chngcntr

1 Taylor

Let $f \in C^n[a,b]$ and assume $f^{(n+1)}$ exists in (a,b). Then for any $c, x \in [a,b]$ there is some ζ between c and x s.t.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)(x-c)^k}{k!} + E_n(x) \tag{1}$$

where

$$E_n(x) = \frac{f^{(n+1)}(\zeta)(x-c)^{n+1}}{(n+1)!}$$

Equation (1) is called the Taylor expansion of f around c.

Observation. The famous *mean value theorem* is simply the case $n = 0$ of Taylor's expansion: if $f \in C[a,b]$ and f' exists on (a,b), then for $x, c \in [a,b]$

$$f(x) = f(c) + f'(\zeta)(x-c)$$

where ζ is between c and x. Take $x = b, c = a$ and the theorem appears:

$$f(b) - f(a) = f'(\zeta)(b-a)$$

We typically extend the Taylor approximation of f around a point r, where $r = x + h$ is an approximation some value of interest x. This is useful because said approximation gives

$$f(r) = f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \ldots + \frac{f^{(n)}(x)}{n!}h^n + E_n(h)$$

In other words, this strategy allows us to extend $f(r)$ in terms of x and h, the approximation and its error. Usually, r, h are unknown but h can be bounded.

Taylor: El desarrollo de una f suficientemente diferenciable alrededor de un punto r espa

$$f(x) = f(r) + f'(r)(x-r) + \frac{f''(r)}{2!}(x-r)^2 + \ldots + \frac{f^{(n)}(r)}{n!}(x-r)^n + R_n(x)$$

donde $R_n(x)$ es el resto.

Usualmente, queremos tomar $r = x + h$, donde x es una aproximación de r y h el error de aproximación. Entonces es provechoso expandir $f(r)$ alreededor de su estimación x:

$$f(r) = f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \ldots + \frac{f^{(n)}(x)}{n!}h^n + R_n(h)$$

Esto es **recontra** útil porque nos dice cuánto se diferencia $f(r)$ de nuestra aproximación $f(x)$ (pues expresa $f(r)$ como $f(x)$ más algo).

Usualmente r, h son desconocidos pero h puede acotarse.

El resto R_n del teorema puede expresarse como sigue:

$$f(r) = f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \ldots + \frac{f^{(n)}(x)}{n!}h^n + \frac{f^{(n+1)}(\zeta)}{(k+1)!}h^{n+1}$$

para algún $\zeta \in (x, h)$. Esta forma de expresar el error de aproximación con el polinomio de Taylor se usará mucho.

2 Alg. de Horner: Polynomial evaluation

Consider

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

We wish to compute $p(k)$ for a given $k \in \mathbb{R}$ minimizing the number of operations. Directly computing $a_0 + a_1 k_1 + \ldots$ leads to n sums. The ith term requires computing k^i, which means i product operations, for a totall of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ products. The total number of operations is then

$$\Theta = n + n(n+1)/2$$

The associated complexity is $O(n^2)$.

Horner's method consists of re-writing $p(x)$ so that the number of products is reduced. One writes

$$p(x) = a_0 + xb_0$$

where $b_{n-1} = a_n$ and for $0 \leq i < n-1$:

$$b_{i-1} = a_i + xb_i$$

Let $p(x) = 3 + 5x - 4x^2 + 0x^3 + 6x^4$, giving $n = 4$. Then $b_3 = 6$ and

$$b_2 = a_3 + xb_3 = 6x, \qquad b_1 = a_2 + xb_2 = -4 + x(6x),$$
$$b_0 = a_1 + xb_1 = 5 + x(-4 + x(6x))$$

This finally gives

$$p(x) = 3 + xb_0 = 3 + x(5 + x(-4 + x(6x)))$$

Here, one must perform n sums again but only n products. Thus, there are $\Theta = n + n = 2n$ operations, giving a complexity of $O(n)$ (in the operation space). See the algorithm below:

input $n; a_i, i = 0, \ldots, n; x$
$b_{n-1} \leftarrow a_n$
for $i = n-2$ **to** $i = 0$
    $b_i = a_{i+1} + x * b_{i+1}$
od
$y \leftarrow a_0 + x * b_0$
return y

It is easy to see in this code that the **for** loop performs $n-1$ iterations, in each of which a single sum and a single product are computed. The nth sum and nth product are performed in the computation of y, the final result.

