$\therefore \quad F(x) = \int_0^x \frac{1 - e^{-t^2}}{t^2}\, dt \quad , \quad e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots$

I)

$\rightarrow e^{-t^2} = 1 + (-t^2) + \frac{1}{2!}(-t^2)^2 + \frac{1}{3!}(-t^2)^3 \ldots$

$\rightarrow f(t) = t^{-2}\left[1 - \left(1 + t^2 - \frac{1}{2!}t^4 + \frac{1}{3!}t^6 - \frac{1}{4!}t^8 \ldots\right)\right]$

$= t^{-2}\left(\cancel{1 - 1} + t^2 - \frac{1}{2!}t^4 + \frac{1}{3!}t^6 - \frac{1}{4!}t^8 \ldots\right)$

$= 1 - \frac{1}{2!}t^2 + \frac{1}{3!}t^4 - \frac{1}{4!}t^6 \ldots$

So, $F(x) = \int_0^x \left(1 - \frac{1}{2!}t^2 + \frac{1}{3!}t^4 - \frac{1}{4!}t^6 \ldots\right) dt$

$= x - \frac{1}{3}\frac{1}{2!}x^3 + \frac{1}{5}\frac{1}{3!}x^5 - \frac{1}{7}\frac{1}{4!}x^7 \ldots$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{(2n-1)}}{(2n-1)\,n!}$

II)

Now, $\lim_{n\to\infty} \left| \frac{g_{n+1}(x)}{g_n(x)} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{(n+1)+1} x^{2(n+1)-1}}{[2(n+1)-1]\,(n+1)!} \circ \frac{(2n-1)\,n!}{(-1)^{n+1} x^{(2n-1)}} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2}}{(-1)^{n+1}} \circ \frac{x^{2n+1}}{x^{2n-1}} \circ \frac{(2n-1)}{(2n+1)} \circ \frac{n!}{(n+1)!} \right|$

$= \lim_{n\to\infty} \left| (-1)^1 \circ x^2 \circ \frac{n!}{(n+1)n!} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{n+1} \right| < 1 \quad , \quad |x| < (n+1)^{\frac{1}{2}}$

$\therefore$ converges $\forall\, x \in \mathbb{R}$

III) $\max_{x\in[-1,1]} \left| F(x) - S_{4}(x) \right| = \max_{x\in[-1,1]} \left| F(x) - \sum_{k=1}^{4} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)k!} \right|$

since $S_n(x)$ is alternating

And: ① $S_n \to 0$, $n \to \infty$ (convergence)

② $S_{n+1} \leq S_n$ (ratio test)

$$\leq \max_{x\in[-1,1]} \left| S_5(x) \right|$$

$$\leq \left| (-1)^{5+1} \frac{x^{2(5)-1}}{[2(5)-1](5)!} \right|$$

$$\leq \left| \frac{x^9}{(5!)\,9} \right|$$

$$\leq \frac{1}{1080} \approx 9.259 \cdot 10^{-4}$$