Discrete Maths 9

1.1a $\frac{}{\varepsilon} \Rightarrow \varepsilon \in L$

ε is of the form $a^n b^n$ for $n=0$

Assume $u = a^n b^n$ for some $n \in \mathbb{N}$

$\frac{u}{aub} \Rightarrow aub \in L$

$aub = aa^n b^n b = a^{n+1} b^{n+1}$
$n+1 \in \mathbb{N}$

$\therefore \forall u \in L.\ u = a^n b^n$ for some $n \in \mathbb{N}$

b. Base case: $n=0$
$a^n b^n = \varepsilon \in L$

Assume $a^k b^k \in L$ for some $k \in \mathbb{N}$

$\frac{a^k b^k}{aa^k b^k b} \equiv \frac{a^k b^k}{a^{k+1} b^{k+1}} \therefore a^{k+1} b^{k+1} \in L$

$\forall n \in \mathbb{N}.\ a^n b^n \in L$
$\therefore L = \{a^n b^n \mid n \in \mathbb{N}\}$

c. The set L' is not closed under the rule, as $\frac{a}{aab}$ but $aab \notin L'$

The smallest closed set would be $\{a^n b^n, a^{n+1} b^n \mid n \in \mathbb{N}\}$

1.2a R^+ is reflexive by the axiom

$$\frac{}{(x,x)\in R^+}\ (x\in X) \Rightarrow \forall x\in X.\ (x,x)\in R^+$$

~~Let (y,z)~~ Let $y,z\in X$ s.t. $(y,z)\in R$

RTP: $(y,z)\in R^+$

Let $x=y$

$(x,x)\in R^+$

$\therefore (x,y)\in R^+$

$$\frac{(x,y)\in R^+}{(x,z)\in R^+} \quad \therefore (x,z)\in R^+$$

$\therefore (y,z)\in R^+$

$\therefore R\subseteq R^+$

b. Base case:

$(y,y)\in R^+ \ \forall y\in X$

RTP $\forall x\in X.\ (x,y)\in R^+ \Rightarrow (x,y)\in R^+$

which is clearly true.

$\therefore (y,y)\in S$

Assume $(x',y')\in S \ (\dagger)$ and $(x',y')\in R^+$ and $(y',z')\in R$

$\therefore (x',z')\in R^+$

RTP: $(x',z')\in S$

Equivalently, $\forall x\in X.\ (x,x')\in R^+ \Rightarrow (x,z')\in R^+$

Let $x\in X$ s.t. $(x,x')\in R^+$. RTP: $(x,z')\in R^+$

By $(\dagger)$. $\forall x\in X.\ (x,x')\in R^+ \Rightarrow (x,y')\in R^+$

$\therefore (x,y')\in R^+$

$$\frac{(x,y')\in R^+}{(x,z')\in R^+}\ (x\in X.\ (y',z')\in R) \quad \therefore (x,z')\in R^+$$

$\therefore (x',z')\in S$

Assume $(x', y') \in R^+$ and $(y', z') \in R^+$

$\therefore \forall x \in X. \left((x, x') \in R^+ \Rightarrow (x, y') \in R^+ \right) \wedge \left((x, y') \in R^+ \Rightarrow (x, z') \in R^+ \right)$

$(x', y') \in R^+$
$\therefore (x', z') \in R^+$
$\therefore$ R^+ is transitive.

c. Base case:

$\dfrac{}{(x, x) \in R^+} x \in X \therefore (x, x) \in R^+ \quad \therefore (x, x) \in S$

Assume $(x, y) \in R^+ \wedge (y, z) \in R \wedge (x, y) \in S$
RTP: $(x, z) \in S$
$(y, z) \in R \Rightarrow (y, z) \in S$
$(x, y) \in S \wedge (y, z) \in S \Rightarrow (x, z) \in S$

$\therefore R^+ \subseteq S$

d. • R^+ is reflexive
• R^+ is transitive
• $R^+ \subseteq$ any other reflexive, transitive set containing R
$\therefore$ R^+ is the reflexive transitive closure of R

1.3. ~~Suppose $ab^5 \in L$~~
~~$\therefore ab^5$ must be in the form ab, au^2, or au~~
~~Case 0: $ab^5 = ab$ contradiction~~
~~Case 1: $ab^5 = au^2$~~
~~$\therefore b^5 = u^2$ contradiction~~

Case 2: $ab^5 = au$

$\cancel{\therefore ab^5}$
$\cancel{\exists\, ab^3 u}$

$$\frac{\overline{\quad ab \quad}}{\frac{ab^2}{\frac{ab^4}{\frac{ab^8 = ab^3 b^5}{ab^5}}}}$$

b. Base case

$\frac{}{ab}$. $ab = ab^n$ where $n = 2^1 - 3 \cdot 0$

Rule C:

