Investigation into Gravitational Motion using Numerical Integration via the Runge-Kutta method

Kacper Slodek
(Dated: March 2024)

I. PROBLEM INTRODUCTION

The aim of this exercise is the computation of rocket orbits within the gravitational fields of the Earth and the Moon. This was completed using the Runge-Kutta method of numerical integration, specifically RK4.

A. Runge-Kutta Algorithm

The RK4 numerical integration method involves taking multiple steps within a time step, and computing 4 values for rate of change of a variable, then, the algorithm computes the weighted average of these 'slopes' to estimate the change in the variables over the time step: If we have a function of derivatives $f(x,t)$ [1]:

$$\begin{aligned}
k_1 &= h \cdot f(x_n, t_n) \\
k_2 &= h \cdot f\left(x_n + \frac{k_1}{2}, t_n + \frac{h}{2}\right) \\
k_3 &= h \cdot f\left(x_n + \frac{k_2}{2}, t_n + \frac{h}{2}\right) \\
k_4 &= h \cdot f\left(x_n + k_3, t_n + h\right) \\
\Delta x &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
x_{n+1} &= x_n + \Delta x \\
t_{n+1} &= t_n + dt
\end{aligned}$$

then the same approach can be used to compute y, v_x, v_y for a complete specification of 2D position and velocity of an object.

Iterating over a large amount of steps n with sufficiently small time step dt can provide a great approximation to an ordinary differential equation.

B. Newton's Gravitation

The ordinary differential equation to which the algorithm was to be applied was Newton's Law of gravitation which states:

$$\vec{F} = -\frac{Gm_1 m_2}{|r^2|}\hat{r}, \tag{1}$$

where G is the gravitational constant, m_1, m_2 are the two masses distance $|r|$ apart between which attraction occurs. This along with Newton's second law, $\vec{F} = m\vec{a}$, assuming that inertial and gravitational mass are the same [2]:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = -\frac{GM}{|r^2|}\hat{r}, \tag{2}$$

gives an ordinary differential equation in $\vec{r}$. To solve it numerically, for each simulation 4 initial conditions were needed, corresponding to positions and velocities: x_0, y_0, v_{x0}, v_{y0}.

It was useful to think about the potential and kinetic energy associated with the problem. Since $\vec{F} = -\nabla V$, the gravitational potential due to Earth at position $\vec{r}$ from Earth:

$$V(\vec{r}) = -\frac{GM_E}{|\vec{r}|}, \tag{3}$$

where M_E is the mass of Earth and has been taken as 5.972×10^{24}kg [3].

The kinetic energy of the rocket was as usual for non-relativistic bodies:

$$E_K = \frac{1}{2}m_r \vec{v_r}^2. \tag{4}$$

The total energy, as a sum of potential and kinetic energy must be conserved so:

$$U = V(\vec{r}) + E_K \ \forall\ t. \tag{5}$$

II. RESULTS

A. Circular Motion

The first part of the program concerns the movement of a rocket with a circular orbit, this means we can equate the fictitious centrifugal force with the gravitational force from eq. 1 to obtain an orbital (tangential) speed in circular motion, depending on the distance from the centre of the Earth:

