

# 1 P1

Let us recall that  $P : \mathcal{P}(\Omega) \mapsto [0, 1]$  satisfies the following three conditions:

- $\forall \zeta \in \mathcal{P}(\Omega) : 0 \leq P(\zeta) \leq 1$
- $P(\Omega) = 1$
- $\forall \{\zeta_i\}_{i \in \mathbb{N}} : \zeta_i \cap \zeta_j = \emptyset : P(\bigcup_{i \in \mathbb{N}} \zeta_i) = \sum_{i \in \mathbb{N}} P(\zeta_i)$

In general, I use  $\omega$  to denote  $|\Omega|$ , and we should recall that whenever  $P$  is a constant map (i.e. when events are equiprobable) we have  $\forall A \subseteq \Omega : P(A) = \frac{|A|}{\omega}$ .

To keep notation brief, we shall speak of probabilities  $P(A)$  without specifying that  $A \subseteq \Omega$ .

We recall the following properties, which follow strictly from the aforementioned facts:

- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$
- $A \subseteq B \Rightarrow P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

**Problem 1 (Problem 2 of the sheet)** Prove that  $A \subseteq B \subseteq \Omega \Rightarrow P(B - A) = P(B) - P(A)$  and also  $P(A) \leq P(B)$ .

Let  $A \subseteq B \subseteq \Omega$ .

$$\begin{aligned} B - A &= \{x \in B : x \notin A\} \\ &= \{x \in \Omega : x \in B \wedge x \notin A\} \\ &= \{x \in \Omega : x \in B \wedge x \in A^c\} \\ &= B \cap A^c \end{aligned}$$

Since  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ , we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Then

$$\begin{aligned} P(B - A) &= P(B \cap A^c) \\ &= P(B) + P(A^c) - P(B \cup A^c) \\ &= P(B) + [1 - P(A)] - P(B \cup A^c) \end{aligned}$$

It is easy to see that, since  $A \subseteq B$ ,  $B \cup A^c = \Omega$ . We readily obtain

$$\begin{aligned} P(B - A) &= P(B) + [1 - P(A)] - P(\Omega) \\ &= P(B) + 1 - P(A) - 1 \\ &= P(B) - P(A) \end{aligned}$$

*quod erat demonstrandum.* And since  $P(B - A) = P(B) - P(A) \geq 0$ , we must have  $P(B) \geq P(A)$ .

**Problem 2 (3)**

For ease of mind, let us write here that

$$P(A_i) = \begin{cases} .22 & i = 1 \\ .25 & i = 2 \\ .28 & i = 3 \end{cases}$$

We are also given

$$P(A_1 \cap A_2) = .11 \quad P(A_1 \cap A_3) = .005 \quad P(A_2 \cap A_3) = .07 \quad P(A_1 \cap A_2 \cap A_3) = .01$$

(1 :  $P(A_1 \cup A_2)$ ) Observe that  $P(A_1 \cap A_2) \neq P(A_1)P(A_2)$ , which entails the events are not independent. We then have

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= .22 + .25 - .11 \\ &= .36 \end{aligned}$$

(2:  $P(A_1^c \cap A_2^c \cap A_3)$ ) Observe that

$$\begin{aligned} P(A_1^c \cap A_2^c \cap A_3) &= P([A_1^c \cap A_2^c] \cap A_3) \\ &= P([A_1 \cup A_2]^c \cap A_3) \\ &= P((A_1 \cup A_2 \cup A_3)^c) \\ &= P(\Omega^c) \\ &= P(\emptyset) \\ &= 0 \end{aligned}$$

(3:  $P((A_1^c \cap A_2^c) \cup A^c)$ )

### Problem 3 (4)

Se nos dice que cinco empresas deben firmar contratos de un grupo de 3 contratos posibles. Se nos dice además que cada empresa firma a lo sumo un contrato.

Mucho ojo: no se nos dice que cada contrato se da a lo sumo a una empresa, sino que cada empresa firma a lo sumo un contrato. Es decir que varias empresas podrían firmar el mismo contrato.

Si contamos la opción "no firma ningún contrato", hay 4 opciones para cada una de las 5 empresas; es decir, hay  $\omega = 4^5$  puntos en el espacio muestral.