$\frac{au}{au^2}$ Assume $au = ab^n$ where $n = 2^k - 3m \geq 0$. $k, m \in \mathbb{N}$

$au^2 = ab^{2n}$ where $2n = 2^{k+1} - 3(2m) \geq 0$. $k+1, 2m \in \mathbb{N}$

Rule 1:

$\frac{ab^3u}{au}$ Assume $ab^3u = ab^n$ where $n = 2^k - 3m \geq 0$. $k, m \in \mathbb{N}$

$au = ab^{n-3}$ where $n-3 = 2^k - 3(m+1) \geq 0$. $k, m+1 \in \mathbb{N}$

$\therefore \forall u \in L$. $u = ab^n$ for some $n = 2^k - 3m \geq 0$. $k, m \in \mathbb{N}$

c. Assume $ab^3 \in L$

$\therefore 3 = 2^k - 3m$ for some $k, m \in \mathbb{N}$

$\therefore 2^k = 3(m+1)$ but 2^k does not have 3 as a factor

Contradiction

$\therefore ab^3 \notin L$

d. $L = \{ ab^n \mid n = 2^k - 3m \geq 0 \; k, m \in \mathbb{N} \}$

2.1 a. $(0^*10^*10^*)^*$

b. ~~[illegible]~~

$(1^*01^*01^*)^*(1^*01^*)$

2.2 a. RTP $(u, a) \in \mathcal{U} \iff u = a$

"$\implies$" first

Assume $(u, a) \in \mathcal{U}$

$\therefore$ a must be in the form $a, \epsilon, r^+, r|s, r|s, rs, r^*$.

Given $a \in \Sigma$, the only possible form is "a" and so the only relevant rule/axiom is the axiom

$$\frac{}{(a, a)}$$

$\therefore u = a$

"$\impliedby$" next.

Assume $u = a$

$$\frac{}{(a, a) = (u, a)}$$

b. RTP: $(u, \epsilon) \in \mathcal{U} \iff u = \epsilon$

"$\implies$" first.

Assume $(u, \epsilon) \in \mathcal{U}$

$\therefore \epsilon$ must be in the form $a, \epsilon, r^+, r|s, r|s, rs, r^*$

$\therefore$ the only relevent axiom/rule is the axiom

$$\frac{}{(\epsilon, \epsilon)}$$

$\therefore u = \epsilon$

"$\impliedby$" next

Assume ~~[illegible]~~ $u = \epsilon$. $\quad \frac{}{(\epsilon, \epsilon) = (u, \epsilon)}$.

$\Downarrow$ 2

c. $\emptyset$ is not in the form $a, \epsilon, r^*, r|s, r|s, rs,$ or r^*
so there is no rule/axiom which might allow it to
match any string

d. RTP $(u, r|s) \in U \iff$ ~~[illegible]~~ $(u,r) \in U \vee (u,s) \in U$

~~Assum~~
"$\Rightarrow$" first Assume $(u, r|s) \in U$
$r|s$ must be in the form $a, \epsilon, r^*, r|s, r|s, rs, s^*$
so the only relevant rules/axioms are the rules

$$\frac{(u,r)}{(u,r|s)}, \quad \frac{(u,s)}{(u,r|s)}$$

$\therefore (u,r) \in U \vee (u,s) \in U$

"$\Leftarrow$" next
Assume $(u,r) \in U \vee (u,s) \in U$

Case 0: $(u,r) \in U$

$$\frac{(u,r)}{(u,r|s)}$$

Case 1: $(u,s) \in U$

$$\frac{(u,s)}{(u,r|s)}$$

e. RTP $(u, rs) \in U \iff \exists v,w \in \Sigma^* .\ u = vw \wedge (v,r) \in U \wedge (w,s) \in U$
"$\Rightarrow$" first
Assume $(u, rs) \in U$
rs must be ~~[illegible]~~ of the form $a, \epsilon, r^*, rs, r|s, rs, r^*$
$\therefore$ the only relevant rules/axioms is the rule

$$\frac{(v,r) \qquad (w,s)}{(vw, rs)}$$

$\therefore \exists v,w \in \Sigma .\ u = vw \wedge (v,r) \in U \wedge (w,s) \in U$

"$\Leftarrow$" next

Assume $\exists v, w \in \Sigma.\ u = vw \wedge (v,r) \in \mathcal{U} \wedge (w,s) \in \mathcal{U}$

