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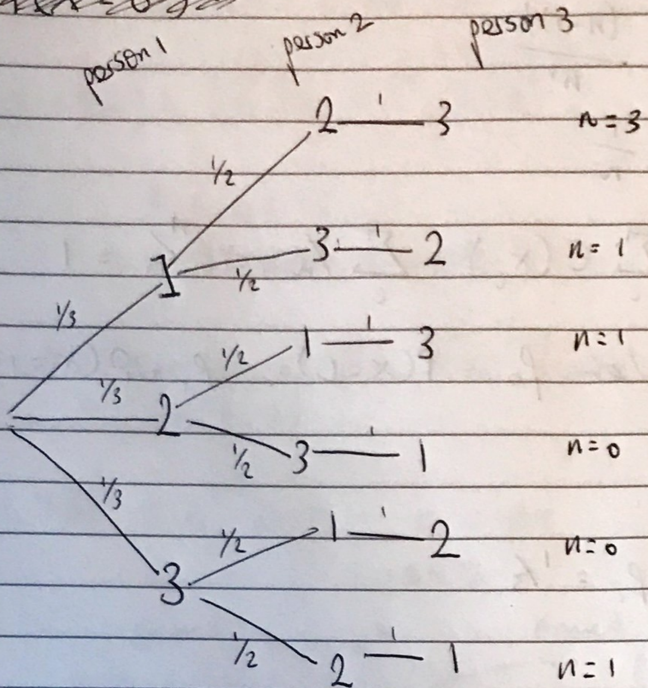
Probability Exercises 2

L3.4. 2a. ~~$P(X=0)=$~~



All 6 outcomes are equally likely.

$\therefore P(X=0) = \frac{2}{6} = \frac{1}{3}$

$P(X=1) = \frac{3}{6} = \frac{1}{2}$

$P(X=2) = \frac{0}{6} = 0$

$P(X=3) = \frac{1}{6}$

$\therefore E[X] = \frac{1}{2}\cdot 1 + \frac{1}{6}\cdot 3 = \frac{1}{2} + \frac{1}{2} = 1$

b. Let $X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ hat goes to the right person,} \\ 0 & \text{otherwise} \end{cases}$

for ~~[illegible]~~ $1 \leq i \in \mathbb{N} \leq n$

$\therefore X = \sum_i X_i$

$P(X_i = 1) = E(X_i)$ by definition

$$= \frac{(n-1)!}{n!}$$

$$= \frac{1}{n}$$

$\therefore E(X) = \sum_i E(X_i) = \sum_i 1/n = n/n = 1$

6a. Let $p_0 = P(X=0)$, $p_1 = P(X=1)$

$p_0 = 2p_1$

$0p_0 + 1p_1 = p_1 = 1/3$

$\therefore p_0 = 2/3$

Sanity check: $p_0 = 2p_1$, $p_0 + p_1 = 1$

$\therefore p_0 + p_1 = 1 = 3p_1$

$\therefore p_1 = 1/3$, $p_2 = 2/3$

$P(X=0) = 2/3$

$P(X=1) = 1/3$

b.

