I.)

1.) $|I| = \max_{|x|=1} |Ix| = \max_{|x|=1} |x| = 1$

II.) $\therefore \quad ||A|| \, ||A^{-1}|| \geq ||AA^{-1}||$

$$||Ax|| \, ||A^{-1}x|| \geq ||I|| \quad (x=1)$$

$$\geq 1$$

$\rightarrow \kappa(A) \geq 1$

2.) $19x_1 + 20x_2 = b_1$
$20x_1 + 21x_2 = b_2$ $\rightarrow \begin{bmatrix} 19 & 20 \\ 20 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$

$\therefore \quad \kappa(A) = |A||A^{-1}|$, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

where $A = \begin{bmatrix} 19 & 20 \\ 20 & 21 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -21 & 20 \\ 20 & -19 \end{bmatrix}$

$\rightarrow \kappa(A) = \max_{|x|=1} |Ax| \max_{|x|=1} |A^{-1}x|$

$= (\max A)(\max A^{-1})$

$= \left(\max \sum_n a_{nj}\right)\left(\max \sum_m a^{-1}_{mj}\right)$

$= \max(39, 41) \max(41, 39) = 41^2$

$41^2 >> 1$

so $\Downarrow$

41^2 ill conditioned

I.) $A = \begin{pmatrix} 1 & 1 \\ \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & \alpha & 0 \\ 1 & 0 & \alpha \end{pmatrix}$

$A^T A X = A^T B$

$$\rightarrow \begin{bmatrix} 1 & \alpha & 0 \\ 1 & 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \alpha & 0 \\ 0 & \alpha \end{bmatrix} X = \begin{bmatrix} 1 & \alpha & 0 \\ 1 & 0 & \alpha \end{bmatrix} \begin{bmatrix} -\alpha \\ 1+\alpha \\ 1-\alpha \end{bmatrix}$$

$$\begin{bmatrix} 1(1) + \alpha(\alpha) + 0 & 1(1) + 0 + 0 \\ 1(1) + 0 + 0 & 1(1) + 0 + \alpha^2 \end{bmatrix} X = \begin{bmatrix} 1(-\alpha) + \alpha(1+\alpha) + 0 \\ 1(-\alpha) + 0 + \alpha(1-\alpha) \end{bmatrix}$$

$$\begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 1+\alpha^2 \end{bmatrix} X = \begin{bmatrix} \alpha^2 \\ -\alpha^2 \end{bmatrix}$$

Now, $\begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 1+\alpha^2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 + 1(-1) \\ 1 + -1(1+\alpha^2) \end{bmatrix}$

$$= \begin{bmatrix} \alpha^2 \\ -\alpha^2 \end{bmatrix}$$

$\therefore \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3.) $M = A^T A = \begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 1+\alpha^2 \end{bmatrix}$

$$\rightarrow M^{-1} = \frac{1}{(1+\alpha^2)^2 - 1} \begin{bmatrix} 1+\alpha^2 & -1 \\ -1 & 1+\alpha^2 \end{bmatrix}$$

$$= \frac{1}{1+2\alpha^2+\alpha^4-1} \begin{bmatrix} 1+\alpha^2 & -1 \\ -1 & 1+\alpha^2 \end{bmatrix}$$

$$= \frac{1}{2\alpha^2+\alpha^4} \begin{bmatrix} 1+\alpha^2 & -1 \\ -1 & 1+\alpha^2 \end{bmatrix}$$

$$\kappa(M)_1 = |M|_1 |M^{-1}|_1$$

$$= \left(\max(1+1+\alpha^2, 1+1+\alpha^2)\right)\left(\max\left(\frac{\alpha^2}{2\alpha^2+\alpha^4}, \frac{\alpha^2}{2\alpha^2+\alpha^4}\right)\right)$$

$$= (\alpha^2+2)\left(\frac{\alpha^2}{2\alpha^2+\alpha^4}\right)$$

$$= \frac{2\alpha^2+\alpha^4}{2\alpha^2+\alpha^4}$$

$$= 1$$

& $\kappa(M)_\infty = |M|_\infty |M^{-1}|_\infty$

$$= \max[\alpha^2+2, \alpha^2+2] \cdot \max[\alpha^2, \alpha^2] \frac{1}{2\alpha^2+\alpha^4}$$

$$= 1$$

EC1) $M = \begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 1+\alpha^2 \end{bmatrix}$

$$M^{-1} = \frac{1}{2\alpha^2+\alpha^4} \begin{bmatrix} 1+\alpha^2 & -1 \\ -1 & 1+\alpha^2 \end{bmatrix}$$

$$\rightarrow \quad |M - \lambda I| = \left| \begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 1+\alpha^2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$= \det \begin{bmatrix} 1+\alpha^2-\lambda & 1 \\ 1 & 1+\alpha^2-\lambda \end{bmatrix}$$

$$= (1+\alpha^2-\lambda)^2 - 1$$

so $|M - \lambda I| = 0 \rightarrow \lambda_m = (\alpha^2, \alpha^2+2)$

$$\rightarrow \quad \kappa(M)_2 = |M|_2 \, |M^{-1}|_2$$

$$= \max \sqrt{\lambda_m} \cdot \max \sqrt{\lambda_{M^{-1}}}$$

$$= \max\left(\sqrt{\alpha^2}, \sqrt{\alpha^2+2}\right) \cdot \max\left(\sqrt{\frac{1}{\alpha^2}}, \sqrt{\frac{1}{\alpha^2+2}}\right)$$

$$= \sqrt{\alpha^2+2} \cdot \sqrt{\frac{1}{\alpha^2}}$$