(1) How many linear codes of length 7 with 8 words are there?

(2) How many linear codes of length 7 with 32 words are there?

(3) How many cyclic codes of length 7 with 8 words are there?

(4) How many cyclic codes of length 23.500.002 with 8 words are there?

(1) We know $\overrightarrow{0} \in C$ for any linear code. Let us analyze how we can build a code satisfying the constraints.

Each time I add a vector α to the code, I must add all of its linear combinations. Thus, any set of words $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ "generates" a code

$$C = \left\{\overrightarrow{0}\right\} \cup \{\text{All non-null linear combinations of } \alpha_1, \ldots, \alpha_k\}$$

It is simple to observe that there are 2^k linear combinations of k vectors. But this includes the one that is zero. So any set of k words generates $2^k - 1$ words.

So, if we want to create a code with 8 words, we must choose 3 vectors because $2^3 - 1 = 7$ and there will be 7 words plus $\overrightarrow{0} = 8$ words in the code.

We are told that the length of the words is seven. So we must calculate in how many ways I can choose 3 non-zero words of length 7. Think about it and you'll see that there are

$$\frac{\left(2^7 - 1\right)\left(2^7 - 2\right)\left(2^7 - 3\right)}{3!}$$

ways to do this. So this is the number of codes with 8 words of length 7.

(2) Same reasoning tells we must chose 5 words, becaue $2^5 = 32$. There are

$$\frac{\prod_{i=1}^{5}\left(2^7 - i\right)}{5!}$$

ways to do this.

(3) Cyclic codes are linear so what we said before still applies. But they also impose the following condition: if a word w is in the code, all its rotations are

in the code. It is very simple to observe that a word of length n has n rotations (including itself and excluding repetead words). So whenever we add a word of length n to the code, we are in fact adding n words.

Using this reasoning, if I add a single word of length 7 to the code, I'm already adding 7 words, which (counting the zero vector) already spans a code of 8 words of length 7. There are $2^7 - 1$ words to chose from. So this is the number codes requested.

(4) Using the same reasoning as in *(3)*, there is no cyclic code with 8 words of length 23.500.002.