$$|\vec{v_c}(\vec{r})| = \sqrt{\frac{GM_E}{|\vec{r}|}}. \tag{6}$$

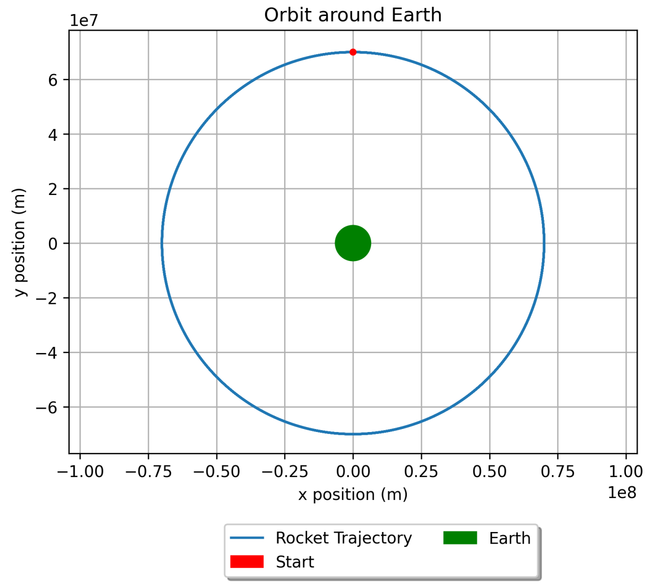

FIG. 1: Circular orbit of a rocket around Earth, initial conditions $\vec{r}_0 = (0, 7 \times 10^7)$m, $\vec{v}_0 = (2386, 0)\text{ms}^{-1}$, time step $dt = 10$s and number of steps $n = 10^5$.

To simplify the initial conditions, the initial displacement was only in the y direction, so that the velocity only needs to be specified in direction orthogonal to displacement, i. e. in the x direction. Thus, the displacement was set to $\vec{r}_0 = (0, 7 \times 10^7)$m and velocity to $\vec{v}_0 = (2386, 0)\text{ms}^{-1}$ and the motion was simulated using the RK4 algorithm with time step $dt = 10$s and number of steps $n = 10^5$. The trajectory (y displacement against x displacement) was plotted and this path of constant radius could be seen on Figure 1. Both potential and kinetic energy were tracked throughout the simulation and as expected, both kinetic and potential energy were constant in circular motion with total energy being constant as well, this could be seen on Figure 2. It can be clearly seen that since the total energy of the system in negative, this is a bound orbit, meaning the rocket will always stay in the Earth orbit, unless it increases its total energy to be > 0.

B. Elliptical Motion

Similarly, the elliptical orbit was simulated, characterised by a semi-major axis $a = 7 \times 10^7$m and eccentricity of $e = 0.5$. From this, using the Vis-viva equation to obtain the orbital velocity [4]:

$$|\vec{v}|(\vec{r}) = \sqrt{GM(\frac{2}{|\vec{r}|} - \frac{1}{a})}. \tag{7}$$

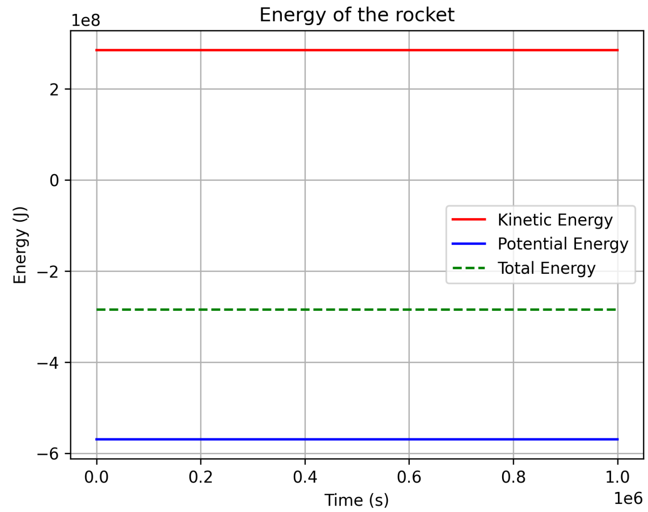

FIG. 2: Kinetic, potential and total energy in a circular orbit of a rocket around Earth, mass of rocket = 100kg.

