Bennit Sloan MCh 357 HW5

7) For $n, m \geq 0$ & $n \neq m$ show $\int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = 0$

$\because$ For $n \geq 0$, $n \in \mathbb{Z}$, $T_n(x) = \cos(n\cos^{-1} x)$, $-1 \leq x \leq 1$,

& $2\cos A \cos B = \cos(A+B) + \cos(A-B)$ ✱

$$\rightarrow \int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{\cos(n\cos^{-1}(x))\cos(m\cos^{-1}(x))}{(1-x^2)^{1/2}} dx$$

let $u = \cos^{-1}(x) \longrightarrow = \int_{-1}^{1} \frac{\cos(nu)\cos(mu)}{(1-x^2)^{1/2}}$

Now, $\frac{dx}{du} = (1-x)^{1/2} \rightarrow du = (1-x)^{1/2} dt$

so, $= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(nu)\cos(mu)\, du$

✱ $\rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[\cos[(n+m)u] + \cos[(n-m)u]\right] du$

$$= \frac{1}{2}\left[(n+m)^{-1}\sin[(n+m)u] + (n-m)^{-1}\sin[(n-m)u]\right]\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

But $\sin(\pm\frac{\pi}{2}) = 0$. $\therefore \int_{-1}^{1} \frac{T_n(x) T_m(x)}{(1-x^2)^{1/2}} = 0$

11.a) $M_1(f)$: $r = 1$, where $M_1(f) = (b-a) f\left(\frac{a+b}{2}\right)$

$\mathbb{P}_1$) For exactness ($f(x)=1$): $\int_a^b f(x)\,dx = \int_a^b (1)\,dx = b-a$

& $(b-a) f\left(\frac{a+b}{2}\right) = (b-a)(1) = b-a$ ✓

$\mathbb{P}_2$) For exactness ($f(x)=x$): $\int_a^b f(x)\,dx = \left(\frac{x^2}{2}\right)\Big|_a^b = \frac{b^2-a^2}{2}$

& $(b-a) f\left(\frac{a+b}{2}\right) = (b-a)\left(\frac{a+b}{2}\right) = \frac{b^2-a^2}{2}$ ✓

$\mathbb{P}_3$) For exactness ($f(x)=x^2$): $\int_a^b f(x)\,dx = \int_a^b x^2\,dx = \frac{1}{3}x^3\Big|_a^b = \frac{b^3-a^3}{3}$

& $(b-a) f\left(\frac{a+b}{2}\right) = (b-a)\left(\frac{a+b}{2}\right)^2 = (b-a)\left[\frac{(a+b)^2}{4}\right]$ X

$\boxed{\therefore\ r=1}$ ($\mathbb{P}_2$)

11.6) $T_1(f)$: $r=1$

$\mathbb{P}_1$) For exactness : $\int_a^b (1)\,dx = b-a$

$f(x)=1$

& $(b-a)\left[\frac{f(a)+f(b)}{2}\right] = \frac{2(b-a)}{2} = b-a$ ✓

$\mathbb{P}_2$) For exactness : $\int_a^b x\,dx = \frac{b^2-a^2}{2}$

$f(x)=x$

& $(b-a)\left[\frac{f(a)+f(b)}{2}\right] = (b-a)\left(\frac{a+b}{2}\right)$

$= \frac{b^2-a^2}{2}$ ✓

$\mathbb{P}_3$) For exactness : $\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$

$f(x)=x^2$

& $(b-a)\left[\frac{f(a)-f(b)}{2}\right] = (b-a)\left(\frac{a^2+b^2}{2}\right)$ ✗

$\boxed{\therefore\ r=1}$ $(\mathbb{P}_2)$

12.) $\int_0^1 f(x)\,dx \approx \frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right)$

$\mathbb{P}_1$) $\int_a^b 1\,dx = b-a = 1$

& $\frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) = 1/4 + 3/4 = 1$ ✓

$\mathbb{P}_2$) $\int_a^b x\,dx = \left(\frac{x^2}{2}\right)\Big|_0^1 = 1/2$

& $\frac{1}{4}f(0) + \frac{3}{4}f\left(\frac{2}{3}\right) = \frac{3}{4}\,\frac{2}{3} = 1/2$ ✓

$\mathbb{P}_3$) $\int_a^b x^2\,dx = \left(\frac{x^3}{3}\right)\Big|_0^1 = 1/3$

& $\frac{1}{4}f(0) + 3/4\,f\left(\frac{2}{3}\right) = \frac{3}{4}\,\frac{4}{9} = 1/3$ ✓

$\mathbb{P}_4$) $\int_a^b x^3\,dx = \left(\frac{x^4}{4}\right)\Big|_0^1 = 1/4$

& $\frac{1}{4}f(0) + 3/4\,f\left(\frac{2}{3}\right) = \frac{3}{4}\,\frac{8}{27} = 2/9$ ✗

$\boxed{\therefore\ n = 2}$ ($\mathbb{P}_3$)

14.9) $\frac{I - I_n}{I - I_{2n}} \approx 2^P$ $\quad I - I_n \approx \frac{c}{n^P}$

Simpson's Rule $\sqrt{x}$

5.4)

n	E	ratio
2	2.86E-2	
4	1.014E-2	2.82
8	3.587E-3	2.83
16	1.268E-3	2.83
32	4.485E-4	2.83

For $n=2$) $\frac{I - I_n}{I - I_{2n}} \rightarrow \frac{2.86 \times 10^{-2}}{1.014 \times 10^{-2}} = 2^P$

$\log_2 (2.82) = P$

$\boxed{P = 1.4959}$