Mth 351 HW 1 Bennet Sloan

Summer '20

1.) $I = \int_0^1 \frac{1 - e^{-t}}{t}\, dt$, $\because e^{-t} = 1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 \ldots$

$$\rightarrow f(t) = (t^{-1})\left[1 - \left(1 - t + \frac{1}{2!}t^2 - \frac{1}{3!}t^3 \ldots\right)\right]$$

$$= (t^{-1})\left(\cancel{1} - \cancel{1} + t - \frac{1}{2!}t^2 + \frac{1}{3!}t^3 \ldots\right)$$

$$= 1 - \frac{1}{2!}t + \frac{1}{3!}t^2 \ldots$$

So, $I = \int_0^1 f(t)\, dt$

$$= \int_0^1 \left(1 - \frac{1}{2!}t + \frac{1}{3!}t^2 \ldots\right) dt$$

$$= \left[t - \left(\frac{1}{2}\right)\left(\frac{1}{2!}\right)t^2 + \left(\frac{1}{3}\right)\left(\frac{1}{3!}\right)t^3 \ldots\right]_0^1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n \circ n!}$$

Now, $|I - S_n| = \left| \sum_{n=0}^{\infty} (-1)^n \frac{1}{n \circ n!} - \sum_{K=0}^{n} (-1)^K \frac{1}{K \circ K!} \right|$

$$\left.\begin{aligned} &\leq S_{n+1} \\ &\leq \frac{1}{(n+1)(n+1)!} \end{aligned}\right\}$$ since S_n is alternating &:

1) $S_n \rightarrow 0$, $n \rightarrow \infty$

2) $S_{n+1} \leq S_n$

For $|I - S_n| < 10^{-6}$:

$$\frac{1}{(8+1)(8+1)!} < 10^{-6} < \frac{1}{(7+1)(7+1)!} \quad \therefore n = 8$$

2.) $C_i(x) = \frac{1}{x}\int_0^x \frac{1-\cos(t)}{t^2}\,dt$, $\cos(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\rightarrow f(t) = (t^{-2})\left[1-\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\frac{1}{6!}t^6\ldots\right)\right]$$

$$=(t^{-2})\left(\cancel{1}\cancel{-1}+\frac{1}{2!}t^2-\frac{1}{4!}t^4+\frac{1}{6!}t^6\ldots\right)$$

$$=\frac{1}{2!}-\frac{1}{4!}t^2+\frac{1}{6!}t^4\ldots$$

so, $C_i(x) = \frac{1}{x}\int_0^x \left(\frac{1}{2!}-\frac{1}{4!}t^2+\frac{1}{6!}t^4\ldots\right)dt$

$$=\frac{1}{x}\left[\frac{1}{2!}t-\frac{1}{3}\frac{1}{4!}t^3+\frac{1}{5}\frac{1}{6!}t^5\ldots\right]_0^x$$

$$=\frac{1}{x}\left(\frac{1}{2!}x-\frac{1}{3}\frac{1}{4!}x^3+\frac{1}{5}\frac{1}{6!}x^5\ldots\right)$$

$$=\frac{1}{2!}-\frac{1}{3}\frac{1}{4!}x^2+\frac{1}{5}\frac{1}{6!}x^4\ldots$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n-1}\frac{x^{2(n-1)}}{2n!}$$

Now, $\lim_{n\to\infty}\left|\frac{C_{n+1}(x)}{C_n(x)}\right| = \lim_{n\to\infty}\left|\left[(-1)^{n+1+1}\left(\frac{1}{2(n+1)-1}\right)\left(\frac{x^{2(n+1-1)}}{2(n+1)!}\right)\right.\right.$

$$\left.\left.\circ\ \frac{1}{(-1)^{n+1}}(2n-1)\frac{2n!}{x^{2(n-1)}}\right]\right|$$

$$=\lim_{n\to\infty}\left|\frac{-(2n-1)x^2}{(2n+1)(2n+2)(2n+1)}\right|$$

$$=\lim_{n\to\infty}\left|\frac{2n-1}{2n+1}\frac{x^2}{(2n+1)(2n+2)}\right|$$

$= 0 < 1$ $\therefore$ converges $\forall\, n \in \mathbb{R}$

2. continued)

Let $C_i(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} \frac{x^{2(n-1)}}{2n!}$

Then For $\left| C_i(x) - S_n(x) \right| \leq 10^{-6}$, $|x| \leq 1$

$\rightarrow \left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1} \frac{x^{2(n-1)}}{2n!} - \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{2k-1} \frac{x^{2(k-1)}}{2k!} \right| \leq 10^{-6}$

$\rightarrow \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} \frac{x^{2(k-1)}}{2k!} \right| \leq 10^{-6}$

$\rightarrow \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} \frac{1}{2k-1} \frac{1}{2k!} \right| \leq 10^{-6}$

$\rightarrow \frac{1}{(2n-1)(2n)!} \leq 10^{-6}$

$(2n-1)(2n)! \geq 10^{6}$

$(2(5)-1)((2)(5))! < 10^{6} < (2(6)-1)(2(5))!$

$\rightarrow n \geq 5$ $\rightarrow$ But cos even $\therefore n = 6$