1 Info

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Temas:

- Cinemática y dinámica (mecánica)
- Campos eléctricos y magnéticos
- Circuitos
- Termodinámica

2 Measurements and magnitudes

Measurements seek to compare a prediction with an observation, so as to test a hypothesis. A magnitude is a number accompanied by a unit. Some magnitudes are:

- Length, measured in meters (m)
- Time, measured in seconds (s)
- Mass, measusred in kilograms (kg)
- Current, measured in ampers (A)
- Temperature, measured in kelvins (k)
- Matter, measured in moles (mol)

We consider 10^3 (e.g. kilometer) and 10^{-3} (e.g. milimiters) to be within human scale. We call mass, seconds and kilograms the *mechanical units*. We define the *force unit*, or Newton, as

$$[F] = N = \text{kg}\frac{m}{s^2}$$

and the Pascal unit as

$$[P] = \text{Pa} = \frac{N}{m^2}$$

We use scientific notation and terms which express quantities as powers of ten. For instance, 10^{12} is the tera, 10^3 the giga, etc.

The magnitudes hereby described are suited for algebraic manipulation. For instance, $m \times m = m^2$, and $s \times \frac{m}{s} = m$.

3 Vectors

Vectors are used to express position, displacement, velocity, force, acceleration, fields, etc. A vector $\overrightarrow{A}$ (or sometimes $\overrightarrow{a}$) in the general sense has a direction (line), an orientation, and a length (or magnitude). A vector also has an application point, which denotes the point of origin of the vector. When saying $\overrightarrow{a} = \overrightarrow{b}$, we mean that $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in direction, magnitude and orientation, irrespective of their application point.

The scalar product is defined as the usual mapping in the space $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$. Intuitively, the scalar product $\lambda\overrightarrow{a}$ "streches" or "shrinks" a vector, depending on wheter $|\lambda| < 1$ or not, and the positivty or negativity of λ determines whether the vector inverts its direction or not. In general, $\left|\lambda\overrightarrow{a}\right| = |\lambda| \left|\overrightarrow{a}\right|$.

The sum of vectors, $\overrightarrow{a} + \overrightarrow{b}$, is a mapping $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$. As usual, and in a graphical sense, the sum corresponds to the application of the parallelogram rule.

Parallelogram rule. Make $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in their point of application. From the tip of $\overrightarrow{a}$, draw a copy of $\overrightarrow{b}$, and from the tip of $\overrightarrow{b}$ a copy of $\overrightarrow{a}$. The corner of the thus generated parallelogram is the tip of $\overrightarrow{a} + \overrightarrow{b}$.

Alternatively, from the tip of $\overrightarrow{a}$ write $\overrightarrow{b}$. Then $\overrightarrow{a} + \overrightarrow{b}$ is the vector which goes from the point of application of $\overrightarrow{a}$ to the tip of $\overrightarrow{b}$.

The sum of vectors is commutative, associative, and distributive with respect to scalar product.

If $\overrightarrow{A}$ is a vector, we use A_x and A_y to denote the projection of the vector over the axis x or y, respectively. Using A_x and A_y one forms a rectangular triangle with sides A_x, A_y and a hypotenuse of length $\left|\overrightarrow{A}\right|$.

Let θ be the angle formed by $\overrightarrow{A}$ with the x-axis. Then, using trigonometry,

$$\cos\theta = \frac{A_x}{\left|\overrightarrow{A}\right|}, \qquad \sin\theta = \frac{A_y}{\left|\overrightarrow{A}\right|}$$

from which one can find A_x, A_y assuming one knows θ. From this follows that $\left|\overrightarrow{A}\right|$ and θ fully determine all the information about the vector, insofar as the allow us to determine A_x, A_y. Conversely, knowing A_x and A_y is also sufficient to determine $\overrightarrow{A}$, insofar as

$$\left|\overrightarrow{A}\right| = \sqrt{A_x^2 + A_y^2}, \qquad \frac{A_y}{A_x} = \frac{\left|\overrightarrow{A}\right|\sin\theta}{\left|\overrightarrow{A}\right|\cos\theta} = \tan\theta \Rightarrow \theta = \arctan\left(\frac{A_y}{A_x}\right)$$

As convention, we use $\hat{\imath}$ to denote the versor (vector of length 1) with direction parallel to the x-axis, and $\hat{\jmath}$ the versor with direction parallel to the y-axis.

Notice that, for any vector $\overrightarrow{A}$, A_x is $\hat{\imath}$ times A_x, and A_y is $\hat{\jmath}$ times A_y, which means

$$\overrightarrow{A} = A_x\hat{\imath} + A_y\hat{\jmath}$$

When writing $\overrightarrow{A}$ in this way, we say we write it in term of its components x, y. In terms of linear algebra, it's not hard to see that we are simply expressing that $\hat{\imath}, \hat{\jmath}$ form a basis of $\mathbb{R}^2$. Thus, it is equivalent to write

$$A_x = \left|\overrightarrow{A}\right|\cos\theta, \qquad A_y = \left|\overrightarrow{A}\right|\sin\theta$$

and

$$\overrightarrow{A} = \left|\overrightarrow{A}\right|\left(\cos\theta\ \hat{\imath} + \sin\theta\ \hat{\jmath}\right)$$

From this follows as well that

$$\begin{aligned}\overrightarrow{A} + \overrightarrow{B} &= \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + (B_x\hat{\imath} + B_y\hat{\jmath}) \\ &= \hat{\imath}\,(A_x + B_x) + \hat{\jmath}\,\left(A_y + B_y\right)\end{aligned}$$

which means the sum of vectors has as components the sum of the components.

The scalar product of two vectors, $\overrightarrow{A} \cdot \overrightarrow{B}$, is a scalar defined as

$$\overrightarrow{A} \cdot \overrightarrow{B} = \left|\overrightarrow{A}\right| \left|\overrightarrow{B}\right| \cos\theta$$

where θ is the angle formed by the two vectors. The scalar product is positive if $\cos\theta$ is positive, which occurs for $0 < \theta \leq 90$. It is negative if $\cos\theta$ is negative, i.e. if $90 < \theta \leq 180$. Clearly, $\overrightarrow{A} \cdot \overrightarrow{B} = 0 \iff \theta = 90$.

In general, from the definition follows that

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y$$

The vectorial product $\overrightarrow{A} \times \overrightarrow{B}$ is a vector perpendicular to the plane formed by $\overrightarrow{A}$ and $\overrightarrow{B}$. Its module is $\left|\overrightarrow{A}\right| \left|\overrightarrow{B}\right| \sin\theta$, and its direction is given by what's called the right-hand rule.

3.1 Excercises

(2) Sean los vectores $\overrightarrow{A} = 2\hat{\imath} + 3\hat{\jmath}$ $\overrightarrow{B} = 4\hat{\imath} - 2\hat{\jmath}$ y $\overrightarrow{C} = -\hat{\imath} + \hat{\jmath}$. Determinar la magnitud y el ángulo (representación polar) de los vectores resultantes $\overrightarrow{D} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$ y $\overrightarrow{E} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$. Resolver analítica y gráficamente.

(Analytical solution.) We'll use A_x, A_y to denote the components of the vector $\overrightarrow{A}$, and same for all other vectors. We know the components of $\overrightarrow{D}$ are

$$D_x = A_x + B_x + C_x = 2 + 4 - 1 = 5, \qquad D_y = 3 - 2 + 1 = 2$$

from which readily follows that $|D| = \sqrt{5^2 + 2^2} = \sqrt{29} \approx 5.385$. Similarly,

$$E_x = 2 + 4 + 1 = 7, \qquad E_y = 3 - 2 - 1 = 0$$

from which follows that $|E| = \sqrt{7^2} = 7$.

Now, we must recall that

$$\theta_{\overrightarrow{Z}} = \arctan\left(\frac{Z_y}{Z_x}\right)$$

for any $\overrightarrow{Z}$.

We need not memorize this: it is trigonometrically clear that $Z_x = \cos\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$ and $Z_y = \sin\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$, and therefore

$$\frac{Z_y}{Z_x} = \tan\theta$$

And arctan is the inverse of tan. Anyhow, for $\overrightarrow{E}$ and $\overrightarrow{D}$ we have

$$\theta_{\overrightarrow{E}} = \arctan\left(\frac{E_y}{E_x}\right) = \arctan(0) = 0$$

$$\theta_{\overrightarrow{D}} = \arctan\left(\frac{D_y}{D_x}\right) = \arctan\left(\frac{2}{5}\right) \approx 0.38$$

(3) Can two vectors of different magnitud be combined and yield zero? What about three?

