Master Theorem.

分治法的递归模型

分治递类型问题的 recurrence 模型.

$$T(n) = aT\left(\frac{n}{b}\right) + f(n).$$

子问题大小

问题规模大小

原问题的子问题个数.

合并的时间.

二分查找: $T(n) = T\left(\frac{n}{2}\right) + \Theta(1)$.

$a=1,\ b=2,\ f(n)=\Theta(1)$.

$n^{\log_a b} = n^{\log_2 1} = n^0 = 1$.

$\Theta(n^{\log_2 1}) = \Theta(1)$.

Solve recurrence by substitution.

例:

$$T(n) = \begin{cases} 1 & n=0 \\ T(n-1)+1 & n>0. \end{cases}$$

substitute

Expand scratch.

令 $n = n-1$.

$T(n-1) = T(n-2)+1$

$T(n-2) = T(n-3)+1$.

Build solution:

pattern.

$T(n) = T(n-1) + 1$

$T(n) = [T(n-2)+1]+1$

$T(n) = [T(n-3)+1]+1$.

⋮ continue for k times

$T(n) = T(n-k) + k$.

We have "$T(n) = T(n-k) + k$".
Assume $n-k=0$. $\rightarrow n=k$.

$$T(n) = T(n-n) + n$$
$$T(n) = T(0) + n = n+1.$$

Build solution

例 2

$$T(n) = T(n-1) + n$$
$$T(n) = [T(n-2) + (n-1)] + n.$$
$$T(n) = [T(n-3) + (n-2)] + (n-1) + n$$
$$T(n) = [T(n-4) + (n-3)] + (n-2) + (n-1) + n.$$

continue k times.

T(

Expand scratch:

$T(n-1) = T(n-2) + (n-1)$
$T(n-2) = T(n-3) + (n-2)$.
$T(n-3) = T(n-4) + (n-3)$
etc.

例 3. $T(1)=4$, $T(n)=2T(\frac{n}{2})+4n$

Build Solution:

$T(n)=2T(\frac{n}{2})+4n$

$T(n)=2\left[2T(\frac{n}{2^2})+4(\frac{n}{2})\right]+4n$

$=2^2T(\frac{n}{2^2})+4n+4n$

$T(n)=2^2\left[2T(\frac{n}{2^3})+4(\frac{n}{2^2})\right]+4n+4n.$

$=2^3T(\frac{n}{2^3})+\underbrace{4n+4n+4n}_{3\text{ times}}$

$\vdots$

$T(n)=2^kT(\frac{n}{2^k})+k\cdot(4n)$

Expand:

$T(\frac{n}{2})=2T(\frac{n}{2^2})+4(\frac{n}{2})$

$T(\frac{n}{2^2})=2T(\frac{n}{2^3})+4(\frac{n}{4}).$

Base case here:

$T(1)=4.$

$\therefore$ 令 $\frac{n}{2^k}=1$, $n=2^k$, $\lg(n)=\lg(2^k)$

$\rightarrow k=\lg n$

$T(n)=2^{\lg n}\cdot T(1)+\lg n\,(4n)$

$=n\cdot 4+(4n)(\lg n)$

$=4n+(4n)(\lg n)$

$=\Theta(n\lg n).$

例4.

$$T(n) = T(\lfloor \tfrac{n}{2} \rfloor) + T(\lceil \tfrac{n}{2} \rceil) + f(n)$$

Carl 上课讲的 . merge sort.

→ 为了 eliminate floor and ceiling.

