Investigation into numerical modelling of free-fall under varying drag

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I. PROBLEM INTRODUCTION

This exercise explores the various methods of modelling the motion of an object under a constant gravitational force, $F = -mg$, where m is the mass of the object, and the drag force due to the surrounding air, proportional to the square of the current velocity of the object such that $F_{drag} = -k|v_y|v_y$, where k is a constant of the form:

$$k = \frac{C_d \rho_0 A}{2}, \tag{1}$$

where C_d is the drag coefficient, ρ_0 is the density of air and A is the cross-sectional area of the falling object. This, with Newton's second law, $F = ma$ (one-dimensional since we are considering a vertical along one axis), can be described by a differential equation of the form:

$$m\frac{dv_y}{dt} = -mg - k|v_y|v_y, \tag{2}$$

where k is the drag coefficient of the form this can be solved analytically to obtain the equations of motion, for some initial vertical displacement, y_0:

$$y = y_0 - \frac{m}{k}\ln(\cosh(\sqrt{\frac{kg}{m}}t)), \tag{3}$$

and the velocity $v_y = \frac{dy}{dt}$:

$$v_y = -\sqrt{\frac{mg}{k}}\tanh(\sqrt{\frac{kg}{m}}t). \tag{4}$$

The task was to evaluate these functions both analytically, and numerically at multiple points and subsequently produce plots of these functions with varying parameters, namely changing the ratio $\frac{k}{m}$, and later providing a more realistic model for the density of air as it varies with vertical displacement such that $\rho = \rho(y)$.

II. EULER METHOD

Euler method was used as a good approximation to variables with values changing over time, to solve a first-order differential equation instead of calculating a value of a function at a time t, the time interval T was divided into N small fixed increments Δt, such that $T = N\Delta t$. Firstly, for an interval $t \in [0, T]$, a set of time variable values was discretised with $N + 1$ points labelled t_i with $i = [0, 1, 2, ..., N]$. Then the following can be written for the Euler method

$$v_{i+1} = v_i + \frac{dv}{dt}\,\Delta t_i\ , \tag{5}$$

Where v = vertical velocity of an object and $\frac{dv}{dt}$ is the acceleration of the object. This approximation works by adding small increments to the current velocity, equal to the rate of change of the velocity multiplied by time. Similarly, for the position we have:

$$z_{i+1} = z_i + v_i\,\Delta t_i\ . \tag{6}$$

Using a loop, the program can calculate values of position and velocity this way. The process starts with initializing sets of N elements for each variable that will be changing (time, position and velocity in this case) with 0 values. This is done to ensure there will be an equal number of elements for the plots when the calculation is complete.

III. PART A RESULTS - ANALYTICAL SOLUTION WITH CONSTANT DRAG FORCE

Since the drag force constant had to be specified in order to produce plot for equations 3 and 4, the drag coefficient was taken as $C_d = 1.15$ as the average for a sky diver whose height is $1.8m$, the cross sectional area for this height was taken as $A = 0.8m^2$. Finally the density of air at room temperature and ambient pressure was taken as $\rho_0 = 1.2kgm^{-3}$. This gives the drag constant $k \approx 0.552Nsm^{-1}$. The mass of the sky diver was taken o be $m = 90kg$ and the gravitational constant as $g = 9.81ms^{-2}$. Using this information, a theoretical terminal velocity, u_T can be calculated as:

$$u_T = -\sqrt{\frac{mg}{k}}, \tag{7}$$

Which can be obtained by taking the limit of equation 4 as the projectile falls: $\lim_{t\to\infty}\tanh(t) = 1$. With the chosen values for drag force constant, this terminal velocity was $u_T \approx 40ms^{-1}$. The analytical solutions were finally calculated and plotted as seen in Figure 1.