The zero vector is the only vector with magnitude zero. Let $\overrightarrow{A}, \overrightarrow{B}$ arbitrary vectors. Then

$$\left|\overrightarrow{A} + \overrightarrow{B}\right| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

which is zero if and only if

$$(A_x + B_x)^2 + (A_y + B_y)^2 = 0$$

This only holds if $A_x + B_x = A_y + B_y = 0$. But

$$A_x + B_x = 0 \Rightarrow A_x = -B_x, \qquad A_y + B_y = 0 \Rightarrow A_y = -B_y$$

But then

$$|A| = \sqrt{A_x^2 + A_y^2} = \sqrt{(-B_x)^2 + (-B_y)^2} = \sqrt{B_x^2 + B_y^2} = |B|$$

$\therefore\ \left|\overrightarrow{A} + \overrightarrow{B}\right| = 0 \iff \left|\overrightarrow{A}\right| = \left|\overrightarrow{B}\right|.$

It is simple to see that three vectors of different magnitude can add to zero.

Assume $A + B + C = 2\hat{\imath} + \hat{\jmath}$ and $A = 6\hat{\imath} - 3\hat{\jmath}, B = 2\hat{\imath} + 5\hat{\jmath}$. Find the components of C. Solve analytically and graphically.

We know

$$6 + 2 + C_x = 2, \qquad -3 + 5 + C_y = 1$$

from which follows that $C_x = -6, C_y = -1$.

(5) A and B have a magnitud of $3m, 4m$ respectively. The angle between them is $\theta = 30$ degrees. Find their scalar product.

Their scalar product is

$$(|B| \cos \theta) |A|$$

Recall that

$$\text{Angle in degrees} = \text{Angle in radians} \cdot \frac{180}{\pi}$$

Thus, thirty degrees equates to $30\frac{\pi}{180} \approx 0.523$ radians. Then the scalar product is

$$4 \cos(0.523) \times 3 \approx 10.395$$

(6) Find the angle between $A = 4\hat{\imath} + 3\hat{\jmath}$ and $B = 6\hat{\imath} - 3\hat{\jmath}$.

Recall that

$$A \cdot B = |A|\,|B| \cos\theta$$

where θ is the angle between the vectors. This readily entails that

$$\frac{A \cdot B}{|A|\,|B|} = \cos\theta$$

or equivalently that

$$\theta = \arccos\left(\frac{A \cdot B}{|A|\,|B|}\right)$$

Now, $A \cdot B = 4 \times 6 + 3 \times -3 = 24 - 9 = 15$ and $|A|\,|B| = 5 \cdot 6.708 = 33.541$.

Therefore,

$$\theta = \arccos\left(\frac{15}{33.541}\right) = \arccos(0.447) = 1.107$$

(7) Let $\overrightarrow{v} = \left(\frac{1}{3}, \frac{2}{3}\right)$ be the vector of components. Find the components of the vector of module 5 whose direction and orientation (sentido) are those of the given vector.

Assume $\overrightarrow{x} = (x_1, x_2)$ is of magnitude 5. Any vector whose direction and orientation are the same than those of $\vec{v}$ is "a stretching" of $\vec{v}$. In other words, for $\vec{x}$ to satisfy the requirements, we must have

$$\vec{x} = \lambda\vec{y} \tag{1}$$

for some $\lambda \in \mathbb{R}$. (Furthermore, $\lambda > 0$ since otherwise orientation is not preserved.)

Now, from equation (1) follows that

$$||\vec{x}|| = \lambda||\vec{y}|| \tag{2}$$

since the magnitude of a scaled vector is the scaled magnitude of the vector. Equation (2) simplifies to

$$||\vec{x}|| = \lambda\sqrt{1/9 + 4/9} = \frac{\lambda\sqrt{5}}{3} \tag{3}$$

From this readily follows that $\frac{3}{\sqrt{5}}||\vec{x}|| = \lambda$. But it is a hypothesis that $||\vec{x}|| = 5$. Therefore,

$$\lambda = \frac{3}{\sqrt{5}} \cdot 5 = \frac{15}{\sqrt{5}} \tag{4}$$

In other words,

$$\vec{x} = \frac{15}{\sqrt{5}}\vec{v} \tag{5}$$

which is ugly but can be simplified.

(8) Write the expression of the vector product $\vec{c} = \vec{u} \times \vec{v}$ in the following cases:

1. $\vec{u}, \vec{v}$ are coplanar. Provide a graphical interpretation.
2. $\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{v} = -3\hat{i} + \hat{j} + 2\hat{k}$. Find the module of the resulting vector $\vec{c}$ in two different ways.

(1) Two vectors are coplanar if there is a plane which contains them both. Since the vector product $\vec{u} \times \vec{v}$ is a vector orthogonal to both $\vec{u}$ and $\vec{v}$

(12) Un avión vuela 200 km hacia el NE en una dirección que forma un ángulo de 30 hacia el este de la dirección norte. En ese punto cambia su dirección de vuelo hacia el NO. En esta dirección vuela 60 km formando un ángulo de 45 con la direccióon norte.

(a) Calcular la máxima distancia hacia el este del punto de partida a la que llegó el avión.

(b) Calcular la máxima distancia hacia el norte del punto de partida a la que llegó el avión.

(c) Calcular la distancia a la que se encuentra el avión del punto de partida al cabo de su recorrido.

(d) Determinar vectorialmente el camino que debería hacer para volver al punto de partida. Resolver gráfica y analíticamente.

Sea $\vec{A}$ el vector que describe el primer recorrido, $\vec{B}$ el vector que describe el segundo recorrido. Al final del problema, el avión se encuentra en la posición indicada por $\vec{A} + \vec{B}$.

Como $\vec{A}$ describe un movimiento con un ángulo de $\theta = 60$ grados $(90-30)$ respecto al eje y (norte), y una magnitud de 200km, podemos determinarlo recordando que

$$A_x = \left|\vec{A}\right| \cos\theta, \qquad A_y = \left|\vec{A}\right| \sin\theta$$

En radianes, $\theta = \frac{\pi}{180} \times 60 = 1.047$

$$A_x = 200 \times \cos(1.047) = 100.034, \qquad A_y = 22 \times \sin(1.047) = 173.185$$

En conclusión, $\vec{A} = 100.034\hat{\imath} + 173.185\hat{\jmath}$. Mismo razonamiento nos da que el ángulo del segundo vector es de 130 grados, lo cual en radianes nos da $\alpha = \pi/180 \times 130 = 2.268$. Por ende,

$$B_x = 60\cos(2.268) = -38.524, \qquad B_y = 60\sin(2.268) = 45.998$$

Es decir que $\vec{B} = -38.524\hat{\imath} + 45.998\hat{\jmath}$. De esto se sigue que $\vec{C} = \vec{A} + \vec{B} = (61.51, 219.183)$.

(a) Claramente, es la coordenada x del vector $\vec{A}$, 100.034.

(b) Claramente, es la coordenada y del vector $\vec{C}$: 218.183.

(c) Claramente, es la magnitud de $\vec{C}$, es decir $\|\vec{C}\| = \sqrt{61.51^2 + 219.183^2} = 227.65$.

(d) El camino para volver es dado por $\vec{C} \times (-1)$.

4 Cynematics

4.1 Unidimensional movement

The study of movement requires two variables: position (x, in units of length) and time (t, in seconds). We begin our study with unidimensional movement, i.e. movement which occurs along a single axis.

Experimentally, a way to study unidimensional movement could consist in taking a sequence of photographs (from the same position and angle) of the moving object at times $t_1, \ldots, t_n$. Some coordinate system must be imposed upon the space along which the object moves, e.g. setting an axis with origin at the initial position of the object, the same direction as the movement of the object, and some appropriate units. The photographs would then provide a sequence of positions $x_1, \ldots, x_n$.

Clearly, $\{t_n\}, \{x_n\}$ could be understood as defining a discrete function $\varphi(n)$, which on its turn might be interpolated to obtain a continuous approximation $\phi(t)$. To the limit, the continuous approximation converges to what we call a movement function.

Movement function. A movement function $x(t)$ is a continuous, smooth function.

Examples. $x(t) = c$ (reposo), $x(t) = at + b$ (MRU), $x(t) = at^2 + bt + c$ (MRUV).

4.2 Coincidence, displacement, temporal intervals

If A, B are objects with movement functions $x_A(t), x_B(t)$, we say A, B coincide (se encuentran) when $x_A(t) = x_B(t)$.

We define displacement (desplazamiento) (relative to positions x_1, x_2) as $\Delta x = x_2 - x_1$. Notice that Δx is not the same as distance: if one travels from A to B and then to B from A, the distance traveled is to times the distance from A to B, but $\Delta x = 0$.

We also define a temporal interval, relative to two times t_1, t_2, as $\Delta t = t_2 - t_1$, where $t_2 > t_1$.

4.3 Velocity

We define *median velocity* (velocidad media) as

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{1}$$

where $x_2 = x(t_2), x_1 = x(t_1)$. Clearly, $\overline{v}$ is the slope of the line which intersects $x(t)$ at points t_1, t_2. The sign of $\overline{v}$ then determines the direction (sentido) of movement. The unit of $\overline{v}$ is then L/T (length over time), for instance kilometers per hour. Median velocity indicates the rate of change of distance in time.

Clearly, an object in reposo has a median velocity of zero. An object with movement function $x(t) = at + b$ (MRU) has median velocity a. The case of interest is an object with a quadratic movement function (MRUV).

