Final Project: Mathematical Modeling and Simulation

Objectives

The objective of this project is to model and simulate the oscillation of a pendulum considering friction using analytical and numerical methods.

1. Pendulum Oscillation Modelling

Consider the pendulum with a length *l* and a mass *m* (Fig. 4).

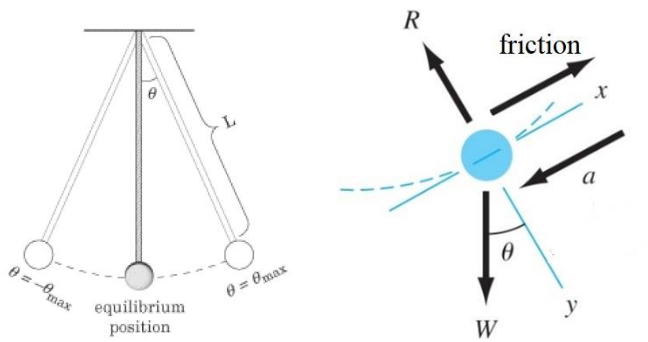

Figure 4. Free body diagram of a simple pendulum

Using equilibrium of forces in the polar system, the forces acting on the system are the string's tension force, gravity and friction. The gravitational force has two components, one along the y axis (string direction) and one along the x-axis. The y component of the gravitational force is equal in magnitude as the tension force but in opposite directions, thus these two forces will cancel out. Therefore, the resultant forces acting on the pendulum are:

$$F_g = -mgsin \tag{1a}$$

$$F_f = -\lambda \frac{ds}{dt} \tag{1b}$$

where F_g is the gravitational force along the *x* axis, F_f is the friction force, λ is the friction coefficient, *s* is the displacement as a vector tangential to the motion curve, *m* is the mass of the pendulum and *g* is the gravitational acceleration. We know that:

$$s = r\theta \qquad \frac{ds}{dt} = r\frac{d\theta}{dt} \qquad \frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2} \tag{2}$$

where r is the distance from the pivot point (fulcrum) to the centre of gravity of the pendulum. From Newton's second law we have:

$$F_g + F_f = m\frac{d^2s}{dt^2}$$

$$m\frac{d^2s}{dt^2} = -mgsin\,\theta - \lambda\frac{ds}{dt} \tag{3}$$

Using Eq. 2 we have:

$$m\frac{d^2\theta}{dt^2} = \frac{-mgsin\,\theta}{r} - \lambda\frac{d\theta}{dt}$$

$$m\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + \frac{mgsin\,\theta}{r} = 0 \tag{4}$$

For small θ, consider this approximation:

$\sin\theta \approx \theta$

$$m\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + \frac{mg\theta}{r} = 0 \tag{5}$$

For simplicity, let $k = \sqrt{\frac{mg}{r}}$:

$$m\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + k^2\theta = 0 \tag{6}$$

Eq. 6 shows the second order differential equation that governs the system which is valid for small θ. Eq. 6 has an oscillatory solution as follows.

$$\theta(t) = \exp(\frac{-\lambda t}{2m})(Acos(\alpha t) + Bsin(\alpha t)) \tag{7}$$

where A and B are two constants that need to be found using initial conditions and:

$$\alpha = \sqrt{\frac{k^2}{m} - \frac{\lambda^2}{4m^2}} \tag{8}$$

Eq. 7 is valid only if $\lambda^2 < 4k^2m$.

In order to find the constants A and B in Eq. 7, we assume that the pendulum is released from an angle θ_0 with no initial velocity:

$$\begin{aligned} \theta(0) &= \theta_0 \\ \theta'(0) &= 0 \end{aligned} \tag{9}$$

Therefore:

$$A = \theta_0 \qquad B = \frac{\lambda\theta_0}{2m\alpha} \tag{10}$$

And we get:

$$\theta(t) = \theta_0 \, exp\left(\frac{-\lambda t}{2m}\right)\left(cos(\alpha t) + \frac{\lambda}{2m\alpha} sin(\alpha t)\right) \tag{11}$$

Equation 11 will define the position of the pendulum at any given time. The project assumptions are shown in Table 1.

