SIAM J. SCI. STAT. COMPUT.
Vol. 5, No. 1, March 1984

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0196-5204/84/0501-0004 $01.25/0

FINITE DIFFERENCE METHODS FOR THE STOKES AND NAVIER-STOKES EQUATIONS*

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Abstract. This paper presents a new finite difference scheme for the Stokes equations and incompressible Navier-Stokes equations for low Reynolds number. The scheme uses the primitive variable formulation of the equations and is applicable with nonuniform grids and nonrectangular geometries. Several other methods of solving the Navier-Stokes equations are also examined in this paper and some of their strengths and weaknesses are described. Computational results using the new scheme are presented for the Stokes equations for a region with curved boundaries and for a disk with polar coordinates. The results show the method to be second-order accurate.

Key words. finite differences, Navier-Stokes equations, Stokes equations

AMS(MOS) subject classifications. 65N05, 76D05

1. Introduction. In this paper we examine several common methods for solving the incompressible Navier-Stokes equations by finite differences and we present a new second-order accurate finite difference scheme for these equations. This new scheme is designed to be applied with nonuniform grids and nonorthogonal coordinate systems. Numerical experiments with the Stokes equations illustrate the versatility and accuracy of the scheme.

The steady-state Stokes equations on a domain Ω in $\mathbb{R}^n$ are given by

$$-\nabla^2\mathbf{u}+\nabla p=\mathbf{f}(x), \qquad \nabla\cdot\mathbf{u}=g(x) \tag{1.1}$$

and the steady-state Navier-Stokes equations are

$$-R^{-1}\nabla^2\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f}(x), \qquad \nabla\cdot\mathbf{u}=g(x) \tag{1.2}$$

where R is the Reynolds number. We will consider the systems (1.1) and (1.2) with Dirichlet boundary conditions

$$\mathbf{u}(x)=\mathbf{b}(x) \quad \text{on } \partial\Omega. \tag{1.3}$$

A necessary condition for (1.1) or (1.2) to have a solution is that the data $g(x)$ and $\mathbf{b}(x)$ satisfy the integrability condition

$$\int_\Omega g=\int_{\partial\Omega}\mathbf{b}\cdot\mathbf{n}, \tag{1.4}$$

where $\mathbf{n}$ is the outer unit normal to $\partial\Omega$. For the mathematical theory of the systems (1.1) and (1.2) we refer to Ladyzhenskaya (1963) and Temam (1979).

We will be concerned only with methods that solve the systems (1.1) and (1.2) in the primitive variables $\mathbf{u}$ and p and not with methods such as the vorticity and stream-function reformulation. Also our methods are applicable in two or three dimensions although our examples will be only in two dimensions.

We emphasize that the scheme presented here is designed to be easily applicable with nonrectangular geometries and nonuniform grids. The vast majority of papers on the numerical solution of the incompressible Navier-Stokes equations limit

* Received by the editors May 17, 1982, and in revised form October 20, 1982. This research was sponsored by the United States Army under contract DAAG29-80-C-0041. Portions of this research were performed under NASA contracts NAS-15810 and NASI-16394 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

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themselves to examples using rectangular geometry and uniform grids. By way of contrast, computations with the compressible Navier–Stokes equations routinely use nonrectangular geometries and nonuniform grids.

The scheme proposed in this paper uses central finite differences with a regularizing term. Chorin (1967), (1968), (1969) proposed schemes with standard central finite differences for the time-dependent Navier–Stokes equations. There are several difficulties with central differencing schemes and the scheme presented here avoids them. The first difficulty is that with central differences the grid decomposes into two staggered grids which are coupled only at the boundaries and this can cause the pressure field to be oscillatory (see for example Patterson and Imberger (1980) and Ghia et al. (1977) reporting on the method of Chorin (1968) and the paper by Zoby in Rubin and Harris (1975) reporting on the method of Chorin (1967)). Since the pressure is of significant importance in engineering applications this oscillatory behavior of the pressure is undesirable. Another difficulty is with the possible nonexistence of a solution to the finite difference equations. We discuss this point in § 4. The regularized central difference scheme proposed here removes the grid decoupling and thus the pressure oscillations and, furthermore, the incompressibility constraint is modified to guarantee the existence of a solution.

The most common finite difference methods for the Navier–Stokes equations are the staggered mesh schemes. They avoid the grid decoupling of the standard central differencing schemes but require more care when used with nonrectangular geometry. These schemes are discussed in more detail in § 3.