If $x(t) = at^2 + bt + c$, let m the midpoint of the quadratic expression and take $t_1 = m - c, t_2 = m + c$ with $c > 0$. Clearly, the median velocity from t_1 to m is negative, that from m to t_1 is positive, and that from t_1 to t_2 is zero. This is sufficient to suggest that median velocity does not clearly express the nature of the movement.

For that reason, the length Δ of the interval $[t_1, t_2]$ might be reduced in the limit to zero, so that we get an accurate notion of the instantaneous (or close to instantaneous) change of direction. Needless to say, the limit converges to the slope of the line tangent to $(t_1, x(t_1))$, i.e. the derivative of $x(t)$ at t_1. Thus, we obtain the definition of instantaneous velocity, usually called simply velocity:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = v'(t) \tag{2}$$

Again, $[v(t)] = \frac{L}{T}$. Quite clearly, $v(t) = \overline{v}$ for constant and linear functions, but for the quadratic function $x(t)$ we have

$$x(t) = at^2 + bt + c, \qquad v(t) = 2at + b$$

5 De adelante hacia atrás

Recordemos que si $x(t)$ es función de movimiento, $v(t) = \frac{dx}{dt}$ es la velocidad, y $a(t) = \frac{d^2x}{dt^2}$ es la aceleración. Naturalmente, esto significa que

$$x(t) = \int v(t')dt' + D, \qquad v(t) = \int a(t')dt' + C$$

donde D, C son constantes de movimiento que dependerán de las condiciones iniciales.

(†) **Derivada y puntos de inflexión** Un punto de inflexión de f continua y dos veces derivable en $[a, b]$ es un valor $x_0 \in [a, b]$ t.q. $f^{(2)}(x_0) = 0$. Los puntos de inflexión representan un cambio de comportamiento en f, en particular transiciones de cóncava a convexa y viceversa.

Intuitivamente, y exceptuando casos límite (como f'' constante), si $f''(x_0) = 0$, entonces alrededor de x_0 hay un cambio de signo en f'', lo cual quiere decir que en el entorno alrededor de x_0 la función original pasa de crecer a decrecer, o de decrecer a crecer.

6 Movimiento bidimensional (cinemática 2D)

Se modela con una curva en el plano cartesiano. La curva (el dibujo del movimiento) se denomina trayectoria. La trayectoria *no* es una función, obviamente (e.g. una circunferencia es una trayectoria posible).

La descripción de la posición del objeto en cada instante de tiempo t se descsribe con vectores. En el plano cartesiano, decimos que $\vec{r} = \hat{x}i + y\hat{j}$ es un vector posición si la punta de $\vec{r}$ se corresponde con la posición del objeto (en un tiempo dado).

Sea θ el ángulo formado por $\vec{r}$ y el eje x, de manera tal que $x = |\vec{r}|\cos\theta, y = |\vec{r}|\sin\theta$. Es decir, $\vec{r} = |\vec{r}|\cos\theta\hat{i} + |\vec{r}|\sin\theta\hat{j}$.

Ahora pensemos el objeto en movimiento, y que registramos a lo largo del tiempo t las posiciones $x(t), y(t)$. Claramente, $x(t), y(t)$ son funciones de movimiento unidimensionales. Por lo tanto, podemos definir el vector posición de manera general como

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \tag{1}$$

Ahroa consideremos la trayectoria T (conjunto de puntos en el eje cartesiano) del objeto. Sean $P_1, P_2 \in T$ dos puntos en el plano que pertenecen a la trayectoria. Definimos el desplazamiento del objeto como

$$\Delta\vec{r} = \vec{r_2}\vec{r_1} \tag{2}$$

donde $\vec{r_1}, \vec{r_2}$ son los vectores con puntas en P_1, P_2. De esto se sigue que

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) \tag{3}$$

para dos instantes de tiempo t_1, t_2. Además, es claro que $\Delta\vec{r}$ es el vector que conecta las dos puntas, desde P_1 hasta P_2, y que $\vec{r_2} = \vec{r_1} + \Delta\vec{r}$.

La velocidad media en el intervalo de tiempo $[t_1, t_2]$ se define entonces como

$$\overline{v}_{[t_1,t_2]} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \tag{4}$$

lo cual es claramente un vector que contendrá las velocidades medias en las direcciones x e y. El vector velocidad entonces se define como uno esperaría:

$$
\begin{aligned}
\vec{v}(t) &= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \\
&= \frac{d\vec{r}}{dt} \\
&= \frac{d}{dx}\left(x(t)\hat{\imath} + y(t)\hat{\jmath}\right) \\
&= \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath}
\end{aligned} \tag{5}
$$

Por lo tanto,

$$
\vec{v}(t) = v_x(t)\hat{\imath} + v_y(t)\hat{\jmath} \tag{6}
$$

con v_x, v_y las velocidades correspondientes a las funciones de movimiento $x(t), y(t)$.

(†) **Interpretación gráfica de $\vec{v}$.** Gráficamente, el vector velocidad $\vec{v}(t)$ se representa como sigue. Imagine la trayectoria T y un vector posición $\vec{r}$ que conecta con $P \in T$. Entonces $\vec{t}$ será paralelo a la recta tangente a la trayectoria T en el punto P. Esto *no* es una derivada, porque la trayectoria T no es una función. En otras palabras, el vector velocidad es tangente a la trayectoria.

Definimos entonces el versor

$$
\hat{v} = \frac{\vec{v}}{|\vec{v}|} \tag{7}
$$

que nos da la dirección tangencial de la velocidad.

Así como tenemos un vector velocidad, tenemos el vector aceleración

$$
\vec{a}(t) = \frac{d^2x}{dt^2}\hat{\imath} + \frac{d^2y}{dt^2}\hat{\jmath} \tag{8}
$$

de acuerdo al mismo razonamiento límite que nos dio el resultado (6). El vector aceleración también puede descomponerse en sus direcciones tangencial y normal (respecto a la trayectoria). La aceleración tangencial será la dirección de la velocidad:

$$
\text{aceleración tangencial} \rightarrow \hat{v} = \frac{\hat{v}}{|\vec{v}|}, \qquad \text{aceleración normal} \rightarrow \hat{n} = (\perp \vec{v}) \tag{9}
$$

donde $\perp\hat{v}$ es el vector tal que $\perp\hat{v} \cdot \hat{v} = 0$ (perpendicular).

(†) **Algunas observaciones**. Hablemos en un intervalo de tiempo $[t, t + \Delta t]$.

(a) Cuando $\vec{v}$ cambia de módulo y de sentido, pero no de dirección, la aceleración es puramente tangencial, es decir está en la dirección de la velocidad. (Son paralelos.)

(b) Cuando $\vec{v}$ cambia de dirección y de sentido, pero no de módulo (i.e. el vector apunta hacia otro lado pero tiene el mismo largo), se cumple que la aceleración es perpendicular a la velocidad. Es decir, es puramente normal.

(c) Si $\vec{v}$ cambia de módulo, de sentido y de dirección, el vector $\vec{a}$ tendrá una componente tangencial y normal.

La aceleración puramente tangencial resulta en un movimiento unidimensional, una línea recta. El caso (b) corresponde a un movimiento bidimensional.

6.1 Tiro de proyectil

El tiro de proyectil es un movimiento bidimensional bajo la acción única de la gravedad. Si imaginamos la parábola de un proyectil, surgen preguntas cómo cuál es su altura máxima, su destino final, etc. Asumamos que se ha impuesto un sistema de coordenadas.

Observemos que el objeto *no* tiene aceleración en el eje x, sólo velocidad. En el eje y el proyectil es afectado por la gravedad. Por ende,

$$\vec{a} = o\hat{i} - 9.8\frac{m}{s^2}\hat{j}$$

Luego, en $t = 0$,

$$\vec{v_0} = v_{0x}\hat{i} + v_{0y}\hat{j} = \vec{v}(t = 0)$$

Asumamos que por convención, $\vec{r}(t = 0) = 0$. Por ende,

$$\vec{r}(t = 0) = x_0\hat{i} + y_0\hat{j}$$

Entonces,

$$\vec{v}(t) = v_{0x}\hat{\imath} - (9.8\frac{m}{s^2}t + v_{0y})\hat{\jmath}$$

Integrando una vez más,

$$\vec{r}(t) = (v_{0x}t + x_0)\hat{\imath} + \left(-9.8\frac{m}{s^2}\frac{t^2}{2} + v_{0y}t + y_0\right)\hat{\jmath}$$

Pero $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. Por ende,

$$x(t) = v_{0x}t + x_0, \qquad y(t) = -9.8\frac{m}{s^2}\frac{t^2}{2} + v_{0y}t + y_0$$

Vemos entonces que $x(t)$ es lineal, y su pendiente es la velocidad inicial (en x) dada por v_{0x}. $y(t)$, por otro lado, es cuadrática con máximo. Dicho máximo se corresponde con el punto más alto. Es decir, el punto más alto ocurre en el tiempo t_m tal que

$$v_y(t_m) = 0 \implies t_m = \frac{v_{0y}}{9.8\frac{m}{s^2}}$$

El tiempo de vuelo está dado por el tiempo en que la altura se hace cero, es decir el t_{vuelo} tal que $y(t_{\text{vuelo}}) = 0$. Es la raíz máxima de $y(t)$. La trayectoria será despejando t de x:

$$t = \frac{x - x_0}{v_{0x}}$$

La trayectoria también será una parábola, en este caso particular. Y como en cada punto la aceleración es la aceleración negativa de la gravedad, los vectores aceleración a lo largo del tiempo serán siempre ”flechas rectas hacia abajo”.

