1.) I.) The rate of convergence is slower due to the function $\sin(\sqrt{x})$ being not sufficiently differentiable on $[0,1]$. $E_n \approx \tilde{E}_n \frac{c}{n^{1.5}}$ suggests this.

II.) $I \approx I_{2n} + \frac{I_{2n} - I_n}{2^p - 1}$

$\rightarrow 2^p = 0.354392 \rightarrow p = -1.49658$

$\rightarrow I \approx T_{10} + \frac{T_{10} - T_9}{(-1.49) - 1} \approx 0.62326415$

since $I - I_n \approx \frac{c}{n^p}$, $I - I_{2n} \approx \frac{c}{2^p n^p}$

with trapezoidal, $I \approx T_{2n} + \frac{1}{3}[T_{2n} - T_n]$

similarly

I.)

2.) $f''(x) = Af(x) + Bf(x + \frac{1}{2}h) + Cf(x+h)$

$\rightarrow f(x+\frac{1}{2}h) \sim f(x) + f'(x)(\frac{h}{2}) + \frac{(\frac{h}{2})^2}{2!} f''(x) + \frac{(\frac{h}{2})^3}{3!} f'''(x)$

$\rightarrow f(x+h) \sim f(x) + f'(x)h + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x)$

$\rightarrow f''(x) = Af(x) + B\left[f(x) + f'(x)(\frac{h}{2}) + \frac{(h/2)^2}{2}\right] f''(x) + \frac{(h/2)^3}{6} f'''(x)$

$\quad + C\left[f(x) + f'(x)h + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x)\right]$

$\quad = (A+B+C)f(x) + (B+2C)(\frac{h}{2}) f'(x)$

$\quad + (B+4C)(\frac{h^2}{8}) f''(x) + (B+8C)(\frac{h^3}{48}) f'''(x)$

$$\rightarrow \begin{cases} A+B+C = 0 \\ \frac{h}{2}(B+2C) = 0 \\ \frac{h^2}{8}(B+4C) = 1 \\ \cancel{\frac{h^3}{48}(B+8C) = 0} \end{cases} \rightarrow B+4C = \frac{8}{h^2}$$

$$\rightarrow \begin{cases} A+B+C = 0 \\ B+2C = 0 \\ B+4C = \frac{8}{h^2} \end{cases} \rightarrow B = -2C$$

$\rightarrow (-2C) + 4C = \frac{8}{h^2} \rightarrow C = \frac{4}{h^2},\ B = -\frac{8}{h^2},\ A = C$

$$\rightarrow f''(x) = \frac{4f(x) - 8f(x+\frac{1}{2}h) + 4f(x+h)}{h^2}$$

2.) II.) $f''(x) = \dfrac{4f(x) - 8f(x+\frac{1}{2}h) + 4f(x+h)}{h^2}$

$f(x) = 1 \rightarrow f''(x) = 0 \rightarrow \dfrac{4(1) - 8(1) + 4(1)}{h^2} = 0$

$f(x) = x \rightarrow f''(x) = 0 \rightarrow \dfrac{4(x) - 8(x+\frac{1}{2}h) + 4(x+h)}{h^2} = 0$

$f(x) = x^2 \rightarrow f''(x) = 2 \rightarrow \dfrac{4(x^2) - 8(x+\frac{1}{2}h)^2 + 4(x+h)^2}{h^2} = 2$

III.) $\left| \dfrac{4f(x) - 8f(x+\frac{1}{2}h) + 4f(x+h)}{h^2} - f''(x) \right|$

$\sim \dfrac{h^2}{48}(B + 8C) f'''(x)$

$\sim \dfrac{h^2}{48}\left(\left(-\dfrac{8}{h^2}\right) + 8\left(\dfrac{4}{h^2}\right)\right) f'''(x)$

