1 Info

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Temas:

- Cinemática y dinámica (mecánica)
- Campos eléctricos y magnéticos
- Circuitos
- Termodinámica

2 Measurements and magnitudes

Measurements seek to compare a prediction with an observation, so as to test a hypothesis. A magnitude is a number accompanied by a unit. Some magnitudes are:

- Length, measured in meters (m)
- Time, measured in seconds (s)
- Mass, measusred in kilograms (kg)
- Current, measured in ampers (A)
- Temperature, measured in kelvins (k)
- Matter, measured in moles (mol)

We consider 10^3 (e.g. kilometer) and 10^{-3} (e.g. milimiters) to be within human scale. We call mass, seconds and kilograms the *mechanical units*. We define the *force unit*, or Newton, as

$$[F] = N = \text{kg}\frac{m}{s^2}$$

and the Pascal unit as

$$[P] = \text{Pa} = \frac{N}{m^2}$$

We use scientific notation and terms which express quantities as powers of ten. For instance, 10^{12} is the tera, 10^3 the giga, etc.

The magnitudes hereby described are suited for algebraic manipulation. For instance, $m \times m = m^2$, and $s \times \frac{m}{s} = m$.

3 Vectors

Vectors are used to express position, displacement, velocity, force, acceleration, fields, etc. A vector $\overrightarrow{A}$ (or sometimes $\overrightarrow{a}$) in the general sense has a direction (line), an orientation, and a length (or magnitude). A vector also has an application point, which denotes the point of origin of the vector. When saying $\overrightarrow{a} = \overrightarrow{b}$, we mean that $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in direction, magnitude and orientation, irrespective of their application point.

The scalar product is defined as the usual mapping in the space $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$. Intuitively, the scalar product $\lambda\overrightarrow{a}$ ”streches” or ”shrinks” a vector, depending on wheter $|\lambda| < 1$ or not, and the positivty or negativity of λ determines whether the vector inverts its direction or not. In general, $\left|\lambda\overrightarrow{a}\right| = |\lambda| \left|\overrightarrow{a}\right|$.

The sum of vectors, $\overrightarrow{a} + \overrightarrow{b}$, is a mapping $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$. As usual, and in a graphical sense, the sum corresponds to the application of the parallelogram rule.

Parallelogram rule. Make $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in their point of application. From the tip of $\overrightarrow{a}$, draw a copy of $\overrightarrow{b}$, and from the tip of $\overrightarrow{b}$ a copy of $\overrightarrow{a}$. The corner of the thus generated parallelogram is the tip of $\overrightarrow{a} + \overrightarrow{b}$.

Alternatively, from the tip of $\overrightarrow{a}$ write $\overrightarrow{b}$. Then $\overrightarrow{a} + \overrightarrow{b}$ is the vector which goes from the point of application of $\overrightarrow{a}$ to the tip of $\overrightarrow{b}$.

The sum of vectors is commutative, associative, and distributive with respect to scalar product.

If $\overrightarrow{A}$ is a vector, we use A_x and A_y to denote the projection of the vector over the axis x or y, respectively. Using A_x and A_y one forms a rectangular triangle with sides A_x, A_y and a hypotenuse of length $\left|\overrightarrow{A}\right|$.

Let θ be the angle formed by $\overrightarrow{A}$ with the x-axis. Then, using trigonometry,

$$\cos\theta = \frac{A_x}{\left|\overrightarrow{A}\right|}, \qquad \sin\theta = \frac{A_y}{\left|\overrightarrow{A}\right|}$$

from which one can find A_x, A_y assuming one knows θ. From this follows that $\left|\overrightarrow{A}\right|$ and θ fully determine all the information about the vector, insofar as the allow us to determine A_x, A_y. Conversely, knowing A_x and A_y is also sufficient to determine $\overrightarrow{A}$, insofar as

$$\left|\overrightarrow{A}\right| = \sqrt{A_x^2 + A_y^2}, \qquad \frac{A_y}{A_x} = \frac{\left|\overrightarrow{A}\right|\sin\theta}{\left|\overrightarrow{A}\right|\cos\theta} = \tan\theta \Rightarrow \theta = \arctan\left(\frac{A_y}{A_x}\right)$$

As convention, we use $\hat{\imath}$ to denote the versor (vector of length 1) with direction parallel to the x-axis, and $\hat{\jmath}$ the versor with direction parallel to the y-axis.