6.2 Excercises

(1) Consider

$$x(t) = 1\left[\frac{m}{s^2}\right]t^2 - 3\left[\frac{m}{s}\right]t$$

the movement function of a body travelling in a straight line, with x in meters and t in seconds.

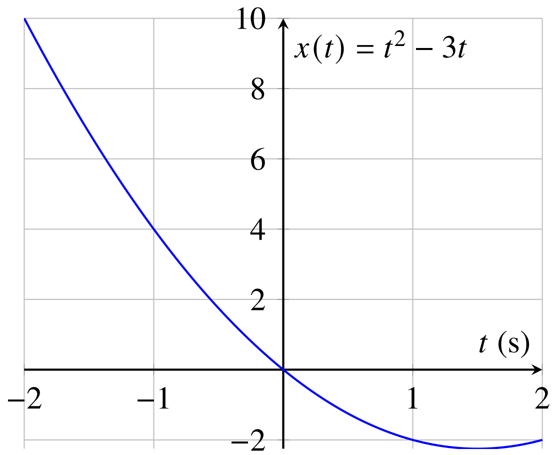
(a) Plot $x(t)$

(b) Determine analytically the median velocity in $[-1,5]$, $[-1,8]$, $[-1,0.9]$, $[-1,0.99]$, $[-1,0.999]$.

(c) Let $\Delta t_n = t_n - t_0$ with $\{t_n\} = \{-1,5,4,1,-0.5,-0.8,-0.9,-0.99,-0.999\}$ all measured in seconds. To what value does the median velocity of the object converge as t_n decreases in the interval $[-1,-1+\Delta t_n]$? What is the geometrical interpretation of this result?

(d) Find the equation for the line tangent to $x(t)$ at $t = -1s$.

(a) Notice that since t is in seconds, $\frac{mt^2}{s^2}$ correctly expresses a quantity in meters, and so does $\frac{mt}{s}$. So we will from now on write simply $x(t) = t^2 - 3t$, understanding that it is a mapping from time in seconds to meters.



(b) The median velocity of an object in the time interval $[t_a, t_b]$ was given by

$$\frac{\Delta x}{\Delta t} = \frac{x(t_b) - x(t_a)}{t_b - t_a} \tag{1}$$

So excercise (b) is as simple as plugging in the corresponding values into equation (1) and I skip it.

(c) Let t be an arbitrary value. Then by definition of $\frac{dx}{dt}$,

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{(t+\Delta t) - t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

the derivative of $x(t)$ at time t. In particular, the limit whose convergence we are asked to study is nothing but the limit above with $t = -1$:

$$\lim_{\Delta t \to 0} \frac{x(-1+\Delta t) - x(-1)}{\Delta t} = x'(-1)$$

So suffices to observe that $x'(t) = 2t - 3$ and $x'(-1) = -5$. In conclusion, the object at time $t = -1$ travels at an instantaneous velocity of -5 meters per second.

(d) The line $\ell(t) = at + b$ tangent to $x(t)$ at $t = -1s$ has slope $a = -5$ and crosses through the point $(-1, 4)$. So we must have $-5(-1) + b = 4 \iff b = 4 - 5 = -1$. So the line is $\ell(t) = -5t - 1$.

(4) Answer the questions.

(a) Can an object have null velocity and yet possess acceleration?

Let $x(t)$ the describe the movement of the object and $v(t) = x'(t)$ its velocity, both as a function of time. Assume for an arbitrary t_0 that $v(t) = 0$. It is very much possible that $v'(t_0) \neq 0$.

Consider, for instance, that $v(t)$ is linear and non-constant, making $v'(t) = a$ a non-null constant. Then there exists a unique root r s.t. $v(r) = 0$, but independently of this fact $v'(r) = a \neq 0$.

Physically, it should be clear that if a non-moving object could not possess acceleration, then it would be impossible for it to pass from a still to a moving state. So, at least at the intutition level, this *reductio ad absurdum* suffices.

(b) Can a moving object have a null displacement in a given interval and yet non-null velocity?

Naturally. Take as example an object moving in circles at a constant, non-null velocity v, and assume it travels a full circle in t seconds. Then all of the intervals in $\{[t_0, t_0 + tk] : k \in \mathbb{N}\}$ are such that they give null displacements. Yet the object *is* moving.

(c) Can an object have an east-bound velocity of while its acceleration is west-bound?

Informally, this is quite clearly the case, insofar as any positively-moving object whose velocity decreases must have a negative acceleration.

(d) Consider an object moving on a straight line, with the east being the positive direction, under a velocity of $v(t) = 20\text{ms}^{-1} - 2\text{ms}^{-2}t$. For $t = 0\text{s}, t = 1\text{s}$, what is the situation?

Its velocity is clearly positive in both cases (20 and 18), evidently decreasing, which points out the fact that its acceleration is negative (-2).

(e) A ball is thrown vertically. What do the *signs* of velocity and acceleration look like as the object ascends, and what does that mean? And when the object descends? What happens at the highest point?

Clearly, its velocity is positive during the ascending phase, and negative during the descending phase. At the highest point, the velocity will be exactly zero.

Conversely, acceleration is always negative due to the force excercised by gravity on the ball.

It is the fact that acceleration is constantly negative what causes the ball not only to lose velocity as it goes up until it begins to fall again, but to then fall more and more rapidly as time goes by.

(5) A particle moves through the x-axis with movement function $x(t) = 3 + 17t - 5t^2$, with x in meters and t in seconds.

(a) What is the position of the particle at times $\{1, 2, 3\}$?

(b) At what point in time does the particle return to the origin?

(c) Find $v(t)$ and determine the instantaneous velocity at times $\{1, 2, 3\}$. When is the velocity null? What is the particles velocity when it crosses the origin?

(d) Plot $x(t), v(t), a(t)$.

(a) Trivial, simply compute $x(1), x(2), x(3)$.

(b) See that

$$x(t) = 0 \iff t = \frac{17}{10} \pm \frac{\sqrt{17^2 + 4 \times 3 \times 5}}{10}$$

which gives approximate solutions $t_1 \approx -0.168, t_2 \approx 3.568$. It makes no sense to speak of negative time and we keep only the positive solution $t = 3.568$. Thus, the particle returns to the origin after approximately 3.568 seconds.

(c) The first derivative of $x(t)$ is

$$v(t) = -10t + 17$$

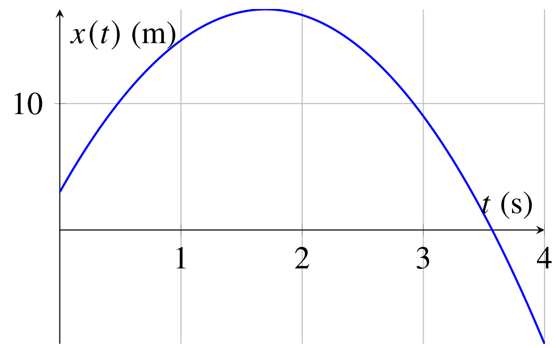
The instantaneous velocity at times 1, 2, 3 are $v(1) = 7, v(2) = -3, v(3) = -13$.

$$17 = 10t \iff t = \frac{17}{10} = 1.7$$

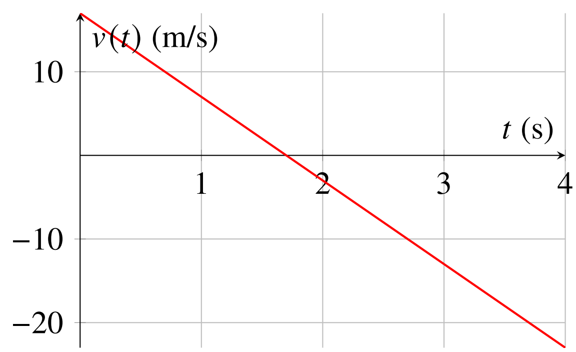
The particle crosses the origin at approximately time 3.568 and its velocity is approximately $v(3.568) = -18.682$.

(d) The acceleration is constant: $a(t) = -10$.