A more polished version includes the last computatoin (the one in the assignment of y) within the loop and makes no use of indexes:

input $n; a_i, i = 0, \ldots, n; x$
$b \leftarrow a_n$
for $i = n - 2$ **to** $i = -1$
    $b = a_{i+1} + x * b$
od
return b

In Python,

```
def horner(coefs, x):
  n = len(coefs)-1
  b = coefs[n]

  for i in reversed(range(-1, n-1)):
    b = coefs[i+1] + x*b

  return b
```

It is trivial to adapt the code so that it returns the coefficients $b_0, \ldots, b_{n-1}$ and not the final result, if needed.

3 Error

Let $r, \overline{r}$ be two real numbers s.t. the latter is an approximation of the first. We define the **error** of the approximation to be $r - \hat{r}$, and

$$\Delta r = |r - \overline{r}|\,, \qquad \delta r = \frac{\Delta r}{|r|}$$

With r unknown the strategy is to work with a known bound of r.

4 Non-linear equations

The general problem is to find members of the set $\mathcal{R}_f$ of roots of $f \in \mathbb{R} \to \mathbb{R}$. The numerical strategy is to iteratively approximate some $r \in \mathcal{R}_f$ until some pre-established threshold in the error of approximation is met.

More formally, the numerical strategy produces a sequence $\{x_k\}_{k \in \mathbb{N}}$ which satisfies

- $\lim_{k\to\infty} \{x_k\} = r$ for some $r \in \mathcal{R}_f$
- Either $e(x_k) < e(x_{k-1})$ or, more strongly, $\lim_{k\to\infty} e(x_k) = 0$, where $e(x_k)$ is some appropriate measure of the error of approximation.

4.1 Bisection

A very simple procedure: if a root exists in $[a, b]$, it iteratively shrinks $[a, b]$ in halves (keeping the halves which contain the root) until the interval is of sufficiently small length or the root is found.

Theorem 1 (Intermediate value)**.** If f is continuous in $[a, b]$ and $f(a)f(b) < 0$, then $\exists r \in \mathcal{R}_f$ s.t. $r \in [a, b]$.

Assume f is continuous. A root exists in $[a, b]$ if $f(a)f(b) < 0$ (**Theorem 1**). If that is the case, the midpoint $(a + b)/2$ is taken as the approximation x_0. It is also trivial to observe that x_0 is *at most* at a distance of $(b - a)/2$ from the real root, so $e_0 = |x_0 - r| \leq (b - a)/2$.

If $f(x_0) = 0$ the procedure must end because a root was found. Otherwise, sufficies to find which half of the interval contains a root computing $f(a)f(c)$ and, if needed, $f(c)f(b)$.

The iterations may stop after reaching a maximum number of steps, when $|f(c)|$ is sufficiently close to zero, or when the error bound $|e_k| \leq (b_k - a_k)/2$ (where $[a_k, b_k]$ is the interval of this iteration) is sufficiently small.

(!) The algorithm not always converges. Take $f(x) = 1/x$. Clearly, it has no root. Yet setting $a = -1, b = 1$ in the initial iteration falsely passes the test. (The problem obviously is that f is not continuous in $[-1, 1]$.) If one sets

```
Input : a, b, δ, M, f
Output : Tupla de la forma: (r, cota de error)
f_a ← f(a)
f_b ← f(b)

if f_a * f_b > 0
    return ?
fi

for i = 1 to i = M do
    c ← a + (b − a)/2
    f_c ← f(c)
    if f_c = 0 then
        return (c, 0)
    fi
    ϵ = (b − a)/2
    if ϵ < δ then
        break
    fi
    if f_a * f_c < 0 then
        b ← c
        f_b = f(b)
    else
        a ← c
        f_a = f(a)
    fi
od
return (c, ϵ)
```

```
def bisection(f : callable, a : float, b : float, delta : float, M : int):

  s, e = f(a), f(b) # function values at (s)tart, (e)nd of interval

  if s*e > 0:
    raise ValueError("Interval [a, b] contains no root.")

  for i in range(M):

    c = a + (b-a)/2
    m = f(c) # value of f at (m)idpoint

    if m == 0:
      return c, 0

    e = (b-a)/2
    if e < delta:
      return c, e

    if s*m < 0:
      b = c
      e = f(b)
    else:
      a = c
      s = f(a)

  return c, e
```