In § 5 results are given of the regularized central scheme being used to solve the Stokes equations on a nonrectangular region and the results are shown to be second-order accurate in both the velocities and the pressure. To our knowledge no other finite difference scheme for the Stokes or Navier–Stokes equations in the primitive variables has been shown to be second-order accurate for nonrectangular geometry. For rectangular geometry with uniform grid one can give proofs of the convergence of schemes, e.g. Chorin (1969), Temam (1979), Kzivickii and Ladyzhenskaya (1966), but these proofs fail for nonrectangular geometry (see § 4). Chorin (1967), (1968) gives results of using central differences for rectangular regions but the results are for only one grid spacing.

Peskin (1977) has used Chorin's method with moving boundaries which are superimposed on a rectangular grid and Viecelli (1971) has a similar method for staggered grid schemes. Liu and Krause (1979) have developed a staggered grid scheme for general geometry. However, the order of accuracy of these schemes has not been demonstrated. Thompson (1980) used central differencing with nonrectangular geometry but also used the elliptic equation for the pressure (see § 2), which makes the accuracy uncertain.

The outline of the remaining sections of the paper is as follows. In § 2 we discuss the strengths and weaknesses of some common approaches to solving the systems (1.1) and (1.2) and in § 3 we discuss finite difference schemes for these systems. The finite difference integrability condition is discussed in § 4, and computational results are given in § 5. The numerical examples of § 5 demonstrate that the new scheme given here can be used to give second-order accurate solutions to the Stokes equations for nonrectangular geometries. To our knowledge no other finite difference schemes for the Stokes or incompressible Navier–Stokes equations in the primitive variables have been shown to be second-order accurate for nonrectangular geometries. Computations using the new scheme for the incompressible Navier–Stokes equations are currently being made and will be reported when complete.

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2. Solution techniques. In this section we review some approaches to solving the Navier–Stokes and Stokes equations numerically. Few researchers have treated the system (1.2) in the given form, most have altered it in some way. Before examining the altered forms of (1.2) we look at the system in the given form.

The Stokes equations (1.1) and the Navier–Stokes equations (1.2) are elliptic systems of $n+1$ equations in $n+1$ dependent variables. The definition of an elliptic system, as given by Douglis and Nirenberg (1957), requires that the determinant of the principal symbol of the system not vanish for nonzero values of dual variables. For the Navier–Stokes equations the determinant of the principal symbol is

$$\det\begin{pmatrix} (1/R)|\xi|^2 I_n & i\boldsymbol{\xi} \\ i\boldsymbol{\xi}^T & 0 \end{pmatrix} = R^{-(n-1)}|\xi|^{2n}, \tag{2.1}$$

which is nonzero for $|\xi| \neq 0$. Moreover, since the determinant is a polynomial of degree $2n$ in the variables $\xi = (\xi_1, \cdots, \xi_n)$ the system requires n boundary conditions at each point of the boundary (Agmon, Douglis and Nirenberg (1964)). These boundary conditions will usually be Dirichlet or Neumann conditions on the velocity $\mathbf{u}$.

One of the most common ways of modifying the Navier–Stokes equations (1.2) is to replace it by the system

$$\begin{aligned} -R^{-1}\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f}(x), \\ \nabla^2 p = \nabla\cdot\mathbf{f} + R^{-1}\nabla^2 g - \sum_{i,j} u^i_{x_j}u^j_{x_i} &- \mathbf{u}\cdot\nabla g. \end{aligned} \tag{2.2}$$

The last equation of (2.2) is obtained by taking the divergence of the first equation of (1.2) and then using the last equation of (1.2) to eliminate the divergence of velocity. The system (2.2) has the advantage over (1.2) in that, when discretized, it can be solved using standard methods for inverting the discrete Laplacian. However, the system (2.2) has a grave disadvantage in that it requires $n+1$ boundary conditions, one for each elliptic equation, as opposed to (1.2) which requires only n boundary conditions. Thus any attempt to solve (1.2) via (2.2) would require some means of determining the correct additional boundary condition. Without the correct boundary condition solutions of (2.2) will not be solutions of (1.2).

Roache (1972, p. 194) suggests that the additional boundary condition be given by the normal derivative of pressure as determined by the first equation of (1.2) or (2.2) evaluated on the boundary. This, however, is not satisfactory as a boundary condition since it is not independent of the system of differential equations. Roache's suggestion leaves the system (2.2) underdetermined.

