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Mth 256 - HW 4

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1.) Consider the second-order linear DE

$$(1 - t\cot(t))\,y'' - ty' + y = 0$$

for $0 < t < \pi$. Let $y_1(t) = t$ & $y_2(t) = \sin(t)$

a.) Are y_1 & y_2 both solutions?

For $y_1 = t$ | For $y_2 = \sin(t)$

$y_1' = 1$ | $y_2' = \cos(t)$

$y_1'' = 0$ | $y_2'' = -\sin(t)$

So $(1 - t\cot(t))0 \cancel{-t} \cancel{+t} = 0$ | So $\left(\cancel{1} - t\frac{\cancel{\cos(t)}}{\cancel{\sin(t)}}\right)\cancel{(-\sin(t))} - \cancel{t\cos(t)} + \cancel{\sin(t)} = 0$

$\therefore$ y_1 is a solution | $\therefore$ y_2 is a solution $\boxed{\text{Yes}}$

b.) Are y_1 & y_2 LI? If so, find gen. soln. If not, find $Ay_1 + By_2 = 0$

$\because$ y_1 & y_2 will be LI iff $W(t) \neq \vec{0}$.

So, $W(t) = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} t & \sin t \\ 1 & \cos t \end{bmatrix} = (t)(\cos t) - (\sin t)(1) \neq 0$

Therefore y_1 & y_2 are Linearly Independent.

Moreover, y_1 & y_2 are a Fundamental set of solutions

so $y(t) = c_1 y_1(t) + c_2 y_2(t) \longrightarrow \boxed{y(t) = c_1 t + c_2 \sin(t)}$

2.) Suppose that α is a constant and $\{y_1, y_2\}$ is a set of solutions to the DE:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Find the Wronskian $W(x)$ of $\{y_1, y_2\}$ given $W(0) = 1$

Standard form: $y'' - \frac{2x}{(1-x^2)} y' + \frac{\alpha(\alpha+1)}{(1-x^2)} y = 0$

$\therefore$ Abel's Thm. $W(y_1, y_2) = c\, e^{-\int\left[\frac{-2x}{1-x^2}\right]dx}$

For $-\int \frac{-2x}{1-x^2}\,dx$ let $u = 1-x^2$

$du = -2x\,dx$

$-\int \frac{1}{u}\,du$

$-\ln|u|$

$-\ln|1-x^2|$

So, $W(y_1, y_2) = c\, e^{-\ln|1-x^2|}$

$= c \frac{1}{|1-x^2|}$

$= \frac{c}{|1-x^2|}$

using $W(0) = 1 = \frac{c}{|1-0^2|}$

$c = 1$

$\therefore W(y_1, y_2) = \frac{1}{|1-x^2|}$

2.) Suppose that α is a constant and $\{y_1, y_2\}$ is a set of solutions to the DE

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

Find $W(x)$ of $\{y_1, y_2\}$ given that $W(0) = 1$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$W'(x) = y_1 y_2'' + y_1' y_2' - y_1' y_2' - y_1'' y_2$$

$$= y_1 y_2'' - y_1'' y_2$$

From DE: $y'' = \frac{2x}{(1-x^2)} y' - \frac{\alpha(\alpha+1)}{(1-x^2)} y$

So, $W'(x) = y_1 y_2'' - y_1'' y_2$

$$= y_1 \left(\frac{2x}{1-x^2} y_2' - \frac{\alpha(\alpha+1)}{(1-x^2)} y_2 \right)$$

$$- \left(\frac{2x}{1-x^2} y_1' - \frac{\alpha(\alpha+1)}{(1-x^2)} y_1 \right) y_2$$

$$= \frac{2x}{(1-x^2)} (y_1 y_2' - y_1' y_2)$$

Thus, $W'(x) = \frac{2x}{1-x^2} W(x)$

Solving, $\frac{dW}{W} = \frac{2x}{1-x^2} dx$

$$\ln|W| = -\ln|1-x^2| + c$$

For $W(0) = 1$: $W = \frac{c}{|1-x^2|}$

let $u = 1-x^2$

$du = -2x\,dx$

So, $\int \frac{2x}{1-x^2} dx$

$= -\int \frac{1}{u} du$

$= -\ln|u| + C$

$1 = \frac{c}{|1-0|} \longrightarrow c = 1 \quad \therefore W(x) = \frac{1}{|1-x^2|}$