If leapfrog time differencing is used to create a differential–difference approximation to (8.93)–(8.96), a necessary and sufficient condition for the stability of the Lamb-wave mode is

$$c_{\mathrm{s}}\Delta t k_{\max} < 1, \tag{8.97}$$

where $k_{\max}$ is the magnitude of the maximum horizontal wave number resolved by the numerical model. This condition is also sufficient to guarantee the stability of the gravity-wave modes, since for these modes

$$\sin^2(\omega\Delta t) = \frac{(N\Delta t k)^2}{m^2 + \Gamma^2 + N^2/c_{\mathrm{s}}^2} \leq (c_{\mathrm{s}}\Delta t k_{\max})^2.$$

In many geophysical applications the vertical resolution is much higher than the horizontal resolution, in which case (8.97) allows a much larger time step than that permitted by the stability condition for the leapfrog approximation to the full nonhydrostatic compressible equations (given by (8.47) with $U = 0$).

8.6 Primitive Equation Models

The exact equations governing global and large-scale atmospheric flows are often approximated by the so-called *primitive equations*. The primitive equations differ from the exact governing equations in that the quasi-hydrostatic assumption is invoked, small “curvature” and Coriolis terms involving the vertical velocity are neglected in the horizontal momentum equations, and the radial distance between any point within the atmosphere and the center of the Earth is approximated by the mean radius of the Earth. Taken together, these approximations yield a system that conserves both energy and angular momentum (Lorenz 1967, p. 16).

The primitive equations governing inviscid adiabatic atmospheric motion may be expressed using height as the vertical coordinate as follows. Let x, y, and z be spatial coordinates that increase eastward, northward, and upward, respectively. Let $\mathbf{u} = (\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t)$ be the horizontal velocity vector, f the Coriolis parameter, $\mathbf{k}$ an upward-directed unit vector parallel to the z-axis, and ∇_z the gradient with respect to x and y along surfaces of constant z. Then the rate of change of horizontal momentum in the primitive equation system is governed by

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k}\times\mathbf{u} + \frac{1}{\rho}\nabla_z p = \mathbf{0},$$

where

$$\frac{d(\,)}{dt} = \frac{\partial(\,)}{\partial t} + \mathbf{u}\cdot\nabla_z(\,) + w\frac{\partial(\,)}{\partial z}.$$

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \nabla_z \cdot (\rho \mathbf{u}) + \frac{\partial \rho w}{\partial z} = 0, \tag{8.98}$$

and the thermodynamic equation may be written

$$\frac{dT}{dt} - \frac{\omega}{c_p \rho} = 0, \tag{8.99}$$

where $\omega = \mathrm{d}p/\mathrm{d}t$ is the change in pressure following a fluid parcel. The preceding system of equations for the unknown variables $\mathbf{u}$, w, p, ω, ρ, and T may be closed using the hydrostatic relation (8.92) and the equation of state $p = \rho RT$.

8.6.1 Pressure and σ Coordinates

The primitive equations are often solved in a coordinate system in which geometric height is replaced by a new vertical coordinate $\zeta(x, y, z, t)$. Simple functions that have been used to define ζ include the hydrostatic pressure and the potential temperature. The most commonly used vertical coordinates in current operational models are generalized functions of the hydrostatic pressure.

The primitive equations may be expressed with respect to a different vertical coordinate as follows. Suppose that $\zeta(x, y, z, t)$ is the new vertical coordinate and that ζ is a monotone function of z for all fixed x, y, and t with a unique inverse $z(x, y, \zeta, t)$. Defining ∇_ζ as the gradient operator with respect to x and y along surfaces of constant ζ and applying the chain rule to the identity

$$p[x, y, z(x, y, \zeta, t), t] = p(x, y, \zeta, t)$$

yields

$$\nabla_z p + \frac{\partial p}{\partial z} \nabla_\zeta z = \nabla_\zeta p.$$

Using the hydrostatic relation (8.92) and defining the geopotential $\phi = gz$,

$$\nabla_z p = \nabla_\zeta p + \rho \nabla_\zeta \phi,$$

and the horizontal momentum equations in the transformed coordinates become

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k} \times \mathbf{u} + \nabla_\zeta \phi + \frac{RT}{p} \nabla_\zeta p = \mathbf{0}, \tag{8.100}$$

where

$$\frac{d(\,)}{dt} = \frac{\partial(\,)}{\partial t} + \mathbf{u} \cdot \nabla_\zeta(\,) + \dot{\zeta} \frac{\partial(\,)}{\partial \zeta} \tag{8.101}$$

and $\dot{\zeta} = \mathrm{d}\zeta/\mathrm{d}t$. The thermodynamic equation in the transformed coordinates is identical to (8.99), except that the total time derivative is computed using (8.101). The hydrostatic equation may be written as

$$\frac{\partial \phi}{\partial \zeta} = -\frac{RT}{p}\frac{\partial p}{\partial \zeta}.$$

The continuity equation in the transformed coordinate system can be determined by transforming the partial derivatives in (8.98) (Kasahara 1974). It is perhaps simpler to derive the continuity equation directly from first principles. Let $\mathscr{V}$ be a fixed volume defined with respect to the time-independent spatial coordinates x, y, and z, and let $\mathbf{n}$ be the outward-directed unit vector normal to the surface $\mathscr{S}$ enclosing $\mathscr{V}$. Since the rate of change of mass in the volume $\mathscr{V}$ is equal to the net mass flux through $\mathscr{S}$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho \, \mathrm{d}V &= -\int_{\mathscr{S}} \rho \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}A \\ &= -\int_{\mathscr{V}} \nabla \cdot (\rho \mathbf{v}) \, \mathrm{d}V, \end{aligned} \tag{8.102}$$

where $\mathbf{v}$ is the three-dimensional velocity vector. Equation (5.111), which states the general relationship between the divergence in Cartesian coordinates and curvilinear coordinates, implies that

$$\nabla \cdot (\rho \mathbf{v}) = \frac{1}{J}\nabla_\zeta \cdot (J\rho\mathbf{u}) + \frac{1}{J}\frac{\partial}{\partial \zeta}\left(J\rho\dot{\zeta}\right),$$

where J is the Jacobian of the transformation between (x, y, z) and (x, y, ζ), which in this instance is simply $\partial z/\partial \zeta$. In the transformed coordinates

$$\mathrm{d}V = \frac{\partial z}{\partial \zeta}\, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\zeta,$$

and since the boundaries of $\mathscr{V}$ do not depend on time, (8.102) may be expressed as

$$\int\int\int_{\mathscr{V}} \left[\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial \zeta}\right) + \nabla_\zeta \cdot \left(\rho\mathbf{u}\frac{\partial z}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(\rho\dot{\zeta}\frac{\partial z}{\partial \zeta}\right)\right] \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}\zeta = 0.$$