Another boundary condition which is commonly used along boundaries corresponding to physical surfaces is to set the normal derivative of the pressure to zero, which is valid in the limit for high Reynolds number flow. With this boundary condition and (1.3) the system (2.2) has the proper number of boundary conditions, however, its solutions are not solutions of (1.2).

As one would expect, the methods using (2.2) or similar systems have difficulty with the accuracy of the pressure field and with satisfying the incompressiblity condition on the velocities (see for example the work by Boney, Hefner, Hirsh and Zoby reported in Rubin and Harris (1975)).

The above mentioned difficulties are seen in computations with the time-dependent Navier–Stokes equations as well. Roache (1972) has a discussion of the difficulties of obtaining a zero divergence for the velocity field when using the above approach for time-dependent flows (see also Harlow and Welch (1964)).

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Because of these difficulties, it seems best not to use the derived system (2.2) but to use the original system (1.2). This is also the approach used by Chorin (1968).

Another approach to solving the Navier–Stokes equations (1.2) is the artificial compressiblity method. The basic idea of this method is to solve a time-dependent system of equations, whose steady-state solutions solve (1.2), until a steady state is reached. Methods have been proposed by Chorin (1967) and Yanenko (1967). The convergence rate of these methods is dependent on the choice of finite difference method used to solve the system. Moreover, as will be discussed in § 4, it may happen that the finite difference equations do not have a steady-state solution, so the method cannot converge. Taylor and Ndefo (1970) reported difficulty in getting Yanenko's method to converge, most likely because there was no solution.

Another common method is to use the "parabolized" Navier–Stokes equations which the second-derivatives in the stream-wise direction are removed. Because of its limited applicability and uncertain justification we will not discuss this method here except to note that often an analogue of (2.2) is derived and thus some of our observations on (2.2) also apply to the parabolized equations. Raithby and Schneider (1979) discuss these difficulties for three-dimensional flow problems.

3. Finite difference schemes. In this section we discuss the staggered mesh and central finite difference schemes for (1.1) and (1.2) and introduce a new scheme. The second-order accurate staggered mesh scheme for a uniform Cartesian grid assigns the values of each of the velocity components and the pressure to different interlaced grids. In two dimensions with velocity components u and v, one may assign values of u to grid locations $((i+\frac{1}{2})h, jh)$, values of v to $[ih, (j+\frac{1}{2})h)$, and values of p to (ih, jh) (see e.g. Harlow and Welch (1965), Patankar and Spalding (1972), Raithby and Schneider (1979), Brandt and Dinar (1979)). This method works very well as long as the geometry is rectangular and the grid is uniform. Nonuniform grids and grid mapping techniques cannot be conveniently handled, although Liu and Krause (1979) have developed a staggered mesh scheme for use with general geometries.

The staggered mesh schemes also have some difficulty at boundaries. For example, when both velocity components are specified at a boundary then that velocity component whose mesh lines do not lie on the boundary requires some special treatment.

The central difference scheme on a uniform rectangular mesh assigns values of all the variables to each grid point. The divergence and gradient operators are approximated using central differences and the Laplacian is approximated by the standard five-point discrete Laplacian. Central difference schemes have been used by Chorin (1967), (1968) in time-dependent calculations.

An important concept for finite difference schemes for elliptic systems such as (1.1) and (1.2) is that of regularity (see Bube and Strikwerda (1983), and also Frank (1968), Brandt and Dinar (1979)). Regular schemes give rise to regularity estimates analogous to those in the theory of elliptic systems of differential equations. Solutions to regular difference schemes will in general be smoother than solutions to nonregular schemes and also will be more accurate approximations to the solutions of the differential equations.

The central difference scheme is nonregular (Bube and Strikwerda (1983)), which results in nonsmooth solutions. The lack of smoothness is most noticeable in the pressure. The staggered mesh scheme is regular. The advantage of the central difference scheme is that it is easily implemented with nonuniform grids as introduced by coordinate changes.

It should be emphasized that none of the difficulties mentioned above are insurmountable. Both the staggered mesh and central differencing schemes have been used

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and often quite successfully. However we will consider a new scheme which incorporates both regularity and ease of implementation with coordinate grid mapping techniques.

Before introducing the new scheme we will discuss the concept of regularity for difference schemes as given in Bube and Strikwerda (1983). A difference operator A may be written as

$$Af(x)=\sum_{\mu} a_{\mu}(h, x)T^{\mu}f(x),$$

where T^{μ} is the translation operator given by

$$T^{\mu}f(x_{\nu})=f(x_{\nu+\mu})$$

for multi-indices ν and μ.