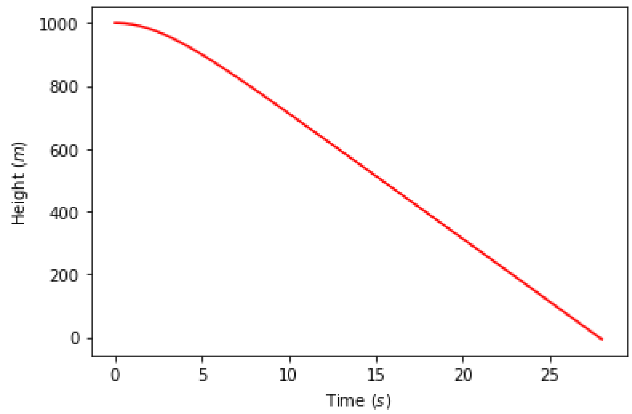

FIG. 1: Plot of the analytical function of vertical displacement postulated in equation 3, with initial displacement $y_0 = 1000m$ and drag force constant $k \approx 0.55Nsm^{-1}$.

The displacement behaved as predicted with a linear decrease of height as the velocity reached its limit. Conversely, the velocity of the skydiver decreased until it reached its terminal velocity ($u_T \approx 40ms^{-1}$).

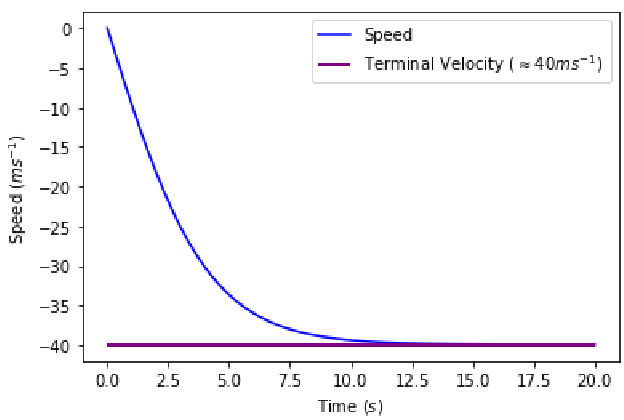

FIG. 2: Plot of the analytical function of vertical speed postulated in equation 3, with initial displacement $y_0 = 1000m$ and drag force constant $k \approx 0.55Nsm^{-1}$. The terminal velocity reached was $u_T \approx 40ms^{-1}$ as portrayed by the purple line.

IV. PART B RESULTS - NUMERICAL MODELLING WITH CONSTANT DRAG FORCE

Knowing the shape of both displacement and velocity of analytical solutions to the problem, the aforementioned Euler method was used to discretise equations 3 and 4 such that:

$$y_{n+1} = y_n + \Delta t\ v_{y_n}, \tag{8}$$

and

$$v_{y_{n+1}} = v_{y_n} - \Delta t\ (g + \frac{k}{m}|v_{y_n}|v_{y_n}). \tag{9}$$

This enabled the iteration over a number of a number of N points and appending a list of both displacement and velocity values iteratively to obtain an approximation to the skydiver problem, as illustrated in figures 3 and 4.

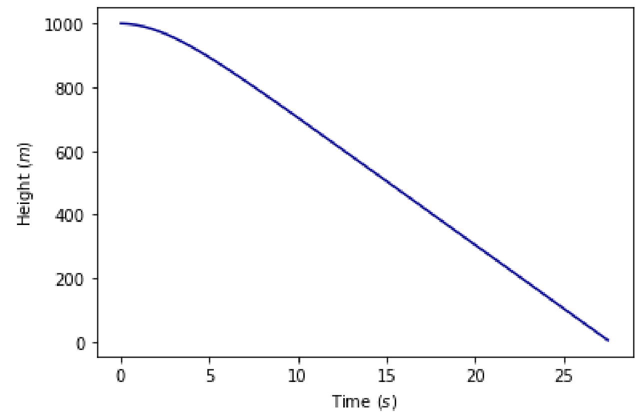

FIG. 3: Plot of the numerical approximation of the vertical displacement of the skydiver with $y_0 = 1000m$, and Euler step size $\Delta t = 0.25s$. The rest of the parameters same as in part A.