Using the hydrostatic equation (8.92) to eliminate ρ from the preceding equation, and noting that the integrand must be identically zero because the volume $\mathscr{V}$ is arbitrary, the continuity equation becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \zeta}\right) + \nabla_\zeta \cdot \left(\mathbf{u}\frac{\partial p}{\partial \zeta}\right) + \frac{\partial}{\partial \zeta}\left(\dot{\zeta}\frac{\partial p}{\partial \zeta}\right) = 0.$$

Now consider possible choices for ζ. In most respects, the simplest system is obtained by choosing $\zeta = p$; this eliminates one of the two terms that make up the pressure gradient in (8.100) and reduces the continuity equation to the simple diagnostic relation

$$\nabla_p \cdot \mathbf{u} + \frac{\partial \omega}{\partial p} = 0.$$

The difficulty with pressure coordinates arises at the lower boundary because the pressure at the surface of the Earth is a function of horizontal position and time. As a consequence, constant-pressure surfaces intersect the lower boundary of the domain in an irregular manner that changes as a function of time. To simplify the lower-boundary condition, Phillips (1957) suggested choosing $\zeta = \sigma = p/p_s$, where p_s is the surface pressure. The upper and lower boundaries in a σ-coordinate model coincide with the coordinate surfaces $\sigma = 0$ and $\sigma = 1$, and $\dot{\sigma} = 0$ at both the upper and lower boundaries.

The σ-coordinate equations include prognostic equations for $\mathbf{u}$, T, and p_s and diagnostic equations for $\dot{\sigma}$, ϕ, and ω. The prognostic equations for the horizontal velocity and the temperature are

$$\frac{d\mathbf{u}}{dt} + f\mathbf{k}\times\mathbf{u} + \nabla_\sigma\phi + \frac{RT}{p_s}\nabla_\sigma p_s = \mathbf{0} \tag{8.103}$$

and

$$\frac{dT}{dt} = \frac{\kappa T}{\sigma p_s}\omega, \tag{8.104}$$

where

$$\frac{d(\,)}{dt} = \frac{\partial(\,)}{\partial t} + \mathbf{u}\cdot\nabla_\sigma(\,) + \dot{\sigma}\frac{\partial(\,)}{\partial\sigma}.$$

The continuity equation in σ coordinates takes the form of a prognostic equation for the surface pressure:

$$\frac{\partial p_s}{\partial t} + \nabla_\sigma\cdot(p_s\mathbf{u}) + \frac{\partial}{\partial\sigma}(p_s\dot{\sigma}) = 0. \tag{8.105}$$

Recalling that $\dot{\sigma}$ is zero at $\sigma = 0$ and $\sigma = 1$, one can integrate (8.105) over the depth of the domain to obtain

$$\frac{\partial p_s}{\partial t} = -\int_0^1 \nabla_\sigma\cdot(p_s\mathbf{u})\,\mathrm{d}\sigma. \tag{8.106}$$

A diagnostic equation for the vertical velocity $\dot{\sigma}$ is obtained by integrating (8.105) from the top of the domain to level σ, which yields

$$\dot{\sigma}(\sigma) = -\frac{1}{p_s}\left[\sigma\frac{\partial p_s}{\partial t} + \int_0^\sigma \nabla_\sigma\cdot(p_s\mathbf{u})\,\mathrm{d}\tilde{\sigma}\right]. \tag{8.107}$$

A diagnostic equation for ω can be derived by noting that

$$\omega = \frac{d}{dt}(\sigma p_s) = \dot{\sigma}\,p_s + \sigma\frac{\partial p_s}{\partial t} + \sigma\mathbf{u}\cdot\nabla_\sigma p_s,$$

and thus

$$\omega(\sigma) = \sigma \mathbf{u} \cdot \nabla_\sigma p_\mathrm{s} - \int_0^\sigma \nabla_\sigma \cdot (p_\mathrm{s}\mathbf{u})\, \mathrm{d}\tilde{\sigma}. \tag{8.108}$$

The geopotential is determined by integrating the hydrostatic equation

$$\frac{\partial \phi}{\partial(\ln \sigma)} = -RT \tag{8.109}$$

from the surface to level σ, which gives

$$\phi(\sigma) = gz_\mathrm{s} - R\int_1^\sigma T\, \mathrm{d}(\ln \tilde{\sigma}), \tag{8.110}$$

where $z_\mathrm{s}(x, y)$ is the elevation of the topography.

The primary disadvantage of the σ-coordinate system is that it makes the accurate computation of horizontal pressure gradients difficult over steep topography. This problem arises because surfaces of constant σ tilt in regions where there are horizontal variations in surface pressure, and such variations are most pronounced over steep topography. When $\nabla_\sigma p_\mathrm{s} \neq 0$, some portion of the vertical pressure gradient is projected onto each of the two terms $\nabla_\sigma \phi$ and $(RT/p_\mathrm{s})\nabla_\sigma p_\mathrm{s}$. The vertical pressure gradient will not exactly cancel between these terms because of numerical errors, and over steep topography the noncanceling residual can be comparable to the true horizontal pressure gradient because the vertical gradient of atmospheric pressure is several orders of magnitude larger than the horizontal gradient. The pressure-gradient error in a σ-coordinate model is not confined to the lower levels near the topography, but it may be reduced at upper levels using a hybrid vertical coordinate that transitions from σ coordinates to p coordinates at some level (or throughout some layer) in the interior of the domain (Sangster 1960; Simmons and Burridge 1981). Although they are widely used in operational weather and climate models (Williamson and Olson 1994; Ritchie et al. 1995; Kiehl et al. 1996), these hybrid coordinates complicate the solution of the governing equations and will not be considered here.

Several other approaches have also been suggested to minimize the errors generated over topography in σ-coordinate models. Phillips (1973) and Gary (1973) suggested performing the computations using a perturbation pressure defined with respect to a hydrostatically balanced reference state. Finite-difference schemes have been proposed that guarantee exact cancelation of the vertical pressure gradient between the last two terms in (8.103) whenever the vertical profiles of temperature and pressure have a specified functional relation, such as $T = a\ln(p) + b$ (Corby et al. 1972; Nakamura 1978; Simmons and Burridge 1981). Mesinger (1984) suggested using "η coordinates," in which the mountain slopes are discretized as vertical steps at the grid interfaces with flat terrain between each step. More details and additional techniques for the treatment of pressure-gradient errors over mountains in quasi-hydrostatic atmospheric models are presented in the review by Mesinger and Janić (1985).

8.6.2 Spectral Representation of the Horizontal Structure

Global primitive-equation models often use spherical harmonics to represent the latitudinal and longitudinal variation of the forecast variables. In the following sections we present the basic numerical procedures for creating a spectral approximation to the σ-coordinate equations in a global atmospheric model. The approach is similar to that in Hoskins and Simmons (1975) and Bourke (1974), which may be consulted for additional details. The latitudinal and longitudinal variations in each field will be approximated using spherical harmonics, and the vertical variations will be represented using grid-point methods.

