Bemel Sleah

1.) $\because$ $I(x) = \int_0^x \frac{1 - e^{-t^2}}{t}\, dt$, $e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$

I.) $f(x) = t^{-1}\left[1 - \left(1 + (-t^2) + \frac{1}{2!}(-t^2)^2 + \frac{1}{3!}(-t^2)^3 + \frac{1}{4!}(-t^2)^4\right)\right]$

$= t^{-1}\left[t^2 - \frac{1}{2!} t^4 + \frac{1}{3!} t^6 - \frac{1}{4!} t^8 \right]$

$= t - \frac{1}{2!} t^3 + \frac{1}{3!} t^5 - \frac{1}{4!} t^7$

$\to I(x) = \int_0^x \left(t - \frac{1}{2!} t^3 + \frac{1}{3!} t^5 - \frac{1}{4!} t^7 \right) dt$

$= \left[\frac{1}{2} t^2 - \frac{1}{2!}\frac{1}{4} t^4 + \frac{1}{3!}\frac{1}{6} t^6 - \frac{1}{4!}\frac{1}{8} t^8 \right]_0^x$

$= \frac{1}{2} x^2 - \frac{1}{2!}\frac{1}{4} x^4 + \frac{1}{3!}\frac{1}{6} x^6 - \frac{1}{4!}\frac{1}{8} x^8$

II.) $S(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)n!}$

Ratio test

$$\lim_{n\to\infty} \left| \frac{S_{n+1}(x)}{S_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{(n+1)+1} x^{2(n+1)}}{(2(n+1))(n+1)!} \cdot \frac{(2n)n!}{(-1)^{n+1} x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \cdot \frac{(2n)n!}{(2n+2)(n+1)n!} \cdot \frac{x^{2n+2}}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{2n}{2n^2+4n+2} \cdot x^2 \right| \cdot \frac{1/2n}{1/2n}$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{n + 2 + \frac{1}{n}} \right| = |x^2| \lim_{n\to\infty} \left| \frac{1}{n+2+\frac{1}{n}} \right| = 0 \quad \forall x \in R$$

$\therefore$ converges absolutely

II. continued)

we want to find $\max_{x\in[-1,1]} \left| I(x) - S_4(x) \right|$

$$\rightarrow \max_{x\in[-1,1]} \left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)n!} - \sum_{k=1}^{4} (-1)^{k+1} \frac{x^{2k}}{(2k)k!} \right|$$

$\because$ $S_n(x)$ is alternating

$\because$ $S_n \to 0$ as $n \to \infty$ since it converges

$\because$ $S_{n+1} \leq S_n$ since it converges absolutely

$$\rightarrow \max_{x\in[-1,1]} \left| I(x) - S_4(x) \right| \leq \max_{x\in[-1,1]} \left| S_5 \right|$$

$$\leq \max_{x\in[-1,1]} \left| (-1)^{k+1} \frac{x^{2k}}{(2k)k!} \right|, \quad k=5$$