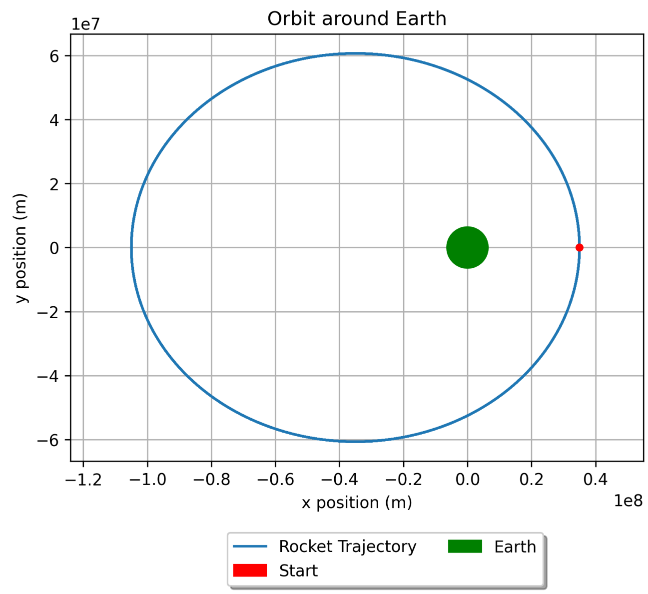

FIG. 3: Elliptical orbit of a rocket around Earth, initial conditions $\vec{r}_0 = (3.5 \times 10^7, 0)$m, $\vec{v}_0 = (0, -4133)\text{ms}^{-1}$, time step $dt = 10$s over two orbital periods.

It can be seen that this equation is a generalised form of equation 6 if $a = |\vec{r}|$. The resultant motion can be seen in Figure 3. The energy of this orbit was also investigated over two periods (from Kepler's 2nd Law $T^2 \propto a^3$). While this time the kinetic and potential energies fluctuated, the total energy was still constant and less than zero showing that the orbit is bound. Plotted energy can be found on Figure 4.

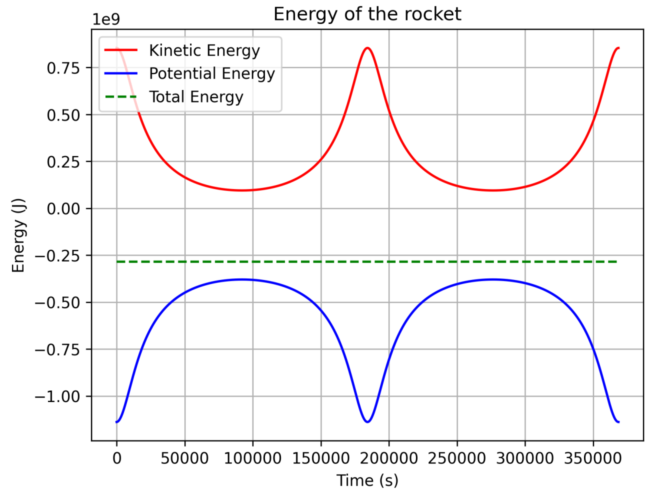

FIG. 4: Kinetic, potential and total energy in an elliptical orbit of a rocket around Earth over two periods. Total energy $U < 0$ so orbit is bound. Mass of rocket = 100kg.

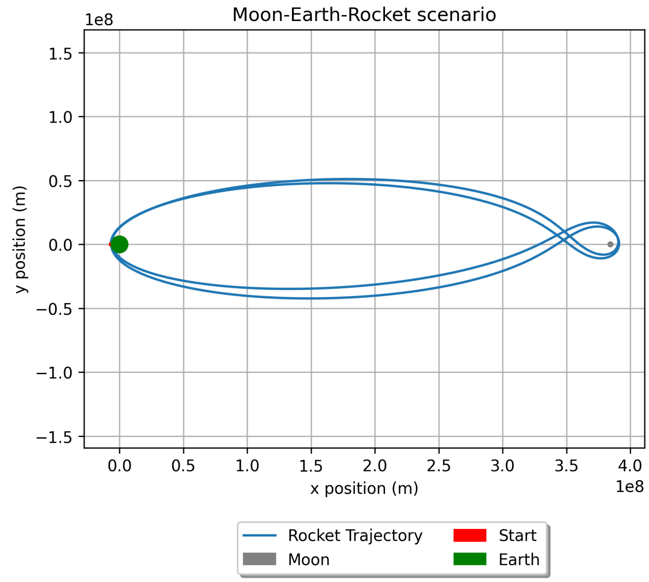

FIG. 5: Orbit of a rocket around Earth and Moon, initial conditions $\vec{r}_0 = (-6.87 \times 10^6, 0)$m, $\vec{v}_0 = (0, -10.6)\text{kms}^{-1}$, time step $dt = 10$s resulting collision with Earth.

