1.) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

1a.) B is an upper-triangular matrix,

so $\sigma(B) = \{1, 4, 6\}$ from $(1-\lambda)(4-\lambda)(6-\lambda)$

1b.) B has 3 distinct roots, so yes.

since $\dim(B) = |\sigma(B)|$ multiplicity not counted

1c.) $(B - \lambda_1 I)v_1 = 0 \rightarrow (B - 1I)v_1 = 0$

$\rightarrow \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Let $x_1 = 1$ arbitrarily, then $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda = 4$: $(B - \lambda_2 I)v_2 = 0 \rightarrow (B - 4I)v_2 = 0$

$\rightarrow \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \xrightarrow{RREF} \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

Let $x_2 = 1$ arbitrarily, then $v_2 = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

For $\lambda = 6$: $(B - \lambda_3 I)v_3 = 0 \rightarrow (B - 6I)v_3 = 0$

$\rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0 \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0$

Let $x_3 = 1$ arbitrarily, then $v_3 = \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$

so, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 16 \\ 25 \\ 10 \end{bmatrix}$

Let $P = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}$

Then $P^{-1} = \begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1/3 & -5/6 \\ 0 & 0 & 1/10 \end{bmatrix}$

Since $B = PDP^{-1}$, then $D = P^{-1}BP$

so $D = \begin{bmatrix} 1 & -2/3 & 1/15 \\ 0 & 1/3 & -5/6 \\ 0 & 0 & 1/10 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}$

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} e_1\lambda_1 & e_2\lambda_2 & e_3\lambda_3 \end{bmatrix}$

2.) Assume $A = P^{-1}BP$

Similarity $B = PAP^{-1}$

Then characteristic poly. of A is:

$\underline{\det(A - \lambda I)} = \det(P^{-1}BP - \lambda I)$

$= \det(P^{-1}BP - \lambda I P^{-1}P)$ where $P^{-1}P = I$

$= \det(P^{-1}BP - P^{-1}\lambda I P)$ since $IP^{-1} = P^{-1}I$

$= \det\left[P^{-1}(B - \lambda I)P\right]$

$= \det(P^{-1})\det(B - \lambda I)\det(P)$

$= \det(P^{-1})\det(P)\det(B - \lambda I)$

$= \det(P^{-1} \cdot P)\det(B - \lambda I) = \underline{\det(B - \lambda I)}$

3.) Let $T: \mathbb{P}_2 \to \mathbb{P}_2$ be defined by $T(f) = \frac{d}{dx}\left(x f(x)\right)$

Let $\mathcal{B}$ & $\mathcal{C}$ be bases of $\mathbb{P}_2$ where

$\mathcal{B} = \{1, x, x^2\}$, $\mathcal{C} = \{1, 1+x, 1+x+x^2\}$

3a.) Let $f(x), g(x) \in \mathbb{P}_2$

Then $T(f) = \frac{d}{dx}\left(xf(x)\right)$, $T(g) = \frac{d}{dx}\left(xg(x)\right)$

Consider $T(f+kg) = \frac{d}{dx}\left[x\left(f(x)+kg(x)\right)\right]$, $k \in \mathbb{R}^1$

$$= \frac{d}{dx}\left[x\left(f(x)\right)\right] + \frac{d}{dx}\left[x\left(kg(x)\right)\right]$$

$$= \frac{d}{dx}\left[xf(x)\right] + k\frac{d}{dx}\left[xg(x)\right]$$

so T is closed under addition and scalar multiplication.

3b.) $T(\mathcal{B}_1) = T(1) = \frac{d}{dx}[x] = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$

$T(\mathcal{B}_2) = T(x) = \frac{d}{dx}[x^2] = 2x = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}^T$

$T(\mathcal{B}_3) = T(x^2) = \frac{d}{dx}[x^3] = 3x^2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}^T$

so $[T]_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

3c.) $T(c_1) = T(1) = \frac{d}{dx}[x] = 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$

$T(c_2) = T(1+x) = \frac{d}{dx}[x+x^2] = 1+2x = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}^T$

$T(c_3) = T(1+x+x^2) = \frac{d}{dx}[x+x^2+x^3] = 1+2x+3x^2 = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}^T$

so, $[T]_{CC} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & -1 & 3 \end{bmatrix}$

3d.) $[T]_{\mathcal{BB}}$ is the RREF of $[T]_{CC}$,

both will have eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

since they are/can be diagonal matrices.

3e.) $\sigma(T) = \{1, 2, 3\}$.

Since $A\vec{v} = \lambda\vec{v}$, and $T(1) = 1$, $T(x) = 2x$, $T(x^2) = 3x^2$

Then eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3f.) Yes, T is invertible since $0 \notin \sigma(T)$

And $[T^{-1}]_{\mathcal{BB}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

3g.) For $T(f) = \frac{d}{dx}[x f(x)]$

$T^{-1}(f) = \frac{1}{x}\int f(x)\,dx$

4) $f(\lambda) = |A - \lambda I|$, degree $f(\lambda) = n$ since $A_{n \times n}$

Therefore $f(\lambda) = 0$ has n roots by F.T.A.

$f(\lambda)$ can be written $\prod_{i=1}^{n} (\lambda_i - \lambda)$ where

λ_i is a root of $f(\lambda) = 0$.

So, $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\ldots(\lambda_n - \lambda)$

Therefore $f(\lambda) = |A - \lambda I|$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ \vdots & a_{22} - \lambda & & \\ & & \ddots & \\ a_{n1} & & & a_{nn} - \lambda \end{vmatrix}$$

$$= (a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix}$$

+ terms of degree at most $(n-2)$

Since the next terms in the Laplace Expansion are cofactors of smaller matrices, $n-2$ is the next highest degree of these cofactor determinants.