Part 1: High-frequency trading in a limit order book

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Introduction

The "High-frequency trading in a limit order book" paper by Avellaneda&Stoikov is a comprehensive exploration of the strategies employed by dealers in securities markets for optimal submission of bid and ask orders in the limit order book. It delves into the dealer's risks, including

- **Inventory risk** arising from uncertainty in the asset's value by the end of the trading period.
- **Asymmetric information risk** arising from informed traders.

The paper is broken down into sections discussing the dynamics of the mid-market price, the agent’s utility objective, the arrival rate of orders, and an analysis of the agent's optimal bid and ask quotes. The study finishes with a numerical simulation and performance comparison of the dealer's inventory strategy introduced by the authors against a classical symmetric strategy.

By melding utility frameworks from previous studies with microstructure insights from econophysics, the paper establishes a method for deriving optimal bid and ask quotes in two steps: First, the dealer computes a personal indifference valuation for the stock based on the current inventory, then they calibrate their bid and ask quotes considering the probability of trade execution in relation to their distance from the mid-price. This approach allows dealer to maximise their P&L profile by the end of the limted period T with low variance.

The article attracted my interest due to my understanding of market makers' operations in Web3. Unlike in TradFi where they primarily earn from spreads, in Web3, their main role is to impact markets and drive price changes.

Mid-price and Reservation price

The paper begins by distinguishing between two kinds of prices and outlining the differences that exist between them in terms of application to Avellaneda&Stoikov market model. The mid-price isl be used solely to value the agent’s assets at the end of the investment period, while the reservation price is subjective to a dealer at each point of time based on their inventory position and allows them to execute better trades.

Mid-Price

In the paper the mid-price is calculated using the formula

$$\mathrm{d}S_u = \sigma \mathrm{d}W_u$$

- S_t = s - initial value .
- Wt is a standard one dimensional Brownian motion.
- σ is a constant.

It's important to note that the paper addresses the random process for generating the mid-price as the prices are being produced. If we were to simulate the algorithm with predetermined prices, the mid-price would be calculated as the average of the bid and ask prices, denoted as $(bid + ask)/2$.

Reservation price

A reservation price serves as a modification to the mid-price, **reflecting the amount of inventory the agent holds**. If the agent has a long position in the stock (q>0), the reservation price is set lower than the mid-price, signifying an intention to liquidate the inventory by selling the stock. If the agent has a short position in the stock (q<0), the reservation price exceeds the mid-price, as the agent is prepared to purchase the stock at a premium.

To formalise the reservation price for agent v the authors introduce a concept of a utility function of a dealer which is to maximise their P&L by the end of a given period. For simplier explanation Avellaneda&Stoikov firstly introduce the utility function for a dealer without a capability of actually execute trades.

$$v(x, s, q, t) = E_t \left[- \exp(-\gamma \,(x + qS_T)\right] \text{(3)}$$

- $-\gamma$ is a risk-aversion parameter
- $(x + qS_t)$ is the total wealth of the agent

Hence, the bid reservation price and ask reservation price can be determined using the following calculations.

- Bid price

$$v\left(x - r^b(s, q, t), s, q + 1, t\right) = v(x, s, q, t) \text{ (4)}$$

In this case, the agent if indifferent between buying an asset and keeping the current position.

- Ask price

$$v\left(x + r^b(s, q, t), s, q - 1, t\right) = v(x, s, q, t) \text{ (5)}$$

In this case, the agent if indifferent between selling an asset and keeping the current position.

After some calculations with equations (3), (4), (5), authors arrive to equations or reservation bid and reservation ask prices.

$$r^a(s, q, t) = s + (1 - 2q)\frac{\gamma\sigma^2(T - t)}{2} \qquad r^b(s, q, t) = s + (-1 - 2q)\frac{\gamma\sigma^2(T - t)}{2}$$

And the reservation indifference price $(r^a + r^b)/2$

$$r(s, q, t) = s - q\gamma\sigma^2(T - t)$$

Core factors influencing the reservation prices

- A risk-aversion parameter γ
- Time until the trading session ends $(T - t)$
- Distance of a dealer's current inventory position to a target position q

But how do we take into account the fact that the time of each trading session is limited and the **dealer wants to hold q assets by the end of each time session to avoid inventory risks**?