Let v, w s.t. ______ $\because$ ______

$$\frac{(v,r) \qquad (w,s)}{(vw, rs) = (u, rs)}$$

f. RTP: $(u, r^*) \in \mathcal{U} \iff u = \varepsilon \vee (u,r) \in \mathcal{U} \vee (\exists u_1, u_2, \ldots .\ u = u_1 u_2 \ldots$
$\wedge (u_1, r) \in \mathcal{U} \wedge (u_2, r) \in \mathcal{U} \ldots)$

"$\Rightarrow$" first

Assume $(u, r^*) \in \mathcal{U}$

r^* must be in the form $a, \epsilon, r^*, r|s, r|s, rs, r^*$

$\therefore$ the only relevant rules / axioms are

$$\overline{(\varepsilon, \epsilon)} \quad \text{and} \quad \frac{(u, r) \quad (v, r^*)}{(uv, r^*)}$$

Case 0: $\overline{(\varepsilon, \epsilon)}$ $\therefore u = \varepsilon$

Case 1: $\dfrac{(u_1, r) \qquad (v, r^*)}{(u_1 v, r^*)}$

$\therefore u = u_1 v$ s.t. $(u_1, r) \in \mathcal{U} \wedge (v, r^*) \in \mathcal{U}$

By symmetry, $v = u_2 v'$ s.t. $(u_2, r) \in \mathcal{U} \wedge (v', r^*) \in \mathcal{U}$
or $(v, r) \in \mathcal{U}$
or $v = \varepsilon$

$\therefore u = u_1 u_2 u_3 \ldots$ s.t. $(u_1, r) \in \mathcal{U}, (u_2, r) \in \mathcal{U}, \ldots$

$\square$

"$\Leftarrow$" next

Assume $u = \varepsilon \vee (u, r) \in \mathcal{U} \vee (\exists u_1, u_2, \ldots, u = u_1 u_2 \ldots \wedge (u_1, r) \in \mathcal{U} \wedge$

Case 0: $u = \varepsilon$

$\overline{(\varepsilon, r^*)}$

Case 1: $(u, r) \in \mathcal{U}$

$$\frac{(u, r) \qquad \overline{(\varepsilon, r^*)}}{(u, r^*)}$$

Case 2:

Let $u_1, u_2, \ldots$ s.t. $u = u_1 u_2 \ldots$

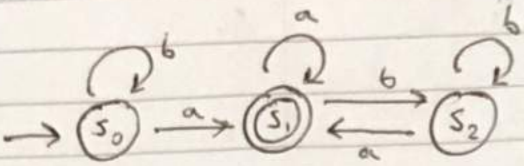
and $(u_1, r) \in \mathcal{U} \wedge (u_2, r) \in \mathcal{U} \ldots$

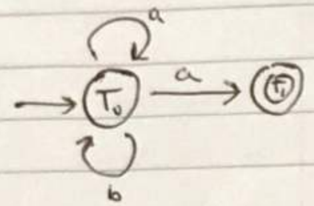
$$\frac{(u_{n-2}, r) \qquad \dfrac{(u_{n-1}, r) \qquad \dfrac{(u_n, r) \qquad \overline{(\varepsilon, r^*)}}{(u_n, r^*)}}{(u_{n-1} u_n, r^*)}}{(u_{n-2} u_{n-1} u_n, r^*)}$$

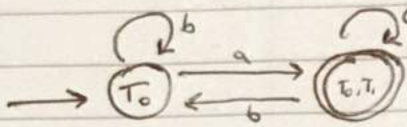
$$\vdots$$

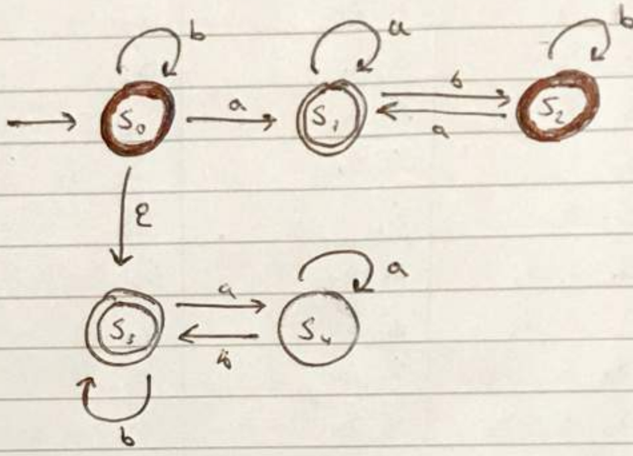
$$\overline{(u_1 u_2 \ldots u_{n-1} u_n, r^*) = (u, r^*)}$$