Si cada evento es equiprobable, la probabilidad de que la tercera empresa reciba un contrato es

$$\frac{|\{x \in \Omega : \text{3era empresa firma contrato}\}|}{\omega}$$

El numerador puede calcularse restando a  $\omega$  la cantidad de casos en que la tercera empresa no firma un contrato. Claramente, la cantidad de tales casos es  $4^4$ ; es decir, hay  $4^5 - 4^4 = 768$  casos en que la tercera empresa firma algún contrato. Lo cual nos da una probabilidad de  $768/4^5 = .75$ .

*Solución alternativa.* Sea  $\alpha$  una palabra sobre el alfabeto  $\{0, \dots, 3\}$  de longitud 5. ¿Cuántas tales palabras hay? Naturalmente,  $4^5 = \omega$ . Asumamos que  $\alpha_3 \neq 0$ ; es decir que el tercer símbolo de  $\alpha$  es no-nulo. ¿Cuántas palabras así hay? Naturalmente,  $4^4 \times 3 = 768$ . Es decir,  $\frac{768}{4^5} = .75$  de las palabras tienen  $\alpha_3 \neq 0$ , lo cual coincide con nuestro resultado anterior.

**Problem 4 (5)**

From a set of 25 buses, 8 present flaws. 5 are randomly (in this context, uniformly) chosen. We are therefore dealing with equiprobable events.

There are  $\omega = \binom{25}{5}$  possible ways of selecting the 5 buses. There are  $\binom{8}{4}$  possible ways of selecting 4 out of the 8 flawed buses and  $\binom{25-8}{1} = \binom{17}{1}$  ways of selecting the remaining bus from the set of non-flawed buses. From this follows that the desired probability is

$$\frac{\binom{8}{4} \cdot \binom{17}{1}}{\binom{25}{5}} = \frac{70 \cdot 17}{53130} = .022$$

With regards to the probability that at least 4 have flaws, we must take into account the cases where 4 have flaws and the cases where 5 have flaws, which are clearly disjoint. The probability then is

$$.022 + \frac{\binom{8}{5}}{53130} = .022 + .001 = .023$$

**Note.** This is obviously a problem involving the hypergeometric distribution, closely related to the binomial distribution. But since we have not studied distributions yet, we cannot use this fact.

**Problem 5 (6)**

Let  $A, B, C, D, E$  denote the five faculty members. Two papers from a set of five are drawn to decide who will be chosen.

Observe that the order does not matter; i.e. drawing  $A$  and then  $B$  is the same than drawing  $B$  and then  $A$ . It follows that

$$\Omega = \mathcal{P}_2(\{A, \dots, E\})$$

where  $\mathcal{P}_i(\zeta) = \{S \in \mathcal{P}(\zeta) : |S| = i\}$ . Naturally,  $\omega = 5 \times 4 \cdot \frac{1}{2} = 10$ , where we divide by 2 to exclude equivalent pairs (e.g.  $A, B$  and  $B, A$ ).

Alternatively, we could have reasoned that  $\omega = \binom{5}{2} = 10$ , the number of 2-element subsets of a 5-element set.

(a) We are asked for the probability of the event  $\{A, B\}$ . It should be obvious that all events  $S \in \Omega$  are equiprobable, which entails  $P(\{A, B\}) = \frac{1}{10} = .1$ .

Alternatively, we could have reasoned the following. There are two ways in which  $A$  and  $B$  may be chosen:  $A$  is chosen first and then  $B$ , or  $B$  is chosen first and then  $A$ . This gives  $\frac{1}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4} = \frac{2}{5 \cdot 4} = \frac{2}{20} = .1$

(b) We are asked for the probability that the selection contains  $C$  or  $D$ . It is straightforward to reason that there are  $\binom{3}{2} = 3$  sets that do not contain neither  $C$  nor  $D$ . From which readily follows that there are  $10 - 3 = 7$  sets containing  $C$ ,  $D$  or both.  $\therefore$  The desired probability is  $\frac{7}{10}$ .

(c) Let us change the notation a bit. Let  $\{1, \dots, 5\}$  be the professors we used to call  $A, \dots, E$ . Let  $a_i := \{3, 6, 7, 10, 14\}$  be the set of years of teaching of each professor, assuming  $a_1$  corresponds to  $A$ ,  $a_2$  to  $B$ , etc. We are asked for the probability that the selected pair  $\{j, k\}$  satisfies  $a_j + a_k \geq 15$ .

There are two ways to solve this problem: one slow but direct, one pretty but a bit more clever.

