MTH351 **Introduction to Numerical Analysis** **Summer 2020**

Quiz 4 - To be discussed in Lecture Friday July 31

Give yourself 30 - 45 minutes to work on these problems. Write down your own solutions and be ready to discuss them in small group during lecture on Friday July 31. You will have a chance to rewrite your answer if needed. You will need to upload your corrected answers by Noon, Saturday August 1.

Problem 1: Recall that the Chebyshev nodes $x_0^*, x_1^*, \ldots, x_n^*$ are determined on the interval $[-1,1]$ as the zeros of $T_{n+1}(x) = \cos((n+1)\arccos(x))$ and are given by

$$x_j = \cos\left(\frac{2j+1}{n+1}\frac{\pi}{2}\right),\ j = 0, 1, \ldots n.$$

Consider now interpolating the function $f(x) = 1/(1+x^2)$ on the interval $[-5,5]$. We have seen in lecture that if equispaced nodes are used, the error grows unboundedly as more points are used. The purpose of this problem is to show that if the Chebyshev nodes are used, this problem disapears.

First denote by $y_j^* = 5 \times x_j^*$ and let $P_n(y)$ denote the polynomial of degree n that interpolates f at these points. Then the error estimate for polynomial interpolation gives for $y \in [-5,5]$

$$|f(y) - P_n(y)| = \frac{|f^{n+1}(c)|}{(n+1)!}|(y-y_0^*)(y-y_1^*)\ldots(y-y_n^*)|$$

for some $c \in [-5,5]$.

Part I: Show that

$$(y-y_0^*)(y-y_1^*)\ldots(y-y_n^*) = \frac{5^{n+1}}{2^n}T_{n+1}(x),\ \text{where } x = y/5$$

Part II: It can be shown that there exists $R > 0$ such that $|f^{(n)}(y)| \leq R^n$ for all $y \in [-5,5]$. Assuming this, show that

$$\lim_{n\to\infty} \max\{|f(y) - P_n(y)|, y \in [-5,5]\} = 0$$

Problem 2: Let $I(f) = \int_a^b f(x)dx$. We are interested in approximating this integral within a certain error tolerance. First some notation. Let n be a positive integer and define $x_j = a + j \times h$ where $h = (b-a)/n$. Recall that the Midpoint rule approximates the integral of f by a Riemann sum that evaluates the function at each of the middle point of the subintervals $[x_{j-1}, x_j]$. That is $M_n(f) = h\sum_{j=1}^n f(\bar{x}_j$ where $\bar{x}_j = (x_{j-1}+x_j)/2 = a + (j-1/2)h$.

In lecture we showed (see corrected lecture notes) that if f has two continous derivatives on the interval $[a,b]$, then for some $c \in [a,b]$,

$$I(f) - M_n(f) = \frac{1}{24}h^3 f''(c)$$

Consider now the evaluation of $I = \int_0^3 e^{-x^2}dx$. Determine n so that the error in approximating this integral using M_n is less than 5×10^{-6}.