The symbol of A is given by

$$a(h, x, \zeta)=\sum_{\mu} a_{\mu}(h, x)\, e^{i\mu\cdot\zeta}.$$

For example, the first-order central difference operator in the kth coordinate direction has symbol

$$\frac{e^{i\zeta_k}-e^{-i\zeta_k}}{2h_k}=ih_k^{-1}\sin\zeta_k,$$

and the standard second-order accurate Laplacian in n variables has the symbol

$$-\sum_{k=1}^{n} 4h_k^{-2}\sin^2\tfrac{1}{2}\zeta_k.$$

A finite difference scheme for the Stokes equations is regular elliptic if the determinant of the matrix of symbols of the scheme vanishes only for ζ equal to zero modulo 2π. For the Stokes equations with central differencing, and $\Delta x=\Delta y=h$, this determinant is

$$\det\begin{pmatrix} 4h^{-2}(\sin^2\frac{1}{2}\zeta_1+\sin^2\frac{1}{2}\zeta_2) & 0 & ih^{-1}\sin\zeta_1 \\ 0 & 4h^{-2}(\sin^2\frac{1}{2}\zeta_1+\sin^2\frac{1}{2}\zeta_2) & ih^{-1}\sin\zeta_2 \\ ih^{-1}\sin\zeta_1 & ih^{-1}\sin\zeta_2 & 0 \end{pmatrix} \tag{3.1}$$

$$=4h^{-4}(\sin^2\tfrac{1}{2}\zeta_1+\sin^2\tfrac{1}{2}\zeta_2)(\sin^2\zeta_1+\sin^2\zeta_2).$$

This determinant vanishes for the dual variables ζ_1 and ζ_2 equal to π, and thus the scheme is not regular. One sees that the nonregularity comes from the form of the differencing used for the gradient and divergence terms. Our new scheme is a modification of the central differencing scheme so as to make the scheme regular.

The new scheme we consider will be called the regularized central difference scheme. In this scheme the derivatives of pressure are approximated as

$$\frac{\partial p}{\partial x_k}\simeq\delta_{k0}p-\alpha h_k^2\delta_{k-}\delta_{k+}^2 p \tag{3.2}$$

and the first derivatives of the velocity in the divergence equation are approximated as

$$\frac{\partial u^k}{\partial x_k}\simeq\delta_{k0}u^k-\alpha h_k^2\delta_{k+}\delta_{k-}^2 u^k, \tag{3.3}$$

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where α is a nonzero constant and δ_{k0}, δ_{k+} and δ_{k-} are the centered, forward, and backward divided differences, respectively. The Laplacian is approximated with the usual five-point scheme. For a Cartesian grid in two dimensions the determinant of the symbol is

$$\det\begin{pmatrix} 4h^{-2}(\sin^2\frac{1}{2}\zeta_1+\sin^2\frac{1}{2}\zeta_2) & 0 & d(\zeta_1) \\ 0 & 4h^{-2}(\sin^2\frac{1}{2}\zeta_1+\sin^2\frac{1}{2}\zeta_2) & d(\zeta_2) \\ -\overline{d(\zeta_1)} & -\overline{d(\zeta_2)} & 0 \end{pmatrix}$$

$$=4h^{-2}(\sin^2\tfrac{1}{2}\zeta_1+\sin^2\tfrac{1}{2}\zeta_2)(|d(\zeta_1)|^2+|d(\zeta_2)|^2),$$

where

$$\begin{aligned} d(\zeta) &= ih^{-1}\sin\zeta-\alpha h^{-1}e^{1/2i\zeta}(2i\sin\tfrac{1}{2}\zeta)^3 \\ &= 2ih^{-1}\sin\tfrac{1}{2}\zeta(\cos\tfrac{1}{2}\zeta+4\alpha\, e^{1/2i\zeta}\sin^2\tfrac{1}{2}\zeta). \end{aligned}$$

since $d(\zeta)$ is not zero for any nonzero value of ζ, when α is nonzero, the scheme is regular. Note that for α equal to one-sixth the approximations (3.2) and (3.3) are third-order accurate.