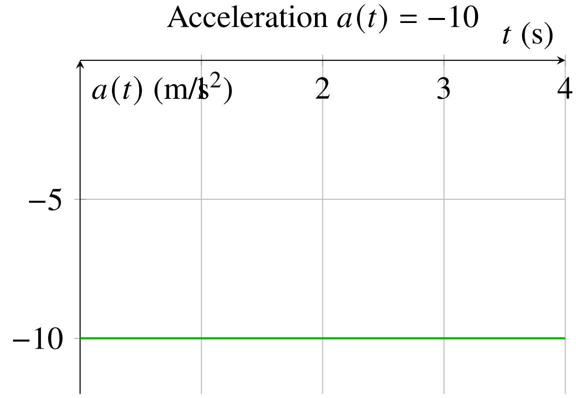
Position $x(t) = 3 + 17t - 5t^2$



Velocity $v(t) = 17 - 10t$



Acceleration $a(t) = -10$



(**6**) (a) Determine the instantaneous acceleration of the object plotted in the exercise for $t = 3, t = 11$.

(b) Compute the distanace traveled by the object in the time intervals $[0, 5]$ and $[0, 9]$ and $[0, 15]$.

(c) Knowing that $x(t = 6) = 0$, find the position of the object at $t = 0$.

(d) Give an expression for the objects position for all t.

(e) Plot $x(t), v(t), a(t)$.

The velocity of the object is constantly 20m/s from time $t = 0$ to time $t = 6$. Then it linearly increases until it reaches 44m/s at $t = 9$, from whereon it linearly decreases until it reaches 0ms at time $t = 15$.

The linear expression $\ell_1(t) = a_1 t + b_1$ which satisfies $\ell(6) = 20, \ell(9) = 44$ is such that

$$6a_1 + b_1 = 20, \qquad 9a_1 + b_1 = 44$$

The associated system of equations yields that $b_1 = 20 - 6a_1$, from which follows that $9a_1 + (20 - 6a_1) = 44$, entailing

$$a_1 = \frac{24}{3} = 8$$

From this readily follows that $b_1 = 20 - 6 \times 8 = -28$.

Similarly, the linear expression $\ell_2(t) = a_2 t + b_2$ which gives a line s.t. $\ell_2(9) = 44, \ell_2(15) = 0$ must satisfy

$$9a_2 + b_2 = 44, \qquad 15a_2 + b_2 = 0$$

Then $b_2 = -15a_2$ and $9a_2 - 15a_2 = 44$, entailing $a_2 = -\frac{22}{3}$. From this follows that $b_2 = 110$ via simple calculations. Thus,

$$v(t) = \begin{cases} 20 & 0 \leq t \leq 6 \\ 8t - 28 & 6 < t \leq 9 \\ -\frac{22}{3}t + 110 & 9 < t \leq 15 \end{cases}$$

It should be intuitive to graps that the distance travelled $d(a, b)$ in the interval $[a, b]$ is

$$d(a,b) = \int_a^b |v(t)|\ dt$$

If $v(t)$ is in meters per second, and t is in seconds, the total number of meters travelled in a time interval $[a,b]$ is the summation of the meters per second travelled in every instant! This will involve the anti-derivative of $v(t)$, i.e. the movement function $x(t)$, which we might as well compute at once.

$$\begin{aligned} x(t) &= \int v(t)\ dt \\ &= \begin{cases} 20t + C_1 & 0 \le t \le 6 \\ 8\frac{t^2}{2} - 28t + C_2 & 6 < t \le 9 \\ -\frac{22}{3}\frac{t^2}{2} + 110t + C_3 & 9 < t \le 15 \end{cases} \\ &= \begin{cases} 20t + C_1 & 0 \le t \le 6 \\ 4t^2 - 28t + C_2 & 6 < t \le 9 \\ -\frac{22}{6}t^2 + 110t + C_3 & 9 < t \le 15 \end{cases} \end{aligned}$$

The constants C_1, C_2, C_3 must satisfy the restriction of continuity and of preserving the necessary values. In particular, we need $20(0) + C_1 = 20$. Since we know $x(6) = 0$, we need $120 + C_1 = 0$, i.e. $C_1 = -120$. We also need

$$4(6^2) - 28(6) + C_2 = 0$$

to ensure continuity, so

$$C_2 = 24$$

Then we can know what $x(9)$ is and deduce C_3, which ends up being -597.

$$\therefore\ \ x(t) = \begin{cases} 20t - 120 & 0 \le t \le 6 \\ 4t^2 - 28t + 24 & 6 < t \le 9 \\ -\frac{22}{6}t^2 + 100t - 597 & 9 < t \le 15 \end{cases}$$

In any case, we could have computed the distance travelled without $x(t)$ (I computed $x(t)$ because it's part of the exercise):

$$
\begin{aligned}
d(0,5) &= \int_0^5 v(t)\ dt \\
&= 20 \times t]_0^5 \\
&= 20 \times (5) \\
&= 100
\end{aligned}
$$

$$
\begin{aligned}
d(0,9) &= \int_0^6 v(t)\ dt + \int_6^9 v(t)\ dt \\
&= 20 \times t\Big]_0^6 + (4t^2 - 28t)\Big]_6^9 \\
&= 120 + [(4 \times 81 - 28 \times 9) - (4 \times 38 - 28 \times 6)] \\
&= 120 + 96 \\
&= 216
\end{aligned}
$$

etc.

(7) A car and a truck leave at the same instant, the car initially being a certain distance behind the truck. The latter has a constant acceleration of $1.2 m/s^2$, while the car accelerates at $1.8 m/s^2$. The car reaches the truck when the latter has covered 45 meters.

(a) How much time does it take for the car to reach the truck?

(b) What is the initial distance between both vehicles?

(c) What is the velocity of each in the moment the cross paths?

(d) Plot $x(t), v(t), a(t)$.

(a) Since both vehicles have constant accelerations, they have linear velocities and therefore quadratic movement functions. They will meet when the parabolas corresponding to these functions intersect.

Let $x_1(t)$ denote the movement function of the car, $x_2(t)$ that of the truck. We then wish to find the solutions to $x_1(t) = x_2(t)$. Now,

$$v_1(t) = \int a_1(t) = \int 1.8 \; dt = 1.8t + C_1 \tag{2}$$

$$v_2(t) = \int a_2(t) = \int 1.2 \; dt = 1.2t + C_2 \tag{3}$$

are the velocities of the car (v_1) and the truck (v_2). Since at $t = 0$ the velocities of both vehicles is zero (they start from rest), it is necessary that $C_1 = C_2 = 0$.

We know that the car reaches the truck when the latter has covered 45 meters, so the question is what is the time t_0 when the distance covered by the truck is that one? In other words, we need to find t_0 such that

$$\begin{aligned} & \int_0^{t_0} v_2(t) \; dt = 45 \\ \iff & \int_0^{t_0} 1.2t \; dt = 45 \\ \iff & \left[0.6t^2\right]_0^{t_0} = 45 \\ \iff & 0.6t_0^2 = 45 \\ \iff & t_0 = \sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3} \end{aligned}$$

Thus, the vehicles meet at time $t_0 = 5\sqrt{3}$.

(b) The initial distance between both vehicles is given by $|x_1(0) - x_2(0)|$. From the velocities v_1, v_2 we can determine that

$$x_1(t) = 0.9t^2 + C_1', \qquad x_2(t) = 0.6t^2 + C_2' \tag{4}$$

Let us fix our coordinate system so that the starting position of the truck corresponds to the origin. Then $C_2' = 0$. Knowing that both vehicles coincide at time $t_0 = 5\sqrt{3}$, we also know $X_1(t_0) = x_2(t_0)$, i.e.

$$0.9(25 \times 3) + C_1' = 0.6(25 \times 3) \tag{5}$$

which entails $67.5 + C_1' = 45$, from which follows that $C_1' = -22.5$. Thus, the original distance of both vehicles is 22.5m.

(c) This consists simply of computing $v_1(t_0), v_2(t_0)$. Trivial.

(d) Meh.

(8) A car travels parallel to a train rail. The car stops at a red light in the exact instant when a train passes with a constant velocity of 12m/s. The car remains at halt for 6s and then continues with a constant acceleration of $2m/s^2$.

(a) Determine the time it takes for the car to reach the train, with $t = 0$ being the instant in which the car halted.

(b) Determine the distance traveled by the car from the red light until it reached the train.

(c) Determine the car's velocity at the instant it reaches the train.

(a) Let $a_1(t)$ be the acceleration of the car, defined as

$$a_1(t) = \begin{cases} 0 & 0 \leq t < 6 \\ 2 & t \geq 6 \end{cases} \tag{6}$$

Let $v_2(t) = 12m/s$ be the constant velocity of the train. Let the point of halt be the origin of our coordinate system, so that at time $t = 0$ (when the car halted) both the train and the car are at position zero. Observe then that it follows that $x_2(t) = 12t$ (in meters) via integration of $v_2(t)$ and the necessary condition of the constant of integration being zero.

