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Maths Supervision 4

5a. $\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$

$\therefore A(1+x) + B(1-x) = 1$

$\therefore (A-B)x + (A+B) = 1$

Comparing coefficients of x and 1:

$A - B = 0 \iff A = B$

$A + B = 2A = 1 \iff A = \frac{1}{2}$

$\implies B = \frac{1}{2}$

$\therefore \frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$

b. $\frac{3x}{2x^2+x-1} = \frac{3x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$

$\therefore A(x+1) + B(2x-1) = 3x$

$\therefore (A+2B)x + (A-B) = 3x$

Comparing coefficients of 1 and x:

$A - B = 0 \iff A = B$

$A + 2B = 3A = 3 \iff A = 1$

$\implies B = 1$

$\therefore \frac{3x}{2x^2+x-1} = \frac{1}{2x-1} + \frac{1}{x+1}$

c. $\frac{2(1-x^2)}{1+x-x^2-x^3} = \frac{2(1-x^2)}{(1-x^2)(x+1)} = \frac{2}{x+1}$ Assuming $x \neq 1$

d. $\dfrac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4}$

Take the denominator, $d: 3+2x-2x^2-2x^3-x^4$:
By observation, $x=\pm 1$ are two solutions to $d=0$
$\therefore$ $(x-1)$ and $(x+1)$ are factors of d.

By the tabular method of polynomial division:

	$-x^2$	$-2x$	-3
x^2	$-x^4$	$-2x^3$	$-3x^2$
-1	x^2	$2x$	3

$$d = (x-1)(x+1)(-x^2-2x-3)$$

$\therefore \dfrac{1}{3+2x-2x^2-2x^3-4x^4} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{-x^2-2x-3}$

$\therefore A(x+1)(-x^2-2x-3)+B(x-1)(-x^2-2x-3)+(Cx+D)(x^2-1)=x^4+x^2+4x+6$
$\therefore A(-x^3-3x^2-5x-3)+B(-x^3-x^2-x+3)+(Cx^3+Dx^2-Cx-D)=x^4+x^2+4x+6$
$\therefore$

$$x^4+x^2+4x+6 = -(-x^4-2x^3-2x^2+2x+3)-2x^3-x^2+6x+9$$

$$\therefore \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4} = -1 + \frac{-2x^3-x^2+6x+9}{3+2x-2x^2-2x^3-x^4}$$

$$= -1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{-x^2-2x-3}$$

$\therefore A(x+1)(-x^2-2x-3)+B(x-1)(-x^2-2x-3)+(Cx+D)(x^2-1) = -2x^3-x^2+6x+9$
$\therefore A(-x^3-3x^2-5x-3)+B(-x^3-x^2-x+3)+(Cx^3+Dx^2-Cx-D) = -2x^3-x^2+6x+9$
$\therefore (-A-B+C)x^3+(-3A-B+D)x^2+(-5A-B-C)x+(-3A+3B-D) = -2x^3-x^2+6x+9$

Comparing coefficients of x^3, and x

$-A - B + C = -2$
$-5A - B - C = 6$

$(-A - B + C) + (-5A - B - C) = -2 + 6$
$-6A - 2B = 4$
$3A + B = -2$

Comparing coefficients of x^2

$-3A - B + D = -1$
$-(3A + B) + D = -1$
$\therefore 2 + D = -1$
$\therefore D = -3$

Comparing coefficients of 1.

$-3A + 3B - D = 9$
$\therefore -3A + 3B + 3 = 9$
$\therefore -3A + 3B = 6$

$\therefore (-3A + 3B) + (3A + B) = 6 - 2$
$\therefore 4B = 4$
$\therefore B = 1$