Plots in figures 1 and 3 look remarkably similar in shape, suggesting that Euler method is very good approximation for this problem, specifically, the analytical solution dictates that the time it took to 'reach the ground' ($y = 0m$) was $t = 27.8593s$, whereas the numerical solution reaches this value of displacement after $t = 27.75s$. This gives an error of just $t_{err} \approx 0.11s$.

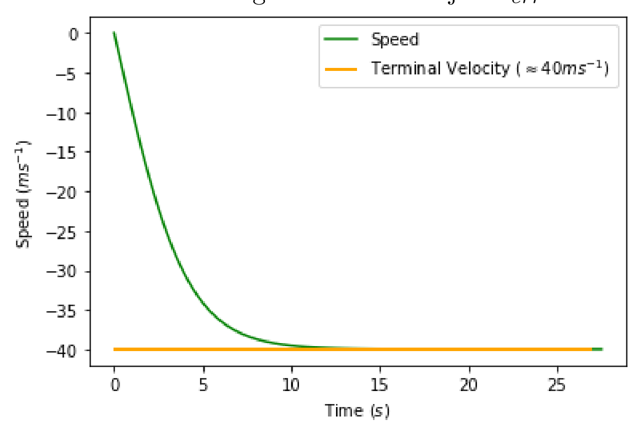

FIG. 4: Plot of the numerical approximation of the downwards speed of the skydiver with $y_0 = 1000m$, and Euler step size $\Delta t = 0.25s$. The rest of the parameters same as in part A.

Comparing figure 4 to figure 2, it can be seen that the shape of the downwards speed curve obtained numerically was very close to to the analytical speed curve. It can be observed that both curves reach a similar terminal velocity, at a very similar time (around $t = 10s$). Varying the time step, Δt, affected the accuracy of the numerical approximation, as the local error in the Euler method is proportional to h^2 and the global error is proportional to h. This effect can be seen in figure 5, along the analytical solution.

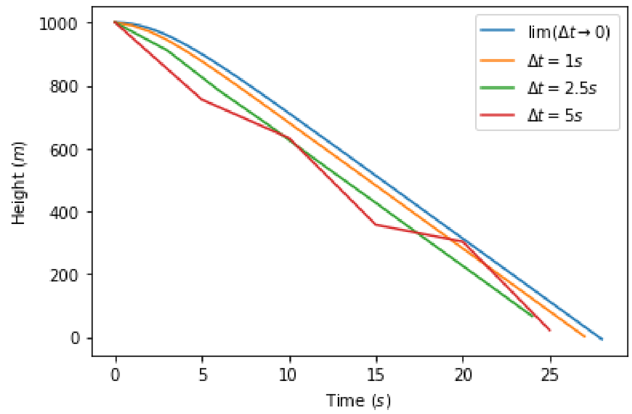

FIG. 5: Numerical modelling of the skydiver height using various time step sizes. While relatively low values like $\Delta t = 1s$ gives a good approximation to analytical solution (blue line), increasing this value to $\Delta t = 5s$ drastically changes the shape of the curve, with visible bumps.

Interestingly, varying the time step decreased the apparent time to hit the ground, which meant that the approximation always underestimates the value of height at the end, which is a result of the global error of the Euler method. The effect of varying $\frac{k}{m}$ ratio was studied, which can be seen as directly corresponding to the terminal velocity via equation 7, such that $(\frac{1}{u_T})^2 \propto \frac{k}{m}$. The varying effect on height of skydiver can be seen in figure 6.

