

# 1 Info

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## Temas:

- Cinemática y dinámica (mecánica)
- Campos eléctricos y magnéticos
- Circuitos
- Termodinámica

# 2 Measurements and magnitudes

Measurements seek to compare a prediction with an observation, so as to test a hypothesis. A magnitude is a number accompanied by a unit. Some magnitudes are:

- Length, measured in meters ( $m$ )
- Time, measured in seconds ( $s$ )
- Mass, measured in kilograms (kg)
- Current, measured in amperes ( $A$ )
- Temperature, measured in kelvins ( $k$ )
- Matter, measured in moles (mol)

We consider  $10^3$  (e.g. kilometer) and  $10^{-3}$  (e.g. millimeters) to be within human scale. We call mass, seconds and kilograms the *mechanical units*. We define the *force unit*, or Newton, as

$$[F] = N = \text{kg} \frac{m}{s^2}$$

and the Pascal unit as

$$[P] = \text{Pa} = \frac{N}{m^2}$$

We use scientific notation and terms which express quantities as powers of ten. For instance,  $10^{12}$  is the tera,  $10^3$  the giga, etc.

The magnitudes hereby described are suited for algebraic manipulation. For instance,  $m \times m = m^2$ , and  $s \times \frac{m}{s} = m$ .

### 3 Vectors

Vectors are used to express position, displacement, velocity, force, acceleration, fields, etc. A vector  $\vec{A}$  (or sometimes  $\vec{a}$ ) in the general sense has a direction (line), an orientation, and a length (or magnitude). A vector also has an application point, which denotes the point of origin of the vector. When saying  $\vec{a} = \vec{b}$ , we mean that  $\vec{a}$  and  $\vec{b}$  coincide in direction, magnitude and orientation, irrespective of their application point.

The scalar product is defined as the usual mapping in the space  $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ . Intuitively, the scalar product  $\lambda \vec{a}$  "stretches" or "shrinks" a vector, depending on whether  $|\lambda| < 1$  or not, and the positivity or negativity of  $\lambda$  determines whether the vector inverts its direction or not. In general,  $|\lambda \vec{a}| = |\lambda| |\vec{a}|$ .

The sum of vectors,  $\vec{a} + \vec{b}$ , is a mapping  $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ . As usual, and in a graphical sense, the sum corresponds to the application of the parallelogram rule.

**Parallelogram rule.** Make  $\vec{a}$  and  $\vec{b}$  coincide in their point of application. From the tip of  $\vec{a}$ , draw a copy of  $\vec{b}$ , and from the tip of  $\vec{b}$  a copy of  $\vec{a}$ . The corner of the thus generated parallelogram is the tip of  $\vec{a} + \vec{b}$ .

Alternatively, from the tip of  $\vec{a}$  write  $\vec{b}$ . Then  $\vec{a} + \vec{b}$  is the vector which goes from the point of application of  $\vec{a}$  to the tip of  $\vec{b}$ .

The sum of vectors is commutative, associative, and distributive with respect to scalar product.

If  $\vec{A}$  is a vector, we use  $A_x$  and  $A_y$  to denote the projection of the vector over the axis  $x$  or  $y$ , respectively. Using  $A_x$  and  $A_y$  one forms a rectangular triangle with sides  $A_x$ ,  $A_y$  and a hypotenuse of length  $|\vec{A}|$ .

Let  $\theta$  be the angle formed by  $\vec{A}$  with the  $x$ -axis. Then, using trigonometry,

$$\cos \theta = \frac{A_x}{|\vec{A}|}, \quad \sin \theta = \frac{A_y}{|\vec{A}|}$$

from which one can find  $A_x, A_y$  assuming one knows  $\theta$ . From this follows that  $|\vec{A}|$  and  $\theta$  fully determine all the information about the vector, insofar as they allow us to determine  $A_x, A_y$ . Conversely, knowing  $A_x$  and  $A_y$  is also sufficient to determine  $\vec{A}$ , insofar as

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}, \quad \frac{A_y}{A_x} = \frac{|\vec{A}| \sin \theta}{|\vec{A}| \cos \theta} = \tan \theta \Rightarrow \theta = \arctan \left( \frac{A_y}{A_x} \right)$$

As convention, we use  $\hat{i}$  to denote the versor (vector of length 1) with direction parallel to the  $x$ -axis, and  $\hat{j}$  the versor with direction parallel to the  $y$ -axis.

Notice that, for any vector  $\vec{A}$ ,  $A_x$  is  $\hat{i}$  times  $A_x$ , and  $A_y$  is  $\hat{j}$  times  $A_y$ , which means

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

When writing  $\vec{A}$  in this way, we say we write it in terms of its components  $x, y$ . In terms of linear algebra, it's not hard to see that we are simply expressing that  $\hat{i}, \hat{j}$  form a basis of  $\mathbb{R}^2$ . Thus, it is equivalent to write

$$A_x = |\vec{A}| \cos \theta, \quad A_y = |\vec{A}| \sin \theta$$

and

$$\vec{A} = |\vec{A}| (\cos \theta \hat{i} + \sin \theta \hat{j})$$

From this follows as well that

$$\begin{aligned} \vec{A} + \vec{B} &= (A_x \hat{i} + A_y \hat{j}) + (B_x \hat{i} + B_y \hat{j}) \\ &= \hat{i} (A_x + B_x) + \hat{j} (A_y + B_y) \end{aligned}$$

which means the sum of vectors has as components the sum of the components.