Notice that, for any vector $\overrightarrow{A}$, A_x is $\hat{\imath}$ times A_x, and A_y is $\hat{\jmath}$ times A_y, which means

$$\overrightarrow{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

When writing $\overrightarrow{A}$ in this way, we say we write it in term of its components x, y. In terms of linear algebra, it's not hard to see that we are simply expressing that $\hat{\imath}, \hat{\jmath}$ form a basis of $\mathbb{R}^2$. Thus, it is equivalent to write

$$A_x = \left|\overrightarrow{A}\right|\cos\theta, \qquad A_y = \left|\overrightarrow{A}\right|\sin\theta$$

and

$$\overrightarrow{A} = \left|\overrightarrow{A}\right|\left(\cos\theta\ \hat{\imath} + \sin\theta\ \hat{\jmath}\right)$$

From this follows as well that

$$\begin{aligned}\overrightarrow{A} + \overrightarrow{B} &= \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath}\right)\\ &= \hat{\imath}\left(A_x + B_x\right) + \hat{\jmath}\left(A_y + B_y\right)\end{aligned}$$

which means the sum of vectors has as components the sum of the components.

The scalar product of two vectors, $\overrightarrow{A} \cdot \overrightarrow{B}$, is a scalar defined as

$$\overrightarrow{A} \cdot \overrightarrow{B} = \left|\overrightarrow{A}\right| \left|\overrightarrow{B}\right| \cos\theta$$

where θ is the angle formed by the two vectors. The scalar product is positive if $\cos\theta$ is positive, which occurs for $0 < \theta \leq 90$. It is negative if $\cos\theta$ is negative, i.e. if $90 < \theta \leq 180$. Clearly, $\overrightarrow{A} \cdot \overrightarrow{B} = 0 \iff \theta = 90$.

In general, from the definition follows that

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y$$

The vectorial product $\overrightarrow{A} \times \overrightarrow{B}$ is a vector perpendicular to the plane formed by $\overrightarrow{A}$ and $\overrightarrow{B}$. Its module is $\left|\overrightarrow{A}\right| \left|\overrightarrow{B}\right| \sin\theta$, and its direction is given by what's called the right-hand rule.

3.1 Excercises

(2) Sean los vectores $\overrightarrow{A} = 2\hat{\imath} + 3\hat{\jmath}$ $\overrightarrow{B} = 4\hat{\imath} - 2\hat{\jmath}$ y $\overrightarrow{C} = -\hat{\imath} + \hat{\jmath}$. Determinar la magnitud y el ángulo (representación polar) de los vectores resultantes $\overrightarrow{D} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$ y $\overrightarrow{E} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$. Resolver analítica y gráficamente.

(Analytical solution.) We'll use A_x, A_y to denote the components of the vector $\overrightarrow{A}$, and same for all other vectors. We know the components of $\overrightarrow{D}$ are

$$D_x = A_x + B_x + C_x = 2 + 4 - 1 = 5, \qquad D_y = 3 - 2 + 1 = 2$$

from which readily follows that $|D| = \sqrt{5^2 + 2^2} = \sqrt{29} \approx 5.385$. Similarly,

$$E_x = 2 + 4 + 1 = 7, \qquad E_y = 3 - 2 - 1 = 0$$

from which follows that $|E| = \sqrt{7^2} = 7$.

Now, we must recall that

$$\theta_{\overrightarrow{Z}} = \arctan\left(\frac{Z_y}{Z_x}\right)$$

for any $\overrightarrow{Z}$.