k	0	1
k^2	0	1
$P(X=k)$	$2/3$	$1/3$
$k^2 P(X=k)$	0	$1/3$

$\xrightarrow{\Sigma} E[X^2] = 1/3$

$V[X] = E[X^2] - (E[X])^2 = 1/3 - 1/9 = 2/9$

L5. 1. Let X = number of home games won
$\sim Bin(35, 0.7)$

$$P(X=20) = \binom{35}{20}(0.7)^{20}(0.3)^{15}$$

$$\approx 0.0372$$

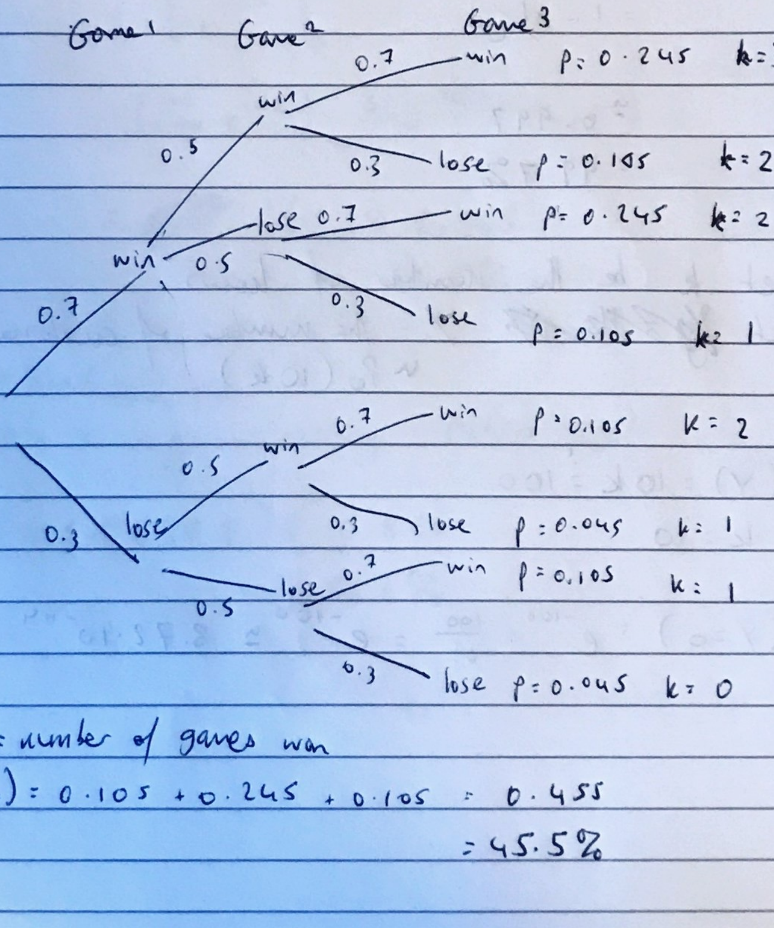
$$= 3.72\%$$

b. Let Y = number of the first home win
$\sim Geo(0.7)$

$$P(Y=\cancel{4}) = 0.3^3 \cdot 0.7 = 0.0189$$

$$= 1.89\%$$

c.



Let k = number of games won

$$P(k=2) = 0.105 + 0.245 + 0.105 = 0.455$$

$$= 45.5\%$$

8. Let Z = number of games won
$\sim Bin(70, 0.05)$
which is approximately $Po(3.5)$

$\therefore P(Z = w) \simeq \frac{3.5^w}{w!} e^{-3.5}$

2a Let X = number of customers ~~ ~~ in one hour
$\sim Po(10)$

$P(X \geq 3) = 1 - P(X \leq 2)$
$= 1 - e^{-10}\left(\frac{10^0}{0!} + \frac{10^1}{1!} + \frac{10^2}{2!}\right)$

$= 1 - e^{-10}(1 + 10 + 50)$
$= 1 - 61/e^{10}$

$\simeq 0.997$
$= 99.7\%$

b Let k be the number of hours
Let ~~ ~~ Y = the number of customers in k hours
$\sim Po(10k)$

$E(Y) = 10k \stackrel{!}{=} 100$
$\therefore k = 10$

$P(Y = 0) = e^{-100} \cdot \frac{100^0}{0!} = e^{-100} \simeq 3.72 \cdot 10^{-44}$

c. Since $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \cancel{10} 10e^{10x}\,dx$ does not converge,

whereas $\int_0^{\infty} 10e^{-10x}\,dx = 1$, I will assume that there is a typo in the question, and that instead

$$f(x) = \begin{cases} 10e^{-10x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Let Z = waiting time

$$P(Z > 0.1) = \int_{0.1}^{\infty} 10e^{-10x}\,dx$$

$$= \left[-e^{-10x}\right]_{0.1}^{\infty}$$

$$= \left(\lim_{k\to\infty} -e^{-10k}\right) - (-e^{-1})$$

$$= 1/e \simeq 0.368$$

$$= 36.8\%$$

L6.1 $X \sim Bin(100, 1/2)$

which is approximately $N(50, 25)$

$$P(42.5 \leq X \leq 57.5) \simeq 0.866$$

$$= 86.6\%$$

2a. ~~Let C_A = the number of cars of type A tested~~

~~C_B = " " " " B~~

~~$C_A \sim Bin(10, 0.2)$~~

~~$C_B \sim Bin(10, 0.2)$~~

$$T = \sum_x (0.2 \cdot A_x + 0.8 \cdot 0) + \sum_y (0.2 \cdot B_y + 0.8 \cdot 0)$$

$$= \sum_x 0.2 A_x + \sum_y 0.2 B_y$$

By linearity of expectation,

$$E[T] = 0.2 \left(\sum_x E[A_x] + \sum_y E[B_y] \right)$$

$$= 0.2 \left(\sum_x 4 + \sum_y 5 \right)$$

$$= 0.2(40 + 50)$$

$$= 18$$

b. $T = \sum^{3} A_x$ (regardless of which 3 x values)

