Turn this in by making a pdf scan of your work and submitting it to Canvas by 11:59 pm on the due date.

How to write up homework. Please put your name and assignment number on the first page. You may use this page as a cover sheet. Show your work and explain it. In general, just writing down an answer without showing the steps to arriving at the answer is not sufficient for credit.

Current reading is Chapter 4 of Linear Algebra Done Wrong (LADW).

1. Consider the matrix

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

   (a) What are the eigenvalues of B?

   (b) Is B diagonalizable?

   (c) Find an eigenvector corresponding to each eigenvalue. If B is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$.

2. Prove that if A and B are similar matrices, then the characteristic polynomial of A is equal to the characteristic polynomial of B.

3. Recall that $\mathbb{P}_2$ denotes the vector space of polynomials of degree ≤ 2. The standard basis for $\mathbb{P}_2$ is $\mathcal{B} = \{1, x, x^2\}$.
   Another basis for $\mathbb{P}_2$ is $\mathcal{C} = \{1, 1 + x, 1 + x + x^2\}$.
   Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be defined by $T(f) = \dfrac{d}{dx}\Big(xf(x)\Big)$.

   (a) Show that T is a linear transformation.

   (b) Find $[T]_{\mathcal{B}\mathcal{B}}$, the matrix of T with respect to the standard basis.

   (c) Find $[T]_{\mathcal{C}\mathcal{C}}$, the matrix of T with respect to the basis $\mathcal{C}$.

   (d) How are $[T]_{\mathcal{C}\mathcal{C}}$ and $[T]_{\mathcal{B}\mathcal{B}}$ related? Should they have the same eigenvalues?

   (e) What are the eigenvalues of T? What are the eigenvectors of T? (Note that an eigenvector of T belongs to $\mathbb{P}_2$, not $\mathbb{R}^3$.)

   (f) Is T invertible? If it is, find $[T^{-1}]_{\mathcal{B}\mathcal{B}}$.

   (g) Can you give a "calculus discription" of T^{-1}?

4. On the last page of this assignment there is a proof of the following theorem. At several places in the proof there are why-questions. Give answers to these questions. If you mention cofactors, you may want to review pages 90-92 in LADW (the beginning of Section 5 of Chapter 3).

   Format of answer: just provide answers to why-questions W1, W2, and W3. You do not need to rewrite the entire proof.

   Theorem. Let A be an $n \times n$ matrix and let λ_1, λ_2, $\ldots$, λ_n be its eigenvalues (counting multiplicities). Then the trace of A satisfies

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i.$$

Proof. The characteristic polynomial A is

$$f(\lambda) = \det(A - \lambda I_n) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda).$$

(W1) Why is this true?

The coefficient of the λ^{n-1} term of $f(\lambda)$ is

$$(-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n). \tag{1}$$

We may also compute $f(\lambda)$ by cofactor expansion of the determinant:

$$f(\lambda) = \det(A - \lambda I_n) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & . \; . \; . & a_{1n} \\ a_{21} & a_{22} - \lambda & . \; . \; . & a_{2n} \\ . & . & . & . \\ . & . & \;\;\; . & . \\ . & . & \;\;\;\;\;\; . & . \\ a_{n1} & a_{n2} & . \; . \; . & a_{nn} - \lambda \end{pmatrix}$$

$$= (a_{11} - \lambda)\det \begin{pmatrix} a_{22} - \lambda & . \; . \; . & a_{2n} \\ . & . & . \\ . & \;\;\; . & . \\ . & \;\;\;\;\;\; . & . \\ a_{n2} & . \; . \; . & a_{nn} - \lambda \end{pmatrix} + \text{terms of degree at most } (n-2)$$

(W2) Why is the degree at most $(n-2)$?

$$= (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda) + \text{terms of degree at most } (n-2).$$

(W3) Why is the degree at most $(n-2)$?

Then coefficient on the λ^{n-1} term of $f(\lambda)$ is also

$$(-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn}). \tag{2}$$

By comparing (1) and (2) we see that

$$a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots \lambda_n,$$

which establishes $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$. This completes the proof. $\square$