We need not memorize this: it is trigonometrically clear that $Z_x = \cos\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$ and $Z_y = \sin\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$, and therefore

$$\frac{Z_y}{Z_x} = \tan\theta$$

And arctan is the inverse of tan. Anyhow, for $\overrightarrow{E}$ and $\overrightarrow{D}$ we have

$$\theta_{\overrightarrow{E}} = \arctan\left(\frac{E_y}{E_x}\right) = \arctan(0) = 0$$

$$\theta_{\overrightarrow{D}} = \arctan\left(\frac{D_y}{D_x}\right) = \arctan\left(\frac{2}{5}\right) \approx 0.38$$

(3) Can two vectors of different magnitud be combined and yield zero? What about three?

The zero vector is the only vector with magnitude zero. Let $\overrightarrow{A}, \overrightarrow{B}$ arbitrary vectors. Then

$$\left|\overrightarrow{A} + \overrightarrow{B}\right| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

which is zero if and only if

$$(A_x + B_x)^2 + (A_y + B_y)^2 = 0$$

This only holds if $A_x + B_x = A_y + B_y = 0$. But

$$A_x + B_x = 0 \Rightarrow A_x = -B_x, \qquad A_y + B_y = 0 \Rightarrow A_y = -B_y$$

But then

$$|A| = \sqrt{A_x^2 + A_y^2} = \sqrt{(-B_x)^2 + (-B_y)^2} = \sqrt{B_x^2 + B_y^2} = |B|$$

$\therefore \ \left|\overrightarrow{A} + \overrightarrow{B}\right| = 0 \iff \left|\overrightarrow{A}\right| = \left|\overrightarrow{B}\right|$.

It is simple to see that three vectors of different magnitude can add to zero.

Assume $A + B + C = 2\hat{\imath} + \hat{\jmath}$ and $A = 6\hat{\imath} - 3\hat{\jmath}, B = 2\hat{\imath} + 5\hat{\jmath}$. Find the components of C. Solve analytically and graphically.

We know

$$6 + 2 + C_x = 2, \qquad -3 + 5 + C_y = 1$$

from which follows that $C_x = -6, C_y = -1$.

(5) A and B have a magnitud of $3m, 4m$ respectively. The angle between them is $\theta = 30$ degrees. Find their scalar product.

Their scalar product is

$$(|B| \cos \theta) |A|$$

Recall that

$$\text{Angle in degrees} = \text{Angle in radians} \cdot \frac{180}{\pi}$$

Thus, thirty degrees equates to $30 \frac{\pi}{180} \approx 0.523$ radians. Then the scalar product is

$$4 \cos(0.523) \times 3 \approx 10.395$$

(6) Find the angle between $A = 4\hat{i} + 3\hat{j}$ and $B = 6\hat{i} - 3\hat{j}$.

Recall that

$$A \cdot B = |A|\,|B| \cos\theta$$

where θ is the angle between the vectors. This readily entails that

$$\frac{A \cdot B}{|A|\,|B|} = \cos\theta$$

or equivalently that

$$\theta = \arccos\left(\frac{A \cdot B}{|A|\,|B|}\right)$$

Now, $A \cdot B = 4 \times 6 + 3 \times -3 = 24 - 9 = 15$ and $|A|\,|B| = 5 \cdot 6.708 = 33.541$.

Therefore,

$$\theta = \arccos\left(\frac{15}{33.541}\right) = \arccos(0.447) = 1.107$$

(7) Let $\overrightarrow{v} = \left(\frac{1}{3}, \frac{2}{3}\right)$ be the vector of components. Find the components of the vector of module 5 whose direction and orientation (sentido) are those of the given vector.

Assume $\overrightarrow{x} = (x_1, x_2)$ is of magnitude 5. Any vector whose direction and orientation are the same than those of $\vec{v}$ is "a stretching" of $\vec{v}$. In other words, for $\vec{x}$ to satisfy the requirements, we must have

$$\vec{x} = \lambda\vec{y} \tag{1}$$

for some $\lambda \in \mathbb{R}$. (Furthermore, $\lambda > 0$ since otherwise orientation is not preserved.)