23 Let $r = b^* a \, (b^* \cdot a)^*$
Let $s = (a|b)^* a$

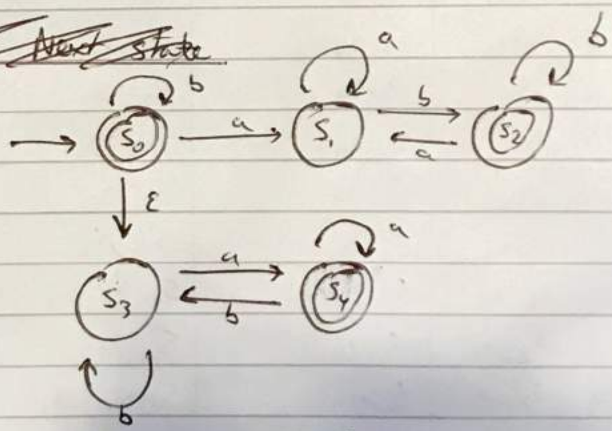
$NFA^{\varepsilon}(r) =$  $= DFA(r)$

$NFA^{\varepsilon}(s) =$ 

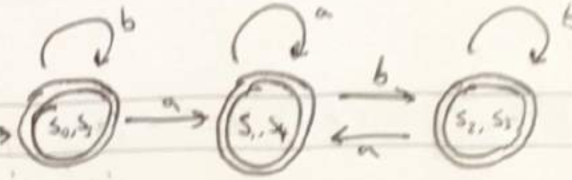
$DFA(s) =$ 

$NFA^{\varepsilon}(\neg r \vee \neg s) =$ 

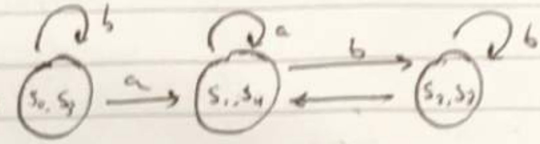
~~State Input Next state~~

$NFA^{\varepsilon}(\neg r \vee \neg s) =$ 

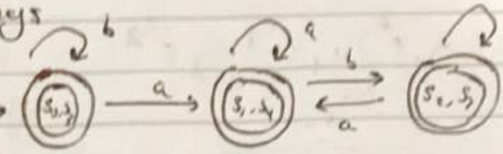
State	Next state (a)	Next state (b)
$\emptyset$	$\emptyset$	$\emptyset$
S_0	S_1, S_4	S_0, S_3
S_1	S_1	S_2
S_2	S_1	S_2
S_3	S_4	S_3
S_4	S_4	S_3
S_0, S_1	S_1, S_4	S_0, S_2, S_3
S_0, S_2	S_1, S_4	S_0, S_2, S_3
S_0, S_3	S_1, S_4	S_0, S_3
S_0, S_4	S_1, S_4	S_0, S_3
S_1, S_2	S_1	S_2
S_1, S_3	S_1, S_4	S_2, S_3
S_1, S_4	S_1, S_4	S_2, S_3
S_2, S_3	S_1, S_4	S_2, S_3
S_2, S_4	S_1, S_4	S_2, S_3
S_3, S_4	S_4	S_3
S_0, S_1, S_2	S_1, S_4	S_0, S_2, S_3
S_0, S_1, S_3	S_1, S_4	S_0, S_2, S_3
S_0, S_1, S_4	S_1, S_4	S_0, S_2, S_3
S_0, S_2, S_3	S_1, S_4	S_0, S_2, S_3
S_0, S_2, S_4	S_1, S_4	S_0, S_2, S_3
S_0, S_3, S_4	S_1, S_4	S_0, S_3
S_1, S_2, S_3	S_1, S_4	S_2, S_3
S_1, S_2, S_4	S_1, S_4	S_2, S_3
S_1, S_3, S_4	S_1, S_4	S_2, S_3
S_2, S_3, S_4	S_1, S_4	S_2, S_3
S_0, S_1, S_2, S_3	S_1, S_4	S_0, S_2, S_3
S_0, S_1, S_2, S_4	S_1, S_4	S_0, S_2, S_3
S_0, S_1, S_3, S_4	S_1, S_4	S_0, S_2, S_3
S_0, S_2, S_3, S_4	S_1, S_4	S_0, S_2, S_3
~~[illegible]~~	~~[illegible]~~	
S_1, S_2, S_3, S_4	S_1, S_4	S_2, S_3
S_0, S_1, S_2, S_3, S_4	S_1, S_4	S_0, S_2, S_3

$\therefore DFA(r \vee {\sim}s) =$ 

$\therefore DFA({\sim}(r \vee {\sim}s))$

$= DFA({\sim}r \,\&\, s) =$ 

has no ~~termination~~ accepting states and so accepts no strings

$DFA({\sim}r \vee s) =$ 

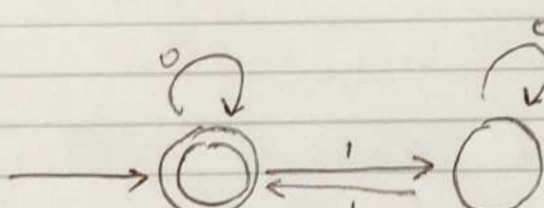
$\therefore DFA({\sim}({\sim}r \vee s))$

