1 Info

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Temas:

- Cinemática y dinámica (mecánica)
- Campos eléctricos y magnéticos
- Circuitos
- Termodinámica

2 Measurements and magnitudes

Measurements seek to compare a prediction with an observation, so as to test a hypothesis. A magnitude is a number accompanied by a unit. Some magnitudes are:

- Length, measured in meters (m)
- Time, measured in seconds (s)
- Mass, measusred in kilograms (kg)
- Current, measured in ampers (A)
- Temperature, measured in kelvins (k)
- Matter, measured in moles (mol)

We consider 10^3 (e.g. kilometer) and 10^{-3} (e.g. milimiters) to be within human scale. We call mass, seconds and kilograms the *mechanical units*. We define the *force unit*, or Newton, as

$$[F] = N = \text{kg}\frac{m}{s^2}$$

and the Pascal unit as

$$[P] = \text{Pa} = \frac{N}{m^2}$$

We use scientific notation and terms which express quantities as powers of ten. For instance, 10^{12} is the tera, 10^3 the giga, etc.

The magnitudes hereby described are suited for algebraic manipulation. For instance, $m \times m = m^2$, and $s \times \frac{m}{s} = m$.

3 Vectors

Vectors are used to express position, displacement, velocity, force, acceleration, fields, etc. A vector $\overrightarrow{A}$ (or sometimes $\overrightarrow{a}$) in the general sense has a direction (line), an orientation, and a length (or magnitude). A vector also has an application point, which denotes the point of origin of the vector. When saying $\overrightarrow{a} = \overrightarrow{b}$, we mean that $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in direction, magnitude and orientation, irrespective of their application point.

The scalar product is defined as the usual mapping in the space $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$. Intuitively, the scalar product $\lambda \overrightarrow{a}$ ”streches” or ”shrinks” a vector, depending on wheter $|\lambda| < 1$ or not, and the positivty or negativity of λ determines whether the vector inverts its direction or not. In general, $\left|\lambda \overrightarrow{a}\right| = |\lambda| \left|\overrightarrow{a}\right|$.

The sum of vectors, $\overrightarrow{a} + \overrightarrow{b}$, is a mapping $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$. As usual, and in a graphical sense, the sum corresponds to the application of the parallelogram rule.

Parallelogram rule. Make $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in their point of application. From the tip of $\overrightarrow{a}$, draw a copy of $\overrightarrow{b}$, and from the tip of $\overrightarrow{b}$ a copy of $\overrightarrow{a}$. The corner of the thus generated parallelogram is the tip of $\overrightarrow{a} + \overrightarrow{b}$.

Alternatively, from the tip of $\overrightarrow{a}$ write $\overrightarrow{b}$. Then $\overrightarrow{a} + \overrightarrow{b}$ is the vector which goes from the point of application of $\overrightarrow{a}$ to the tip of $\overrightarrow{b}$.

The sum of vectors is commutative, associative, and distributive with respect to scalar product.

If $\overrightarrow{A}$ is a vector, we use A_x and A_y to denote the projection of the vector over the axis x or y, respectively. Using A_x and A_y one forms a rectangular triangle with sides A_x, A_y and a hypotenuse of length $\left|\overrightarrow{A}\right|$.

Let θ be the angle formed by $\overrightarrow{A}$ with the x-axis. Then, using trigonometry,

$$\cos\theta = \frac{A_x}{\left|\overrightarrow{A}\right|}, \qquad \sin\theta = \frac{A_y}{\left|\overrightarrow{A}\right|}$$

from which one can find A_x, A_y assuming one knows θ. From this follows that $\left|\overrightarrow{A}\right|$ and θ fully determine all the information about the vector, insofar as the allow us to determine A_x, A_y. Conversely, knowing A_x and A_y is also sufficient to determine $\overrightarrow{A}$, insofar as

$$\left|\overrightarrow{A}\right| = \sqrt{A_x^2 + A_y^2}, \qquad \frac{A_y}{A_x} = \frac{\left|\overrightarrow{A}\right| \sin\theta}{\left|\overrightarrow{A}\right| \cos\theta} = \tan\theta \Rightarrow \theta = \arctan\left(\frac{A_y}{A_x}\right)$$

As convention, we use $\hat{\imath}$ to denote the versor (vector of length 1) with direction parallel to the x-axis, and $\hat{\jmath}$ the versor with direction parallel to the y-axis.

