Mth 351 Quiz 3 Bennet Sloan

1.) $\therefore f(x) = \ln(x)$, $1 \le x \le 5$

using $x_j = 1 + j \cdot h$, where $h = 4/n$.

For linear interpolation, $\varepsilon = \frac{h^2 M}{8}$.

Now
$$M = \max_{1 \le x \le 5} |f''(x)|$$
$$= \max_{1 \le x \le 5} \left| -\frac{1}{x^2} \right|$$
$$= \max_{1 \le x \le 5} \left(\frac{1}{x^2} \right)$$
$$= 1$$

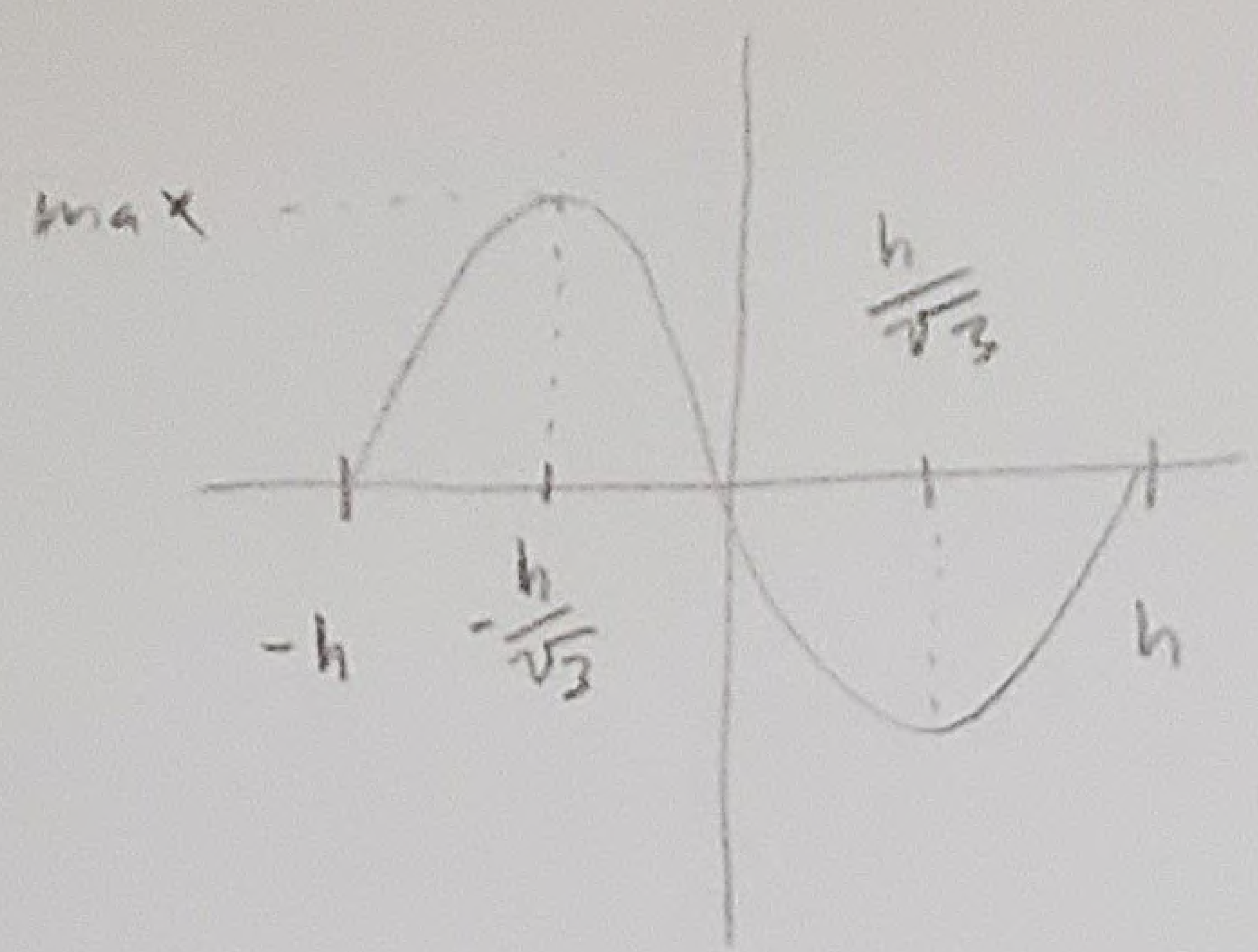
For $\varepsilon < 5 \cdot 10^{-6}$,
$$\frac{h^2 M}{8} < 5 \cdot 10^{-6}$$
$$\frac{h^2}{8} < 5 \cdot 10^{-6}$$
$$\frac{(4/n)^2}{8} < 5 \cdot 10^{-6}$$
$$\frac{16}{n^2} < 4 \cdot 10^{-5}$$
$$n^2 > 4 \cdot 10^{5}$$
$$n > (4 \cdot 10^5)^{1/2} \approx 632.455$$