C. Moon-Earth-Rocket System

The second part of the program concerned a scenario in which a rocket would be shot from low Earth orbit (here set to Earth radius + 500km) to pass near the Moon and return close to Earth once again. Both the equations for force felt by the rocket and the gravitational potential had to be modified by adding another term due to the Moon. By putting the Earth at the origin, and Moon at a distance 3.8×10^8m to the right of the centre of the Earth, the rocket trajectory could be simulated in a similar fashion to previous orbits.

A function was written to iterate through a range of initial velocities to estimate which one would result in a closest approach to the Moon. This was done by iterating through a number λ that would multiply the total escape velocity of the system (such that $v_{y0} = \lambda v_{esc}$), simulating the trajectory each time and keeping track of the closest approach. This resulted in a closest approach to the Moon of $\sim$ 4000km as seen on Figure 5. One flight from Earth orbit and back took $\sim$ 9.4h The total energy of this arrangement (Figure 6) was less than 0, suggesting that the rocket would stay in the system, and indeed it does, however on the second loop, it collides with Earth, perhaps to be recovered like the NASA Artemis 1 mission [5]. Interestingly, the closest approach of the aforementioned Artemis 1 rocket was only 130km from Moon's surface, compared to the value of 3960km from simulated arrangement.

Thus, the arrangement was modified such that the x velocity of the rocket could also be modified. Via a similar method to the first case, a fraction of total escape velocity was found to be used as the initial velocity such that $\vec{v}_0 = (\lambda_1 v_{esc}, \lambda_2 v_{esc})$. This can be seen in Figure 7. This has improved the closest approach to $\sim$ 227km, much closer to the Artemis 1 approach. The energy profile of these initial conditions looked similar to the previous one with much smaller absolute increases of energies (found in Figure 8). This suggest a much smaller acceleration while 'slingshotting', indeed in this scenario the rocket

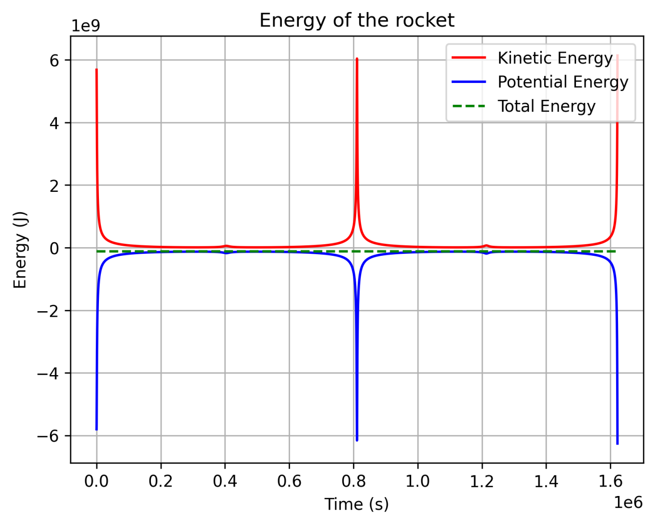

FIG. 6: Kinetic, potential and total energy in an orbit of a rocket around Earth and Moon over two periods. Total energy $U < 0$ so orbit is bound. Mass of rocket = 100kg.

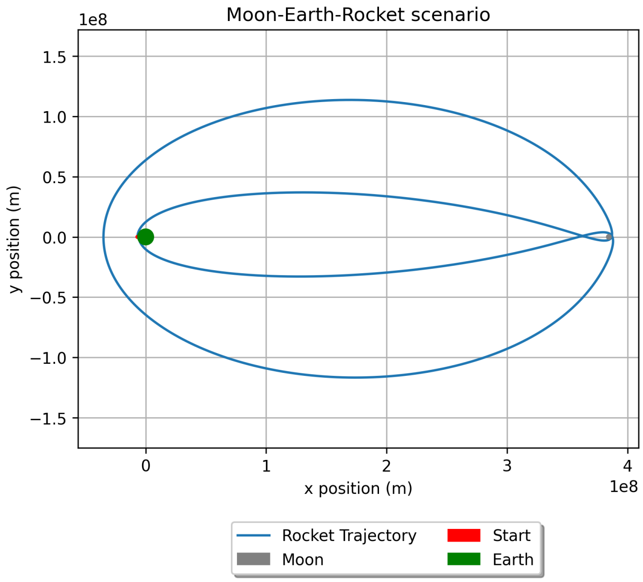

FIG. 7: Orbit of a rocket around Earth and Moon, initial conditions $\vec{r}_0 = (-6.87 \times 10^6, 0)$m, $\vec{v}_0 = (0.51, -10.67)$kms^{-1}, time step $dt = 10$s resulting collision with Earth.