As was the case for the global shallow-water model described in Sect. 6.4.4, the spectral representation of the horizontal velocity field is facilitated by expressing the horizontal momentum equations in terms of the vertical vorticity ζ and the divergence δ. To integrate this system easily using semi-implicit time differencing, it is also helpful to divide the temperature into a horizontally uniform reference state and a perturbation such that $T = \overline{T}(\sigma) + T'$. Using the identity (6.78) and taking the divergence of (8.103) yields

$$\frac{\partial \delta}{\partial t} - \mathbf{k} \cdot \nabla \times (\zeta + f)\mathbf{u} + \nabla \cdot \left(\dot{\sigma} \frac{\partial \mathbf{u}}{\partial \sigma} + RT' \nabla (\ln p_s) \right) + \nabla^2 \left(\phi + \frac{\mathbf{u} \cdot \mathbf{u}}{2} + R\overline{T} \ln p_s \right) = 0. \tag{8.111}$$

Again using (6.78) and taking the vertical component of the curl of (8.103), one obtains

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot (\zeta + f)\mathbf{u} + \mathbf{k} \cdot \nabla \times \left(\dot{\sigma} \frac{\partial \mathbf{u}}{\partial \sigma} + RT' \nabla (\ln p_s) \right) = 0. \tag{8.112}$$

Following the notation used in Sect. 6.4.4, let χ be the velocity potential and ψ the stream function for the horizontal velocity. Let λ be the longitude, θ the latitude, and $\mu = \sin\theta$. Define the operator

$$\mathscr{H}(M, N) = \frac{1}{a} \left(\frac{1}{1 - \mu^2} \frac{\partial M}{\partial \lambda} + \frac{\partial N}{\partial \mu} \right),$$

where a is the mean radius of the Earth. Then using the formula for the horizontal divergence in spherical coordinates,

$$\begin{aligned} \nabla M &= \frac{1}{a \cos\theta} \frac{\partial M}{\partial \lambda} \mathbf{i} + \frac{1}{a} \frac{\partial M}{\partial \theta} \mathbf{j} \\ &= \frac{1}{a(1 - \mu^2)^{1/2}} \left(\frac{\partial M}{\partial \lambda} \mathbf{i} + (1 - \mu^2) \frac{\partial M}{\partial \mu} \mathbf{j} \right), \end{aligned}$$

and the relations (6.82)–(6.85), one may express the prognostic equations for the σ-coordinate system in the form

$$\frac{\partial \nabla^2 \chi}{\partial t} = \mathscr{H}(B, -A) - 2\Omega\left(\frac{U}{a} - \mu\nabla^2\psi\right)$$
$$-\nabla^2\left(\phi + \frac{U^2+V^2}{2(1-\mu^2)} + R\overline{T}\ln p_\mathrm{s}\right), \tag{8.113}$$

$$\frac{\partial \nabla^2 \psi}{\partial t} = -\mathscr{H}(A, B) - 2\Omega\left(\frac{V}{a} + \mu\nabla^2\chi\right), \tag{8.114}$$

$$\frac{\partial T'}{\partial t} = -\mathscr{H}(UT', VT') + T'\nabla^2\chi - \dot{\sigma}\frac{\partial T}{\partial \sigma} + \frac{\kappa T\omega}{\sigma p_\mathrm{s}}, \tag{8.115}$$

$$\frac{\partial}{\partial t}(\ln p_\mathrm{s}) = -\frac{U}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}(\ln p_\mathrm{s}) - \frac{V}{a}\frac{\partial}{\partial\mu}(\ln p_\mathrm{s}) - \nabla^2\chi - \frac{\partial\dot{\sigma}}{\partial\sigma}, \tag{8.116}$$

where

$$U = u\cos\theta = (1-\mu^2)\mathscr{H}(\chi, -\psi),$$
$$V = v\cos\theta = (1-\mu^2)\mathscr{H}(\psi, \chi),$$
$$A = U\nabla^2\psi + \dot{\sigma}\frac{\partial V}{\partial\sigma} + \frac{RT'}{a}(1-\mu^2)\frac{\partial}{\partial\mu}(\ln p_\mathrm{s}).$$
$$B = V\nabla^2\psi - \dot{\sigma}\frac{\partial U}{\partial\sigma} - \frac{RT'}{a}\frac{\partial}{\partial\lambda}(\ln p_\mathrm{s}).$$

The preceding system of equations is formulated using $\ln p_\mathrm{s}$ instead of p_s as the prognostic variable to make the term $(RT/p_\mathrm{s})\nabla_\sigma p_\mathrm{s}$ into a binary product of the prognostic variables and thereby facilitate the alias-free evaluation of the pressure-gradient force via the spectral transform method.

At each σ level, the unknown functions ψ, χ, T', and ϕ are approximated using a truncated series of spherical harmonics. The unknown function $\ln p_\mathrm{s}$ is also approximated by a spherical harmonic expansion. Expressions for the time tendencies of the expansion coefficients for each spherical harmonic are obtained using the transform method in a manner analogous to that for the global shallow-water model described in Sect. 6.4.4. As an example, suppose that the stream function and velocity potential at a given σ level are expanded in spherical harmonics as in (6.90) and (6.91). Then, using the notation defined in Sect. 6.4.4, the equation for $\partial\psi_{m,n}/\partial t$ is once again given by (6.95) except that $\hat{A}_m$ and $\hat{B}_m$ now satisfy

$$U\nabla^2\psi + \dot{\sigma}\frac{\partial V}{\partial\sigma} + \frac{RT'}{a}(1-\mu^2)\frac{\partial}{\partial\mu}(\ln p_\mathrm{s}) = \sum_{m=-M}^{M}\hat{A}_m e^{\mathrm{i}m\lambda} \tag{8.117}$$

and

$$V\nabla^2\psi - \dot{\sigma}\frac{\partial U}{\partial\sigma} - \frac{RT'}{a}\frac{\partial}{\partial\lambda}(\ln p_\mathrm{s}) = \sum_{m=-M}^{M}\hat{B}_m e^{\mathrm{i}m\lambda}. \tag{8.118}$$

The spectral form of the tendency equations for the velocity potential, the perturbation temperature, and the surface pressure may be found in Bourke (1974) and will not be given here. Note that the vertical advection terms in (8.117) and (8.118) involve the product of three spatially varying functions (since $\dot{\sigma}$ itself depends on the product of two spatially varying functions). The standard transform method cannot be used to transform these triple products between wave-number and physical space without incurring some numerical error. This "aliasing" error is nevertheless very small (Hoskins and Simmons 1975).

8.6.3 Vertical Differencing

The most significant modifications required to extend the shallow-water algorithm to a σ-coordinate model are those associated with the computation of the vertical derivatives. The vertical derivatives are computed using finite differences at that stage of the integration cycle when all the unknown variables are available on the physical mesh. As in (8.117) and (8.118), the results from these finite-difference computations are then combined with the other binary products computed on the physical mesh, and the net forcing is transformed back to wave-number space.

