

# 1 Info

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## Temas:

- Cinemática y dinámica (mecánica)
- Campos eléctricos y magnéticos
- Circuitos
- Termodinámica

# 2 Measurements and magnitudes

Measurements seek to compare a prediction with an observation, so as to test a hypothesis. A magnitude is a number accompanied by a unit. Some magnitudes are:

- Length, measured in meters ( $m$ )
- Time, measured in seconds ( $s$ )
- Mass, measured in kilograms ( $kg$ )
- Current, measured in amperes ( $A$ )
- Temperature, measured in kelvins ( $K$ )
- Matter, measured in moles ( $mol$ )

We consider  $10^3$  (e.g. kilometer) and  $10^{-3}$  (e.g. millimeters) to be within human scale. We call mass, seconds and kilograms the *mechanical units*. We define the *force unit*, or Newton, as

$$[F] = N = kg \frac{m}{s^2}$$

and the Pascal unit as

$$[P] = Pa = \frac{N}{m^2}$$

We use scientific notation and terms which express quantities as powers of ten. For instance,  $10^{12}$  is the tera,  $10^3$  the giga, etc.

The magnitudes hereby described are suited for algebraic manipulation. For instance,  $m \times m = m^2$ , and  $s \times \frac{m}{s} = m$ .

### 3 Vectors

Vectors are used to express position, displacement, velocity, force, acceleration, fields, etc. A vector  $\vec{A}$  (or sometimes  $\vec{a}$ ) in the general sense has a direction (line), an orientation, and a length (or magnitude). A vector also has an application point, which denotes the point of origin of the vector. When saying  $\vec{a} = \vec{b}$ , we mean that  $\vec{a}$  and  $\vec{b}$  coincide in direction, magnitude and orientation, irrespective of their application point.

The scalar product is defined as the usual mapping in the space  $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ . Intuitively, the scalar product  $\lambda \vec{a}$  "stretches" or "shrinks" a vector, depending on whether  $|\lambda| < 1$  or not, and the positivity or negativity of  $\lambda$  determines whether the vector inverts its direction or not. In general,  $|\lambda \vec{a}| = |\lambda| |\vec{a}|$ .

The sum of vectors,  $\vec{a} + \vec{b}$ , is a mapping  $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ . As usual, and in a graphical sense, the sum corresponds to the application of the parallelogram rule.

**Parallelogram rule.** Make  $\vec{a}$  and  $\vec{b}$  coincide in their point of application. From the tip of  $\vec{a}$ , draw a copy of  $\vec{b}$ , and from the tip of  $\vec{b}$  a copy of  $\vec{a}$ . The corner of the thus generated parallelogram is the tip of  $\vec{a} + \vec{b}$ .

Alternatively, from the tip of  $\vec{a}$  write  $\vec{b}$ . Then  $\vec{a} + \vec{b}$  is the vector which goes from the point of application of  $\vec{a}$  to the tip of  $\vec{b}$ .

The sum of vectors is commutative, associative, and distributive with respect to scalar product.

If  $\vec{A}$  is a vector, we use  $A_x$  and  $A_y$  to denote the projection of the vector over the axis  $x$  or  $y$ , respectively. Using  $A_x$  and  $A_y$  one forms a rectangular triangle with sides  $A_x$ ,  $A_y$  and a hypotenuse of length  $|\vec{A}|$ .

Let  $\theta$  be the angle formed by  $\vec{A}$  with the  $x$ -axis. Then, using trigonometry,

$$\cos \theta = \frac{A_x}{|\vec{A}|}, \quad \sin \theta = \frac{A_y}{|\vec{A}|}$$

from which one can find  $A_x, A_y$  assuming one knows  $\theta$ . From this follows that  $|\vec{A}|$  and  $\theta$  fully determine all the information about the vector, insofar as they allow us to determine  $A_x, A_y$ . Conversely, knowing  $A_x$  and  $A_y$  is also sufficient to determine  $\vec{A}$ , insofar as

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}, \quad \frac{A_y}{A_x} = \frac{|\vec{A}| \sin \theta}{|\vec{A}| \cos \theta} = \tan \theta \Rightarrow \theta = \arctan \left( \frac{A_y}{A_x} \right)$$

As convention, we use  $\hat{i}$  to denote the versor (vector of length 1) with direction parallel to the  $x$ -axis, and  $\hat{j}$  the versor with direction parallel to the  $y$ -axis.