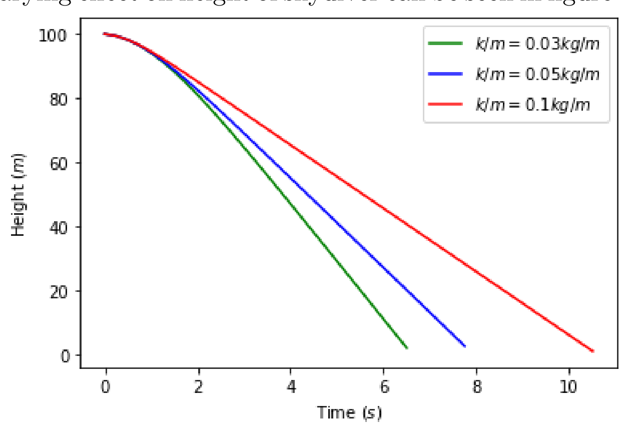

FIG. 6: Numerical modelling of skydiver height using varying $\frac{k}{m}$ values. Larger $\frac{k}{m}$ makes the time-of-flight shorter.

This investigation confirms the intuition that a more massive object (smaller $\frac{k}{m}$) falls to the ground faster (which is not the case under gravity alone). A clearer effect of varying this ratio however can be seen when looking at plots of downward speed against time where the terminal velocity approached for different ratios, as in figure 7.

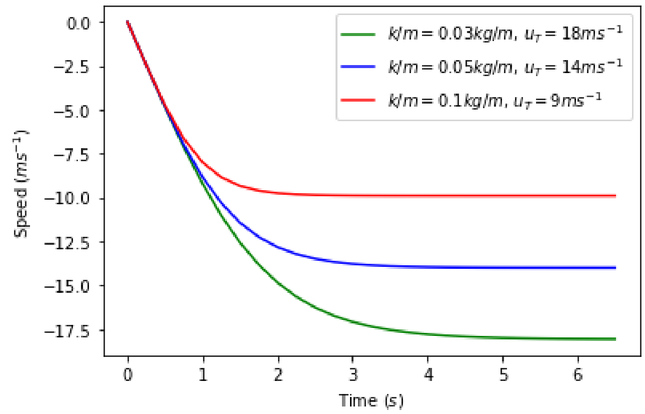

FIG. 7: Numerical modelling of skydiver downwards speed using varying $\frac{k}{m}$ values. Larger $\frac{k}{m}$ decrease the terminal velocity and time to reach terminal velocity, which each curve can be seen approach.

Figure 7 shows that for the same drag force coefficient, k, more massive objects will reach a higher terminal velocity, and it will take them longer, compared to less massive objects.

V. PART C RESULTS - NUMERICAL MODELLING OF SKYDIVER WITH VARYING DRAG FORCE

The problem being investigated - Felix Baumgartner jump started from a much higher altitude, and so the air density was different at different parts of the jump. This can be modelled using the equation:

$$\rho(y) = \rho_0 e^{-y/h}, \tag{10}$$

where h is a constant which was be taken as $h = 7.64km^2$. This means that the drag force coefficient now depends on the height:

$$k(y) = \frac{C_d A \rho_0}{2} e^{-y/h} \tag{11}$$

This, in combination with the Euler method for part B, and a new initial height, $y_0 = 41400m$ (mimicking Baumgartner's jump) was used to model a plot of height and velocity over time, seen in figures 8 and 9.

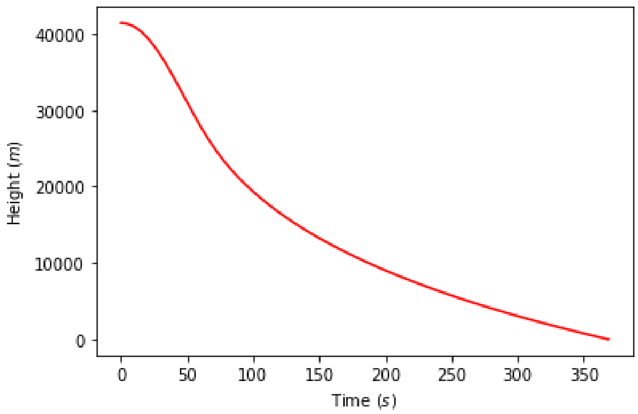

FIG. 8: Numerical modelling of skydiver height under altitude-dependent drag force.

