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Mth 342
Lab M

1.) Let $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_K\}$ where

$\lambda_1, \dots, \lambda_r$ are distinct eigenvalues.

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K$ be corresponding eigenvectors.

Then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_K$ are Linearly Independent.

Consider the system of eigenspaces:

$$E_K = \ker(A - \lambda_K I), \quad \lambda_K \in \sigma(A)$$

Let $\mathcal{B}_K$ be a basis for E_K, $K = 1, \dots, K$

Then $\operatorname{span}\{\vec{v}_i\} = \mathcal{B}_i$ for $i = 1, \dots, K$

Because $\vec{v}_i$ corresponds to a non-degenerate λ_i.

Since $\vec{v}_1, \dots, \vec{v}_K$ are Linearly Independent,

Then $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_K$ are also L. I.

$\therefore$ The system of eigenspaces with basis $\mathcal{B}_K$

has basis $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_K$ which means

E_K is Linearly Independent.

2.) $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(4-\lambda)(1-\lambda) + 2 = 0$

$4 - 4\lambda - \lambda + \lambda^2 + 2 = 0$

$(\lambda-2)(\lambda-3) = 0$ so $\lambda = 2, 3$

Let $A = PBP^{-1}$, P the matrix of eigenvectors

Then $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

Let $\vec{v}_3$ be the eigenvector associated with $\lambda = 3$

Then $\vec{v}_3$ is: $(A - 3I)\vec{v}_3 = 0$

$\begin{bmatrix} 4-3 & -2 \\ 1 & 1-3 \end{bmatrix} \vec{v}_3 = 0 \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \vec{v}_3 = 0$

So $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

2 continued.)

Let $\vec{V}_2$ be the eigenvector associated with $\lambda = 2$.

Then $(A - 2I)\vec{V}_2 = 0 \rightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \vec{V}_2 = 0$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \vec{V}_2 = 0$$

So $\vec{V}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \vec{V}_3 & \vec{V}_2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

Then $A^{1/2} = P B^{1/2} P^{-1}$

$$A^{1/2} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{1/2} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^{1/2} = \begin{bmatrix} 2\sqrt{3} - \sqrt{2} & -2\sqrt{3} + 2\sqrt{2} \\ \sqrt{3} - \sqrt{2} & -\sqrt{3} + 2\sqrt{2} \end{bmatrix}$$

Since $A = PBP^{-1}$, B is diagonal of $\sigma(A)$,

There are 4 square roots of A from the 4 $B^{1/2}$ that can be made of the $\pm\sqrt{\lambda_i}$ for $i = 1, 2$.