Fall-2023 5304 LecN1 Notes

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Topics: Introduction; Types of problems seen in this course; Math background; Matrices; Eigenvalues and Eigencevectors; Null space and range; Rank; Types of matrices; Special Matrices.

1 Matrix Multiplication

Matrix-Matrix Multiplication

Matrix-Vector Multiplication, 2 Ways

Form 1: Dot product view

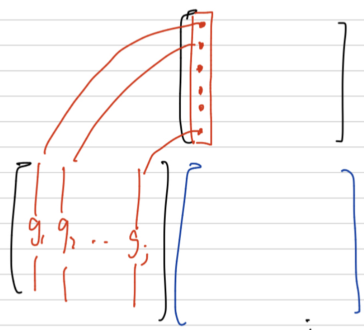
Form 2: Linear combination view In this view, vector can be treated as a place storing the coefficients of the column of matrix A.

Vector-Matrix Multiplication, 2 Ways

Vector-Vector Multiplication, 2 Ways

Form 1: Inner product view This is a scalar.
Form 2: Outer product view Will produce rank-1 matrix.

A x B = C, get jth col of C



matrix-vector product:

linear combination form.

$$a(:,j) = \sum_{k=1}^{j} g^T(k,j) \cdot \underbrace{g(:,k)}_{\text{column k of g.}}$$

e.g.:

$$B \begin{bmatrix} 6 & 3 & 8 \\ 7 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

得到C的第j列：
$$6\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 7\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 29 \\ 77 \\ 125 \end{bmatrix}$$ ✓

2 Rank

Rank + Nullity Theorem

Some Conclusions/Facts

1, A full rank matrix X, and a nonzero vector v, $Xv \neq 0$
2, If A is a nonsingular, square matrix, full rank, then A^{-1} is full rank. $R(A) = R(A^{-1}) = n$
3, If we have a matrix A and a full rank matrix Q, then R(A) = R(QA) = R(AQ). i.e., useing Q left-multiply a matrix or right-multiply a matrix will not change the rank of the original matrix.

3 Special Matrices

Diaguonal Matrix

Vandermond

Hermitan

Def: $A^H = A$

Unitary

Def: $Q^TQ = QQ^T = I$, or $Q^{-1} = Q^T$

Orthogonal

Def: $Q^TQ = I$ [orthonormal columns]
A matrix is orthogonal iff its columns are orthonormal, meaning they are orthogonal and of unit length.
A set of vectors is said to be orthonormal if they are all normal, and each pair of vectors in the set is orthogonal.

举一个向量集合正交归一（orthonormal）的例子，我们可以考虑三维空间中最简单和最常见的例子，即三维笛卡尔坐标系中的单位向量。

在三维空间中，一个常见的正交归一向量集合是由以下三个向量构成的：

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

这三个向量满足正交归一集合的所有条件：

1. 每个向量都是单位向量，因为它们的长度（或范数）是1。
2. 任意两个不同的向量之间都是正交的，因为它们的点积（内积）是0。

让我们来验证一下这些性质：

- 向量的长度（或范数）为1：
  $||\mathbf{e}_1|| = \sqrt{1^2 + 0^2 + 0^2} = 1$
  $||\mathbf{e}_2|| = \sqrt{0^2 + 1^2 + 0^2} = 1$
  $||\mathbf{e}_3|| = \sqrt{0^2 + 0^2 + 1^2} = 1$
- 向量之间相互正交：
  $\mathbf{e}_1 \cdot \mathbf{e}_2 = 1 \times 0 + 0 \times 1 + 0 \times 0 = 0$
  $\mathbf{e}_1 \cdot \mathbf{e}_3 = 1 \times 0 + 0 \times 0 + 0 \times 1 = 0$
  $\mathbf{e}_2 \cdot \mathbf{e}_3 = 0 \times 0 + 1 \times 0 + 0 \times 1 = 0$

因此，向量集合 $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ 是正交归一的。在实际应用中，例如在计算机图形学和信号处理中，正交归一基是非常有用的，因为它简化了很多数学运算和分析。

Properties:
det(Q) = ± 1.

Geometric understanding:
Rotation matrices are orthogonal.

Application:
- decompositions in numerical linear algebra: QR, SVD. - Orthogonality is essential in understanding and solving least-squares problems.

Note:
- its difference with unitary (orthogonal is not necessarily square)

Symmetric and Skew-Symmetric

Def od SM:
$A^T = A$

Application:
- SPD

3.1 Skew-Symmetric

Def:
$A^T = -A$

Skew-Symmetric