Now, from equation (1) follows that

$$\|\vec{x}\| = \lambda\|\vec{y}\| \tag{2}$$

since the magnitude of a scaled vector is the scaled magnitude of the vector. Equation (2) simplifies to

$$\|\vec{x}\| = \lambda\sqrt{1/9 + 4/9} = \frac{\lambda\sqrt{5}}{3} \tag{3}$$

From this readily follows that $\frac{3}{\sqrt{5}}\|\vec{x}\| = \lambda$. But it is a hypothesis that $\|\vec{x}\| = 5$. Therefore,

$$\lambda = \frac{3}{\sqrt{5}} \cdot 5 = \frac{15}{\sqrt{5}} \tag{4}$$

In other words,

$$\vec{x} = \frac{15}{\sqrt{5}}\vec{v} \tag{5}$$

which is ugly but can be simplified.

(8) Write the expression of the vector product $\vec{c} = \vec{u} \times \vec{v}$ in the following cases:

1. $\vec{u}, \vec{v}$ are coplanar. Provide a graphical interpretation.
2. $\vec{u} = 2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $\vec{v} = -3\hat{\imath} + \hat{\jmath} + 2\hat{k}$. Find the module of the resulting vector $\vec{c}$ in two different ways.

(1) Two vectors are coplanar if there is a plane which contains them both. Since the vector product $\vec{u} \times \vec{v}$ is a vector orthogonal to both $\vec{u}$ and $\vec{v}$

(12) Un avión vuela 200 km hacia el NE en una dirección que forma un ángulo de 30 hacia el este de la dirección norte. En ese punto cambia su dirección de vuelo hacia el NO. En esta dirección vuela 60 km formando un ángulo de 45 con la direccióon norte.

(a) Calcular la máxima distancia hacia el este del punto de partida a la que llegó el avión.

(b) Calcular la máxima distancia hacia el norte del punto de partida a la que llegó el avión.

(c) Calcular la distancia a la que se encuentra el avión del punto de partida al cabo de su recorrido.

(d) Determinar vectorialmente el camino que debería hacer para volver al punto de partida. Resolver gráfica y analíticamente.

Sea $\vec{A}$ el vector que describe el primer recorrido, $\vec{B}$ el vector que describe el segundo recorrido. Al final del problema, el avión se encuentra en la posición indicada por $\vec{A} + \vec{B}$.

Como $\vec{A}$ describe un movimiento con un ángulo de $\theta = 60$ grados $(90-30)$ respecto al eje y (norte), y una magnitud de 200km, podemos determinarlo recordando que

$$A_x = \left|\vec{A}\right| \cos\theta, \qquad A_y = \left|\vec{A}\right| \sin\theta$$

En radianes, $\theta = \frac{\pi}{180} \times 60 = 1.047$

$$A_x = 200 \times \cos(1.047) = 100.034, \qquad A_y = 22 \times \sin(1.047) = 173.185$$

En conclusión, $\vec{A} = 100.034\hat{\imath} + 173.185\hat{\jmath}$. Mismo razonamiento nos da que el ángulo del segundo vector es de 130 grados, lo cual en radianes nos da $\alpha = \pi/180 \times 130 = 2.268$. Por ende,

$$B_x = 60\cos(2.268) = -38.524, \qquad B_y = 60\sin(2.268) = 45.998$$

Es decir que $\vec{B} = -38.524\hat{\imath} + 45.998\hat{\jmath}$. De esto se sigue que $\vec{C} = \vec{A} + \vec{B} = (61.51, 219.183)$.

(a) Claramente, es la coordenada x del vector $\vec{A}$, 100.034.

(b) Claramente, es la coordenada y del vector $\vec{C}$: 218.183.

(c) Claramente, es la magnitud de $\vec{C}$, es decir $\|\vec{C}\| = \sqrt{61.51^2 + 219.183^2} = 227.65$.

(d) El camino para volver es dado por $\vec{C} \times (-1)$.

4 Cynematics

4.1 Unidimensional movement

The study of movement requires two variables: position (x, in units of length) and time (t, in seconds). We begin our study with unidimensional movement, i.e. movement which occurs along a single axis.

Experimentally, a way to study unidimensional movement could consist in taking a sequence of photographs (from the same position and angle) of the moving object at times $t_1, \ldots, t_n$. Some coordinate system must be imposed upon the space along which the object moves, e.g. setting an axis with origin at the initial position of the object, the same direction as the movement of the object, and some appropriate units. The photographs would then provide a sequence of positions $x_1, \ldots, x_n$.

