MTH 342 OSU Winter 2019 Thursday, Jan. 24, Lab E, done in class.

Complete this and submit it to Canvas by the posted due date: Monday, Jan. 28.

Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be the linear transformation $Tf(x) = f'(x)$. We will write $\mathcal{B}_n = \{1, x, x^2, \ldots, x^n\}$ for the standard basis for $\mathbb{P}_n$. Note that $\mathcal{B}_n$ has $n+1$ elements.

1. Can you find a right inverse of T? This means find $S_2 : \mathbb{P}_1 \to \mathbb{P}_2$ such that $T \circ S_2 = I_{\mathbb{P}_1}$. Give a "calculus description" of S_2.

2. Can you find a left inverse of T? This means find $S_1 : \mathbb{P}_1 \to \mathbb{P}_2$ such that $S_1 \circ T = I_{\mathbb{P}_2}$.

3. Can you find a different right inverse of T? (Another, different, S_2.)

4. Let $A = [T]_{\mathcal{B}_1\mathcal{B}_2}$ be the standard matrix for T. For the right inverse S_2 you found in part 1, find its standard matrix $B = [S_2]_{\mathcal{B}_2\mathcal{B}_1}$ and check that

$$[T]_{\mathcal{B}_1\mathcal{B}_2}[S_2]_{\mathcal{B}_2\mathcal{B}_1} = [T \circ S_2]_{\mathcal{B}_1\mathcal{B}_1}.$$

In other words, check that $AB = I_{2\times 2}$.

5. Repeat part 4 for the different right inverse that you found in part 3.

6. A matrix M is called right invertible if the corresponding linear transformation is right invertible. Write down the most general right inverse of $A = [T]_{\mathcal{B}_1\mathcal{B}_2}$. (This will be a 3×2 matrix with free parameters.)

7. Why doesn't A have a left inverse? Is T left invertible? Explain.

1.) $T: \mathbb{P}_2 \to \mathbb{P}_1 \;\Big|\; Tf(x) = f'(x)$

Find $S_2: \mathbb{P}_1 \to \mathbb{P}_2$ such that $T \circ S_2 = I_{P_1}$?

$$\begin{tikzcd}
V \arrow[r, "S_2"] \arrow[dr, "T\circ S_2 = I_V"'] & U \arrow[d, "T"'] \arrow[dr, "S_1 \circ T = I_U"] & \\
& V \arrow[r, "S_1"'] & U
\end{tikzcd}$$

$v \in \mathbb{P}_1$

$u \in \mathbb{P}_2$

Let $f(x) \in \mathbb{P}_1$

Suppose $S_2(f(x)) = \int_0^x f(x)\, dt$

Then $T \circ S_2(f(x)) = T(S_2(f(t)))$

$= T\left(\int_0^x f(t)\, dt\right)$

$= \frac{d}{dx} \int_0^x f(t)\, dt$

$= f(x) = I_{\mathbb{P}_1} f(x)$

Therefore $T \circ S_2 = I_{\mathbb{P}_1}$

For $S_2(f(x)) = \int_0^x f(t)\, dt$

2.) It is not possible for T to have a left inverse.

Consider $S_1 \circ T(c) = 0$

For c, any ~~&~~ all scalars.

3.) Yes, S_2 is not unique.

Consider:

$$S_2(f(x)) = \int_0^x f(t)\,dt$$

$$T \circ S_2(f(x)) = \frac{d}{dx}\int_0^x f(t)\,dt$$

$$= f(x)$$

4.) $\left.\begin{aligned} T(1) &= 0 \\ T(x) &= 1 \\ T(x^2) &= 2x \end{aligned}\right\} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\left.\begin{aligned} S_2(1) &= x \\ S_2(x) &= \tfrac{1}{2}x^2 \\ S_2(x^2) &= \tfrac{1}{3}x^3 \end{aligned}\right\} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$$[T]_{S_1,S_2}[S_2]_{S_2,S_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5.) $\int_b^x 1 \, dt = x - b$

$\int_b^x t \, dt = \frac{1}{2}x^2 - \frac{1}{2}b^2$

so, $B = \begin{bmatrix} -b & -\frac{b^2}{2} \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -b & -\frac{b^2}{2} \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

6.) If $A = [T]_{B_1 B_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Then the general right inverse is:

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ 1 & b_{22} \\ b_{31} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For any $b_{11}, b_{12}, b_{22}, b_{31}$

7.) A does not have a left inverse because T is not injective due to the property of the derivative of a constant. Thus T is <u>not</u> left invertible