Min-Cost Arborescence

Edmonds Algorithm

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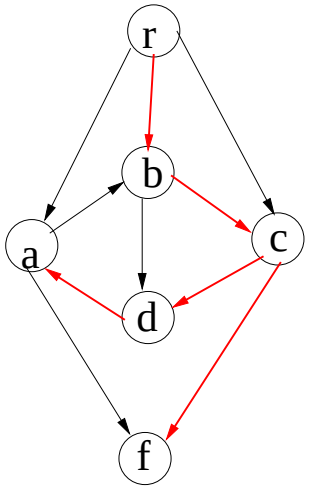
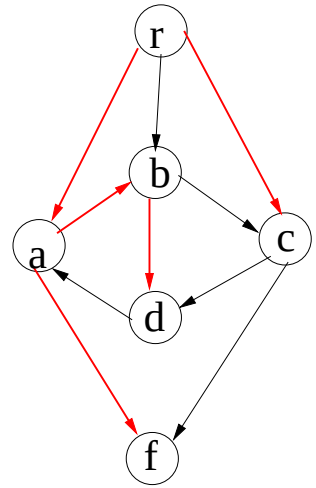
Defintion: arborescence (a.k.a. connected branching)

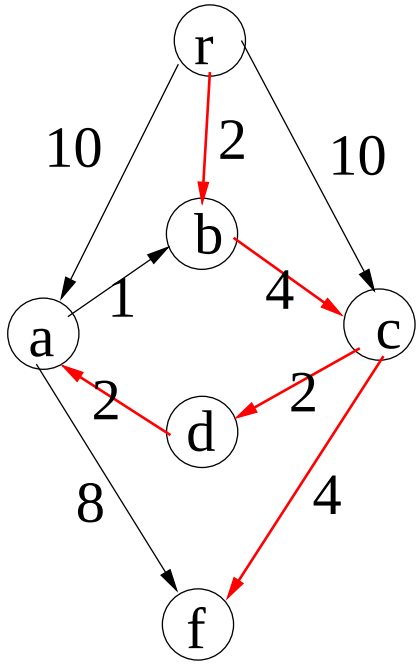




An *arborescence w.r.t.* r, $T(V,F)$, of a directed graph $G(V,E)$ is a spanning tree of G if we ignore the direction of edges; and there is a path in T from r to each node in $V - \{r\}$ if we take the direction of edges into account.

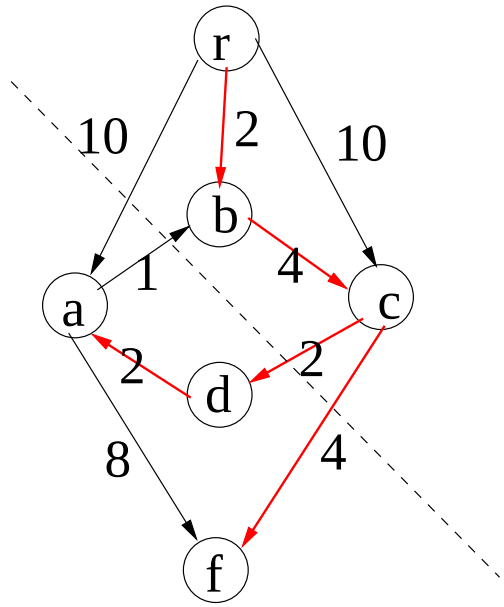
- A subgraph $T(V,F)$ is an arborescence w.r.t. root r iff T has no cycles, and for each node $v \neq r$, there is exactly one edge in F that enters v.
- A directed graph G has an arborescence rooted at r iff there is a directed path from r to each of $V - \{r\}$.

