Let $X(t), Y(t)$ denote two EEG signals viewed as random variables. Their cross-correlation is defined as

$$(X \star Y)(\tau) = \int_{-\infty}^{\infty} X(t)Y(t + \tau) \, dt$$

As a simple note, we observe that for $\tau = 0$ their cross correlation becomes $\int_{-\infty}^{\infty} X(t)Y(t) \, dt = \mathbb{E}\left[X \cdot Y\right]$. If $X, Y$ are zero-meaned then this entails $(X \star Y)(0) \propto \text{Cov}(X, Y)$. In short, the zero-lag cross-correlation of two zero-mean signals is proportional to their covariance.

For discrete signals, like digital EEGs, the definition is analogous, though this time we normalize:

$$C(t) = \frac{1}{(N - m)\sigma_X \sigma_Y} \sum_{j=0}^{N-(m+1)} \hat{X}_j \hat{Y}_{j+m}, \qquad t = m\tau$$

where

- $\hat{X}_i$ is the fluctuation of $X_i$ around its mean $(X_i - \bar{X})$;

- $\tau$ is the constant sampling time, i.e. the inverse of the sampling rate.

- $m$ is the discrete lag, computed as a function of the continuous lag $t$.

  Consider for example a sampling rate of 500Hz, which gives a sampling time of 0.002 seconds per sample. If we wish to compute the discrete cross-correlation with a time phase of two seconds, then 2sec $= m \cdot 0.002$sec/sample giving $m = 1000$ samples. In short, a two second shift would correspond to a discrete shift of 1000 samples.

To further understand the concept which I wish to present, let us pose a few questions beforehand.

(1) First, let us ask how the existence of certain *common* frequency in both signals affects their cross-correlation function. Say, for the sake of the example, that both signals are strongly composed of a 10Hz frequency oscillation, meaning that their power spectrums show «high» energy at said frequency.

When evaluating their cross-correlation, we effectively «slide» signal $Y$ across $X$ checking for similarity; and since by assumption both contain a periodic 10Hz oscillation, they will align and misalign cyclically at the exact same rate. Importantly, even if both signals contain noise, random noise in $X$ will not generally correlate with random noise in $Y$, so the 10Hz oscilation will be exposed in the correlation function.

*Insight from question (1):* If $X$ and $Y$ share a frequency $f$, the cross-correlation function also oscilates at frequency $f$.

(2) Now we deepen the question. We may assume that both signals share a frequency $f$, but their oscilations in that frequency may or may not be aligned. How does this affect the cross-correlation function?

It should be clear that if the oscilations in this frequency are perfectly aligned (i.e. shifted by an angle of $\phi = 0$), the maximum correlation occurs at lag $\tau = 0$. If the oscilations are out of phase, the maximum correlation will occur at the lag $\tau$ which compensates for the delay.

*Insight from question (2):* Aligned, shared oscilations peak at $\tau = 0$.

With these observations in hand, understanding our topic of interest will be simpler. Let us proceed: we'll get back to these insights shortly.

We now define $S_{XY}$ as the power spectrum of the cross-correlation function of two signals $X$ and $Y$. This is known as the cross-spectral density or simply cross-spectrum:

$$S_{XY}(\nu) = \int_{-\infty}^{\infty} C(t)e^{-i2\pi\nu t}\, dt$$

To further disect the meaning of the power spectrum let us decompose it a bit. Applying Euler's formula ($e^{-i\theta} = \cos\theta - i\sin\theta$), we obtain:

$$\begin{aligned}
S_{XY}(\nu) &= \int_{-\infty}^{\infty} C(t)\left[\cos(2\pi\nu t) - i\sin(2\pi\nu t)\right]\, dt \\
&= \underbrace{\int_{-\infty}^{\infty} C(t)\cos(2\pi\nu t)\, dt}_{\text{Real Part: Co-spectrum}} - i\underbrace{\int_{-\infty}^{\infty} C(t)\sin(2\pi\nu t)\, dt}_{\text{Imaginary Part: Quad-spectrum}}
\end{aligned}$$

It is time to tie our previous insights with this expression. We can view the formula above as asking, at each frequency component, how the cross-correlation behaves relative to cosine and sine waves. Importantly, the distinction is not simply between "synchrony" and "asynchrony," but between *alignment* and *orthogonality*. To make this clearer, note that for each frequency component we essentially have three cases:

- **In-Phase ($\phi = 0$):** The signals are perfectly synchronous. The cross-correlation peaks at zero and resembles a **positive cosine**. Its Fourier transform is purely real (only co-spectrum) and positive.

- **Anti-Phase ($\phi = \pi$):** The signals are perfectly oppositional (one peaks when the other troughs). The cross-correlation is a **negative cosine**. Here, again, the activity appears only in the co-spectrum (real part), but this time with a negative sign.

- **Quadrature ($\phi = \pi/2$):** The signals are shifted by a quarter cycle and thus are perfectly asynchronous. The cross-correlation resembles a **sine wave** (it is zero at lag 0). Its Fourier transform is purely imaginary.

What am I trying to say? That the co-spectrum measures the linear relationship (whether parallel or anti-parallel) of both signals across the range of frequencies. The quad-spectrum measures the orthogonal relationship.

In EEG analysis, we are often interested specifically in *zero-lag* synchrony. This is captured entirely by the real part. Therefore, the «cross-spectrum» formula in these contexts often simplifies to the co-spectrum:

$$\text{Co}_{XY}(v) \approx \text{Re}\left[ \sum_j C(t_j) e^{-i2\pi v t_j} \right] = \sum_j C(t_j) \cos(2\pi v t_j)$$

We discard the imaginary part not because it is error, but because it represents time-delayed (orthogonal) interactions rather than zero-lag synchronization.

Some methodologies define a specific synchronization metric, $\mu_0^{XY}(v)$, which not only isolates the co-spectrum but squares it. Squaring transforms the amplitude into power and ensures that both in-phase (positive) and anti-phase (negative) synchrony contribute positively to the strength of the connection:

$$\mu_0^{XY}(v) = \left| \tau \sum_j c(t_j) \cos(2\pi v t_j) \right|^2$$