*Direct solution.* It is easy to see that, of all pairs  $j, k$ , only the following satisfy the requirement:

- $1, 5 \mapsto a_1 + a_5 = 17$
- $2, 4 \mapsto a_2 + a_4 = 16$
- $2, 5 \mapsto a_2 + a_5 = 20$ .

- $3, 4 \mapsto a_3 + a_4 = 17$
- $3, 5 \mapsto a_3 + a_5 = 21$
- $4, 5 \mapsto a_4 + a_5 = 24$

So only 6 out of the 10 possible pairs satisfy the relationship, giving us the desired probability:  $\frac{6}{10} = \frac{3}{5} = .6$ .

*Pretty solution.* Draw the  $4 \times 5$  boolean matrix  $\mathcal{A}$  whose coefficients  $\mathcal{A}_{ij}$  are 1 if  $a_i + a_j \geq 15$ , 0 otherwise. Since upper and lower diagonal entries are equivalent (the matrix is symmetric), and because  $i \neq j$  in our experiment, the diagonal of the matrix should not be considered. This gives the representation

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

where 6 out of 10 relevant entries are 1.

### Problem 6 (7)

Let  $M := \{m_1, \dots, m_4\}$  and  $W := \{w_1, \dots, w_4\}$  be alphabets denoting the men and women, respectively. The sample space  $\Omega$  consists of all permutations in  $M \cup W$ , which readily entails  $\omega = 8!$ .

(a) Consider the event  $\zeta$  when at least one women  $w \in W$  is among the first three elements in the sampled permutation. Then  $w$  could be the first, the second or the third element in the permutation, and we impose no condition on the rest of the elements. These events are evidently disjoint. So, if we denote with  $e_i$  the event where  $w$  is the  $i$ th element of the permutation, we have

$$P(\zeta) = \frac{P(e_1) + P(e_2) + P(e_3)}{8!}$$

Each  $e_i$  may occur in  $7!$  ways, since we fix  $w$  at the  $i$ th element and we must only choose from the remaining 7 assistants.

$$\therefore P(\zeta) = \frac{3 \cdot 7!}{8!} = \frac{3}{8}$$

**Note.** If you are interested in being very formal, this are the rigorous steps taken above.

- (1)  $\zeta = e_1 \cup e_2 \cup e_3 \Rightarrow P(\zeta) = P(e_1 \cup e_2 \cup e_3)$ .
- (2)  $(\forall i, j : e_i \cap e_j = \emptyset) \Rightarrow P(\zeta) = P(e_1) + P(e_2) + P(e_3)$ .
- (3) Since events are equiprobable,  $P(e_i) = |e_i|/\omega$ .
- (4)  $|e_i| = 7!$  because  $e_i$  is a permutation of 7 elements.
- (5)  $P(\zeta) = |e_1|/\omega + |e_2|/\omega + |e_3|/\omega = \frac{3 \times 7!}{8!}$ .

(b) Let  $\varrho$  denote the event where, after the first five meetings, all female assistants have been met. Let  $(p_1, \dots, p_8) \in \varrho$  be an arbitrary permutation, and denote it with  $\vec{p}$ . (Remember that an event is a subset of the sample space, and hence a set, so the expression  $\vec{p} \in \varrho$  is well defined.)

The definition of  $\varrho$  entails that one and only one  $m_j$  exists in  $p_1, \dots, p_5$ , since all  $w_1, \dots, w_4$  must lie in this sequence. So the number of ways in which we may construct  $\vec{p}$  (i.e. the cardinality of  $\varrho$ ) is readily determined by the number of ways in which we can place exactly one  $m_j$  among the first elements of  $\vec{p}$ .

There are 5 positions to place  $m_j$ , and 4 elements in  $M$  to choose from. Assuming  $m_j$  was placed at the  $k$ th position, we know there are  $4!$  ways of placing the 4 women among the remaining positions in  $p_1, \dots, p_5$ . So there are  $5 \times 4 \times 4!$  ways to construct  $p_1, \dots, p_5$  for  $\vec{p} \in \varrho$ .