Theorem 2. If $\{[a_i, b_i]\}_{i=0}^{\infty}$ are the intervals generated by the bisection method on iterations $i = 0, 1, \ldots$, then:

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ is a member of $\mathcal{R}_f$.
2. If $c_n = \frac{1}{2}(a_n + b_n), r = \lim_{n\to\infty} c_n$, then $|r - c_n| \leq \frac{1}{2^{n+1}}(b_0 - a_0)$

Proof. **(1)** It is clear that $a_i \leq a_{i+1}$ and $b_i \geq b_{i+1}$, since the interval on each iteration shrinks in one direction.

$\therefore$ a_n, b_n are monotonous.

But clearly a_n is bounded by b_0 and b_n is bounded by a_0.

$\therefore$ a_n, b_n are monotonous and bounded.

$\therefore$ Their limits exist.

It is also clear that the interval shrinks to half its size on each iteration:

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}), \qquad n \geq 1 \tag{1}$$

By recurrence on (1),

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0), \qquad n \geq 0 \tag{2}$$

Then

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} \frac{1}{2^n}(b_0 - a_0) = 0 \tag{3}$$

$\therefore$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Since the limit of a_n, b_n exists and f is by assumption continuous, the composition limit theorem applies and:

$$\begin{aligned}
&\lim_{n\to\infty}(f(a_n) \cdot f(b_n)) \\
=&\lim_{n\to\infty} f(a_n) \cdot \lim_{n\to\infty} f(b_n) && \{\text{Product of limits}\} \\
=&f\left(\lim_{n\to\infty} a_n\right) \cdot f\left(\lim_{n\to\infty} b_n\right) && \{\text{Composition limit theorem}\} \\
=&[f(r)]^2 && \left\{r = \lim_{n\to\infty} a_n\right\}
\end{aligned} \tag{4}$$

The invariant of the algorithm is $f(a_n)f(b_n) < 0$. But due to the last result,

$$\lim_{n\to\infty} f(a_n)f(b_n) \leq 0 \iff [f(r)]^2 \leq 0 \iff f(r) = 0$$

$\therefore$ $r = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ is a root.

(2) Follows directly from result (2)

$$\begin{aligned} |r - c_n| &= \left| r - \frac{1}{2}(b_n - a_n) \right| \\ &\leq \left| \frac{1}{2}(b_n - a_n) \right| \\ &= \left| \frac{1}{2^{n+1}}(b_0 - a_0) \right| && \{\text{Result (2)}\} \end{aligned}$$

4.2 Newton's method

Assume $r \in \mathcal{R}_f$ and $r = x + h$, with x an approximation of r and h its error. Assume f'' exists and is continuous in some I around x s.t. $r \in I$. What we explained on Taylor expansions around a point gives:

$$0 = f(r) = f(x+h) = f(x) + f'(x)h + O(h^2)$$

If x is sufficiently close to r, h is small and h^2 even smaller, so that $O(h^2)$ is unconsiderable:

$$0 \approx f(x) + hf'(x)$$

Therefore,

$$h \approx -\frac{f(x)}{f'(x)} \tag{1}$$

From this follows that $r = x + h$ is approximated by

$$r \approx x - \frac{f(x)}{f'(x)}$$

Since the approximation in (5) truncated the terms of $O(h^2)$ complexity, this new approximation is closer to r than x originally was. In other words, $x - f(x)/f'(x)$ is a better approximation to r than x itself.

Thus, if x_0 is an original approximation, we can define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

to produce a sequence of approximations. This is the fundamental idea of Newton's method.

```
Input: x_0, M, δ, ϵ;
v ← f(x_0)
if |v| < ϵ then return x_0 fi
for k = 1 to k = M do
    x_1 ← x_0 − v / f'(x_0)
    v ← f(x_1)
    if |x_1 − x_0| < δ ∨ v < ϵ then
        return x_1
    fi
    x_0 ← x_1
od
return x_0
```

The predicate $|x_1 - x_0| < \delta$ checks whether our algorithm is adjusting x in a negligible degree. If that is the case, we should stop.