It can be seen that the height curve (figure 8) looks drastically different with this model of varying drag, compared to constant drag (figure 3). The slope of the curve seems to have an inflection point at around $t = 50s$, which can be seen as the minimum of figure 9. The maximum speed reached was $v_{max} = 324.4ms^{-1}$.

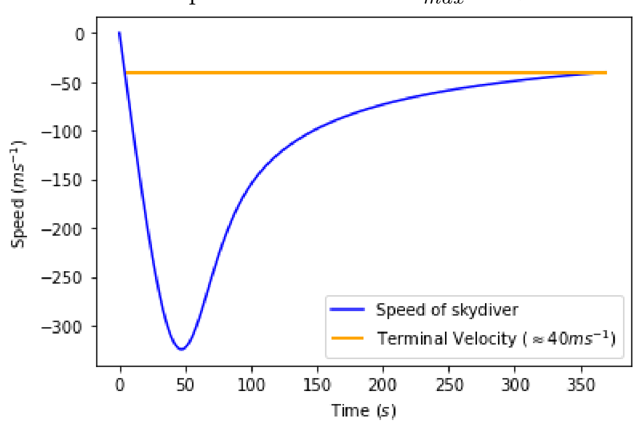

FIG. 9: Numerical modelling of skydiver downwards speed under altitude-dependent drag force. Minimum occurs at around $t = 50s$

Interestingly, the velocity goes through a much more negative minimum compared to constant drag model, specifically, it reaches a velocity of $v_y = 324.5ms^{-1}$. As time passes however the velocity approaches the terminal velocity value as the altitude decreases, the air density ρ increases and so the drag force coefficient increases, making the object establish its terminal velocity described by equation 7. This maximum velocity the simulated skydiver reaches is $v_{max} = 324.4ms^{-1}$ compared to the actual value for the Red Bull Stratos jump, $v_{max} = 1,357.64kmh^{-1} = 377.1ms^{-1}$. The discrepancy most likely arises from choice of initial parameters like the cross-sectional area and the mass of the skydiver.

The effect of starting height, y_0, was investigated next. This showed that increasing the starting height, drastically increases the maximum speed the skydiver goes through (due to exponential decay of air density postulated in equation 10), which can be seen in figure 10.

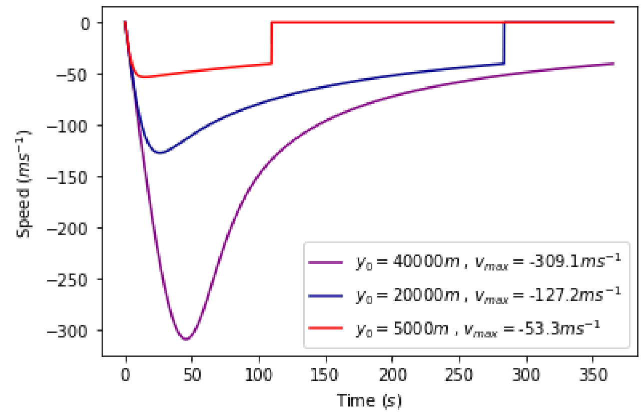

FIG. 10: Numerical plot of skydiver downwards speed for a range of values of initial height y_0. Increasing this value increases the maximum speed the skydiver goes through massively: doubling the initial height increases the maximum speed by a factor of ≈ 2.4.

Moreover the time to hit the ground was larger, for a larger value of y_0, as expected. Interestingly however, for a lower initial height, the distance-time plot looks almost linear (similar to figure 1), but as the initial height is increased, the plot looks more and more curved, revealing the underlying exponential in value of k from equation 11, which can be seen in figure 11.