Notice that, for any vector  $\vec{A}$ ,  $A_x$  is  $\hat{i}$  times  $A_x$ , and  $A_y$  is  $\hat{j}$  times  $A_y$ , which means

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

When writing  $\vec{A}$  in this way, we say we write it in terms of its components  $x, y$ . In terms of linear algebra, it's not hard to see that we are simply expressing that  $\hat{i}, \hat{j}$  form a basis of  $\mathbb{R}^2$ . Thus, it is equivalent to write

$$A_x = |\vec{A}| \cos \theta, \quad A_y = |\vec{A}| \sin \theta$$

and

$$\vec{A} = |\vec{A}| (\cos \theta \hat{i} + \sin \theta \hat{j})$$

From this follows as well that

$$\begin{aligned} \vec{A} + \vec{B} &= (A_x \hat{i} + A_y \hat{j}) + (B_x \hat{i} + B_y \hat{j}) \\ &= \hat{i} (A_x + B_x) + \hat{j} (A_y + B_y) \end{aligned}$$

which means the sum of vectors has as components the sum of the components.

The scalar product of two vectors,  $\vec{A} \cdot \vec{B}$ , is a scalar defined as

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

where  $\theta$  is the angle formed by the two vectors. The scalar product is positive if  $\cos \theta$  is positive, which occurs for  $0 < \theta \leq 90$ . It is negative if  $\cos \theta$  is negative, i.e. if  $90 < \theta \leq 180$ . Clearly,  $\vec{A} \cdot \vec{B} = 0 \iff \theta = 90$ .

In general, from the definition follows that

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

The vectorial product  $\vec{A} \times \vec{B}$  is a vector perpendicular to the plane formed by  $\vec{A}$  and  $\vec{B}$ . Its module is  $|\vec{A}| |\vec{B}| \sin \theta$ , and its direction is given by what's called the right-hand rule.

### 3.1 Exercises

(2) Sean los vectores  $\vec{A} = 2\hat{i} + 3\hat{j}$ ,  $\vec{B} = 4\hat{i} - 2\hat{j}$  y  $\vec{C} = -\hat{i} + \hat{j}$ . Determinar la magnitud y el ángulo (representación polar) de los vectores resultantes  $\vec{D} = \vec{A} + \vec{B} + \vec{C}$  y  $\vec{E} = \vec{A} + \vec{B} - \vec{C}$ . Resolver analítica y gráficamente.

(Analytical solution.) We'll use  $A_x, A_y$  to denote the components of the vector  $\vec{A}$ , and same for all other vectors. We know the components of  $\vec{D}$  are

$$D_x = A_x + B_x + C_x = 2 + 4 - 1 = 5, \quad D_y = 3 - 2 + 1 = 2$$

from which readily follows that  $|\vec{D}| = \sqrt{5^2 + 2^2} = \sqrt{29} \approx 5.385$ . Similarly,

$$E_x = 2 + 4 + 1 = 7, \quad E_y = 3 - 2 - 1 = 0$$

from which follows that  $|\vec{E}| = \sqrt{7^2} = 7$ .

Now, we must recall that

$$\theta_{\vec{Z}} = \arctan\left(\frac{Z_y}{Z_x}\right)$$

for any  $\vec{Z}$ .

We need not memorize this: it is trigonometrically clear that  $Z_x = \cos \theta_{\vec{Z}} |\vec{Z}|$  and  $Z_y = \sin \theta_{\vec{Z}} |\vec{Z}|$ , and therefore

$$\frac{Z_y}{Z_x} = \tan \theta$$

And arctan is the inverse of tan. Anyhow, for  $\vec{E}$  and  $\vec{D}$  we have

$$\theta_{\vec{E}} = \arctan\left(\frac{E_y}{E_x}\right) = \arctan(0) = 0$$

$$\theta_{\vec{D}} = \arctan\left(\frac{D_y}{D_x}\right) = \arctan\left(\frac{2}{5}\right) \approx 0.38$$

(3) Can two vectors of different magnitude be combined and yield zero? What about three?

