Bennet
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Mth 342 Lab Q 3/8/19

1.) Let $V: \mathbb{P}_2(\mathbb{R})$, $(f,g) = \int_0^1 f(x)g(x)\,dx$

Consider $V = \text{span}(1, x, x^2) = \text{span}(v_1, v_2, v_3)$

Let $\mathbb{P}_k'$ be an orthonormalized vector space.

Then for $\vec{u}_k = v_k - \text{Proj}_{\mathbb{P}_{k-1}}(v_k)$, $(\vec{u}_k, \vec{u}_k) = 0$.

Set $\vec{e}_1 = \frac{v_1}{\|v_1\|} = \frac{1}{((1,1))^{1/2}} = \frac{1}{(\int_0^1 1\,dx)^{1/2}} = 1$

Then $\vec{u}_2 = v_2 - \text{Proj}_{\mathbb{P}_1}(v_2) = v_2 - (v_2, e_1)e_1$

$= x - \int_0^1 x\,dx = x - \frac{1}{2}$

And $\|\vec{u}_2\| = \left[\left(x-\frac{1}{2}, x-\frac{1}{2}\right)\right]^{1/2}$

$= \left[\int_0^1 x^2 - x + \frac{1}{4}\,dx\right]^{1/2} = \sqrt{\frac{1}{12}}$

so $\vec{e}_2 = \frac{\vec{u}}{\|\vec{u}\|} = \frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}$

Now $\vec{u}_3 = v_3 - \text{Proj}_{\mathbb{P}_2}(v_3)$

$= x^2 - (x^2, 1) - \left(x^2, \frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}\right)\frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}$

$= x^2 - \int_0^1 x^2\,dx - \int_0^1 \frac{x^3 - x^2/2}{\sqrt{\frac{1}{12}}}\,dx\left(\frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}\right)$

$= x^2 - 1 + \frac{x-\frac{1}{2}}{1/6}$

$= x^2 + 6x - 4$

And $\|\vec{u}_3\| = \|x^2+6x-4\|$

$$= \left[\left(x^2+6x-4,\ x^2+6x-4\right)\right]^{1/2}$$

$$= \left[\int_0^1 x^4+12x^3+28x^2-48x+16\,dx\right]^{1/2}$$

$$= \sqrt{\frac{68}{15}}$$

so $\vec{e}_3 = \dfrac{\vec{u}_3}{\|\vec{u}_3\|} = \dfrac{x^2+6x-4}{\left(\frac{68}{15}\right)^{1/2}}$

Therefore an orthonormal basis for $P_2(\mathbb{R})$ is:

$$V = \operatorname{span}\left(1,\ x-\frac{1}{2},\ \frac{x^2+6x-4}{\sqrt{\frac{68}{15}}}\right)$$