Fall-23 5304 LecN7 Notes

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Topics: The Gram-Schmidt algo and the QR Factorization; Least-squares problems; Applications; Data fitting.

Math tools: Eigenvalues and Eigencevectors

1 Least-Squares Systems and Data fitting

Problem BG

1, overdetermined system, more euqations than unknowns, no sol

如果一个方程组无解，那么这个方程组被称为不一致。例如下面的方程组：

$x_1 + x_2 = 2$
$x_1 - x_2 = 1$
$x_1 + x_2 = 3$

根据线性代数的知识，m个方程n个未知量 $m > n$ 时通常无解，但是虽然不能求出 $Ax = b$ 的解，那何不退而求其次，去寻找与解近似的向量 x 。

那么如何定义与解相似，一般使用欧氏距离来进行度量，即两点间的距离，这其实很好理解，越相似，欧氏距离越近，这样求出的 x 被称为最小二乘解。

2, Function approximation

3, Goal: Find the best approximation to the system of equations.

Geometric Interpretation

Good illustration: Data fitting

2 Gram-Schmidt

2.1 Concept

2.2 Classical Gram-Schmidt

The cost of classical Gram-Schmidt is $2mn^2 - \frac{2}{3}n^3$ flops.

Verification of Algo:
1, A = QR. by checking whether norm(A - Q*R) is a small number.
2, Orthogonality: Q is orthogonal. by checking whether $Q^T Q = I$, norm(Q'* Q - eye(n)) should be small.

2.3 Modified Gram-Schmidt

Cost of QR by using classical Gram-Schmidt:

Cost

3 QR Factorization

3.1 BG

The norm here is referring to the 2-norm.

Concept

Definition 8.6 QR-factorization

Let A be an $m \times n$ matrix with independent columns. A ***QR-factorization*** *of A expresses it as $A = QR$ where Q is $m \times n$ with orthonormal columns and R is an invertible and upper triangular matrix with positive diagonal entries.*

The importance of the factorization lies in the fact that there are computer algorithms that accomplish it with good control over round-off error, making it particularly useful in matrix calculations. The factorization is a matrix version of the Gram-Schmidt process.

A = QR where Q is an orthogonal matrix and R is an invertible upper triangular matrix. If A is non singular, then this factorization is unique.

A 可以表示为：

$$\begin{aligned}
\mathbf{a}_1 &= \langle \mathbf{e}_1, \mathbf{a}_1 \rangle \mathbf{e}_1 \\
\mathbf{a}_2 &= \langle \mathbf{e}_1, \mathbf{a}_2 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_2 \rangle \mathbf{e}_2 \\
\mathbf{a}_3 &= \langle \mathbf{e}_1, \mathbf{a}_3 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_3 \rangle \mathbf{e}_2 + \langle \mathbf{e}_3, \mathbf{a}_3 \rangle \mathbf{e}_3 \\
&\vdots \\
\mathbf{a}_k &= \sum_{j=1}^{k} \langle \mathbf{e}_j, \mathbf{a}_k \rangle \mathbf{e}_j
\end{aligned}$$

所以有：

$$Q = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

$$R = \begin{pmatrix} \langle \mathbf{e}_1, \mathbf{a}_1 \rangle & \langle \mathbf{e}_1, \mathbf{a}_2 \rangle & \langle \mathbf{e}_1, \mathbf{a}_3 \rangle & \ldots \\ 0 & \langle \mathbf{e}_2, \mathbf{a}_2 \rangle & \langle \mathbf{e}_2, \mathbf{a}_3 \rangle & \ldots \\ 0 & 0 & \langle \mathbf{e}_3, \mathbf{a}_3 \rangle & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Examples

Cost

Application

QR分解主要有三个用途：

- （1）首先它可以用于求解n个方程n个未知变量的系统 $Ax = b$，如果A非奇异，那么就可以将A分解为QR，回代则求解出x，但是QR分解线性方程组比LU分解方法的计算代价大三倍还多，用的不多。
- （2）用于求解最小二乘问题
- （3）用于特征值计算，这个之后的帖子再写