Reservation price (infinite horizon)

Intuitively, the closer our agent is to time T, the less risky his inventory in stock is, since it can be liquidated at the mid- price S_t.

$$\bar{v}(x,s,q)=E\left[\int_0^{\infty}-\exp(-\omega t)\exp(-\gamma\left(x+qS_t\right))\mathrm{d}t\right]$$

where ω is interpreted as an upper bound on the inventory position our agent is allowed to take

- $\bar{v}(x,s,q)$ represents the value of the agent's portfolio, which depends on the wealth x, the mid-price s, and the inventory of stocks q.
- The integral $\int_0^{\infty}$ represents summing over time from the current moment (0) to infinity. However, the expectation E and the exponential decay factor exp(-ωt) ensure that the most significant contribution is from the time near T.
- The exponential term $\exp(-\gamma\left(x+qS_t\right))$ reflects the risk-averse nature of the agent. Here, γ is a risk aversion parameter, and $x+qS_t$ represents the total wealth of the agent, including the value of the inventory of stocks. This term decreases with increasing wealth, indicating that holding a large inventory becomes less attractive as the agent's wealth increases.

Hence, the reservation prices can be calculated as follows

$$\bar{r}^a(s,q)=s+\frac{1}{\gamma}\ln\left(1+\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)\ \bar{r}^b(s,q)=s+\frac{1}{\gamma}\ln\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)$$

Limit orders and trading intensity

After determining the reservation prices, also referred to as "fair prices", it's essential for the dealer to understand at what level relative to the market price to place the orders. For this, the concept of limit orders needs to be considered, as well as comprehending how frequently our orders will actually be executed

Limit orders

Authors determine distances of agent's order to market price as follows

$$\delta^b=s-p^b \qquad \delta^a=p^a-s$$

- p^b and p^a represent the bid price pb and the ask price accordingly

Authors make the assumption that market buy orders will execute our agent's sell limit orders at a Poisson rate $\mathrm{Poisson}(\delta^a)$, which is a diminishing function its parameter. Similarly, sell orders will execute against the agent's buy limit order at a Poisson rate $\mathrm{Poisson}(\delta^b)$, a function that also decreases. This suggests that **the more distant the agent's quotes are from the mid-price, the less frequently he will be on the receiving end of buy and sell orders.**

The objective function of a dealer who can put limit bids&asks to the market

$$u(s,x,q,t)=\max_{\delta^a,\delta^b}E_t\left[-\exp(-\gamma\left(X_T+q_TS_T\right))\right]$$

But how do we know the right shape of curves $\mathrm{Poisson}(\delta^a)$ and $\mathrm{Poisson}(\delta^b)$? The authors advise referring to the results of econophysics studies.

The trading intensity

Upon examining the outcomes of research on market microstructure, the authors put forth a function that estimates the likelihood that a dealer's market order will be executed depended on it's distance from the mid_price δ

$$\text{Poisson}(\delta) = A * exp(-k * \delta)$$

Solution: Finding optimal bids&asks

The authors deduced the ideal bid and ask quotes through a two-tiered process. The first phase involves solving the Hamilton–Jacobi–Bellman Partial Differential Equation to procure the reservation bid and ask prices, denoted as $r^b(s,q,t)$ and $r^a(s,q,t)$ respectively. Subsequently, the second phase is to determine the ideal distances, notated as $\delta^b(s,q,t)$ and $\delta^a(s,q,t)$, between the mid-price and the optimal bid and ask quotes.

Afer taking into account the previously explained assumption that $\text{Poisson}(\delta)$= A * $exp(-k * \delta)$

Authors come up with **formulas for the indifferent reservation price and spread.**

$$r(s,t) = s - q\gamma\sigma^2(T-t) \qquad \delta^a + \delta^b = \gamma\sigma^2(T-t) + \frac{2}{\gamma}\ln\left(1 + \frac{\gamma}{k}\right)$$

Simulation

To validate their proposed model, the authors implemented a simulation that models a dealer's actions with prices formulated based on Brownian motion random process.

The simulation exercise incorporated a range of risk aversion parameters [0.01, 0.1, and 1] among other parameters. Avellaneda&Stoikov executed 1000 simulation runs to contrast the inventory strategy (their newly proposed method) and the symmetric strategy (a method that does not factor in the current inventory balance when setting orders).

The outcomes showed that the inventory strategy exhibited significantly less variance compared to the symmetric strategy. This approach resulted in **lower standard deviations of profits and final inventory, though it yielded slightly less robust profits compared to the corresponding symmetric strategy.**

Conclusion

In conclusion, Avellaneda and Stoikov proposed a novel method for dealers to place optimal orders aimed at maximizing net profits while avoiding inventory risk. To mitigate these risks, the **authors present an algorithm that places bids at every timestep, considering the current asset balance.**

All in all, according to Avellaneda&Stoikov, the process of calculation of optimal bid and ask prices consists of two phases

1. Calculate the reservation price based on what is the target inventory
2. Calculate the optimal bid and ask spread

- Bid offer price = reservation price - optimal spread / 2
- Ask offer price = reservation price + optimal spread / 2

The conducted simulation demonstrated that **this algorithm exhibits lower variance in q at the end of the period, thus substantiating its effectiveness in lowering inventory risk.**

