Assuming a body b at time t is at one and only one place, and that the movement of b is continuous—i.e. moving from a to b entails moving through all intermediate points—, then there exists a mathematical function $x(t)$ which describes the movement of b.

If a body moves **linearly** (i.e. with uniform rectilinear movement), suffices to measure two points (t_1, x_1), (t_2, x_2) to describe its moevement. In particular, if

$$x(t) = at_1 + b \quad \text{and measures} \begin{cases} x_1 & = at_1 + b \\ x_2 & = at_2 + b \end{cases}$$

we have

$$a = \frac{x_2 - x_1}{t_2 - t_1} \; ; \; b = \frac{t_2 x_1 - t_1 x_2}{t_2 - t_1}$$

If a body moves parabolically, knowing that three points are sufficient to fully describe a parabole, we need only measure three instances (t_1, x_1), (t_2, x_2), (t_3, x_3)

and with a bit of algebra we can find the parameters a, b, c of $x(t) = ax^2 + bx + c$.

0.1 Distance and displacement

The **distance traveled** is the length of the path a body has traveled in a given time frame. The **displacement** is the measure of how much its position has modified respect to its initial position.

If $x = x(t)$ is the movement, then in the time interval $[t_1, t_2]$ its displacement is $\Delta x = x(t_2) - x(t_1) = x_2 - x_1$. Its distance traveled is denoted with d_{12}.

0.2 Velocity and derivatives

The velocity $\overline{v}$ of a body is the quotient between its displacement and the time frame of said displacement. If $x(t)$ is the movement function and $x_1 = x(t_1), x_2 = x(t_2)$, then the **median velocity** in $\Delta t = t_2 - t_1$ is

$$\overline{v}(t_1, t_2) = \overline{v}(t_1, \Delta t) = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

The unity of said average velocity is given by the unity of distance $[\ell]$ and the unity of time $[t]$:

$$[V] = \frac{[\ell]}{[t]}$$

For instance, m/s or km/h, etc.

Note. If $x(t_1) = x(t_2)$ (the body returns at its initial point), then $\overline{v} = 0$. But the body has moved. This reflects that the **median velocity** is not the average velocity, which is defined as d/t. The **median velocity** describes how much the position of the object has changed with respect to time!

Consider that:

$$
\begin{aligned}
x(t) &= c & \Rightarrow && \overline{v} = \frac{c-c}{t_2 - t_1} = 0 \\
x(t) &= at + b & \Rightarrow && \overline{v} = \frac{a(t_2 - t_1)}{t_2 - t_1} = a \\
x(t) &= ax_2 + bx + c & \Rightarrow && \overline{v} = (2t_1 + \Delta t) + b
\end{aligned}
$$

0.3 Instantaneous velocity and acceleration

Given $\Delta t = t_2 - t_1$, the instantaneous velocity is

$$v(t) := \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{\partial x(t)}{\partial t}$$

The **median acceleration** is

$$\overline{a}(t_1, \Delta t) := \frac{v(t_2) - v(t_1)}{\Delta t} = \frac{\Delta v}{\Delta t}$$

and the immediate acceleration is

$$a(t) := \lim_{\Delta t \to 0} \overline{a}(t, \Delta t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{\partial v(t)}{\partial t}$$

Using the definition of v as the derivative of the movement function with respect to time, we have

$$a = \frac{\partial v(t)}{\partial t} = \frac{\partial d}{\partial dt}\left(\frac{\partial x(t)}{\partial t}\right) = \frac{\partial^2 x(t)}{\partial t^2}$$

0.4 Piecewise acceleration

The movement function $x(t)$ is continuous and has a continuous, differentiable derivative $v(t)$. The velocity is continuous and differentiable, but its derivative $a(t)$ is not necessarily continuous nor differentiable.

For instance, we could have

$$a(t) = \begin{cases} 1\frac{m}{s^2} & t < 1s \\ 2\frac{m}{s^2} & t \geq 1s \end{cases}$$

Assuming we know $v(t=0) = 1\frac{m}{s}, x(t=2s) = 2m$, we can find the velocity

$$v(t) = \int a(t)\ dt = \begin{cases} 1\frac{m}{s^2}t + C_1 & t < 1s \\ 2\frac{m}{s^2}t + C_2 & t \geq 1s \end{cases}$$

Since $v(t=0) = 1\frac{m}{s}$, we have $C_1 = 1\frac{m}{s}$. And since the velocity function is **continuous**, in particular it is continuous for $t = 1s$, and this means

$$\lim_{t \to 1^- s} v(t)v(t) = \lim_{t \to 1^+ s} v(t) = \lim_{t \to 1} v(t)$$

Then

$$\begin{aligned} \lim_{t \to 1^- s} v(t) = \lim_{t \to 1^+ s} &\iff 1\frac{m}{s^2}1s + 1\frac{m}{s} = 2\frac{m}{s^2}1s + C_2 \\ &\iff C_2 = 0 \end{aligned}$$

Having found the velocity, we could find the movement function by integrating the velocity.