$= DFA(r \,\&\, {\sim}s)$ similarly accepts no strings

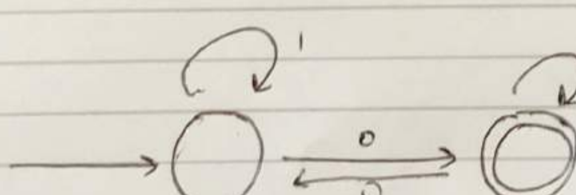
$\therefore \mathcal{L}({\sim}r \,\&\, s) = \emptyset = \mathcal{L}(r \,\&\, {\sim}s)$

$\therefore r \equiv s, \quad \mathcal{L}(r) = \mathcal{L}(s)$

3.1 a.



b.



3.2. RTP: $(q \overset{u}{\Rightarrow} q') \iff (q, u, q')$ can be inductively defined by the rules

" $\Longrightarrow$ " first

assume $q \overset{u}{\Rightarrow} q'$

Let $u = u_0 \varepsilon^{e_0} u_1 \varepsilon^{e_1} u_2 \varepsilon^{e_2} \ldots$ ~~...~~

where $\forall n.\, u_n \in \Sigma \wedge e_n \in \mathbb{N}$

Let $q_0 \xrightarrow{u_0} q_1 \underbrace{\xrightarrow{\varepsilon} q_2 \xrightarrow{\varepsilon} q_3 \xrightarrow{\cdots} q_k}_{e_0} \xrightarrow{u_1} q_{k+1} \underbrace{\xrightarrow{\varepsilon} q_{k+2} \xrightarrow{\cdots} q_j}_{e_1} \xrightarrow{u_2} q_{j+1} \xrightarrow{\cdots} q_n$

where $q = q_0$, $q' = q_n$

$$\frac{}{(q, \varepsilon, q_0)}$$
$$\frac{}{(q, u_0, q_1)}$$
$$e_0 \left\{ \begin{array}{l} \overline{(q, u_0, q_2)} \\ \overline{(q, u_0, q_3)} \\ \vdots \\ \overline{(q, u_0, q_k)} \end{array} \right.$$
$$\frac{}{(q, u_0 u_1, q_{k+1})}$$
$$e_1 \left\{ \begin{array}{l} \overline{(q, u_0 u_1, q_{k+2})} \\ \vdots \\ \overline{(q, u_0 u_1, q_j)} \end{array} \right.$$
$$\frac{}{(q, u_0 u_1 u_2, q_{j+1})}$$
$$\vdots$$
$$\frac{}{(q, u, q_n)}$$
$$\frac{}{(q, u, q')}$$

" $\Longleftarrow$ " next.

RTP: $\forall (q,u,q')$ defined by the rules, $q \stackrel{u}{\Longrightarrow} q'$

Base case:

$$\frac{}{(q,\varepsilon,q)} \quad q \Longrightarrow q$$

Rule 0:

Assume $q \stackrel{u}{\Longrightarrow} q' \wedge q' \stackrel{\varepsilon}{\longrightarrow} q''$ $(\dagger)$

$$\frac{(q,u,q')}{(q,u,q'')}$$

RTP $q \stackrel{u}{\Longrightarrow} q''$ which is true by $(\dagger)$

Rule 1:

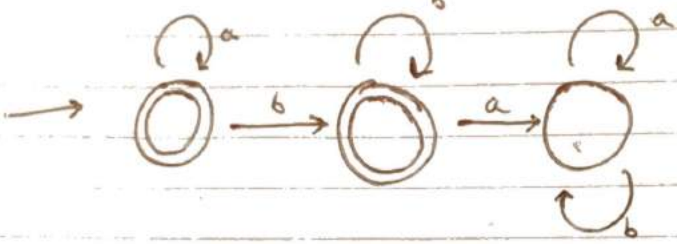
Assume $q \stackrel{u}{\Longrightarrow} q' \wedge q' \stackrel{a}{\longrightarrow} q''$ $(\dagger)$

$$\frac{(q,u,q')}{(q,ua,q'')}$$

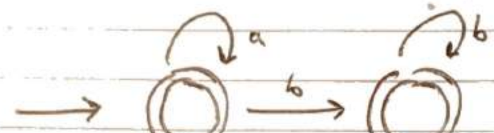
RTP $q \stackrel{ua}{\Longrightarrow} q''$ which is true by $(\dagger)$