Clearly, $\{t_n\}, \{x_n\}$ could be understood as defining a discrete function $\varphi(n)$, which on its turn might be interpolated to obtain a continuous approximation $\phi(t)$. To the limit, the continuous approximation converges to what we call a movement function.

Movement function. A movement function $x(t)$ is a continuous, smooth function.

Examples. $x(t) = c$ (reposo), $x(t) = at + b$ (MRU), $x(t) = at^2 + bt + c$ (MRUV).

4.2 Coincidence, displacement, temporal intervals

If A, B are objects with movement functions $x_A(t), x_B(t)$, we say A, B coincide (se encuentran) when $x_A(t) = x_B(t)$.

We define displacement (desplazamiento) (relative to positions x_1, x_2) as $\Delta x = x_2 - x_1$. Notice that Δx is not the same as distance: if one travels from A to B and then to B from A, the distance traveled is to times the distance from A to B, but $\Delta x = 0$.

We also define a temporal interval, relative to two times t_1, t_2, as $\Delta t = t_2 - t_1$, where $t_2 > t_1$.

4.3 Velocity

We define *median velocity* (velocidad media) as

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{1}$$

where $x_2 = x(t_2), x_1 = x(t_1)$. Clearly, $\overline{v}$ is the slope of the line which intersects $x(t)$ at points t_1, t_2. The sign of $\overline{v}$ then determines the direction (sentido) of movement. The unit of $\overline{v}$ is then L/T (length over time), for instance kilometers per hour. Median velocity indicates the rate of change of distance in time.

Clearly, an object in reposo has a median velocity of zero. An object with movement function $x(t) = at + b$ (MRU) has median velocity a. The case of interest is an object with a quadratic movement function (MRUV).

If $x(t) = at^2 + bt + c$, let m the midpoint of the quadratic expression and take $t_1 = m - c, t_2 = m + c$ with $c > 0$. Clearly, the median velocity from t_1 to m is negative, that from m to t_1 is positive, and that from t_1 to t_2 is zero. This is sufficient to suggest that median velocity does not clearly express the nature of the movement.

For that reason, the length Δ of the interval $[t_1, t_2]$ might be reduced in the limit to zero, so that we get an accurate notion of the instantaneous (or close to instantaneous) change of direction. Needless to say, the limit converges to the slope of the line tangent to $(t_1, x(t_1))$, i.e. the derivative of $x(t)$ at t_1. Thus, we obtain the definition of instantaneous velocity, usually called simply velocity:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = v'(t) \tag{2}$$

Again, $[v(t)] = \frac{L}{T}$. Quite clearly, $v(t) = \overline{v}$ for constant and linear functions, but for the quadratic function $x(t)$ we have

$$x(t) = at^2 + bt + c, \qquad v(t) = 2at + b$$

4.4 Excercises

(1) Consider

$$x(t) = 1\left[\frac{m}{s^2}\right]t^2 - 3\left[\frac{m}{s}\right]t$$

the movement function of a body travelling in a straight line, with x in meters and t in seconds.

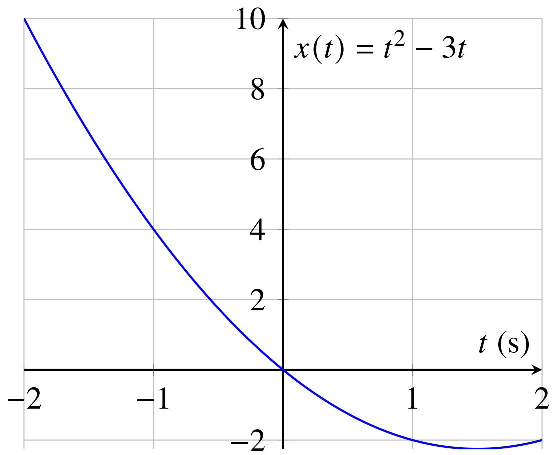
(a) Plot $x(t)$

(b) Determine analytically the median velocity in $[-1, 5]$, $[-1, 8]$, $[-1, 0.9]$, $[-1, 0.99]$, $[-1, 0.999]$.