Notice that, for any vector $\overrightarrow{A}$, A_x is $\hat{\imath}$ times A_x, and A_y is $\hat{\jmath}$ times A_y, which means

$$\overrightarrow{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

When writing $\overrightarrow{A}$ in this way, we say we write it in term of its components x, y. In terms of linear algebra, it's not hard to see that we are simply expressing that $\hat{\imath}, \hat{\jmath}$ form a basis of $\mathbb{R}^2$. Thus, it is equivalent to write

$$A_x = \left|\overrightarrow{A}\right| \cos\theta, \qquad A_y = \left|\overrightarrow{A}\right| \sin\theta$$

and

$$\overrightarrow{A} = \left|\overrightarrow{A}\right| \left(\cos\theta\ \hat{\imath} + \sin\theta\ \hat{\jmath}\right)$$

From this follows as well that

$$\begin{aligned} \overrightarrow{A} + \overrightarrow{B} &= \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath}\right) \\ &= \hat{\imath}\left(A_x + B_x\right) + \hat{\jmath}\left(A_y + B_y\right) \end{aligned}$$

which means the sum of vectors has as components the sum of the components.

The scalar product of two vectors, $\overrightarrow{A} \cdot \overrightarrow{B}$, is a scalar defined as

$$\overrightarrow{A} \cdot \overrightarrow{B} = \left|\overrightarrow{A}\right|\left|\overrightarrow{B}\right| \cos\theta$$

where θ is the angle formed by the two vectors. The scalar product is positive if $\cos\theta$ is positive, which occurs for $0 < \theta \leq 90$. It is negative if $\cos\theta$ is negative, i.e. if $90 < \theta \leq 180$. Clearly, $\overrightarrow{A} \cdot \overrightarrow{B} = 0 \iff \theta = 90$.

In general, from the definition follows that

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y$$

The vectorial product $\overrightarrow{A} \times \overrightarrow{B}$ is a vector perpendicular to the plane formed by $\overrightarrow{A}$ and $\overrightarrow{B}$. Its module is $\left|\overrightarrow{A}\right|\left|\overrightarrow{B}\right| \sin\theta$, and its direction is given by what's called the right-hand rule.

3.1 Excercises

(2) Sean los vectores $\overrightarrow{A} = 2\hat{\imath} + 3\hat{\jmath}$ $\overrightarrow{B} = 4\hat{\imath} - 2\hat{\jmath}$ y $\overrightarrow{C} = -\hat{\imath} + \hat{\jmath}$. Determinar la magnitud y el ángulo (representación polar) de los vectores resultantes $\overrightarrow{D} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$ y $\overrightarrow{E} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$. Resolver analítica y gráficamente.

(Analytical solution.) We'll use A_x, A_y to denote the components of the vector $\overrightarrow{A}$, and same for all other vectors. We know the components of $\overrightarrow{D}$ are

$$D_x = A_x + B_x + C_x = 2 + 4 - 1 = 5, \qquad D_y = 3 - 2 + 1 = 2$$

from which readily follows that $|D| = \sqrt{5^2 + 2^2} = \sqrt{29} \approx 5.385$. Similarly,

$$E_x = 2 + 4 + 1 = 7, \qquad E_y = 3 - 2 - 1 = 0$$

from which follows that $|E| = \sqrt{7^2} = 7$.

Now, we must recall that

$$\theta_{\overrightarrow{Z}} = \arctan\left(\frac{Z_y}{Z_x}\right)$$

for any $\overrightarrow{Z}$.

We need not memorize this: it is trigonometrically clear that $Z_x = \cos\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$ and $Z_y = \sin\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$, and therefore

$$\frac{Z_y}{Z_x} = \tan\theta$$

And arctan is the inverse of tan. Anyhow, for $\overrightarrow{E}$ and $\overrightarrow{D}$ we have

$$\theta_{\overrightarrow{E}} = \arctan\left(\frac{E_y}{E_x}\right) = \arctan(0) = 0$$

$$\theta_{\overrightarrow{D}} = \arctan\left(\frac{D_y}{D_x}\right) = \arctan\left(\frac{2}{5}\right) \approx 0.38$$

(3) Can two vectors of different magnitud be combined and yield zero? What about three?