$\square$

3.3



DFA

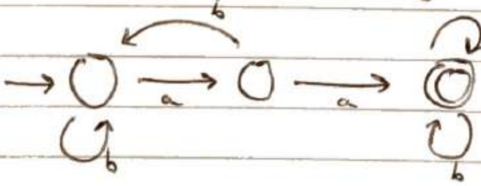


NFA

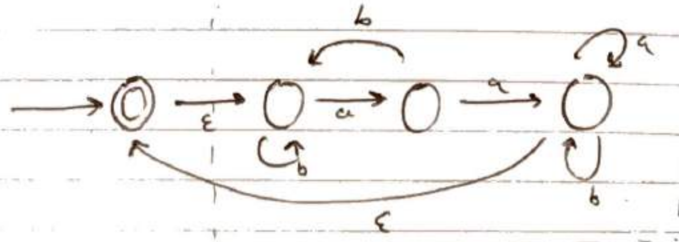
4.1 There may have been other ways of reaching the start state of M which we would not want to satisfy Star(M)

Adding the extra state with an epsilon transition makes the acceptance "one-way" i.e. just because you end up at the start state it does not guarantee that the string was accepted.

Eg. M:

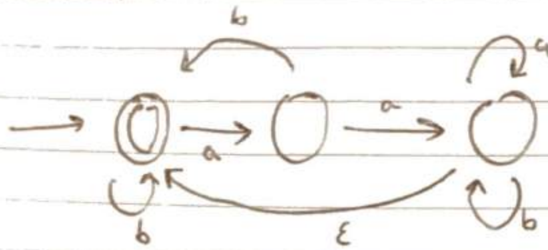


~~matches aa(a|b)~~
matches $(a|b)^*aa(a|b)^*$



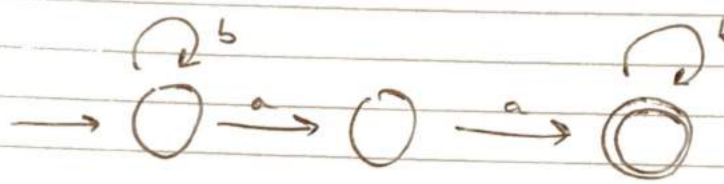
matches
$((a|b)^*aa(a|b)^*)^*$
as required

However,



is satisfied by "b" and "ab" and so clearly is not
$((a|b)^* aa (a|b)^*)^*$

4.2



4.3 Let S be a regular language of finite size s.t. $S = \{s_0, s_1, s_2, s_3, \ldots s_n\}$ for some $n \in \mathbb{N}$

$\forall n \in \mathbb{N}$. s_n is finite. let l_n be the length of s_n.

$s_n = \sigma_0 \sigma_1 \sigma_2 \cdots \sigma_{l_n - 1}$ where $\forall k \in \mathbb{N}.\ \sigma_k \in \Sigma$

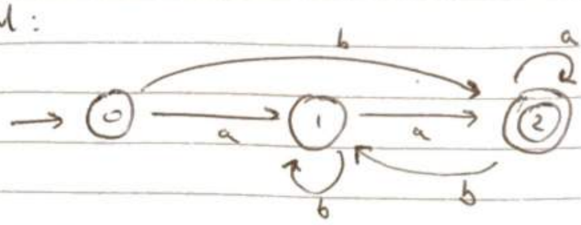
$\therefore r_n =$ the regular expression matching s_n
$= \sigma_0 \sigma_1 \sigma_2 \ldots \sigma_{l_n - 1}$

$\therefore$ Let $r = r_0 | r_1 | r_2 | r_3 | \ldots | r_n$

$x \in \mathcal{L}(r) \iff x \in S$
$\therefore S = \mathcal{L}(r)$
$\therefore S$ is regular

4.4

M :



$r = r_{0,2}^{\{0,1,2\}}$

$L(r_{0,2}^{\{0,1,2\}}) = L(r_{0,2}^{\{0,2\}} \mid r_{0,1}^{\{0,2\}} (r_{1,1}^{\{0,2\}})^* r_{1,2}^{\{0,2\}})$

$r_{0,2}^{\{0,2\}} = b$

$r_{0,1}^{\{0,2\}} = a$

$L(r_{1,1}^{\{0,2\}}) = L(r_{1,1}^{\{0\}} \mid r_{1,2}^{\{0\}} (r_{2,2}^{\{0\}})^* r_{2,1}^{\{0\}})$

$r_{1,1}^{\{0\}} = b$

$r_{1,2}^{\{0\}} = a$

$r_{2,2}^{\{0\}} = a$

$r_{2,1}^{\{0\}} = b$

$L(r_{1,2}^{\{0,2\}}) = L(r_{1,2}^{\{0\}} \mid r_{1,0}^{\{0\}} (r_{0,0}^{\{0\}})^* (r_{0,2}^{\{0\}}))$

