Investigation into Fresnel diffraction using numerical integration methods in Python

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I. PROBLEM INTRODUCTION

The aim of this exercise was to simulate the Fresnel diffraction pattern, which is a solution to the Helmholtz equation [1]:

$$E(x,y,z) = \frac{1}{i\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x',y',0)\frac{e^{ikr}}{r}\frac{z}{r}(1+\frac{i}{kr})dx'dy', \tag{1}$$

where r is the distance from the point on aperture to point on screen. Firstly, for a 1D aperture and screen, then for 2D aperture and screen and finally allowing shapes different than rectangles to be the aperture shape, depending on the function supplied. The formula for a Fresnel diffraction with constant field at aperture at a point in space (x,y,z) was used:

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0\exp(\frac{ik}{2z}[(x-x')^2+(y-y')^2])dx'dy', \tag{2}$$

where k is the wavenumber defined as $k = 2\pi/\lambda$ with λ as the wavelength. As the experiment regarded a screen a constant distance from the aperture, the screen distance z was varied to explore its effects on the diffraction pattern. The since Eq 1 yields a complex number, to calculate the intensity of the diffraction, the following formula was used:

$$I(x,y,z) = \epsilon_0 c E^*(x,y,z)E(x,y,z), \tag{3}$$

where ϵ_0 is the permittivity of free space and c is the speed of EM radiation in vacuum. E^* is the complex conjugate of the electric field.

II. SINGLE DIMENSION DIFFRACTION

Firstly, to evaluate the shape of the pattern from a 1D aperture, the electric field expression from Eq. 1 was simplified to

$$E(x,z) = \frac{kE_0}{2\pi z}\int_{x_1'}^{x_2'} \exp(\frac{ik}{2z}(x-x')^2)dx. \tag{4}$$

The aperture width, $x_2' - x_1'$ was specified to be $1\times10^{-5}m$, $\lambda = 1\times10^{-6}m$ and initially the screen width was $z = 2cm$. With these parameters, a python program was written to evaluate this complex integral using the scipy quadrature function, into which the kernel of the integral was passed, along with the limits of the aperture to compute the electric field. This was then done over a range of screen coordinate to illustrate the results, in this example for $x \in [-5,5]mm$. From the comparison in Figure 1 it can be seen that the intensity of the diffraction decreases with screen distance, albeit not linearly. Moreover, the intensity pattern 'spreads out' with increasing distance to the screen. These two observations suggest that $I \to \infty$ as $z \to 0$, showing an infinitely high spike of intensity at the centre of screen, and $I \to 0$ as $z \to \infty$ where the waveform would spread out into a constant zero value everywhere infinitely far from the screen, as expected by the physical intuition. The effects of varying aperture size, $x_2' - x_1'$ were investigated for a range of values, resulting in the plots seen in Figure 2. It can be seen that increasing aperture width size leads to an increase in maximum intensity at the origin of the screen, as well as the oscillations on the further ends of the screen decay so quickly that one can not see the oscillations, making the smallest aperture size diffraction look almost like a Gaussian distribution.

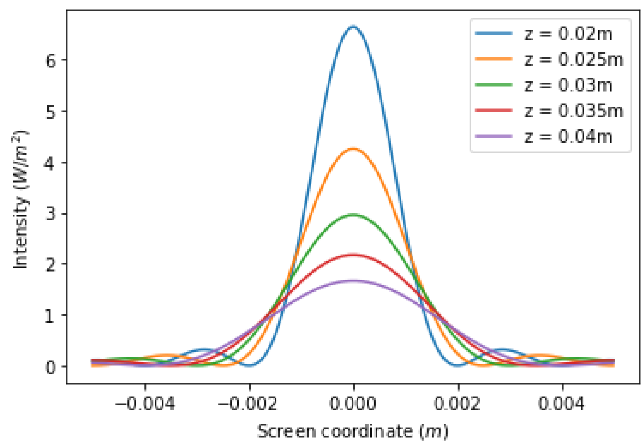

FIG. 1: The 1D Fresnel diffraction pattern for a range of screen distances $z \in [0.02, 0.04]m$

