MTH 342 OSU Winter 2019 Friday, Jan. 25, Lab F, done in class.
Complete this and submit it to Canvas by the posted due date: Tuesday, Jan. 29.

1. Give an example of a matrix A that has a left inverse but does not have a right inverse. (If $BA = I$ then B is a left inverse of A.)

2. Give an example of a matrix A that has a right inverse but does not have a left inverse. (If $AB = I$ then B is a right inverse of A.)

Let V and W be vector spaces. If $T \in \mathcal{L}(V, W)$ is invertible then T is called an isomorphism and V and W are isomorphic, written $V \simeq W$. The inverse of T is written T^{-1}.

Theorem. Let $T : V \to W$ be an isomorphism, and let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V. Then the system $\{T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)\}$ is a basis for W.

3. Prove this theorem.

1.) Give an example of a matrix A with a left inverse but no right inverse.

If A is a standard matrix for a linear transformation T,

Then T is injective & not surjective.

Suppose $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Then $MA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_{2\times 3}$

so, $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} b & \frac{1}{2}c \\ e & \frac{1}{2}f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$b = 1$, $\frac{1}{2}f = 1$, $e = 0$, $c = 0$

$f = 2$

$M = \begin{bmatrix} a & 1 & 0 \\ d & 0 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

while $AM \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ For: $T: \mathbb{P}_1 \to \mathbb{P}_2$. $Tf(x) = \int f(x)$

2.) If a matrix A has a right inverse, but no left inverse, Then T is surjective but not injective.

Consider $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$AM = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_{3\times2}$

Then $A\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} c & d \\ 2e & 2f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $c = 1$, $f = \frac{1}{2}$, $d = e = 0$

And $M = \begin{bmatrix} a & b \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, but $MA \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Assuming A is a standard matrix for a linear transformation T.

$T: \mathbb{P}_2 \to \mathbb{P}_1$, $Tf(x) = f'(x)$

3.) Prove: Let $T: V \to W$ be an isomorphism, and let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for V. Then the system $\{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)$ is a basis for W.

Let $\vec{v} \in V$

Then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

Consider $T(\vec{v}) = T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \dots + T(c_n\vec{v}_n)$

$= c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \dots + c_nT(\vec{v}_n)$

Because T is a linear transformation

Furthermore if $T: V \to W$ is an isomorphism, Then is injective with $Ker(T) = 0$.

Therefore since $\{v_1 \dots v_n\}$ is a linearly independent set in V, $\{T(\vec{v}_1) \dots T(\vec{v}_n)\}$ must be a linearly independent set in W.

where $Ker(T) = 0$ implies $0 = T(0)$

And injectivity of T implies linear Independence.