Bennet Sloan

Mth 342

HW 3

1a.) If $AB = 0$, then either $B = 0$ or

A is not invertible.

Proof: Let A, B be any matrices in $\mathbb{R}^n$

consider $AB = 0$.

case 1: If A is invertible then:

$$AB = 0$$
$$A^{-1}AB = A^{-1}0$$
$$IB = 0$$
$$B = 0$$

so $B = 0$ if A is invertible.

case 2: If A is not invertible:

A is not invertible iff the column vectors

of A are Linearly Dependent.

By matrix multiplication, $AB = A[B_1 \ldots B_n]$

where $B_1 \ldots B_n$ are the columns of B.

Moreover, $A[B_1 \ldots B_n] = B_{11}A_1 + B_{21}A_2 + \ldots + B_{n1}A_n$, (column 1)

$B_{1n}A_1 + B_{2n}A_2 + \ldots + B_{nn}A_n$ (Last column

Therefore the (i,j) entry of AB is:

$$\sum_{k=1}^{n} a_{ik} b_{kj} = (AB)_{ij}$$

so, if $(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$

Consider: $(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj} = 0$

If $B \neq 0$, then $a_{i1}, a_{i2}, a_{i3}, \dots, a_{in}$ must satisfy the equation for some nonzero scalar c such that $\sum c \, a_{ik} = 0$ where $c = b_{ij}$.

16.) If A is symmetric & invertible, then A^{-1} is symmetric.

Proof:

Let $A = A^T$ be a symmetric matrix

If A is invertible then there exists:

A^{-1} & $(A^T)^{-1}$

And, $(A^{-1})^T = (A^T)^{-1}$

Consider $A = A^T$

$$A A^{-1} = A^T A^{-1}$$

$$\underline{I} = A^T A^{-1}$$

$$(A^T)^{-1} \underline{I} = (A^T)^{-1} A^T A^{-1}$$

$$(A^T)^{-1} = \underline{I} A^{-1}$$

$$(A^T)^{-1} = A^{-1}$$

Therefore A^{-1} is symmetric.

2.) Let $C[0,1]$ denote the vector space of all continuous functions defined on $[0,1]$

For each $a \in [0,1]$ the evaluation map:

$$E_a : C[0,1] \to \mathbb{R} \quad , \quad E_a(f) = f(a)$$

is a linear transformation. Describe $Ker(E_a)$.

→ If f is a continues function in $\mathbb{R}$

Then the kernel will be its zero solution.

That means $Ker(E_a) = \{ f \mid f \in C[0,1] , f(a) = 0 \}$

Therefore (a) is the root(s) of $f \in C[0,1]$.

3.) Let V & W be vector spaces and

let $T: V \to W$ be a linear transformation.

Let $V_1, V_2, V_3, \ldots, V_K \in V$.

3a.) show that if $T(V_1), T(V_2), \ldots, T(V_K)$ are LI

then so are $V_1, V_2, \ldots, V_K$.

Proof: Let $T(V_1), T(V_2), \ldots, T(V_K)$ be linearly Independent,

And $C_1, C_2, \ldots, C_K$ be scalars.

Consider $C_1V_1 + C_2V_2 + \ldots + C_KV_K = 0$

$$T(C_1V_1 + C_2V_2 + \ldots + C_KV_K) = T(0)$$

$$T(C_1V_1) + T(C_2V_2) + \ldots + T(C_KV_K) = 0$$

$$C_1T(V_1) + C_2T(V_2) + \ldots + C_KT(V_K) = 0$$

Since $T(V_1), T(V_2), \ldots, T(V_K)$ are defined as L.I.

Then $C_1, C_2, \ldots, C_K$ must all be zero,

Since any scalar multiple of a L.I. vector in the

set is still linearly Independent.

3b.) Show by counterexample the converse of 3a is false

Converse: If vectors $V_1, V_2, \ldots, V_k$ are L.I.

Then the vectors $T(v_1), T(v_2), \ldots, T(v_k)$ are L.I.