$$\leq \max_{x\in[-1,1]} \left| (-1)^{6} \frac{x^{10}}{(10)10!} \right|$$

$$\leq \left(\frac{1}{(10)\,10!} \right)$$

$$\leq 2.7557\ldots \cdot 10^{-8}$$

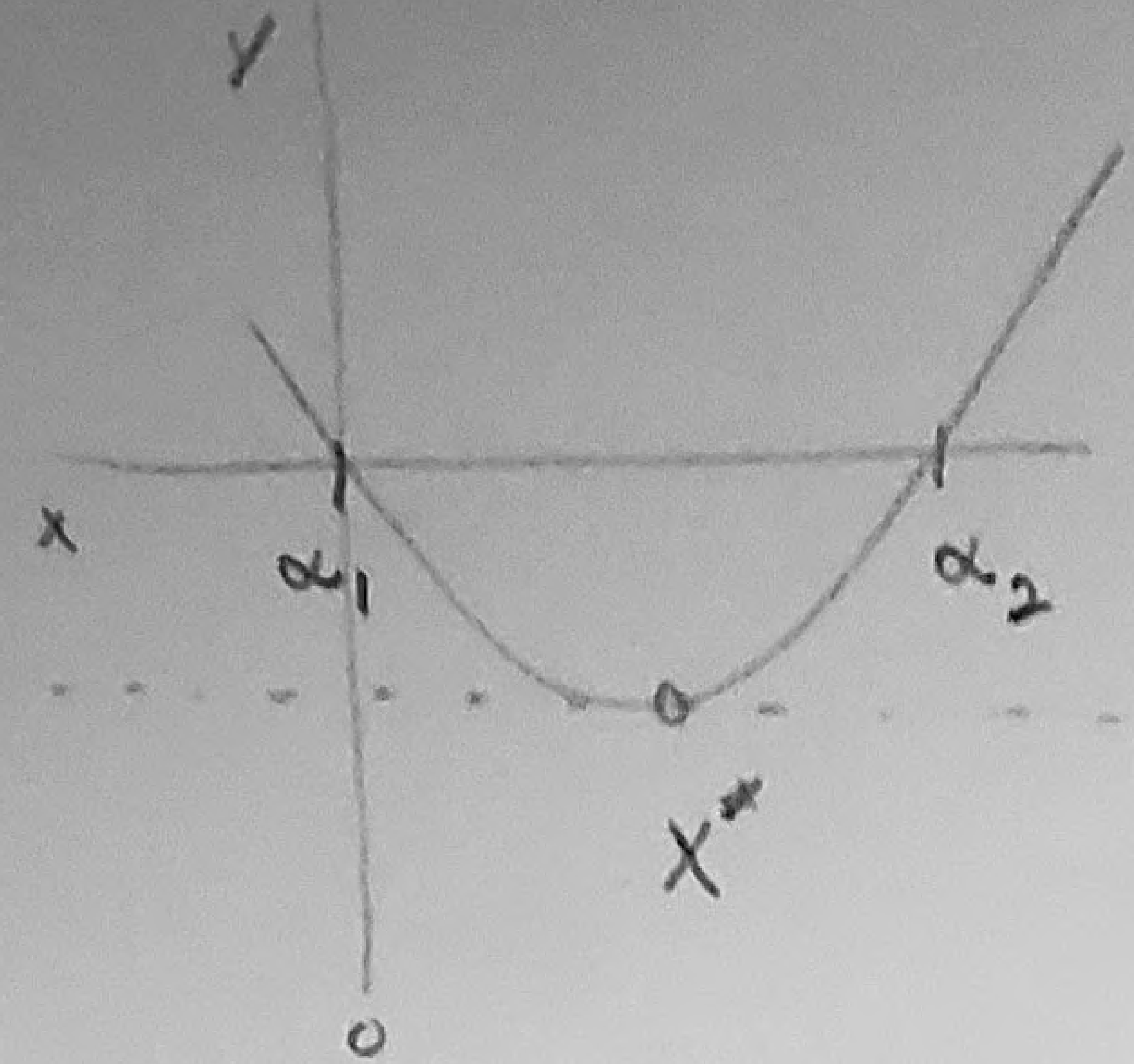
III.) Since the error we found in II was bounded by $\frac{1}{(2n)n!}$, we want to find:

$$\frac{1}{(2n)n!} < 5 \cdot 10^{-10}$$

Now, $\frac{1}{(2)(13)(13!)} < 5 \cdot 10^{-10}$

so $\boxed{n=13}$ (will be of degree 26)

2.) I.)



At this point the tangent line of $f(x)$ is parallel the x axis and will not converge if used as an initial guess for Newton's Method.

