \{(1,b),(2,c)\},$
$\{(1,c),(2,a)\}, \{(1,c),(2,b)\}, \{(1,c),(2,c)\}\}$

$A_3 \Rightarrow A_2$: $\{\{(a,1),(b,1),(c,1)\}, \{(a,1),(b,1),(c,2)\}, \{(a,1),(b,2),(c,1)\},$
$\{(a,1),(b,2),(c,2)\}, \{(a,2),(b,1),(c,1)\}, \{(a,2),(b,1),(c,2)\},$
$\{(a,2),(b,2),(c,1)\}, \{(a,2),(b,2),(c,2)\}\}$

$A_3 \Rightarrow A_3$: $\{\{(a,a),(b,a),(c,a)\}, \{(a,a),(b,a),(c,b)\}, \{(a,a),(b,a),(c,c)\},$
$\{(a,a),(b,b),(c,a)\}, \{(a,a),(b,b),(c,b)\}, \{(a,a),(b,b),(c,c)\},$
$\{(a,a),(b,c),(c,a)\}, \{(a,a),(b,c),(c,b)\}, \{(a,a),(b,c),(c,c)\},$
$\{(a,b),(b,a),(c,a)\}, \{(a,b),(b,a),(c,b)\}, \{(a,b),(b,a),(c,c)\},$
$\{(a,b),(b,b),(c,a)\}, \{(a,b),(b,b),(c,b)\}, \{(a,b),(b,b),(c,c)\},$
$\{(a,b),(b,c),(c,a)\}, \{(a,b),(b,c),(c,b)\}, \{(a,b),(b,c),(c,c)\},$
$\{(a,c),(b,a),(c,a)\}, \{(a,c),(b,a),(c,b)\}, \{(a,c),(b,a),(c,c)\},$
$\{(a,c),(b,b),(c,a)\}, \{(a,c),(b,b),(c,b)\}, \{(a,c),(b,b),(c,c)\},$
$\{(a,c),(b,c),(c,a)\}, \{(a,c),(b,c),(c,b)\}, \{(a,c),(b,c),(c,c)\}\}$

2. RTP: $\forall$ sets A, B $\forall R \in A \times B$. R is a partial function $\iff R \circ R^{op} \subseteq id_B$

Let A, B arbitrary sets

Let $R \in A \times B$

Start with "$\Rightarrow$"

Assume R is a partial function

Equivalently $\forall a, b, c.\ (a,b) \in R \wedge (a,c) \in R \Rightarrow b = c$ (†)

RTP: $R \circ R^{op} \subseteq id_B$

$$R \circ R^{op} = \{(b,b') \mid \exists a.\ (b,a) \in R^{op} \wedge (a,b') \in R\}$$
$$= \{(b,b') \mid \exists a.\ (a,b) \in R \wedge (a,b') \in R\}$$
$$\subseteq \{(b,b') \mid b = b'\} \quad \text{by } (\dagger)$$
$$= id_B$$

Next prove "$\Leftarrow$"

Assume $R \circ R^{op} \subseteq id_B$

~~Equivalently~~ $\therefore \{(b,b') \mid \exists a. (a,b) \in R \wedge (a,b') \in R\} \subseteq id_B$

$\therefore \forall a, b, b'. \ (a,b) \in R \wedge (a,b') \in R \Rightarrow b = b'$

~~RTP $\forall a \in A \Rightarrow \exists b \in B$ s.t. $(a,b) \in R$~~

~~Let $a \in A$~~

$\square$

3a. RTP: $\forall$ sets A, id_A is a partial function

Let A arbitrary set

RTP: $\forall a, b, c. \ (a,b) \in id_A \wedge (a,c) \in id_A \Rightarrow b = c$

~~Assume~~ Let $a, b, c \in A$

Assume $(a,b) \in id_A \Rightarrow b = a$

$(a,c) \in id_A \Rightarrow c = a$

$\therefore b = c$ $\square$

b. Let A, B, C arbitrary sets

Let $S : A \to B$, $T : B \to C$

RTP $T \circ S : A \to C$

Equivalently, $\forall a \in A, c \in C, c' \in C. \ (a,c) \in T \circ S \wedge (a,c') \in T \circ S$
$\Rightarrow c = c'$