Counterexample: Any non-injective function.

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x,0)$

If $v_1 = (1,0)$ & $v_2 = (0,1)$

Then v_1, v_2 are L.I. vectors in $\mathbb{R}^2$.

However, $T(v_1) = T(1,0) = (1,0)$

And $T(v_2) = T(0,1) = (0,0)$

Since $T(v_1)$ & $T(v_2)$ are Linearly Dependent,

The statement is disproven.

3c.) Show the converse of 3a holds if

T is injective.

Let $(v_1, v_2, \dots, v_k)$ be linearly Independent vectors.

If $T: V \to W$ is injective,

Then $T(x) = T(y)$ implies $x = y$.

Consider $T(0) = T(c)$,

Since only zero can satisfy this equality,

$\ker(T) = \{\vec{0}\}$.

Therefore if:

$$c_1 T(v_1) + c_2 T(v_2) + \dots + c_k T(v_k) = 0$$

Then scalars $c_1, c_2, \dots c_k$ must all be zero.

Therefore $T(v_1), T(v_2), \dots T(v_k)$ must be

linearly Independent as well.

4.) Let U & V be subspaces of W.

4a.) The subspace sum is: $U+V = \{x+y \mid x \in U, y \in V\}$

Show $U+V$ is also a subspace.

i.) If U & V are subspaces then $0 \in U$, $0 \in V$.

Therefore $0+0 \in U+V \rightarrow 0 \in U+V$.

ii) Let $\vec{w}_1, \vec{w}_2 \in U+V$.

Then $\vec{w}_1 = \vec{u}_1 + \vec{v}_1$ And $\vec{w}_2 = \vec{u}_2 + \vec{v}_2$

Where $\vec{u}_1, \vec{u}_2 \in U$ And $\vec{v}_1, \vec{v}_2 \in V$

So, $\vec{u}_1 + \vec{u}_2 \in U$ And $\vec{v}_1 + \vec{v}_2 \in V$

Consider $(\vec{w}_1) + (\vec{w}_2)$

$(\vec{u}_1 + \vec{v}_1) + (\vec{u}_2 + \vec{v}_2)$

$(\vec{u}_1 + \vec{u}_2) + (\vec{v}_1 + \vec{v}_2) \in U+V$

Therefore $\vec{w}_1 + \vec{w}_2 \in U+V$

iii) Let $a \in F$, $\vec{w} \in U+V$ where $\vec{w} = \vec{u} + \vec{v}$

Consider $a\vec{w} = a(\vec{u} + \vec{v})$

$= a\vec{u} + a\vec{v} \in U+V$ $\therefore U+V$ is a subspace of W

Therefore $a\vec{w} \in U+V$

4.b.) Let $\vec{w} \in W$, but $\vec{w} \notin V$. Show for all $v \in V$, $v + w \notin V$.

If $\vec{v} + \vec{w} \in V$, $\pm\vec{v} \in V$ since V is a subspace

Then $-\vec{v} + \vec{v} = 0 \in V$

Consider $-\vec{v} + \vec{v} + \vec{w}$

$\vec{0} + \vec{w}$

$\vec{w} \notin V \quad \therefore \vec{v} + \vec{w} \notin V$

4.c.) Show that if $U \not\subset V$ & $V \not\subset U$ then $U \cup V$ is not a subspace.

Let $\vec{u} \in U$ but $\vec{u} \notin V$

And $\vec{v} \in V$ but $\vec{v} \notin U$

Then, $\vec{u}, \vec{v} \in U \cup V$

Consider $\vec{u} + \vec{v}$ And $\vec{u} + \vec{v}$

$\vec{u} + (-\vec{v} + v) \qquad (-\vec{u} + u) + \vec{v}$ (additive inverse)

$\vec{u} \notin V \qquad \vec{v} \notin U$

$\therefore \vec{u} + \vec{v} \notin U$ & $\vec{u} + \vec{v} \notin V$

So $\vec{u} + \vec{v}$ is not a subspace.