Integration of equation (6) gives

$$v_1(t) = \begin{cases} C_1 & 0 \leq t < 6 \\ 2t + C_2 & t \geq 6 \end{cases} \tag{7}$$

where the constants of integration must satisfy two constraints: (a) $v_1(0) = 0$ and v_1 must be continuous. From this follows that $C_1 = 0$ and that $2(6) + C_2 = 0$, i.e. $C_2 = -12$. Therefore,

$$v_1(t) = \begin{cases} 0 & 0 \leq t < 6 \\ 2t - 12 & t \geq 6 \end{cases} \tag{8}$$

Via integration of v_1,

$$x_1(t) = \begin{cases} C_1' & 0 \leq t < 6 \\ t^2 - 12t + C_2' & t \geq 6 \end{cases} \tag{9}$$

Again, C_1' must of course be zero, and $x_1(6)$ must also be zero, meaning that $C_2 = -36 + 12(6) = 36$. Therefore,

$$x_1(t) = \begin{cases} 0 & 0 \le t < 6 \\ t^2 - 12t + 36 & t \ge 6 \end{cases} \tag{10}$$

The car reaches the train at the time $t_0 > 6$ which satisfies $x_1(t_0) = x_2(t_0)$, so we solve

$$t^2 - 12t + 36 = 12t \iff t^2 - 24t + 36 = 0$$

which has solutions

$$\frac{24}{2} \pm \frac{\sqrt{24^2 - 4 \times 36}}{2} = 12 \pm \frac{\sqrt{432}}{2} \approx 12 \pm 10.392$$

Keeping only the positive solution, we have that $t_0 \approx 22.392$.

(b) The distance traveled by the car from the red light until it reached the train is the distance traveled from $t = 0$ to $t = t_0$, i.e.

$$\int_0^{t_0} v_1(t)\, dt = |x_1(t_0) - x_1(t_0)| = x_1(t_0) \approx 268.697$$

where the equality above holds only because velocity is always positive (i.e. the car moves only in one direction).

(c) Simply computing $v_1(t_0)$ gives the answer.

(9) A ball is thrown vertically and upwards from the floor with initial velocity v_0. Write the equations for the movement of the ball and plot graphically the vectors $\vec{y}(t), \vec{v}(t), \vec{a}(t)$. Identify the conditions for the instant of maximum height and the instant it reaches the floor.

The move is strictly vertical, so $\vec{r}(t) = 0\hat{\imath} + y(t)\hat{\jmath}$ and we need only determine the unidimensional vertical movement function $y(t)$. Now, the ball is affected only by gravitiy, i.e. it is subjected to a constant acceleration of $\vec{a}(t) = 0\hat{\imath} - 9.8\frac{m}{s^2}\hat{\jmath}$. From this we can derive the vertical velocity:

$$v_y(t) = -9.8 \int dt = -9.8t + C$$

The constant of integration must satisfy the initial velocity being v_0, so we must have

$$v_y(t) = v_0 - 9.8t$$

From this follows that

$$r_y(t) = \int v_0 - 9.8t \; dt = v_0 t - \frac{9.8}{2}t^2 + C'$$

If we assume the position on the floor (vertically) is zero, we must have $C' = 0$, and

$$r_y(t) = v_0 t - 4.9t^2$$

In summary,

$$\vec{r}(t) = 0\hat{\imath} - (v_0 t - 4.9t^2)\hat{\jmath}, \qquad \vec{v}(t) = 0\hat{\imath} - 9.8t\hat{\jmath}, \qquad \vec{a}(t) = 0\hat{\imath} - 9.8\hat{\jmath}$$

Maximum hight will occur at time $t \neq 0$ when the vertical velocity of the ball is exactly zero. So, we solve

$$v_0 - 9.8t = 0 \iff \frac{v_0}{9.8} = t$$

It will reach the floor at time $t \neq 0$ when the vertical position of the ball is zero, so we solve

$$v_0 t - 4.9t^2 = 0 \iff t(v_0 - 4.9t) = 0$$

The root $t = 0$ is not a solution that interests us, so we only care about the root that solves $v_0 - 4.9t = 0$, i.e. $t = \frac{v_0}{4.9}$.

(10) A rock is thrown vertically and upwards. On its path, it crosses point A with velocity v and point B, which is 3m higher than A, at velocity $v/2$. Determine v an the maximum height reached by the rock above point B.

Let $\Delta y = y_B - y_A = 3\text{m}$ the distance between A and B. We know the rock has a constant acceleration $-g$ due to gravity. This tells us that velocity is linear and movement is quadratic. In general,

$$a(t) = -g, \qquad v(t) = v_0 - gt, \qquad x(t) = y_0 + v_0 t - g\frac{t^2}{2}$$

Now, given an arbitrary velocity v occurring when the object is at position y,

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = a$$

assuming a constant acceleration a. However, since the position y is a function of time, the chain rule gives

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$$

From this follows $a = v\frac{dv}{dy}$ or equivalently $a\ dy = v\ dv$. Integrating from initial positions and velocities y_0, v_0;

$$\int_{y_0}^{y} a\ dy = \int_{v_0}^{v} v\ dv$$
$$\Rightarrow\ v^2 - v_0^2 = 2a(y - y_0)$$

In other words, for any given velocity v occurring at position y, the relationship above holds. In particular, if we treat A as our initial position,

$$\frac{v^2}{4} - v^2 = 2a(y_B - y_A)$$

Since $y_B - y_A = 3\text{m}$,

$$\frac{v^2}{4} - v^2 = 6a$$

Now, the acceleration in our problem is simply $-g$, and then the equation above gives solutions for v given by

$$\begin{aligned} &\frac{-3v^2}{4} = -6g \\ \iff &v^2 = 8g \\ \iff &v = \pm\sqrt{8g} \end{aligned}$$

Now, since velocities can only be positive, we keep $v = \sqrt{8g}$ as the only solution.

Now the question is, what is the maximum height? Well, in the path from B to the top, velocity drops from $\frac{v}{2}$ to zero. So again, if we take as reference B,

$$0 - \left(\frac{v}{2}\right)^2 = 2a(y_{\text{top}} - y_B)$$

Let $h = y_{\text{top}} - y_B$. Then

$$h = -\frac{v^2}{8a}$$

But $a = -g$ and $v = \sqrt{8g}$, giving

$$h = \frac{8g}{8g} = 1$$

So the maximum height above B is 1 meter.

(11) The movement in the plane of a particle is given by $x(t) = at^2$, $y(t) = bt^3$, with $a = 3\frac{m}{s^2}$, $b = 2\frac{m}{s^3}$.

(a) Find the trajectory of the particle. Plot it.

(b) Compute the acceleration at $t = 12$s.

(c) What is the angle formed by the velocity vectors and the acceleration at that instant?

(d) Determine the instant t_1 where acceleration is parallel to the line $y = x$, and the instant t_2 in which velocity is parallel to said line.

(e) Determine the median velocity in the interval (t_1, t_2).

(a) The trajectory of the particle is

$$\begin{aligned} S &= \{(x(t), y(t)) : t \in \mathbb{R}\} \\ &= \left\{(3t^2, 2t^3) : t \in \mathbb{R}\right\} \end{aligned}$$

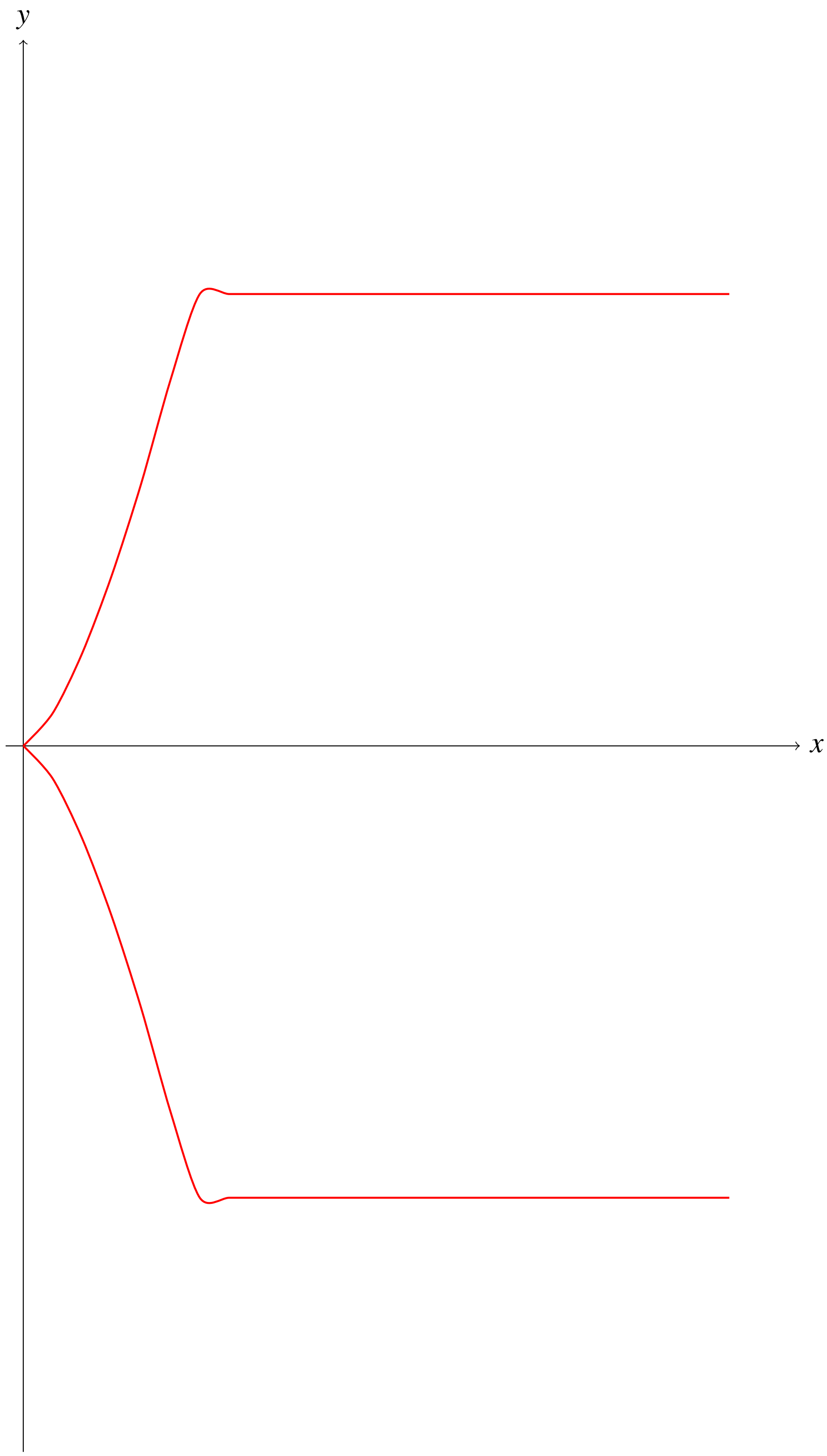
Since $x = 3t^2, t^2 = \frac{x}{3}$. And since $y = 2t^3$, we have $y^2 = 4t^6 = 4(t^2)^3 = 4\left(\frac{x}{3}\right)^3 = \frac{4}{27}x^3$. In summary,

