Morgan Saville

Maths Supervision Work 1

5a. Seperable

$$\frac{1}{y^2+1}\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

$$\therefore \int \frac{1}{y^2+1}\frac{dy}{dx}\,dx = \int \frac{1}{\cos^2 x}\,dx$$

$$\therefore \arctan(y) = \tan(x) + C$$

$$\therefore y = \tan(\tan x + C)$$

$$y(0) = \tan(\tan 0 + C)$$
$$= \tan C$$
$$= 0$$

$\therefore C = \arctan 0 = k\pi$ for some integer k, w.l.o.g. take $k = 0$

$\therefore C = 0$

$$y = \tan(\tan(x))$$

b. Seperable

$$\frac{dy}{dx} + 4xy = 2x(y^2+1)$$

$$\therefore \frac{dy}{dx} = 2x(y^2+1) - 4xy$$
$$= 2x(y^2+1-2y)$$
$$= 2x(y-1)^2$$

$$\therefore \frac{1}{(y-1)^2}\frac{dy}{dx} = 2x$$

$$\therefore \int \frac{1}{(y-1)^2}\frac{dy}{dx}\,dx = \int 2x\,dx$$

$$\therefore -\frac{1}{y-1} = x^2 + C$$

$$\therefore y - 1 = -\frac{1}{x^2+C}$$

$$y = 1 - \frac{1}{x^2+C}$$

$y(0) = 1 - \frac{1}{C} = 0$

$\therefore \frac{1}{C} = 1$

$\therefore C = 1$

$\therefore y = 1 - \frac{1}{x^2+1}$

c. Linear

$x\frac{dy}{dx} + (x-1)y = e^{-x}$

$\therefore \frac{dy}{dx} + \frac{x-1}{x}y = \frac{1}{x}e^{-x}$

Integrating factor $= e^{\int \frac{x-1}{x}dx} = e^{\int (1-\frac{1}{x})dx} = e^{x-\ln|x|} = \frac{e^x}{|x|}$

$\therefore \frac{e^x}{|x|}\frac{dy}{dx} + \frac{e^x}{|x|}\cdot\frac{x-1}{x}y = \frac{e^x}{|x|}\cdot\frac{1}{x}e^{-x}$

$\therefore \frac{d}{dx}\left(\frac{e^x}{|x|}y\right) = \frac{1}{x|x|}$

$\therefore \frac{e^x}{|x|}y = \int \frac{1}{x|x|}dx = C - \frac{1}{|x|}$

$\therefore e^x y = C|x| - 1$

$\therefore y = \frac{C|x|-1}{e^x}$

$y(1) = \frac{C-1}{e} = 0$

$\therefore C = 1$

$\therefore y = \frac{|x|-1}{e^x}$

d. ~~Linear~~ Seperable

$(1+x^3)\frac{dy}{dx} - x^2 y = x^2$ $\quad (1+x^3)\frac{dy}{dx} = x^2(1+y)$

$\therefore \frac{dy}{dx} = \frac{-x^2}{1+x^3} y = \frac{x^2}{1+x^3}$ $\quad \therefore \frac{1}{1+y}\frac{dy}{dx} = \frac{x^2}{1+x^3}$ or $y=-1$

$\therefore \int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int \frac{x^2}{1+x^3}\,dx$ or $y=-1$

$\therefore \ln|1+y| = \frac{1}{3}\ln|1+x^3| + C$ or $y=-1$

$\therefore 1+y = \pm e^{\ln|(1+x^3)^{1/3}| + C}$ or $y=-1$

$= A\,|(1+x^3)^{1/3}|$

$\therefore y = A\,|(1+x^3)^{1/3}| - 1$

$y(0) = A\,|1^{1/3}| - 1$

$= A - 1$

$= 0$

$\therefore A = 1$

$\therefore y = |(1+x^3)^{1/3}| - 1$

6. $(y-x)\frac{dy}{dx} + (2x+3y) = 0$

$\therefore \left(\frac{y}{x} - 1\right)\frac{dy}{dx} + \left(2 + 3\frac{y}{x}\right) = 0$

$\therefore \frac{dy}{dx} = \frac{-2 - 3\frac{y}{x}}{y/x - 1} = \frac{2 + 3\frac{y}{x}}{1 - y/x}$

Let $u = y/x$ $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}ux = u + x\frac{du}{dx}$

$\therefore u + x\frac{du}{dx} = \frac{2+3u}{1-u}$

$$\therefore x\frac{du}{dx} = \frac{2+3u}{1-u} - \frac{u-u^2}{1-u} = \frac{2+2u+u^2}{1-u}$$

$$\therefore \frac{1-u}{2+2u+u^2}\frac{du}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{1-u}{2+2u+u^2}\frac{du}{dx}\,dx = \int \frac{1}{x}\,dx$$

Let $I = \int \frac{1-u}{2+2u+u^2}\,du$

$$= \int \frac{2}{2+2u+u^2}\,du - \tfrac{1}{2}\ln|2+2u+u^2|$$

$$= \int \frac{2}{(u+1)^2+1}\,du - \tfrac{1}{2}\ln|2+2u+u^2|$$

$$= 2\arctan(u+1) - \tfrac{1}{2}\ln|2+2u+u^2| + 2C$$

$$\therefore 2\arctan(u+1) - \tfrac{1}{2}\ln|2+2u+u^2| = \ln|x| + 2C$$

$$\therefore 2\arctan(u+1) = 2C + \ln|x(2+2u+u^2)^{1/2}|$$

$$= 2C + \ln|(2x^2+2x+1)^{1/2}|$$

$$= 2C + \ln|x+1|$$

$$\therefore u+1 = \tan\left(C + \tfrac{1}{2}\ln|x+1|\right)$$

$$\therefore u = \tan\left(C + \tfrac{1}{2}\ln|x+1|\right) - 1$$

$$\therefore y = x\tan\left(C + \tfrac{1}{2}\ln|x+1|\right) - x$$

$$\therefore 2\arctan(u+1) = 2C + \ln|(2x^2+2xy+y^2)^{1/2}|$$

$$\therefore \arctan(u+1) = C + \tfrac{1}{4}\ln|2x^2+2xy+y^2|$$

$$\therefore \arctan(y/x+1) = C + \tfrac{1}{4}\ln|2x^2+2xy+y^2|$$