Fall-2023 5304 LecN6 Notes

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Topics: PD; SPD; LDLT; Cholesky Factorization.

1 Positive Definite Matrices

Definition of PD

A real, square matrix is said to be PD if:
$\langle Au, u \rangle > 0$ for all $u \neq 0$ and $u \in R^n$

Properties

1, A is nonsingular.
This can be proof by contradiction.

2, The eigenvalues of A are real and positive.
This can be proved by the definition of eigenvalues.

2 Symmetric positive definite matrices, SPD

Definition of SPD

A square matrix is SPD if it is **symmetric** and all its eigenvalues λ are **positive**, that is $\lambda > 0$.

Properties

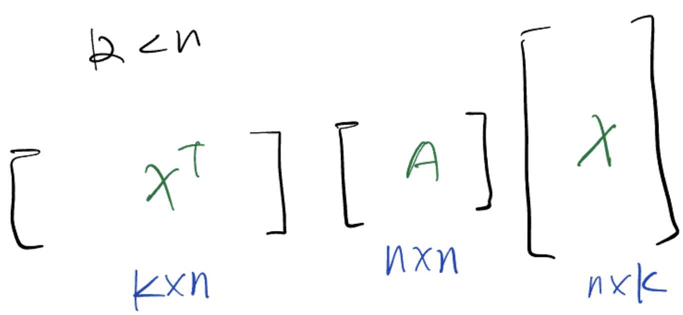
1, Diagonal entries of SPD is positive.

Proof starting from the definition: Known that A is SPD, then for all nonzero vector u, we have $\langle Au, u \rangle > 0$. Then, utilize identity matrix to extract an element of A. (MV Product, dot product view). Good enough.

2, Each A_k is SPD
will fill this part later.

3, a conclusion
will fill this part later.

4, If A is SPD, then for any nxk matrix X of rank k, the matrix X^TAX is SPD.
Memory:



Proof (will use the notation of the inner product and def of SPD):

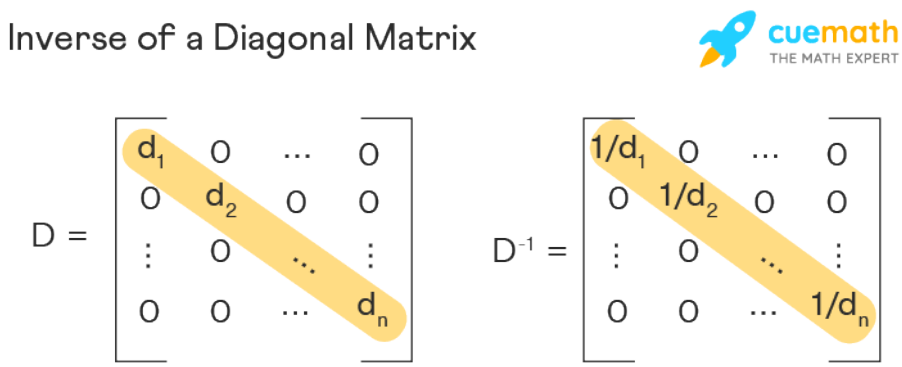
Application: Covariance Matrices in Statistics

3 Pred: SPD, Semi-Definite, Neg definite, and Indefinite Matrices

4 The LDL^T Factorization from LU

For the LU Factorization, all we need is all A_k matrices for k=1 to n-1 have to be nonsingular. As A is SPD, implying that all A_k matrices from k=1 to n are nonsingular. Thus LU fac exists and is unique.

Inverse of a Diagonal Matrix is easy to compute:



Nice results because of symmetry
  L is M
Proof that $A = LU = LDL^T$

Thus, a SPD matrix A could be written in this form: $A = LDL^T$ where L is a lower triangular matrix with 1s on the diagonal, and D is a diagonal matrix (of U).

5 The Cholesky Factorization GGT from LDLT

Diagonal of D shoud be positive so that we can take a square root to D to reach the Cholesky.

Remark. Suppose that S has an LDL^T decomposition with

$$D = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix}.$$

Then we define

$$\sqrt{D} = \begin{pmatrix} \sqrt{d_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{d_n} \end{pmatrix},$$

so that $(\sqrt{D})^2 = D$, and we set $L_1 = L\sqrt{D}$. Then

$$L_1 L_1^T = L(\sqrt{D})(\sqrt{D})^T L^T = LDL^T = S,$$

so $L_1 L_1^T$ is the Cholesky decomposition of S.

Proof: Diagonal of D shoud is positive

What can we say about G? G is lower triangular
The Cholesky: Any matrix that is SPD could be written as $A = GGT$ where G is a lower triangular matrix, and positive entries on diagonal.

Algo of LDLT

Idea: Just work on upper part of the matrix because of symmetry.

Rank 1 update: You're updating the matrix by something like uv^T where u and v are the same.