# Options pricing methologies

Review: Litasov Aleksandr

## | Introduction: Intuition of options pricing

Before delving into the mathematics of option pricing, it's essential to first grasp the rationale behind it and the fundamental ideas that back the pricing of options. To get a clear perspective on this, we need to explore some key principles.

#### 1. Risk-free portfolio

Using the option and its dynamic hedge, one can theoretically create a 'risk-free' portfolio. For instance:

- If one sels a call option and hold a certain number of the underlying shares (determined by the option's delta), movements in the stock's price will be offset by changes in the option's value.
- Thus, whether the stock price rises or falls, the total value of this portfolio remains relatively stable over infinitesimally small time intervals.

#### 2. Equation to risk-free rate

Since this portfolio is 'risk-free', its return should be equivalent to the risk-free rate of interest available in the market. If it were not, traders can exploit this difference for risk-free profit, creating an arbitrage opportunity.

#### 3. No arbitrage principle in options pricing

If options were mispriced relative to the underlying stock or currency, traders could construct arbitrage strategies to achieve risk-free profits which shouldn't happen on a market.

## | Types of options

### American vs European

The main difference between American and European options is that while European options can only be exercised at expiration, American options can be exercised at any time before or on the expiration day. This added flexibility often results in American options having a higher price than the European.

## | Methods of pricing

---

### 1. Binomial tree model

---

The binomial tree method is a core model for pricing options, rooted in the principle of no-arbitrage. Let's delve into its fundamentals of it

Note: In this article, I'll cover the essential mathematics needed to describe the model, bypassing detailed proofs crucial for a complete understanding of the model's math.

## || One-step model

Let's begin by supposing we have a portfolio valued at $S_o \cdot \Delta - f$

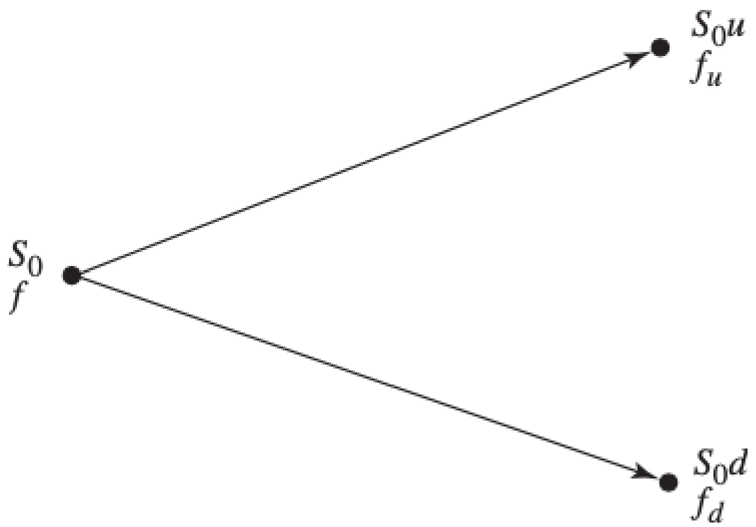
where

$S_0$ - the price of the stock

$\Delta$ - number of stocks/currencies held in the portfolio

$f$ - option's price

In a simplified single-step binomial model, the stock value can either increase by $u$ or decrease by $d$.




As a result, the value of the portfolio can be whether $[S_o \cdot u \cdot \Delta - f_u]$ or $[S_o \cdot d \cdot \Delta - f_d]$

To create a risk-free portfolio we need to equialise the outcomes

$$S_o \cdot u \cdot \Delta - f_u = S_o \cdot d \cdot \Delta - f_d \Rightarrow \Delta = \frac{f_u - f_d}{S_o \cdot u - S_o \cdot d}$$

To establish the fair value of the option, we need to **equate the present value of the risk-free portfolio with the expense of building that risk-free portfolio**. By denoting risk-free rate as $r$,

$$S_o \cdot \Delta - f = (S_o \cdot u \cdot \Delta - f_u) \cdot e^{-rT}$$

resulting in

$$f = e^{-rT}[pf_u + (1-p)f_d] - discounted\ expected\ payoff\ (1)$$

$$p = \frac{e^{-rT} - d}{u - d} - probability\ of\ an\ asset\ going\ up\ in\ a\ riskfree\ world$$