The scalar product of two vectors,  $\vec{A} \cdot \vec{B}$ , is a scalar defined as

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

where  $\theta$  is the angle formed by the two vectors. The scalar product is positive if  $\cos \theta$  is positive, which occurs for  $0 < \theta \leq 90$ . It is negative if  $\cos \theta$  is negative, i.e. if  $90 < \theta \leq 180$ . Clearly,  $\vec{A} \cdot \vec{B} = 0 \iff \theta = 90$ .

In general, from the definition follows that

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

The vectorial product  $\vec{A} \times \vec{B}$  is a vector perpendicular to the plane formed by  $\vec{A}$  and  $\vec{B}$ . Its module is  $|\vec{A}| |\vec{B}| \sin \theta$ , and its direction is given by what's called the right-hand rule.

### 3.1 Exercises

(2) Sean los vectores  $\vec{A} = 2\hat{i} + 3\hat{j}$ ,  $\vec{B} = 4\hat{i} - 2\hat{j}$  y  $\vec{C} = -\hat{i} + \hat{j}$ . Determinar la magnitud y el ángulo (representación polar) de los vectores resultantes  $\vec{D} = \vec{A} + \vec{B} + \vec{C}$  y  $\vec{E} = \vec{A} + \vec{B} - \vec{C}$ . Resolver analítica y gráficamente.

(Analytical solution.) We'll use  $A_x, A_y$  to denote the components of the vector  $\vec{A}$ , and same for all other vectors. We know the components of  $\vec{D}$  are

$$D_x = A_x + B_x + C_x = 2 + 4 - 1 = 5, \quad D_y = 3 - 2 + 1 = 2$$

from which readily follows that  $|\vec{D}| = \sqrt{5^2 + 2^2} = \sqrt{29} \approx 5.385$ . Similarly,

$$E_x = 2 + 4 + 1 = 7, \quad E_y = 3 - 2 - 1 = 0$$

from which follows that  $|\vec{E}| = \sqrt{7^2} = 7$ .

Now, we must recall that

$$\theta_{\vec{Z}} = \arctan\left(\frac{Z_y}{Z_x}\right)$$

for any  $\vec{Z}$ .

We need not memorize this: it is trigonometrically clear that  $Z_x = \cos \theta_{\vec{Z}} |\vec{Z}|$  and  $Z_y = \sin \theta_{\vec{Z}} |\vec{Z}|$ , and therefore

$$\frac{Z_y}{Z_x} = \tan \theta$$

And arctan is the inverse of tan. Anyhow, for  $\vec{E}$  and  $\vec{D}$  we have

$$\theta_{\vec{E}} = \arctan\left(\frac{E_y}{E_x}\right) = \arctan(0) = 0$$

$$\theta_{\vec{D}} = \arctan\left(\frac{D_y}{D_x}\right) = \arctan\left(\frac{2}{5}\right) \approx 0.38$$

(3) Can two vectors of different magnitude be combined and yield zero? What about three?

The zero vector is the only vector with magnitude zero. Let  $\vec{A}, \vec{B}$  arbitrary vectors. Then

$$|\vec{A} + \vec{B}| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

which is zero if and only if

$$(A_x + B_x)^2 + (A_y + B_y)^2 = 0$$

This only holds if  $A_x + B_x = A_y + B_y = 0$ . But

$$A_x + B_x = 0 \Rightarrow A_x = -B_x, \quad A_y + B_y = 0 \Rightarrow A_y = -B_y$$

But then

$$|A| = \sqrt{A_x^2 + A_y^2} = \sqrt{(-B_x)^2 + (-B_y)^2} = \sqrt{B_x^2 + B_y^2} = |B|$$

$$\therefore |\vec{A} + \vec{B}| = 0 \iff |\vec{A}| = |\vec{B}|.$$

It is simple to see that three vectors of different magnitude can add to zero.

Assume  $A + B + C = 2\hat{i} + \hat{j}$  and  $A = 6\hat{i} - 3\hat{j}$ ,  $B = 2\hat{i} + 5\hat{j}$ . Find the components of  $C$ . Solve analytically and graphically.

We know

$$6 + 2 + C_x = 2, \quad -3 + 5 + C_y = 1$$

from which follows that  $C_x = -6$ ,  $C_y = -1$ .

(5)  $A$  and  $B$  have a magnitude of  $3m, 4m$  respectively. The angle between them is  $\theta = 30$  degrees. Find their scalar product.

Their scalar product is

$$(|B| \cos \theta) |A|$$

Recall that

$$\text{Angle in degrees} = \text{Angle in radians} \cdot \frac{180}{\pi}$$

Thus, thirty degrees equates to  $30 \frac{\pi}{180} \approx 0.523$  radians. Then the scalar product is

$$4 \cos(0.523) \times 3 \approx 10.395$$



(6) Find the angle between  $A = 4\hat{i} + 3\hat{j}$  and  $B = 6\hat{i} - 3\hat{j}$ .