$$y^2 = \frac{4}{27}x^3$$

or equivalently

$$y = \pm\frac{2}{3}\sqrt{x^3}$$

This entirely defines S.



(b) Clearly,

$$v_x(t) = 2at, \qquad v_y(t) = 3bt^2$$

meaning that

$$a_x(t) = 2a, \qquad a_y(t) = 6bt$$

So the acceleration at time 12 is

$$\vec{a}(12) = 2a\hat{\imath} + 72b\hat{\jmath} = 6\hat{\imath} + 144\hat{\jmath}$$

(c) Recall that the angle between two vectors $\vec{u}, \vec{w}$ is

$$\theta = \arccos\left(\frac{u \cdot w}{|u|\,|w|}\right)$$

because the dot product is $u \cdot w = |u|\,|w| \cos\theta$. Now,

$$\vec{v}(12) = 72\hat{\imath} + 864\hat{\jmath}$$

It is simple to compute:

$$|\vec{v}(12)| = 866.994, \qquad |\vec{a}(12)| = 144.125$$

and

$$\vec{v}(12) \cdot \vec{a}(12) = 6 \cdot 72 + 144 \cdot 864 = 124848$$

Then

$$\theta = \arccos\left(\frac{124848}{866.994 \times 144.125}\right) = \arccos(0.999) = 0.044$$

In degrees, these are $0.044 \times \frac{180}{\pi} = 2.521$.

(d) It is quite simple to reason and see that any vector parallel to $y = x$ is such that its x and y coordinates are the same. So the find t_1 s.t. $\vec{v}(t_1)$ is parallel to $y = x$, we need only find t_1 s.t.

$$\vec{v}_x(t_1) = \vec{v}_y(t_1)$$

But this holds if and only if

$$\begin{aligned} 2at = 3bt^2 &\iff 6t = 6t^2 \\ &\iff t = t^2 \\ &\iff t \in \{0, 1\} \end{aligned}$$

But 0 is a trivial solution, so we keep only 1. Same goes for acceleration:

$$\begin{aligned} a_x(t_2) = a_y(t_2) &\iff 6 = 12t_2 \\ &\iff \frac{1}{2} = t_2 \end{aligned}$$

(e) Median velocity was defined as $\Delta x/\Delta t$. So the median velocity in $(t_1, t_2) = (\frac{1}{2}, 1)$ is

$$\frac{\Delta x}{\Delta t} = \frac{x\,(1) - x\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = \frac{2.25}{0.5} = 4.5$$

(12) A bullet is shot horizontally from a canon placed on a platform of height 44m. Iis exit velocity is $25m/s$. Assume the terrain is horizontal and perfectly plain.

(a) Write the movement equations.

(b) Draw the vectors $\vec{r}(t), \vec{v}(t), \vec{a}(t)$ at the highest point of the curve and when the ball reaches the ground.

(c) How much time does the ball remain in the air before hitting the ground?

(d) What is its reach, i.e. at what distance from the cannon does it hit the ground?

(e) What is the magnitude of the vertical component of $\vec{v}$ when the bullet reaches the ground?

(f) Repeate (c) for the case when the ball is dropped in free fall from the platform.

(g) Consider now that the exit velocit has vertical direction and posit the movement equations.

Recall that, when a projectile is shot, it has no horizontal acceleration, only horizontal velocity. Its vertical acceleration is given by gravity alone. Thus, it is always the case for a projectile that

$$\vec{a} = 0\hat{i} - g\hat{j}$$

where g is in m/s^2. From this readily follows:

$$\vec{v}(t) = v_{0x}\hat{i} - (gt + v_{0y})\hat{j}, \qquad \vec{r}(t) = (v_{0x}t + x_0)\hat{i} + (-\frac{g}{2}t^2 + v_{0y}t + y_0)\hat{j}$$

Fix $x_0 = 0$. We know, from the conditions of the problem, that $y_0 = 44$, $v_{0x} = 25$ (in meters per second) and $v_{0y} = 0$ (since the ball is shot horizontally, it has no vertical velocity at the start.) Thus,

$$\vec{v}(t) = 25\hat{i} - gt\hat{j}, \qquad \vec{r}(t) = 25t\hat{i} + \left(-\frac{g}{2}t^2 + 44\right)\hat{j}$$

(b) Observe that the y-coordinate of $\vec{r}(t)$, henceforth denoted $\vec{r}_y(t)$, is zero if and only if

$$
\begin{aligned}
& \vec{r}_y(t) = 0 \\
\iff\quad & -\frac{g}{2}t^2 + 44 = 0 \\
\iff\quad & t^2 = \frac{2}{g}44 \\
\iff\quad & t = \pm\sqrt{\frac{88}{g}} \\
\iff\quad & t \approx \sqrt{\frac{88}{9.8}} \\
\iff\quad & t \approx \sqrt{8.97} \\
\iff\quad & t \approx \sqrt{8.97} \\
\iff\quad & t \approx 2.996
\end{aligned}
$$

where we kept only the positive solution because $t \geq 0$. So the object touches the aground after approximately three seconds. This means $\vec{r}_y(t)$ is a quadratic function with roots ± 2.996, whose midpoint is zero (i.e. the curve is symmetric around the y-axis). This is the formal way of saying something obvious: the ball reaches its maximum height at $t = 0$ (since then it falls). Then it is simple to see:

$$
\vec{r}(0) = (0, 44)^\top, \qquad \vec{v}(0) = (25, 0)^\top, \qquad \vec{a}(0) = (0, -g)^\top
$$

It is easy to imagine what these vectors look like when graphed.

(c) From our computation in (b) we already know the ball remains in the air almost three seconds before hitting the ground.

(d) Let $t_0 = 2.996$ the time at which the bullet hits the ground. Observe that $\vec{r}_x(t_0) = 25(t_0) = 75.9$. So the projectile hits the ground at 75.9 meters from its starting position in the x-axis.

(e) Note that $\vec{v}(t_0) = (25, gt_0)^\top \approx (25, -29.3608)^\top$. The magnitude of the vertical component is the second component of the vector given.

7 Dirección de fuerzas

7.1 Fuerza de roce (o fricción)

Involucra contacto paralelo de un cuerpo con una superficie. Hay estático y dinámico. Trataremos el estático primero.

En el roce estático, el cuerpo no se mueve ($\vec{v} = 0$). Digamos que queremos empujar una mesa apoyada en el piso. Si al hacer fuerza empujando, la mesa aún no se mueve, es porque existe otra fuerza (la de rozamiento estático) que equilibra o supera la que estamos haciendo.

Sabemos, por las leyes de Newton, que la suma de las fuerzas es igual a la masa por aceleración:

$$\begin{aligned}\sum \vec{F} &= m\vec{a} \\ &= \vec{N} + \vec{P} + \vec{F}_{\text{ext}} + \vec{F}_{\text{roce estatico}}\end{aligned}$$

Aquí, la fuerza normal $\vec{N}$ equilibra $\vec{P}$, i.e. $\vec{N} + \vec{P} = 0$. Como el objeto no se mueve, la aceleración es cero y por lo tanto $m\vec{a} = 0$. Entonces necesariamente la fuerza externa y la fuerza de roce estático también suman cero (se equilibran).