A convenient and widely used vertical discretization for the σ-coordinate equations is illustrated in Fig. 8.7 for a model with N vertical levels. The upper and lower boundaries are located at $\sigma = 0$ and

$$\sigma = 1 = \sum_{k=1}^{N} \Delta\sigma_k ,$$

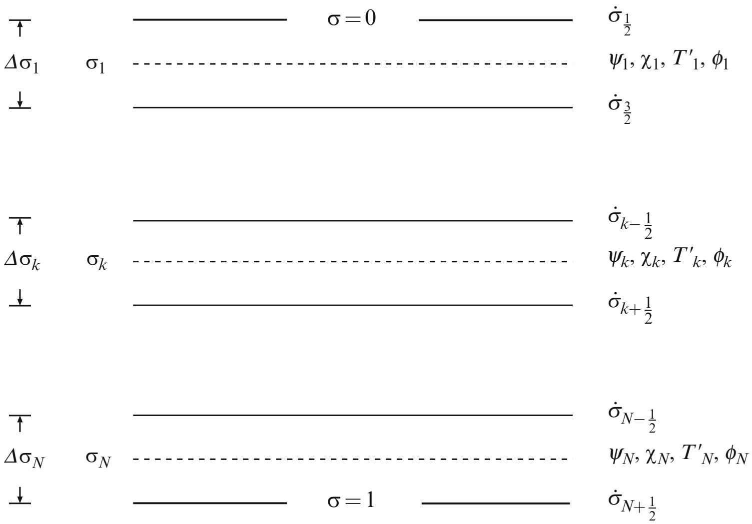

Fig. 8.7 Vertical distribution of the unknown variables on a σ-coordinate grid. The thickness of the σ layers need not be uniform. The center of each layer is at level $\sigma = \sigma_k$ and is indicated by the *dashed lines*. Note that the vertical index k increases with σ and decreases with geometric height

where $\Delta\sigma_k$ is the width of the kth σ layer. The stream function, velocity potential, temperature, and geopotential are defined at the center of each σ layer, and the velocity $\dot{\sigma}$ is defined at the interface between each layer. The vertical derivatives appearing in (8.113)–(8.115) involve variables, such as the temperature, that are defined at the center of each σ layer. These derivatives are approximated such that

$$\begin{aligned}\left(\dot{\sigma}\frac{\partial T}{\partial \sigma}\right)_k &\approx \langle \dot{\sigma}_k \delta_\sigma T_k \rangle^\sigma \\ &= \dot{\sigma}_{k+\frac{1}{2}}\left(\frac{T_{k+1}-T_k}{\Delta\sigma_{k+1}+\Delta\sigma_k}\right) + \dot{\sigma}_{k-\frac{1}{2}}\left(\frac{T_k - T_{k-1}}{\Delta\sigma_k + \Delta\sigma_{k-1}}\right). \quad (8.119)\end{aligned}$$

The preceding differencing is the generalization of the "averaging scheme" discussed in Sect. 4.4 to the nonuniform staggered mesh shown in Fig. 8.7.

The vertical derivative of $\dot{\sigma}$ in (8.116) is approximated as

$$\left(\frac{\partial\dot{\sigma}}{\partial\sigma}\right)_k \approx \delta_\sigma \dot{\sigma}_k = \frac{\dot{\sigma}_{k++\frac{1}{2}} - \dot{\sigma}_{k--\frac{1}{2}}}{\Delta\sigma_k}. \quad (8.120)$$

Defining $G_k = \nabla_\sigma \cdot \mathbf{u}_k + \mathbf{u}_k \cdot \nabla_\sigma(\ln p_\mathrm{s})$, the preceding expression implies that the vertically discretized approximation to the surface-pressure-tendency equation (8.106) is

$$\frac{\partial}{\partial t}(\ln p_\mathrm{s}) = -\sum_{k=1}^{N} G_k \Delta\sigma_k, \quad (8.121)$$

and that (8.107) is approximated as

$$\dot{\sigma}_{k+\frac{1}{2}} = \left(\sum_{j=1}^{k}\Delta\sigma_j\right)\sum_{j=1}^{N} G_j\Delta\sigma_j - \sum_{j=1}^{k} G_j \Delta\sigma_j. \quad (8.122)$$

The hydrostatic equation (8.109) is approximated as

$$\frac{\phi_{k+1}-\phi_k}{\ln\sigma_{k+1} - \ln\sigma_k} = -\frac{R}{2}(T_{k+1}+T_k),$$

except in the half-layer between the lowest σ level and the surface, where

$$\frac{\phi_N - \phi_\mathrm{s}}{\ln\sigma_N} = -RT_N.$$

Defining $\alpha_N = -\ln\sigma_N$ and $\alpha_k = 1/2\ln(\sigma_{k+1}/\sigma_k)$ for $1 \le k < N$, the discrete analogue of (8.110) becomes

$$\phi_k = \phi_\mathrm{s} + R\left(\sum_{j=k}^{N}\alpha_j T_j + \sum_{j=k+1}^{N}\alpha_{j-1}T_j\right). \quad (8.123)$$

Finally, as suggested by Corby et al. (1972), the vertical discretization for the ω equation (8.108) is chosen to preserve the energy-conservation properties of the vertically integrated continuous equations. Such conservation is achieved if

$$\frac{\omega_k}{\sigma_k p_s} = \mathbf{u}_k \cdot \nabla_\sigma (\ln p_s) - \frac{\alpha_k}{\Delta\sigma_k} \sum_{j=1}^{k} G_j \Delta\sigma_j - \frac{\alpha_{k-1}}{\Delta\sigma_k} \sum_{j=1}^{k-1} G_j \Delta\sigma_j. \tag{8.124}$$

8.6.4 Energy Conservation

Why does (8.124) give better energy-conservation properties than the simpler formula that would result if both α_k and α_{k-1} were replaced by $\Delta\sigma_k/(2\sigma_k)$? To answer this question it is necessary to review the energy-conservation properties of the continuous σ-coordinate primitive equations. Our focus is on the vertical discretization, so it is helpful to obtain a conservation law for the vertically integrated total energy per unit horizontal area. Using the hydrostatic equation (8.92) and the definition $\sigma p_s = p$, one may express the vertical integral of the sum of the kinetic[10] and internal energy per unit volume as

$$\begin{aligned}\int_{z_s}^{\infty} \rho \left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + c_v T\right) \mathrm{d}z &= -\frac{1}{g}\int_{p_s}^{0} \left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + c_v T\right) \mathrm{d}p \\ &= \frac{p_s}{g}\int_0^1 \left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + c_v T\right) \mathrm{d}\sigma.\end{aligned}$$

Using the hydrostatic equation twice and integrating by parts, the vertical integral of the potential energy becomes

$$\int_{z_s}^{\infty} \rho g z \,\mathrm{d}z = \int_0^{p_s} \frac{\phi}{g}\,\mathrm{d}p = \frac{1}{g}\int_0^{\phi_s p_s} \mathrm{d}(\phi p) - \int_{\infty}^{\phi_s} \frac{p}{g}\,\mathrm{d}\phi = \frac{\phi_s p_s}{g} + \int_0^{p_s} \frac{p}{g\rho}\,\mathrm{d}p.$$

