ENGR 2420: Lab 3 Prelab Solution

October 8, 2020

Prelab

1. **Forward-Active Bipolar Transistor Characteristics**.

   (a) We know that $I_\mathrm{e} = I_\mathrm{c}/\alpha$, that $I_\mathrm{b} = I_\mathrm{c}/\beta$, and that $I_\mathrm{e} = I_\mathrm{c} + I_\mathrm{b}$. By substituting the two former relations into the third, we have that

$$\begin{aligned} I_\mathrm{e} &= I_\mathrm{c} + I_\mathrm{b} \\ \frac{I_\mathrm{c}}{\alpha} &= I_\mathrm{c} + \frac{I_\mathrm{c}}{\beta} \\ \frac{I_\mathrm{c}}{\alpha} &= \left(1 + \frac{1}{\beta}\right) I_\mathrm{c} \\ \frac{1}{\alpha} &= \frac{1+\beta}{\beta} \\ \alpha &= \frac{\beta}{1+\beta}. \end{aligned}$$

   Alternatively, we can solve this equation for β, thereby obtaining

$$\begin{aligned} \frac{1}{\beta} &= \frac{1}{\alpha} - 1 \\ \frac{1}{\beta} &= \frac{1-\alpha}{\alpha} \\ \beta &= \frac{\alpha}{1-\alpha}. \end{aligned}$$

   (b) We know that the base current related to the base and emitter voltages through

$$I_\mathrm{b} = \frac{I_\mathrm{s}}{\beta} \cdot e^{(V_\mathrm{b} - V_\mathrm{e})/U_\mathrm{T}}.$$

   We can invert this relationship, thereby obtaining

$$V_\mathrm{b} = V_\mathrm{e} + U_\mathrm{T} \log \frac{\beta I_\mathrm{b}}{I_\mathrm{s}},$$

   which we can differentiate with respect to I_b to obtain the incremental resistance of the base termianl, r_b, which is given by

$$r_\mathrm{b} = \frac{\partial V_\mathrm{b}}{\partial I_\mathrm{b}} = 0 + U_\mathrm{T} \cdot \frac{1}{\beta I_\mathrm{b}/I_\mathrm{s}} \cdot \frac{\beta}{I_\mathrm{s}} = \frac{U_\mathrm{T}}{I_\mathrm{b}}.$$

(c) We know that the collector current is related to the base and emitter voltages through

$$I_\mathrm{c} = I_\mathrm{s} e^{(V_\mathrm{b} - V_\mathrm{e})/U_\mathrm{T}},$$

which we can differentiate with respect to V_b to obtain an expression for the incremental transconductance gain, g_m, of the bipolar transistor. Doing so, we have that

$$g_\mathrm{m} = \frac{\partial I_\mathrm{c}}{\partial V_\mathrm{b}} = \underbrace{I_\mathrm{s} e^{(V_\mathrm{b} - V_\mathrm{e})/U_\mathrm{T}}}_{I_\mathrm{c}} \cdot \frac{1}{U_\mathrm{T}} = \frac{I_\mathrm{c}}{U_\mathrm{T}}.$$

(d) From part c, we know that

$$r_\mathrm{b} = \frac{U_\mathrm{T}}{I_\mathrm{b}} = \beta \cdot \frac{U_\mathrm{T}}{\beta I_\mathrm{b}} = \beta \cdot \frac{U_\mathrm{T}}{I_\mathrm{c}} = \frac{\beta}{g_\mathrm{m}}.$$

2. **Emitter-Degenerated Bipolar Transistor**.

(a) From Section 3.3 of the bipolar transistor handout, we know have that the incremental resistance of the base terminal of a bipolar transistor whose emitter is degenerated by a resistor, R, is given by

$$R_\mathrm{b} = r_\mathrm{b} + (\beta + 1)\, R = \frac{U_\mathrm{T}}{I_\mathrm{b}} + (\beta + 1)\, R \approx \frac{U_\mathrm{T}}{I_\mathrm{b}} + \beta R = \beta R \left(1 + \frac{U_\mathrm{T}}{\beta I_\mathrm{b} R}\right),$$

where the approximation follows if $\beta \gg 1$.