Problem: Min-Cost Arborescence (MCA)



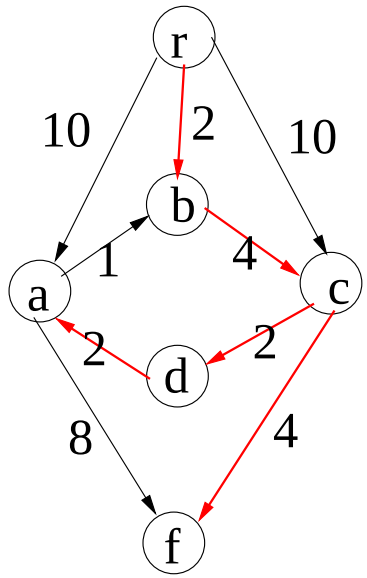
Given a weighted directed graph $G(V, E)$ with a distinguished root node r and with a positive cost w_e on each edge e, compute an arborescence rooted at r of minimum total cost.

MST cut property (of undirected graphs) not applicable



ab is not part of MCA

MST cycle property (of undirected graphs) not applicable



max weighted edge *bc* of cycle *abcd* is part of MCA

Outline

LP for the problem (P)

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij}$$

$$\sum_{(i,j)\in E, i\in S, j\in V-S} x_{ij} \geq 1 \qquad \text{for every } S \subseteq V, r \in S$$

$$x_{ij} \in \{0,1\} \qquad \text{for every } (i,j) \in E$$

Relaxed LP

$\min \sum_{(i,j)\in E} c_{ij}x_{ij}$

$\sum_{(i,j)\in E, i\in S, j\in V-S} x_{ij} \geq 1 \qquad$ for every $S \subseteq V, r \in S \qquad$ ——— (1)

$0 \leq x_{ij} \leq 1 \qquad$ for every $(i,j) \in E \qquad$ ———- (2)

(1) essentially ensures that for every $r \in S, S \subset V$, $\delta^+(S)$ has positive weight

Edmonds has proven that the relaxed LP polytope is same as P — not proved in class

Separation Oracle

- explicitly detecting a violated one in (2) —> takes linear time
- detecting a violated one in (1) is equivalent to finding a r-t min-cut for some $t \in V - \{r\}$ such that the cut capacity is less than one —> takes $O((n-1)n^3)$ time

i.e., although the number of constraints in the LP are exponential, (weakly) polynomail algorithm exists through Ellipsoid method!

Outline

Properties

For each vertex $v \in V - \{r\}$, let S_v be the arc set entering v; suppose choosing a least cost arc in S_v for every vertex v, caused an arborescence $T(V, F)$, then T is a min-cost arborescence.

- consider G' with modified costs: subtract w_{e_v} from each arc in S_v for any/every vertex v of $G(V, E)$ wherein e_v is the least cost arc entering v
  - $T(V, F)$ is an optimal arborescence for G iff $T'(V, F)$ is an optimal arborescence for G'
  - further, the cost of T' is zero

Properties (cont)

In other words,

- when the direction of arcs is ignored, if T' is a spanning tree of G', then T' is a min-cost arborescence.
- otherwise, T' must be having at least one zero cost cycle.

Properties (cont)

Let C be a zero cost cycle in G', then there is an optimal arborescence T'' rooted at r that has exactly one arc entering C in G'.

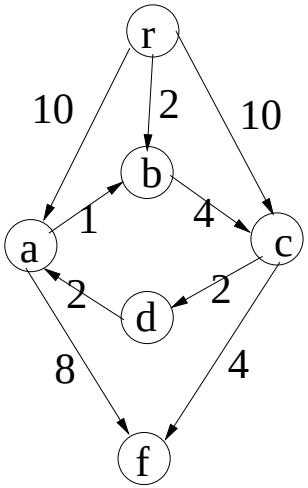
- among all paths that lead to C from r in G', choose the shortest one, say P, in forming T''; let P incident to $v \in C$; include all arcs of C except the one that has v as the head
- a T that has two paths to C from r is an optimal arborescence $\Rightarrow T''$ is an optimal arborescence