Recall that

$$A \cdot B = |A| |B| \cos \theta$$

where  $\theta$  is the angle between the vectors. This readily entails that

$$\frac{A \cdot B}{|A| |B|} = \cos \theta$$

or equivalently that

$$\theta = \arccos \left( \frac{A \cdot B}{|A| |B|} \right)$$

Now,  $A \cdot B = 4 \times 6 + 3 \times -3 = 24 - 9 = 15$  and  $|A| |B| = 5 \cdot 6.708 = 33.541$ .

Therefore,

$$\theta = \arccos \left( \frac{15}{33.541} \right) = \arccos (0.447) = 1.107$$

(7) Let  $\vec{v} = \left(\frac{1}{3}, \frac{2}{3}\right)$  be the vector of components. Find the components of the vector of module 5 whose direction and orientation (sentido) are those of the given vector.

Assume  $\vec{x} = (x_1, x_2)$  is of magnitude 5. Any vector whose direction and orientation are the same than those of  $\vec{v}$  is "a stretching" of  $\vec{v}$ . In other words, for  $\vec{x}$  to satisfy the requirements, we must have

$$\vec{x} = \lambda \vec{v} \quad (1)$$

for some  $\lambda \in \mathbb{R}$ . (Furthermore,  $\lambda > 0$  since otherwise orientation is not preserved.)

Now, from equation (1) follows that

$$\|\vec{x}\| = \lambda \|\vec{v}\| \quad (2)$$

since the magnitude of a scaled vector is the scaled magnitude of the vector. Equation (2) simplifies to

$$\|\vec{x}\| = \lambda \sqrt{1/9 + 4/9} = \frac{\lambda \sqrt{5}}{3} \quad (3)$$

From this readily follows that  $\frac{3}{\sqrt{5}} \|\vec{x}\| = \lambda$ . But it is a hypothesis that  $\|\vec{x}\| = 5$ . Therefore,

$$\lambda = \frac{3}{\sqrt{5}} \cdot 5 = \frac{15}{\sqrt{5}} \quad (4)$$

In other words,

$$\vec{x} = \frac{15}{\sqrt{5}} \vec{v} \quad (5)$$

which is ugly but can be simplified.

(8) Write the expression of the vector product  $\vec{c} = \vec{u} \times \vec{v}$  in the following cases:

1.  $\vec{u}, \vec{v}$  are coplanar. Provide a graphical interpretation.
2.  $\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}$  and  $\vec{v} = -3\hat{i} + \hat{j} + 2\hat{k}$ . Find the module of the resulting vector  $\vec{c}$  in two different ways.

(1) Two vectors are coplanar if there is a plane which contains them both. Since the vector product  $\vec{u} \times \vec{v}$  is a vector orthogonal to both  $\vec{u}$  and  $\vec{v}$

**(12)** Un avión vuela 200 km hacia el NE en una dirección que forma un ángulo de 30 hacia el este de la dirección norte. En ese punto cambia su dirección de vuelo hacia el NO. En esta dirección vuela 60 km formando un ángulo de 45 con la dirección norte.

- (a) Calcular la máxima distancia hacia el este del punto de partida a la que llegó el avión.
- (b) Calcular la máxima distancia hacia el norte del punto de partida a la que llegó el avión.
- (c) Calcular la distancia a la que se encuentra el avión del punto de partida al cabo de su recorrido.
- (d) Determinar vectorialmente el camino que debería hacer para volver al punto de partida. Resolver gráfica y analíticamente.

Sea  $\vec{A}$  el vector que describe el primer recorrido,  $\vec{B}$  el vector que describe el segundo recorrido. Al final del problema, el avión se encuentra en la posición indicada por  $\vec{A} + \vec{B}$ .

Como  $\vec{A}$  describe un movimiento con un ángulo de  $\theta = 60$  grados ( $90 - 30$ ) respecto al eje y (norte), y una magnitud de 200km, podemos determinarlo recordando que

$$A_x = |\vec{A}| \cos \theta, \quad A_y = |\vec{A}| \sin \theta$$

En radianes,  $\theta = \frac{\pi}{180} \times 60 = 1.047$

$$A_x = 200 \times \cos(1.047) = 100.034, \quad A_y = 200 \times \sin(1.047) = 173.185$$

En conclusión,  $\vec{A} = 100.034\hat{i} + 173.185\hat{j}$ . Mismo razonamiento nos da que el ángulo del segundo vector es de 130 grados, lo cual en radianes nos da  $\alpha = \pi/180 \times 130 = 2.268$ . Por ende,

$$B_x = 60 \cos(2.268) = -38.524, \quad B_y = 60 \sin(2.268) = 45.998$$

Es decir que  $\vec{B} = -38.524\hat{i} + 45.998\hat{j}$ . De esto se sigue que  $\vec{C} = \vec{A} + \vec{B} = (61.51, 219.183)$ .

- (a) Claramente, es la coordenada  $x$  del vector  $\vec{A}$ , 100.034.
- (b) Claramente, es la coordenada  $y$  del vector  $\vec{C}$ : 218.183.
- (c) Claramente, es la magnitud de  $\vec{C}$ , es decir  $\|\vec{C}\| = \sqrt{61.51^2 + 219.183^2} = 227.65$ .
- (d) El camino para volver es dado por  $\vec{C} \times (-1)$ .

## 4 Cynematics

### 4.1 Unidimensional movement

The study of movement requires two variables: position ( $x$ , in units of length) and time ( $t$ , in seconds). We begin our study with unidimensional movement, i.e. movement which occurs along a single axis.

