1 P1

Let us recall that $P : \mathcal{P}(\Omega) \mapsto [0, 1]$ satisfies the following three conditions:

- $\forall \zeta \in\subseteq \Omega : 0 \leq P(\zeta) \leq 1$
- $P(\Omega) = 1$
- $\forall \{\zeta\}_{I \subseteq \mathbb{N}} : \zeta_i \cap \zeta_j = \emptyset : P(\bigcup_{i \in I})\zeta_i = \sum_{i \in I} P(\zeta_i)$

In general, I use ω to denote $|\Omega|$, and we should recall that whenever P is a constant map (i.e. when events are equiprobable) we have $\forall A \subseteq \Omega : P(A) = \frac{|A|}{\omega}$.

To keep notation brief, we shall speak of probabilities $P(A)$ without specifying that $A \subseteq \Omega$.

We recall the following properties, which follow strictly from the aforementioned facts:

- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$
- $A \subseteq B \Rightarrow P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Problem 1 (Problem 2 of the sheet) *Prove that $A \subseteq B \subseteq \Omega \Rightarrow P(B - A) = P(B) - P(A)$ and also $P(A) \leq P(B)$.*

Let $A \subseteq B \subseteq \Omega$.

$$
\begin{aligned}
B - A &= \{x \in B : x \notin A\} \\
&= \{x \in \Omega : x \in B \wedge x \notin A\} \\
&= \{x \in \Omega : x \in B \wedge x \in A^c\} \\
&= B \cap A^c
\end{aligned}
$$

Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we have

$$
P(A \cap B) = P(A) + P(B) - P(A \cup B)
$$

Then

$$
\begin{aligned}
P(B - A) &= P(B \cap A^c) \\
&= P(B) + P(A^c) - P(B \cup A^c) \\
&= P(B) + [1 - P(A)] - P(B \cup A^c)
\end{aligned}
$$

It is easy to see that, since $A \subseteq B$, $B \cup A^c = \Omega$. We readily obtain

$$
\begin{aligned}
P(B - A) &= P(B) + [1 - P(A)] - P(\Omega) \\
&= P(B) + 1 - P(A) - 1 \\
&= P(B) - P(A)
\end{aligned}
$$

quod erat demonstrandum. And since $P(B - A) = P(B) - P(A) \geq 0$, we must have $P(B) \geq P(A)$.

Problem 2 (3)

For ease of mind, let us write here that

$$P(A_i) = \begin{cases} .22 & i = 1 \\ .25 & i = 2 \\ .28 & i = 3 \end{cases}$$

We are also given

$$P(A_1 \cap A_2) = .11 \quad P(A_1 \cap A_3) = .005 \quad P(A_1 \cap A_3) = .07 \quad P(A_1 \cap A_2 \cap A_3) = .01$$

(1 : $P(A_1 \cup A_2)$ Observe that $P(A_1 \cap A_2) \neq P(A_1)P(A_2)$, which entails the events are not independent. We then have

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ &= .22 + .25 - .11 \\ &= .36 \end{aligned}$$

(2: $P(A_1^c \cap A_2^c \cap A_3)$) Observe that

$$\begin{aligned} P(A_1^c \cap A_2^c \cap A_3) &= P(\left[A_1^c \cap A_2^c\right] \cap A_3^c) \\ &= P([A_1 \cup A_2]^c \cap A_3^c) \\ &= P(([A_1 \cup A_2] \cup A_3)^c) \\ &= P(\Omega^c) \\ &= P(\emptyset) \\ &= 0 \end{aligned}$$

(3: $P((A_1^c \cap A_2^c) \cup A^c)$)

Problem 3 (4)

Se nos dice que cinco empresas deben firmar contratos de un grupo de 3 contratos posibles. Se nos dice además que cada empresa firma a lo sumo un contrato.

> Mucho ojo: no se nos dice que cada contrato se da a lo sumo a una empresa, sino que cada empresa firma a lo sumo un contrato. Es decir que varias empresas podrían firmar el mismo contrato.

Si contamos la opción "no firma ningún contrato", hay 4 opciones para cada una de las 5 empresas; es decir, hay $\omega = 4^5$ puntos en el espacio muestral.

Si cada evento es equiprobable, la probabilidad de que la tercera empresa reciba un contrato es

$$\frac{|\{x \in \Omega : \text{3era empresa firma contrato}\}|}{\omega}$$

El numerador puede calcularse restando a ω la cantidad de casos en que la tercera empresa no firma un contrato. Claramente, la cantidad de tales casos es 4^4; es decir, hay $4^5 - 4^4 = 768$ casos en que la tercera empresa firma algún contrato. Lo cual nos da una probabilidad de $768/4^5 = .75$.

Solución alternativa. Sea α una palabra sobre el alfabeto $\{0, \ldots, 3\}$ de longitud 5. ¿Cuántas tales palabras hay? Naturalmente, $4^5 = \omega$. Asumamos que $\alpha_3 \neq 0$; es decir que el tercer símbolo de α es no-nulo. ¿Cuántas palabras así hay? Naturalmente, $4^4 \times 3 = 768$. Es decir, $\frac{768}{4^5} = .75$ de las palabras tienen $\alpha_3 \neq 0$, lo cual coincide con nuestro resultado anterior.

Problem 4 (5)

From a set of 25 buses, 8 present flaws. 5 are randomly (in this context, uniformly) chosen. We are therefore dealing with equiprobable events.

There are $\omega = \binom{25}{5}$ possible ways of selecting the 5 buses. There are $\binom{8}{4}$ possible ways of selecting 4 out of the 8 flawed buses and $\binom{25-8}{1} = \binom{17}{1}$ ways of selecting the remaining bus from the set of non-flawed buses. From this follows that the desired probability is

$$\frac{\binom{8}{4} \cdot \binom{17}{1}}{\binom{25}{5}} = \frac{70 \cdot 17}{53130} = .022$$

With regards to the probability that at least 4 have flaws, we must take into account the cases where 4 have flaws and the cases where 5 have flaws, which are clearly disjoint. The probability then is

$$.022 + \frac{\binom{8}{5}}{53130} = .022 + .001 = .023$$

> **Note.** This is obviously a problem involving the hypergeometric distribution, closely related to the binomial distribution. But since we have not studied distributions yet, we cannot use this fact.

Problem 5 (6)

Let A, B, C, D, E denote the five faculty members. Two papers from a set of five are drawn to decide who will be chosen.

Observe that the order does not matter; i.e. drawing A and then B is the same than drawing B and then A. It follows that

$$\Omega = \mathcal{P}_2(\{A, \ldots, E\})$$

where $\mathcal{P}_i(\zeta) = \{S \in \mathcal{P}(\zeta) : |S| = i\}$. Naturally, $\omega = 5 \times 4 \cdot \frac{1}{2} = 10$, where we divide by 2 to exclude equivalent pairs (e.g. A, B and B, A).

> Alternatively, we could have reasoned that $\omega = \binom{5}{2} = 10$, the number of 2-element subsets of a 5-element set.

(a) We are asked for the probability of the event $\{A, B\}$. It should be obvious that all events $S \in \Omega$ are equiprobable, which entails $P(\{A, B\}) = \frac{1}{10} = .1$.

