Maths Supervision

19i. $\vec{v} = \frac{d\vec{x}}{dt}$ and so $\vec{v}$ is parallel to $d\vec{x}$

$\vec{c} \times \vec{v}$ is perpendicular to $\vec{v}$ and so is perpendicular to $d\vec{x}$

$\therefore (\vec{c} \times \vec{v}) \cdot d\vec{x} = 0$

$\therefore \int_C (\vec{c} \times \vec{v}) \cdot d\vec{x} = 0 \quad \forall C$

$\therefore W = 0$

ii a. $d\vec{x} = ((-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k})dt$

$W = \int_0^{\pi} ((\sin t)\hat{i} + (-\cos t)\hat{j} + (-1)\hat{k}) \cdot d\vec{x}$

$= \int_0^{\pi} (-\sin^2 t - \cos^2 t - 1)\, dt$

$= \int_0^{\pi} -2 \, dt$

$= -2\pi$

b. $W = \int_0^{\pi} ((\cos t)\hat{i} + (\sin t)\hat{j} + 0\hat{k}) \cdot d\vec{x}$

$= \int_0^{\pi} ((-\sin t \cos t) + (\sin t \cos t))\, dt$

$= \int_0^{\pi} 0 \, dt$

$= 0$

20. $\vec{V}$ is conservative iff $\nabla \times \vec{V} = 0$.

i. $$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y+y & xy^2+x & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + \left(\frac{\partial}{\partial x}(xy^2+x) - \frac{\partial}{\partial y}(x^2y+y)\right)\hat{k}$$

$$= \left((y^2+1) - (x^2+1)\right)\hat{k}$$

$$= (y^2 - x^2)\hat{k}$$ which is not always 0

$\therefore \vec{V}$ is not conservative

ii. $$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (ye^{xy}+2x+y) & (xe^{xy}+x) & 0 \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + \left(\frac{\partial}{\partial x}(xe^{xy}+x) - \frac{\partial}{\partial y}(ye^{xy}+2x+y)\right)\hat{k}$$

$$= \left((e^{xy} + xye^{xy} + 1) - (e^{xy} + xye^{xy} + 1)\right)\hat{k}$$

$$= \left(e^{xy}(1 + xy - 1 - xy) + 1 - 1\right)\hat{k}$$

$$= \left(e^{xy}(0) + 0\right)\hat{k}$$

$$= 0$$

$\therefore \vec{V}$ is conservative.

Let $f(x,y)$ such that $\vec{V} = \nabla f$

$$\therefore \frac{\partial f}{\partial x} = ye^{xy} + 2x + y$$

$$\frac{\partial f}{\partial y} = xe^{xy} + x$$

$\therefore f\big|_x = \int (xe^{xy} + x)\,dy = e^{xy} + xy + C(x)$

$\therefore \frac{\partial f}{\partial x} = ye^{xy} + y + C'(x) = ye^{xy} + y + 2x$

$\therefore C'(x) = 2x$

$\therefore C(x) = x^2 + A$

$\therefore f(x,y) = e^{xy} + xy + x^2 + A$

a.i. $\displaystyle\int_{C_1} P\,dx + Q\,dy = \int_0^1 (t^2 t + t)\,dt + (t t^2 + t)\,dt$

$\displaystyle = 2\int_0^1 (t^3 + t)\,dt$

$\displaystyle = 2\left[\frac{t^4}{4} + \frac{t^2}{2}\right]_0^1$

$\displaystyle = 2\left(\frac{1}{4} + \frac{1}{2}\right) - 0$

$= 3/2$

b.i. $\cancel{\displaystyle\int_{C_2} P\,dx + Q\,dy = \int_0^1 (0^2 t + t)\,dt + (0t^2 + 0)\,dt}$

$\displaystyle\int_{C_2} P\,dx + Q\,dy = \int_0^1 (0^2 t + t)\,\cancel{dt}^{\,0} + (0t^2 + 0)\,dt + (t^2 \cdot 1 + 1)\,dt + (t \cdot 1^2 + t)\,\cancel{dt}^{\,0}$

$\cancel{\displaystyle = \int_0^1 (t^2 + 3t + 1)\,dt = \left[\frac{t^3}{3} + \frac{3t^2}{2} + t\right]_0^1}$

$\cancel{\displaystyle = \frac{1}{3} + \frac{3}{2} + 1 - 0}$

$\cancel{\displaystyle = \frac{17}{6} \neq 3/2}$

$\displaystyle = \int_0^1 (t^2 + 1)\,dt = \left[\frac{t^3}{3} + t\right]_0^1 = 1/3 + 1 = \frac{4}{3} \neq 3/2$

a.ii. $\int_{C_1} P\,dx + Q\,dy = \int_0^1 (te^{t^2} + 2t + t)\,dt + (te^{t^2} + t)\,dt$

$= \int_0^1 (2te^{t^2} + 4t)\,dt$

$= \left[e^{t^2} + 2t^2\right]_0^1$

$= e + 2 - 1$

$= e + 1$

b.ii. $\int_{C_2} P\,dx + Q\,dy = \int_0^1 (te^0 + 0 + t)\,0 + (0e^0 + 0)\,dt + (e^t + 2t + 1)\,dt + (te^t + t)\,0$

$\cancel{= \int_0^1 (te^t + e^t + 5t + 1)\,dt}$

$\cancel{= \left[e^t + \frac{5}{2}t^2 + t\right]_0^1 + \int_0^1 te^t\,dt}$

$\cancel{= e + 5/2 + 1 - 1 + \int_0^1 te^t\,dt}$

	D	I
+	t	e^t
−	1	e^t
+	0	e^t

$\cancel{= e + 5/2 + \left[te^t - e^t\right]_0^1}$

$\cancel{= e + 5/2 + 1}$

$\cancel{= e + \frac{7}{2}}$

$= \int_0^1 (e^t + 2t + 1)\,dt = \left[e^t + t^2 + t\right]_0^1 = e + 2 - 1 = e + 1$