$\therefore 3A + 1 = 2$
$\therefore A = -1$

$\therefore 1 - 1 + C = -2$
$\therefore C = -2$

$$\therefore \frac{x^4+x^2+4x+6}{3+2x-2x^2-2x^3-x^4} = \cancel{-1 + \frac{1}{6x-6} + \frac{5}{4x+4} - \frac{x+3}{3x^2+6x+9}}$$

$$= -1 - \frac{1}{x-1} + \frac{1}{x+1} + \frac{2x+3}{x^2+2x+3}$$

7a. $y + e^y \sin y = 1/x$

$$\therefore x = \frac{1}{y+e^y\sin y} \quad \text{assuming } y + e^y \sin y \neq 0$$

$$\therefore \frac{dx}{dy} = -\frac{1}{(y+e^y\sin y)^2}\frac{d}{dy}(y+e^y\sin y)$$

$$= -\frac{1}{(y+e^y\sin y)^2}\cdot(1+e^y\sin y+e^y\cos y)$$

$$= -\frac{1+e^y(\sin y+\cos y)}{(y+e^y\sin y)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{(y+e^y\sin y)^2}{1+e^y(\sin y+\cos y)}$$

$$= \frac{-1}{x^2(1+e^y(\sin y+\cos y))} \quad (*)$$

b. $\frac{d}{dx}(y+e^y\sin y) = \frac{d}{dx}\left(\frac{1}{x}\right)$

$$\therefore \frac{dy}{dx} + e^y(\sin y+\cos y)\frac{dy}{dx} = -\frac{1}{x^2}$$

$$\therefore \frac{dy}{dx}\left(1+e^y(\sin y+\cos y)\right) = -\frac{1}{x^2}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{x^2(1+e^y(\sin y+\cos y))} \quad (**)$$

c. Observe $(*) = (**)$

5a. $y = (x-3)^2 + 2x$

y-intercept: $y|_{x=0} = (-3)^2 = 9$

x-intercept: $y = 0 \Rightarrow (x-3)^2 + 2x = 0$

$\Rightarrow x^2 - 4x + 9 = 0$

$\therefore$ there are no x-intercepts

$\frac{dy}{dx} = 2x - 4$

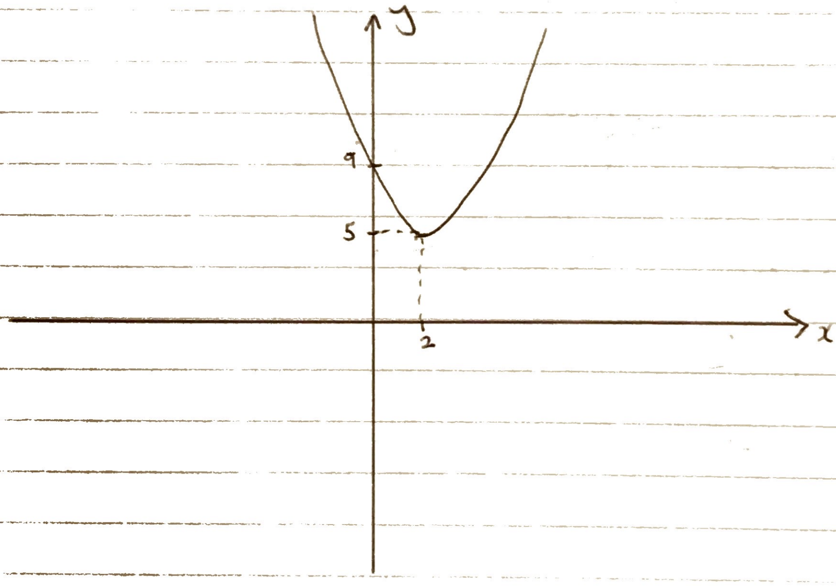
Setting $\frac{dy}{dx} = 0$, we get

$x = 2$

$\frac{d^2y}{dx^2} = 2 > 2$

$y|_{x=2} = 5$

$\therefore (2, 5)$ is a local minimum



b. $y = \dfrac{x}{1+x^2}$

y-intercept: ~~$y=0$~~ $y|_{x=0} = 0$

x-intercept: $y=0 \Rightarrow x=0$

asymptote at $1+x^2=0$ $\therefore$ there are no asymptotes

$\dfrac{dy}{dx} = \dfrac{1+x^2-2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

Setting $\dfrac{dy}{dx}=0$ we get

$1-x^2=0 \Rightarrow x=\pm 1$

$\dfrac{d^2y}{dx^2} = \dfrac{(1+x^2)^2\cdot(-2x) - (1-x^2)\cdot 2\cdot(1+x^2)\cdot 2x}{(1+x^2)^4}$