(c) Let $\Delta t_n = t_n - t_0$ with $\{t_n\} = \{-1, 5, 4, 1, -0.5, -0.8, -0.9, -0.99, -0.999\}$ all measured in seconds. To what value does the median velocity of the object converge as t_n decreases in the interval $[-1, -1 + \Delta t_n]$? What is the geometrical interpretation of this result?

(d) Find the equation for the line tangent to $x(t)$ at $t = -1s$.

(a) Notice that since t is in seconds, $\frac{mt^2}{s^2}$ correctly expresses a quantity in meters, and so does $\frac{mt}{s}$. So we will from now on write simply $x(t) = t^2 - 3t$, understanding that it is a mapping from time in seconds to meters.



(b) The median velocity of an object in the time interval $[t_a, t_b]$ was given by

$$\frac{\Delta x}{\Delta t} = \frac{x(t_b) - x(t_a)}{t_b - t_a} \tag{1}$$

So excercise (b) is as simple as plugging in the corresponding values into equation (1) and I skip it.

(c) Let t be an arbitrary value. Then by definition of $\frac{dx}{dt}$,

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

the derivative of $x(t)$ at time t. In particular, the limit whose convergence we are asked to study is nothing but the limit above with $t = -1$:

$$\lim_{\Delta t \to 0} \frac{x(-1+\Delta t) - x(-1)}{\Delta t} = x'(-1)$$

So suffices to observe that $x'(t) = 2t - 3$ and $x'(-1) = -5$. In conclusion, the object at time $t = -1$ travels at an instantaneous velocity of -5 meters per second.

(d) The line $\ell(t) = at + b$ tangent to $x(t)$ at $t = -1s$ has slope $a = -5$ and crosses through the point $(-1, 4)$. So we must have $-5(-1) + b = 4 \iff b = 4 - 5 = -1$. So the line is $\ell(t) = -5t - 1$.

(4) Answer the questions.

(a) Can an object have null velocity and yet possess acceleration?

Let $x(t)$ the describe the movement of the object and $v(t) = x'(t)$ its velocity, both as a function of time. Assume for an arbitrary t_0 that $v(t) = 0$. It is very much possible that $v'(t_0) \neq 0$.

Consider, for instance, that $v(t)$ is linear and non-constant, making $v'(t) = a$ a non-null constant. Then there exists a unique root r s.t. $v(r) = 0$, but independently of this fact $v'(r) = a \neq 0$.

Physically, it should be clear that if a non-moving object could not possess acceleration, then it would be impossible for it to pass from a still to a moving state. So, at least at the intutition level, this *reductio ad absurdum* suffices.

(b) Can a moving object have a null displacement in a given interval and yet non-null velocity?

Naturally. Take as example an object moving in circles at a constant, non-null velocity v, and assume it travels a full circle in t seconds. Then all of the intervals in $\{[t_0, t_0 + tk] : k \in \mathbb{N}\}$ are such that they give null displacements. Yet the object *is* moving.

(c) Can an object have an east-bound velocity of while its acceleration is west-bound?

Informally, this is quite clearly the case, insofar as any positively-moving object whose velocity decreases must have a negative acceleration.

(d) Consider an object moving on a straight line, with the east being the positive direction, under a velocity of $v(t) = 20\text{ms}^{-1} - 2\text{ms}^{-2}t$. For $t = 0\text{s}, t = 1\text{s}$, what is the situation?

Its velocity is clearly positive in both cases (20 and 18), evidently decreasing, which points out the fact that its acceleration is negative (-2).

(e) A ball is thrown vertically. What do the *signs* of velocity and acceleration look like as the object ascends, and what does that mean? And when the object descends? What happens at the highest point?

Clearly, its velocity is positive during the ascending phase, and negative during the descending phase. At the highest point, the velocity will be exactly zero.

Conversely, acceleration is always negative due to the force excercised by gravity on the ball.

It is the fact that acceleration is constantly negative what causes the ball not only to lose velocity as it goes up until it begins to fall again, but to then fall more and more rapidly as time goes by.