Let $a \in A$, $c \in C$, $c' \in C$

Assume $(a,c) \in T \circ S$

let b s.t. $(a,b) \in S \wedge (b,c) \in T$

~~Let b~~

Assume $(a,c') \in T \circ S$

Let b' s.t. $(a,b') \in S \wedge (b',c') \in T$

S is a partial function $\therefore b = b'$

$\therefore (b,c) \in T \wedge (b,c') \in T \wedge T$ is a partial function

$\therefore c = c'$ $\square$

3.2.1 Reflexivity:

RTP: $\forall R: A \rightharpoonup B.\ R \subseteq R$

Let R be an arbitrary partial function from A to B. RTP $R \subseteq R$ which is true by modus ponens

Hint: In 1.1.1, you have shown those properties for all sets (so in particular for partial functions). Therefore, $\subseteq$ is a partial order for any set (whose elements are sets).

Transitivity.

~~RTP: $\forall R: A \rightharpoonup B$,~~

~~Let A, B~~

Let R, S, T arbitrary partial functions from A to B.

RTP $R \subseteq S \wedge S \subseteq T \Rightarrow R \subseteq T$

Assume $R \subseteq S$, equivalently, $\forall x.\ x \in R \Rightarrow x \in S$ (†)

Assume $S \subseteq T$, equivalently, $\forall x.\ x \in S \Rightarrow x \in T$ (‡)

RTP $\forall x.\ x \in R \Rightarrow x \in T$

Let $x \in R$

$\therefore x \in S$ by (†)

$\therefore x \in T$ by (‡)

□

Antisymmetry:

Let R, S arbitrary partial functions from A to B.

RTP. $R \subseteq S \wedge S \subseteq R \Rightarrow S = R$

Assume $R \subseteq S$, equivalently. $\forall x.\ x \in R \Rightarrow x \in S$

Assume $S \subseteq R$ equivalently. $\forall x.\ x \in S \Rightarrow x \in R$

$\therefore \forall x.\ x \in R \Leftrightarrow x \in S$

$\therefore R = S$

□

✓

2. Let $F \subseteq PFun(A, B)$
Assume $F \neq \emptyset$

$\bigcap F = \{(a,b) \mid \forall R \in F,\ (a,b) \in R\}$
RTP: $\forall a,b,c.\ (a,b) \in \bigcap F \wedge (a,c) \in \bigcap F \Rightarrow b=c$
Let $a \in A,\ b \in B,\ c \in B$
Assume $(a,b) \in \bigcap F$
$\therefore \forall R \in F,\ (a,b) \in R$
Assume $(a,c) \in \bigcap F$
$\therefore \forall R \in F,\ (a,c) \in R$
~~RTP: b=c~~
Let $R \in F$
$(a,b) \in R \wedge (a,c) \in R$
$\therefore b=c$

3. Let $f, g \in PFun(A, B)$
Assume $\exists h \in PFun(A, B)$ (There exists h, such that what?)
Let $h \in PFun(A, B)$
Assume $f \subseteq h \supseteq g$
RTP: $f \cup g \in PFun(A, B)$
Equivalently, $\forall a,b,c.\ (a,b) \in f \cup g \wedge (a,c) \in f \cup g \Rightarrow b=c$
Let $a \in A,\ b, c \in B$.
Assume $(a,b) \in f \cup g$
$\therefore (a,b) \in f \vee (a,b) \in g$
$\therefore (a,b) \in h$ in either case
Assume $(a,c) \in f \cup g$
$\therefore (a,c) \in f \vee (a,b) \in g$
$\therefore (a,c) \in h$ in either case
$\therefore b=c$
$\square$

4.1.1 $A_2 \Rightarrow A_2 : \{ \{(1,1),(2,1)\}, \{(1,1),(2,2)\}, \{(1,2),(2,1)\}, \{(1,2),(2$