$\sim \dfrac{h^2}{2} f'''(x)$

3.) $a = 10^{-10}$

I) $A = \begin{bmatrix} 1 & 1 \\ 1-a & 1+a \end{bmatrix}$

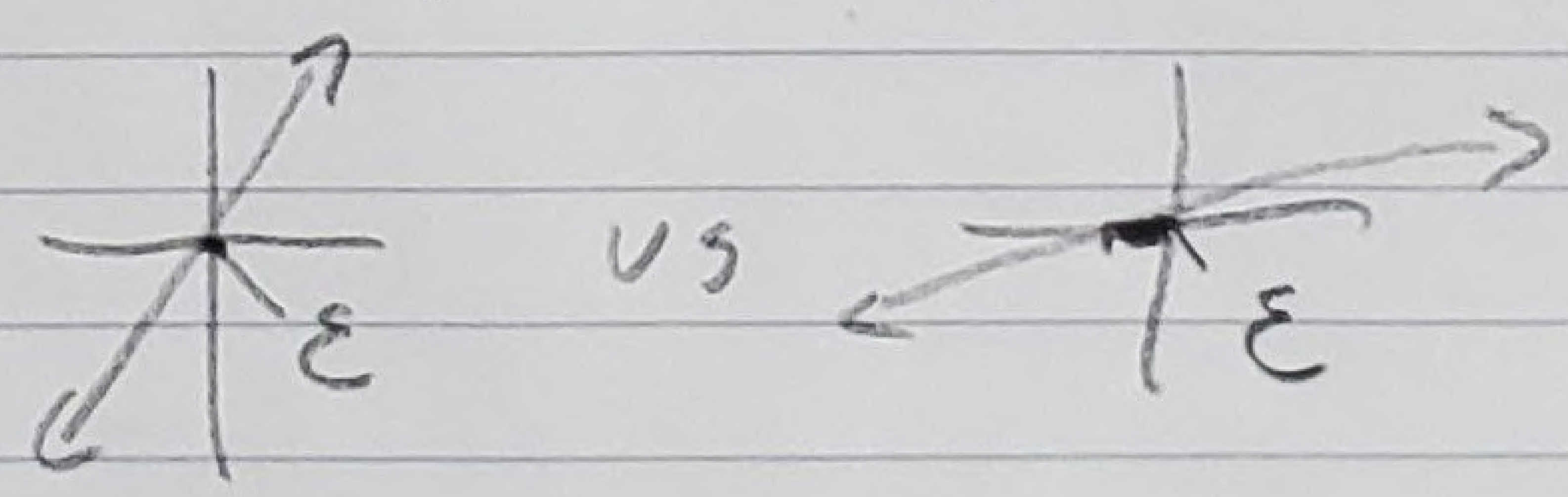
$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} 1+a & -1 \\ -(1-a) & 1 \end{bmatrix} = \frac{1}{2a}\begin{bmatrix} 1+a & -1 \\ a-1 & 1 \end{bmatrix}$$

$$\kappa_1(A) = |A||A^{-1}| = \max(2-a, 2+a)\max\left(\frac{1}{2a}+1, 0\right)$$
$$= (2+a)\left(\frac{1}{2a}+1\right)$$

II $\begin{bmatrix} 1 & 1 \\ 1-a & 1+a \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 + 1/2 \\ 1/2(1-a) + 1/2(1+a) \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 - 1/2a + 1/2 + 1/2a \end{bmatrix}$
$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

III) As $\hat{B}$ is constant the error will show the conditioning.

consider



even for same $\hat{B}$, the error will explode in the later one to near parallel nature. $K \gg 1$ will explode in error, too.

I.)

a.) $P^T P = I$

$$\rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0(0)+1(1)+00 & 0(0)+1(0)+0(1) & 0(1)+1(0)+0(0) \\ 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) & 0(1)+0(0)+1(0) \\ 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) & 1(1)+0(0)+0(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times 3} \quad \therefore P^T = P^{-1}$$

II.) $PLUX = B \rightarrow P^{-1}PLUX = P^{-1}B \rightarrow LUX = P^{-1}B$

$$LUX = P^{-1}B \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 2 \end{array}\right] \rightarrow RREF \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\therefore X = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$