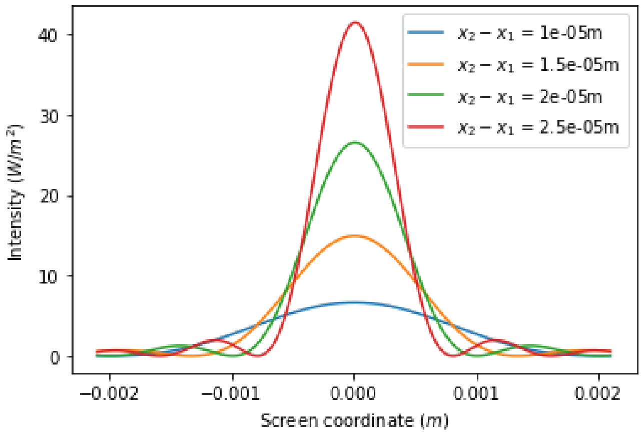

FIG. 2: The 1D Fresnel diffraction pattern for a range of aperture widths $\Delta x' \in [1.0, 2.5] \times 10^{-5}m$

III. TWO-DIMENSIONAL DIFFRACTION

The next task was to program and investigate the 2D Fresnel diffraction pattern. Modyfing equation 1 for a 2D screen yields:

$$E(x, y, z) = \frac{kE_0}{2\pi z} \int_{y_1'}^{y_2'} \int_{x_1'(y')}^{x_2'(y')} \exp\left[\frac{ik}{2z}((x-x')^2 + (y-y')^2)\right] dx'dy' \tag{5}$$

This form of Equation 4 enables two different limits for both the x' and y' coordinates of the screen. Firstly, it was set that both of these quantities are just constant, such that the aperture is rectangular. Using the 'dblquad' Scipy function to compute the double integral, iterated using a nested loop, such that for each point on the screen, (x, y) there was an assigned value of intensity $I(x, y)$ using Equation 2 which then can be plotted on a heat-map style diagram to show the shape of the diffraction patterns. Other parameters were generally the same as in the 1D case above.

A. Square Aperture

Setting $x_2' - x_1' = y_2' - y_1'$ creates square aperture, with the expected pattern appearing as a central bright square with rectangular fringes each side of diminishing intensity as evident in Figure 3; the intensity falls to 0 zero quickly on the diagonals of the screen, as expected from the square aperture. Interestingly, the central maximum appears circular in a square envelope.

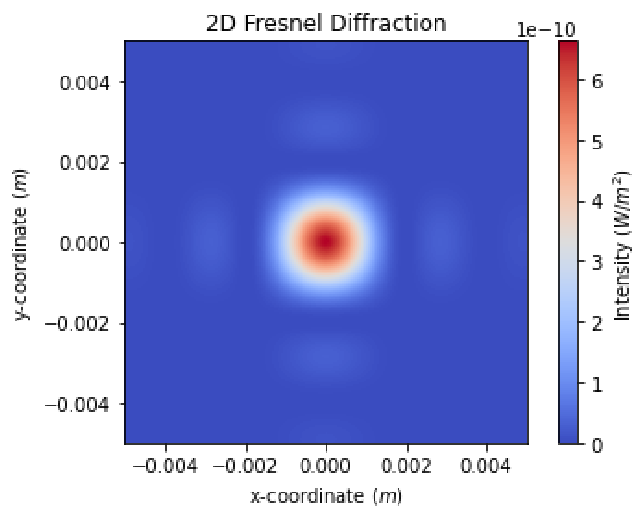

FIG. 3: Intensity plot of the Fresnel diffraction using square aperture of width $1 \times 10^{-5}m$. Central square of maximum intensity can be seen with side fringes appearing at the sides of the pattern.

B. Rectangular Aperture

Similarly, setting $x_2' - x_1' = a(y_2' - y_1')$, where a is a constant, creates a rectangular aperture, which results in a rectangular diffraction pattern, which while similar to the square aperture, appears stretched in x if $a > 1$ or in y if $a < 1$. Interestingly, more fringes are visible in the direction orthogonal to 'stretched' direction compared to the square aperture.