(b) From Section 3.3 of the bipolar transistor handout, we also know that the incremental transconductance gain of the bipolar transistor whose emitter is degenerated by a resistor, R, is given by

$$\begin{aligned} G_\mathrm{m} &= \frac{g_\mathrm{m}}{1 + R/r_\mathrm{e}} \\ &= \frac{g_\mathrm{m}}{1 + g_\mathrm{m} R/\alpha} \\ &= \frac{\alpha}{R} \cdot \frac{1}{1 + \alpha/g_\mathrm{m} R} \\ &= \frac{\alpha}{R} \cdot \frac{1}{1 + \alpha U_\mathrm{T}/I_\mathrm{c} R} \\ &\approx \frac{1}{R} \cdot \frac{1}{1 + U_\mathrm{T}/I_\mathrm{c} R}, \end{aligned}$$

where the approximation follows if $\alpha \approx 1$, which from part a of question 1 is true if $\beta \gg 1$.

(c) Before we make the change in V_b by δV_b, we know by KCL that the emitter current is given both by

$$I_\mathrm{e} = \frac{I_\mathrm{s}}{\alpha} \cdot e^{(V_\mathrm{b} - V_\mathrm{e})/U_\mathrm{T}},$$

and by

$$I_\mathrm{e} = \frac{V_\mathrm{e}}{R}.$$

The change in V_b results in a change in I_e by δI_e and in V_e by δV_e. After the change, we can thus express the emitter current using the transistor characterisic as

$$\begin{aligned}
I_\mathrm{e} + \delta I_\mathrm{e} &= \frac{I_\mathrm{s}}{\alpha} \cdot e^{((V_\mathrm{b}+\delta V_\mathrm{b})-(V_\mathrm{e}+\delta V_\mathrm{e}))/U_\mathrm{T}} \\
&= \frac{I_\mathrm{s}}{\alpha} \cdot e^{(V_\mathrm{b}-V_\mathrm{e})/U_\mathrm{T}} \cdot e^{(\delta V_\mathrm{b}-\delta V_\mathrm{e})/U_\mathrm{T}} \\
&= I_\mathrm{e} e^{(\delta V_\mathrm{b}-\delta V_\mathrm{e})/U_\mathrm{T}} \\
&\approx I_\mathrm{e}\left(1 + \frac{\delta V_\mathrm{b} - \delta V_\mathrm{e}}{U_\mathrm{T}}\right) \\
&= I_\mathrm{e} + \frac{\delta V_\mathrm{b} - \delta V_\mathrm{e}}{U_\mathrm{T}/I_\mathrm{e}},
\end{aligned}$$

where the approximation follows if $|\delta V_\mathrm{b} - \delta V_\mathrm{e}| \ll U_\mathrm{T}$, by expanding the exponential in a Taylor series and truncating after the linear term. By subtracting I_e from both sides of this equation, we obtain an expression for δI_e, given by

$$\delta I_\mathrm{e} = \frac{\delta V_\mathrm{b} - \delta V_\mathrm{e}}{U_\mathrm{T}/I_\mathrm{e}}.$$

Likewise, we can express the emitter current after the change using Ohm's law as

$$I_\mathrm{e} + \delta I_\mathrm{e} = \frac{V_\mathrm{e} + \delta V_\mathrm{e}}{R} = \underbrace{\frac{V_\mathrm{e}}{R}}_{I_\mathrm{e}} + \frac{\delta V_\mathrm{e}}{R}.$$

By subtracting I_e from both sides of this equation, we obtain a second expression for δI_e, given by

$$\delta I_\mathrm{e} = \frac{\delta V_\mathrm{e}}{R}.$$

By equating these two expressions for δI_e, we can solve for δV_e to find that

$$\delta V_\mathrm{e} = \frac{I_\mathrm{e}}{I_\mathrm{e} + (U_\mathrm{T}/R)} \cdot \delta V_\mathrm{b}.$$

(d) The change in V_e will be half the change in V_b when $I_\mathrm{e} = U_\mathrm{T}/R$. At this emitter current level, the collector current, which is I_on, is be given by

$$I_\mathrm{on} = \alpha\frac{U_\mathrm{T}}{R} \approx \frac{U_\mathrm{T}}{R},$$

where the approximation follows if $\alpha \approx 1$. Using this result, we can express our final result from part a in terms of I_on as

$$R_\mathrm{b} \approx \beta R\left(1 + \frac{U_\mathrm{T}}{\beta I_\mathrm{b} R}\right) = \beta R\left(1 + \frac{I_\mathrm{on}}{\beta I_\mathrm{b}}\right).$$