Theorem 3. If f'' continuous around $r \in \mathcal{R}_f$ and $f'(r) \neq 0$, then there is some $\delta > 0$ s.t. if $|r - x_0| \leq \delta$, then:

- $|r - x_n| \leq \delta$ for all $n \geq 1$.
- $\{x_n\}$ converges to r
- The convergence is quadratic, i.e. there is a constant $c(\delta)$ and a natural N s.t. $|r - x_{n+1}| \leq c\,|r - x_n|^2$ for all $n \geq N$.

Proof. Let $e_n = r - x_n$ be the error in the nth approximation. Assume f'' is continuous and $f(r) = 0$, $f'(r) \neq 0$. Then

$$\begin{aligned} e_{n+1} &= r - x_{n+1} \\ &= r - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) \\ &= r - x_n + \frac{f(x_n)}{f'(x_n)} \\ &= \frac{e_n f'(x_n) + f(x_n)}{f'(x_n)} \end{aligned} \tag{3}$$

Thus, the error at any given iteration is a function of the error at the previous iteration. Now consider the expansion of $f(r)$ as

$$f(r) = f(x_n - e_n) = f(x_n) + e_n f'(x_n) + \frac{e_n^2 f''(\zeta_n)}{2} \tag{4}$$

for ζ_n between x_n and r. This equation gives

$$e_n f'(x_n) + f(x_n) = -\frac{1}{2} f''(\zeta_n) e_n^2 \tag{5}$$

The expression in (5) is the numerator in (3), whereby we obtain via substitution:

$$e_{n+1} = -\frac{1}{2} \frac{f''(\zeta_n) e_n^2}{f'(x_n)} \tag{6}$$

Equation (6) ensures that the error scales quadratically. Now we wish to bound the error expression in (6). To bound e_{n+1}, we take $\delta > 0$ to define a neighbourhood of length δ around r. For any x in this neighbourhood, (6) reaches its maximum when the numerator is maximized and the denominator is minimized:

$$c(\delta) = \frac{1}{2} \frac{\max_{|x-r| \leq \delta} |f''(x)|}{\min_{|x-r| \leq \delta |f'(x)|}}$$

In other words, $c(\delta)$ is the maximum value which e_{n+1} can take if ζ_n, x_n are assumed to belong to the neighbourhood. Now we make two assumptions:

1. x_0 belongs to the neighbourhood, i.e. $|x_0 - r| \leq \delta$
2. δ is sufficiently small so that $\varrho := \delta c(\delta) < 1$.

Note that, since ζ_0 is between x_0 and r, assumption (1) ensures that ζ_0 is also in the neighbourhood, i.e. $|r - \zeta_0| \leq \delta$. Then we have:

$$|e_0| = \frac{1}{2} |f''(\zeta_0)/f'(x_0)| \leq c(\delta)$$

Then:

$$\begin{aligned}
|x_1 - r| &= |e_1| \\
&= \left| e_0^2 \cdot \frac{1}{2} f''(\zeta_0)/f'(x_0) \right| \\
&\leq |e_0^2| c(\delta) && \left\{ \frac{1}{2} f''(\zeta_0)/f'(x_0) \leq c(\delta) \right\} \\
&\leq |e_0| \delta c(\delta) && \{ |e_0| \leq \delta \} \\
&= |e_0| \, \varrho && \{ \varrho = \delta c(\delta) \} \\
&< |e_0| && \{ \varrho < 1 \} \\
&\leq \delta
\end{aligned}$$

$\therefore |e_1| < |e_0| \leq \delta$, which means the error decreases. This argument may be repeated inductively, giving:

$$
\begin{aligned}
|e_1| &\le \varrho\,|e_0| \\
|e_2| &\le \varrho\,|e_1| \le \varrho^2\,|e_0| \\
|e_3| &\le \varrho\,|e_2| \le \varrho^3\,|e_0| \\
&\vdots
\end{aligned}
$$

In general, $|e_n| \le \varrho^n\,|e_0|$. And since $0 \le \varrho < 1$, we have $\varrho^n \to 0$ when $n \to \infty$, entailing that $|e_n| \to 0$ when $n \to \infty$.

Theorem 4. If f'' is continuous in $\mathbb{R}$, and if f is increasing, convex, and has a root, then said root is unique and Newton's method converges to it from any starting point.

