Mth 351 Quiz 2 Bennet Sloan

1.) Given $|x_{n+1} - \alpha| \leq 5|x_n - \alpha|^2$, $1 \leq \alpha \leq 2$

Let $|x_k - \alpha| \equiv \varepsilon_k$, $k \in \mathbb{Z}_0^+$

Then, $\varepsilon_{n+1} \leq 5\varepsilon_n^2$

$\rightarrow$ $\varepsilon_1 \leq 5\varepsilon_0^2$

$\varepsilon_2 \leq 5\varepsilon_1^2 \rightarrow \varepsilon_2 \leq 5(5\varepsilon_0^2)^2$

$\varepsilon_3 \leq 5\varepsilon_2^2 \rightarrow \varepsilon_3 \leq 5[5(5\varepsilon_0^2)^2]^2$

$\varepsilon_4 \leq 5\varepsilon_3^2 \rightarrow \varepsilon_4 \leq 5\{5[5(5\varepsilon_0^2)^2]^2\}^2$

$\varepsilon_5 \leq 5\varepsilon_4^2 \rightarrow \varepsilon_5 \leq 5[[5\{5[5(5\varepsilon_0^2)^2]^2\}^2]]^2$

$\therefore \varepsilon_{n+1} \leq \left[5^{\left(\sum_{k=0}^{n} 2^k\right)}\right]\left[\varepsilon_0^{\left(2^{n+1}\right)}\right]$

For $\varepsilon_5 \leq 10^{-4}$,

$$\varepsilon_5 \leq \left[5^{\sum_{k=0}^{4} 2^k}\right]\left[\varepsilon_0^{2^{4+1}}\right]$$

$$\varepsilon_5 \leq \left(5^{2^0+2^1+2^2+2^3+2^4}\right)\left(\varepsilon_0^{2^5}\right)$$

$$\varepsilon_5 \leq 5^{31}\varepsilon_0^{32}$$

$\rightarrow 5^{31}\varepsilon_0^{32} < 10^{-4} \rightarrow \varepsilon_0 < \left(\frac{10^{-4}}{5^{31}}\right)^{\frac{1}{32}} \approx \boxed{0.1577149}$

2.) I. → Method 3 is the bisection method, since the error is halved every iteration.

→ Method 2 is Newton's Method, since it superconverges the fastest to α.

→ Method 4 behaves similarily to Method 2, and is therefore the approximation to Newtons Method, the secant method.

→ Method 1 converges only slightly faster than linear convergence, akin to fixed point method of the form $x = x - f(x)$,

II. → Method 3 (Bisection) → $|\varepsilon_{n+1}| = |\varepsilon_n \frac{1}{2}|$, $\varepsilon_1 = 5 \cdot 10^{-2}$

→ $(5 \cdot 10^{-2}) 2^{-n} < 10^{-15}$

→ $2^n > \frac{5 \cdot 10^{-2}}{10^{-15}}$ → $2^n > 5 \cdot 10^{13}$

$\boxed{n = 43}$

→ method 4 (secant)

$\boxed{n = 7}$

since quadratic and convergence rate increases as with Newton's Method, $\frac{\varepsilon_6}{\varepsilon_5} \leq 10^{-5}$, so with $\varepsilon_6 \approx 6.6 \cdot 10^{-11}$,