$A_2 \Rightarrow A_3 : \{ \{(1,a),(2,a)\}, \{(1,a),(2,b)\}, \{(1,a),(2,c)\},$
$\{(1,b),(2,a)\}, \{(1,b),(2,b)\}, \{(1,b),(2,c)\},$
$\{(1,c),(2,a)\}, \{(1,c),(2,b)\}, \{(1,c),(2,c)\}\}$

$A_3 \Rightarrow A_2 : \{ \{(a,1),(b,1),(c,1)\}, \{(a,1),(b,1),(c,2)\}, \{(a,1),(b,2),(c,$
$\{(a,1),(b,2),(c,2)\}, \{(a,2),(b,1),(c,1)\}, \{(a,2),(b,1),(c$
$\{(a,2),(b,2),(c,1)\}, \{(a,2),(b,2),(c,2)\}\}$

$A_3 \Rightarrow A_3 : \{ \{(a,a),(b,a),(c,a)\}, \{(a,a),(b,a),(c,b)\}, \{(a,a),(b,a),(c,c)\},$
$\{(a,a),(b,b),(c,a)\}, \{(a,a),(b,b),(c,b)\}, \{(a,a),(b,b),(c,c)\},$
$\{(a,a),(b,c),(c,a)\}, \{(a,a),(b,c),(c,b)\}, \{(a,a),(b,c),(c,c)\},$
$\{(a,b),(b,a),(c,a)\}, \{(a,b),(b,a),(c,b)\}, \{(a,b),(b,a),(c,c)\},$
$\{(a,b),(b,b),(c,a)\}, \{(a,b),(b,b),(c,b)\}, \{(a,b),(b,b),(c,c)\},$
$\{(a,b),(b,c),(c,a)\}, \{(a,b),(b,c),(c,b)\}, \{(a,b),(b,c),(c,c)\},$
$\{(a,c),(b,a),(c,a)\}, \{(a,c),(b,a),(c,b)\}, \{(a,c),(b,a),(c,c)\},$
$\{(a,c),(b,b),(c,a)\}, \{(a,c),(b,b),(c,b)\}, \{(a,c),(b,b),(c,c)\},$
$\{(a,c),(b,c),(c,a)\}, \{(a,c),(b,c),(c,b)\}, \{(a,c),(b,c),(c,c)\}\}$

4.1.2. RTP: $\forall a \in A. \exists b \in B. aRb \iff id_A \subseteq R^{op} \circ R$

Start with "$\Rightarrow$"

Assume $\forall a \in A \exists b \in B. aRb$ (†)

~~Let $a \in A$~~

~~Let $b \in B$ st aRb~~

More directly: Let a∈A. Then there exists b∈B with aRb. As aRb, bR^{op}a, so a(R^{op} o R) a.

RTP $id_A \subseteq R^{op} \circ R$

$R^{op} \circ R : \{(a,a') \mid \exists b \in B. (a,b) \in R \wedge (b,a') \in R^{op}\}$

$= \{(a,a') \mid \exists b \in B. (a,b) \in R \wedge (a',b) \in R\}$

This statement does not mean what you want.

$\forall a,a'.\ a = a' \wedge \exists b \in B. (a,b) \in R \Rightarrow (a',b) \in R$

By (†): $\forall a = a' \in A \Rightarrow \exists b \in B. (a,b) \in R \wedge (a',b) \in R$

$id_A \subseteq R^{op} \circ R$

Either the second b would be out of the scope of the exists, or you are saying something trivial.

Next prove "$\Leftarrow$"

Assume $id_A \subseteq R^{op} \circ R$

Equivalently, $a = a' \in A \Rightarrow \exists b \in B. (a,b) \in R \wedge (a',b) \in R$

Equivalently, $\forall a \in A. \exists b \in B. (a,b) \in R$

□

3a) RTP: $\forall a \in A.\ \exists a' \in A.\ (a, a') \in id_A$

Equivalently $\forall a \in A.\ \exists a' \in A.\ a = a'$

which is satisfied by $a' = a$

b.) RTP: $\forall$ sets A, B, C $\forall f : A \to B,\ g : B \to C.\ g \circ f : \cancel{A \to C}$ is a function

Let A, B, C arbitrary sets

Let $f : A \to B,\ g : B \to C$

RTP: $\cancel{A}$ $g \circ f$ is a function.