Table 1. Project assumptions

m (pendulum mass)	0.3 Kg
r (distance from the pivot point (fulcrum) to the centre of gravity of the pendulum)	1 m
λ (friction coefficient)	0.02 <λ< 0.1

2. Project requirements

In this section, students will compare numerical results for Eq. (4) and (6) with the analytical solution (Eq. (11)). Please note that Eq. (4) and (6) represent non-linear and linearized pendulum governing equations, respectively.

A. Displacement Analysis

1. Choose three values for θ_o less than 20°.
2. Based on the mathematical model Eq. (11)), evaluate the position of the pendulum for a time of 8s.
3. Develop an algorithm for the Runge-Kutta 4th-order method in Matlab (See pseudocode at the end of this document). Solve the second order ODE (Eq. (4) and (6)) numerically with the three θ_o amplitudes. Simulate the pendulum for 8s with a time step of $h = 0.1s$. Choose 4 values in the specified range for λ (see Table 1).
   In a single graph for each θ_o amplitude and each λ, plot the oscillatory curve resulted from the analytical method (Eq. (11)), and the oscillatory curves resulted from the Runge-Kutta 4th-order method (Eq. (4) and (6)). Repeat the same process with the other θ_o amplitudes and λ (Note that Eq. 11 has to be evaluated for different values of θ_o and λ). Comment your results.

B. Velocity Analysis

4. Plot the velocities that were resulted using the ODE (Eq. (4) and (6)) (numerical method) for all three θ_o amplitudes and four values for λ and the analytical results (You need to find the derivative of Eq. (11) to obtain the analytical velocity of the pendulum). Comment on the results.

Include the Runge-Kutta 4th-order method code (part 3) in your submission of this lab report.

Project deliverables

1. A brief project report including an explanation of your modeling including code, brief analysis including plotting and data results.
2. Project source code, ideally all of the three different methods (Q2, Q4, Q5) are written in 3 separate files (MATLAB code files)

Grading

This project is worth 15% of the final course grade. Report Quality must include the following:

Final Report, including:	
▪ Introduction and description of the problem (Problem statement)	**5%**
▪ Description of objectives and modeling process	**5%**
▪ Results	**30%**
▪ Discussions	**30%**
▪ Conclussions	**15%**
▪ MATLAB code files with proper comments	**10%**

100%

Project Selection:

This is a group project, please submit your report in Canvas.

Project submission:

Submission of the project is **April 11th, 2023 11:59 pm** via Canvas.

Definition of Runge-Kutta 4th-order method for 2nd order ODE:

$$\frac{d^2x}{dt^2} = f(t,x,v), \quad \frac{dx}{dt} = v, \quad x(t_0) = x_0, \quad v(t_0) = v_0$$

$$t_{i+1} = t_i + h, \qquad h = stepsize$$

$$dx_1 = h \times v_i$$

$$dv_1 = h \times f(t_i, x_i, v_i)$$

$$dx_2 = h\left(v_i + \frac{1}{2}dv_1\right)$$

$$dv_2 = h \times f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}dx_1, v_i + \frac{1}{2}dv_1\right)$$

$$dx_3 = h\left(v_i + \frac{1}{2}dv_2\right)$$

$$dv_3 = h \times f\left(t_i + \frac{1}{2}h, x_i + \frac{1}{2}dx_2, v_i + \frac{1}{2}dv_2\right)$$

$$dx_4 = h(v_i + dv_3)$$

$$dv_4 = h \times f(t_i + h, x_i + dx_3, v_i + dv_3)$$

$$x_{i+1} = x_i + \frac{1}{6}(dx_1 + 2dx_2 + 2dx_3 + dx_4)$$

$$v_{i+1} = v_i + \frac{1}{6}(dv_1 + 2dv_2 + 2dv_3 + dv_4)$$

References

[1]https://a1384-235428.cluster8.canvas-user
content.com/courses/1384~1159/files/1384~235428/course%20files/apb11o/labs/L105/L105_pend.htm

[2] https://www.wikihow.com/Build-and-Use-a-Pendulum