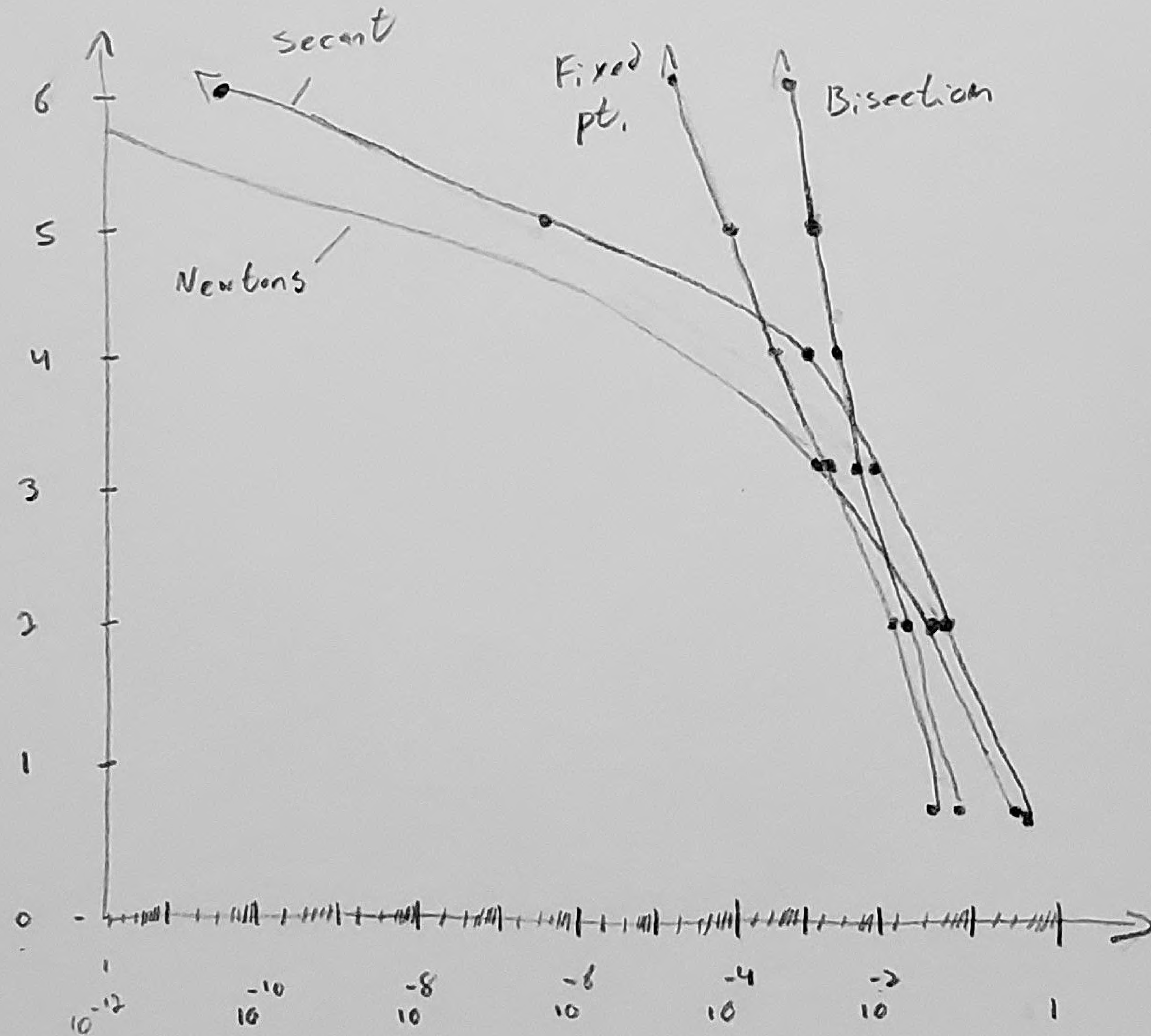
Then $\frac{\varepsilon_7}{\varepsilon_6} \leq 10^{-5}$ → $\varepsilon_7 \leq 10^{-5} \varepsilon_6 \leq 10^{-16}$

II cont) → Method 1 (Fixed point)

ε decreases one decade every 2 iterations.

So, 15 - 4 = 11 decades means $n = 6 + 2(10) = \boxed{27}$

In this case we assume $F'(r) \neq 0$



Bisection is always linear, Fixed point may be linear if $F'(r) \neq$

Newtons & secant converge at least quadratically and increase in convergence rate as they approach α.