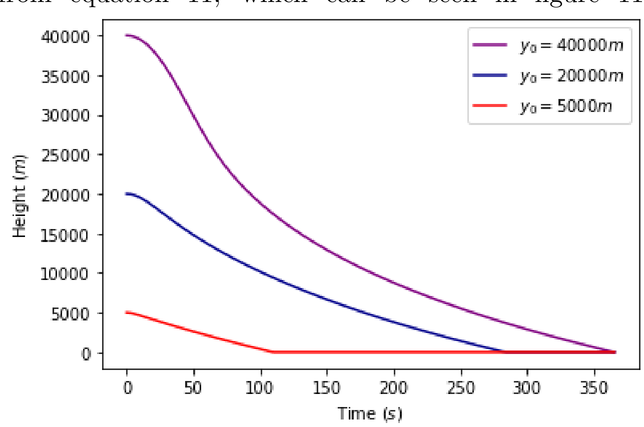

FIG. 11: Numerical modelling of skydiver height for a range of initial height. The lowest initial height appears almost linear, compared to the highly curved plot when the initial height has been increased eightfold.

The final part of the investigation regards changing the C_dA/m ratio. It has been observed that decreasing this ratio, increases the maximum velocity the skydiver goes through, at a slightly later time, just like in the case of figure 10. This phenomenon can be seen in figure 12.

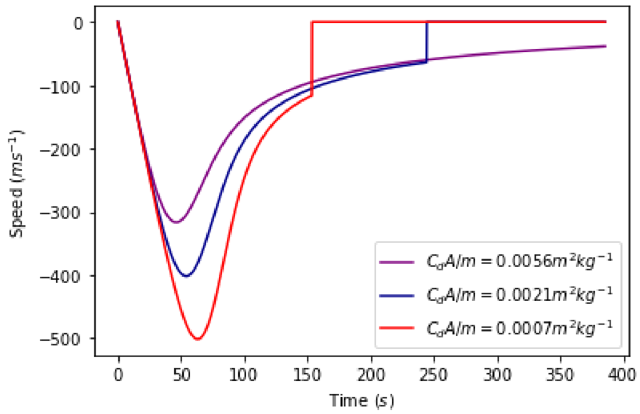

FIG. 12: Numerical modelling of a skydiver downwards speed for a range of values of C_dA/m. Increasing the value of this ratio causes the skydiver to go through a smaller maximum speed.

On the other hand, increasing the ratio C_dA/m decreases the time to reach the ground, as well as makes the later part of the height plot less 'flat'. Interestingly, the initial portion of all of the height plots appears the same until some time around $t = 50s$, like on plot in figure 13.

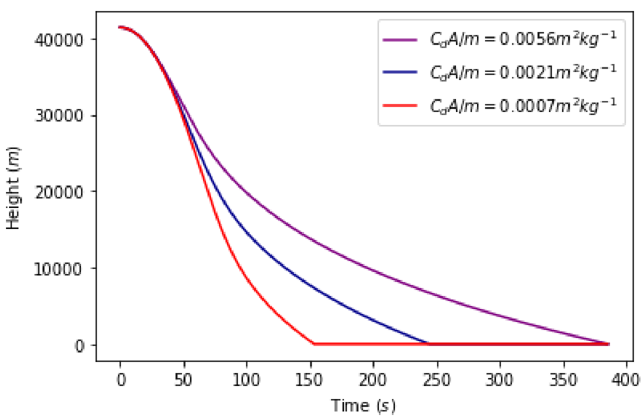

FIG. 13: Numerical modelling of a skydiver height for a range of values of C_dA/m. Decreasing the value of C_dA/m ratio 'flattens' the height curve and causes the 'skydiver' to 'hit' the 'ground'.

