Fall-2023 5304 PrEx04

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This exercise is related to LecN5 and LecN6.

1 Q1: True or False

1.1 a: If A and B are SPD then A+B is also SPD

True. Using the operation property of inner product
$\langle (A+B)u, u\rangle = \langle Au, u\rangle + \langle Bu, u\rangle > 0$

1.2 b: When A is SPD then its inverse is also SPD

True. There are 2 ways to show.

Way1: 2nd part def of SPD and property of eigen val.

A matrix is SPD if it is symmetric and its eigenvalues are positive. Then, use the fact that the eigenvalues of A^{-1} are (1/eigenvalues of A).

Way2: Use 1st part def of SPD and the property of inner product.

?. 想知道 if $(A^{-1}u,u)>0$ $\forall u \neq 0$. 已知 A is SPD.

令 $A^{-1}u = v \rightarrow u = Av$，且 $v = A^{-1}u \neq 0$

因为：① A full rank (SPD 一定满秩)
且 A square $\longrightarrow$ A^{-1} full rank.
② $u \neq 0$. 一个 full rank 矩阵乘以一个非零向量不可能为 0

则 $(A^{-1}u,u) = \underbrace{(v,Av) = (Av,v)}_{\text{交换律}} > 0$，又∵ $v \neq 0$ (前面已证).

前提.

所以有：when A is SPD then its inverse is also SPD.

1.3 c: If A $= GG^T$ is the Cholesky factorization of A, what can you say about det(A)

$\det(A) = \det(GG^T) = \det(G)\det(G^T) = \det(G)\det(G) = \det(G)^2$
just the square of the product of the diagonal entries of G.
Notes: A can do GGT implying that A is SPD.

1.4 d: If X is a full-rank $n \times k$ matrix iff $X^T X$ is SPD

True.

Direction 1 (2 ways): If X is a full-rank $n \times k$ matrix, then $X^T X$ is SPD.
Way1:

利用该结论: If A is SPD, then for any $n \times k$ matrix X of rank k, the matrix $X^T A X$ is SPD.

因为单位阵是 SPD (单位阵是对称的且有 n 重特征值 $1 > 0$)
所以当 $A = I$ 时, 我们可以推出 $X^T X$ is SPD.

Way2: Use property of inner product.

$(X^T \underbrace{Xu}_{\text{vector}}, u) = (Xu, Xu)$ $\quad (Ax, y) = (x, A^H y)$

$(Xu, Xu) > 0$ for $\forall u \neq 0$. because X full rank

Direction 2: If $X^T X$ is SPD, then X is a full-rank $n \times k$ matrix.

X^TX is SPD $\to$ X^TX full rank, if X full rank?

X^TX 与 X 有相同的零空间.

因为 if $Xy=0 \to X^TXy=0$.

if $X^TXy=0 \overset{?}{\to} Xy=0$

$X^TXy=0$

$y^TX^TXy=0$

$(Xy, Xy)=0$. ∴ $Xy=0$

∵ X full rank, ∴ X^TX full rank. (nullity theorem)

1.5 e: The Cholesky factorization of A exists iff A is SPD.

True.

Direction 1: If A is SPD, then the Cholesky factorization of A exists.
This is proved in class.

Direction 2: If the Cholesky factorization of A exists, then A is SPD.
Premise: G and GT always full rank because:

Yes - If the Cholesky facrtorization of X exists then $X = GG^T$ - where G is lower triangular with positive diagonal entries -- G is therefore of full column rank and thus $X = GG^T$ is SPD fronm one of the results seen in class.

undo thanks | 1 Updated 35 minutes ago by Yousef Saad

Then, We can use the conclusion from d that: if X is a full-rank matrix iff X^TX is SPD.
Let X $= G^T$, then $(G^T)^TG^T = GG^T$ is SPD, then $A = GG^T$ is also SPD.

2 Q2

2 Consider the following matrix A whose inverse is also given where τ just stands for 10^{-4}:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -\tau \\ 0 & 1 & 1 \end{pmatrix}; \qquad A^{-1} = \frac{1}{\tau} \begin{pmatrix} 2+\tau & -2 & 2-\tau \\ -1 & 1 & -1+\tau \\ 1 & -1 & 1 \end{pmatrix}$$

Find a vector v such that $\|Av\|_\infty = \tau$ and $\|v\|_\infty = 2$.

Deduce a lower bound for $\|A^{-1}\|_\infty$ and for $\kappa_\infty(A)$.

Calculate $\kappa_\infty(A)$ (3 digits accuracy OK).

2.1 Find vector v, 2 ways

Way1: tao is 0 and A becomes singular

let $\tau = 0$ 时. A becomes singular. Think about which v can give you $\|Av\|_\infty = 0$

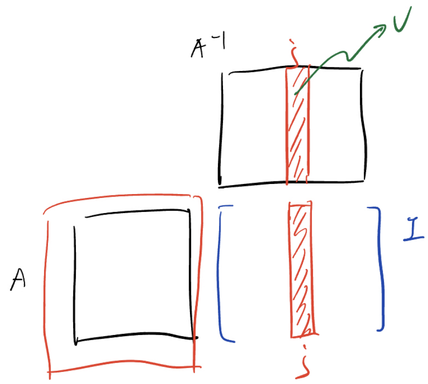
$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

. 观察 A 可以发现 columns 之间的线性关系: $2r_1 = r_2 - r_3$ (因为singular)

$\therefore v = [-2, 1, -1]^T$. where $\|v\|_\infty = 2$.

Way2: utilize matrix-vector product

$\because AA^{-1}=I$, 利用这个 I.



then. $A \cdot (A^{-1}$的第2列$) = e_2$

$$A \cdot \boxed{\frac{1}{v}} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A \cdot \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}$$

$\therefore v = [-2, 1, -1]^T$

2.2 Deduce lower bound for norm-inf invA and kapa-inf A

$$\underbrace{\|v\|_\infty}_{2} = \|A^{-1} \cdot A\, v\|_\infty \leq \|A^{-1}\|_\infty \underbrace{\|A\, v\|_\infty}_{\tau}$$

$$\|A^{-1}\|_\infty \geq \frac{2}{\tau}.$$

$\hookrightarrow$ lower bound.

$$\text{kapa}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty \geq \frac{2}{\tau} \cdot \|A\|_\infty$$

$$= \frac{2}{\tau} \cdot (3+\tau)$$

$$= \frac{6}{\tau} + 2.$$

2.3 Calculate kapa-inf A

$$\text{kapa}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

$$= (3+\tau) \cdot \frac{1}{\tau} \cdot 6$$

$$= \frac{18}{\tau} + 6.$$