$\boxed{\therefore n = 633}$

2.) I.) $\psi_2(x) = (x+h)x(x-h)$

$= x(x^2-h^2)$

using $(x_0, x_1, x_2) = (-h, 0, h)$:



Now, $\frac{d}{dx} x(x^2-h^2) = (1)(x^2-h^2) + (x)(2x)$

$= 3x^2 - h^2$

For $\frac{d}{dx}\psi_2(x) = 0$: $3x^2 - h^2 = 0$

$3x^2 = h^2$

$x = \pm\frac{1}{\sqrt{3}}h$

For $\max\limits_{-h \le x \le h} \psi_2(x)$: $\psi_2\left(-\frac{h}{\sqrt{3}}\right) = \left(-\frac{h}{\sqrt{3}}+h\right)\left(-\frac{h}{\sqrt{3}}\right)\left(-\frac{h}{\sqrt{3}}-h\right)$

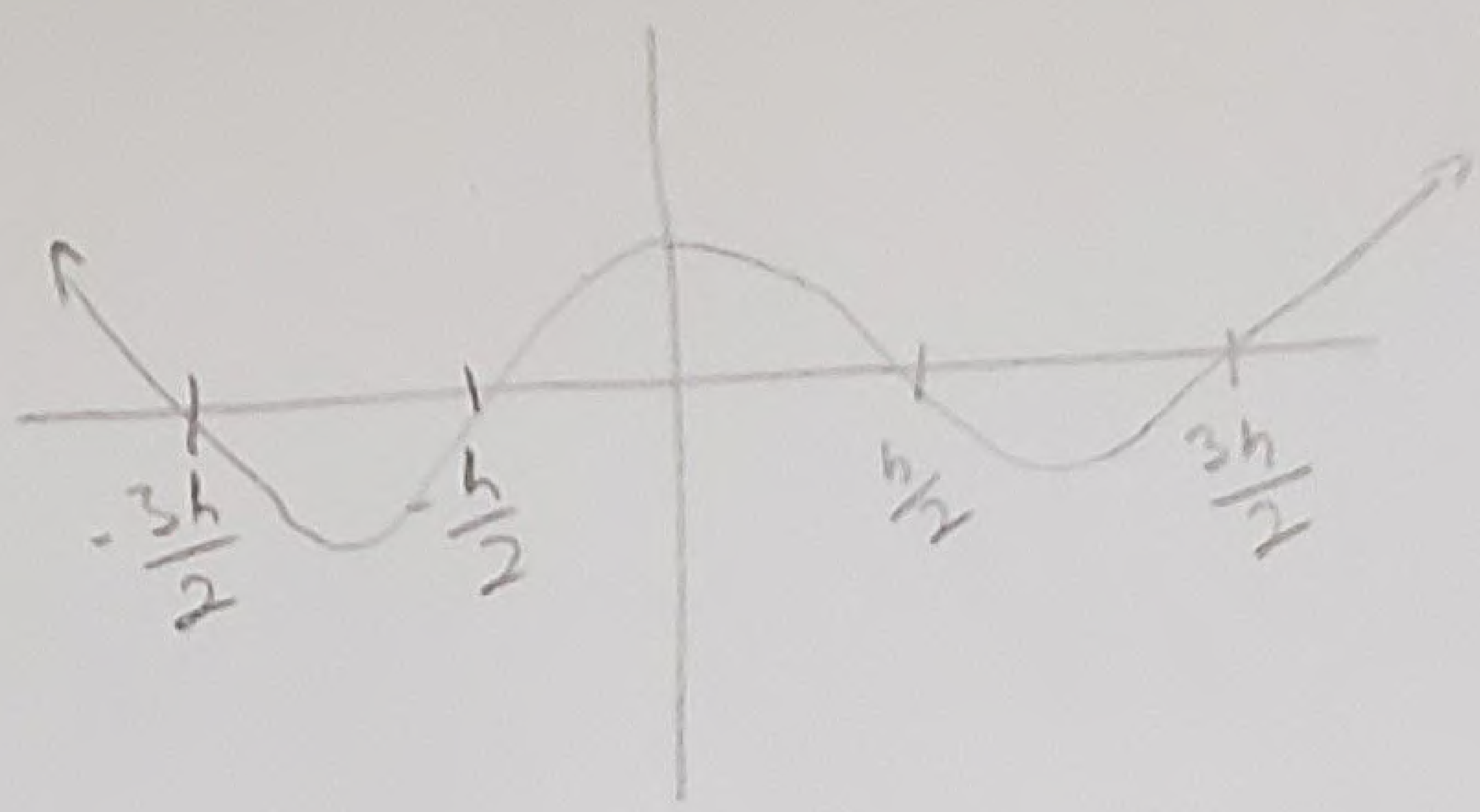
$= \left(\frac{h^2}{3} - h^2\right)\left(-\frac{h}{\sqrt{3}}\right)$