VI. DISCUSSION AND IMPROVEMENTS

Comparing figures 1 and 3 shows that Euler method was a good approximation of the analytical solution, with the global error being only $t_{err} = 0.11s$. This might seem small but it could cause problems if the method was used for more precise modelling where split-second precision is important like a rocket launch. Figure 5 shows that the quality of the approximation degrades greatly as the time step Δt is increased. Ideally, the time step would be adjusted based on the time scale of the simulation, using the formula $\Delta t = T_{max}/N$, if the approximate higher time scale boundary of the simulation, T_{max} is known, and number of points to evaluate, N, can be varied to obtain a good level of accuracy. An improvement to the method of approximating the rate of change of velocity for this simulation could be the Runge-Kutta method (particularly the RK4), which calculates the slope at four different points of the curve, integrating the ODE with fourth-order accuracy, in this case, the equation would be:

$$v_{y_{n+1}} = v_{y_n} + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad (12)$$

where k_1, k_4 are the slope values at the beginning and the end of the step and k_2, k_3 are slopes in the middle of the time step which are weighted more. RK4 being a fourth-order method, means that the local error is proportional to $(\Delta t)^5$ and the global error is proportional to $(\Delta t)^4$; much better than the Euler method. A comparison of Euler and RK4 method, along with the analytical solution can be seen in Figures 14 and 15.

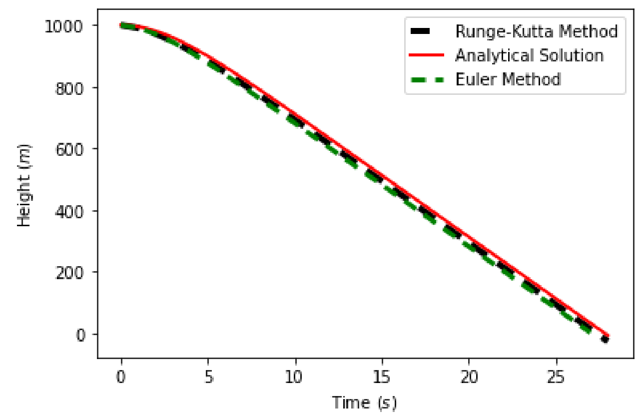

FIG. 14: Numerical modelling of a skydiver height using Euler method, RK4 and the analytical solution. The time step used is $\Delta t = 1s$, quite a high value. Despite this, RK4 seems better at the approximation.

The real difference between the methods can be seen in figure 15, where the speed values really deviate around $t = 5s$. Runge-Kutta tracks the analytical solution almost perfectly compared to the Euler method, which overestimates the speed.

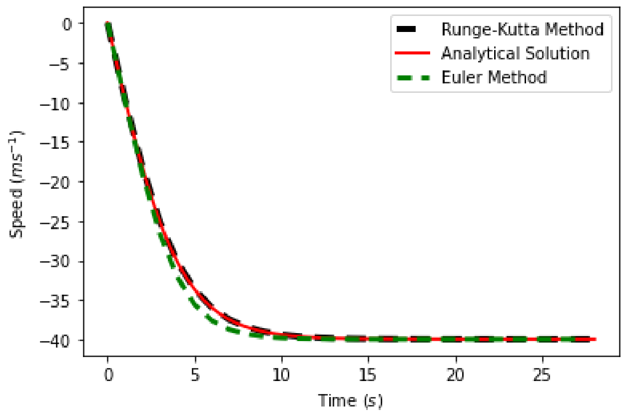

FIG. 15: Numerical modelling of a skydiver height using Euler method, RK4 and the analytical solution. The time step used is $\Delta t = 1s$. RK4 follows the analytical solution to a much higher degree of accuracy compared to the Euler method.

The difference in these values could be much more significant for different differential equation. The Euler method could eventually approach a similar degree of accuracy as RK4, but it would require decreasing the time step into a smaller and smaller values. This however would increase the number of times the program would have to calculate the individual values, increasing the time to complete the simulation. For more complex ODEs or larger time scales, Euler method would fail to provide both a good approximation and an efficient one, leaving the Runge-Kutta method as a superior one to use in problems such as this one.