A su vez, hay un valor máximo que la fuerza de roce estático puede tomar:

$$\left|\vec{F}^{\text{max}}_{\text{r.e.}}\right| = \mu_e \left|\vec{N}\right|$$

donde $\mathcal{M}_e$ se llama coeficiente de roce estático.

Pasemos al caso dinámico, cuando el objeto empieza a moverse. Existe cuando $\vec{v} \neq 0$, tiene una constante $\mathcal{M}_a < \mathcal{M}_e$ que lo acota. Esta fuerza no necesita de otras fuerzas aplicadas paralelas a la superficie: se opone al movimiento y disminuye $\vec{v}$. Es la fricción que "frena" a un objeto que se mueve/desliza. Siempre se cumple

$$\left|\vec{F}_{\text{r.d.}}\right| = \mu_a \left|\vec{N}\right|$$

7.2 Movimiento oscilatorio armónico

Digamos que tenemos un resorte horizontal (e.g. contra la pared) de constante k y longitud ℓ_0 con una masa m en el extremo.

Ley de Hooke. La fuerza de un resorte de constante k es $\vec{F}_k = -k\Delta\ell$ donde ℓ es la distancia entre el extremo y la posición de equilibrio del extremo.

Las fuerzas involucradas en un resorte que ha sido estirado o movido fuera de su equilibrio son: la fuerza normal $\vec{N}$, la fuerza $\vec{P}$, la fuerza del resorte $vecF_k$. Por ende, la segunda ley de Newton nos da

$$\begin{aligned} &\sum \vec{F} = m\vec{a} \\ \Rightarrow \quad &\vec{N} + \vec{P} + \vec{F}_k = m\vec{a} \end{aligned}$$

Pero la aceleración es una aceleración en x (se mueve horizontalmente). Es decir, $\vec{a} = a_x\hat{\imath} + 0\hat{\jmath} = a_x\hat{\imath}$. Sabemos además que $\vec{N} + \vec{P} = 0$. Por lo tanto,

$$\vec{F}_k = m\vec{a} = ma_x\hat{\imath}$$

Pero $a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$. Por ende

$$\vec{F}_k = m\frac{d^2x}{dt^2}\hat{\imath}$$

o bien, por la ley de Hooke,

$$-k(x - x_0) = m\frac{d^2x}{dt^2}\hat{\imath}$$

donde x es la posición actual de la masa en el extremo del resorte y x_0 su posición original (de equilibrio). Sea $\widetilde{m} = \frac{d^2x}{dt^2}\hat{\imath}$. Entonces

$$a_x = \widetilde{x} + \frac{k}{m}(x - x_0) = 0$$

El problema es que la ecuación

$$v = \int a_x = \int \widetilde{x} + \frac{k}{m}(x(t) - x_0)\ dt$$

no se puede resolver, porque $v(t)$ depende de $x(t)$. Si no sabemos $x(t)$ no podemos continuar.

Sea $\mu = x - x_0$, tal que $\mu^{(1)} = x^{(1)}$ y $\mu^{(2)} = x^{(2)} = \widetilde{x}$. Entonces

$$\widetilde{x} + \frac{k}{m}(x - x_0) = 0$$

puede escribirse como

$$\mu^{(2)} + \frac{k}{m}\mu = 0$$

de lo cual se sigue

$$\mu^{(2)} = -\frac{k}{m}\mu \tag{1}$$

Es decir, necesitamos una función u tal que su segunda derivada sea una constante por sí misma. Podemos ver que

$$\mu(t) = A\sin(\omega t + \phi), \qquad \mu(t) = A'\sin(\omega t) + B'(\cos\omega t)$$

donde ϕ es un ángulo inicial (fase) satisfacen la ecuación (1). Tomemos

$$\mu(t) = x(t) - x_0 = A\sin(\omega t + \phi)$$

Entonces

$$\mu^{(1)}(t) = \mu^{(1)}(t) = v(t) = \omega A\cos(\omega t + \phi)$$

y

$$\begin{aligned}\mu^{(2)}(t) &= a(t)\\ &= -\omega^2 A\sin(\omega t + \phi)\\ &= -\omega^2\mu(t)\\ &= -\frac{k}{m}\mu(t)\end{aligned}$$

Es decir,

$$\omega^2 = \sqrt{\frac{k}{m}}$$

Decimos que $T = \frac{2\pi}{\omega}$ es el período.

Repasemos. Por la segunda ley de newton, teníamos

$$\mu^{(2)}(t) + \frac{k}{m}\mu(t) = 0$$

La solución que hallamos es $\mu(t) = A\sin(\omega t + \phi)$, con $\omega = \sqrt{\frac{k}{m}}$. El período es $T = \frac{2\pi}{\omega}$. ¿Pero quién es A?

A determina la amplitud del movimiento, en unidades de longitud, y se corresponde con el apartamiento máximo, i.e. la máxima distancia entre la masa y su punto de equilibrio x_0. Ahora bien, como

$$v(t) = A\omega\cos(\omega t + \phi)$$

tenemos que $v_{\max} = A\omega$ (cuando el coseno es uno). Además,

$$\mu(0) = A\sin(\phi_0)$$

y a esto le denominamos fase inicial. Describe algo que sucede en el instante $t = 0$.

Los puntos exxtrmeos del movimiento, con distanca A, son donde la velocidad es cero. El punto de equilibrio es el punto donde la velocidad es máxima.

7.3 Energía

La energía es la capacidad de un sistema para realizar un proceso (cambio). Matemáticamente, es un escalar independiente del tiempo. Un sistema puede ser un sistema de partículas, una masa particular, entre otras cosas. La unidad de la energía es $[E] = [F]\,[d]$, fuerza por distancia, y se denomina Jule o simplemente J. Usaremos conceptos como *trabajo* (W de *work*), energía potencial, energía cinemática, energía mecánica, calor, etc.

7.3.1 El trabajo (invertido/realizado) de una fuerza

Supongamos un sistema simple con un cuerpo puntual, un recorrido unidimensional, con desplazamiento bajo la acción de una fuerza. Sea $\vec{\Delta x} = \Delta x\hat{\imath} = (x_f - x)\hat{\imath}$, con x_f la posición final. Sabemos que

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} = F\cos\theta\hat{\imath} + F\sin\theta\hat{\jmath}$$

El **trabajo** invertido por la fuerza en mover el objeto es

$$W := F_x\vec{\Delta x} \tag{1}$$

es decir, es fuerza por distancia. Claramente,

$$W = F\cos\theta\vec{\Delta x} = \vec{F}\cdot\vec{\Delta x}$$

Es decir, W es el component de la fuerza en la dirección del movimiento, producto por el deslpazamiento. (Notar que $\vec{F}\vec{\Delta x}$ es un producto escalar: $\vec{A}\cdot\vec{B} = |A|\,|B|\cos\theta$ con θ el ángulo entre ambos.)

Las unidades del trabajo son unidades de fuerza por unidades de longitud: $[W] = [F]\,[\Delta x]$.

Ahora bien, es posible que $\vec{F}$ varíe con la posición, i.e. que $\vec{F}$ sea una función x. Entonces, podemos calcular diferenciales de trabajo, es decir el cambio en el trabajo de acuerdo a la posición:

$$\frac{dW}{dx} = \vec{F}(x)$$

o bien

$$dw = \vec{F}(x)\ d\vec{x} \tag{2}$$

Integrando (2) obtenemos

$$W = \int_{x_0}^{x_f} \vec{F}(x)\ d\vec{x} \tag{3}$$

Más complejo aún, ¿qué pasa si la trayectoria del objeto es 2D (curva)? También aquí la fuerza varía dependientemente de la trayectoria. Sea $d\vec{s}$ el vector que indica el cambio de posición instantáneo del objeto en su trayectoria. Se deduce:

$$W = \int_{s_0}^{s_f} \vec{F}(s)\, d\vec{s} \tag{4}$$

A la integral de (4) se le llama *integral de línea*.

Si hay muchas fuerzas aplicadas, $\vec{F}_1, \vec{F}_2, \ldots$, el trabajo total será

$$W_{\text{total}} = W_{F_1} + WF2 + \ldots = \sum_i W_{F_i}$$

Respecto al signo del trabajo, recordemos que $W = \vec{F} \cdot \vec{\Delta x}$. Si $0 \leq \theta < \frac{\pi}{2}$, entonces la fuerza está ”tironeando” del cuerpo, pues el ángulo está entre 0 y 45 grados. El trabajo será positivo, y diremos que la fuerza *entrega* trabajo.

Si $\frac{\pi}{2} < \theta \leq \pi$, el producto escalar será negativo y por ende lo será el trabajo. Este caso se corresponde por ejemplo con un cuerpo que se mueve hacia la derecha y una fuerza que lo ”tira” hacia arriba a la izquierda, u directamente a la izquierda. Decimos entonces que la fuerza *frena* al objeto.

Si $\theta = \frac{\pi}{2}$, es decir el ángulo es de 90 grados, el producto escalar es cero, y el trabajo es cero. La fuerza normal es un ejemplo.