SYMMETRIC POSITIVE DEFINITE (SPD) MATRICES

SPD LINEAR SYSTEMS

- **Symmetric positive definite matrices.**
- **The LDL^T decomposition; The Cholesky factorization**

Positive-Definite Matrices

- A real matrix is said to be positive definite if

$$(Au, u) > 0 \text{ for all } u \neq 0 \; u \in \mathbb{R}^n$$

- Let A be a real positive definite matrix. Then there is a scalar $\alpha > 0$ such that

$$(Au, u) \geq \alpha \|u\|_2^2.$$

- Consider now the case of Symmetric Positive Definite (SPD) matrices.
- Consequence 1: A is nonsingular
- Consequence 2: the eigenvalues of A are (real) positive

A few properties of SPD matrices

➤ Diagonal entries of A are positive

➤ Recall: the k-th principal submatrix A_k is the $k \times k$ submatrix of A with entries a_{ij}, $1 \le i, j \le k$ (Matlab: $A(1:k, 1:k)$).

✍1 Each A_k is SPD

✍2 Consequence: $Det(A_k) > 0$ for $k = 1, \cdots, n$. In fact A is SPD iff this condition holds.

✍3 If A is SPD then for any $n \times k$ matrix X of rank k, the matrix $X^T AX$ is SPD.

A SPD $\to$ $(Au, u) > 0$ for all $u \neq 0$.

$X^T A X$

let v be $\forall$ nonzero vector $\to$ $(X^T A X v, v)$

$\to (AXv, Xv)$

$Xv \neq 0$ v is nonzero because X full rank.

$\to (AXv, Xv) > 0$. (正)

Q.E.D.

➤ The mapping : $x, y \quad \rightarrow \quad (x, y)_A \equiv (Ax, y)$

defines a proper inner product on $\mathbb{R}^n$. The associated norm, denoted by $\|.\|_A$, is called the energy norm, or simply the A-norm:

$$\|x\|_A = (Ax, x)^{1/2} = \sqrt{x^T A x}$$

A norm, satisfies 3 properties.

➤ Related measure in Machine Learning, Vision, Statistics: the Mahalanobis distance between two vectors:

$$d_A(x, y) = \|x - y\|_A = \sqrt{(x - y)^T A (x - y)}$$

Appropriate distance (measured in # standard deviations) if x is a sample generated by a Gaussian distribution with covariance matrix A and center y.

More terminology

➤ A matrix is Positive Semi-Definite if: P.S.D $(Au, u) \geq 0$ *for all* $u \in \mathbb{R}^n$

➤ Eigenvalues of symmetric positive semi-definite matrices are real nonnegative, i.e., ...

indefinite.
PSD
SPD.

➤ ... A can be singular [If not, A is SPD]

➤ A matrix is said to be Negative Definite if $-A$ is positive definite. Similar definition for Negative Semi-Definite

➤ A matrix that is neither positive semi-definite nor negative semi-definite is indefinite

✍4 Show that if $A^T = A$ and $(Ax, x) = 0 \; \forall x$ then $A = 0$

✍5 Show: $A \neq 0$ is indefinite iff $\exists \; x, y : (Ax, x)(Ay, y) < 0$

The LDL^T and Cholesky factorizations

6 The (standard) LU factorization of an SPD matrix A exists

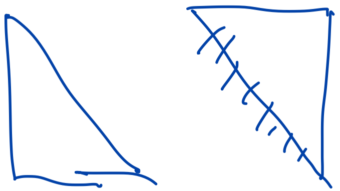
because of Symmetric

- Let $A = LU$ and $D = diag(U)$ and set $M \equiv (D^{-1}U)^T$.

→ nice result.

Then
$$A = LU = LD(D^{-1}U) = LDM^T$$

- Both L and M are unit lower triangular
- Consider $L^{-1}AL^{-T} = DM^TL^{-T}$
- Matrix on the right is upper triangular. But it is also symmetric. Therefore $M^TL^{-T} = I$ and so $M = L$



$A = L\underset{\sim}{U}$

D

$u = D(\underline{D^{-1}u})$

$M^T \Rightarrow 1$

$M =$

same shape as L

$A = LDM^T.$

"M", "L" are the same because of "symmetric"

↓

how to show?

$A = LDM^T$

$A^T = MDL^T$

$A = A^T$ ($\because$ symmetric)

$\therefore M = L$.

Then. $A = LDL^T$.