Recalling that $p = \rho R T$,

$$\int_{z_s}^{\infty} \rho g z \,\mathrm{d}z = \frac{\phi_s p_s}{g} + \frac{p_s}{g}\int_0^1 RT\,\mathrm{d}\sigma,$$

and the total vertically integrated energy per unit area is

$$\mathscr{E} = \frac{\phi_s p_s}{g} + \frac{p_s}{g}\int_0^1 \left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + c_p T\right) \mathrm{d}\sigma.$$

The total time derivatives in the momentum and thermodynamic equations must be written in flux form to obtain a conservation law governing $\mathscr{E}$. For any scalar γ,

[10] Note that as a consequence of the primitive-equation approximation, the vertical velocity does not appear as part of the kinetic energy.

$$
\begin{aligned}
p_\mathrm{s}\frac{d\gamma}{dt} &= p_\mathrm{s}\frac{\partial\gamma}{\partial t} + p_\mathrm{s}\mathbf{u}\cdot\nabla_\sigma\gamma + p_\mathrm{s}\dot{\sigma}\frac{\partial\gamma}{\partial\sigma} + \gamma\left[\frac{\partial p_\mathrm{s}}{\partial t} + \nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u}) + \frac{\partial}{\partial\sigma}(p_\mathrm{s}\dot{\sigma})\right] \\
&= \frac{\partial}{\partial t}(p_\mathrm{s}\gamma) + \nabla_\sigma\cdot(p_\mathrm{s}\gamma\mathbf{u}) + \frac{\partial}{\partial\sigma}(p_\mathrm{s}\gamma\dot{\sigma}),
\end{aligned}
\tag{8.125}
$$

where the quantity in square brackets is zero by the pressure-tendency equation (8.105). Adding $p_\mathrm{s}c_p$ times the thermodynamic equation (8.104) to the dot product of $p_\mathrm{s}\mathbf{u}$ and the momentum equation (8.103) and using (8.125), one obtains

$$
\frac{\partial E}{\partial t} + \nabla_\sigma\cdot(E\mathbf{u}) + \frac{\partial}{\partial\sigma}(E\dot{\sigma}) + \mathbf{u}\cdot p_\mathrm{s}\nabla_\sigma\phi + \mathbf{u}\cdot RT\nabla_\sigma p_\mathrm{s} - \frac{RT\omega}{\sigma} = 0, \tag{8.126}
$$

where $E = p_\mathrm{s}(\mathbf{u}\cdot\mathbf{u}/2 + c_pT)$. Defining

$$
F = -\phi\nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u}) + \mathbf{u}\cdot RT\nabla_\sigma p_\mathrm{s} - \frac{RT\omega}{\sigma}, \tag{8.127}
$$

one may express (8.126) as

$$
\frac{\partial E}{\partial t} + \nabla_\sigma\cdot[(E + p_\mathrm{s}\phi)\mathbf{u}] + \frac{\partial}{\partial\sigma}(E\dot{\sigma}) + F = 0. \tag{8.128}
$$

The forcing F may be written as the vertical divergence of a flux as follows. Substituting for ω using (8.108),

$$
F = -\phi\nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u}) + \frac{RT}{\sigma}\int_0^\sigma \nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u})\,\mathrm{d}\tilde{\sigma},
$$

and then substituting for RT/σ from the hydrostatic equation,

$$
\begin{aligned}
F &= -\phi\nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u}) - \frac{\partial\phi}{\partial\sigma}\int_0^\sigma \nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u})\,\mathrm{d}\tilde{\sigma} \\
&= -\frac{\partial}{\partial\sigma}\left[\phi\int_0^\sigma \nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u})\,\mathrm{d}\tilde{\sigma}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^1 F\,\mathrm{d}\sigma &= -\phi_\mathrm{s}\int_0^1 \nabla_\sigma\cdot(p_\mathrm{s}\mathbf{u})\,\mathrm{d}\sigma \\
&= \phi_\mathrm{s}\int_0^1 \left(\frac{\partial p_\mathrm{s}}{\partial t} + \frac{\partial}{\partial\sigma}(\dot{\sigma}p_\mathrm{s})\right)\mathrm{d}\sigma \\
&= \frac{\partial}{\partial t}(\phi_\mathrm{s}p_\mathrm{s}).
\end{aligned}
$$

The preceding expression may be used to derive a conservation law for $\mathscr{E}$ by integrating (8.128) over the depth of the domain and applying the boundary condition $\dot{\sigma} = 0$ at the upper and lower boundaries to obtain

$$
\frac{\partial\mathscr{E}}{\partial t} + \frac{1}{g}\nabla_\sigma\cdot\int_0^1 (E + p_\mathrm{s}\phi)\mathbf{u}\,\mathrm{d}\sigma = 0. \tag{8.129}
$$

Of course, (8.129) also implies that if the horizontal domain is periodic, or if there is no flow normal to the lateral boundaries, the σ-coordinate primitive equations conserve the domain-integrated total energy

$$\int\int\left[\frac{\phi_\mathrm{s} p_\mathrm{s}}{g}+\frac{p_\mathrm{s}}{g}\int_0^1\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}+c_p T\right)\mathrm{d}\sigma\right]\mathrm{d}x\,\mathrm{d}y. \tag{8.130}$$

The domain-integrated total energy is not, however, exactly conserved by global spectral models. As discussed in Sect. 6.2.3, a Galerkin spectral approximation to a prognostic equation for an unknown function γ will generally conserve the domain integral of γ^2, provided that the domain integral of γ^2 is also conserved by the continuous equations and time-differencing errors are neglected. Unfortunately, the conservation of the squares of the prognostic variables in (8.113)–(8.116) does not imply exact conservation of the total energy. Practical experience has, nevertheless, shown that the deviations from exact energy conservation generated by the spectral approximation of the horizontal derivatives is very small. The nonconservation introduced by the semi-implicit time differencing used in most global primitive-equation models has also been shown to be very small (Hoskins and Simmons 1975). Nonconservative formulations of the vertical finite differencing can, however, have a significantly greater impact on the global energy conservation. This appears to be a particularly important issue if long-time integrations are conducted using global climate models with poor vertical resolution.

The energy-conservation properties of the vertical discretization given by (8.119)–(8.124) will therefore be isolated from the nonconservative effects of the spectral approximation and the time differencing by considering a system of differential–difference equations in which only those terms containing vertical derivatives are discretized. Except for the terms involving vertical derivatives, the total-energy equation for the semidiscrete system must be identical to (8.128) because the time and horizontal derivatives are exact.