By 3.1.3 $g \circ f$ is a $\cancel{f}$ partial function

RTP: $\forall a \in A.\ \exists c \in C$ s.t. $(a, c) \in \cancel{f \circ g}\ g \circ f$

Let $a \in C$

Let $b \in B$ s.t. $(a, b) \in f$

Let $c \in C$ st. $(b, c) \in g$

$\therefore (a, c) \in g \circ f$ $\square$

4.2.1. $f = \{(a, a+1) \mid a \in A\}$ where $A = \mathbb{N}$

$g = \{(a, 2a) \mid a \in A\}$

$f \circ g = \{(a, 2a+1) \mid a \in A\}$

$g \circ f = \{(a, 2a+2) \mid a \in A\}$

$(0, 1) \in f \circ g$

$(0, 1) \notin g \circ f$

$\therefore f \circ g \neq g \circ f$

2a. $\chi_{A \cup B}(x) = (x \in A \cup B)$ ~~...~~ $(x \in A) \vee (x \in B)$

$= \chi_A$ OR χ_B

χ_A	χ_B	$\chi_{A \cup B}$	$\max(\chi_A, \chi_B)$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

$\therefore \chi_{A \cup B}(x) = \max(\chi_A(x), \chi_B(x))$

$\chi_{A\cap B}(x) = (x \in A \cap B) = (x \in A) \wedge (x \in B) = \chi_A(x) \text{ AND } \chi_B(x)$

χ_A	χ_B	$\chi_{A\cap B}$	$\min(\chi_A, \chi_B)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

$\therefore \chi_{A\cap B}(x) = \min(\chi_A(x), \chi_B(x))$

$\chi_{A^c}(x) = (x \in A^c) = (x \notin A) = \text{NOT } (x \in A) = \text{NOT } \chi_A(x)$

χ_A	χ_{A^c}	$1 - \chi_A$
0	1	1
1	0	0

b. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$\chi_{A\Delta B} = (x \in A \Delta B) = (x \in A) \text{ XOR } (x \in B) = \chi_A \text{ XOR } \chi_B$

χ_A	χ_B	$\chi_{A\Delta B}$	$\chi_A +_2 \chi_B$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	0

~~$f: \mathbb{Z} \to \mathbb{N}$~~

5.1.1 ~~a. $f: \mathbb{N}\to\mathbb{N}$ $f(x) = x^2$ has no retraction~~

~~$f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = 2x$ has one retraction~~

~~$f: \{0,1\} \to \{0\}$ $f(x) = 0$ has two retractions, namely $g(x) = 0$ and $g(x) = 1$~~

~~a. $f: \mathbb{Z} \to \mathbb{N}$ $f(x) = |x|$ has~~

~~$f: \{0\} \to \{0,1\}$ if $f(x)=0$ has many retractions including $g(x) = 0$ and $g(x) = x$~~

~~b. $f: \mathbb{Z} \to \mathbb{N}$ $f(x) = |x|$ has no section~~

~~$f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = x+1$ has one section~~

	Retractions	Sections
0	$f: \mathbb{N} \to \mathbb{N}: f(x) = x^2$	$f: \mathbb{N} \to \mathbb{N}: f(x) = 2^x$
1	$f: \mathbb{Z} \to \mathbb{Z}: f(x) = x+1$	$f: \mathbb{Z} \to \mathbb{Z}: f(x) = 1 - x$
Many	$f: \{0,1\} \to \{0,1\}$ $f(x) = 0$	$f: \mathbb{Z} \to \mathbb{N}: f(x) = \lvert x\rvert$

Why? Explain brifly each example.