$\therefore T \sim Po(12)$

$$P(T \geq 20) = 1 - \sum_{i=0}^{19} e^{-12} \frac{12^i}{i!}$$

$$\simeq 0.213$$

$$= 21.3\%$$

c. $T = \sum_{y}^{3} B_y$

$\therefore T \sim N(15, 9)$

$$P(T \geq 20) = 1 - P(T < 20)$$
$$\approx 1 - 0.952$$
$$= 0.0478$$
$$= 47.8\%$$

L7.1 By definition, $F(\alpha, \beta) = \int_{-\infty}^{\beta} \int_{-\infty}^{\alpha} f(x,y)\, dx\, dy$

for some joint probability density function f

$P(a \leq X \leq b, c \leq Y \leq d) = \int_{c}^{d} \int_{a}^{b} f(x,y)\, dx\, dy$ by definition

$$= \int_{-\infty}^{d} \int_{a}^{b} f(x,y)\, dx\, dy - \int_{-\infty}^{c} \int_{a}^{b} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{d} \left(\int_{-\infty}^{b} f(x,y)\, dx - \int_{-\infty}^{a} f(x,y)\, dx \right) dy - \int_{-\infty}^{c} \left(\int_{-\infty}^{b} f(x,y)\, dx - \int_{-\infty}^{a} f(x,y)\, dx \right) dy$$

$$= \int_{-\infty}^{d}\int_{-\infty}^{b} f(x,y)\, dx\, dy - \int_{-\infty}^{d}\int_{-\infty}^{a} f(x,y)\, dx\, dy - \int_{-\infty}^{c}\int_{-\infty}^{b} f(x,y)\, dx\, dy + \int_{-\infty}^{c}\int_{-\infty}^{a} f(x,y)\, dx\, dy$$

$= F(b,d) - F(a,d) - F(b,c) + F(a,c)$ as required

2. $\cancel{F_X(x) = \int_{-\infty}^{\infty} F_{X,Y}(x,y)\,dy}$

$\cancel{= \int_0^{\infty} 1 - e^{-x} - e^{-y} + e^{-x-y}\,dy \quad \text{if } x \geq 0}$

$\cancel{= \left[y - ye^{-x} + e^{-y} - e^{-x-y} \right]_0^{\infty}}$

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y)$$

$$= \lim_{y \to \infty} 1 - e^{-x} - e^{-y} + e^{-x-y}$$

$$= 1 - e^{-x}$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y)$$

$$= \lim_{x \to \infty} 1 - e^{-x} - e^{-y} + e^{-x-y}$$

$$= 1 - e^{-y}$$

$$F_X(x)\,F_Y(y) = (1 - e^{-x})(1 - e^{-y})$$

$$= 1 - e^{-x} - e^{-y} + e^{-x-y}$$

$$= F_{X,Y}(x,y)$$

$\therefore$ X and Y are independent

3. $P(X_i = 1) = E[X_i]$ by definition

$$E[X_i] = \frac{m - E[\#\text{ red balls already removed}]}{N-(i-1)}$$

$$= \frac{m - E\left[\sum_{j=1}^{i-1} X_j\right]}{N-(i-1)}$$

$$= \frac{m - \sum_{j=1}^{i-1} E[X_j]}{N-(i-1)}$$ by linearity of expectation

Proof by induction that $E[X_i] = {}^m/_N \quad \forall 1 \le i \le n$ *

Base case: $i = 1$

$$E[X_1] = \frac{m - \sum_{j=1}^{0} E[X_j]}{N-(1-1)} = \frac{m}{N}$$

$\therefore$ * holds for $i = 1$

Inductive step: Assume * holds for $i = k$

$$E[X_{k+1}] = \frac{m - \sum_{j=1}^{k} E[X_j]}{N-k} = \frac{N-(k-1)}{N-k} \cdot \frac{m - \sum_{j=1}^{k-1} E[X_j] - E[X_k]}{N-(k-1)}$$