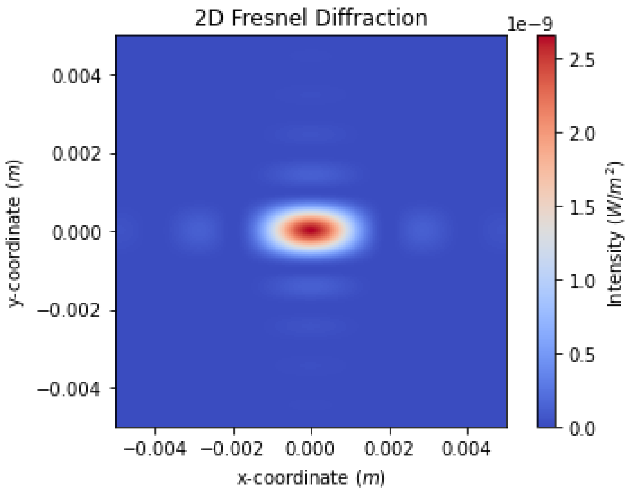

FIG. 4: Intensity plot of the Fresnel diffraction using rectangular aperture of sides $1 \times 10^{-5}m$ and $2 \times 10^{-5}m$. Central rectangle of maximum intensity can be seen with side fringes appearing at the sides of the pattern.

C. Circular Aperture

Finally, diffraction from circular aperture was investigated. Since the 'dblquad' function used in the 2D case did not accept complex integrals, the limits of the inner integral had to be replaced by functions which parameterise the circle with Cartesian coordinates [2]:

$$x_1'(y') = -\sqrt{R^2 - y'^2}, \;\; x_2'(y') = \sqrt{R^2 - y'^2}, \qquad (6)$$

where R is the radius of the circular aperture set to $R = 1 \times 10^{-5}m$. This yielded in a circular disk diffraction with further circular fringes as the distance increases from centre of the screen radially out.

IV. DISCUSSION

The Fresnel diffraction patterns were simulated using numerical integration through various SciPy functions, however bigger pictures (more than 250x250 pixels) took a very long time to calculate. To remedy this some improvements could be made:

Firstly, the computational efficiency of the simulation could be improved, with techniques like caching intermediate results for quicker callback to ensure the time taken for each 2D imaged is minimised.

Secondly, the visualisation of the diffraction patterns could be improved by incorporating advanced plotting libraries or interactive visualization tools. This could include features such as 3D rendering of diffraction patterns or real-time adjusting of parameters like aperture size.

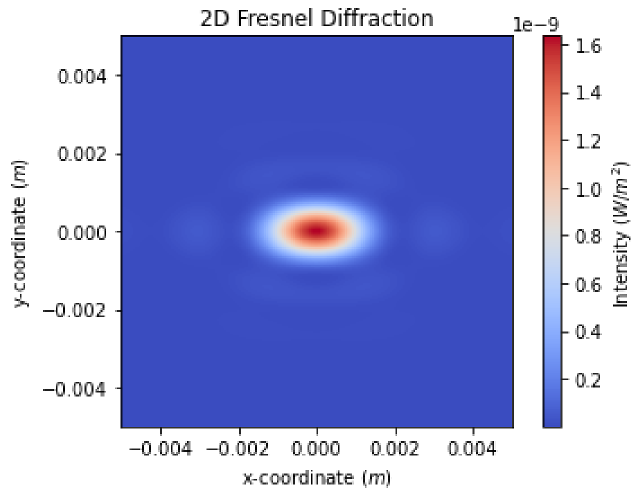

FIG. 5: Intensity plot of the Fresnel diffraction using circular aperture of radius $R = 1 \times 10^{-5}m$. Central disk of maximum intensity can be seen with concentric fringes appearing at the sides of the pattern.

Moreover, the execution time of the program could be decreased by using parallel processing by dividing the Fresnel integral into parts and assigning different threads of the CPU to execute each part simultaneously. This would enable a larger range of possible simulations in a sensible time frame.

A potential extension to this program would be to allow multi-aperture systems where the intensity would be recalculated after passing through multiple, possibly different apertures to investigate phenomena like optical arrays.

Finally, incorporating advanced wavefront modulation techniques, such as phase masks or spatial light modulators, into our simulations could enable the exploration of novel diffraction phenomena and optical manipulation strategies.

REFERENCES

[1] Max Born. *Principles of Optics*. 7th. Cambridge University Press, 1999.

[2] Shuyun Teng, Liren Liu, and De'an Liu. "Analytic expression of the diffraction of a circular aperture". In: *Optik* 116.12 (2005), pp. 568–572.