$= \dfrac{-2x(1+x^2)^2 - 4x(1-x^2)(1+x^2)}{(1+x^2)^4}$

$\therefore \left.\dfrac{d^2y}{dx^2}\right|_{x=1} = \dfrac{-2(2^2)}{2^4} = -\tfrac{1}{2} < 0$

$\left.\dfrac{d^2y}{dx^2}\right|_{x=-1} = \dfrac{2(2^2)}{2^4} = \tfrac{1}{2} > 0$

$y|_{x=1} = \tfrac{1}{2}$

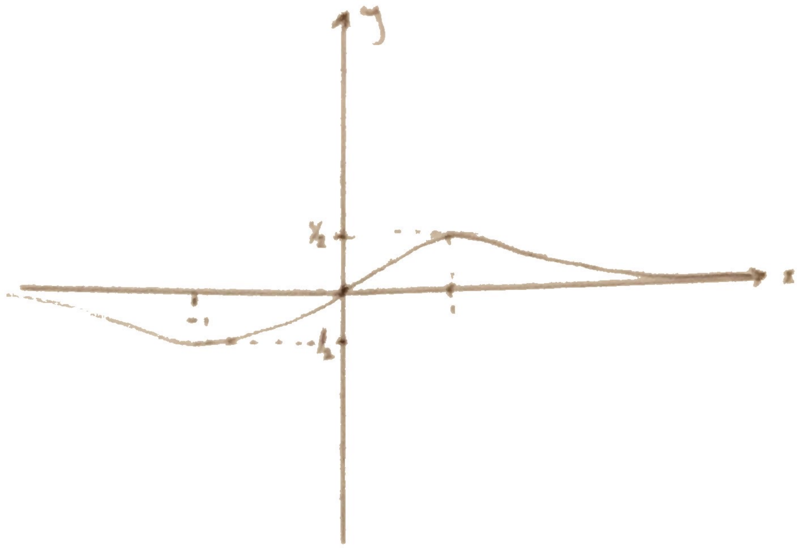
$y|_{x=-1} = -\tfrac{1}{2}$

$\therefore (1, \tfrac{1}{2})$ is a local maximum

$(-1, -\tfrac{1}{2})$ is a local minimum

$\lim_{x\to\infty} y = \lim_{x\to\infty} \dfrac{1}{2x} = 0^+$

$\lim_{x\to-\infty} y = \lim_{x\to-\infty} \dfrac{1}{2x} = 0^-$



c $y = xe^x$

y-intercept $y|_{x=0} = 0$

x-intercept $y = 0 \Rightarrow x = 0$

$\frac{dy}{dx} = xe^x + e^x = e^x(x+1)$

Setting $\frac{dy}{dx} = 0$, we get

$x = -1$

$\frac{d^2y}{dx^2} = e^x(x+2)$

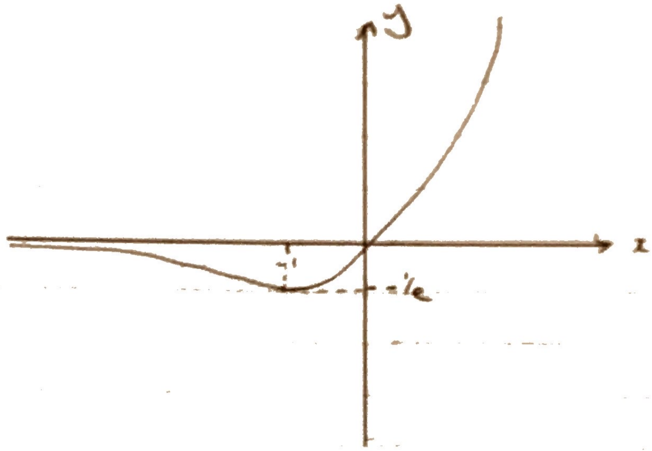
$\frac{d^2y}{dx^2}\Big|_{x=-1} = e^{-1} = 1/e > 0$