2 a. 2: $[0..n] \to [-n..n]: x \mapsto x$

$[0..n] \to [-n..n]: x \mapsto -x$

There are more, e.g. $\{(1,1), (2,-2), (3,3), \ldots\}$

b. $\text{retraction}(x) = \begin{cases} \log_2 x & \text{if } \log_2 x \text{ is an integer} \quad (n \text{ cases}) \\ \text{anything} & \text{otherwise} \quad (2^n + 1 - n \text{ cases}) \end{cases}$

$\therefore$ there are $(1+n)(2^n + 1 - n)$ retractions

3i. $A = \mathbb{N}$
$B = \mathbb{N}$
$f(x) = 2x$
The retraction : $x \mapsto \begin{cases} x/2 & \text{if } x \text{ is even} \\ \text{anything} & \text{otherwise} \end{cases}$

There is no section, because ~~there's~~ on odd inputs, there is no natural number which, when doubled, gives that input.

2.i.a. Let A be some arbitrary set

As bijection = invertible, you can also show that (id o id = id)

RTP: $\forall a' \in A\ \exists! a \in A.\ (a, a') \in id_A$
Let $a' \in A.\quad (a', a') \in id_A \therefore \exists a \in A.\ (a, a') \in id_A$
RTP: $\forall a_1, a_2 \in A.\ ((a_1, a') \in id_A \wedge (a_2, a') \in id_A) \Rightarrow a_1 = a_2$
Let $a_1, a_2 \in A$
Assume $(a_1, a') \in id_A \therefore a_1 = a'$
✓ Assume $(a_2, a') \in id_A \therefore a_2 = a'$
$\therefore a_1 = a_2$ □

b.
RTP. $\forall$ sets A, B, C, $\forall$ bijections $f : A \to B$, $g : B \to C$. $g \circ f$ is a bijection
Let A, B, C arbitrary sets
Let $f : A \to B$, $g : B \to C$ be bijections
RTP: $\forall c \in C.\ \exists! a \in A.\ g \circ f(a) = c$
Let $c \in C$
$\therefore$ Let $!b \in B.\ g(b) = c$
$\therefore$ Let $!a \in A.\ f(a) = b$
$\therefore g \circ f(a) = c \wedge a$ is unique
□

(or show f^{-1} o g^{-1} is the inverse of gof)

5. By associativity,

$$g \circ f \circ h = g \circ (f \circ h) = g \circ id_B = g$$
$$= (g \circ f) \circ h = id_A \circ h = h$$

$\therefore g = h$

5.2.1a. Assume $r \circ s = id_A$

RTP $(s \circ r) \circ (s \circ r) = s \circ r$

$$(s \circ r) \circ (s \circ r) = s \circ r \circ s \circ r$$
$$= s \circ (r \circ s) \circ r$$
$$= s \circ id_A \circ r$$
$$= s \circ r$$

$\square$

6. Let $e : B \to B$ s.t. $e \circ e = e$

RTP: $\exists$ set A, $s : A \to B$, $r : B \to A$ s.t. $s \circ r = e \wedge r \circ s = id_A$

Let $A = \{(e(x), ☺) \mid x \in B\}$

$r : B \to A : r(x) = (e(x), ☺)$

$s : e \circ Pr_1$

(even though the happy face is a nice touch, you can also just take A = {e(x) : x∈B} and s:A->B the subset inclusion, i.e. s = id_A seen as a function from A to B.)

RTP: $s \circ r = e^{(\dagger)} \wedge r \circ s = id_A \ (\ddagger)$

$(\dagger)$ first:

$$(s \circ r)(x) = (e \circ Pr_1)((e(x), ☺))$$
$$= e(e(x))$$
$$= e(x)$$

$\therefore s \circ r = e$

Next $(\ddagger)$

~~$(r \circ s)(x) = (e(e \circ Pr_1(x)))$~~

Let $x \in A = (e(y), ☺)$

$(r \circ s)(x) = (e(e(e(Pr_1((e(y), ☺))))), ☺)$
$= (e(e(y)), ☺)$
$= (e(y), ☺)$
$= x$

$\therefore r \circ s = id_A$

$\square$

c. $p \circ q \circ p = p$
$\therefore p \circ q \circ p \circ q = p \circ q$
$\therefore (p \circ q) \circ (p \circ q) = p \circ q$
$\therefore p \circ q$ is idempotent