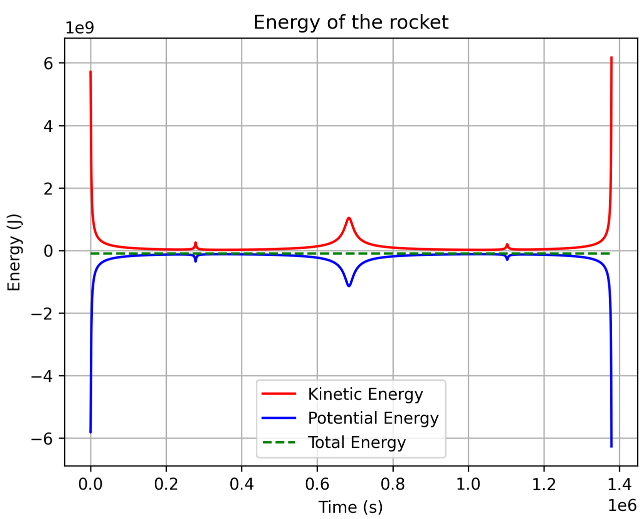

FIG. 8: Kinetic, potential and total energy in an orbit of a rocket around Earth and Moon over two periods with initial x-velocity Total energy $U < 0$ so orbit is bound. Mass of rocket = 100kg.

goes on a much larger 'radius' orbit after passing over the Moon first, and mirrors its movement on the second lap, returning to Earth, for two potential photo-shoots with Luna.

III. DISCUSSION

The discrepancy between the closest approach distance and Artemis 1 mission closest approach (as theoretically the simulation should have given closest possible approach) could have been due to a variety of factors. Firstly, the initial position in orbit was set and not varied as well, which could have brought better results. Secondly, in real missions, the rocket would loose material when accelerating from orbit, and it would be non-instantaneous. The simulation could be altered by incorporating the change in velocity over a time step using the rocket equation:

$$\Delta v = v_e \ln \left(\frac{m_0}{m_f} \right), \tag{8}$$

Where:

- Δv is the change in velocity of the rocket.
- v_e is the effective exhaust velocity of the rocket engine, representing the speed at which the propellant leaves the rocket.
- m_0 is the initial mass of the rocket, including the mass of the rocket itself and the mass of the propellant.
- m_f is the final mass of the rocket, which is the mass of the rocket after the propellant has been expended.

To model the change in velocity from Earth orbit to the Moon. This would obviously also change how the kinetic and potential energy of the system evolves, but since both $KE, PE \propto m_r$ the total energy would not change.
Moreover, the fact that both Moon is moving around the Earth, and Earth around the Sun means that the trajectories could have looked very differently. This could have been added to the code with the Moon's position moving along a circle with the equations for force and gravitational potential being updated every time step. Using the Moon's motion is how the Artemis 1 program has achieved such a close approach.

REFERENCES

[1] C. Runge. "Ueber die numerische Auflsung von Differentialgleichungen". In: *Mathematische Annalen* 46 (June 1895), pp. 167–178.

[2] Fulvio Melia. "Inertia, gravity and the meaning of mass". In: *Physica Scripta* 99.1 (Dec. 2023), p. 015024.

[3] E. V. Pitjeva and E. M. Standish. "Proposals for the masses of the three largest asteroids, the Moon-Earth mass ratio and the Astronomical Unit". In: (Oct. 2017).

[4] T. Logsdon. *Orbital Mechanics: Theory and Applications*. Wiley, 1997.

[5] BBC News. "Nasa's Orion capsule on target for splashdown". In: *BBC News* (Dec. 2022). Archived from the original on December 11, 2022. Retrieved December 11, 2022. URL: `https://www.bbc.co.uk/news/science-environment-63907649`.