$$= \frac{N-(k-1)}{N-k}\left(E[X_k] - \frac{E[X_k]}{N-(k-1)}\right)$$

$$= E[X_k]\left(\frac{N-(k-1)}{N-k}\right)\left(1 - \frac{1}{N(k-1)}\right)$$

$$= \frac{m}{N}\left(\frac{1}{N-k}\right)(N-(k-1)-1)$$

$$= \frac{m}{N} \cdot \frac{N-k}{N-k} = \frac{m}{N}$$

$\therefore *$ holds for $i = k+1$

$\therefore *$ holds $\forall i \in \cancel{\mathbb{N}}\ \mathbb{Z}^+$

$$\therefore P(X_i = 1) = \frac{m}{N}$$
$$P(X_i = 0) = 1 - \frac{m}{N}$$
$$X_i \sim Ber(m/N)$$

This implies that the number of red balls drawn is $\frac{nm}{N}$

4. $Cov[X, Y] = \overline{(X - \bar{X})(Y - \bar{Y})}$ where $\bar{X} = E[X]$, $\bar{Y} = E[Y]$, etc. for concision

$$= \overline{XY - \bar{X}Y - X\bar{Y} + \bar{X}\bar{Y}}$$

$$= E[XY - \bar{X}Y - X\bar{Y} + \bar{X}\bar{Y}]$$

$$= E[XY] - E[\bar{X}Y] - E[X\bar{Y}] + E[\bar{X}\bar{Y}]$$

$$= E[XY] - (\bar{X})E[Y] - (\bar{Y})E[X] + E[X]E[Y]$$

$$= E[XY] - 2E[X]E[Y] + E[X]E[Y]$$

$$= E[XY] - E[X]E[Y]$$

9. Consider 2 independent tosses of a fair coin

Let $X = \begin{cases} 1 & \text{if the first toss is heads} \\ 0 & \text{otherwise} \end{cases}$

$Y = \begin{cases} 1 & \text{if the second toss is heads} \\ 0 & \text{otherwise} \end{cases}$

$Z = (X+Y) \bmod 2$
or equivalently
$X + Y - XY$

$p_X(0) = 1/2 \quad p_X(1) = 1/2$
$p_Y(0) = 1/2 \quad p_Y(1) = 1/2$
$p_Z(0) = 1/2 \quad p_Z(1) = 1/2$

$P_{X,Y}(0,0) = 1/4 = P_X(0)\,P_Y(0)$
$P_{X,Y}(0,1) = 1/4 = P_X(0)\,P_Y(1)$
$P_{X,Y}(1,0) = 1/4 = P_X(1)\,P_Y(0)$
$P_{X,Y}(1,1) = 1/4 = P_X(1)\,P_X(1)$

$\therefore P_{X,Y} = P_X P_Y \quad \therefore$ X and Y are independent

$P_{X,Z}(0,0) = 1/4 = P_X(0)\,P_Z(0)$
$P_{X,Z}(0,1) = 1/4 = P_X(0)\,P_Z(1)$
$P_{X,Z}(1,0) = 1/4 = P_X(1)\,P_Z(0)$
$P_{X,Z}(1,1) = 1/4 = P_X(1)\,P_Z(1)$

$\therefore P_{X,Z} = P_X P_Z \quad \therefore$ X and Z are independent

$P_{Y,Z}(0,0) = 1/4 = P_Y(0)\, P_Z(0)$
$P_{Y,Z}(0,1) = 1/4 = P_Y(0)\, P_Z(1)$
$P_{Y,Z}(1,0) = 1/4 = P_Y(1)\, P_Z(0)$
$P_{Y,Z}(1,1) = 1/4 = P_Y(1)\, P_Z(1)$

$\therefore P_{Y,Z} = P_Y \cdot P_Z$

$\therefore$ Y and Z are independent

$P_{X,Y,Z}(0,0,0) = 1/4 \neq 1/8 = P_X(0)\, P_Y(0)\, P_Z(0)$

$\therefore$ X, Y, and Z are not (mutually) independent