Bennet Sloan Quiz 4 Mth 351

1.) $x_j = \cos\left(\frac{(2j+1)(\pi)}{(n+1)(2)}\right)$, $j \in \mathbb{Z}$. $f(x) = \frac{1}{1+x^2}$, $x \in [-5,5]$

I.) $|f(y) - P_n(y)| = \frac{|f^{n+1}(c)|}{(n+1)!} \left|(y-y_0^*)(y-y_1^*)\dots(y-y_n^*)\right|$, $c \in [-5,5]$

$\rightarrow |f(y) - P_n(y)| = 5^{n+1}(x-x_0^*)(x-x_1^*)\dots(x-x_n^*)$, $x = \frac{1}{5}y$.

For $T_0(x) = 1$

$T_1(x) = x$, $x_0^* = 0$

$\cos(2\theta) \rightarrow T_2(x) = 2x^2-1$, $x_0^* = \frac{1}{\sqrt{2}}$, $x_1^* = \frac{-1}{\sqrt{2}}$

$T_3(x) = 4x^3 - 3x$, $x_2^* = \cos(\pi - \frac{\pi}{6}) = \frac{-\sqrt{3}}{2}$

so, $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

$= (x-x_0^*)(x-x_1^*)\dots(x-x_n^*)2^n$

so $(y-y_0^*)(y-y_1^*)(y-y_2^*)\dots(y-y_n^*)$, $x = \frac{y}{5}$

$= \frac{5^{n+1}}{2^n} T_{n+1}(x)$

II.) $\therefore \quad |f(y) - P_n(y)| \leq \frac{R^{n+1}}{(n+1)!} \frac{S^{n+1}}{2^n} |T_{n+1}(x)|$

$\Rightarrow \quad |f(y) - P_n(y)| \leq \frac{R^{n+1}}{(n+1)!} \frac{S^{n+1}}{2^n} \left[\cos\left[(n+1)\cos^{-1}(x)\right]\right]$

$\leq \frac{R^{n+1}}{(n+1)!} \frac{S^{n+1}}{2^n} \cos(n+1)\cos^{-1}(x), \quad x \leq 1$

For max: $\max |f(y) - P_n(y)| = \lim_{n\to\infty} \frac{R^{n+1} S^{n+1}}{(n+1)!\, 2^n}$

$= SR \lim_{n\to\infty} \frac{\left(\frac{SR}{2}\right)^n}{(n+1)!}$

$= SR \lim_{n\to\infty} \frac{P^n}{(n+1)!}$

$= 0$

2.) $I(f) = \int_a^b f(x)\,dx$, $I(f) - M_n(f) = \frac{1}{24} h^3 f''(c)$

Let $I = \int_0^3 e^{-x^2}\,dx$, $h = \left(\frac{b-a}{n}\right) \to h = \frac{3-0}{n}$

$= 3/n$

Then $|\mathcal{E}_n| = \frac{1}{24}\left(\frac{3}{n}\right)^3 f''(c)$

$\leq \frac{1}{24}\left(\frac{3}{n}\right)^3 \max_{c\in(a,b)} f''(c)$

Now, $f'(x) = \frac{d}{dx}\left[e^{-x^2}\right] = -2xe^{-x^2}$

$f''(x) = \frac{d}{dx}\left(-2xe^{-x^2}\right) = 4x^2e^{-x^2} - 2xe^{-x^2}$

$= (4x^2-2)\,e^{-x^2}$

$f'''(x) = -4x(2x^2-3)\,e^{-x^2}$

For $f'''(x) = 0$, $f''' = -4x(2x^2-3)e^{-x^2} = 0$

$x = 0, \pm\sqrt{3/2}$

So, $f'''(1/2) > 0$ & $f'''(2) < 0$

$\therefore$ $x = \sqrt{3/2}$ is max

$\to f''(\sqrt{3/2}) = \frac{4\times\frac{3}{2}-2}{e^{3/2}} = \frac{4}{e^{3/2}}$

2 cont.) Now, $|E_n| \le \frac{1}{24}\left(\frac{3}{n}\right)^3 \max_{c\in a,b} f''(c)$

$$\le \frac{1}{24}\left(\frac{3}{n}\right)^3 \frac{4}{e^{3/2}}$$

For $\varepsilon = 5\times10^{-6}$: $\quad n^3 \ge \frac{108}{(24)(5\times10^{-6})(e^{3/2})}$

$\therefore \boxed{n \ge 59}$