Recall that f is convex if $f''(x) > 0$ for all x. Graphically, it is convex if the line connecting two arbitrary points of f lies above the curve of f between those two points.

4.3 Secant method

In Netwon's method,

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
$$

The function of interest is f. We cannot escape computing $f(x_n)$, but it would be desirable to avoid the computation of $f'(x_n)$, which may potentially be expensive. Since

$$
f'(x) = \lim_{h \to x} \frac{f(x) - f(h)}{x - h}
$$

it is natural to suggest

$$
f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{1}
$$

Graphically, this means we are not using the line tangent to the point $(x_n, f(x_n))$ but the line secant to the points $(x_n, f(x_n)$ and $(x_{n-1}, f(x_{n-1}))$. The point x_{n+1} is then the value of x where this secant line has a root.

4.4 Fixed point iteration

The key observation is this: if $r \in \mathcal{R}_f$, then $g(x) = x - kf(x)$ has r as fixed point, for any $k \in \mathbb{R}$. Inversely, if g has a fixed point in r, then $r \in \mathcal{R}_f$.

Theorem 5. (1) Let $g \in C[a,b]$ and assume $g(x) \in [a,b]$ for all $x \in [a,b]$. Then there is a fixed point of g in $[a,b]$.

(2) If, on top of previous conditions, g is differentiable in (a,b) and there is some $k < 1$ s.t. $|g'(x)| \leq k$ for all $x \in (a,b)$, then the fixed point referred in (1) is unique.

Theorem 6 (Mean value theorem)**.** Let $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b) with $a < b$. Then there is some $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The interpretation is simple: consider the line secant to f on a, b. The theorem ensures that there is some point c s.t. the line tangent to c is parallelt to said secant (equal slopes).

Proof. (1) If a or b are fixed points the proof is done so assume otherwise. Since $g(x) \in [a,b]$, we have $g(a) > a$ and $g(b) < b$.

Take $\varphi(x) = g(x) - x$, which is continuous and defined in $[a,b]$. Then

$$\varphi(a) = g(a) - a > 0, \qquad \varphi(b) = g(b) - b < 0$$

Then $\varphi(a)\varphi(b) < 0$. Then, by the intermediate value theorem, φ has a root in (a,b). In otherwords, there is at least one p s.t.

$$\varphi(p) = g(p) - p = 0$$

$\therefore g(p) = p$ is a fixed point of g.

(2) Assume two distinct fixed points p, q exist in $[a,b]$. The mean value theorem ensures the existence of some ζ between p, q (and thus in $[a,b]$) s.t.t

$$g'(\zeta) = \frac{g(a) - g(b)}{a - b} \iff g'(\zeta)(a-b) = g(a) - g(b) \tag{1}$$

By hypothesis, $|g'(x)| \leq k < 1$. Since p, q are assumed to be fixed points, equation (1) gives:

$$\begin{aligned} |p - q| &= |g(p) - g(q)| \\ &= |g'(\zeta)|\,|p - q| \\ &\leq k\,|p - q| < |p - q| \end{aligned}$$

But this is absurd. The contradiction arises from assuming p, q to be distinct. Therefore, the fixed point is unique.

The fixed point algorithm begins with an approximation p_0. Then,

$$p_n = g(p_{n-1})$$

If g continuous and the sequence converges, then it converges to a fixed point, since:

$$p := \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

Input: p, M, δ
$p_{\text{previous}} = p$
for $i = 1$ **to** $i = M$ **do**
    $p \leftarrow g(p)$
    if $\left|p - p_{\text{previous}}\right| < \delta$ **then**
        return p
    fi
    $p_{\text{previous}} = p$
od
return p

Theorem 7. Let $g \in C[a, b]$ be a self-map of $[a, b]$ differentiable in (a, b). Assume there is a constant $0 < k < 1$ s.t. $|g'(x)| \leq k$ for all $x \in (a, b)$.

For all $p_0 \in [a, b]$, the sequence $p_n = g(p_{n-1})$ converges to the unique f ixed point p in (a, b).

