Fall-2023 5304 LecN6 Notes

Wan

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Topics: PD; SPD; LDLT; Cholesky Factorization.

1 Positive Definite Matrices

Definition of PD

A real, square matrix is said to be PD if:
$\langle Au, u\rangle > 0$ for all $u \neq 0$ and $u \in R^n$

Properties

1, A is nonsingular.
This can be proof by contradiction.

2, The eigenvalues of A are real and positive.
This can be proved by the definition of eigenvalues.

2 Symmetric positive definite matrices, SPD

Definition of SPD

A square matrix is SPD if it is **symmetric** and all its eigenvalues λ are **positive**, that is $\lambda > 0$.

Properties

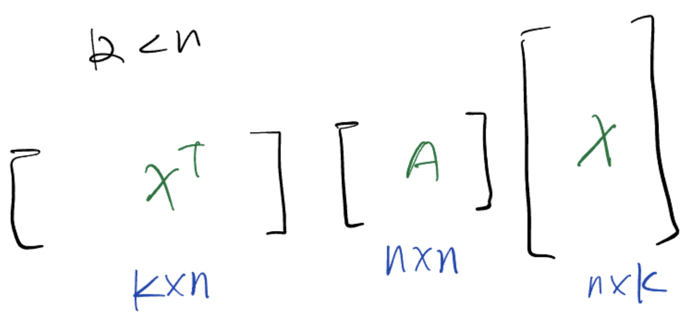
1, Diagonal entries of SPD is positive.

Proof starting from the definition: Known that A is SPD, then for all nonzero vector u, we have $\langle Au, u\rangle > 0$. Then, utilize identity matrix to extract an element of A. (MV Product, dot product view). Good enough.

2, Each A_k is SPD
will fill this part later.

3, a conclusion
will fill this part later.

4, If A is SPD, then for any nxk matrix X of rank k, the matrix X^TAX is SPD.
Memory:



Proof (will use the notation of the inner product and def of SPD):

Application: Covariance Matrices in Statistics

3 Pred: SPD, Semi-Definite, Neg definite, and Indefinite Matrices

4 The LDL^T Factorization from LU

Theorem

Theorem. *Let S be a positive-definite symmetric matrix. Then S has unique decompositions*

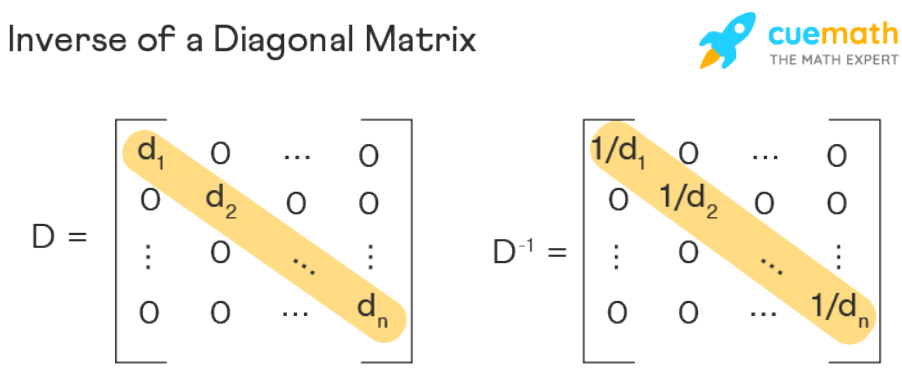
$$S = LDL^T \quad \textit{and} \quad S = L_1 L_1^T$$

where:

- *L is lower-unitriangular,*
- *D is diagonal with positive diagonal entries, and*
- *L_1 is lower-triangular with positive diagonal entries.*

For the LU Factorization, all we need is all A_k matrices for k=1 to n-1 have to be nonsingular. As A is SPD, implying that all A_k matrices from k=1 to n are nonsingular. Thus LU fac exists and is unique.

Inverse of a Diagonal Matrix is easy to compute:



Nice results because of symmetry

L is M

Proof that $A = LU = LDL^T$

Thus, a SPD matrix A could be written in this form: $A = LDL^T$ where L is a lower triangular matrix with 1s on the diagonal, and D is a diagonal matrix (of U).

5 The Cholesky Factorization GGT from LDLT

GGT, scope of application: SPD matrices (??? what about semi-definte matrix???).

Diagonal of D shoud be positive so that we can take a square root to D to reach the Cholesky.

Remark. Suppose that S has an LDL^T decomposition with

$$D = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix}.$$

Then we define

$$\sqrt{D} = \begin{pmatrix} \sqrt{d_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{d_n} \end{pmatrix},$$

so that $(\sqrt{D})^2 = D$, and we set $L_1 = L\sqrt{D}$. Then

$$L_1 L_1^T = L(\sqrt{D})(\sqrt{D})^T L^T = LDL^T = S,$$

so $L_1 L_1^T$ is the Cholesky decomposition of S.

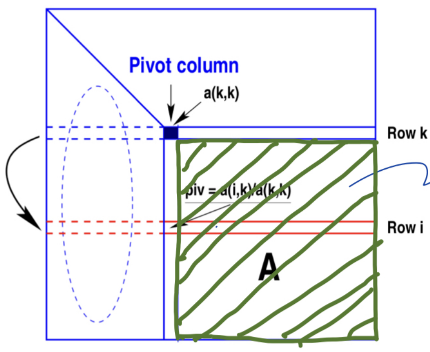
Proof: Diagonal of D shoud is positive

What can we say about G? G is lower triangular
The Cholesky: Any matrix that is SPD could be written as $A = GGT$ where G is a lower triangular matrix, and positive entries on diagonal.

6 Algos of LDLT and GGT

row-oriented LDLT

Idea: Adapted from Gaussian but just work on upper part of the matrix because of symmetry because the working matrix A(k+1:n, k+1:n) in standard LU remains symmetric. See graph below:



remains symmetric.

WHY?

Proof: A(k+1:n, k+1:n) is symmetric (induction argument).

What is the cost(FLOPs) of this algo?
Cost of standard LU:
Suppose k's index starting from 0, then in standard LU decomposition, when clearing the 1st col, each row replacement invoves n-1 multiplications (scale piv row) and n-1 substraction (update curr row), 2(n-1) flops in total. Hence, when k increase, problem size decrese, it takes $2(n-2)^2$ flops to clear the 2nd col etc.. Thus, we have:

$$(n-1)\cdot 2(n-1) + (n-2)\cdot 2(n-2) + \dots + 1$$

$$= 2\left((n-1)^2 + (n-2)^2 + \dots + 1\right) = 2\cdot \frac{n(n-1)(2n-1)}{6} \approx \frac{2}{3}n^3.$$

平方和求和公式.

平方和与立方和

平方和与立方和的求和公式如下：

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad \text{(A. 3)}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4} \qquad \text{(A. 4)}$$

Cost of LDLT (just half of LU): $\frac{1}{2} * \frac{2}{3} * n^3 = \frac{1}{3} * n^3$

Rank 1 update: You're updating the matrix by something like uv^T where u and v are the same.

2, row GGT, outer product form

3, column-oriented LDLT