$r_{1,2}^{\{0\}} = a$

$r_{1,0}^{\{0\}} = \emptyset$

$r_{0,0}^{\{0\}} = \emptyset$

$r_{0,2}^{\{0\}} = b$

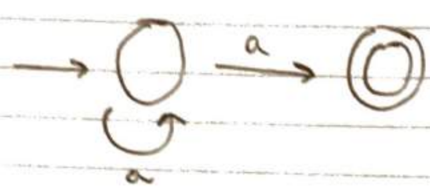
$\therefore\ r_{1,2}^{\{0,2\}} = a \mid \emptyset\emptyset^* b = a$

$r_{1,1}^{\{0,2\}} = b \mid aa^* b = a^* b$

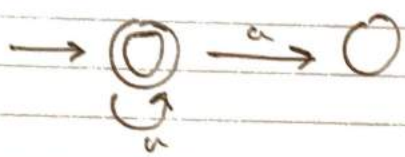
$r_{0,2}^{\{0,1,2\}} = b \mid a(a^* b)^* a$

$\therefore\ r = b \mid a(a^* b)^* a$

4.5 $\Sigma = \{a\}$



M



$Not(M)$

$\mathcal{L}(M) = \mathcal{L}(aa^*)$
$\mathcal{L}(Not(M)) = \mathcal{L}(a^*)$

$\{u \in \Sigma^* \mid u \notin \mathcal{L}(M)\} = \{\varepsilon\} \neq \mathcal{L}(a^*) = \mathcal{L}(Not(M))$

For example

$a \in \mathcal{L}(Not(M))$
$a \notin \{u \in \Sigma^* \mid u \notin \mathcal{L}(M)\}$

4.6 $r = (a|b)^* ab (a|b)^*$

matches any string with a consecutive "ab"

$\therefore$ In $\sim r$, any "a" must be followed by another "a" or be at the end of the string

$\therefore$ All the "b"s must appear before the first "a"

$\therefore \sim r = b^* a^*$

4.7. Let $u \in \Sigma^*$

RTP: $u \in L(M) \iff u \in L(M_1) \wedge u \in L(M_2)$

"$\Rightarrow$" first.

Assume $u \in L(M)$

Equivalently, $(s_1, s_2) \overset{u}{\Longrightarrow} (f_1, f_2)$ in M for some $f_1 \in F_1, f_2 \in F_2$

Let that path be

$$(s_1, s_2) \xrightarrow{u_1} (q_1^1, q_2^1) \xrightarrow{u_2} (q_1^2, q_2^2) \xrightarrow{u_3} (q_1^3, q_2^3) \rightarrow \cdots \xrightarrow{u_n} (f_1, f_2)$$

Let $s_1 = q_1^0$, $s_2 = q_2^0$, $f_1 = q_1^n$, $f_2 = q_2^n$

where $u = u_1 u_2 u_3 \cdots u_n$

without loss of generality, consider the individual transition

$$(q_1^k, q_2^k) \xrightarrow{u_{k+1}} (q_1^{k+1}, q_2^{k+1})$$

$\therefore \delta((q_1^k, q_2^k), u_{k+1}) = (q_1^{k+1}, q_2^{k+1}) = (\delta_1(q_1^k, u_{k+1}), \delta_2(q_2^k, u_{k+1}))$

$\therefore \delta_1(q_1^k, u_{k+1}) = q_1^{k+1} \wedge \delta_2(q_2^k, u_{k+1}) = q_2^{k+1}$

$\therefore q_1^k \xrightarrow{u_{k+1}} q_1^{k+1}$ in M_1

and $q_2^k \xrightarrow{u_{k+1}} q_2^{k+1}$ in M_2

Similarly for the other transitions:

$s_1 \xrightarrow{u_1} q_1^1 \xrightarrow{u_2} q_1^2 \rightarrow \dots \xrightarrow{u_n} f_1$ in M_1

$s_2 \xrightarrow{u_1} q_2^1 \xrightarrow{u_2} q_2^2 \rightarrow \dots \xrightarrow{u_n} f_2$ in M_2

$\therefore s_1 \overset{u}{\Rightarrow} f_1$ in M_1 $\therefore u \in L(M_1)$

$s_2 \overset{u}{\Rightarrow} f_2$ in M_2 $\therefore u \in L(M_2)$

$\square$

" $\Leftarrow$ " next

Assume $u \in L(M_1) \wedge u \in L(M_2)$

$\therefore s_1 \overset{u}{\Rightarrow} f_1$ for some $f_1 \in F_1$ in M_1
$s_2 \overset{u}{\Rightarrow} f_2$ for some $f_2 \in F_2$ in M_2