Since the regularized central difference scheme is a variant of the central difference scheme it is easy to implement with coordinate maps. At those boundary points where the correction term would require points beyond the boundary we use the correction term which interchanges the forward and backward operators. This scheme also requires the use of extrapolation to compute the pressure values on the boundary. It has been found that third order extrapolation gave quite good results, e.g.

$$p_{0j}=3p_{1j}-3p_{2j}+p_{3j} \tag{3.4}$$

at the boundary $x=0$ in two dimensions.

The use of the extrapolation boundary condition (3.4) should not be confused with the discussion in § 2 about the extra boundary condition for the system of differential equations (2.2). The boundary condition (3.4) is required only by the finite difference scheme and is not required by the system of differential equations (1.1) or (1.2). Therefore the solution of the system (1.1) or (1.2) will not satisfy any extra nontrivial condition analogous to (3.4), even though the difference approximations will satisfy (3.4). Other extrapolations may be used in place of (3.4); however, if the order of extrapolation is too low the accuracy may be degraded at the boundary.

A number of first-order accurate schemes for the Stokes and Navier–Stokes equations have been presented e.g. Kzivickii and Ladyzhenskaya (1966) and Temam (1979, p. 48). In this paper we are concerned only with second-order accurate schemes.

4. The integrability condition. Each of the schemes for the Stokes equations which have been discussed in the previous section can be written as

$$(4.1)\qquad \begin{aligned} &\text{(a)} \qquad L_h\mathbf{u}_h+\mathbf{G}_hp_h=\mathbf{f}_h, \\ &\text{(b)} \qquad \mathbf{D}_h\cdot\mathbf{u}_h=g_h, \end{aligned} \qquad \text{on } \Omega_h,$$

with Dirichlet boundary conditions

$$\mathbf{u}_h=\mathbf{b}_h \quad \text{on } \partial\Omega_h.$$

The difference operators L_h, $\mathbf{G}_h$, and $\mathbf{D}_h$ are approximations to the differential operators in (1.1). The discrete functions $\mathbf{f}_h$, g_h, and $\mathbf{b}_h$ are approximations to $\mathbf{f}$, g and $\mathbf{b}$ on the mesh Ω_h, where h is some measure of the fineness of the mesh Ω_h.

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Now let us compare the system (4.1) with the system (1.1). First note that if $\mathbf{G}_h$ is a consistent approximation to the gradient then the discrete pressure p_h is determined only up to a constant. This means that the system of linear equations (4.1) does not have full column rank. If there are as many equations in (4.1) as there are unknowns, and this is the case for each scheme we have considered, then the system (4.1) does not have full row rank either. This implies that there is a constraint which the data must satisfy to guarantee a solution; in particular, the discrete integrability condition analogous to (1.4) must be satisfied.

There are at least two ways to satisfy the discrete integrability condition. The first method would be to analyze the matrix corresponding to (4.1) and determine the null space of the adjoint matrix. If the data is constrained to be orthogonal to this null space then a solution will exist. This approach is impractical for many situations, especially if coordinate changes have been employed, since then the matrices are not easy to analyze.

A second approach, which will be adopted here, is to replace (4.1b) by

$$\mathbf{D}_h \cdot \mathbf{u}_h = g_h + \delta_h \tag{4.1b$'$}$$

where δ_h is a constant chosen to guarantee a solution. The value of δ_h must be determined as part of the solution. As shown in the examples in § 5 δ_h is at least $O(h^2)$ for the regularized central scheme. We will refer to the equations (4.1a, b′, c) as (4.2′).

Another way of looking at the condition (4.1b′) is as follows. As shown by Temam (1979) and others, any discrete divergence operator D_h, defined only on the interior of the grid, has a corresponding gradient operator $\mathbf{G}'_h$ defined by

$$(D_h\mathbf{u}, \phi) + (\mathbf{u}, \mathbf{G}'_h\phi) = 0 \tag{4.2}$$

for all grid vector functions $\mathbf{u}$ and scalar functions ϕ which vanish on the boundary. If one wishes to satisfy (4.1b) at each point of the interior then (4.2) with ϕ taken to be one at each interior point gives the requirement that

$$(g, 1) + (\mathbf{u}, \mathbf{G}'_h 1) = 0 \tag{4.3}$$

must be satisfied. This formula is the analogue of the integrability condition (1.4), and the second term in (4.3) will usually involve only the values of $\mathbf{u}$ on and near the boundary. If the constraint (4.3) is not satisfied then the data must be modified so that (4.3) is satisfied. The use of (4.1b′) in place of (4.1b) is one way by which (4.3) can be satisfied. An advantage of using (4.1b′) over approaches which would modify the boundary data of $\mathbf{u}$ is that (4.1b′) requires no explicit knowledge of $\mathbf{G}'_h$. Note that it is not necessary for $\mathbf{G}'_h$ to be the same as $\mathbf{G}_h$.