**Example**

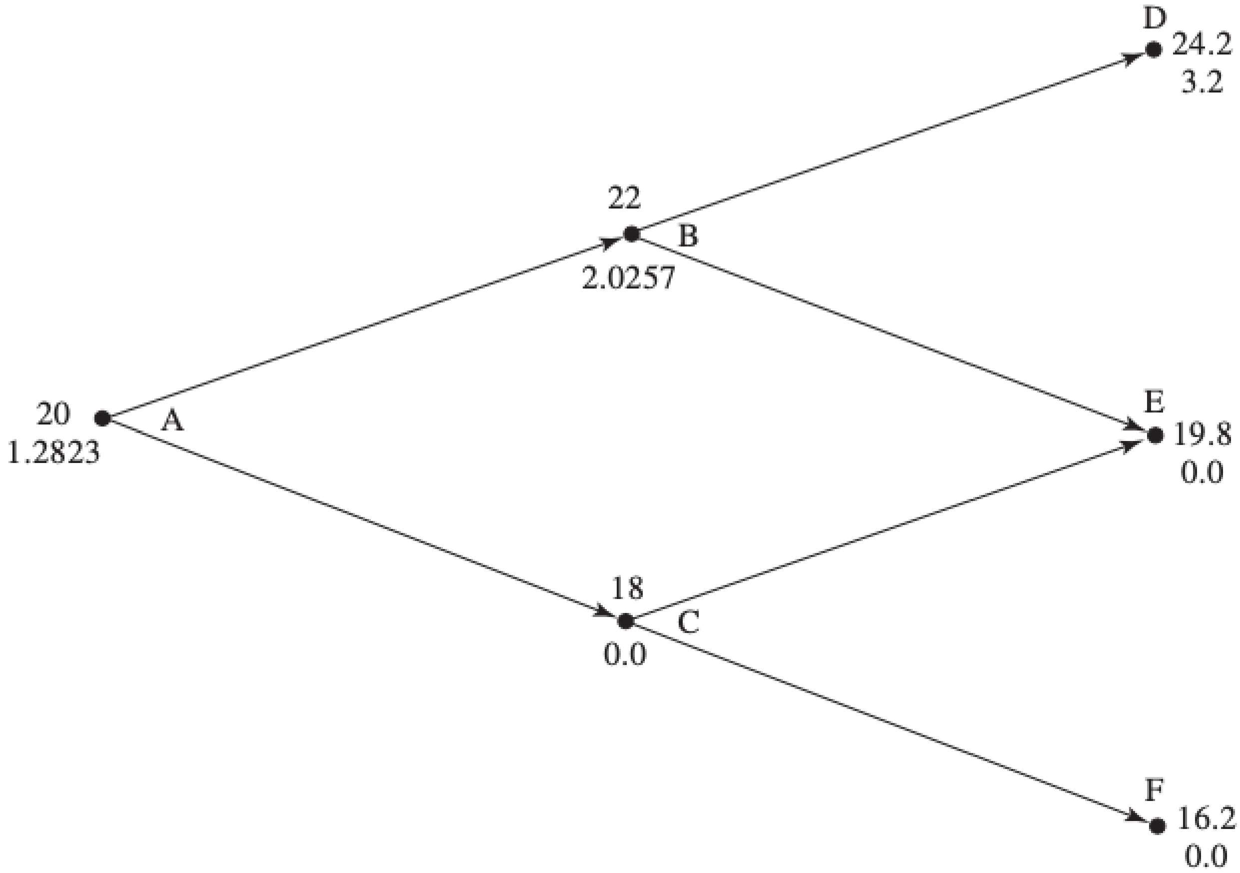
In case $u = 1.1, d = 0.9, r = 0.12, T = 0.25, f_u = 1, and\, f_d = 0$

$$p = \frac{e^{-0.12 \cdot 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

$$\Rightarrow f = e^{-0.12 \cdot 0.25}[0.6523 \cdot 1 + (1 - 0.6523) \cdot 0] = 0.633$$

### || Two-step model (European options)

In the two-step model the node structure is used. For example, D point represents the stock going up in the first timestamp and up in the second timestamp. On the other hand, the F point represents going down two times.




To determine the option's price at point B, we must utilize the same approach as in the one-step model, referencing prices from points D and C. The price at a point C is equal to 0 due to the fact that it's lower than 20.

With the known prices at points B and C, we can then ascertain the starting price at point A, following the procedure outlined in the one-step model.

The final price of an european call option is

$$f = e^{-0.12 \cdot 0.25}[0.6523 \cdot 2.0257 + (1 - 0.6523) \cdot 0] = 1.2823$$

The computation for the put option adheres to the same procedure, bearing in mind that its value is 0 when the price ascends and grows as the price descends.

### || Two-step model (American options)

For American options, the two-step model becomes slightly more complicated due to the fact that the option can be exercised at any point before expiration. This requires us to compare the immediate exercise value of the option at each node with the value of holding and potentially exercising the option in the future.

$$f = max(expected\ payoff, strike - stock\ price)$$

Except for that, the model is exactly the same as the European version of it.

