MTH 342 OSU Winter 2019
Wed. Jan. 16, Lab C, done in class

Complete this and turn it in at the beginning of class on Friday, Jan. 18. Turn in a paper copy. Put your name and solutions on a separate sheet(s) of paper. Staple multiple pages. You may use this as a cover sheet.

Decide which of these are linear transformations. Explain why it is (with justification using the definition of a linear transformation) or provide a counterexample that shows it is not a linear transformation.

1. For fixed $\vec{g} \in \mathbb{R}^n$, define $T_{\vec{g}}: \mathbb{R}^n \to \mathbb{R}$ by

$$T_{\vec{g}}(\vec{v}) = \vec{v} \cdot \vec{g} \qquad \text{(dot product).}$$

   Is $T_{\vec{g}}$ a linear transformation?

2. Let $M_{n\times n}$ be the vector space of all $n \times n$ matrices with real entries. The trace of a square matrix is the sum of its diagonal entries:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}, \qquad \text{where } A = (a_{ij}).$$

   Is $\operatorname{tr}: M_{n\times n} \to \mathbb{R}$ a linear transformation?

3. Is the determinant $\det: M_{2\times 2} \to \mathbb{R}$ a linear transformation?

4. Let $V = C[0, 1]$, the set of all real valued continuous functions defined on the closed unit interval $[0, 1]$. When addition and scalar multiplication are defined pointwise (the usual way of adding functions and multiplying functions by real numbers), V becomes a vector space.

   (a) For fixed $a \in [0, 1]$, define $E_a: V \to \mathbb{R}$ by $E_a(f) = f(a)$. In other words, E_a is an evaluation map; it evaluates the function f at a. Is E_a a linear transformation?

   (b) For fixed $g \in C[0, 1]$ define $T_g: V \to \mathbb{R}$ by

$$T_g(f) = \int_0^1 f(x)g(x)dx.$$

   Is T_g a linear transformation?

Find the standard matrix for the following linear transformations:

1. The linear tranformation $T_{\vec{g}}$ from item 1 above. Assume $\vec{v} = (v_1, v_2, \ldots, v_n)^T$, so that you can compute $T_{\vec{g}}(\vec{v}) = \vec{v} \cdot \vec{g}$ by multiplying $\vec{v}$ by a matrix on the left.

2. For fixed $\vec{w} \in \mathbb{R}^3$, define $C_{\vec{w}} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$C_{\vec{w}}(\vec{v}) = \vec{v} \times \vec{w} \qquad \text{(cross product).}$$

Then $C_{\vec{w}}$ is a linear transformation. What is the standard matrix for $C_{\vec{w}}$? Assume $\vec{w} = (w_1, w_2, w_3)^T$.

3. $T : \mathbb{P}_3 \to \mathbb{P}_2$ given by $Tf(x) = f'(x)$. Use the standard basis $\{1, x, x^2, ..., x^n\}$ for $\mathbb{P}_n$.

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1.) Let $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$, c be any scalar

Then, $T_{\vec{g}}(\vec{v}_1+\vec{v}_2) = (\vec{v}_1+\vec{v}_2)\cdot\vec{g}$

$= \vec{v}_1\cdot\vec{g} + \vec{v}_2\cdot\vec{g}$

$= T_{\vec{g}}(v_1) + T_{\vec{g}}(v_2)$

And, $T_{\vec{g}}(c\vec{v}_1) = T(c\vec{v}_1)\cdot\vec{g}$

$= m(\vec{v}_1\cdot g)$

$= mT_{\vec{g}}(\vec{v}_1)$ $\therefore T_{\vec{g}}$ is linear

2.) Let $A, B \in M_{n\times n}$, c be any scalar

Then, $tr_n(A+B) = \sum_{j=1}^{n}(a_{ij}+b_{ij}) = \sum_{i=1}^{n} a_{ij} + \sum_{j=1}^{n} b_{ij}$

$= tr_n(A) + tr_n(B)$

Also, $tr_n(cA) = \sum_{j=1}^{n} ca_{ij} = c\sum_{i=1}^{n} a_{ij} = c\,tr_n(A)$

$\therefore tr : M_{n\times n} \to \mathbb{R}$ is linear

3.) No, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $\det(2A) = 4$, while $2\det(A) = 2$

4.) a.) Let $f, g \in V$, c be any scalar

Then, $E_a(f + cg) = (f + cg)(a)$

$= f(a) + cg(a)$

$= E_a(f) + cE_a(g)$

$\therefore E_a$ is linear

4b.) Let $f, h \in V$, c be any scalar

Then, $T_g(f + ch) = \int_0^1 (f + ch)(x)\, g(x)\, dx$

$= \int_0^1 (f(x)g(x) + ch(x)g(x))\, dx$

$= \int_0^1 f(x)g(x)\, dx + \int_0^1 ch(x)g(x)\, dx$

$= \int_0^1 f(x)g(x)\, dx + c\int_0^1 h(x)g(x)\, dx$

$= T_g(f) + cT_g(h) \quad \therefore T_g$ is linear

5.1) Let $\vec{v} = (v_1, \dots, v_n)^T$ & $g = (g_1, g_2, \dots g_n)^T$

Then, if $e^i = (0, \dots, 1, 0, \dots 0)^T$, $i = 1 \dots n$

standard basis is $\{e_1, e_2, \dots, e_n\}$

so, $T_{\vec{g}}(e^1) = g_1$, $T_{\vec{g}}(e^2) = g_2$, $T_{\vec{g}}(e^n) = g_n$

Thus the matrix of $T_{\vec{g}}$ is:

$$[T_{\vec{g}}] = [g_1, g_2, \dots, g_n]$$

And $T_{\vec{g}}(\vec{v}) = [g_1, g_2, \dots g_n] \cdot \vec{v} = \vec{v} \cdot \vec{g}$

5.2) Let $\vec{w} \in \mathbb{R}^3$, $\vec{w} = (w_1, w_2, w_3)$

$C_{\vec{w}} : \mathbb{R}^3 \to \mathbb{R}$, $C_{\vec{w}}(\vec{v}) = \vec{v} \times \vec{w}$

Then $C_{\vec{w}}(\vec{e}_1) = (0, -w_3, w_2)$

$C_{\vec{w}}(\vec{e}_2) = (w_3, 0, -w_1)$

$C_{\vec{w}}(\vec{e}_3) = (-w_2, -w_1, 0)$

Matrix of $C_{\vec{w}}(\vec{v})$ is:

$$[C_{\vec{w}}] = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ -w_2 & -w_1 & 0 \end{bmatrix}$$

5.3) $T : \mathbb{P}_3 \to \mathbb{P}_2$, $Tf(x) = f'(x)$

standard basis of $\mathbb{P}_3$: $\{1, x, x^2, x^3\}$

standard basis of $\mathbb{P}_2$: $\{1, x, x^2\}$

$$\left.\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2x \\ T(x^3) &= 3x^2 \end{aligned}\right\} [T] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$