Proof. The mean value theorem ensures that

$$\begin{aligned} |p_n - p| &= |g(p_{n-1}) - g(p)| \\ &= |g'(\zeta_n)||(p_{n-1} - p)| \\ &\leq k\,|p_{n-1} - p| \end{aligned}$$

with $\zeta_n \in (a, b)$. More succintly, with $e_n := p_n - p$,

$$|e_n| \leq k\,|e_{n-1}| \leq k\,|e_{n-2}| \leq \ldots \leq k\,|e_0|$$

By recurrence,

$$|e_n| \leq k^n \, |e_0|$$

Since $0 < k < 1$, $k^n \to 0$ when $n \to \infty$, which entails $|e_n| \to 0$ when $n \to \infty$. It follows that $\{p_n\} \to p$ when $n \to \infty$.

Now let us consider the error of this method. Take $p_n = p + e_n$ and consider the Taylor expanssion of g around p evaluated at $p_n = p + e_n$:

$$g(p_n) = g(p + e_n) = \sum_{i=1}^{m-1} \frac{g^{(i)}(p)}{i!} e_n^i + \frac{f^{(m)}(\zeta_n)}{(n+1)!} e_n^m \tag{2}$$

See that in (2), n corresponds to the iteration we are dealing with, and thus ζ_n and e_n depend on it. On the contrary, m is the degree to which we expand the series of g around p evaluated at p_n. We also assume that ζ_n lies between p_n and p.

By definition, $g(p_n) = p_{n+1}$ so (2) is nothing but an expression for this value. Assume $g^{(k)}(p) = 0$ for $k = 1, 2, \ldots, m - 1$, but $g^{(m)}(p) \neq 0$. Then

$$\begin{aligned} e_{n+1} &= p_{n+1} - p \\ &= g(p_n) - g(p) \\ &= \frac{g^{(m)}\,(\zeta_n)}{m!} e_n^m \end{aligned}$$

More succintly,

$$e_{n+1} = \frac{g^{(m)}\,(\zeta_n)}{m!} e_n^m$$

Then

$$\lim_{n \to \infty} \left| \frac{e_{n+1}}{e_n^m} \right| = \frac{|g^m(p)|}{m!}$$

which is a constant. In conclusion, if the derivatives of g are null in p up to the order $m - 1$, the method as an order of convergence of at least m. Three results follow from this fact.

5 P2

(1) Let $f(x) = (x+2)(x+1)^2 x(x-1)^3(x-2)$. To which root does the biscection method converge on the following intervals?

$$[-1.5, 2.5], \qquad [-0.5, 2.4], \qquad [-0.5, 3], \qquad [-3, -0.5]$$

(a) The midpoint of $I_0 = [-1.5, 2.5]$ is $c_0 := (2.5 - 1.5)/2 = 1/2$. Since $f(a)f(c) < 0$, we have $I_1 = [-1.5, 0.5]$. The midpoint of I_1 is $c_1 = -0.5$, so I_2 will be $[-0.5, 0.5]$. The only root in this interval is $r = 0$, so the algorithm converges to it.

(b) The midpoint of $I_0 = [-0.5, 2.4]$ is $c := (2.4 - 0.5)/2 = 0.95$. Then $I_1 = [-1.5, 0.95]$. Same logic gives $c_1 = -0.725$ and then $I_2 = [-0.725, 0.95]$. The only root here is zero again.

(c, d) Same.

(2) We wish to find a root of f in $[a, b]$ using bisection method and ensuring that the error is not greater than $\epsilon \in \mathbb{R}^+$.

(a) Estimate the number of iterations sufficient to meet the criterion.

(b) What is the number of iterations for $a = 0, b = 1, \epsilon = 10^{-5}$?

Let $e_n = x_n - r$. It is trivial to note that $|e_n| \leq \frac{b_n - a_n}{2}$. Furthermore, the length of I_1 is half the length of I_0, that of I_2 is half that of I_1, etc. In other words,

$$|e_0| \leq \frac{b-a}{2}, \qquad |e_1| \leq \frac{b-a}{2^2}, \qquad |e_2| \leq \frac{b-a}{2^3}, \ldots$$

In general,

$$|e_n| \leq \frac{b-a}{2^{n+1}}$$

Imposing

$$|e_n| \leq \frac{b-a}{2^{n+1}} \leq \epsilon$$

we satisfy our criterion, but we wish to express this bound in terms of n. Now, clearly,

$$\begin{aligned}
&\frac{b-a}{2^{n+1}} \leq \epsilon \\
\iff &\frac{b-a}{\epsilon} \leq 2^{n+1} \\
\iff &\log_2\left(\frac{b-a}{\epsilon}\right) - 1 \leq n \\
\impliedby &\log_2\left(\frac{b-a}{\epsilon}\right) \leq n \\
\iff &\frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2} \leq n
\end{aligned}$$

which is our final answer.