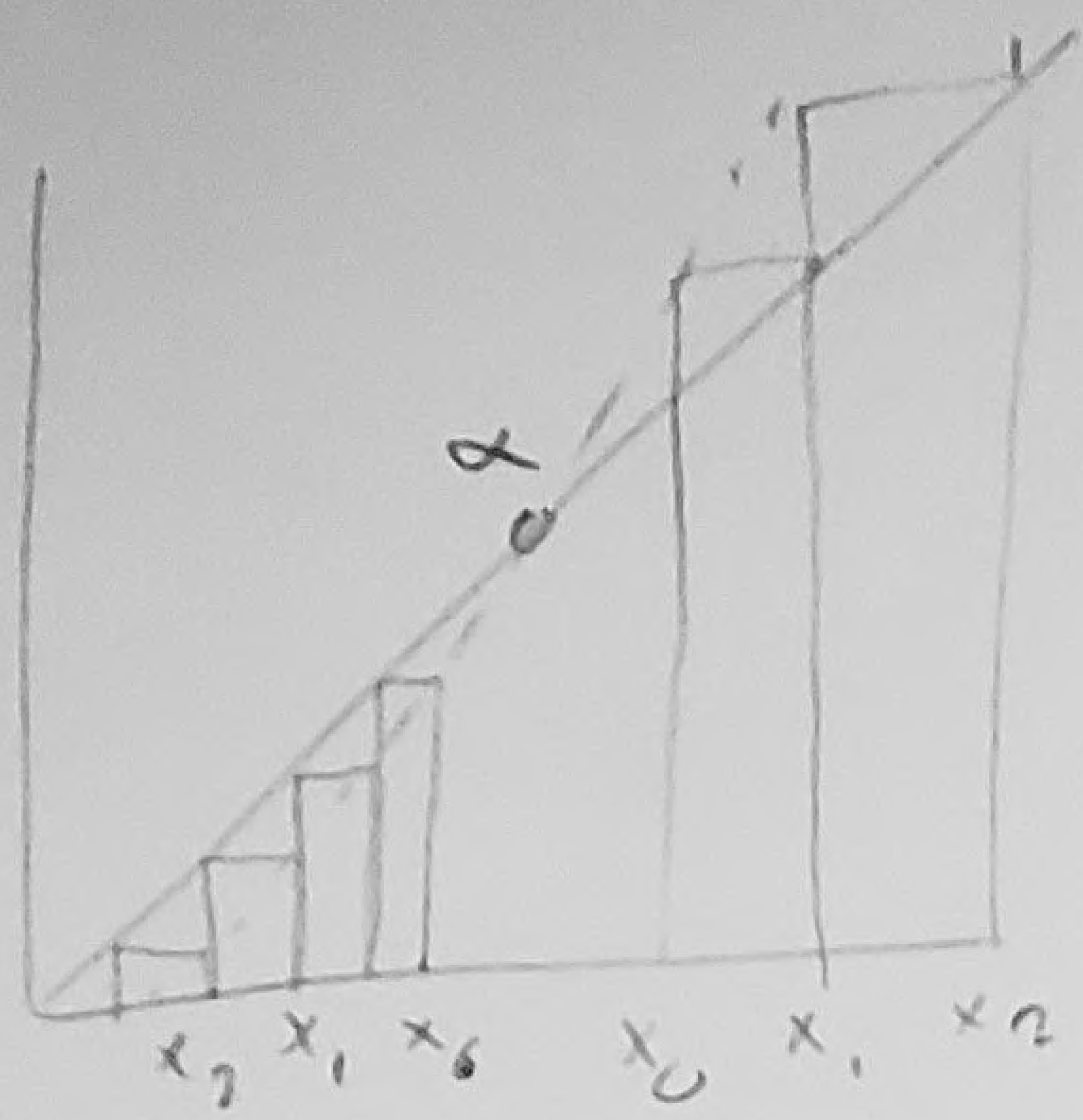
II.) Since this is a local minimum of a continuous, differentiable function, we can find x^* with $f'(x^*) = 0$.

so, $f'(x) = \frac{d}{dx}\left[e^{x/3} - 1 - x\right] = \frac{1}{3}e^{x/3} - 1$

Now, $f'(x^*) = 0 \rightarrow \frac{1}{3}e^{x^*/3} - 1 = 0 \rightarrow \frac{x^*}{3} = \ln(3)$