Experimentally, a way to study unidimensional movement could consist in taking a sequence of photographs (from the same position and angle) of the moving object at times  $t_1, \dots, t_n$ . Some coordinate system must be imposed upon the space along which the object moves, e.g. setting an axis with origin at the initial position of the object, the same direction as the movement of the object, and some appropriate units. The photographs would then provide a sequence of positions  $x_1, \dots, x_n$ .

Clearly,  $\{t_n\}, \{x_n\}$  could be understood as defining a discrete function  $\varphi(n)$ , which on its turn might be interpolated to obtain a continuous approximation  $\phi(t)$ . To the limit, the continuous approximation converges to what we call a movement function.

**Movement function.** A movement function  $x(t)$  is a continuous, smooth function.

**Examples.**  $x(t) = c$  (reposito),  $x(t) = at + b$  (MRU),  $x(t) = at^2 + bt + c$  (MRUV).

### 4.2 Coincidence, displacement, temporal intervals

If  $A, B$  are objects with movement functions  $x_A(t), x_B(t)$ , we say  $A, B$  coincide (se encuentran) when  $x_A(t) = x_B(t)$ .

We define displacement (desplazamiento) (relative to positions  $x_1, x_2$ ) as  $\Delta x = x_2 - x_1$ . Notice that  $\Delta x$  is not the same as distance: if one travels from  $A$  to  $B$  and then to  $B$  from  $A$ , the distance traveled is to times the distance from  $A$  to  $B$ , but  $\Delta x = 0$ .

We also define a temporal interval, relative to two times  $t_1, t_2$ , as  $\Delta t = t_2 - t_1$ , where  $t_2 > t_1$ .

### 4.3 Velocity

We define *median velocity* (velocidad media) as

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \quad (1)$$

where  $x_2 = x(t_2), x_1 = x(t_1)$ . Clearly,  $\bar{v}$  is the slope of the line which intersects  $x(t)$  at points  $t_1, t_2$ . The sign of  $\bar{v}$  then determines the direction (sentido) of movement. The unit of  $\bar{v}$  is then  $L/T$  (length over time), for instance kilometers per hour. Median velocity indicates the rate of change of distance in time.

Clearly, an object in reposo has a median velocity of zero. An object with movement function  $x(t) = at + b$  (MRU) has median velocity  $a$ . The case of interest is an object with a quadratic movement function (MRUV).

If  $x(t) = at^2 + bt + c$ , let  $m$  the midpoint of the quadratic expression and take  $t_1 = m - c, t_2 = m + c$  with  $c > 0$ . Clearly, the median velocity from  $t_1$  to  $m$  is negative, that from  $m$  to  $t_1$  is positive, and that from  $t_1$  to  $t_2$  is zero. This is sufficient to suggest that median velocity does not clearly express the nature of the movement.

For that reason, the length  $\Delta$  of the interval  $[t_1, t_2]$  might be reduced in the limit to zero, so that we get an accurate notion of the instantaneous (or close to instantaneous) change of direction. Needless to say, the limit converges to the slope of the line tangent to  $(t_1, x(t_1))$ , i.e. the derivative of  $x(t)$  at  $t_1$ . Thus, we obtain the definition of instantaneous velocity, usually called simply velocity:

$$v(t) = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = v'(t) \quad (2)$$

Again,  $[v(t)] = \frac{L}{T}$ . Quite clearly,  $v(t) = \bar{v}$  for constant and linear functions, but for the quadratic function  $x(t)$  we have

$$x(t) = at^2 + bt + c, \quad v(t) = 2at + b$$

## 4.4 Exercises

(1) Consider

$$x(t) = 1 \left[ \frac{m}{s^2} \right] t^2 - 3 \left[ \frac{m}{s} \right] t$$

the movement function of a body travelling in a straight line, with  $x$  in meters and  $t$  in seconds.

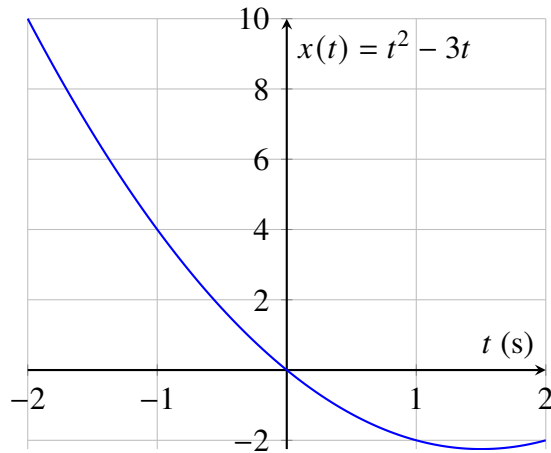
(a) Plot  $x(t)$

(b) Determine analytically the median velocity in  $[-1, 5]$ ,  $[-1, 8]$ ,  $[-1, 0.9]$ ,  $[-1, 0.99]$ ,  $[-1, 0.999]$ .

(c) Let  $\Delta t_n = t_n - t_0$  with  $\{t_n\} = \{-1, 5, 4, 1, -0.5, -0.8, -0.9, -0.99, -0.999\}$  all measured in seconds. To what value does the median velocity of the object converge as  $t_n$  decreases in the interval  $[-1, -1 + \Delta t_n]$ ? What is the geometrical interpretation of this result?

