1.) Let $\{\lambda_1, \lambda_2, \lambda_3\} = \sigma(A_{3\times3})$

1a.) By the fundamental theorem of Algebra:

Since $|\sigma(A)| = 3$, $f(\lambda)$ is degree 3

if multiplicity is counted. Therefore by polynomial

factorization, $f(\lambda) = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)$

$$= \prod_{i=1}^{n}(-1)(\lambda-\lambda_i)$$

$$= \prod_{i=1}^{n}(\lambda_i-\lambda)$$

so if $n=3$, $\prod_{i=1}^{3}(\lambda_i-\lambda) = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)$

1b.) $(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = (\lambda_1\lambda_2-\lambda\lambda_1-\lambda\lambda_2+\lambda^2)(\lambda_3-\lambda)$

$$= \lambda_1\lambda_2\lambda_3-\lambda\lambda_1\lambda_2-\lambda\lambda_1\lambda_3+\lambda^2\lambda_1-\lambda\lambda_2\lambda_3+\lambda^2\lambda_2+\lambda^2\lambda_3-\lambda^3$$

$$= -\lambda^3+\lambda^2\underline{(\lambda_1+\lambda_2+\lambda_3)}-\lambda(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)+\lambda_1\lambda_2\lambda_3$$

The coefficient of the λ^2 term is $\sum_{i=1}^{n}\lambda_i$.

1c.) Let $A = \begin{bmatrix}3 & * & *\\ * & 4 & *\\ * & * & 5\end{bmatrix}$ $*$ = any number.

$f(\lambda) = \det\begin{bmatrix}3-\lambda & & \\ & 4-\lambda & \\ & & 5-\lambda\end{bmatrix} = (3-\lambda)\begin{vmatrix}4-\lambda & \\ & 5-\lambda\end{vmatrix} + (\text{degree} \leq 1)$

$$= (3-\lambda)(4-\lambda)(5-\lambda) + (\text{degree} \leq 1)$$

so $(3-\lambda)(4-\lambda)(5-\lambda) = 12-3\lambda-4\lambda+\lambda^2(5-\lambda) = (\lambda^2-7\lambda+12)(5-\lambda)$

$$= 5\lambda^2-\lambda^3-35\lambda+7\lambda^2+60-12\lambda$$

λ^2 coefficient is $(5+7)$

$$= -\lambda^3+\lambda^2(5+7)+\lambda(-12-35)+60$$

1.d.) From 1b λ^2 coefficient was $\sum_{i=1}^{n} \lambda_i$

using $A = \begin{bmatrix} 3 & & \\ & 4 & \\ & & 5 \end{bmatrix}$ this is $(3+4+5) = \underline{11}$

From 1d it is $(5+7) = \underline{11}$.

so if $tr(A) = 11$, $\sum_{i=1}^{n} \lambda_i = tr(A)$

2.) The computation is not a proof because it is not shown to true in general, only for the specific A given. To prove

$$\sum_{i=1}^{n} \lambda_i = tr(A)$$

for any $A_{n \times n}$, first show the coefficient of λ^{n-1} is $\sum_{i=1}^{n} \lambda_i$ for any n.

Then, show $f(\lambda)$ can be represented as a polynomial of degree n plus a polynomial degree $n-2$.

Equating the coefficient of λ^{n-1} in both general cases will prove $\sum_{i=1}^{n} \lambda_i = tr(A)$ in general.

3) Let A be an upper triangular matrix

Then the eigenvalues of A are roots of:

$$f(\lambda) = \det(A - \lambda I)$$

Since $A - \lambda I$ will also be triangular,

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)$$

So the roots are $a_{11}, a_{22}, a_{33} \ldots$

which is the diagonal entries of A.