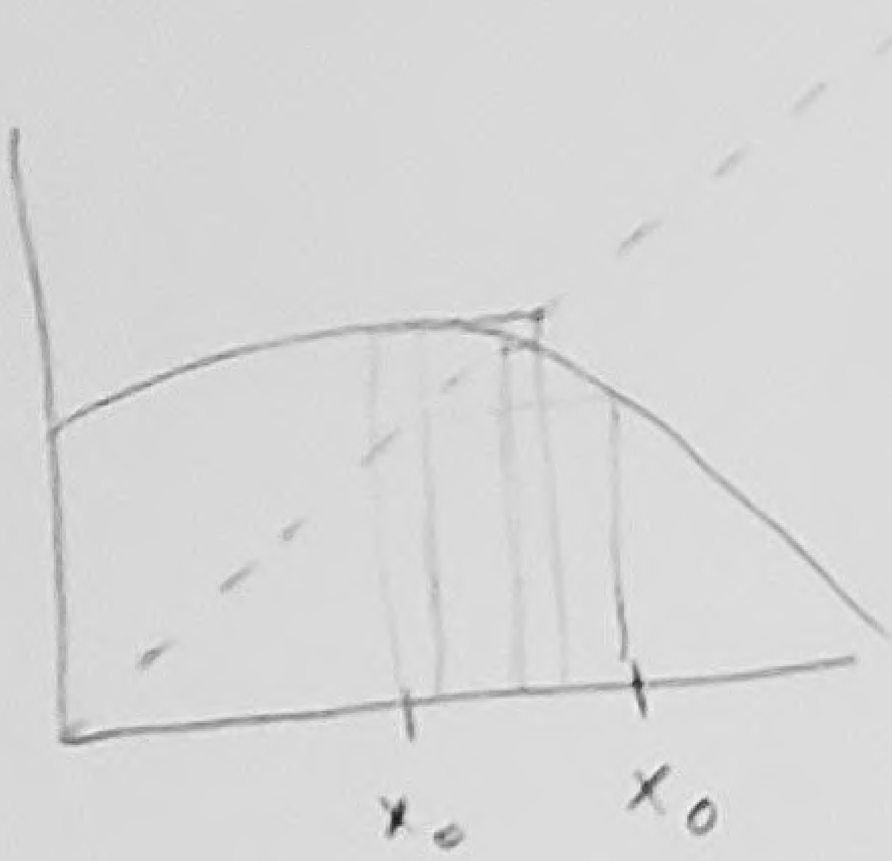
$\rightarrow x^* = \ln(3^3)$

3.) 1 does not converge to α, because the sequence will diverge from α for any given x_0



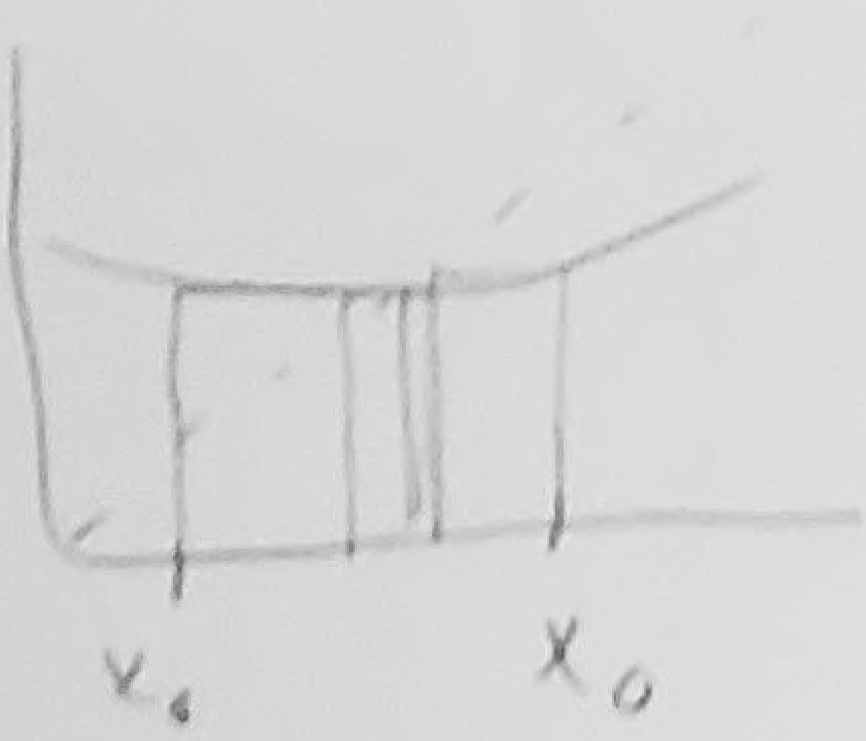
$(g'(\alpha) > 1)$

2 will converge slower as $-1 < g'(\alpha) < 0$ and may oscillate.



$(-1 < g'(\alpha) < 0)$

3 will converge the quickest and not oscillate



$0 < g'(\alpha) < 1$