(d) Find the equation for the line tangent to  $x(t)$  at  $t = -1s$ .

(a) Notice that since  $t$  is in seconds,  $\frac{m}{s^2}$  correctly expresses a quantity in meters, and so does  $\frac{mt}{s}$ . So we will from now on write simply  $x(t) = t^2 - 3t$ , understanding that it is a mapping from time in seconds to meters.



(b) The median velocity of an object in the time interval  $[t_a, t_b]$  was given by

$$\frac{\Delta x}{\Delta t} = \frac{x(t_b) - x(t_a)}{t_b - t_a} \quad (1)$$

So exercise (b) is as simple as plugging in the corresponding values into equation (1) and I skip it.

(c) Let  $t$  be an arbitrary value. Then by definition of  $\frac{dx}{dt}$ ,

$$\lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

the derivative of  $x(t)$  at time  $t$ . In particular, the limit whose convergence we are asked to study is nothing but the limit above with  $t = -1$ :

$$\lim_{\Delta t \rightarrow 0} \frac{x(-1 + \Delta t) - x(-1)}{\Delta t} = x'(-1)$$

So suffices to observe that  $x'(t) = 2t - 3$  and  $x'(-1) = -5$ . In conclusion, the object at time  $t = -1$  travels at an instantaneous velocity of  $-5$  meters per second.

(d) The line  $\ell(t) = at + b$  tangent to  $x(t)$  at  $t = -1$  s has slope  $a = -5$  and crosses through the point  $(-1, 4)$ . So we must have  $-5(-1) + b = 4 \iff b = 4 - 5 = -1$ . So the line is  $\ell(t) = -5t - 1$ .



#### (4) Answer the questions.

(a) Can an object have null velocity and yet possess acceleration?

Let  $x(t)$  describe the movement of the object and  $v(t) = x'(t)$  its velocity, both as a function of time. Assume for an arbitrary  $t_0$  that  $v(t_0) = 0$ . It is very much possible that  $v'(t_0) \neq 0$ .

Consider, for instance, that  $v(t)$  is linear and non-constant, making  $v'(t) = a$  a non-null constant. Then there exists a unique root  $r$  s.t.  $v(r) = 0$ , but independently of this fact  $v'(r) = a \neq 0$ .

Physically, it should be clear that if a non-moving object could not possess acceleration, then it would be impossible for it to pass from a still to a moving state. So, at least at the intuition level, this *reductio ad absurdum* suffices.

(b) Can a moving object have a null displacement in a given interval and yet non-null velocity?

Naturally. Take as example an object moving in circles at a constant, non-null velocity  $v$ , and assume it travels a full circle in  $t$  seconds. Then all of the intervals in  $\{[t_0, t_0 + tk] : k \in \mathbb{N}\}$  are such that they give null displacements. Yet the object *is* moving.

(c) Can an object have an east-bound velocity of while its acceleration is west-bound?

Informally, this is quite clearly the case, insofar as any positively-moving object whose velocity decreases must have a negative acceleration.

(d) Consider an object moving on a straight line, with the east being the positive direction, under a velocity of  $v(t) = 20\text{ms}^{-1} - 2\text{ms}^{-2}t$ . For  $t = 0\text{s}$ ,  $t = 1\text{s}$ , what is the situation?

Its velocity is clearly positive in both cases (20 and 18), evidently decreasing, which points out the fact that its acceleration is negative (-2).

(e) A ball is thrown vertically. What do the *signs* of velocity and acceleration look like as the object ascends, and what does that mean? And when the object descends? What happens at the highest point?

Clearly, its velocity is positive during the ascending phase, and negative during the descending phase. At the highest point, the velocity will be exactly zero.

Conversely, acceleration is always negative due to the force exercised by gravity on the ball.

It is the fact that acceleration is constantly negative what causes the ball not only to lose velocity as it goes up until it begins to fall again, but to then fall more and more rapidly as time goes by.

(5) A particle moves through the  $x$ -axis with movement function  $x(t) = 3 + 17t - 5t^2$ , with  $x$  in meters and  $t$  in seconds.

(a) What is the position of the particle at times  $\{1, 2, 3\}$ ?

(b) At what point in time does the particle return to the origin?

(c) Find  $v(t)$  and determine the instantaneous velocity at times  $\{1, 2, 3\}$ . When is the velocity null? What is the particles velocity when it crosses the origin?

(d) Plot  $x(t)$ ,  $v(t)$ ,  $a(t)$ .

(a) Trivial, simply compute  $x(1), x(2), x(3)$ .

(b) See that

$$x(t) = 0 \iff t = \frac{17}{10} \pm \frac{\sqrt{17^2 + 4 \times 3 \times 5}}{10}$$

which gives approximate solutions  $t_1 \approx -0.168$ ,  $t_2 \approx 3.568$ . It makes no sense to speak of negative time and we keep only the positive solution  $t = 3.568$ . Thus, the particle returns to the origin after approximately 3.568 seconds.

