Computer physics - Assignment 6 - Task 2

$$\lambda_{i+1} = \frac{\langle x p_i | p_i \rangle}{\langle p_i | p_i \rangle} \qquad \gamma^2_{i+1} = \begin{cases} 0 & , \text{if } i=0 \\ \dfrac{\langle p_i | p_i \rangle}{\langle p_{i-1} | p_{i-1} \rangle} & , \text{otherwise} \end{cases}$$

$$\langle f | g \rangle = \int_a^b dx \; \omega(x) f(x) g(x)$$

$$p_{i+1}(x) = (x - \lambda_{i+1}) p_i(x) - \gamma^2_{i+1} p_{i-1}(x) \qquad p_0(x) = 1, \quad p_{-1}(x) = 0$$

n=1

$$p_1(x) = (x - \lambda_1) p_0(x) - \gamma^2_1 \; p_{-1}(x)$$

$$= x - \frac{\int_{-1}^{1} dx \; \omega(x) p_0^2(x) x}{\int_{-1}^{1} dx \; \omega(x) p_0^2(x)} \cdot 1 - \gamma^2_1 \cdot 0$$

$$= x - \frac{\left[\frac{1}{2} x^2\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}} - 0 = x - \frac{\frac{1}{2} - \frac{1}{2}}{1 + 1} = x$$

n=2

$$p_2(x) = (x - \lambda_2) p_1(x) - \gamma^2_2 \; p_0(x)$$

$$= \left(x - \frac{\int_{-1}^{1} dx \; x^3}{\int_{-1}^{1} dx \; x^2}\right)(x) - \frac{\int_{-1}^{1} dx \; x^2}{\int_{-1}^{1} dx \; 1} \cdot 1$$

$$= \left(x - \frac{\left[\frac{1}{4} x^4\right]_{-1}^{1}}{\left[\frac{1}{3} x^3\right]_{-1}^{1}}\right) x - \frac{\left[\frac{1}{3} x^3\right]_{-1}^{1}}{\left[x\right]_{-1}^{1}}$$

$$= \left(x - \frac{\frac{1}{4} - \frac{1}{4}}{\frac{1}{3} + \frac{1}{3}}\right) x - \frac{\frac{1}{3} + \frac{1}{3}}{1 + 1}$$

$$= x^2 - \frac{1}{3}$$

$\underline{n=3}$

$$p_3(x) = (x - \lambda_3)\, p_2(x) - \cancel{g}\, \gamma_3^2 \cdot p_1(x)$$

$$= \left(x - \frac{\int_{-1}^{1} (x^2 - \frac{1}{3})^2 \cdot x}{\int_{-1}^{1} (x^2 - \frac{1}{3})^2} \right) \left(x^2 - \frac{1}{3} \right) - \frac{\int_{-1}^{1} (x^2 - \frac{1}{3})^2}{\int_{-1}^{1} x}$$

$$= \left(x - \frac{\int_{-1}^{1} (x^4 - \frac{1}{6}x^2 + \frac{1}{9}) \cdot x}{\int_{-1}^{1} (x^4 - \frac{1}{6}x^2 + \frac{1}{9}) \cancel{x}} \right) \left(x^2 - \frac{1}{3} \right) - \frac{\int_{-1}^{1} (x^4 - \frac{1}{6}x^2 - \frac{1}{9})}{\int_{-1}^{1} x} \cdot x$$

$$= \left(x - \frac{\left[\frac{1}{6}x^6 - \frac{1}{24}x^4 + \frac{1}{18}x^2\right]_{-1}^{1}}{\left[\frac{1}{5}x^5 - \frac{1}{18}x^3 + \frac{1}{9}x\right]_{-1}^{1}} \right) \left(x^2 - \frac{1}{3} \right) - \frac{\left[\frac{1}{5}x^5 - \frac{1}{18}x^3 - \frac{1}{9}x\right]_{-1}^{1}}{\left[\frac{1}{2}x^2\right]_{-1}^{1}} \cdot x$$

$$= \left(x - \frac{\frac{1}{6} - \frac{1}{24} + \frac{1}{27} - \left(\frac{1}{6} - \frac{1}{24} + \frac{1}{18}\right)}{\frac{1}{5} - \frac{1}{18} + \frac{1}{9} - \left(-\frac{1}{5} + \frac{1}{18} \dots\right)} \right) \left(x^2 - \frac{1}{3} \right) - \frac{\left[\frac{1}{5}x^5 - \frac{1}{18}x^3 - \frac{1}{9}x\right]_{-1}^{1}}{\frac{1}{2} + \frac{1}{2}} \cdot x$$

$$= x \left(x^2 - \frac{1}{3} \right) - \frac{\left(\frac{1}{5} - \frac{1}{18} - \frac{1}{9}\right) \cdot 2}{1} \cdot x$$

$$= x^3 - \frac{1}{3}x^2 - \frac{1}{15}x$$

I am ~~not~~ 100% sure that i am unable to find my error, but the polynomials are exept for the first one not the Legendrepolynomials, so I skip the computation of p_4 and go straight to the next part.

$$p_1(x_n^{(1)}) = 0 \iff x_n^{(1)} = 0$$

$$p_2(x_n^{(2)}) = 0$$

$$x_n^{(2)2} - \frac{1}{3} = 0$$

$$x_n^{(2)2} = \frac{1}{3} \iff x_1^{(2)} = \sqrt{\frac{1}{3}} \quad x_2^{(2)} = -\sqrt{\frac{1}{3}}$$

negativ sqrt

$$p_3(x_n^{(3)}) = 0$$

$$x_n^{(3)3} - \frac{1}{3}x_n^{(3)2} - \frac{1}{15}x_n^{(3)} = 0 \iff x_1^{(3)} = 0$$

$$x_n^{(3)2} - \frac{1}{3}x_n^{(3)} - \frac{1}{15} = 0 \iff x_{3,2}^{(3)} = -\frac{1}{3 \cdot 2} \pm \sqrt{\left(\frac{1}{3 \cdot 2}\right)^2 - \frac{1}{15}}$$