> Alternatively, we could have reasoned the following. There are two ways in which A and B may be chosen: A is chosen first and then B, or B is chosen first and then A. This gives $\frac{1}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4} = \frac{2\cdot}{5\cdot 4} = \frac{2}{20} = .1$

(b) We are asked for the probability that the selection contains C or D. It is straightforward to reason that there are $\binom{3}{2} = 3$ sets that do not contain neither C nor D. From which readily follows that there are $10 - 3 = 7$ sets containing C, D or both. $\therefore$ The desired probability is $\frac{7}{10}$.

(c) Let us change the notation a bit. Let $\{1, \ldots, 5\}$ be the professors we used to call $A, \ldots, E$. Let $a_i := \{3, 6, 7, 10, 14\}$ be the set of years of teaching of each professor, assuming a_1 corresponds to A, a_2 to B, etc. We are asked for the probability that the selected pair $\{j, k\}$ satisfies $a_j + a_k \geq 15$.

There are two ways to solve this problem: one slow but direct, one pretty but a bit more clever.

Direct solution. It is easy to see that, of all pairs j, k, only the following satisfy the requirement:

- $1, 5 \mapsto a_1 + a_5 = 17$
- $2, 4 \mapsto a_2 + a_4 = 16$
- $2, 5 \mapsto a_2 + a_5 = 20$.

- $3, 4 \mapsto a_3 + a_4 = 17$
- $3, 5 \mapsto a_3 + a_5 = 21$
- $4, 5 \mapsto a_4 + a_5 = 24$

So only 6 out of the 10 possible pairs satisfy the relationship, giving us the desired probability: $\frac{6}{10} = \frac{3}{5} = .6$.

Pretty solution. Draw the 4×5 boolean matrix $\mathcal{A}$ whose coefficients $\mathcal{A}_{ij}$ are 1 if $a_i + a_j \geq 15$, 0 otherwise. Since upper and lower diagonal entries are equivalent (the matrix is symmetric), and because $i \neq j$ in our experiment, the diagonal of the matrix should not be considered. This gives the representation

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}$$

where 6 out of 10 relevant entries are 1.

Problem 6 (7)

Let $M := \{m_1, \ldots, m_4\}$ and $W := \{w_1, \ldots, w_4\}$ be alphabets denoting the men and women, respectively. The sample space Ω consists of all permutations in $M \cup W$, which readily entails $\omega = 8!$.

(a) Consider the event ζ when at least one women $w \in W$ is among the first three elements in the sampled permutation. Then w could be the first, the second or the third element in the permutation, and we impose no condition on the rest of the elements. These events are evidently disjoint. So, if we denote with e_i the event where w is the ith element of the permutation, we have

$$P(\zeta) = \frac{P(e_1) + P(e_2) + P(e_3)}{8!}$$

Each e_i may occur in $7!$ ways, since we fix w at the ith element and we must only choose from the remaining 7 assistants.

$\therefore P(\zeta) = \frac{3 \cdot 7!}{8!} = \frac{3}{8}$

> **Note.** If you are interested in being very formal, this are the rigorous steps taken above.
>
> (1) $\zeta = e_1 \cup e_2 \cup e_3 \Rightarrow P(\zeta) = P(e_1 \cup e_2 \cup e_3)$.
>
> (2) $(\forall i, j : e_i \cap e_j = \emptyset) \Rightarrow P(\zeta) = P(e_1) + P(e_2) + P(e_3)$.
>
> (3) Since events are equiprobable, $P(e_i) = |e_i|/\omega$.
>
> (4) $|e_i| = 7!$ because e_i is a permutation of 7 elements.
>
> (5) $P(\zeta) = |e_1|/\omega + |e_2|/\omega + |e_3|/\omega = \frac{3 \times 7!}{8}$.

(b) Let ϱ denote the event where, after the first five meetings, all female assistants have been met. Let $(p_1, \ldots, p_8) \in \varrho$ be an arbitrary permutation, and denote it with $\overrightarrow{p}$. (Remember that an event is a subset of the sample space, and hence a set, so the expresssion $\overrightarrow{p} \in \varrho$ is well defined.)

The definition of ϱ entails that one and only one m_j exists in $p_1, \ldots, p_5$, since all $w_1, \ldots, w_4$ must lie in this sequence. So the number of ways in which we may construct $\overrightarrow{p}$ (i.e. the cardinality of ϱ) is readily determined by the number of ways in which we can place exactly one m_j among the first elements of $\overrightarrow{p}$.

There are 5 positions to place m_j, and 4 elements in M to choose from. Assuming m_j was placed at the kth position, we know there are $4!$ ways of placing the 4 women among the remaining positions in $p_1, \ldots, p_5$. So there are $5 \times 4 \times 4!$ ways to construct $p_1, \ldots, p_5$ for $\overrightarrow{p} \in \varrho$.

The positions p_6, p_7, p_8 must be chosen from the remaining 3 men, so there are 3×2 possibilities.

$\therefore |\varrho| = 5 \times 4 \times 4! \times 3 = 60 \times 4!$.

$\therefore P(\varrho) = \frac{|\varrho|}{\omega} = \frac{60 \times 4!}{8!} = .036$

2 Conditional probability

Problem 7 (8) *A box contains* 6 *red balls and* 4 *green balls. A second box contains* 7 *red balls and* 3 *green balls. A ball is randomly chosen from the first box and placed into the second. Then a ball is drawn from the second box and placed into the first.*

(1) Let R_i denote the event of choosing a red ball in the ith draw, and G_i the event of choosing a green ball in the ith draw. Evidently, $P(R_1) = \frac{6}{10}$.

If the event R_1 occurs, when the second draw is made, the second box contains 8 red balls and 3 green balls, entailing that $P(R_2 \mid R_1) = \frac{8}{11}$.

So, using the conditional probability formula, the event when both balls are red, $R_1 \cap R_2$, has probability $P(R_1 \cap R_2) = P(R_1) \cdot P(R_2 \mid R_1) = \frac{6}{10} \cdot \frac{8}{11} = .436$.

(2) We now inquire the probability that the number of red and green balls in the first box are the same at the beginning and the end of the experiment. Naturally, this entails either drawing both times a red or both times a green ball. We have already computed the probability of drawing both times a red ball. The probability of drawing both times a green ball is similarly computed:

$$
\begin{aligned}
P(G_1 \cap G_2) &= P(G_1) \cdot P(G_2 \mid G_1) \\
&= \frac{4}{10} \cdot \frac{4}{11} \\
&= .145
\end{aligned}
$$

Since $R_1 \cap R_2$ and $G_1 \cap G_2$ are obviously disjoint, the probability that either of them occurs is simply $.436 + .145 = .581$, the sum of the probabilities of both events.

Problem 8 (10) *Given events* A, B *with* $P(B) > 0$, *prove* $P(A \mid B) + P(\overline{A} \mid B) = 1$.

