Turn this in by making a pdf scan of your work and submitting it to Canvas by 11:59 pm on the due date.

How to write up homework. Please put your name and assignment number on the first page. You may use this page as a cover sheet. Show your work and explain it. In general, just writing down an answer without showing the steps to arriving at the answer is not sufficient for credit.

Current reading is Chapter 1 of Linear Algebra Done Wrong (LADW). For problem 2 it may be helpful to read subsections 8.1 and 8.2 on page 69 of LADW.

1. Find the standard matrix for each linear transformation below:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection through the line $y = mx$ ($m \in \mathbb{R}$). Hint: use the method given in lecture on Thursday, Jan. 17. This was the example from pages 20 and 21 in LADW, which does the special case of reflection through the line $y = (1/3)x$. Generalize this to $y = mx$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by reflection through the plane $y = mx$ ($m \in \mathbb{R}$).

2. Let $M_{3\times 3}$ be the vector space of all 3×3 matrices with real entries.

   Let $E = \{A \in M_{3\times 3} \mid \text{all row and column sums of } A \text{ are equal}\}$. For example $L \in E$ where

$$L = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 0 & 0 \\ -1 & -2 & 5 \end{bmatrix}$$

   because all rows and columns of L sum to 2. A basis for E is

$$\mathcal{B} = \{B_1, B_2, B_3, B_4, B_5\}$$

   where

$$B_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

   and $B_5 = I$, the identity matrix. Do the following problems (which are similar to the Lab D problems):

(a) Find $[L]_{\mathcal{B}}$, the coordinate vector of L relative to the basis $\mathcal{B}$. Hint: to express L in terms of the basis vectors, start by writing

$$L - c_5 B_5 = Z$$

where c_5 is a scalar and $Z \in W$. Here W is the subspace spanned by $\{B_1, B_2, B_3, B_4\}$ (so all rows and columns of Z sum to 0.)

(b) The trace on $M_{3\times 3}$ restricted to the subspace E is still a linear transformation. Represent

$$\mathrm{tr} : E \to \mathbb{R}$$

as a matrix $[\mathrm{tr}]_{\mathcal{SB}}$. This is the matrix of the trace (as a linear transformation from E to $\mathbb{R}$) relative to the bases $\mathcal{B}$ and $\mathcal{S}$. Here $\mathcal{S} = \{[1]\}$ is the standard basis for $\mathbb{R}$.

i. What is the size of $[\mathrm{tr}]_{\mathcal{SB}}$?

ii. Compute $[\mathrm{tr}]_{\mathcal{SB}}$.

iii. Check that $[\mathrm{tr}]_{\mathcal{SB}}[L]_{\mathcal{B}} = [\mathrm{tr}(L)]_{\mathcal{S}}$.

1.) Find the standard matrix.

a.) $T: \mathbb{R}^2 \to \mathbb{R}^2$, reflection through $y = mx$, $m \in \mathbb{R}$

Let $T(1,0) = (h,k)$

Then, $\frac{(h,k)+(1,0)}{2} \in y = mx$

Also, $k\frac{1}{2} = m(h+1)\frac{1}{2}$

$k = m(h+1)$ ✗

Furthermore, the line containing $(1,0)$ & orthogonal to $y=mx$ is:

$y = -\frac{1}{m}x + \frac{1}{m}$, satisfied by (h,k).

Together, $k = -\frac{1}{m}h + \frac{1}{m}$

$m(h+1) = -\frac{1}{m}h + \frac{1}{m}$ ✗

$h\left(m + \frac{1}{m}\right) = \frac{1}{m} - m$

$h = \frac{(1-m^2)m}{m(m^2+1)} = \frac{1-m^2}{m^2+1}$

So, $k = \frac{-1}{m}\frac{1-m^2}{1+m^2} + \frac{1}{m} = \frac{1}{m}\left(1 - \frac{1-m^2}{1+m^2}\right)$

$= \frac{(1+m^2)-(1-m^2)}{m(1+m^2)} = \frac{2m^2}{m(1+m^2)} = \frac{2m}{1+m^2}$

Therefore $T(1,0) = (h,k)$

$$= \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2} \right)$$

Also let $T(0,1) = (H,K)$

Then $\frac{K+1}{2} = m \cdot \frac{H}{2}$

$K+1 = mH$ ☆

And the line orthogonal to $y = mx \in (0,1)$ is:

$y = -\frac{1}{m}x + 1$, satisfied by (H,K).