It is interesting to note that for the staggered mesh scheme on a uniform grid one can easily satisfy the discrete integrability condition since the calculus of finite differences mimics the differential calculus very closely, see e.g. Kzivickii and Ladyzhenskaya (1966). Similarly, Chorin (1969) proves the convergence of a central difference scheme for the time-dependent Navier–Stokes equations on a periodic rectangular mesh. An essential element of these proofs is that one has a convenient form of the finite difference analogue of the divergence theorem of the differential calculus. Liu and Krause (1979) develop a staggered grid scheme for nonrectangular grids and the success of their scheme is due to their careful treatment of the integrability constraint. Also, Ghia, Hankey and Hodge (1977) mention being unable to obtain a solution to the discrete Navier–Stokes equations for certain situations. We conjecture that this difficulty was caused by the discrete integrability condition not being satisfied.

There is the possibility that the null space of the discrete operator (4.1) has dimension greater than one. The regularized central scheme with the third-order extrapolation (3.4) appears to have only a one-dimensional null space. However, for α equal to zero numerical experiments indicate that there are solutions which are effectively null vectors in that they solve (4.1) with $\mathbf{f}_h$ and g_h smaller than the norm of the solution by a factor proportional to h or h^2. The dimension of the space of nearly null vectors and null vectors appears to be four for the central differencing scheme. These vectors correspond to the four zeros of the determinant of the symbol of the difference operator.

These nearly null vectors and null vectors, other than the usual constant pressure null solution, make solving the discrete system very difficult. On the other hand the regular discrete systems can be solved easily by the iterative procedure given in Strikwerda (1983).

5. Computational results. In this section we present the results of testing the new scheme described in § 3. In the examples discussed here the discrete Stokes equations were solved using test problems which illustrate various features of the schemes. For each example an exact analytical solution is known and the approximate solutions were compared to the exact solutions to study the accuracy of the method. The value of α, the regularity parameter, was one-sixth in all cases. We restrict ourselves here to the Stokes equations for reasons of simplicity. For low Reynolds numbers the nonlinearity of the Navier–Stokes equations usually does not present difficulties as great as those addressed in this paper. For higher Reynolds numbers the nonlinear effects cause additional computational problems which we do not wish to address here. The schemes presented here are being used in computations for the incompressible Navier–Stokes equations and the results will be reported when complete.

The iterative procedure which was used to solve the system of finite difference equations is described at length in Strikwerda (1983). The method consists of alternatively updating the velocity components by successive over-relaxation and updating the pressure by subtracting from the pressure at each grid point a multiple of the discrete divergence of the velocity field. This update of the pressure is of the same form as that used by Chorin (1968). The iterative method was stopped when the changes to the velocity field were sufficiently small and when the changes of the pressure were sufficiently close to being constant. The quantity δ_h was computed as the average value of the discrete divergence of the velocity minus the average value of g. The magnitude of δ_h is one measure of the truncation error of the scheme.

For the first test problem the Stokes equations were solved on the unit square with a uniform grid. The exact solution is

$$\begin{aligned} u &= (2\pi)^{-1} \sin \pi x \cos \pi y, \\ v &= (2\pi)^{-1} \cos \pi x \sin \pi y, \qquad (5.1) \\ p &= \cos \pi x \cos \pi y \end{aligned}$$

with $\mathbf{f} = 0$ and $g = \cos \pi x \cos \pi y$. For this example both the accuracy and symmetry of the solution were checked. The symmetry was checked to study the effect of the nonsymmetric regularizing term on the symmetry of the solution. The symmetry was

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measured by computing the quantities sym (u) and sym (p) given by

$$\begin{aligned}\text{sym}\,(u)&=\left(\sum_{i,j=0}^{N}(u_{ij}+u_{N-i,N-j})^2\right)^{1/2}\Big/\|u\|_2,\\ \text{sym}\,(p)&=\left(\sum_{i,j=0}^{N}(p_{ij}-p_{N-i,N-j})^2\right)^{1/2}\Big/\|p-\bar{p}\|_2\end{aligned}\tag{5.2}$$

for an $(N+1)\times(N+1)$ grid. The quantity $\bar{p}$ is the average value of the p_{ij} and the norm is the l^2-norm, e.g.