➤ Alternative proof: exploit uniqueness of LU factorization without pivoting + symmetry: $A = LDM^T = MDL^T \rightarrow M = L$

➤ The diagonal entries of D are positive [Proof: consider $L^{-1}AL^{-T} = D$]. In the end:

$$\boxed{A = LDL^T = GG^T \text{ where } G = LD^{1/2}}$$

diagonal of U

➤ Cholesky factorization is a specialization of the LU factorization for the SPD case. Several variants exist.

WHY diagonal ~~of POS~~ D is positive?

Don't use eigenval. → overthinking

Use result of exer 3

~~diag~~ Goal: diagonal D >0 entries

SHOW:

$$A = LDL^T \rightarrow D = L^{-1} A \underbrace{L^{-T}}_{\text{inverse + transpose}}$$

use previous result with $X = L^{-1}$.

$$A = LDL^T = \underbrace{LD^{\frac{1}{2}}}_{b} \underbrace{D^{\frac{1}{2}} L^T}_{a} = GG^T. \quad (\underbrace{G}_{\text{lower triangular}})$$

a is transpose(b).

Cholesky factorization :

Any matrix A can be factorize to

GG^T.

GE Algo: 有一些东西 stay symmetric, why?

$$a(i,j) = a(i,j) - piv \cdot a(k,j)$$

$$= a(i,j) - (a(i,k)/a(k,k)) \cdot a(k,j)$$

$$= a(i,j) - (a(i,k) * a(k,j))/a(k,k)$$

~~then~~: $a(i,j) = a(j,i) \rightarrow$ "symmetric".

[induction argument]

Symmetric, 有什么优化?

1, can work just part of the matrix.
(when updating)

First algorithm: row-oriented LDLT

Adapted from Gaussian Elimination. Main observation: The working matrix $A(k+1 : n, k+1 : n)$ in standard LU remains symmetric.
$\longrightarrow$ Work only on its upper triangular part & ignore lower part

normally is $a(i,k)$.

1. For $k = 1 : n-1$ Do:
2. For $i = k+1 : n$ Do:
3. $piv := a(k,i)/a(k,k)$
4. $a(i, i:n) := a(i, i:n) - piv * a(k, i:n)$
5. End
6. End

➤ This will give the U matrix of the LU factorization. Therefore $D = diag(U)$, $L^T = D^{-1}U$.

cost of this?

half of original cost!

$$\left(\frac{2}{3} \times \frac{1}{2}\right) n^3 = \frac{1}{3} n^3$$

LU factorization: Cost =

$$\boxed{\frac{2}{3} n^3} \ldots$$

Row-Cholesky (outer product form)

Scale the rows as the algorithm proceeds. Line 4 becomes

$LD^{\frac{1}{2}}D^{\frac{1}{2}}L^T$

$$a(i,:) := a(i,:) - [a(k,i)/\sqrt{a(k,k)}] \ * \ [a(k,:)/\sqrt{a(k,k)}]$$

ALGORITHM : 1. *Outer product Cholesky*

Trick: square root to get 2 pieces.

1. *For* $k = 1 : n$ *Do:*
2. $\quad A(k, k : n) = A(k, k : n)/\sqrt{A(k,k)}$;
3. $\quad$ *For* $i := k + 1 : n$ *Do :*
4. $\quad\quad A(i, i : n) = A(i, i : n) - A(k, i) * A(k, i : n)$;
5. $\quad$ *End*
6. *End*

这个 algo work 的条件: $\forall$ pivot > 0

➤ Result: Upper triangular matrix U such $A = U^T U$.

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

✍7 Is A symmetric positive definite?

✍8 What is the LDL^T factorization of A ?

✍9 What is the Cholesky factorization of A ?

Exer 7:

~~AD~~:

1) determinant of A_k. ([illegible]).

2) just do LU factorization

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 2 \\ 0 & 2 & 5 \end{bmatrix} \quad \overset{U}{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}}$$

$$D = \begin{bmatrix} 1 & & \\ & 4 & \\ & & 4 \end{bmatrix}$$

$$G^T = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \left(U - \sqrt{D} \right)$$

$$\sqrt{D} = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

$$U\sqrt{D} = \Bigg[$$

Column Cholesky. Let $A = GG^T$ with G = lower triangular. Then equate j-th columns:

column wise instead of row wise

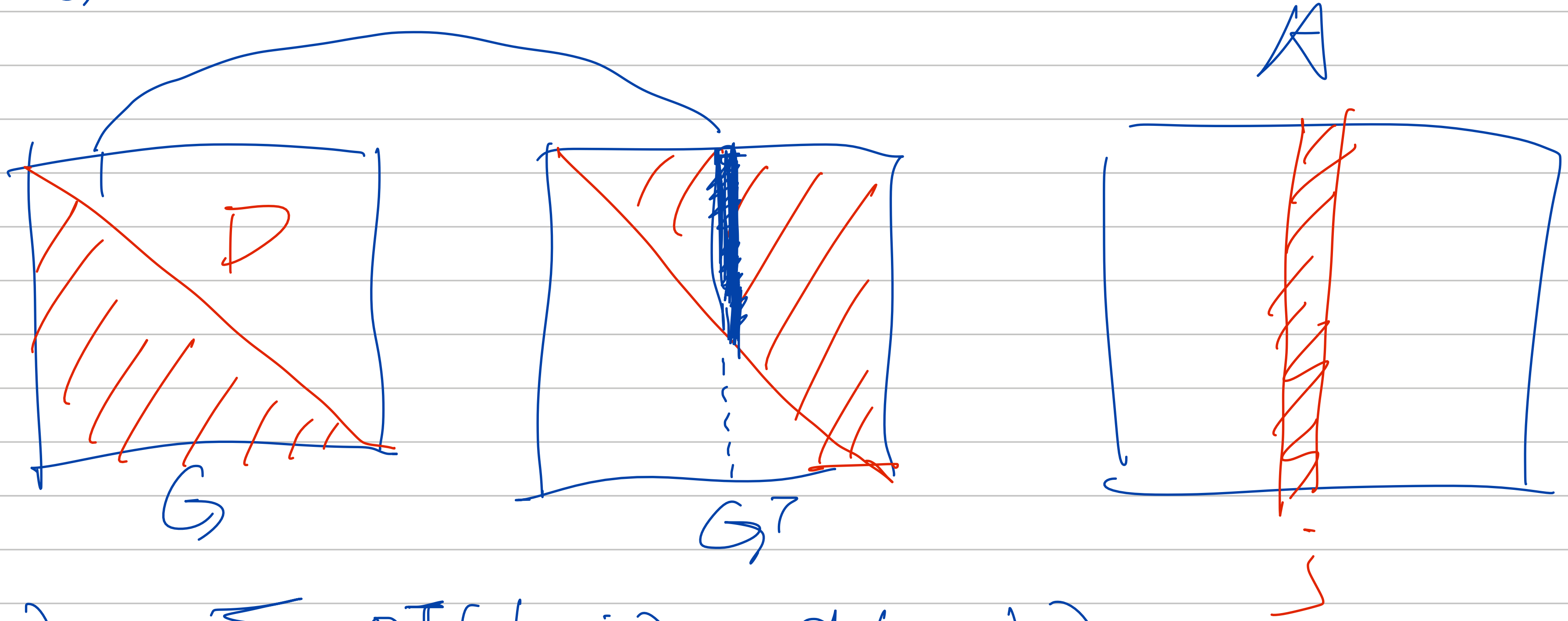
$A = GG^T$

$\to A$ is positive definite.

$$a(:,j) = \sum_{k=1}^{j} g(:,k) g^T(k,j) \to$$

$$\begin{aligned} A(:,j) &= \sum_{k=1}^{j} G(j,k)G(:,k) \\ &= G(j,j)G(:,j) + \sum_{k=1}^{j-1} G(j,k)G(:,k) \to \\ G(j,j)G(:,j) &= A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k) \end{aligned}$$

$A = G G^T,$



$$a(i,j) = \sum g^T(k,j) \cdot g(:,1)$$

$$a(:,j) = \sum_{k=1}^{j} g(j,k) \cdot g(:,k)$$

- Assume that first $j-1$ columns of G already known.
- Compute unscaled column-vector:

$$v = A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k)$$

- Notice that $v(j) \equiv G(j,j)^2$.
- Compute $\sqrt{v(j)}$ and scale v to get j-th column of G.

1. For $j = 1 : n$ do
2. For $k = 1 : j - 1$ do
3. $A(j : n, j) = A(j : n, j) - A(j, k) * A(j : n, k)$
4. EndDo
5. If $A(j, j) \leq 0$ ExitError("Matrix not SPD")
6. $A(j, j) = \sqrt{A(j, j)}$
7. $A(j + 1 : n, j) = A(j + 1 : n, j)/A(j, j)$
8. EndDo

✍10 Try algorithm on:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$