Let these paths respectively be

$s_1 \xrightarrow{u_1} q_1^1 \xrightarrow{u_2} q_1^2 \rightarrow \dots \xrightarrow{u_n} f_1$ in M_1
$s_2 \xrightarrow{u_1} q_2^1 \xrightarrow{u_2} q_2^2 \rightarrow \dots \xrightarrow{u_n} f_2$ in M_2

where $s_1 = q_1^0$, $s_2 = q_2^0$, $f_1 = q_1^n$, $f_2 = q_2^n$
where $u = u_1 u_2 u_3 \dots u_n$

Considering two individual transitions:

$q_1^k \xrightarrow{u_{k+1}} q_1^{k+1}$ in M_1

$q_2^k \xrightarrow{u_{k+1}} q_2^{k+1}$ in M_2

$\therefore \delta_1(q_1^k, u_{k+1}) = q_1^{k+1} \wedge \delta_2(q_2^k, u_{k+1}) = q_2^{k+1}$

$\therefore (\delta_1(q_1^k, u_{k+1}), \delta_2(q_2^k, u_{k+1})) = (q_1^{k+1}, q_2^{k+1})$
$= \delta((q_1^k, q_2^k), u_{k+1})$

$\therefore$ the transition $(q_1^k, q_2^k) \xrightarrow{u_{k+1}} (q_1^{k+1}, q_2^{k+1})$ exists in M

Likewise for all $0 \leq k \in \mathbb{N} \leq n$

the path $(s_1, s_2) \xrightarrow{u_1} (q_1^1, q_2^1) \xrightarrow{u_2} (q_1^2, q_2^2) \rightarrow \dots \xrightarrow{u_n} (f_1, f_2)$
exists in M

$\therefore (s_1, s_2) \overset{u}{\Longrightarrow} (f_1, f_2)$ in M

$\therefore u \in L(M)$

$\square$

5.1 Suppose such an M exists

Let $M = DFA(Q, \{a,b,c\}, \delta_M, s_M, F)$

Let $M' = DFA(Q, \{a,b\}, \delta_{M'}, \delta_M(s_M, c), F)$

where $\delta_{M'} = \{(q, \sigma) \in \delta_M \mid \sigma \in \{a,b\}\}$

$\forall n \in \mathbb{N} \geq 0, \quad ca^n b^n \in \mathcal{L}(M)$

$\therefore s_M \xrightarrow{c} \delta_M(s_M, c) \xRightarrow{a^n b^n} f$ for some $f \in F$ in M

$\therefore \delta_M(s_M, c) \xRightarrow{a^n b^n} f$ for some $f \in F$ in M'

$\therefore \{a^n b^n \mid n \in \mathbb{N} \geq 0\} \subseteq \mathcal{L}(M')$

$\forall x \in \{a,b,c\}^*. \ x \in L(M) \Rightarrow x \in \{c^m a^n b^n \mid m \geq 1 \ \& \ n \geq 0\} \ \vee$
$x \in \{a^m b^n \mid m, n \geq 0\}$

$\therefore \forall x \in \{a,b,c\}^*. \ x \in L(M) \wedge x = cy$ for some $y \in \{a,b,c\}^*$
$\Rightarrow x \in \{c^m a^n b^n \mid \geq 1 \ \& \ n \geq 0\}$
$\Rightarrow x = c^m a^n b^n$ for some $m \geq 1 \ \& \ n \geq 0$

$\therefore \forall x \in \{a,b,c\}^*, y \in \{a,b\}^*. \ x \in L(M) \wedge x = cy \Rightarrow y = a^n b^n \quad n \geq 0$

$\mathcal{L}(M') \subseteq \{y \mid \forall x \in \{a,b,c\}^*, y \in \{a,b\}^*. \ x \in L(M) \wedge x = cy\}$

$\therefore \mathcal{L}(M') \subseteq \{a^n b^n \mid n \in \mathbb{N} \geq 0\}$

$\therefore \mathcal{L}(M') = \{a^n b^n \mid n \in \mathbb{N} \geq 0\}$

$\forall l \in \mathbb{N} \geq 1$, $a^l b^l$ is of length $\geq l$

Let $u_1 = a^k$ with $k < l$
Let $v = a^{l-k}$
Let $u_2 = b^l$

such that $a^l b^l = u_1 v u_2 \in \mathcal{L}(M')$
$|u_1 v| = l \leq l$
$|v| \geq 1$

However, $u_1 v^0 v_2 = u_1 v_2 = a^k b^l \notin \mathcal{L}(M')$
$\therefore \mathcal{L}(M')$ is not regular
$\therefore$ there can exist no such M

$\square$