Resources

- https://math.nyu.edu/~avellane/HighFrequencyTrading.pdf
- https://blog.hummingbot.org/2021-04-avellaneda-stoikov-market-making-strategy/
- https://www.youtube.com/watch?v=S7eig5VXFpY&t=816s

Part 2: Enhancing Trading Strategies with Order Book Signals

Introduction

The article by A 1varo Carteaa, Ryan Donnellyb, Sebastian Jaimungalc proposes an approach to enhace trading strategies by implementing of a volume imbalance factor which considers difference between buy and sell orders in a limit order book. When the volume imbalance leans toward buying (selling), it is highly likely that the forthcoming MO corresponds to a buy (sell) order. Moreover, right after a buy (sell) MO, if the volume imbalance is skewed towards buying (selling), the midprice adjustment is notably positive (negative) in its size and direction.

Weaknesses of Stoikov's strategy

According to the article, Stoikov's strategy have several important weak points

1. Authors state that depending on equity, only 0.1% - 8.4% orders have a chance to be executed in the book deeper than the best price. This results into suffering from the adverse selection problem. Also, such strategy performs almost as always like posting at best quotes due to the fact that a small part of orders is executed deeper than the best levels.
2. Market-making tactics, which aim to profit from the spread by consistently setting two-sided LOs at top quotes, also face significant challenges from adverse selection.

The main outcome is that by adding information about volumes authors mainly solve the adverse selection problem.

Volume imbalance

Authors express the volume imbalance at a given time `t` by the formula:

$$\rho_t = \frac{V_b - V_a}{V_b + V_a}$$

where

$\rho_t \in [-1, 1]$.

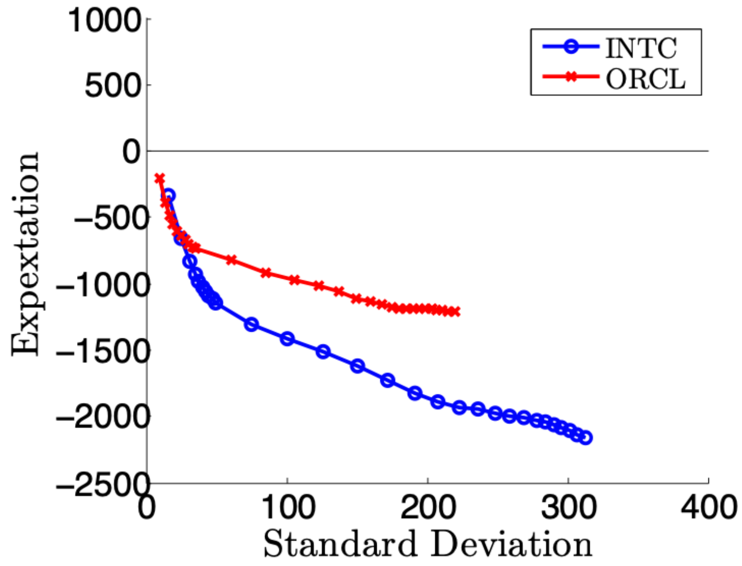
V_b and V_a represent the volumes of the limit orders at the best bid and best ask at time `t` respectively.

The volume imbalance is a pivotal metric as it captures the traders' desire towards purchasing or selling assets.

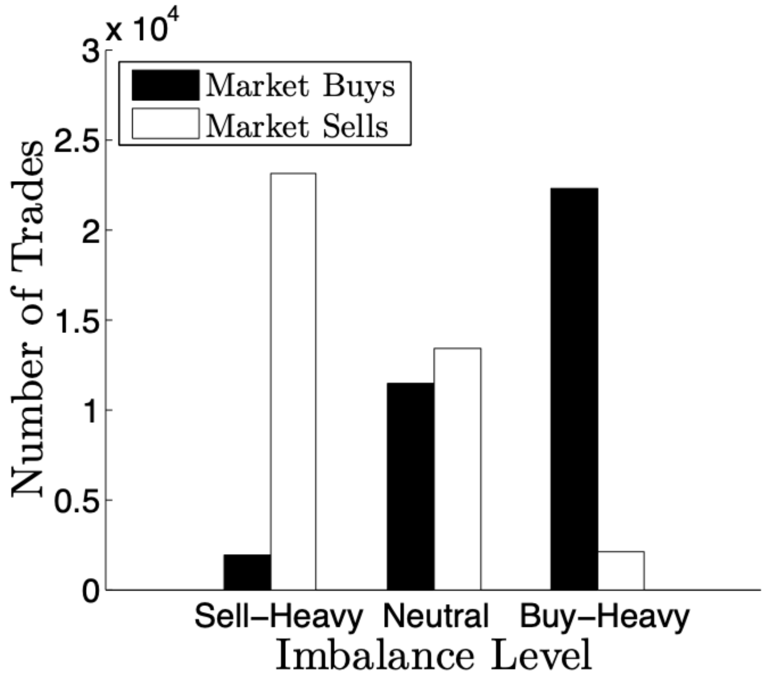
Considering example of trading strategies

The authors also examine the tactics of market makers using zero-intelligence strategies, wherein they simply place their two-sided orders at optimal prices, aiming to profit from spreads. The distinction between the two approaches lies in the order size. In the initial strategy, the order size is set to 1, whereas in the subsequent strategy, the order size is adjusted to match the size of a market order.