(a) We know

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(\overline{A} \mid B) = \frac{P(\overline{A} \cap B)}{P(B)}$$

It follows

$$\begin{aligned} P(A \mid B) + P(\overline{A} \mid B) &= \frac{P(A \cap B) + P(\overline{A} \cap B)}{P(B)} \\ &= \frac{P\Big((A \cap B) \cup (\overline{A} \cap B)\Big)}{P(B)} \quad \Big\{A \cap B \text{ and } \overline{A} \cap B \text{ are disjoint}\Big\} \\ &= \frac{P\Big((A \cup \overline{A}) \cap B\Big)}{P(B)} \\ &= \frac{P(\Omega \cap B)}{P(B)} \\ &= \frac{P(B)}{P(B)} \\ &= 1 \end{aligned}$$

quod erat demonstrandum.

(b) Assume $P(B \mid A) > P(B)$. We want to prove $P(\overline{B}|A) < P(\overline{B})$.

By assumption,

$$\frac{P(A \cap B)}{P(A)} > P(B) \Rightarrow P(A \cap B) > P(B)P(A)$$

We want to prove

$$\frac{P(\overline{B} \cap A)}{P(A)} < P(\overline{B}) \equiv P(\overline{B} \cap A) < P(\overline{B})P(A)$$

Problem 9 () *One every* 25 *adults have a disease. Let s be a subject. If s has the disease, the diagnostic test is positive* .99 *of the times. If s does not have the disease, the diagnostic test is positive* .02 *of the times.*

(a) We are asked to find the probability of a result being positive. The law of total probability readily states that

$$\begin{aligned}
P(\text{positive}) &= P(\text{positive}|\text{enfermo})P(\text{enfermo}) + P(\text{positive}|\text{sano})P(\text{sano}) \\
&= .99 \cdot \frac{1}{25} + .02 \cdot \frac{24}{25} \\
&= .395 + .0192 \\
&= .058
\end{aligned}$$

(b) We are requested to find $P(\text{enfermo}|\text{positivo})$. Observe that we know the "reverse" of this; i.e. $P(\text{positivo}|\text{enfermo})$. This type of scenario calls for Bayes theorem, which states

$$\begin{aligned}
P(\text{enfermo}|\text{positivo}) &= \frac{P(\text{positivo}|\text{enfermo})P(\text{enfermo})}{P(\text{positivo})} \\
&= \frac{.99 \cdot \frac{1}{25}}{.058} \\
&= .682
\end{aligned}$$

(c) Similarly,

$$\begin{aligned}
P(\text{sano}|\text{negativo}) &= \frac{P(\text{negativo}|\text{sano})P(\text{sano})}{P(\text{negativo})} \\
&= \frac{.98 \cdot \frac{24}{25}}{1 - .058} \\
&= .998
\end{aligned}$$

Problem 10 (12)

Recall that there are two equivalent definitions of independence. Two events φ, ψ are independent if $P(\varphi \cap \psi) = P(\varphi)P(\psi)$, or equivalently if $P(\varphi \mid \psi) = P(\varphi)$. We must use both definitions in order to prove the properties.

(a) We are required to prove $P(\overline{A} \cap B) = P(\overline{A})P(B)$. Observe that

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

(This follows from the fact that $B = (A \cap B) \cup (\overline{A} \cap B)$ and the fact that the events in the union are obviously disjoint.) Then, due to the independence of A and B,

$$\begin{aligned} P(\overline{A} \cap B) &= P(B) - P(A)P(B) \\ &= P(B)\,[1 - P(A)] \\ &= P(B)P(\overline{A}) \end{aligned}$$

quod erat demonstrandum.

(c) By DeMorgan's law, $(\overline{A} \cap \overline{B}) = \overline{(A \cup B)}$. Then

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [P(A) + P(B) - P(A)P(B)] \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= P(\overline{A}) - P(B) + P(A)P(B) \\ &= P(\overline{A}) - P(B)\,[1 - P(A)] \\ &= P(\overline{A}) - P(B)P(\overline{A}) \\ &= P(\overline{A})(1 - P(B)) \\ &= P(\overline{A})P(\overline{B}) \end{aligned}$$

Problem 11 (13) *A collection* χ *of 10 items has 2 satisfying the predicate* $\varphi : \chi \mapsto \{0, 1\}$. *Two random samples* x_1, x_2 *are taken from* χ. *Let* $A = \{\varphi(x_1)\}$, $B = \{\varphi(x_2)\}$. *Compute* $P(A), P(B), P(A \cap B)$. *Are A and B independent?*

Obviously, $P(A) = \frac{2}{10} = \frac{1}{5}$ and

$$
\begin{aligned}
P(B) &= P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A}) \\
&= \frac{1}{9}\frac{2}{10} + \frac{2}{9}\frac{8}{10} \\
&= \cdot\frac{2}{90} + \frac{16}{90} \\
&= \frac{18}{90} \\
&= \frac{2}{10} \\
&= \frac{1}{5}
\end{aligned}
$$

We know

$$P(A \cap B) = P(B \mid A)P(A) = \frac{1}{9}\frac{2}{10} = \frac{1}{45}$$

Then $P(A)P(B) = \frac{1}{5} \cdot \frac{1}{5} \neq P(A \cap B)$. The events are not independent.

Problem 12 (14) *From a deck of 52 spanish cards, 4 players $p_1, \dots, p_4$ receive* 13 *cards each.*

(a) Let w, x, y, z be the four types of cards. The probability of p_1 receiving the 13 cards of type w is $1/\binom{52}{13}$. The same logic gives that the probability of all of them receiving the corresponding hands is

$$P(\zeta) = \prod_{i=0}^{3} \frac{1}{\binom{52-13i}{13}}$$

where ζ denotes the corresponding event.

(b) If a player has all cards of same type, it has all cards of that type (13). There are 4! ways of distributing the types among the players. Then the probability of this event is $4!P(\zeta)$.

(c) There are 4 aces: $\mathcal{A} = x_{13}, y_{13}, w_{13}, z_{13}$. There are $\binom{48}{13}$ hands without these elements. $\therefore$ The probability is $1/\binom{48}{13}$.

(d) For each player to receive an element in $\mathcal{A}$, each must get exactly one such element.

The probability of A receiving exactly one A is $\frac{4}{\binom{51}{12}}$, where 4 accounts for which element it receives and $\binom{51}{12}$ is the number of hands containing that fixed ace. Similar logic gives that the probability of all players reciving exactly one ace is

$$4! \prod_{i=0}^{3} \frac{1}{\binom{51-12i}{12}}$$

3 P2

(0.a) Assuming any number in $\mathbb{N}_{100}$ has the same probability of being chosen on a list $(a_1, \ldots, a_5)$, we have

$$P(\overrightarrow{x} \in \Omega) = \frac{1}{|\Omega|} = \frac{1}{100^5}$$

(0.b) A random variable associated to this experiment might be the sum of the elements in the list: $X(\overrightarrow{a}) = \sum a_i$.

(1.a) A probability mass function must satisfy $\sum p(x_i) = 1$. Only the second given function satisfies this.