(c) The first derivative of  $x(t)$  is

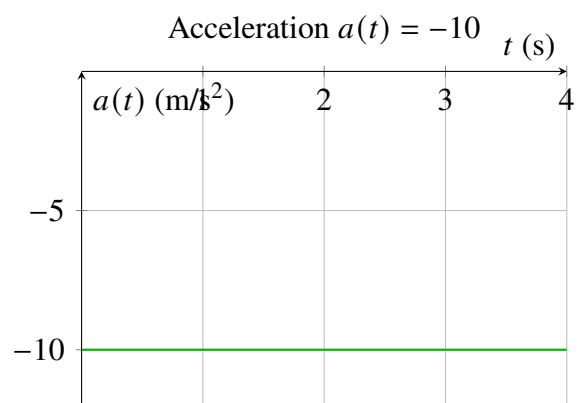
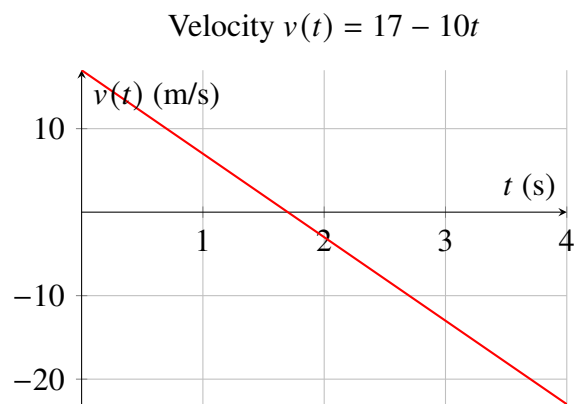
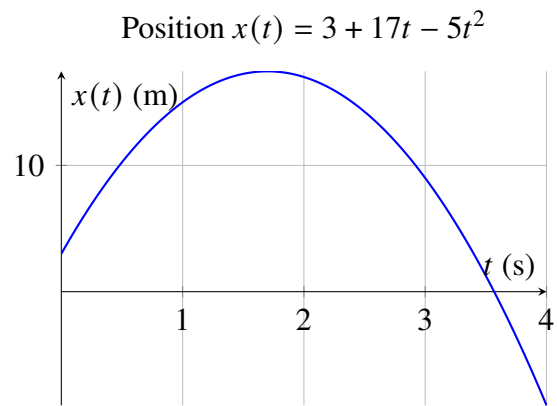
$$v(t) = -10t + 17$$

The instantaneous velocity at times 1, 2, 3 are  $v(1) = 7$ ,  $v(2) = -3$ ,  $v(3) = -13$ .

$$17 = 10t \iff t = \frac{17}{10} = 1.7$$

The particle crosses the origin at approximately time 3.568 and its velocity is approximately  $v(3.568) = -18.682$ .

(d) The acceleration is constant:  $a(t) = -10$ .



- (6) (a) Determine the instantaneous acceleration of the object plotted in the exercise for  $t = 3, t = 11$ .
- (b) Compute the distance traveled by the object in the time intervals  $[0, 5]$  and  $[0, 9]$  and  $[0, 15]$ .
- (c) Knowing that  $x(t = 6) = 0$ , find the position of the object at  $t = 0$ .
- (d) Give an expression for the object's position for all  $t$ .
- (e) Plot  $x(t), v(t), a(t)$ .

The velocity of the object is constantly 20m/s from time  $t = 0$  to time  $t = 6$ . Then it linearly increases until it reaches 44m/s at  $t = 9$ , from whereon it linearly decreases until it reaches 0m/s at time  $t = 15$ .

The linear expression  $\ell_1(t) = a_1t + b_1$  which satisfies  $\ell(6) = 20, \ell(9) = 44$  is such that

$$6a_1 + b_1 = 20, \quad 9a_1 + b_1 = 44$$

The associated system of equations yields that  $b_1 = 20 - 6a_1$ , from which follows that  $9a_1 + (20 - 6a_1) = 44$ , entailing

$$a_1 = \frac{24}{3} = 8$$

From this readily follows that  $b_1 = 20 - 6 \times 8 = -28$ .

Similarly, the linear expression  $\ell_2(t) = a_2t + b_2$  which gives a line s.t.  $\ell_2(9) = 44, \ell_2(15) = 0$  must satisfy

$$9a_2 + b_2 = 44, \quad 15a_2 + b_2 = 0$$

Then  $b_2 = -15a_2$  and  $9a_2 - 15a_2 = 44$ , entailing  $a_2 = -\frac{22}{3}$ . From this follows that  $b_2 = 110$  via simple calculations. Thus,

$$v(t) = \begin{cases} 20 & 0 \leq t \leq 6 \\ 8t - 28 & 6 < t \leq 9 \\ -\frac{22}{3}t + 110 & 9 < t \leq 15 \end{cases}$$

It should be intuitive to grasp that the distance travelled  $d(a, b)$  in the interval  $[a, b]$  is

$$d(a, b) = \int_a^b |v(t)| dt$$

If  $v(t)$  is in meters per second, and  $t$  is in seconds, the total number of meters travelled in a time interval  $[a, b]$  is the summation of the meters per second travelled in every instant! This will involve the anti-derivative of  $v(t)$ , i.e. the movement function  $x(t)$ , which we might as well compute at once.