The obvious result of these strategy is net losses due to vulnerability to adverse selection.



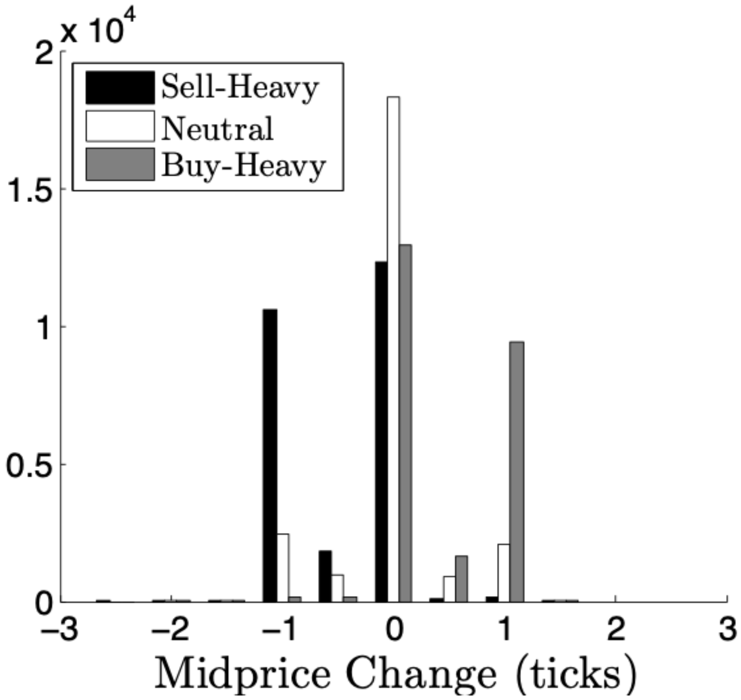
Authors also prove that the probabilities of incoming buy/sell market orders are highly dependeded on the on volume imbalance parameter [Sell heavily / Neutral / Buy heavily]



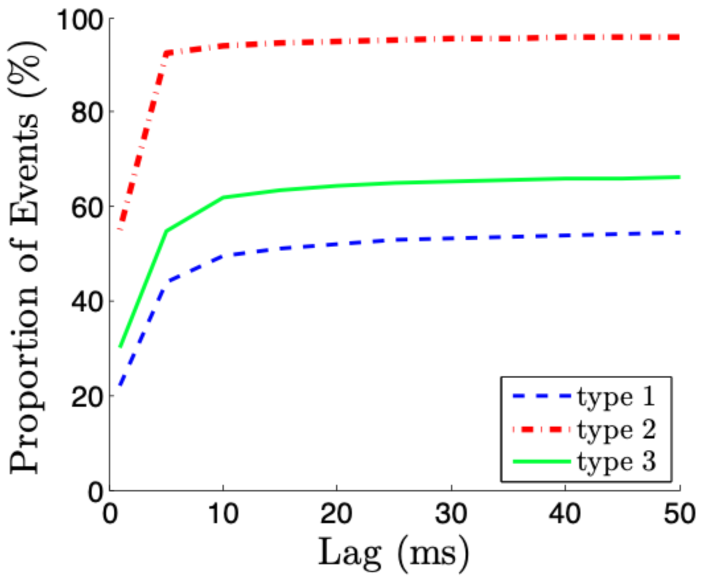
Implementation of volume imbalance

Typically, a strategy that leverages extra insights to anticipate trade occurrences and price trends can effectively mitigate adverse selection risks and benefit from price dynamics.

Furthermore, authors state that empirical data proof dependecies between volume imbalance and future price change



Moreover, the most of changes in Limit Order Book happen just after the execution of a MO



where

- Type 1: placed or cancelled.
- Type 2: placed or cancelled at a price equal to or better than the current best bid or ask.
- Type 3: placed or cancelled and causes the midprice to change.

Strategy construction and backtesting

The authors, through their modeling, present a trading strategy that considers volume imbalances in the Limit Order Book. This strategy comes with parameters that ought to be tailored specifically for each market. Authors further provide good results of its' backtesting.

Conclusion

Thus, A ̀lvaro Carteaa, Ryan Donnellyb, and Sebastian Jaimungalc offer a new perspective on devising trading strategies. By considering volume imbalances, their approach leads to increased profitability and more effectively mitigates the adverse selection issue.