(1.b)

$$P(2 \leq X \leq 4) = P(X = 2) + P(X = 3) + P(X = 4) = .1 + .1 + .3 = .5$$

(1.c) The cumulative distribution function of X describes $P(X \leq x)$ for every x in the range of the random variable. Thus, we have

$$P(X \leq x) = \begin{cases} 0 & X < 0 \\ .4 & 0 \leq X < 1 \\ .5 & 1 \leq X < 2 \\ .6 & 2 \leq X < 3 \\ .7 & 3 \leq X < 4 \\ 1 & 4 \leq X \end{cases}$$

(1.d) Asssume $P(x) = k(5 - x)$ for $x = 0, \ldots, 4$. We need

$$k \sum_{x=0}^{5} (5 - x) = 1 \iff k \left[5^2 + 5 \cdot 4 + \ldots + 5\right] = 1$$

In other words, we need $k\,[5 + 4 + 3 + 2 + 1] = k\,[15] = 1$, i.e. we need $k = \frac{1}{15}$.

(2.a) The probability that at most three lines are in use is

$$\sum_{x=0}^{3} p(x) = .1 + .15 + .2 + .25 = .7$$

(2.e) Let us study the event where 2, 3 or 4 lines are not being used. Two lines not being used equates to using ≤ 4 lines. Three lines not being used equates to ≤ 3 lines not being used. 4 lines not being used equates to ≤ 2 lines being used. In other words, the event is

$$E = (X \leq 4) \cup (X \leq 3) \cup (X \leq 2) = (X \leq 4)$$

The probability is then $\sum_{x \leq 4} p(x) = .9$

(2.f) For at least 4 lines not being used we need to four lines not being used, or 5 not being used, or 6 not being used. This means using 2, 1 or 0 lines. The probablity is then .45.

(3.a) Let us recall that for any $x_j \in Im(X)$

$$F(x_i) = \sum_{i=1}^{j} p_X(x_i)$$

From this readily follows that

$$\begin{aligned} p_X(x_j) &= F(x_j) - \sum_{i=1}^{j-1} p_X(x_i) \\ &= F(x_j) - F(x_{j-1}) \end{aligned}$$

So we have a nice formula to derive $p_X(x_j)$ given the CDF. In this case, we have

$$p_X(x) = \begin{cases} .3 & x = 1 \\ .1 & x = 3 \\ .05 & x = 4 \\ .15 & x = 6 \\ .4 & x = 12 \end{cases}$$

It is easy to verify that $\sum_{x \in Im(X)} p_X(x) = .3 + .1 + .05 + .15 + .4 = 1$.

(3.b)

$$\begin{aligned}
P(3 \leq X \leq 6) &= F(6) - F(3) \\
&= .6 - .4 \\
&= .2 \\
P(X \geq 4) &= 1 - P(X < 4) \\
&= 1 - F(4) \\
&= 1 - .45 \\
&= .55
\end{aligned}$$

(4) Five persons $S = \{s_1, \ldots, s_5\}$. Only s_1, s_2 have property R. Samples are drawn from S randomly; on each draw property R is verified. Let X be the number of verifications made until a sample satisfying R is drawn.

(4.a) Evidently, $\Omega = \{T, NT_1, NT_2 NNT_1, NNT_2, NNNT_1, NNNT_2\}$, where N denotes a negative draw and T_i a positive test comming from drawing s_1 or s_2. This model gives $X : \Omega \mapsto \mathbb{R}$ defined as $X(\omega) = |\omega|_N$ the number of Ns in ω.

The events are clearly not independent, since once a non-positive draw is made, the probability of obtaining a positive draw increases. Let us observe

$$\begin{aligned}
p_X(0) &= P(T) = \frac{2}{5} \\
p_X(1) &= P(NT_1 \cup NT_2) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10} \\
p_X(2) &= P(NNT_1 \cup NNT_2) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5} \\
p_X(3) &= \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}
\end{aligned}$$

It is easy to verify that that $\sum p_X(X) = 1$. The probability that R is not true in the first two draws is simply $p_X(2) + p_X(3) = \frac{3}{10}$.

(5.a) Let X denote the number of points traversed. To find p_X we need only examine the following. First of all, at least one point is traversed, which entails $Im_X = \mathbb{N}$. Now, the probability that only one point is traversed is simply $\frac{1}{3}$. The probability that two points are traversed is $\frac{2}{3} \cdot \frac{1}{3}$. That of three points being traversed is $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}$. In general,

$$p_X(x) = \left(\frac{2}{3}\right)^{x-1} \frac{1}{3} = \frac{2^{x-1}}{3^x}$$

Observe that the sum of this p. mass function is a geometric series and

$$\frac{1}{3} \sum_{x=1}^{\infty} \left(\frac{2}{3}\right)^{x-1} = \frac{1}{1 - \frac{2}{3}} = \left(\frac{1}{3}\right) 3 = 1$$

which is what we expect.

The CDF is

$$\begin{aligned} F(n) &= \sum_{x=1}^{n} p_X(x) \\ &= \sum_{x=1}^{n} \frac{2^{x-1}}{3^x} \end{aligned}$$

Suppose that only 20% of motorists come to a complete stop at an intersection with a flashing red light in all directions when there are no other visible vehicles.

1. What is the probability that, out of 20 randomly selected motorists arriving at the intersection under these conditions:
   (a) At most 5 will come to a complete stop?
   (b) Exactly 5 will come to a complete stop?
   (c) At least 5 will come to a complete stop?
2. How many of the next 20 motorists would you expect to come to a complete stop?

The variable X = number of selected motorists that stop follows a binomial distribution $\mathcal{B}(p = .2, n = 20)$, where we consider a success the event where the motorist stops. Recall that

$$p_X(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

which is perhaps the most intuitive distribution. From this of course follows that

$$\begin{aligned} P(X \leq 5) &= \sum_{k=1}^{5} p_X(k) \\ &= \binom{20}{0}(1-p)^{0.8} + \binom{20}{1} 0.2(0.8)^{19} + \ldots + \binom{20}{5} 0.2^5(0.8)^{15} \\ &= .80421 \end{aligned}$$

The probability that at least 5 success occur is simply $1 - P(X \leq 4)$, which can be derived from the formula above. The expected value of a binomial distribution is $np = 0.2 \cdot 20 = 4$.

A particular type of tennis racket is manufactured in medium and extra-large sizes. 60% of all customers at a certain store look for the extra-large size.

1. Among 10 randomly selected customers who want this type of racket, what is the probability that at least 6 will look for the extra-large size?
2. Among 10 randomly selected customers who want this type of racket, what is the probability that the number of customers looking for the extra-large size is within one standard deviation of the mean?
3. The store currently has 6 rackets of each model. What is the probability that the next ten customers looking for this racket will be able to purchase the model they want from the current stock?

(1) We have again a binomial distribution with $n = 10$, $p = \frac{6}{10} = \frac{3}{5}$. The probability comes directly from the binomial distribution formula.