$p \circ q \circ p = p$
$\therefore q \circ p \circ q \circ p = q \circ p$
$\therefore (q \circ p) \circ (q \circ p) = q \circ p$
$\therefore q \circ p$ is idempotent

2c. RTP: $A \cong A$
The bijection is id_A

ii. RTP: $A \cong B \Rightarrow B \cong$; Assume $A \cong B$
Let f be ~~the~~ a bijection from A to B
f^{-1} is a bijection from B to A

iii. RTP $(A \cong B \wedge B \cong C) \Rightarrow A \cong C$
~~Let~~ Assume $A \cong B \wedge B \cong C$
Let f = a bijection from A to B
Let g = a bijection from B to C
$g \circ f$ is a bijection from A to C

Assume $A \cong X \wedge B \cong Y$

Let f = a bijection from A to X, g = a bijection from B to Y

iv). RTP $P(A) \cong P(X)$

~~Let f = a bijection from~~

~~$\{(a, f(a)) \mid a \in A$~~

Let $F: P(A) \to P(B) = x \mapsto \{f(a) \mid a \in x\}$

is a bijection ~~from~~

v). RTP $A \times B \cong X \times Y$

Let $F: A \times B \to X \times Y = \{(f(a), g(b)) \mid \forall (a,b) \in A \times B\}$

is a bijection

vi) RTP $A \uplus B \cong X \uplus Y$

Let $F: A \uplus B \to X \uplus Y: \{f(a) \mid a \in A\} \uplus \{g(b) \mid b \in B\}$

is a bijection

vii.) RTP: $Rel(A,B) \cong Re(X,Y)$

Equivalently, $P(A \times B) \cong P(X \times Y)$

$A \times B \cong X \times Y$ by v)

$P(A \times B) \cong P(X \times Y)$ by iv)

Another way to write this wihch is useful is the following: If f: A -> X and g:B->Y are the assumed bijections, then a relation R:A-|->B goes to (goRof^{-1}) : X-|->Y.
The inverse of this bijection is R' |-> foR'og^{-1}

viii) RTP: $(A \rightharpoonup B) \cong (X \rightharpoonup Y)$

Let $F: (A \rightharpoonup B) \to (X \rightharpoonup Y): x \mapsto \{(f(a), g(b)) \mid \forall (a,b) \in (A \rightharpoonup B)\}$

is a bijection

This way (viii) - (x) follow by (vii) and that partial functions, functions and bijections are closed under composition.

ix) RTP: $(A \Rightarrow B) \cong (X \Rightarrow Y)$

Let $F: (A \Rightarrow B) \to (X \Rightarrow Y): x \mapsto \{(f(a), g(b)) \mid \forall (a,b) \in (A \Rightarrow B)\}$

is a bijection

x) RTP: $Bij(A,B) \cong Bij(X,Y)$

Let $F: Bij(A,B) \to Bij(X,Y): x \mapsto \{(f(a), g(b)) \mid \forall (a,b) \in Bij(A,B)\}$

is a bijection

3a. $\#P([n]) = 2^n$ because for each $x \in [n]$ it is either in the subset or not

Not correct reason. Identify P([n]) with functions [n] -> [2].

$\#[2^n] = 2^n = \#P([n]) \therefore$ there is a bijection

b. $\#([m] \times [n]) = \#[m] \times \#[n]$
$= m \times n$
$= \#[m \times n]$

Why?

c. $\#([m] \uplus [n]) = \#[m] + \#[n]$
$= m + n$
$= \#[m+n]$

d. $\#([m] \rightharpoonup [n]) = (n+1)^m$ because for each of the m elements in $[m]$, it has $n+1$ options for a mapping: one of the n elements of $[n]$, or to be undefined
$= \#[(n+1)^m]$

e. $\#([m] \Rightarrow [n]) = n^m$ because for each of the m elements in $[m]$ it can be mapped to any of the n elements in $[n]$
$= \#[n^m]$

f. $\#Bij([n], [n]) = n!$ because for the first element of the first set, it has n possible images. The second has $n-1$ and so on until the n^{th} element has only 1 possible image $\therefore \# = \sum_{i=0}^{n-1} (n-i) = n!$

$= \#[n!]$