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Mth 342
HW #6

1.) Let $V = \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

① Homogeneity: Let λ be a scalar.

$$\|\lambda x\|_1 = \sum_{i=1}^{n} |\lambda x_i| = |\lambda| \sum_{i=1}^{n} |x_i| = |\lambda| \|x\|_1$$

② Minkowski inequality: Given V is real,

$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$ is real $\forall x, y \in V$

by triangle inequality of real numbers,

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} \left[|x_i| + |y_i| \right]$$

$$\sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

③ Non-negativity: Since $|x_i| \geq 0$, $\forall x_i \in V$

Then $\sum_{i=1}^{n} |x_i| \geq 0 \quad \forall x_i \in V$

④ Non-degeneracy: Since $|x_i| \geq 0$, $\forall x_i \in V$

Then $\sum_{i=1}^{n} |x_i| = 0$ iff $x = 0$

2.) $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$

2.a.) $P = P_{col(A)} = A(A^*A)^{-1}A^*$

so $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix}$

$= \frac{1}{9} \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & -2 \\ 2 & -2 & 8 \end{bmatrix}$

2.b.) P is a 3×3 matrix

2.c.) col(A) is a subspace of $\mathbb{R}^3$, and the columns of P are the projections of the basis vectors in $\mathbb{R}^3$. So, col(A) is spanned by 2 vectors while P has $\dim(\mathbb{R}^3)$ columns.

2.d.) $\|x - Px\|$ is the distance from a point $\vec{x}$ to its projection $P\vec{x}$. In the case of least square method this is deviation.

3.) Find the best fit parabola for the data:

$$\{(-1,1),(0,0),(1,1),(2,3)\}$$

① $y(-1) = a - b + c = 1$

② $y(0) = c = 0$

③ $y(1) = a + b + c = 1$

④ $y(2) = 4a + 2b + c = 3$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} \rightarrow A\vec{x} = \vec{b}$$

Let $\vec{x}^*$ approx. $\vec{x}$. Then $A\vec{x}^* = \text{proj}_{\text{Ran}A}\vec{b}$

Equivalently $A^*A\vec{x}^* = A^*\vec{b}$

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 4 \end{bmatrix}$$

Scilab $\rightarrow \vec{x}^* = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^* = \begin{bmatrix} 1/2 \\ -\frac{1}{10} \\ 3/10 \end{bmatrix}$

4.) Show $\text{Ker}(A) = \text{Ker}(A^*A)$

This means ① $\text{Ker}(A^*A) \subseteq \text{Ker}(A)$

And ② $\text{Ker}(A) \subseteq \text{Ker}(A^*A)$

① Let $\vec{x} \in \text{Ker}(A)$, Then $A\vec{x} = \vec{0}$.

Consider $A^*A\vec{x} = A^*(A\vec{x}) = A^*(\vec{0}) = \vec{0}$

So $\vec{x} \in \text{Ker}(A^*A\vec{x}) \rightarrow \text{Ker}(A^*A) \subseteq \text{Ker}(A)$.

② Let $\vec{x} \in \text{ker}(A^*A\vec{x})$, Then $A^*A\vec{x} = 0$.

Consider $(A\vec{x}, A\vec{x}) = ||A\vec{x}||^2$

But $(A\vec{x}, A\vec{x}) = (A\vec{x})^*A\vec{x} = \vec{x}^*A^*A\vec{x} = (A^*A\vec{x}, \vec{x})$

And $(A^*A\vec{x}, \vec{x}) = (\vec{0}, \vec{x}) = 0$

So $(A\vec{x}, A\vec{x}) = ||A\vec{x}||^2 = 0$

Therefore $A\vec{x} = \vec{0}$

So $\vec{x} \in \text{ker}(A) \longrightarrow \text{ker}(A) \subseteq \text{ker}(A^*A)$. $\blacksquare$