Also, $K = -\frac{1}{m}H + 1$

$mH - 1 = -\frac{1}{m}H + 1$ ☆

$H(m+\frac{1}{m}) = 2 \rightarrow H = \frac{2m}{m^2+1}$

And $K = -\frac{1}{m}H + 1 = -\frac{1}{m} \cdot \frac{2m}{m^2+1} + 1$

$= -\frac{2}{m^2+1} + 1 = \frac{m^2-1}{m^2+1}$

So, $T(0,1) = (H,K) = \left(\frac{2m}{1+m^2}, \frac{m^2-1}{m^2+1} \right)$

$$\therefore \quad M_T = \left(T(1,0)^T, T(0,1)^T \right)$$

$$= \begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{m^2+1} \end{bmatrix} \quad \boxed{E}$$

1.) Since $y = mx$ involves only two variables;

$$\begin{bmatrix} \dfrac{1-m^2}{1+m^2} & \dfrac{2m}{1+m^2} & 0 \\ \dfrac{2m}{1+m^2} & \dfrac{m^2-1}{m^2+1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.a.) $E = \{ A \in M_{3\times3} \mid \text{row \& col sum} = \}$

$$L = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 0 & 0 \\ -1 & -2 & 5 \end{bmatrix} \quad \begin{matrix} \text{row sum} = 2 \\ \text{col sum} = 2 \end{matrix}$$

$B = \{ B_1, \dots B_5 \}$

Let $L - c_5 B_5 = Z$, $Z \in W \operatorname{span}\{B_1 .. B_4\}$

$$\begin{bmatrix} 1 & 4 & -3 \\ 2 & 0 & 0 \\ -1 & -2 & 5 \end{bmatrix} - \begin{bmatrix} c_5 & & \\ & c_5 & \\ & & c_5 \end{bmatrix} = Z$$

$$\begin{bmatrix} 1-c_5 & 4 & -3 \\ 2 & -c_5 & 0 \\ -1 & -2 & 5-c_5 \end{bmatrix} = Z \in W \operatorname{span}\{B_1 \dots B_4\}$$

so $(1-c_5) + 2 - 1 = 0$, $c_5 = 2$

And $\begin{bmatrix} -1 & 4 & -3 \\ 2 & -2 & 0 \\ -1 & -2 & 3 \end{bmatrix} = L - c_{c_5} B_5$

Let $a_1, \dots, a_4$ be scalars

Then

$$\left(a_1 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. a_3 \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right)$$

$$x \begin{bmatrix} -1 & 4 & -3 \\ 2 & -2 & 0 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & -a_1 + a_2 & -a_2 \\ -a_1 + a_3 & a_1 - a_2 + a_3 + a_4 & a_2 - a_4 \\ -a_3 & a_3 - a_4 & a_4 \end{bmatrix} x \begin{bmatrix} -1 & 4 & -3 \\ 2 & -2 & 0 \\ -1 & -2 & 3 \end{bmatrix}$$

where $a_1 = -1$, $a_2 = 3$, $a_3 = 1$, $a_4 = 3$

so, $[L]_B = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}$ $\blacksquare$

2.b.) $tr : E \to \mathbb{R}$, $S = \{[1]\}$

$tr(B_1) = 2$, $[tr(B_1)]_S = [2]$
$tr(B_2) = -1$, $[tr(B_2)]_S = [-1]$
$tr(B_3) = -1$, $[tr(B_3)]_S = [-1]$
$tr(B_4) = 2$, $[tr(B_4)]_S = [2]$
$tr(B_5) = 3$, $[tr(B_5)]_S = [3]$

so,

$$[tr]_{SB} = \left[[tr(B_1)]_S, \ldots, [tr(B_5)]_S \right]$$
$$= \begin{bmatrix} 2 & -1 & -1 & 2 & 3 \end{bmatrix}$$

i.) Size of $[tr]_{SB}$ is 1×5

ii) $[tr]_{SB} = \begin{bmatrix} 2 & -1 & -1 & 2 & 3 \end{bmatrix}$

iii) Consider $[tr]_{SB}[L]_B = [tr(L)]_S$

so, $\begin{bmatrix} 2 & -1 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} = [-2 - 3 - 1 + 6 + 6] = \underline{6}$

So, $[tr]_{SB}[L]_B = 6$

And, $tr[L] = 6$

Then $[tr[L]]_B = [6]$

$\therefore [tr]_{SB}[L]_B = [tr(L)]_S$ $\blacksquare$