Mth 351 Quiz 5 Bennet Sloan

1.) Let $B_n = [1, x, x^2, x^3 \dots x^n]$, $f(x) \in \mathbb{P}_n$

I.) Define $T_1 : \mathbb{P}_n \to \mathbb{R}^1 \mid \int_{-1}^{1} f(x)\,dx$
& $T_2 : \mathbb{P}_n \to \mathbb{R}^1 \mid f(-\alpha) + f(\alpha)$

$$\to T_1 B_n = \begin{cases} 0 & , n = 2k-1 \\ 2 & , n = 0 \\ 2\int_0^1 B_n\,dx & , n = 2k \end{cases} , k \in \mathbb{Z}$$

$$\to T_2 B_n[\alpha] = \begin{cases} 0 & , n = 2k-1 \\ 2 & , n = 0 \\ 2 B_n[\alpha] & , n = 2k \end{cases} , k \in \mathbb{Z}$$

Consider $T_1 B_0 = T_2 B_0[\alpha] = 2$
& $T_1 B_1 = T_2 B_1[\alpha] = 0$

But $T_1 B_2 = 2/3$
& $T_2 B_2[\alpha] = 2\alpha^2$

$\therefore$ degree of precision ≥ 1

which is only equal when $2/3 = 2\alpha^2$

II.) $\to \alpha = \left(\frac{2}{2 \cdot 3}\right)^{1/2} = 1/\sqrt{3} \in (0, 1]$

For $\alpha = \frac{1}{\sqrt{3}}$: $\to T_1 B_3 = 0$ & $T_2 B_3[\alpha] = 0$
$\to T_1 B_4 = 2\int_0^1 x^4\,dx = 2/5$
& $T_2 B_4 = 2(\alpha^4) = 2\left(\frac{1}{\sqrt{3}}\right)^4$

$T_1 B_4 \neq T_2 B_4$

$\boxed{\therefore r = 3}$

2.) Approx. $f'(x)$ using x, $x+h$, $x+2h$. Let $\theta < 1$,

$\rightarrow$ $f'(x) \approx A f(x) + B f(x+h) + C f(x+2h)$

$$\approx A f(x) + B\left[f(x) + h f'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{3}h^3 f'''(x+\theta h)\right]$$
$$+ C\left[f(x) + 2h f'(x) + \frac{1}{2}h^2 f''(x) + \frac{8}{3}h^3 + (x+2\theta h)\right.$$

$$\approx (A+B+C) + (x) + h(B+2C) f'(x) + \frac{1}{2}h^2 (B-4C) f''(x)$$
$$+ \frac{1}{6}h^3\left[B f'''(x+\theta h) + 8(f'''(x+2\theta h))\right]$$

① $A+B+C = 0$

② $B+2C = \frac{1}{h}$

③ $B = -4C$

③ → ② : $(-4C) + 2C = \frac{1}{h}$

$C = -\frac{1}{2h}$

③ $B = \frac{2}{h}$

① $A + \left(\frac{2}{h}\right) + \left(-\frac{1}{2h}\right) = 0$

$A = -\frac{3}{2h}$

$\rightarrow$ $f'(x) \approx \left(-\frac{3}{2h}\right) f(x) + \left(\frac{2}{h}\right) f(x+h) + \left(-\frac{1}{2h}\right) f(x+2h)$
$$+ \frac{1}{6}h^3\left[\frac{2}{h} f'''(x+\theta h) - \frac{8}{2h} f'''(x+2\theta h)\right.$$

$$\approx -\frac{3}{2h} f(x) + \frac{2}{h} f(x+h) - \frac{1}{2h} f(x+2h) + \frac{h^2}{3} f'''(x+\theta h)$$
$$- \frac{2h^2}{3} f'''(x+2\theta h)$$

$\rightarrow$ $\epsilon = f'(x) + \frac{3}{2h} f(h) - \frac{2}{h} f(x+h) + \frac{1}{2h} f(x+2h)$

$$- \frac{1}{3}h^2 f'''(x+\theta h) + \frac{2}{3}h^2 f'''(x+2\theta h)$$