The positions  $p_6, p_7, p_8$  must be chosen from the remaining 3 men, so there are  $3 \times 2$  possibilities.

$$\therefore |\varrho| = 5 \times 4 \times 4! \times 3 = 60 \times 4!.$$

$$\therefore P(\varrho) = \frac{|\varrho|}{\omega} = \frac{60 \times 4!}{8!} = .036$$

## 2 Conditional probability

**Problem 7 (8)** *A box contains 6 red balls and 4 green balls. A second box contains 7 red balls and 3 green balls. A ball is randomly chosen from the first box and placed into the second. Then a ball is drawn from the second box and placed into the first.*

(1) Let  $R_i$  denote the event of choosing a red ball in the  $i$ th draw, and  $G_i$  the event of choosing a green ball in the  $i$ th draw. Evidently,  $P(R_1) = \frac{6}{10}$ .

If the event  $R_1$  occurs, when the second draw is made, the second box contains 8 red balls and 3 green balls, entailing that  $P(R_2 | R_1) = \frac{8}{11}$ .

So, using the conditional probability formula, the event when both balls are red,  $R_1 \cap R_2$ , has probability  $P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 | R_1) = \frac{6}{10} \cdot \frac{8}{11} = .436$ .

(2) We now inquire the probability that the number of red and green balls in the first box are the same at the beginning and the end of the experiment. Naturally, this entails either drawing both times a red or both times a green ball. We have already computed the probability of drawing both times a red ball. The probability of drawing both times a green ball is similarly computed:

$$\begin{aligned} P(G_1 \cap G_2) &= P(G_1) \cdot P(G_2 | G_1) \\ &= \frac{4}{10} \cdot \frac{4}{11} \\ &= .145 \end{aligned}$$

Since  $R_1 \cap R_2$  and  $G_1 \cap G_2$  are obviously disjoint, the probability that either of them occurs is simply  $.436 + .145 = .581$ , the sum of the probabilities of both events.

**Problem 8 (10)** Given events  $A, B$  with  $P(B) > 0$ , prove  $P(A | B) + P(\bar{A} | B) = 1$ .

(a) We know

$$P(A | B) = \frac{P(A \cap B)}{P(B)}, \quad P(\bar{A} | B) = \frac{P(\bar{A} \cap B)}{P(B)}$$

It follows

$$\begin{aligned} P(A | B) + P(\bar{A} | B) &= \frac{P(A \cap B) + P(\bar{A} \cap B)}{P(B)} \\ &= \frac{P\left((A \cap B) \cup (\bar{A} \cap B)\right)}{P(B)} \quad \left\{A \cap B \text{ and } \bar{A} \cap B \text{ are disjoint}\right\} \\ &= \frac{P\left((A \cup \bar{A}) \cap B\right)}{P(B)} \\ &= \frac{P(\Omega \cap B)}{P(B)} \\ &= \frac{P(B)}{P(B)} \\ &= 1 \end{aligned}$$

*quod erat demonstrandum.*

(b) Assume  $P(B | A) > P(B)$ . We want to prove  $P(\bar{B} | A) < P(\bar{B})$ .

By assumption,

$$\frac{P(A \cap B)}{P(A)} > P(B) \Rightarrow P(A \cap B) > P(B)P(A)$$

We want to prove

$$\frac{P(\bar{B} \cap A)}{P(A)} < P(\bar{B}) \equiv P(\bar{B} \cap A) < P(\bar{B})P(A)$$

**Problem 9 ()** One every 25 adults have a disease. Let  $s$  be a subject. If  $s$  has the disease, the diagnostic test is positive .99 of the times. If  $s$  does not have the disease, the diagnostic test is positive .02 of the times.

(a) We are asked to find the probability of a result being positive. The law of total probability readily states that

$$\begin{aligned} P(\text{positive}) &= P(\text{positive}|\text{enfermo})P(\text{enfermo}) + P(\text{positive}|\text{sano})P(\text{sano}) \\ &= .99 \cdot \frac{1}{25} + .02 \cdot \frac{24}{25} \\ &= .395 + .0192 \\ &= .058 \end{aligned}$$

(b) We are requested to find  $P(\text{enfermo}|\text{positivo})$ . Observe that we know the "reverse" of this; i.e.  $P(\text{positivo}|\text{enfermo})$ . This type of scenario calls for Bayes theorem, which states

$$\begin{aligned} P(\text{enfermo}|\text{positivo}) &= \frac{P(\text{positivo}|\text{enfermo})P(\text{enfermo})}{P(\text{positivo})} \\ &= \frac{.99 \cdot \frac{1}{25}}{.058} \\ &= .682 \end{aligned}$$