$y|_{x=-1} = -1/e$

$\therefore (-1, -1/e)$ is a local minimum

$\lim_{x\to\infty} xe^x = \infty$

$\lim_{x\to-\infty} xe^x = 0$



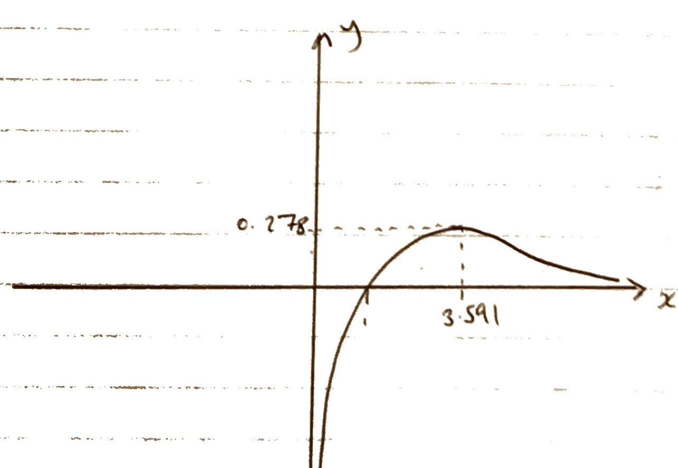
d. $y = \frac{\ln x}{x+1}$

y-intercept : $y|_{x=0} = -\infty$

x-intercept : $y = 0 \Rightarrow \ln x = 0$
$\Rightarrow x = 1$

asymptote at $x=-1$ but that doesn't matter because $\ln x$ is undefined for $x \leq 0$ anyway

$$\lim_{x\to\infty} \frac{\ln x}{x+1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0^+$$



$$\frac{dy}{dx} = \frac{(x+1)\cdot\frac{1}{x} - \ln x}{(x+1)^2} = \frac{x+1-x\ln x}{x(x+1)^2}$$

Setting $\frac{dy}{dx} = 0$, we get $x+1-x\ln x = 0$, $x \simeq 3.591$
$y \simeq 0.278$

e. $y = \frac{1}{1-e^x}$

No y intercept because $x=0$ is an asymptote
No x intercepts because $1 \neq 0$

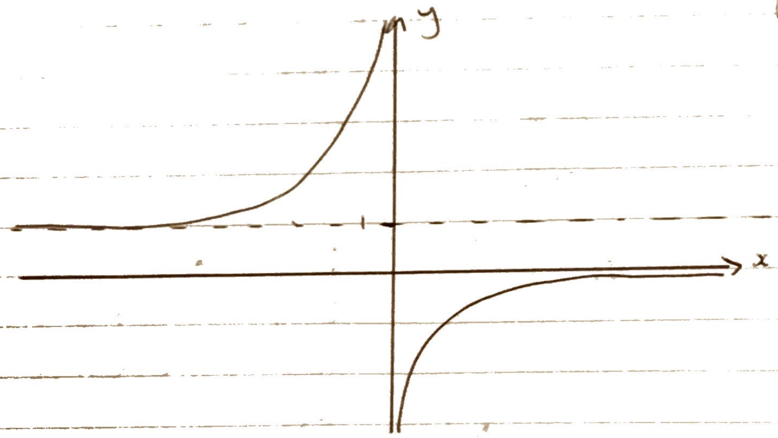
$\lim_{x \to \infty} = 0^-$
$\lim_{x \to -\infty} = 1^+$

$\lim_{x \to 0^+} = -\infty$
$\lim_{x \to 0^-} = \infty$

$\frac{dy}{dx} = -\frac{1}{(1+e^x)^2} \cdot \frac{d}{dx}(1-e^x)$

$= \frac{e^x}{(1+e^x)^2}$

which can never be zero $\therefore$ no ~~tur~~ turning points



y-intercept: $y|_{x=0} = 1$

x-intercept: $y = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = (n+\tfrac{1}{2})\pi$ for any $n \in \mathbb{Z}$

$$\frac{dy}{dx} = e^x(\cos x - \sin x)$$

Setting $\frac{dy}{dx} = 0$, we get $\cos x = \sin x$

$\therefore x = (m+\tfrac{1}{4})\pi$ for any $m \in \mathbb{Z}$

$$\frac{d^2y}{dx^2} = e^x(\cos x - \sin x) - e^x(\cos x + \sin x)$$
$$= -2e^x \sin x$$