## || Dividends payout (stock options) and foreign rate (currency options)

**Dividends payout**

For stocks that distribute dividends, the combined return from both dividends and capital appreciation in a risk-neutral environment is denoted as $r$. From this, the dividends yield a return represented by $q$. This implies that the capital appreciation alone contributes a return of $r - q$. When the initial stock value is $S_0$, its projected value after a specific time interval $\Delta t$ is calculated as $S_0 \times e^{(r-q)\times\Delta t}$.

As a result, the futures price for a dividends paying stock is

$$F = S \cdot e^{(r-q)\cdot T}$$

**Foreign risk-free rate**

For currency options, the expected return from holding a foreign currency in a risk-neutral environment is primarily determined by the difference between the domestic risk-free rate, $r$, and the foreign risk-free rate, $r_f$. In this scenario, the foreign risk-free rate can be seen as playing a role analogous to the dividend yield in equities. This means that the anticipated exchange rate change alone contributes a return of $r - r_f$. If you begin with an amount in domestic currency equivalent to $S_0$ units, the projected value of this foreign currency, when converted back to domestic currency after a duration of $\Delta t$, is expressed as $S_0 \times e^{(r-r_f)\times\Delta t}$.

As a result, the futures price for a currency is

$$F = S \cdot e^{(r-r_f)\cdot T}$$

## || Generalisation

The review has focused on one-step and two-step models. Nevertheless, through mathematical induction, we can extend these models to an indefinite time span. As documented in literature, it's common practice to utilize a 30-step model for option pricing, ensuring minimal error. This would produce a tree with $2^{30}$ nodes.

---

## 2. Black-Scholes model

---

Another popular method for determining European options prices is the Black - Scholes model. The model can be derived from constructing a system of differential equasions related to no-arbitrage portfolio or derived from binomial tree model with large about of steps. Let's look at the end-formulas

For a European call option:

$$C = S_0N(d1) - Ke^{-rT}N(d2)$$

For a European put option:

$$P = Ke^{-rT}N(-d2) - S_0N(-d1)$$

Where:

$$d1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d2 = d1 - \sigma\sqrt{T}$$

Where:

- $S_0$ is the current stock price.
- $K$ is the strike price.
- $r$ is the risk-free rate.
- $\sigma$ is the volatility of the stock price.
- $T$ is the time to maturity.
- $N$ is the cumulative distribution function for a standard normal distribution.

### || Intuition behind the formula

The model assumes that it's possible to create a portfolio that replicates the option's payoffs. By continuously adjusting the position in the underlying stock, the risk from price movements can be offset. Therefore, the option's price can be determined by the cost of setting up and maintaining this replicating portfolio.

- $N(d2)$ can be considered as a probability of options' execution in a risk-neutral world
- $S_0N(d1)e^{rT}$ is the value of an asset at time T (before discounting)
- The higher volatility is, the larger volatity increasing the call option's price, because an asset with higher volatility can reach the exercise price with higher probability
- The time value T increases the asset's value while decreasing the probability of option's execution leadidng to lower price

---

## 3. Monte Carlo simulations

---

Conducting Monte Carlo simulations is another approach to calculate option's price. The idea unlike in previous methods where strict formulas are used is to

1. Simulate a future price on an asset in a risk-neutral world.
2. Calculate payoff from the derivative (repeat n_simulations times).
3. Calculate mean of the payoff (expected payoff). (that's very similar to bootstrap).
4. Discount the expected payoff at risk-free rate to get an option's price.

### || Price behavior simulation

To simulate future price of an asset we assume that

$$dS = \mu^r Sdt + \sigma Sdz$$

where

$dz$ is a Wiener process.

$r$ is the expected return in a risk-neutral world.

and $\sigma$ is the volatility.

Going through some math we arrive at

$$S_T = S_0 \times e^{(r-0.5\times\sigma^2)\times T+\sigma\times\sqrt{T}\times z}$$

## | Conclusion

The methodologies reviewed in this article, namely the binomial tree model, Black-Scholes model, and Monte Carlo simulations, provide unique vantage points for approaching this options pricing.

- Binomial model is a step-by-step, discrete-time view of how the option's price evolves over time.
- Black-Scholes model offers a formula that instantaneously provides the option's price given a set of parameters.
- Lastly, Monte Carlo simulations take us into the probabilistic realm, where the option's life is simulated multiple times

Valuation of American options is a slightly more complex process as the possibility of early exercise must be taken into account and not all model can do it. Options on currencies must take into account rate differentials to value them