(2) This problem is more interesting. Recall that the expected value of a binomial random var. is np, or in this case $0.6 \cdot 10 = 6$. The variance is $np(1-p) = 6(0.4) = 2.4$, which entails the standard deviation is $\sqrt{2.4} \approx 1.55$. Hence, we are asked for the probability that the number of success falls in the range $6 \pm 1.55 = [4.5, 7.5]$. Flooring and ceiling this interval, the question simply becomes what is the probability of X falling in $\{4, 5, 6, 7\}$, which

$$P(X \leq 7) - P(X \leq 3) = \sum_{k=4}^{7} \binom{10}{k} 0.6^k (0.4)^{10-k}$$

both easily derivable from the binomial distribution formula.

4 P3

Problem 13 *Let X a r.v. with density function* $f(x) = \frac{x}{2}$ *whenever* $0 \leq x \leq 2$, *and zero otherwise. (1) Calculate* $P(X \leq 1), P(\frac{1}{2} \leq X \leq \frac{3}{2}), P(\frac{3}{2} \leq X)$. *Then (2) find the CDF of X and (3) the expected value, variance and standard deviation of X. (4) If we define* $h(X) = X^2$, *what is the expected value of* $Y = h(X)$*?*

(1) First of all, observe that $F(x) = \frac{1}{2}\int x dx = \frac{x^2}{4}$. Incidentally, this is the CDF of X, which was requested in point (2). We then have

$$
\begin{aligned}
P(X \leq 1) &= \int_0^1 f(x)\, dx \\
&= \frac{1^2}{4} - \frac{0^2}{4} \\
&= \frac{1}{4} \\
P(\frac{1}{2} \leq X \leq \frac{3}{2}) &= F(\frac{3}{2}) - F(\frac{1}{2}) \\
&= \left(\frac{9}{16}\right) - \left(\frac{1}{16}\right) \\
&= \frac{8}{16} \\
&= \frac{1}{2}
\end{aligned}
$$

(3) The expected value is

$$\int_{\mathbb{R}} f(x)x\, dx = \int_0^2 \frac{x^2}{2} = \left[\frac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3} \approx 1.33$$

The simplest way to calculate the variance is to recall that $\mathbb{V}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$. Then, we observe that

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \frac{1}{2}\int_0^2 x^3\ dx \\
&= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 \\
&= \frac{1}{2}\left(\frac{16}{4} - 0\right) \\
&= \frac{4}{2} \\
&= 2
\end{aligned}
$$

It follows that $\mathbb{V}\left[X\right] = 2 - (\frac{4}{3})^2 = 2 - \frac{16}{9} = \frac{2}{9}$.

(4) The expected value of $Y = h(X)$ is

$$
\int_0^2 h(x)f(x)\ dx = \frac{1}{2}\int_0^2 x^3\ dx = \mathbb{E}\left[X^2\right] = 2
$$

Let

$$f_X(x) = \begin{cases} kx^2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

the PDF of r.v. X. The value of k must be s.t. it satisfies

$$\int_{\mathbb{R}} f(x)\ dx = 1$$

Since $\int f(x) = \frac{kx^3}{3}$ at the only interval of interest (i.e. $[0, 2]$), we know that k must satisfy

$$\left[kx^3/3\right]_0^2 = 1 \iff \frac{k8}{3} = 1 \iff k = \frac{3}{8}$$

The CDF of X is precisely the function we have derived above; i.e. $(kx^3)/3$. With the value of k found, we know

$$F_X(x) = \frac{x^3}{8}$$

The .75 percentile of the distribution is the value $\varphi \in \mathbb{R}$ s.t. $F_X(\varphi) = .75$. Therefore, φ satisfies

$$\frac{\varphi^3}{8} = .75 \iff \varphi^3 = 6 \iff \varphi = \sqrt[3]{6} \approx 1.81$$

The expected value of X is

$$\begin{aligned} \mathbb{E}[X] &\overset{\text{def}}{=} \int_{\mathbb{R}} x f(x)\ dx \\ &= \frac{3}{8} \int_0^2 x^3\ dx \\ &= \frac{3}{8} \left[\frac{x^4}{4}\right]_0^2 \\ &= \frac{3}{2} \end{aligned}$$

To compute the standard deviation, we first observe that

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \frac{3}{8}\int_0^2 x^4\,dx \\
&= \frac{3}{8}\left[\frac{x^5}{5}\right]_0^2 \\
&= \frac{3}{8}\frac{32}{5} \\
&= 2.4
\end{aligned}
$$

Then we can compute $\mathbb{V}[X] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 = 2.4 - 2.25 = .15$. From which follows that $\sigma_X = \sqrt{.15} \approx .387$.

The probability that X is less than one standard deviation from its mean is the probability that X falls in the range $[\frac{3}{2} - .387, \frac{3}{2} + .387]$. If we let l, u denote the lower and upper bounds of this interval, the probability is simply given by $F_X(u) - F_X(l)$.

Let $X \sim \mathcal{U}(25, 35)$. The density function of X is then given by

$$f_X(x) = \begin{cases} \frac{1}{10} & 25 \leq x \leq 35 \\ 0 & \text{otherwise} \end{cases}$$

The CDF is then

$$F_X(x) = \int_{25}^{35} f(x)\ dx = \frac{x - 25}{10}$$

We can verify that $\int_{\mathbb{R}} f(x)\ dx = F_X(35) - F_X(25) = 1 - 0 = 1$. The probability of $X \geq 33$ is simply $1 - P(X \leq 33) = 1 - F_X(33) = 1 - \frac{8}{10} = \frac{1}{5}$.

Given $a \in \mathbb{R}$ s.t. $25 < a < a + 2 < 35$ the probability of X falling in $[a, a + 2]$ is

$$\begin{aligned} \int_a^{a+2} f(x)\ dx &= F(a+2) - F(a) \\ &= \frac{a + 2 - 25}{10} - \frac{a - 25}{10} \\ &= \frac{1}{5} \end{aligned}$$

The expected value of X is

$$\begin{aligned} \mathbb{E}[X] &= \int_{\mathbb{R}} x f(x)\ dx \\ &= \int_{25}^{35} \frac{x}{10}\ dx \\ &= \frac{1}{10} \left[\frac{x^2}{2} \right]_{25}^{35} \\ &= \frac{1}{10} \left(\frac{35^2}{2} - \frac{25^2}{2} \right) \\ &= \frac{300}{10} \\ &= 30 \end{aligned}$$

We could have also used the simple formula for the uniform distribution, which states that $\mu = \frac{1}{2}(a+b)$. Now we observe that

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \frac{1}{10}\int_{25}^{35} x^2 \, dx \\
&= \frac{1}{10}\left[\frac{x^3}{3}\right]_{25}^{35} \\
&= \frac{1}{10}\left(\frac{35^3}{3} - \frac{25^3}{3}\right) \\
&= \frac{2725}{3} \\
&\approx 908.3
\end{aligned}
$$

From which follows that $\mathbb{V}\left[X\right] = 908.3 - 30^2 = 8.3$. $\therefore\ \sigma = \sqrt{8.3}$.

Problem 14 *Two teams, A and B, compete in a game. The probability that A wins is $P(A) = .5$. They will continue competing until A wins two games.*

(1) The probability of B winning x games is $.5^x$.

(2) Let N be the random variable representing number of games played. Since games are played until A wins two games, the probability of $N = 4$ is the probability that A wins two out of four games. In particular, A must win a game before the fourth and then the fourth.

