Fall-2023 5304 LecN5 Notes

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1 Question

BG: Normwise backward error
Question: Find smallest perturbation to apply to A, b so that exact solution of perturbed system is y
After understanding this, then back to Exer 8

2 Exer 8

Comments: Perturb every entry by a small scalar,

3 Topic: LEMMA1, LEMMA2, Theorem1, Theorem2

Lemma1

If $\|E\| < 1$, then $I - E$ is nonsingular and $\left\|(I-E)^{-1}\right\| \leq \frac{1}{1-\|E\|}$

Lemma2

Lemma2 generalizes Lemma1.

4 Topic: Condtion Number-Definition and Peoperties

5 Topic: Norm-based Error Bounds

Forward err and Backward err

Conclusion: 1, forward error expression is $\frac{\|x-y\|}{\|x\|}$ where y is the approximate solution by the perturbation system.

2, backward error expression is $\|r\|$

6 Topic Estimating condition number, 5-17

BG: We need cond num to estimate the illness of a system, $\|A\|$ is easy to compute, but A^{-1} is not. Computing $\left\|A^{-1}\right\|$ is expensive.
Thus, we sometimes just want to know a lower bound for the cond num, which implies the illness of the system is at least as bad as this bound.
There are two ways to do it.

Method1: Select a vector v where norm(v)=1, 5-17

Assumption (2):
$\|v\| = 1$
$\|Av\|$ is small. (A is near-singular)

Conclusion:
$\|v\| = \left\|A^{-1}Av\right\| \leq \left\|A^{-1}\right\| \|Av\|$
Thus,
$\left\|A^{-1}\right\| \geq \frac{\|v\|}{\|Av\|}$

Steps to solve problem:
1, observe given A, find a v, normally there exists linear combination relation in A.
2, calculate Av
3, Normally take 1-norm (col) or inf-norm(row)
Example:

Method2: Near-sigularity, 5-19

Conclusion:
$cond(A) * \|A - B\| \geq \|A\|$

$cond(A) \geq \frac{\|A\|}{\|A-B\|}$

$\frac{1}{cond(A)} \leq \frac{\|A\|}{\|A-B\|}$

Method2 proof:

Method Desp: If you take A and find that it is close to a singular matrix b, then $\|A - B\|$ will as large as it can, so that $\frac{\|A\|}{\|A-B\|}$ is small, implying a good lower bound. then:

7 Esitimating Errs from Residual Norms

BG: Forward error Residual norm is a very common way to stop algo.

Proof: used Theorem 1 when $E = 0$.

8 Appendix: Component-based Error Bounds