The semidiscrete system will therefore conserve total energy, provided that it satisfies the discrete analogues of

$$\int_0^1 \frac{\partial}{\partial\sigma}(E\dot{\sigma})\,\mathrm{d}\sigma = 0 \tag{8.131}$$

and

$$\int_0^1 F\,\mathrm{d}\sigma = \frac{\partial}{\partial t}(\phi_\mathrm{s} p_\mathrm{s}). \tag{8.132}$$

The integrand in (8.131) appears in the total-energy equation (8.128) as a mathematical simplification of a linear combination of the vertical derivative terms in the momentum, thermodynamic, and surface-pressure-tendency equations such that

$$\frac{\partial}{\partial\sigma}(E\dot{\sigma})\,\mathrm{d}\sigma = \mathbf{u}\cdot\dot{\sigma}\frac{\partial\mathbf{u}}{\partial\sigma}+\frac{\mathbf{u}\cdot\mathbf{u}}{2}\frac{\partial\dot{\sigma}}{\partial\sigma}+c_p\dot{\sigma}\frac{\partial T}{\partial\sigma}+c_p T\frac{\partial\dot{\sigma}}{\partial\sigma}.$$

When the vertical derivatives on the right side of the preceding equation are approximated using (8.119) and (8.120), their summation over the depth of the domain is exactly zero. This may be demonstrated for the pair of terms involving T by noting that since $\dot{\sigma}_{1/2} = \dot{\sigma}_{N+1/2} = 0$,

$$\sum_{k=1}^{N} \left[\langle \dot{\sigma}_k \delta_\sigma T_k \rangle^\sigma + T_k \delta_\sigma \dot{\sigma}_k \right] \Delta\sigma_k = \sum_{k=1}^{N} \left[\dot{\sigma}_{k+\frac{1}{2}} \left(\frac{\Delta\sigma_k T_{k+1} + \Delta\sigma_{k+1} T_k}{\Delta\sigma_{k+1} + \Delta\sigma_k} \right) - \dot{\sigma}_{k-\frac{1}{2}} \left(\frac{\Delta\sigma_{k-1} T_k + \Delta\sigma_k T_{k-1}}{\Delta\sigma_k + \Delta\sigma_{k-1}} \right) \right] = 0.$$

A similar relation holds for the two terms involving the horizontal velocity (see Problem 7).

Now consider the discrete analogue of (8.132), or equivalently,

$$\int_0^1 \frac{F}{p_s} \, d\sigma = \phi_s \frac{\partial}{\partial t} (\ln p_s).$$

Defining $G = \nabla_\sigma \cdot \mathbf{u} + \mathbf{u} \cdot \nabla_\sigma (\ln p_s)$ and substituting for F using (8.127) yields

$$\int_0^1 \left(-\phi G + \mathbf{u} \cdot RT \nabla_\sigma (\ln p_s) - \frac{RT\omega}{\sigma p_s} \right) d\sigma = \phi_s \frac{\partial}{\partial t} (\ln p_s).$$

The discrete form of this integral equation may be obtained using (8.121) and (8.124) and is algebraically equivalent to

$$\sum_{k=1}^{N} (\phi_k - \phi_s) G_k \Delta\sigma_k = \sum_{k=1}^{N} RT_k \left(\alpha_k \sum_{j=1}^{k} G_j \Delta\sigma_j + \alpha_{k-1} \sum_{j=1}^{k-1} G_j \Delta\sigma_j \right).$$

It may be verified that the preceding expression is indeed an algebraic identity by substituting for $\phi_k - \phi_s$ from the discrete form of the hydrostatic equation (8.123) and using the relation

$$\sum_{k=1}^{N} \sum_{j=1}^{k} a_k b_j = \sum_{k=1}^{N} \sum_{j=k}^{N} a_j b_k.$$

In addition to conserving total energy, the preceding vertical discretization also conserves total mass (see Problem 8). This scheme does not conserve the integrated angular momentum or the integrated potential temperature. Arakawa and Lamb (1977), Simmons and Burridge (1981), and Arakawa and Konor (1996) described alternative vertical discretizations that conserve angular momentum, potential temperature, or various other vertically integrated functions.

8.6.5 Semi-implicit Time Differencing

Computational efficiency can be enhanced by using semi-implicit time differencing to integrate the preceding primitive-equation model. The semi-implicit method can be implemented in σ-coordinate primitive-equation models as follows. Let **d** be a column vector whose kth element is the function $\nabla_\sigma^2\chi$ at level σ_k. Similarly, define $\bar{\mathrm{t}}$, **t**, and **h** to be column vectors containing the σ-level values of the functions $R\overline{T}$, T, and ϕ. Let $\mathbf{h}_\mathrm{s}$ be a column vector in which every element is ϕ_s. Then the vertically discretized equations for the divergence, temperature, surface-pressure tendency, and geopotential may be written in the form

$$\frac{\partial \mathbf{d}}{\partial t} = \mathbf{f_d} - \nabla_\sigma^2\left(\mathbf{h} + \bar{\mathrm{t}}\ln p_\mathrm{s}\right), \tag{8.133}$$

$$\frac{\partial \mathbf{t}}{\partial t} = \mathbf{f_t} - \mathbf{Hd}, \tag{8.134}$$

$$\frac{\partial}{\partial t}(\ln p_s) = f_p - \mathbf{p}^\mathrm{T}\,\mathbf{d}, \tag{8.135}$$

$$\mathbf{h} = \mathbf{h}_\mathrm{s} + \mathbf{Gt}. \tag{8.136}$$

Here **G** and **H** are matrices and **p** is a column vector, none of which depend on λ, μ, or t. The thermodynamic equation (8.134) is partitioned such that all terms containing the product of $\overline{T}(\sigma)$ and the divergence are collected in **Hd**.

Equation (8.121) implies that

$$\mathbf{p}^\mathrm{T} = (\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_N),$$

and (8.123) requires

$$\frac{\mathbf{G}}{R} = \begin{pmatrix} \alpha_1 & \alpha_1+\alpha_2 & \alpha_2+\alpha_3 & \ldots \\ 0 & \alpha_2 & \alpha_2+\alpha_3 & \ldots \\ 0 & 0 & \alpha_3 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Let $h_{r,s}$ denote the sth element in the rth row of **H**. Then according to (8.115), $h_{r,s}$ is determined by the contribution of the divergence at level s to $\dot\sigma\partial\overline{T}/\partial\sigma - \kappa\overline{T}\omega/(\sigma p_\mathrm{s})$ at level r. Define a step function such that $\mathscr{S}(x) = 1$ if $x \geq 0$ and $\mathscr{S}(x) = 0$ otherwise. Then from (8.119), (8.122), and (8.124),

$$\begin{aligned}\frac{h_{r,s}}{\Delta\sigma_s} = {} & \frac{\kappa\overline{T}_r\mathscr{S}(r-s)}{\Delta\sigma_r}\left[\alpha_r + \mathscr{S}(r-s-1)\alpha_{r-1}\right] \\ & - \left(\frac{\overline{T}_{r+1}-\overline{T}_r}{\Delta\sigma_{r+1}+\Delta\sigma_r}\right)\left(\mathscr{S}(r-s) - \sum_{j=1}^{r}\Delta\sigma_j\right)\end{aligned}$$

$$-\left(\frac{\overline{T}_r - \overline{T}_{r-1}}{\Delta\sigma_r + \Delta\sigma_{r-1}}\right)\left(\mathscr{S}(r-s-1) - \sum_{j=1}^{r-1}\Delta\sigma_j\right).$$

The remaining terms in (8.121) and the vertically discretized versions of (8.113) and (8.115) are gathered into f_p, $\mathbf{f_d}$, and $\mathbf{f_t}$, respectively.