$$\|u\|_2=\left(\sum_{i,j}u_{ij}^2\right)^{1/2}.$$

The second test problem demonstrates the ability of the scheme to produce second-order accurate solutions on a nonrectangular region. The exact solution is

$$u=\xi^2+\eta^2,\quad v=-2\xi\eta+\xi^2,\quad p=4\xi+2\eta\tag{5.3}$$

on the region Ω which is the image of the unit square under the mapping

$$\xi=x\cosh(y),\qquad \eta=y-x^2$$

for (x, y) in the unit square, i.e. $0<x,\ y<1$. Thus the equations being solved on the unit square were

$$\begin{aligned}&x_\xi(x_\xi u_x)_x+x_\xi(y_\xi u_y)_x+y_\xi(x_\xi u_x)_y+y_\xi(y_\xi u_y)_y\\ &\quad+x_\eta(x_\eta u_x)_x+x_\eta(y_\eta u_y)_x+y_\eta(x_\eta u_x)_y+y_\eta(y_\eta u_y)_y-x_\xi p_x-y_\xi p_y=0\end{aligned}$$

for the first equation, with the second being similar, and

$$x_\xi u_x+y_\xi u_y+x_\eta v_x+y_\eta v_y=0$$

for the third equation. The regularizing terms were added only to the terms corresponding to p_x in the first equation, p_y in the second, and u_x and v_y in the third. The regularizing terms were added to only these terms since that was sufficient to guarantee the regularity of the scheme.

In the third test problem the Stokes equations were solved on a disk using polar coordinates with uneven grid spacing in both the radial and angular direction. The exact solution is

$$u=r^3\sin 2\theta,\quad v=2r^3\cos 2\theta,\quad p=6r^2\sin 2\theta\tag{5.4}$$

with $\mathbf{f}$ and g being zero. The uneven grid was given by

$$r_i=.75\rho_i+.25\rho_i^2,\qquad \theta_j=\varphi_j-.25\sin\varphi_j,$$

where ρ_i and φ_j were evenly spaced in the interval $[0, 1]$ and $[0, 2\pi]$ respectively. This uneven spacing was chosen merely to show the versatility of the scheme and is not intended to give a better resolution of the solution.

For completeness we give the Stokes equations in polar coordinates,

$$\begin{aligned}&r^{-1}(ru_r)_r+r^{-2}u_{\theta\theta}-r^{-2}u-2r^{-2}v_\theta-p_r=0,\\ &r^{-1}(rv_r)_r+r^{-2}v_{\theta\theta}-r^{-2}v+2r^{-2}u_\theta-r^{-1}p_\theta=0,\\ &r^{-1}(ru)_r+r^{-1}v_\theta=0.\end{aligned}\tag{5.5}$$

The difference formulas used in the numerical experiments were all second-order accurate. As an example of the formulas, the term $r^{-1}(ru_r)_r$ was differenced as

$$\left(\left(\frac{r_{i+1}+r_i}{r_{i+1}-r_i}\right)(u_{i+1,j}-u_{i,j})-\left(\frac{r_i+r_{i-1}}{r_i-r_{i-1}}\right)(u_{i,j}-u_{i-1,j})\right)\Big/\tfrac{1}{2}(r_{i+1}^2-r_{i-1}^2).$$

The results of the numerical experiments are shown in the following tables. Each table lists the errors incurred for grids with $N+1$ points on a side for values of N of 20, 30, 40 and 60. Tables 1, 2 and 3 list the relative errors for test problems 1, 2 and

TABLE 1
Errors for test problem 1 for grids with $N+1$ points on a side for four values of N. The numbers in parentheses are the decimal exponents, i.e., $.35(-3)=.35\times 10^{-3}$.