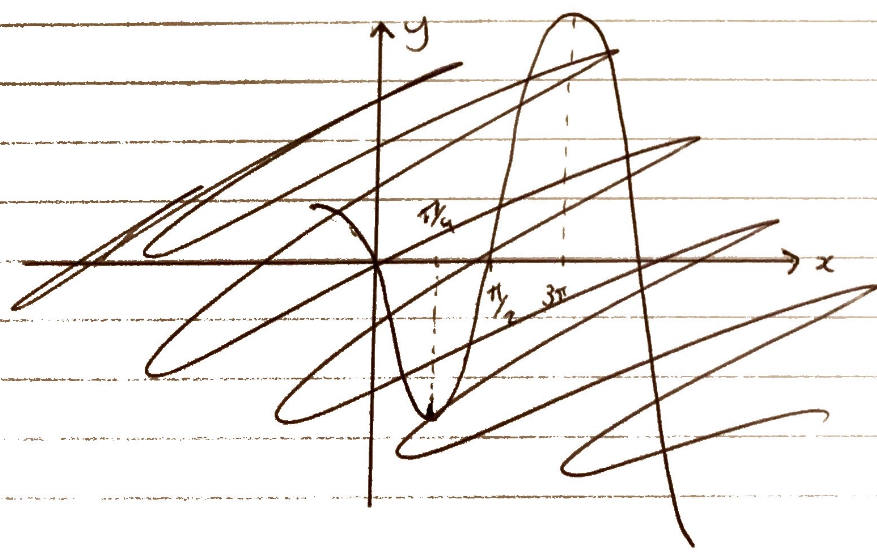
$$\left.\frac{d^2y}{dx^2}\right|_{x=(m+\frac{1}{4})\pi} = (-1)^{m+1}\sqrt{2}\,e^{(m+\frac{1}{4})\pi}$$

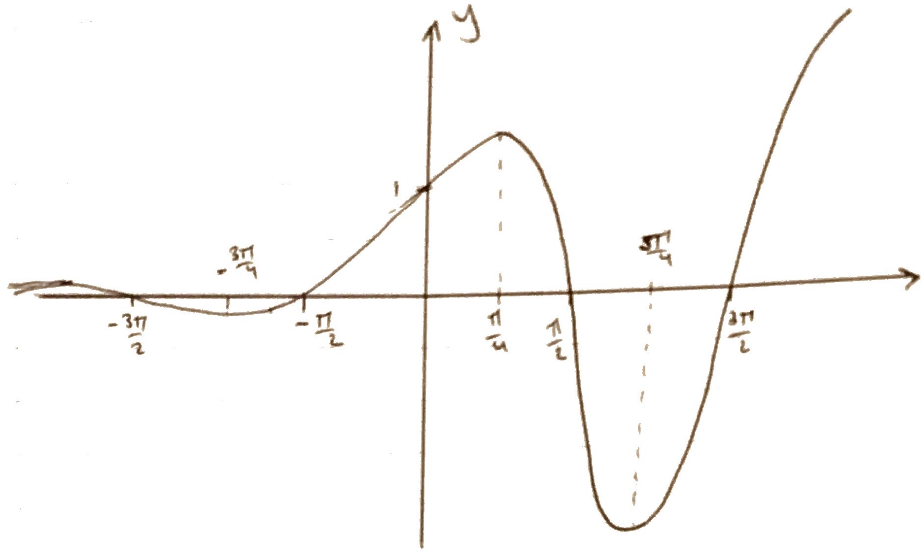
> 0 when m is odd

< 0 otherwise

$$y|_{x=(m+\frac{1}{4})\pi} = \frac{(-1)^m}{\sqrt{2}}e^{(m+\frac{1}{4})\pi}$$

$\therefore \left((m+\tfrac{1}{4})\pi, \frac{(-1)^m}{\sqrt{2}}e^{(m+\frac{1}{4})\pi}\right)$ is a local maximum where m is even, and a local minimum otherwise.





y
1
$-\frac{3\pi}{4}$
$\frac{5\pi}{4}$
$-\frac{3\pi}{2}$
$-\frac{\pi}{2}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$