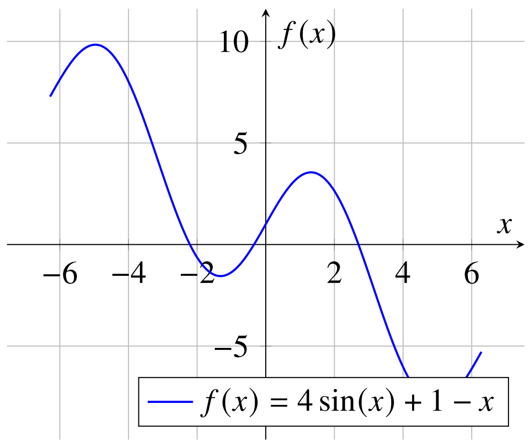
(b) For $a = 0, b = 1, \epsilon = 10^{-5}$, we need

$$n \geq \frac{\ln\left(\frac{1}{10^{-5}}\right)}{\ln 2} \approx 16.609$$

so $n = 17$ would suffice.

(3) Determine graphically some root of $f(x) = 4\sin x + 1 - x$ and perform three iterations of the bisection method to approximate. How many steps are needed to ensure an error less than 10^{-3}?

Let us unveil the full power of LaTex:



I'm too lazy to perform the steps of the algorithm. The number of steps needed again are given by

$$n \geq \frac{\ln\left(\frac{4-2}{10^{-3}}\right)}{\ln 2} \approx 10.96$$

so taking $n = 11$ suffices.

(4) Let $a > 0$. Computing $\sqrt{a}$ is equivalent to finding the root of $f(x) = x^2 - a$.

(a) Show that Newton's sequence for this case is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Prove that f or any $x_0 > 0$, the approximations $\{x_n\}$ satisfy $x_n \geq \sqrt{a}$ for $n \geq 1$.

(c) Prove $\{x_n\}$ is sdecreasing.

(d) Conclude that the sequence converges to $\sqrt{a}$

(a) In Newton's algorithm,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Clearly,

$$f'(x) = \frac{d}{dx}(x^2 - a) = 2x$$

Therefore,

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= x_n - \frac{1}{2}\left(x_n - \frac{a}{x_n}\right) \\ &= \frac{1}{2}x_n + \frac{1}{2}\frac{a}{x_n} \\ &= \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) \qquad \blacksquare \end{aligned}$$

(b) Let $x_0 > 0$. Recall that, among all Pythagorean means, the arithmetic mean is the greatest, asuming positively-valued vectors. In particular, it is greater or equal to the geometric mean:

$$\frac{1}{N}\sum_{i=1}^{n} y_i \geq \sqrt[n]{\prod_{i=1}^{n} y_i}$$

for any set of points $y_1, \ldots, y_n$ all positive. In particular,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \sqrt{x_n \frac{a}{x_n}} = \sqrt{a} \qquad \blacksquare$$

(c)

$$\begin{aligned}
&\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \leq x_n \\
\iff & x_n + \frac{a}{x_n} \leq 2x_n \\
\iff & \frac{a}{x_n} \leq x_n \\
\iff & a \leq x_n^2 \\
\iff & \sqrt{a} \leq x_n
\end{aligned}$$

which is true due to point (b).

(d) Let $e_n = x_n - \sqrt{a}$. We have shown $\{x_n\}$ to be decreasing and bounded below by $\sqrt{a}$. Therefore, it converges to a limit L (with L the infimum of $\{x_n\}$). Then

$$\lim_{n\to\infty} x_n = \frac{1}{2}\lim_{n\to\infty}\left(x_{n-1} + \frac{a}{x_{n-1}}\right) = \frac{1}{2}L + \frac{a}{2L}$$

This induces the equation

$$\begin{aligned}
L = \frac{L}{2} + \frac{a}{2L} \iff & \frac{L}{2} = \frac{a}{2L} \\
\iff & L^2 = a \\
\iff & L = \sqrt{a} \qquad \blacksquare
\end{aligned}$$