令 $n = 2^k$. $T(2^k) = 2T(2^{k-1}) + \boxed{f(n)}$ can be replaced with n.

$$T(2^k) = 2T(2^{k-1}) + n.$$

$$T(\underbrace{2^k}_{m}) = 2T(\underbrace{2^{k-1}}_{\frac{m}{2}}) + \underbrace{2^k}_{m}$$

$T(m) = 2T(\frac{m}{2}) + m.$

$$T(m) = 2T(\tfrac{m}{2}) + m.$$

$T(\frac{m}{2}) = 2T(\frac{m}{2^2}) + \frac{m}{2}.$

$T(\frac{m}{2^2}) = 2T(\frac{m}{2^3}) + \frac{m}{2^2}$

$$T(m) = 2\left[2T(\tfrac{m}{2^2}) + \tfrac{m}{2}\right] + m$$

$$= 2^2 T(\tfrac{m}{2^2}) + m + m$$

$$T(m) = 2^2\left[2T(\tfrac{m}{2^3}) + \tfrac{m}{2^2}\right] + m + m.$$

$$= 2^3 T(\tfrac{m}{2^3}) + \underbrace{m + m + m}_{3 \text{ items}}.$$

$\vdots$ q

$$T(m) = 2^q T(\tfrac{m}{2^q}) + q m.$$

令 $\frac{m}{2^q} = 1$, $m = 2^q$

$\lg m = \lg(2^q)$

$\lg m = q.$

$$T(m) = m T(1) + \lg m \cdot m$$

$$= m + m \lg m$$

$$T(2^k) = 2^k + 2^k \lg(2^k)$$

$$= 2^k + k \cdot 2^k.$$

back substitute.

$n = 2^k$

$\lg n = \lg(2^k)$

$\lg n = k.$

$$T(n) = 2^{\lg n} + \lg n \cdot 2^{\lg n}$$

$$= n + n \lg n. = \Theta(n \lg n)$$

例5：

Assign 2, #7.

7. Solve the following recurrence relation *exactly* assuming that $T(1) = 1$ and that n is an exact power of 2 i.e. $n = 2^k$ for $k \in \mathbb{N}$. Show all your working.

$$T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$$

$T(n) = 2T(\frac{n}{2}) + n\lg n.$

$T(\frac{n}{2}) = 2T(\frac{n}{2^2}) + \frac{n}{2}\lg(\frac{n}{2})$

$T(\frac{n}{2^2}) = 2T(\frac{n}{2^3}) + \frac{n}{2^2}\lg(\frac{n}{2^2})$

$= 2\left[2 \cdot T(\frac{n}{2^2}) + \frac{n}{2}\lg(\frac{n}{2})\right] + n\lg n$

$= 2^2 T(\frac{n}{2^2}) + \underbrace{n\lg(\frac{n}{2}) + n\lg n}$

$T(n) = 2^2\left[2T(\frac{n}{2^3}) + \frac{n}{2^2}\lg(\frac{n}{2^2})\right] + n\lg(\frac{n}{2}) + n\lg n.$

$= 2^3 T(\frac{n}{2^3}) + \underbrace{n\lg(\frac{n}{2^2}) + n\lg(\frac{n}{2}) + n\lg n}.$

$\vdots$ k

$T(n) = 2^k T(\frac{n}{2^k}) + \underbrace{n\lg(\frac{n}{2^{k-1}}) + n\lg(\frac{n}{2^{k-2}}) + \dots + n\lg n}.$

令 $\frac{n}{2^k} = 1$, $n = 2^k$, $\lg n = \lg(2^k) \Rightarrow k = \lg n.$

$T(n) = n \cdot T(1) + 2^k\lg(\frac{2^k}{2^{k-1}}) + 2^k\lg(\frac{2^k}{2^{k-2}}) + \dots + 2^k\lg(2^k)$

$T(n) = n + 2^k\lg(2) + 2^k\lg(2^2) + \dots + 2^k\lg(2^k)$

$= n + 2^k + 2^k \cdot 2 + \dots + 2^k \cdot k$

$= n + 2^k\boxed{(1 + 2 + \dots + k)}$ arithmetic sequence

$= n + 2^k \cdot \frac{k(k+1)}{2}$

$T(n) = n + 2^{\lg n} \cdot \frac{\lg n(\lg n + 1)}{2}$ back substitute.

$= n + \frac{n\lg n \cdot (\lg n + 1)}{2} = \Theta(n\lg^2 n)$