The zero vector is the only vector with magnitude zero. Let $\overrightarrow{A}, \overrightarrow{B}$ arbitrary vectors. Then

$$\left|\overrightarrow{A} + \overrightarrow{B}\right| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

which is zero if and only if

$$(A_x + B_x)^2 + (A_y + B_y)^2 = 0$$

This only holds if $A_x + B_x = A_y + B_y = 0$. But

$$A_x + B_x = 0 \Rightarrow A_x = -B_x, \qquad A_y + B_y = 0 \Rightarrow A_y = -B_y$$

But then

$$|A| = \sqrt{A_x^2 + A_y^2} = \sqrt{(-B_x)^2 + (-B_y)^2} = \sqrt{B_x^2 + B_y^2} = |B|$$

$\therefore \left|\overrightarrow{A} + \overrightarrow{B}\right| = 0 \iff \left|\overrightarrow{A}\right| = \left|\overrightarrow{B}\right|$.

It is simple to see that three vectors of different magnitude can add to zero.

Assume $A + B + C = 2\hat{\imath} + \hat{\jmath}$ and $A = 6\hat{\imath} - 3\hat{\jmath}, B = 2\hat{\imath} + 5\hat{\jmath}$. Find the components of C. Solve analytically and graphically.

We know

$$6 + 2 + C_x = 2, \qquad -3 + 5 + C_y = 1$$

from which follows that $C_x = -6, C_y = -1$.

(5) A and B have a magnitud of $3m, 4m$ respectively. The angle between them is $\theta = 30$ degrees. Find their scalar product.

Their scalar product is

$$(|B| \cos \theta) \, |A|$$

Recall that

$$\text{Angle in degrees} = \text{Angle in radians} \cdot \frac{180}{\pi}$$

Thus, thirty degrees equates to $30\frac{\pi}{180} \approx 0.523$ radians. Then the scalar product is

$$4 \cos(0.523) \times 3 \approx 10.395$$

(6) Find the angle between $A = 4\hat{\imath} + 3\hat{\jmath}$ and $B = 6\hat{\imath} - 3\hat{\jmath}$.

Recall that

$$A \cdot B = |A|\,|B| \cos\theta$$

where θ is the angle between the vectors. This readily entails that

$$\frac{A \cdot B}{|A|\,|B|} = \cos\theta$$

or equivalently that

$$\theta = \arccos\left(\frac{A \cdot B}{|A|\,|B|}\right)$$

Now, $A \cdot B = 4 \times 6 + 3 \times -3 = 24 - 9 = 15$ and $|A|\,|B| = 5 \cdot 6.708 = 33.541$.

Therefore,

$$\theta = \arccos\left(\frac{15}{33.541}\right) = \arccos(0.447) = 1.107$$

(7) Let $\overrightarrow{v} = \left(\frac{1}{3}, \frac{2}{3}\right)$ be the vector of components. Find the components of the vector of module 5 whose direction and orientation (sentido) are those of the given vector.

Assume $\overrightarrow{x} = (x_1, x_2)$ is of magnitude 5. Any vector whose direction and orientation are the same than those of $\vec{v}$ is "a stretching" of $\vec{v}$. In other words, for $\vec{x}$ to satisfy the requirements, we must have

$$\vec{x} = \lambda\vec{y} \tag{1}$$

for some $\lambda \in \mathbb{R}$. (Furthermore, $\lambda > 0$ since otherwise orientation is not preserved.)

Now, from equation (1) follows that

$$||\vec{x}|| = \lambda||\vec{y}|| \tag{2}$$

since the magnitude of a scaled vector is the scaled magnitude of the vector. Equation (2) simplifies to

$$||\vec{x}|| = \lambda\sqrt{1/9 + 4/9} = \frac{\lambda\sqrt{5}}{3} \tag{3}$$

From this readily follows that $\frac{3}{\sqrt{5}}||\vec{x}|| = \lambda$. But it is a hypothesis that $||\vec{x}|| = 5$. Therefore,

$$\lambda = \frac{3}{\sqrt{5}} \cdot 5 = \frac{15}{\sqrt{5}} \tag{4}$$

In other words,

$$\vec{x} = \frac{15}{\sqrt{5}}\vec{v} \tag{5}$$

which is ugly but can be simplified.

(8) Write the expression of the vector product $\vec{c} = \vec{u} \times \vec{v}$ in the following cases:

1. $\vec{u}, \vec{v}$ are coplanar. Provide a graphical interpretation.
2. $\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{v} = -3\hat{i} + \hat{j} + 2\hat{k}$. Find the module of the resulting vector $\vec{c}$ in two different ways.

(1) Two vectors are coplanar if there is a plane which contains them both. Since the vector product $\vec{u} \times \vec{v}$ is a vector orthogonal to both $\vec{u}$ and $\vec{v}$