$A = [c_1, c_2 \ldots c_n]$. $m \times n$.

Orthonormal col: $[f_1, f_2 \ldots f_n]$. 正交化.

归一化.

$q_k = \frac{1}{\|f_k\|} f_k$.

These equations have a matrix form that gives the required factorization:

$$
\begin{aligned}
A &= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \cdots & \mathbf{c}_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 & \cdots & \mathbf{q}_n \end{bmatrix}
\begin{bmatrix}
\|\mathbf{f}_1\| & \mathbf{c}_2 \cdot \mathbf{q}_1 & \mathbf{c}_3 \cdot \mathbf{q}_1 & \cdots & \mathbf{c}_n \cdot \mathbf{q}_1 \\
0 & \|\mathbf{f}_2\| & \mathbf{c}_3 \cdot \mathbf{q}_2 & \cdots & \mathbf{c}_n \cdot \mathbf{q}_2 \\
0 & 0 & \|\mathbf{f}_3\| & \cdots & \mathbf{c}_n \cdot \mathbf{q}_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \|\mathbf{f}_n\|
\end{bmatrix}
\end{aligned}
\tag{8.5}
$$

① 求解矩A的正交矩.

② 单位化.

Example:

Find the QR-factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\|x\|_2 = \sqrt{(x, x)}$$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \end{array}$$

$$\begin{array}{c} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} f_1 & f_2 & f_3. \end{array} \end{array}$$

$$f_1 = C_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\|f_1\|_2^2 = (f_1, f_1) = 2$$

$$f_2 = C_2 - \frac{C_2 \cdot f_1}{\|f_1\|_2^2} \cdot f_1$$

$$= [1, 0, 1, 0]^T - \frac{1}{2} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[\frac{1}{2}, \frac{1}{2}, 1, 0\right]^T.$$

$$\|f_2\|_2^2 = (f_2, f_2) = \frac{3}{2}$$

$$f_3 = c_3 - \frac{c_3 \cdot f_1}{\|f_1\|^2} \cdot f_1 - \frac{c_3 \cdot f_2}{\|f_2\|^2} f_2 .$$

$$= c_3 - \frac{-1}{2} f_1 - \frac{\frac{3}{2}}{\frac{3}{2}} f_2$$

$$= c_3 + \frac{1}{2} f_1 - f_2 .$$

$$= [0, 1, 1, 1]^T + [\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0]^T - [\tfrac{1}{2}, \tfrac{1}{2}, 1, 0] .$$

$$= [0, 0, 0, 1]^T .$$

单位化 F, F → Q

$$F = \begin{array}{c} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} f_1 \quad & \quad f_2 \quad & \quad f_3. \end{matrix} \end{array}$$

$$\|f_1\| = \sqrt{2}, \quad \frac{1}{\|f_1\|} = \frac{1}{\sqrt{2}}$$

$$q_1 = \frac{f_1}{\|f_1\|} = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right]^T.$$

$$\|f_2\| = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}. \quad \frac{1}{\|f_2\|} = \frac{2}{\sqrt{6}}.$$

$$q_2 = \frac{f_2}{\|f_2\|} = \left[\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0\right]^T.$$

$$\|f_3\| = 1.$$

$$q_3 = [0, 0, 0, 1]^T.$$

$$
\begin{array}{cccc}
 & q_1 & q_2 & q_3. \\
Q = & \left[\begin{matrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix}\right].
\end{array}
$$

and $\mathrm{Span}(Q) = \mathrm{span}(A)$.

计算R的简便方法.

$R = Q^T A$

1	1	0
-1	0	1
0	1	1
0	0	1

$$
= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{6}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & 1. \end{bmatrix}}_{R \in 3\times 3.}
$$

也可以按一般方式计算 R.

$$R = \begin{bmatrix} \|f_1\| & c_2 q_1 & c_3 q_1 \\ 0 & \|f_2\| & c_3 q_2 \\ 0 & 0 & \|f_3\| \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{6}}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Solution

Example 8.4.1

Find the QR-factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution. Denote the columns of A as $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$, and observe that $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is independent. If we apply the Gram-Schmidt algorithm to these columns, the result is:

$$\mathbf{f}_1 = \mathbf{c}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{f}_2 = \mathbf{c}_2 - \tfrac{1}{2}\mathbf{f}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \text{ and } \quad \mathbf{f}_3 = \mathbf{c}_3 + \tfrac{1}{2}\mathbf{f}_1 - \mathbf{f}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Write $\mathbf{q}_j = \frac{1}{\|\mathbf{f}_j\|^2}\mathbf{f}_j$ for each j, so $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is orthonormal. Then equation (8.5) preceding Theorem 8.4.1 gives $A = QR$ where

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{2}{\sqrt{6}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{3} & 1 & 0 \\ -\sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} \|\mathbf{f}_1\| & \mathbf{c}_2 \cdot \mathbf{q}_1 & \mathbf{c}_3 \cdot \mathbf{q}_1 \\ 0 & \|\mathbf{f}_2\| & \mathbf{c}_3 \cdot \mathbf{q}_2 \\ 0 & 0 & \|\mathbf{f}_3\| \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{2}} \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & -1 \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

The reader can verify that indeed $A = QR$.