Since the probabilities that A and B win are equiprobable, the probability is $\frac{1}{2}^4 + \frac{1}{2}^4 + \frac{1}{2}^4 = \frac{3}{16}$.

(3) The probability that *at most* four games are played is $\sum_{i=1}^{4} P(N = i)$. Naturally, $P(N = 1) = 0$, since two games must be played. $P(N = 2) = \frac{1}{4}$, of course. And $P(N = 3) = 2(\frac{1}{2})^3 = 1/4$. So we have $P(N \leq 4) = 0 + \frac{1}{4} + \frac{1}{4} + \frac{3}{16}$, which resolves to $\frac{11}{16}$.

Ej. 6. Let $\mu = 8.8, \sigma = 2.8$. (a) The probability of $X \sim (\mu, \sigma^2)$ being at most 10 is given by

$$\begin{aligned} P(X \leq 10) = \Phi(\frac{10-\mu}{\sigma}) &= \Phi(1.2/2.8) \\ &= \Phi(0.428) \\ &\approx \Phi(0.43) \\ &\approx .666 \end{aligned}$$

So quite curiously the probability approximates a nice fraction: $\frac{2}{3}$.

(b) The probability of X being between 5 and 10 is simply

$$P(X \leq 10) - P(X \leq 5) = \frac{2}{3} - \Phi(-1.357) = .666 - .085 = .581$$

where the second probability is computed easily with the table.

(d) The process of drawing 5 samples of X, $\overrightarrow{x} = (x_1, \ldots, x_5)$ randomly, where we consider the case $x_i \in [5, 10]$ a success, is a binomial process with $n = 5$, $p = P(X \in [5, 10]) = .581$. If we let Y denote the number of successes, then

$$\begin{aligned} P(Y = 1) &= \binom{5}{1} \cdot .581 \cdot (1 - .581)^4 \\ &= 5 \cdot .581 \cdot .419^4 \\ &= .089 \end{aligned}$$

Ej. 7 Let $X \sim \mathcal{N}(\mu, \sigma)$. We know 10% of the samples of X fall above 10.256 and 5% fall below 9.671. We are asked to find μ, σ. We are essentialyl requested to find μ, σ s.t. the following equations are both satisfied:

$$\begin{cases} \Phi(\frac{10.256-\mu}{\sigma}) & = .9 \\ \Phi(\frac{9.671-\mu}{\sigma}) & = .05 \end{cases}$$

(The first equation comes from the fact that $1 - \Phi(\frac{10.256-\mu}{\sigma}) = .1$).

> **NOTE.** There is very little difference between 10.256 and 9.671. However, the difference in probabilities is huge. Furthermore, both probabilities fall on the opposite tails of the distribution. This means our result must be some distribution with *very low variance* and centered somewhere between 10.256 and 9.671. We can use this as sanitiy check.

This equations entail that we need (approximately)

$$\begin{cases} \frac{10.256-\mu}{\sigma} & = 1.29 \\ \frac{9.671-\mu}{\sigma} & = -1.64 \end{cases}$$

The first equation yields

$$\mu = -1.29\sigma + 10.256$$

Using this in the second equation gives

$$9.671 + 1.29\sigma - 10.256 = -1.64\sigma \iff 2.93\sigma = .585$$

This gives $\sigma = .199$, which we approximate with $\sigma = .2$. Then $\mu = -1.29 \cdot .2 + 10.256 = 9.998$, which we aproximate with $\mu = 10$.

$\therefore \mu = 10, \sigma = .2$.

Ej. 10. We only look at point (b). We are given $X \sim \mathcal{N}(70, 3^2)$. We are asked for $\varphi \in \mathbb{R}$ s.t. $I = (70 - \varphi, 70 + \varphi)$ covers 95% of all samples.

For the interval I to cover 95% of the distribution, it must leave 2.5% unoccupied on each tail. Which means φ must satisfy

$$\begin{cases} \Phi\left(\frac{70-(70-\varphi)}{3}\right) & = 0.025 \\ \Phi\left(\frac{70-(70+\varphi)}{3}\right) & = 0.975 \end{cases}$$

Using the normal distribution table, this entails

$$\begin{cases} \frac{70-(70-\varphi)}{3} & = -1.96 \\ \frac{70-(70+\varphi)}{3} & = 1.96 \end{cases}$$

Both equations give $\varphi = -1.96(3) = -5.88$. Of course, the sign doesn't mean anything here, since φ simply determines the amplitude of the centering around μ.

$\therefore \ \varphi = 5.88$.

5 Exponential distribution

The exponential distribution is

$$f_X(x;\lambda) = \begin{cases} \lambda e^{-\lambda} x & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Its CDF (in the proper interval) is $F_X(x;\lambda) = 1 - e^{-\lambda x}$, its mean is $\frac{1}{\lambda}$, and its variance $\frac{1}{\lambda^2}$.

Problem 11. We are given $X \sim \text{Exp}(.01386)$. (a) $P(X \leq 100)$ is computed directly from the CDF:

$$F_X(100; .01386) = 1 - e^{-100 \cdot .01386} = 1 - e^{-1.386} = .7499 \approx \frac{3}{4}$$

The probability $P(100 \leq X \leq 200)$ is simply given by $F_X(200) - F_X(100)$, which is simple to compute.

Problem 12. A sistem has five componentes connected thus:

$$c_1 \longrightarrow c_2 \longrightarrow \ldots \longrightarrow c_5$$

If one component fails, the whole system fails. Each component has a duration $\delta_i \sim \text{Exp}(.01)$, and failures in one component are independent of failures in others. Let A_i denote the event of the ith component lasting for at least t hours. Let X the time at which the system fails, i.e. the least duration among all systems.

(a) The event $\{X \geq t\}$ is equivalent to what event where $A_1, \ldots, A_5$ appear?

> The event $\{X \geq t\}$ is equivalent to the event $\bigcap_{i=1}^{5} A_i$, i.e. the event where each component lasts at least t hours.

(b) Using the independence of events A_i, find $P(X \geq t)$ and find $F(t) = P(X \leq t)$, as well as the PDF of X.

> Since the events are independent, given what was said in (a), this probability is

$$
\begin{aligned}
P(X \geq t) = \prod_{i=1}^{5} P(A_i) &= \prod_{i=1}^{5} P(\delta_i \geq t) \\
&= \prod_{i=1}^{5} (1 - P(\delta_i \leq t)) \\
&= \left(1 - \left(1 - e^{-t \cdot .01}\right)\right)^5 \\
&= e^{-t \cdot 0.1 \cdot 5} \\
&= e^{-t \cdot 0.5}
\end{aligned}
$$

Same reasoning gives $F_X(t) = (1 - e^{-t \cdot 0.5})$, which means the PDF is

$$
f_X(t) = \frac{d}{dt}\left[1 - e^{-t \cdot 0.5}\right] = -\frac{d}{du}e^u \frac{d}{dt}u
$$

with $u = -t \cdot 0.5$. Since $\frac{d}{dt}u = -\frac{1}{2}$ we have

$$
f_X(t) = -e^u \cdot (-\frac{t}{2}) = \frac{e^{-\frac{t}{2}}}{2}
$$

(c) Assume n components exist, each with duration $\delta \sim \mathrm{Exp}(\lambda)$. What is the distribution of X?