(c) Similarly,

$$\begin{aligned} P(\text{sano}|\text{negativo}) &= \frac{P(\text{negativo}|\text{sano})P(\text{sano})}{P(\text{negativo})} \\ &= \frac{.98 \cdot \frac{24}{25}}{1 - .058} \\ &= .998 \end{aligned}$$

**Problem 10 (12)**

Recall that there are two equivalent definitions of independence. Two events  $\varphi, \psi$  are independent if  $P(\varphi \cap \psi) = P(\varphi)P(\psi)$ , or equivalently if  $P(\varphi \mid \psi) = P(\varphi)$ . We must use both definitions in order to prove the properties.

(a) We are required to prove  $P(\bar{A} \cap B) = P(\bar{A})P(B)$ . Observe that

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

(This follows from the fact that  $B = (A \cap B) \cup (\bar{A} \cap B)$  and the fact that the events in the union are obviously disjoint.) Then, due to the independence of  $A$  and  $B$ ,

$$\begin{aligned} P(\bar{A} \cap B) &= P(B) - P(A)P(B) \\ &= P(B) [1 - P(A)] \\ &= P(B)P(\bar{A}) \end{aligned}$$

*quod erat demonstrandum.*

(c) By DeMorgan's law,  $(\bar{A} \cap \bar{B}) = \overline{(A \cup B)}$ . Then

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [P(A) + P(B) - P(A)P(B)] \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= P(\bar{A}) - P(B) + P(A)P(B) \\ &= P(\bar{A}) - P(B) [1 - P(A)] \\ &= P(\bar{A}) - P(B)P(\bar{A}) \\ &= P(\bar{A})(1 - P(B)) \\ &= P(\bar{A})P(\bar{B}) \end{aligned}$$

**Problem 11 (13)** A collection  $\chi$  of 10 items has 2 satisfying the predicate  $\varphi : \chi \mapsto \{0, 1\}$ . Two random samples  $x_1, x_2$  are taken from  $\chi$ . Let  $A = \{\varphi(x_1)\}$ ,  $B = \{\varphi(x_2)\}$ . Compute  $P(A), P(B), P(A \cap B)$ . Are  $A$  and  $B$  independent?

Obviously,  $P(A) = \frac{2}{10} = \frac{1}{5}$  and

$$\begin{aligned}
 P(B) &= P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A}) \\
 &= \frac{1}{9} \frac{2}{10} + \frac{2}{9} \frac{8}{10} \\
 &= \frac{2}{90} + \frac{16}{90} \\
 &= \frac{18}{90} \\
 &= \frac{2}{10} \\
 &= \frac{1}{5}
 \end{aligned}$$

We know

$$P(A \cap B) = P(B \mid A)P(A) = \frac{1}{9} \frac{2}{10} = \frac{1}{45}$$

Then  $P(A)P(B) = \frac{1}{5} \cdot \frac{1}{5} \neq P(A \cap B)$ . The events are not independent.

**Problem 12 (14)** From a deck of 52 spanish cards, 4 players  $p_1, \dots, p_4$  receive 13 cards each.

(a) Let  $w, x, y, z$  be the four types of cards. The probability of  $p_1$  receiving the 13 cards of type  $w$  is  $1/\binom{52}{13}$ . The same logic gives that the probability of all of them receiving the corresponding hands is

$$P(\zeta) = \prod_{i=0}^3 \frac{1}{\binom{52-13i}{13}}$$

where  $\zeta$  denotes the corresponding event.

(b) If a player has all cards of same type, it has all cards of that type (13). There are  $4!$  ways of distributing the types among the players. Then the probability of this event is  $4!P(\zeta)$ .

(c) There are 4 aces:  $\mathcal{A} = x_{13}, y_{13}, w_{13}, z_{13}$ . There are  $\binom{48}{13}$  hands without these elements.  $\therefore$  The probability is  $1/\binom{48}{13}$ .

(d) For each player to receive an element in  $\mathcal{A}$ , each must get exactly one such element.

The probability of  $A$  receiving exactly one  $A$  is  $\frac{4}{\binom{51}{12}}$ , where 4 accounts for which element it receives and  $\binom{51}{12}$  is the number of hands containing that fixed ace. Similar logic gives that the probability of all players receiving exactly one ace is

$$4! \prod_{i=0}^3 \frac{1}{\binom{51-12i}{12}}$$