$$\begin{aligned} x(t) &= \int v(t) dt \\ &= \begin{cases} 20t + C_1 & 0 \leq t \leq 6 \\ 8\frac{t^2}{2} - 28t + C_2 & 6 < t \leq 9 \\ -\frac{22}{3}\frac{t^2}{2} + 110t + C_3 & 9 < t \leq 15 \end{cases} \\ &= \begin{cases} 20t + C_1 & 0 \leq t \leq 6 \\ 4t^2 - 28t + C_2 & 6 < t \leq 9 \\ -\frac{22}{6}t^2 + 110t + C_3 & 9 < t \leq 15 \end{cases} \end{aligned}$$

The constants  $C_1, C_2, C_3$  must satisfy the restriction of continuity and of preserving the necessary values. In particular, we need  $20(0) + C_1 = 20$ . Since we know  $x(6) = 0$ , we need  $120 + C_1 = 0$ , i.e.  $C_1 = -120$ . We also need

$$4(6^2) - 28(6) + C_2 = 0$$

to ensure continuity, so

$$C_2 = 24$$

Then we can know what  $x(9)$  is and deduce  $C_3$ , which ends up being  $-597$ .

$$\therefore x(t) = \begin{cases} 20t - 120 & 0 \leq t \leq 6 \\ 4t^2 - 28t + 24 & 6 < t \leq 9 \\ -\frac{22}{6}t^2 + 100t - 597 & 9 < t \leq 15 \end{cases}$$

In any case, we could have computed the distance travelled without  $x(t)$  (I computed  $x(t)$  because it's part of the exercise):

$$\begin{aligned}
 d(0, 5) &= \int_0^5 v(t) \, dt \\
 &= 20 \times t \Big|_0^5 \\
 &= 20 \times (5) \\
 &= 100
 \end{aligned}$$

$$\begin{aligned}
 d(0, 9) &= \int_0^6 v(t) \, dt + \int_6^9 v(t) \, dt \\
 &= 20 \times t \Big|_0^6 + (4t^2 - 28t) \Big|_6^9 \\
 &= 120 + [(4 \times 81 - 28 \times 9) - (4 \times 36 - 28 \times 6)] \\
 &= 120 + 96 \\
 &= 216
 \end{aligned}$$

etc.

## 5 De adelante hacia atrás

Recordemos que si  $x(t)$  es función de movimiento,  $v(t) = \frac{dx}{dt}$  es la velocidad, y  $a(t) = \frac{d^2x}{dt^2}$  es la aceleración. Naturalmente, esto significa que

$$x(t) = \int v(t')dt' + D, \quad v(t) = \int a(t')dt' + C$$

donde  $D, C$  son constantes de movimiento que dependerán de las condiciones iniciales.

(†) **Derivada y puntos de inflexión** Un punto de inflexión de  $f$  continua y dos veces derivable en  $[a, b]$  es un valor  $x_0 \in [a, b]$  t.q.  $f^{(2)}(x_0) = 0$ . Los puntos de inflexión representan un cambio de comportamiento en  $f$ , en particular transiciones de cóncava a convexa y viceversa.

Intuitivamente, y exceptuando casos límite (como  $f''$  constante), si  $f''(x_0) = 0$ , entonces alrededor de  $x_0$  hay un cambio de signo en  $f''$ , lo cual quiere decir que en el entorno alrededor de  $x_0$  la función original pasa de crecer a decrecer, o de decrecer a crecer.

## 6 Movimiento bidimensional (cinemática 2D)

Se modela con una curva en el plano cartesiano. La curva (el dibujo del movimiento) se denomina trayectoria. La trayectoria *no* es una función, obviamente (e.g. una circunferencia es una trayectoria posible).

La descripción de la posición del objeto en cada instante de tiempo  $t$  se describe con vectores. En el plano cartesiano, decimos que  $\vec{r} = x\hat{i} + y\hat{j}$  es un vector posición si la punta de  $\vec{r}$  se corresponde con la posición del objeto (en un tiempo dado).

Sea  $\theta$  el ángulo formado por  $\vec{r}$  y el eje  $x$ , de manera tal que  $x = |\vec{r}| \cos \theta$ ,  $y = |\vec{r}| \sin \theta$ . Es decir,  $\vec{r} = |\vec{r}| \cos \theta \hat{i} + |\vec{r}| \sin \theta \hat{j}$ .

Ahora pensemos el objeto en movimiento, y que registramos a lo largo del tiempo  $t$  las posiciones  $x(t), y(t)$ . Claramente,  $x(t), y(t)$  son funciones de movimiento unidimensionales. Por lo tanto, podemos definir el vector posición de manera general como

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \quad (1)$$

Ahora consideremos la trayectoria  $T$  (conjunto de puntos en el eje cartesiano) del objeto. Sean  $P_1, P_2 \in T$  dos puntos en el plano que pertenecen a la trayectoria. Definimos el desplazamiento del objeto como

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \quad (2)$$

donde  $\vec{r}_1, \vec{r}_2$  son los vectores con puntas en  $P_1, P_2$ . De esto se sigue que

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) \quad (3)$$

para dos instantes de tiempo  $t_1, t_2$ . Además, es claro que  $\Delta\vec{r}$  es el vector que conecta las dos puntas, desde  $P_1$  hasta  $P_2$ , y que  $\vec{r}_2 = \vec{r}_1 + \Delta\vec{r}$ .