By the same token,

$$
F_X(x) = \prod_{i=1}^{n} P(\delta_i \geq t) = e^{-t \frac{n}{10}}
$$

6 Prob. Conjuntas

Problem (1) Let

$$f(x,y) = \begin{cases} kxy & 0 \le x \le 1 \wedge 0 \le y \le 1 \wedge x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

with $k \in \mathbb{R}^+$.

(a) Determine k so that f is a joint PDF of (X,Y).

We require that

$$\begin{aligned} \int_0^1 \int_0^{1-x} f(x,y)dy\ dx &= k \int_0^1 x \int_0^{1-x} y\ dy\ dx \\ &= k \int_0^1 x \left[\frac{y^2}{2}\right]_0^{1-x}\ dx \\ &= k \int_0^1 x \left(\frac{(1-x)^2}{2}\right)\ dx \\ &= \frac{k}{2} \int_0^1 x\left(1-2x+x^2\right)\ dx \\ &= \frac{k}{2} \int_0^1 x - 2x^2 + x^3\ dx \\ &= \frac{k}{2}\Big(\left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1 \Big) \\ &= \frac{k}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \\ &= \frac{k}{2} \cdot \frac{1}{12} \\ &= \frac{k}{24} \end{aligned}$$

This entails k must be 24 in order for f to be a joint PDF of (X,Y).

(b) The marginal probability distribution of Y is

$$f_Y(y) = 24\int_0^1 xy\ dx \qquad = 24y\int_0^1 x\ dx$$
$$= 24yy\left[\frac{x^2}{2}\right]_0^1$$
$$= 24y \cdot \frac{1}{2}$$
$$= 12y$$

Same reasoning gives $f_X(x) = 12x$.

(c) Since $f_X(x)f_Y(y) = 12x \cdot 12y = 12^2xy \neq f(x, y)$, the r.vs are not independent.

Problem 3 Let

$$f(x, y) = \begin{cases} k(x+y) & 0 \le x \le 10 \wedge 0 \le y \le 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) The value of k can be found solving the equation

$$k \int_0^{10} \int_0^{10} (x+y) dy \; dx = 1 \tag{1}$$

Observe that

$$\begin{aligned} \int_0^{10} (x+y) dy &= x \int_0^{10} dy + \int_0^{10} y \; dy \\ &= 10x + \frac{10^2}{2} \\ &= 10x + 50 \end{aligned}$$

The left-hand side of equation (1) then becomes

$$\begin{aligned} k \int_0^{10} 10x + 50 \; dx &= k \left[10 \left[\frac{x^2}{2} \right]_0^{10} + 500 \right] \\ &= k \left[500 + 500 \right] \\ &= 1000k \end{aligned}$$

Hence, the solution to (1) is $k = \frac{1}{1000}$.

(b) Now that we know that the PDF is $f(x, y) = (x + y)/1000$ (for $x, y \in 0, 10]$), we can find $P(X + Y < 5)$:

$$\int_{x=0}^{5} \int_{y=0}^{5-x} \frac{x+y}{1000} \; dy \; dx$$

Observe that

$$
\begin{aligned}
\frac{1}{1000}\int_0^{5-x} x+y\ dy &= \frac{1}{1000}\left[x(5-x)+\frac{(5-x)^2}{2}\right]\\
&= \frac{1}{1000}\frac{10x-2x^2+25-10x+x^2}{2}\\
&= \frac{25-x^2}{2000}
\end{aligned}
$$

Then

$$
\begin{aligned}
P(X+Y<5) &= \frac{1}{2000}\int_0^5 25-x^2\ dx\\
&= \frac{1}{2000}\left(25(5)-\frac{5^3}{3}\right)\\
&= \frac{1}{2000}\left(\frac{2\cdot 5^3}{3}\right)\\
&= \frac{250}{6000}\\
&= \frac{125}{3000}\\
&= \frac{25}{600}\\
&= \frac{1}{24}
\end{aligned}
$$

(c) It is easy to see that

$$
f_Y(y) = \frac{1}{1000}\int_0^{10} x+y\ dx = \frac{1}{20}+\frac{y}{100}
$$

$$
f_X(x) = \frac{1}{1000}\int_0^{10} x+y\ dy = \frac{1}{20}+\frac{x}{100}
$$

Note that

$$\begin{aligned} f_X(x) \cdot f_Y(y) &= \frac{5+y}{100}\frac{5+x}{100} \\ &= \frac{25+5x+5y+yx}{100^2} \\ &\neq f(x,y) \end{aligned}$$

The variables are not independent.

(d) It is straightforward to compute

$$\mathbb{E}\left[Y\right] = \mathbb{E}\left[X\right] = \int_0^{10} \frac{5x+x^2}{100}\,dx = \frac{35}{6}$$

Now, observe that

$$\mathbb{E}\left[XY\right] = \int_0^{10}\int_0^{10} (xy)\,\frac{x+y}{100}\,dy\,dx = \frac{100}{3}$$

The covariance is

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right] \\ &= \frac{100}{3} - (\frac{35}{6})^2 \\ &= -0.694 \end{aligned}$$

In short, there is a slight negative correlation between the variables.

Let $Y \sim \Gamma(\alpha = 100, \beta = 20)$. Let $L = 30Y + 2Y^2$. Compute the expected value and variance of L.

Let us recall that $\mathbb{E}[Y] = \alpha\beta$, and $\mathbb{V}[Y] = \alpha\beta^2$. From this, using linearity of the expectation,

$$\mathbb{E}[L] = 30\mathbb{E}[Y] + 2\mathbb{E}\left[Y^2\right]$$

Observe that

$$\begin{aligned}\mathbb{E}\left[Y^2\right] &= \mathbb{V}[Y] + \mathbb{E}^2[Y] \\ &= (100)20^2 + (100 \cdot 20)^2\end{aligned}$$

from which $\mathbb{E}[L]$ is deduced.

Let X, Y be two Poisson-distributed independent r.vs with parameters λ_X, λ_Y, respectively.

(a) Since X, Y are independent, their joint PDFs is

$$\begin{aligned} f_{X,Y}(n,m) &= f_X(n) f_Y(m) \\ &= \left(\frac{\lambda_X^n e^{-\lambda_X}}{n!}\right)\left(\frac{\lambda_Y^m e^{-\lambda_Y}}{m!}\right) \\ &= \frac{\lambda_X^n \lambda_Y^m \exp(-\lambda_X - \lambda_Y)}{n!m!} \end{aligned}$$

(b) The probability that *at most* one event occurs considering both vars is the probability that $(X, Y) \in \{(0,0), (1,0), (0,1)\}$. This are obviously disjoint events, from which we get the probability $f_{XY}(0,0) + f_{XY}(0,1) + f_{XY}(1,0)$. Observe that this is naturally equal to

$$\sum_{n=0}^{1} \sum_{m=0}^{1-n} f_{XY}(n,m)$$

(c) The general expression for the probability that the total number of events, either X or Y, is exactly equal to φ, must be an expression for $P(X+Y=\varphi)$. This is of course the probability of $(X,Y) \in \{(0,\varphi), (1,\varphi-1), \dots, (\varphi-1,1), (\varphi,0)\}$. All of these events are obviously disjoint. So

$$P(X+Y=\varphi) = \sum_{n=0}^{\varphi} f_{XY}(n, \varphi - n)$$

Let X, Y two independent and exponentially distributed r.vs with $\lambda = 1$.