(Greedy) Algorithm

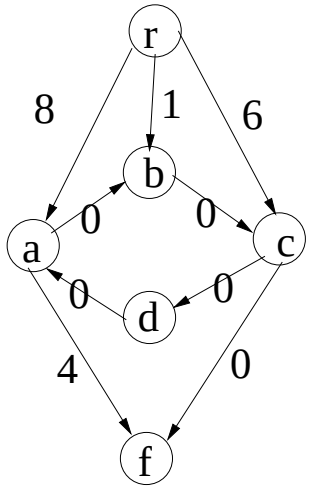
FindMincostArbo(G)

1. obtain G' from G (as detailed above)
2. obtain not necessarily connected T' from G' (as detailed above)
3. if T' is an arborescence, then return T'
4. contract vertices of G' that correspond to some cycle C in T', and let the resultant graph be G''
5. $(V'', F'') \leftarrow$ FindMincostArbo(G'')
6. extend (V'', F'') to an arborescence (V', F') in G' by including all arcs of $F'' \cup C$ except one arc in C whose head is a head of some arc in F''

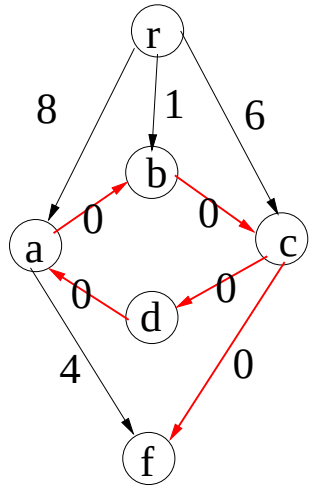
Algorithm in Execution



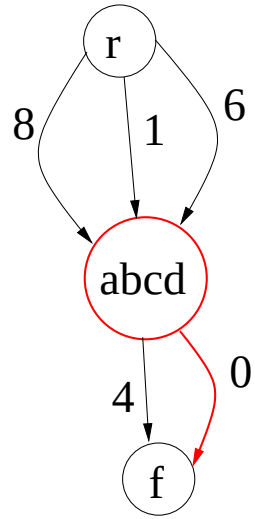
G



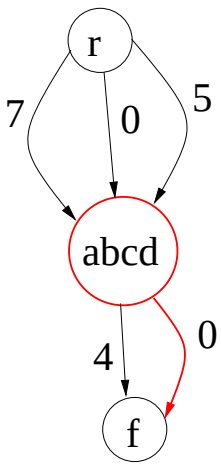
G'_1



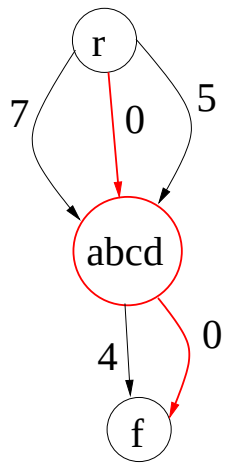
T'_1



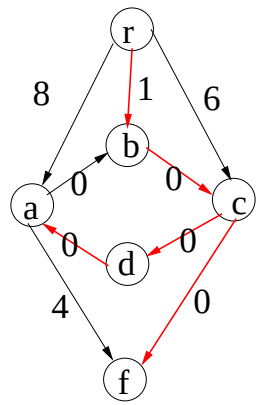
G_2: G'_1 after contracting *abcd* cycle in T'



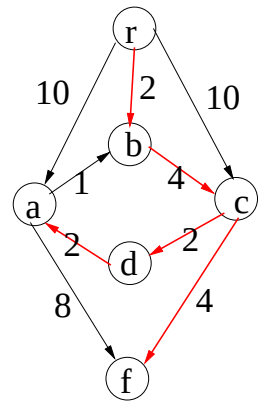
G'_2



T'_2



expand *abcd* and remove *ab* to obtain a MCA in G'_1



corresponding MCA in G

Correctness

induction on the number of nodes in G

Analysis

$O(nm)$ time