La velocidad media en el intervalo de tiempo  $[t_1, t_2]$  se define entonces como

$$\bar{\vec{v}}_{[t_1, t_2]} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \quad (4)$$

lo cual es claramente un vector que contendrá las velocidades medias en las direcciones  $x$  e  $y$ . El vector velocidad entonces se define como uno esperaría:



$$\begin{aligned}
\vec{v}(t) &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \\
&= \frac{d\vec{r}}{dt} \\
&= \frac{d}{dx} (x(t)\hat{i} + y(t)\hat{j}) \\
&= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}
\end{aligned} \tag{5}$$

Por lo tanto,

$$\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} \tag{6}$$

con  $v_x, v_y$  las velocidades correspondientes a las funciones de movimiento  $x(t), y(t)$ .

(†) **Interpretación gráfica de  $\vec{v}$ .** Gráficamente, el vector velocidad  $\vec{v}(t)$  se representa como sigue. Imagine la trayectoria  $T$  y un vector posición  $\vec{r}$  que conecta con  $P \in T$ . Entonces  $\vec{v}$  será paralelo a la recta tangente a la trayectoria  $T$  en el punto  $P$ . Esto *no* es una derivada, porque la trayectoria  $T$  no es una función. En otras palabras, el vector velocidad es tangente a la trayectoria.

Definimos entonces el versor

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} \tag{7}$$

que nos da la dirección tangencial de la velocidad.

Así como tenemos un vector velocidad, tenemos el vector aceleración

$$\vec{a}(t) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} \tag{8}$$

de acuerdo al mismo razonamiento límite que nos dio el resultado (6). El vector aceleración también puede descomponerse en sus direcciones tangencial y normal (respecto a la trayectoria). La aceleración tangencial será la dirección de la velocidad:

$$\text{aceleración tangencial} \rightarrow \hat{v} = \frac{\vec{v}}{|\vec{v}|}, \quad \text{aceleración normal} \rightarrow \hat{n} = (\perp \vec{v}) \tag{9}$$

donde  $\perp \hat{v}$  es el vector tal que  $\perp \hat{v} \cdot \hat{v} = 0$  (perpendicular).

(†) **Algunas observaciones.** Hablemos en un intervalo de tiempo  $[t, t + \Delta t]$ .

(a) Cuando  $\vec{v}$  cambia de módulo y de sentido, pero no de dirección, la aceleración es puramente tangencial, es decir está en la dirección de la velocidad. (Son paralelos.)

(b) Cuando  $\vec{v}$  cambia de dirección y de sentido, pero no de módulo (i.e. el vector apunta hacia otro lado pero tiene el mismo largo), se cumple que la aceleración es perpendicular a la velocidad. Es decir, es puramente normal.

(c) Si  $\vec{v}$  cambia de módulo, de sentido y de dirección, el vector  $\vec{a}$  tendrá una componente tangencial y normal.

La aceleración puramente tangencial resulta en un movimiento unidimensional, una línea recta. El caso (b) corresponde a un movimiento bidimensional.

## 6.1 Tiro de proyectil

El tiro de proyectil es un movimiento bidimensional bajo la acción única de la gravedad. Si imaginamos la parábola de un proyectil, surgen preguntas cómo cuál es su altura máxima, su destino final, etc. Asumamos que se ha impuesto un sistema de coordenadas.

Observemos que el objeto *no* tiene aceleración en el eje  $x$ , sólo velocidad. En el eje  $y$  el proyectil es afectado por la gravedad. Por ende,

$$\vec{a} = a\hat{i} - 9.8\frac{m}{s^2}\hat{j}$$

Luego, en  $t = 0$ ,

$$\vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j} = \vec{v}(t = 0)$$

Asumamos que por convención,  $\vec{r}(t = 0) = 0$ . Por ende,

$$\vec{r}(t = 0) = x_0\hat{i} + y_0\hat{j}$$

Entonces,

$$\vec{v}(t) = v_{0x}\hat{i} - (9.8\frac{m}{s^2}t + v_{0y})\hat{j}$$

Integrando una vez más,

$$\vec{r}(t) = (v_{0x}t + x_0)\hat{i} + \left(-9.8\frac{m}{s^2}\frac{t^2}{2} + v_{0y}t + y_0\right)\hat{j}$$

Pero  $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ . Por ende,

$$x(t) = v_{0x}t + x_0, \quad y(t) = -9.8\frac{m}{s^2}\frac{t^2}{2} + v_{0y}t + y_0$$

Vemos entonces que  $x(t)$  es lineal, y su pendiente es la velocidad inicial (en  $x$ ) dada por  $v_{0x}$ .  $y(t)$ , por otro lado, es cuadrática con máximo. Dicho máximo se corresponde con el punto más alto. Es decir, el punto más alto ocurre en el tiempo  $t_m$  tal que

$$v_y(t_m) = 0 \implies t_m = \frac{v_{0y}}{9.8\frac{m}{s^2}}$$

El tiempo de vuelo está dado por el tiempo en que la altura se hace cero, es decir el  $t_{\text{vuelo}}$  tal que  $y(t_{\text{vuelo}}) = 0$ . Es la raíz máxima de  $y(t)$ . La trayectoria será despejando  $t$  de  $x$ :

$$t = \frac{x - x_0}{v_{0x}}$$

La trayectoria también será una parábola, en este caso particular. Y como en cada punto la aceleración es la aceleración negativa de la gravedad, los vectores aceleración a lo largo del tiempo serán siempre "flechas rectas hacia abajo".