Proof of Euler's Partition Theorem

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Let $\mathbb{D}(n)$ be the number of distinct partitions of n
Let $\mathbb{O}(n)$ by the number of odd partitions of n

Consider the polynomials:

$$P(x) = \sum_{n \in \mathbb{N}_0^+} \mathbb{D}(n) x^n$$

$$Q(x) = \sum_{n \in \mathbb{N}_0^+} \mathbb{O}(n) x^n$$

where $\mathbb{N}_0^+$ is the set of positive integers and 0

Proving that $P(x) = Q(x)\ \forall\ x$ is sufficient to show that $\mathbb{D}(n) = \mathbb{O}(n)\ \forall\ n \in \mathbb{N}_0^+$ by comparing coefficients of powers of x. (*)

Consider the series $(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$

$$= \prod_{n=1}^{\infty} (1+x^n)$$

To expand the brackets, we must consider every possible way of choosing one of the two terms from each factor (i.e. either the 1 or the power of x), taking the product of all of these, and taking the sum of these products

One such product is given by

$$(\boxed{1}+x)(1+\boxed{x^2})(1+\boxed{x^3})(\boxed{1}+x^4)\cdots$$

$$1 \cdot x^2 \cdot x^3 \cdot 1 \cdot \cdots$$

where the selected term is outlined in red.

Each of these products will equal x raised to the power of the sum of the powers selected.

Therefore the coefficient of x^k is the number of ways to add distinct positive integers together to get $k = \mathbb{D}(k)$

$$\therefore P(x) = \sum_{n \in \mathbb{N}_0} \mathbb{D}(n) x^n = \prod_{n=1}^{\infty} (1+x^n)$$

Now consider the series:

$$(1+x+x^2+x^3+\dots)(1+x^3+x^6+x^9+\dots)(1+x^5+x^{10}+x^{15}+\dots)\dots$$

$$= \prod_{n=1}^{\infty} \left(1 + x^{2n-1} + x^{2(2n-1)} + x^{3(2n-1)} + \dots\right)$$

$$= \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} x^{m(2n-1)}$$

Again, to expand the brackets, choose one term from each factor, multiply these together, and add this product over each possible way of choosing.

Since this product for each choice is equal to x raised to the power of the sum of the powers selected, and since the available powers are all possible integer multiples of odd numbers, then the overall coefficient of x^k is the number of ways in which k can be written as the sum of odd integers $= \mathbb{O}(k)$

$$\therefore Q(x) = \sum_{n \in \mathbb{N}_0^+} \mathbb{O}(n) x^n = \prod_{n=1}^{\infty} \sum_{m=0}^{\infty} x^{m(2n-1)}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1 - x^{2n-1}}$$

Observe that

$$P(x) = \prod_{n=1}^{\infty} (1 + x^n)$$

$$= \prod_{n=1}^{\infty} \frac{(1+x^n)(1-x^n)}{1-x^n}$$

$$= \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}$$

$$= Q(x) \quad \forall x$$

$\therefore$ By $(*)$, $\mathbb{D}(n) = \mathbb{O}(n)$ QED.