$= -\frac{2h^2}{3}\left(-\frac{h}{\sqrt{3}}\right)$

$= \frac{2h^3}{3\sqrt{3}}$

II.) $\psi_3(x) = \left(x + \frac{3h}{2}\right)\left(x + \frac{h}{2}\right)\left(x - \frac{h}{2}\right)\left(x - \frac{3h}{2}\right)$

$= \left(x^2 - \left(\frac{3h}{2}\right)^2\right)\left(x^2 - \left(\frac{h}{2}\right)^2\right)$



For $\frac{\partial}{\partial x}\psi_3(x) = \frac{\partial}{\partial x}\left[x^2 - \left(\frac{3h}{2}\right)^2\right]\left[x^2 - \left(\frac{h}{2}\right)^2\right]$

$= (2x)\left[x^2 - \left(\frac{h}{2}\right)^2\right] + (2x)\left[x^2 - \left(\frac{3h}{2}\right)^2\right]$

For $\frac{\partial}{\partial x}\psi_3(x) = 0$:

$(2x)\left[x^2 - \left(\frac{h}{2}\right)^2\right] + (2x)\left[x^2 - \left(\frac{3h}{2}\right)^2\right] = 0$

$2x^3 - (2x)\left(\frac{h}{2}\right)^2 + 2x^3 - (2x)\left(\frac{3h}{2}\right)^2 = 0$

$4x^3 - (2x)\left(\frac{h^2}{4} + \frac{9h^2}{4}\right) = 0$

$4x^3 - x5h^2 = 0$

$x(4x^2 - 5h^2) = 0$

$\to 4x^2 - 5h^2 = 0 \to \left(x + \frac{\sqrt{5}}{2}h\right)\left(x - \frac{\sqrt{5}}{2}h\right) = 0$

so $x = 0, \pm\frac{\sqrt{5}}{2}h$ are min/max points.

II cont.)

For $\max_{x_0 \le x \le x_3} |\Psi_3(x)|$: $\left|\Psi_3\left(\pm\frac{\sqrt{5}}{2}h\right)\right| = \left|\left(\left(\pm\frac{\sqrt{5}}{2}h\right)^2 - \left(\frac{3h}{2}\right)^2\right)\left(\left(\pm\frac{\sqrt{5}}{2}h\right)^2 - \left(\frac{h}{2}\right)^2\right)\right|$

$$= \left|\left(\frac{5h^2}{4} - \frac{9h^2}{4}\right)\left(\frac{5h^2}{4} - \frac{h^2}{4}\right)\right|$$

$$= \left|(-h^2)(h^2)\right|$$

$$= |-h^4|$$

$$= h^4$$

And $|\Psi_3(0)| = \left|\left(0^2 - \left(\frac{3h}{2}\right)^2\right)\left(0^2 - \left(\frac{h}{2}\right)^2\right)\right|$

$$= \left|\left(-\frac{9h^2}{4}\right)\left(-\frac{h^2}{4}\right)\right|$$

$$= \left|\frac{9h^4}{16}\right|$$

$$\therefore \max_{-\frac{3h}{2} \le x \le \frac{3h}{2}} |\Psi_3(x)| = h^4$$

III.) $\frac{\max \Psi_3}{\max \Psi_2}$ with $h = 1 \rightarrow \frac{(1)^4}{\frac{2(1)^3}{3\sqrt{3}}} = \frac{1}{\frac{2}{3\sqrt{3}}} = 0.866$

The error in polynomial interpolation is less than linear interpolation though oscillation may occur