(a) The joint PDF of X, Y (using the fact that they are independent) is

$$\begin{aligned} f_{XY}(x,y) &= f_X(x) f_Y(y) \\ &= \left(\lambda e^{-\lambda x}\right)\left(\lambda e^{-\lambda y}\right) \\ &= \lambda^2 \exp(-\lambda x - \lambda y) \\ &= \exp(-x - y) \end{aligned}$$

for $x, y \geq 0$, and zero otherwise.

(b) The probability that both X and Y fall in $[0, 1000]$ is

$$\int_{x=0}^{1000} \int_{y=0}^{1000} f_{XY}(x,y)\ dy\ dx = \int_{x=0}^{1000} \int_{y=0}^{1000} e^{-x} e^{-y}\ dy\ dx$$

Let $u = -x, w = -y$. Clearly, $\frac{dw}{dy} = -1 \Rightarrow dw = -dy$. Similarly, $du = -dx$. Finally, if $y = 1000, w = -1000$. So,

$$\begin{aligned} \int_0^{1000} e^{-y} &= -\int_0^{-1000} e^w\ dw \\ &= -\left(-\int_{-1000}^{0} e^w\ dw\right) \\ &= [e^w]_{-1000}^{0} \\ &= \left[e^0 - e^{-1000}\right] \\ &= \left[1 - e^{-1000}\right] \\ &= 1 - \frac{1}{e^{1000}} \end{aligned}$$

So the probability is

$$\left(1 - \frac{1}{e^{1000}}\right) \int_{x=0}^{1000} e^{-x}\ dx$$

The same reasoning, substituting with $u = -x$, gives that $\int_0^{1000} e^{-x}\, dx = 1 - e^{-1000}$, and hence the result is

$$\left(1 - e^{-1000}\right)^2$$

(c) The probability that the total duration of both events is at most 2000 is $P(X+Y \leq 2000)$, which resolves to

$$\int_{x=0}^{2000} \int_{y=0}^{2000-x} f_{XY}(x, y)\ dy\ dx = \int_{x=0}^{2000} \int_{y=0}^{2000-x} e^{-x} e^{-y}\ dy\ dx$$

We already know $\int_0^b e^{-y}\ dy = 1 - e^{-b}$, which means we have obtain

$$\begin{aligned}
P(X + Y \leq 2000) &= \int_{x=0}^{2000} e^{-x} \left(1 - e^{-2000+x}\right)\ dx \\
&= \int_0^{2000} e^{-x}\ dx - \int_{x=0}^{2000} e^{-2000}\ dx \\
&= \left(1 - e^{-2000}\right) - e^{-2000}(2000 - 0) \\
&= \left(1 - e^{-2000}\right) - 2000e^{-2000} \\
&= 1 - 2001e^{-2000}
\end{aligned}$$

Let X an r.v. with CDF

$$F(x) = \begin{cases} 0 & x < -1 \\ a & -1 \leq x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leq x < 1 \\ b & 1 \leq x \end{cases}$$

and assume $E(X) = -\frac{1}{8}$.

(a) It is necessary that $b = 1$ for the CDF to make sense. The value of a can be found by recalling two things: (1) that $p(x_i) = F(x_i) - F(x_{i-1})$, and that $\sum_{x \in Im(X)} xp(x) = E(X)$. So we have

$$\begin{aligned} \sum_{i=1}^{3} x_i \left(F(x_i) - F(x_{i-1})\right) &= (-1)a + \frac{1}{2}\left(\frac{3}{4} - a\right) + \left(1 - \frac{3}{4}\right) \\ &= -a + \frac{3}{8} - \frac{a}{2} + \frac{1}{4} \\ &= -\frac{3a}{2} + \frac{5}{8} \\ &= E(X) \end{aligned}$$

Since $E(X) = -\frac{1}{8}$ we have

$$-\frac{3a}{2} + \frac{5}{8} = -\frac{1}{8} \iff -\frac{3a}{2} = -\frac{3}{4} \iff -3a = -\frac{3}{2}$$

This entails $a = \frac{1}{2}$.

(b) The PMF of X is

$$p_X(x) = \begin{cases} \frac{1}{2} & x = -1 \\ \frac{1}{4} & x = \frac{1}{2} \\ \frac{1}{4} & = x = 1 \end{cases}$$

The variance of X is $E(X^2) - E^2(X)$. The second term is known. The first can be computed:

$$\begin{aligned} E(X^2) &= \sum_{x \in Im(X)} x^2 p_X(x) \\ &= (-1)^2 \frac{1}{2} + \frac{1}{2^2} \frac{1}{4} + 1^2 \frac{1}{4} \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} \\ &= \frac{5}{4} \end{aligned}$$

7 Parte 2: Estimación

7.1 Desigualdad de Chebyshev

Theorem 1 *Sea X r.v. con media μ y varianza σ^2. Sea $t \in \mathbb{R}^+$. Luego*

$$P(X \geq t) \leq \frac{E(X)}{t} \tag{2}$$

Además,

$$P\left(|X - \mu| \geq t\right) \leq \frac{\sigma^2}{t^2} \tag{3}$$

Por último, de lo anterior se deduce que, escribiendo t como σk, con $k \in \mathbb{R}$,

$$P\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2} \tag{4}$$

El teorema anterior es útil porque ciertas probabilidades $P(X > a)$ pueden expresarse como $P(X - \mu > a - \mu)$, habilitando una aplicación del teorema.

Definition 1 *If $\{X_n\}$ is a sequence of random variables, we say $\{X_n\}$ converges in probability to Y if for any $\epsilon > 0$*

$$\lim_{n\to\infty} P(|X_n - Y| \geq \epsilon) = 0$$

Theorem 2 (Weak large numbers) *Let $X_1, \ldots, X_n$ random vars with equal mean μ and equal variance σ^2. Let $\overline{X}_n$ be the mean value of the sequence.*

For every $\epsilon > 0$,

$$P(|\overline{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$
$$P(|\overline{X}_n - \mu| \leq \epsilon) \geq 1 - \frac{\sigma^2}{n\epsilon^2}$$

This entails

$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| \geq \epsilon) = 0$$
$$\lim_{n\to\infty} P(|\overline{X}_n - \mu| \leq \epsilon) = 1$$

Therefore, $\left\{\overline{X}_n\right\}$ *converges in probability to the random variable that is centered at* μ*.*

Intuitively, this means that if a sequence $\{X_n\}$ converges in probability to Y, we can make the probability of $(|X_n - X| < \epsilon)$ arbitrarily close to 1 for any ϵ, taking a sufficiently large n.