Let $G = (V, E)$ a bipartite graph with parts X and Y, and let $Z \in \{X, Y\}$. We want to prove that there is a complete matching from X to Y iff $\forall S \subseteq Z : |S| \leq |\Gamma(S)|$.

(⇒) Assume there is a complete matching from X to Y. Then there is an injective function $f : S \subseteq X \to Y$ s.t. any $x \in S$ is associated to a distinct $y \in Y$ and $|f(S)| = |S|$. Since G is bipartite, any $f(S) \subseteq \Gamma(S)$. Then $|S| \leq |\Gamma(S)|$. The proof is equivalent if we take $S \subseteq Y$.

(⇐) We will use the matching algorithm to construct our proof. Take $S \subseteq X$ and assume $|S| \leq |\Gamma(S)|$. Assume that, after running the algorithm, an incomplete matching is found.

Let $S_0 \subseteq S$ be the set of rows tagged at the end of the algorithm. Then all its neighbors $T_1 = \Gamma(S_0)$ were matched with a non-intersecting set of rows S_1. On its turn, rows in S_1 may have neighbors that are not matched, which inspires the definition of $T_2 = \Gamma(S_1) - T_1$. In general, we define

$$
\begin{aligned}
S_i &= \Gamma(T_i) \\
T_{i+1} &= \Gamma(S_i) - \bigcup_{j=0}^{i} T_i
\end{aligned}
$$

This sequence of intermeddling sets, $S_k, T_k, S_{k-1}, T_{k-1}, \ldots, S_1, T_1, S_0$, describes the run of the algorithm. Now, if the algorithm on S_0, it means there were no available neighbors for S_0; i.e. no way to construct the next T_i.

Observe that each row matches a unique column, and then $|S_i| = |T_i|$ for any i. Furthermore, the S_i are disjoint and the T_i are disjoint. Then

$$
\begin{aligned}
|S| &= \sum |S_i| \\
&= |S_0| + \sum |T_i| \\
&= |S_0| + |T|
\end{aligned}
$$

It readily follows that $|S| > |T|$, because by assumption $S_0 \neq \emptyset$. Now, we will prove $T = \Gamma(S)$.

(1) T are the labeled columns, and each labeled column is labeled by a neighboring row. So any $t \in T$ has at least one neighbor in$S$. This means $T \subseteq \Gamma(S)$.

(2) Now assume there is some $y \in \Gamma(S)$ s.t. $y \notin T$. Then y was not labeled. But since $y \in \Gamma(S)$ it could have been labeled by a neighboring row. Then y would have been labeled. Then $\Gamma(S) \subseteq T$.

Now that we know $\Gamma(S) = T$, we have $|S| > |\Gamma(S)|$. This contradicts our assumption. Then there must be a complete matching.