MTH 342 OSU Winter 2019 Wednesday, Feb. 13, Lab J, done in class.
Complete this and submit it to Canvas by the posted due date: Friday, Feb. 15.

1. Find the (complex) eigenvalues of the rotation matrix

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$(\cos\alpha-\lambda)(\cos\alpha-\lambda)-(-\sin\alpha)(\sin\alpha)$$

$$(\cos\alpha-\lambda)^2+\sin^2\alpha$$

$$\cos\alpha^2-2\lambda\cos\alpha+\lambda^2+\sin^2\alpha$$

$$\cos^2\alpha+\sin^2\alpha-2\lambda\cos\alpha+\lambda^2$$

$$1-2\lambda\cos\alpha+\lambda^2$$

$$\lambda^2-2\lambda\cos\alpha+1 \qquad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{2\cos\alpha\pm\sqrt{4\cos^2\alpha-4}}{2}$$

$$\frac{2\cos\alpha\pm\sqrt{4(\cos^2\alpha-1)}}{2}$$

$$\frac{2\cos\alpha\pm\sqrt{-4\sin^2\alpha}}{2}=\frac{2\cos\alpha}{2}\pm\frac{\sqrt{-1}\sqrt{4}\sqrt{\sin^2\alpha}}{2}$$

$$\cos\alpha\pm i\sin\alpha$$

$$\cos\alpha+i\sin\alpha,\quad \cos\alpha-i\sin\alpha$$

$$e^{i\alpha},\quad e^{-i\alpha}$$

2. An $n \times n$ matrix A is called *nilpotent* if $A^k = \mathbf{0}$ (the zero matrix) for some positive integer k.

(a) Find an example of a 2×2 nilpotent matrix A.

If $A\vec{v} = \lambda\vec{v} = 0$ for $\vec{v} \neq 0$

Then $A^2\vec{v} = AA\vec{v} = A(\lambda\vec{v}) = \lambda^2\vec{v}$

Consider $A^k\vec{v} = \lambda^k\vec{v} = 0$

Then $\lambda^k = 0$

So $\lambda = 0$

(Trace & determinant = 0)

$A = \begin{bmatrix} a & b \\ -\frac{a^2}{b} & -a \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{bmatrix}$

(b) Find an example of a 3×3 nilpotent matrix B. Can you find such a matrix B having the property that $B^2 \neq 0$?

If trace B = det $B = 0$ but $B^2 \neq 0$,

Then B could have a zero row and column, Zero diagonal.

ex: $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $B^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq 0$, $B^3 = 0$

(c) Verify that 0 is an eigenvalue of your 2×2 matrix A. This means find a nonzero vector $\vec{v}$ such that $A\vec{v} = 0\vec{v} = \vec{0}$.

$A\vec{v} = 0 \longrightarrow \begin{bmatrix} 1 & 2 \\ -\frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\frac{1}{2}R_1 + R_2 \to R_2 \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $a = -2b$

$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\therefore 0 \in \sigma(A)$

(d) Prove that if an $n \times n$ matrix A is nilpotent, and λ is an eigenvalue of A, then $\lambda = 0$. (An eigenvalue can be zero, but the zero vector is never an eigenvector.)

$\because A^k_{n\times n} = 0$ & $A\vec{v} = \lambda\vec{v}$, $k \geq 2$

Consider $A^k\vec{v} = \left[\prod_1^k A\right]\vec{v} = (A \circ A \circ A \ldots)\vec{v} = (A\vec{v} \circ A\vec{v} \circ \ldots)$

$= (\lambda\vec{v} \circ \lambda\vec{v} \circ \ldots) = (\lambda \circ \lambda \circ \lambda \ldots)\vec{v} = \left[\prod_1^k \lambda\right]\vec{v}$

$= \lambda^k\vec{v}$

If $A^k\vec{v} = \lambda^k\vec{v} = 0$, $\lambda^k = 0$ since $\vec{v} \neq 0$.

The only scalar $\lambda^k = 0$ is $\lambda = 0$.