N	err (u)	err (p)	δ_h	sym (u)	sym (p)
20	.35 (−3)	.17 (−2)	−.44 (−5)	.68 (−3)	.13 (−2)
30	.11 (−3)	.86 (−3)	−.89 (−6)	.22 (−3)	.37 (−3)
40	.41 (−4)	.51 (−3)	−.53 (−6)	.82 (−4)	.15 (−3)
60	.19 (−4)	.23 (−3)	−.50 (−7)	.37 (−4)	.52 (−4)

TABLE 2
Errors for test problem 2

N	err (u)	err (p)	δ_h
20	.10 (−3)	.21 (−2)	−.24 (−3)
30	.45 (−4)	.92 (−3)	−.12 (−3)
40	.25 (−4)	.48 (−3)	−.74 (−4)
60	.11 (−4)	.22 (−3)	−.35 (+4)

TABLE 3
Errors for test problem 3

N	err (u)	err (p)	δ_h
20	.75 (−1)	.93 (−1)	−.33 (−2)
30	.33 (−1)	.34 (−1)	−.53 (−3)
40	.19 (−1)	.18 (−1)	−.15 (−3)
60	.83 (−2)	.75 (−2)	−.27 (−4)

3, respectively, and Table 1 also shows the symmetry errors for problem 1. The relative errors are measured in the l^2-norm, i.e.,

$$\text{err}\,(u)=\left(\sum (u_{ij}-u(x_i,y_i))^2\right)^{1/2}/\|u\|_2.$$

The error in pressure is computed similarly except that the norms are taken modulo additive constants, i.e.

$$\text{err}\,(p)=\|p_h-p_e-\overline{(p_h-p_e)}\|_2/\|p_e-\bar{p}_e\|_2$$

where p_h and p_e are the approximate and exact solutions, respectively. Also shown is the value of δ_h which is described in § 4. Table 4 displays the behavior of the error

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TABLE 4
Computed order of accuracy for u, p, and δ_h for the test problems

N_1/N_2		1	2	3
	u	2.8	2.0	2.0
30/20	p	1.7	2.0	2.5
	δ_h	4.0	1.7	4.5
	u	3.4	2.0	1.9
40/30	p	1.8	2.3	2.2
	δ_h	1.9	1.7	4.4
	u	3.1	2.0	2.0
40/20	p	1.7	2.1	2.4
	δ_h	3.1	1.7	4.5
	u	2.5	2.0	2.0
60/30	p	1.9	2.1	2.2
	δ_h	4.2	1.8	4.3
	u	1.9	2.0	2.0
60/40	p	2.0	1.9	2.2
	δ_h	5.8	1.8	4.2

as the grid resolution is increased. The numbers shown are values of

$$-\frac{\log(\mathrm{err}_1/\mathrm{err}_2)}{\log(N_1/N_2)}$$

where err_1 and err_2 are the errors for grids of N_1+1 and N_2+1 points on a side, respectively. This value should be approximately 2.0 for a second-order scheme. The error reductions are shown for u, p and δ_h. The other velocity component had a similar error behavior in all the examples. All of the solutions were computed by the iterative method given in Strikwerda (1983).

That some of the errors were better than second-order accurate for test problems 1 and 3 can be attributed to the third-order accurate difference formulas used for the gradient and divergence terms. One might expect that some of the errors would behave as third-order errors for some value of N_1 and N_2. However, since the discrete Laplacian is second-order accurate, for N large enough the total scheme should be second-order accurate. It is not clear why δ_h should behave as a fourth-order error as seen in test problem 3 and for some values of N_1 and N_2 in test problem 1. Test problem 2 was no better than second-order accurate since the gradient and divergence were only second-order accurate. The third-order differences were only used on those terms which were necessary to achieve regularity of the scheme. The results show conclusively that the scheme has overall second-order accuracy.

Test problem 1 for $N=40$ is similar to the test problem of Chorin (1968) for the time-dependent Navier–Stokes equations with Reynolds number of 1.0. While the accuracy of the velocity components is of the same order of magnitude for both problems, the results for pressure are more accurate for the regularized central scheme than are Chorin's results by at least an order of magnitude. We attribute this increase in accuracy to both the regularized differencing and the use of the integrability variable δ_h.

In engineering computations it is very important to know the accuracy of one's results. For the incompressible Navier–Stokes calculations the pressure is important since it is used to compute drag and lift forces, however, the pressure is often computed with only indifferent accuracy, e.g. Rubin and Harris (1975). The second-order accuracy obtained for pressure by the regularized central scheme demonstrates the advantage of this scheme over other existing schemes.

6. Conclusion. In this paper we have examined several finite difference methods for the steady Stokes and incompressible Navier–Stokes equations in primitive variables. We have shown that the regularized centered difference scheme is second-order accurate and useful with nonrectangular regions. Although the numerical experiments were done using the Stokes equations, for which exact solutions were available, we believe the regularized central scheme is equally useful with the incompressible Navier–Stokes equations at moderate Reynolds number.

Acknowledgment. The author wishes to thank the referees for their very helpful suggestions and comments.

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