A single equation for the divergence may be obtained by eliminating $\mathbf{t}$, $\mathbf{h}$, and $\ln p_{\mathrm{s}}$ from (8.133)–(8.136) to give

$$\left(\frac{\partial^2}{\partial t^2} - \mathbf{B}\nabla_\sigma^2\right)\mathbf{d} = \frac{\partial \mathbf{f_d}}{\partial t} - \nabla_\sigma^2\left(\mathbf{G}\mathbf{f_t} + f_p\bar{\mathrm{t}}\right), \tag{8.137}$$

where $\mathbf{B} = \mathbf{G}\mathbf{H} + \bar{\mathrm{t}}\mathbf{p}^{\mathrm{T}}$. The solutions to the homogeneous part of this equation comprise the set of gravity waves supported by the vertically discretized model. Hoskins and Simmons (1975) presented plots showing the vertical structure of each of the gravity-wave modes in a five-layer model. For typical atmospheric profiles of $\overline{T}(\sigma)$ the fastest mode propagates at a speed on the order of 300 ms^{-1} and thereby imposes a severe constraint on the maximum stable time step with which these equations can be integrated using explicit time differencing.

Since the fastest-moving gravity waves do not need to be accurately simulated to obtain an accurate global weather forecast, (8.133)–(8.135) can be efficiently integrated using a semi-implicit scheme in which those terms that combine to form the left side of (8.137) are integrated using the trapezoidal method over a time interval of $2\Delta t$. The formulae that result from this semi-implicit approximation are

$$\delta_{2t}\mathbf{d}^n = \mathbf{f_d}^n - \nabla_\sigma^2\left[\langle\mathbf{h}^n\rangle^{2t} + \bar{\mathrm{t}}\,\langle(\ln p_{\mathrm{s}})^n\rangle^{2t}\right], \tag{8.138}$$

$$\delta_{2t}\mathbf{t}^n = \mathbf{f_t}^n - \mathbf{H}\,\langle\mathbf{d}^n\rangle^{2t}, \tag{8.139}$$

$$\delta_{2t}(\ln p_{\mathrm{s}})^n = f_p^n - \mathbf{p}^T\,\langle\mathbf{d}^n\rangle^{2t}. \tag{8.140}$$

Using the relation $\delta_{2t}\gamma^n = (\langle\gamma^n\rangle^{2t} - \gamma^{n-1})/\Delta t$ together with (8.136), (8.139), and (8.140) to eliminate $\langle\mathbf{h}^n\rangle^{2t}$ and $\langle(\ln p_{\mathrm{s}})^n\rangle^{2t}$ from (8.138) gives

$$\begin{aligned}&\left[\mathbf{I} - (\Delta t)^2\mathbf{B}\nabla_\sigma^2\right]\langle\mathbf{d}^n\rangle^{2t}\\ &\quad = \mathbf{d}^{n-1} + \Delta t\mathbf{f_d}^n - \nabla_\sigma^2\left\{\Delta t\left[\mathbf{h}^{n-1} + \bar{\mathrm{t}}(\ln p_{\mathrm{s}})^{n-1}\right] + (\Delta t)^2\left[\mathbf{G}\mathbf{f_t}^n + \bar{\mathrm{t}}f_p^n\right]\right\}.\end{aligned} \tag{8.141}$$

Let $\boldsymbol{\chi}_{r,s}$ be a column vector whose kth element is the coefficient of $Y_{r,s}$ in the series expansion for the velocity potential at level k. Since the spherical harmonics are eigenfunctions of the horizontal Laplacian operator on the sphere, (8.141) is equivalent to a linear-algebraic system for $\boldsymbol{\chi}_{r,s}^{n+1}$ of the form

$$\left[\mathbf{I} + (\Delta t)^2\frac{s(s+1)}{a^2}\mathbf{B}\right]\boldsymbol{\chi}_{r,\mathrm{s}}^{n+1} = \mathbf{f},$$

where $\mathbf{f}$ does not involve the values of any unknown functions at time $(n+1)\Delta t$. The N unknown variables in this relatively small linear system can be determined by Gaussian elimination. Additional efficiency can be achieved by exploiting the fact that the coefficient matrix is constant in time, so its "LU" decomposition into upper and lower triangular matrices need only be computed once.

Some of the forcing terms that are responsible for gravity-wave propagation in the σ-coordinate equations are nonlinear. To obtain the preceding linear-algebraic equation for $\boldsymbol{\chi}_{r,s}^{n+1}$ these terms have been decomposed into a linear part and a nonlinear perturbation by splitting the total temperature into a constant horizontally uniform reference temperature $\overline{T}(\sigma)$ and a perturbation. As discussed in Sect. 8.2.3, this decomposition imposes a constraint on the stability of the semi-implicit solution that, roughly speaking, requires the speed of the fastest-moving gravity wave supported by the actual atmospheric structure to be only modestly faster than the speed of the fasting-moving gravity wave in the reference state. This stability constraint is usually satisfied by choosing an isothermal profile for the reference state, i.e., $\overline{T}(\sigma) = T_0$ (Simmons et al. 1978). A typical value for T_0 is 300 K.

Problems

1. Consider small-amplitude shallow-water motions on a "mid-latitude β-plane." In the following, x and y are horizontal coordinates oriented east–west and north–south, respectively; the Coriolis parameter is approximated as $f_0 + \beta y$ where f_0 and β are constant; g is the gravitational acceleration; $U > 0$ is a constant mean flow from west to east; u' and v' are the perturbation west-to-east and south-to-north velocities; h is the perturbation displacement of the free surface. Define the vorticity ζ and the divergence δ as

$$\zeta = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}, \qquad \delta = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}.$$

Assume that the mean flow is in geostrophic balance,

$$U = -\frac{g}{f_0}\frac{\partial \overline{h}}{\partial y},$$

and that there is a mean north–south gradient in the bottom topography equal to the mean gradient in the height of the free surface, $\partial\overline{h}/\partial y$, so that the mean fluid depth is a constant H. The linearized shallow-water equations for this system are

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta + f\delta + \beta v = 0,$$
$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\delta - f\zeta + \beta(U + u) + g\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = 0,$$
$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)h + H\delta = 0. \quad (8.142)$$

The terms involving β in the preceding vorticity and divergence equations can be approximated[11] as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\zeta + f_0\delta + \frac{\beta g}{f_0}\frac{\partial h}{\partial x} = 0, \quad (8.143)$$
$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\delta - f_0\zeta + g\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = 0. \quad (8.144)$$

(a) Show that waves of the form

$$(\zeta, \delta, h) = (\zeta_0, \delta_0, h_0)\mathrm{e}^{\mathrm{i}(kx+\ell y-\omega t)}$$

are solutions to the preceding system if they satisfy the dispersion relation

$$(\omega - Uk)^2 = c^2(k^2 + \ell^2) + f_0^2 + \frac{k\beta c^2}{\omega - Uk},$$

where $c^2 = gH$.