Likewise, we can express our final result from part b in terms of I_{on} as

$$G_{\mathrm{m}} \approx \frac{1}{R} \cdot \frac{1}{1 + U_{\mathrm{T}}/I_{\mathrm{c}}R} = \frac{1}{R} \cdot \frac{1}{1 + I_{\mathrm{on}}/I_{\mathrm{c}}}.$$

(e) By using the transistor's characteristic, we can express V_{on} as

$$V_{\mathrm{on}} = U_{\mathrm{T}} \log \frac{I_{\mathrm{on}}}{I_{\mathrm{s}}}.$$

(f) By applying KCL at the emitter node, we have that

$$\begin{aligned}
\frac{V_{\mathrm{e}}}{R} &= \frac{I_{\mathrm{s}}}{\alpha} \cdot e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \\
0 &= -V_{\mathrm{e}} + R \cdot \frac{I_{\mathrm{s}}}{\alpha} \cdot e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \\
\frac{V_{\mathrm{b}}}{U_{\mathrm{T}}} &= \frac{V_{\mathrm{b}} - V_{\mathrm{e}}}{U_{\mathrm{T}}} + \frac{I_{\mathrm{s}}}{\alpha U_{\mathrm{T}}/R} \cdot e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \\
\frac{V_{\mathrm{b}}}{U_{\mathrm{T}}} &= \frac{V_{\mathrm{be}}}{U_{\mathrm{T}}} + \frac{I_{\mathrm{s}}}{I_{\mathrm{on}}} \cdot e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \\
\frac{V_{\mathrm{b}}}{U_{\mathrm{T}}} &= \frac{V_{\mathrm{be}}}{U_{\mathrm{T}}} + e^{-V_{\mathrm{on}}/U_{\mathrm{T}}} e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \\
\frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} &= \frac{V_{\mathrm{be}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} + e^{(V_{\mathrm{be}} - V_{\mathrm{on}})/U_{\mathrm{T}}}.
\end{aligned}$$

(g) If $V_{\mathrm{be}} < V_{\mathrm{on}}$ by more than a few U_{T}, then it follows that

$$e^{(V_{\mathrm{be}} - V_{\mathrm{on}})/U_{\mathrm{T}}} \ll \left| \frac{V_{\mathrm{be}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} \right|.$$

In this regime, we can approximate the result from part f as

$$\frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} = \frac{V_{\mathrm{be}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} + e^{(V_{\mathrm{be}} - V_{\mathrm{on}})/U_{\mathrm{T}}} \approx \frac{V_{\mathrm{be}} - V_{\mathrm{on}}}{U_{\mathrm{T}}},$$

which implies that $V_{\mathrm{e}} \approx 0$ and that

$$I_{\mathrm{c}} = I_{\mathrm{s}} e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \approx I_{\mathrm{s}} e^{V_{\mathrm{b}}/U_{\mathrm{T}}}.$$

(h) If $V_{\mathrm{be}} > V_{\mathrm{on}}$ by more than a few U_{T}, then it follows that

$$e^{(V_{\mathrm{be}} - V_{\mathrm{on}})/U_{\mathrm{T}}} \gg \frac{V_{\mathrm{be}} - V_{\mathrm{on}}}{U_{\mathrm{T}}}.$$

In this regime, we can approximate the result from part f as

$$\frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} = \frac{V_{\mathrm{be}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} + e^{(V_{\mathrm{be}} - V_{\mathrm{on}})/U_{\mathrm{T}}} \approx e^{(V_{\mathrm{be}} - V_{\mathrm{on}})/U_{\mathrm{T}}},$$

which implies that

$$\begin{aligned} e^{V_{\mathrm{be}}/U_{\mathrm{T}}} &\approx e^{V_{\mathrm{on}}/U_{\mathrm{T}}} \cdot \frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} \\ &= \alpha \cdot \frac{U_{\mathrm{T}}}{R I_{\mathrm{s}}} \cdot \frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{U_{\mathrm{T}}} \\ &= \frac{\alpha}{I_{\mathrm{s}}} \cdot \frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{R}. \end{aligned}$$

The collector current is then given by

$$I_{\mathrm{c}} = I_{\mathrm{s}} e^{V_{\mathrm{be}}/U_{\mathrm{T}}} \approx I_{\mathrm{s}} \cdot \frac{\alpha}{I_{\mathrm{s}}} \cdot \frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{R} = \alpha \cdot \frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{R} \approx \frac{V_{\mathrm{b}} - V_{\mathrm{on}}}{R},$$

where the aproximation follows from the fact that $\alpha \approx 1$.