The zero vector is the only vector with magnitude zero. Let  $\vec{A}, \vec{B}$  arbitrary vectors. Then

$$|\vec{A} + \vec{B}| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

which is zero if and only if

$$(A_x + B_x)^2 + (A_y + B_y)^2 = 0$$

This only holds if  $A_x + B_x = A_y + B_y = 0$ . But

$$A_x + B_x = 0 \Rightarrow A_x = -B_x, \quad A_y + B_y = 0 \Rightarrow A_y = -B_y$$

But then

$$|A| = \sqrt{A_x^2 + A_y^2} = \sqrt{(-B_x)^2 + (-B_y)^2} = \sqrt{B_x^2 + B_y^2} = |B|$$

$$\therefore |\vec{A} + \vec{B}| = 0 \iff |\vec{A}| = |\vec{B}|.$$

It is simple to see that three vectors of different magnitude can add to zero.

Assume  $A + B + C = 2\hat{i} + \hat{j}$  and  $A = 6\hat{i} - 3\hat{j}$ ,  $B = 2\hat{i} + 5\hat{j}$ . Find the components of  $C$ . Solve analytically and graphically.

We know

$$6 + 2 + C_x = 2, \quad -3 + 5 + C_y = 1$$

from which follows that  $C_x = -6$ ,  $C_y = -1$ .

(5)  $A$  and  $B$  have a magnitude of  $3m, 4m$  respectively. The angle between them is  $\theta = 30$  degrees. Find their scalar product.

Their scalar product is

$$(|B| \cos \theta) |A|$$

Recall that

$$\text{Angle in degrees} = \text{Angle in radians} \cdot \frac{180}{\pi}$$

Thus, thirty degrees equates to  $30 \frac{\pi}{180} \approx 0.523$  radians. Then the scalar product is

$$4 \cos(0.523) \times 3 \approx 10.395$$



(6) Find the angle between  $A = 4\hat{i} + 3\hat{j}$  and  $B = 6\hat{i} - 3\hat{j}$ .

Recall that

$$A \cdot B = |A| |B| \cos \theta$$

where  $\theta$  is the angle between the vectors. This readily entails that

$$\frac{A \cdot B}{|A| |B|} = \cos \theta$$

or equivalently that

$$\theta = \arccos \left( \frac{A \cdot B}{|A| |B|} \right)$$

Now,  $A \cdot B = 4 \times 6 + 3 \times -3 = 24 - 9 = 15$  and  $|A| |B| = 5 \cdot 6.708 = 33.541$ .

Therefore,

$$\theta = \arccos \left( \frac{15}{33.541} \right) = \arccos (0.447) = 1.107$$

(7) Let  $\vec{v} = \left(\frac{1}{3}, \frac{2}{3}\right)$  be the vector of components. Find the components of the vector of module 5 whose direction and orientation (sentido) are those of the given vector.

Assume  $\vec{x} = (x_1, x_2)$  is of magnitude 5. Any vector whose direction and orientation are the same than those of  $\vec{v}$  is "a stretching" of  $\vec{v}$ . In other words, for  $\vec{x}$  to satisfy the requirements, we must have

$$\vec{x} = \lambda \vec{v} \quad (1)$$

for some  $\lambda \in \mathbb{R}$ . (Furthermore,  $\lambda > 0$  since otherwise orientation is not preserved.)

Now, from equation (1) follows that

$$\|\vec{x}\| = \lambda \|\vec{v}\| \quad (2)$$

since the magnitude of a scaled vector is the scaled magnitude of the vector. Equation (2) simplifies to

$$\|\vec{x}\| = \lambda \sqrt{1/9 + 2/3} = \frac{\lambda \sqrt{7}}{3} \quad (3)$$

From this readily follows that  $\frac{3}{\sqrt{7}} \|\vec{x}\| = \lambda$ . But it is a hypothesis that  $\|\vec{x}\| = 5$ . Therefore,

$$\lambda = \frac{3}{\sqrt{7}} \cdot 5 = \frac{15}{\sqrt{7}} \quad (4)$$

In other words,

$$\vec{x} = \frac{15}{\sqrt{7}} \vec{v} \quad (5)$$