(b) Show that if $|\beta/c| \ll k^2$, the individual solutions to this dispersion relation are well approximated by the solutions to either the inertial-gravity-wave dispersion relation

$$(\omega - Uk)^2 = c^2(k^2 + \ell^2) + f_0^2$$

or the Rossby-wave dispersion relation

$$\omega = Uk - \frac{\beta k}{k^2 + \ell^2 + f_0^2/c^2}.$$

2. Suppose that the time derivatives in (8.142)–(8.144) are approximated by leapfrog differencing and the spatial dependence is represented by a Fourier spectral or pseudospectral method. Recall that we have assumed $U > 0$, as would be the case in the middle latitudes of the Earth's atmosphere.

(a) Determine the constraints on Δt required to keep the gravity waves stable and show that $(U + c)k\Delta t \leq 1$ is a necessary condition for stability.

[11] The approximations used to obtain (8.143) and (8.144) are motivated by the desire to obtain a clean dispersion relation rather than a straightforward scale analysis.

(b) Determine the constraints on Δt required to keep the Rossby waves stable. Let K be the magnitude of the maximum vector wave number retained in the truncation, i.e.,

$$K = \max_{k,\ell} \sqrt{k^2 + \ell^2}.$$

Show that $UK\Delta t \leq 1$ is a sufficient condition for the stability of the Rossby waves unless

$$K^2 \leq \frac{\beta}{2U} - \frac{f_0^2}{c^2}.$$

3. Suppose that (8.142)–(8.144) are integrated using the semi-implicit scheme

$$\delta_{2t}\zeta^n + U\frac{\partial \zeta^n}{\partial x} + f_0\delta^n + \frac{\beta g}{f_0}\frac{\partial h^n}{\partial x} = 0,$$

$$\delta_{2t}\delta^n + U\frac{\partial \delta^n}{\partial x} - f_0\zeta^n + g\left\langle \frac{\partial^2 h^n}{\partial x^2} + \frac{\partial^2 h^n}{\partial y^2}\right\rangle^{2t} = 0,$$

$$\delta_{2t}h^n + U\frac{\partial h^n}{\partial x} + H\left\langle \delta^n\right\rangle^{2t} = 0.$$

(a) Determine the conditions under which the gravity waves are stable.

(b) Determine the conditions under which the Rossby waves are stable.

(c) Discuss the impact of semi-implicit differencing on the accuracy of the Rossby-wave and gravity-wave modes.

4. Two-dimensional sound waves in a neutrally stratified atmosphere satisfy the linearized equations

$$\frac{\partial u}{\partial t} + c_s\frac{\partial P}{\partial x} = 0.$$

$$\frac{\partial w}{\partial t} + c_s\frac{\partial P}{\partial z} = 0.$$

$$\frac{\partial P}{\partial t} + c_s\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0,$$

where $P = p'/(\rho_0 c_s)$. Let this system be approximated using forward–backward differencing for the horizontal gradients and trapezoidal differencing for the vertical gradients such that

$$u^{m+1} = u^m - c_s\Delta\tau\frac{\partial P^m}{\partial x},$$

$$w^{m+1} = w^m - c_s\frac{\Delta\tau}{2}\frac{\partial}{\partial z}\left(P^{m+1} + P^m\right),$$

$$P^{m+1} = P^m - c_s\Delta\tau\left(\frac{\partial u^{m+1}}{\partial x} - \frac{1}{2}\frac{\partial}{\partial z}\left(w^{m+1} + w^m\right)\right).$$

Consider an individual Fourier mode with spatial structure $\exp i(kx + \ell z)$ and show that the eigenvalues of the amplification matrix for this scheme are unity and

$$\frac{4 - \tilde{\ell}^2 - 2\tilde{k}^2 \pm 2\sqrt{(\tilde{k} - 2)(\tilde{k} + 2)(\tilde{k}^2 + \tilde{\ell}^2)}}{\tilde{\ell}^2 + 4},$$

where $\tilde{k} = c_s k \Delta\tau$ and $\tilde{\ell} = c_s \ell \Delta\tau$. What is the stability condition that ensures that this method will not have any eigenvalues with absolute values exceeding unity?

5. Compare the errors generated in gravity waves using semi-implicit differencing with those produced in a compressible Boussinesq system in which the true speed of sound c_s is artificially reduced to $\tilde{c}_s$ in an effort to increase efficiency by increasing the maximum stable value for $\Delta\tau$. *Hint:* Consider waves with wave numbers on the order of $N/\tilde{c}_s$ but larger than N/c_s.

6. The oscillations of the damped-harmonic oscillator (8.88) are "overdamped" when $\alpha^2\kappa^2 > c_s^2$. Suppose that the mesh is isotropic with grid interval $\varDelta$, the Courant number for sound-wave propagation on the small time step is $1/2$, and $\alpha = \gamma\varDelta^2/\varDelta\tau$. Estimate the minimum value of α required make the divergence damper overdamp a mode resolved on the numerical mesh. Do the values of α_x and α_z used in the test problem shown in Fig. 8.3d overdamp any of the resolved modes in that test problem?

7. Show that discrete integral of the finite-difference approximation to the vertical divergence of the vertical advective flux of kinetic energy,

$$\sum_{k=1}^{N} \left[\left\langle \langle \mathbf{u}_k \rangle^{\sigma} \cdot \dot{\sigma}_k \, \delta_\sigma \mathbf{u}_k \right\rangle^{\sigma} + \frac{\mathbf{u}_k \cdot \mathbf{u}_k}{2} \, \delta_\sigma \dot{\sigma}_k \right] \varDelta\sigma_k,$$

is zero.

8. Examine the mass-conservation properties of σ-coordinate primitive-equation models.

(a) Show that the vertically integrated mass per unit area in a hydrostatically balanced atmosphere is p_s/g.

(b) Show that the vertical finite-difference scheme for the σ-coordinate primitive-equation model described in Sect. 8.6.3 will conserve total mass if nonconservative effects due to time differencing and the horizontal spectral representation are neglected.

(c) Suppose that instead of predicting $\ln p_s$ as in (8.116), the actual surface pressure were predicted using (8.105). Show that except for nonconservative effects due to time differencing, total mass will be exactly conserved in a numerical model in which the Galerkin spectral method is used to evaluate the horizontal derivatives, and the vertical derivative is approximated by (8.120). This approach is not used in practice because it generates a noisier solution than that obtained using $\ln p_s$ as the prognostic variable (Kiehl et al. 1996, p. 15).