1 Info

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Temas:

- Cinemática y dinámica (mecánica)
- Campos eléctricos y magnéticos
- Circuitos
- Termodinámica

2 Measurements and magnitudes

Measurements seek to compare a prediction with an observation, so as to test a hypothesis. A magnitude is a number accompanied by a unit. Some magnitudes are:

- Length, measured in meters (m)
- Time, measured in seconds (s)
- Mass, measusred in kilograms (kg)
- Current, measured in ampers (A)
- Temperature, measured in kelvins (k)
- Matter, measured in moles (mol)

We consider 10^3 (e.g. kilometer) and 10^{-3} (e.g. milimiters) to be within human scale. We call mass, seconds and kilograms the *mechanical units*. We define the *force unit*, or Newton, as

$$[F] = N = \text{kg}\frac{m}{s^2}$$

and the Pascal unit as

$$[P] = \text{Pa} = \frac{N}{m^2}$$

We use scientific notation and terms which express quantities as powers of ten. For instance, 10^{12} is the tera, 10^3 the giga, etc.

The magnitudes hereby described are suited for algebraic manipulation. For instance, $m \times m = m^2$, and $s \times \frac{m}{s} = m$.

3 Vectors

Vectors are used to express position, displacement, velocity, force, acceleration, fields, etc. A vector $\overrightarrow{A}$ (or sometimes $\overrightarrow{a}$) in the general sense has a direction (line), an orientation, and a length (or magnitude). A vector also has an application point, which denotes the point of origin of the vector. When saying $\overrightarrow{a} = \overrightarrow{b}$, we mean that $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in direction, magnitude and orientation, irrespective of their application point.

The scalar product is defined as the usual mapping in the space $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$. Intuitively, the scalar product $\lambda \overrightarrow{a}$ ”streches” or ”shrinks” a vector, depending on wheter $|\lambda| < 1$ or not, and the positivty or negativity of λ determines whether the vector inverts its direction or not. In general, $\left|\lambda \overrightarrow{a}\right| = |\lambda| \left|\overrightarrow{a}\right|$.

The sum of vectors, $\overrightarrow{a} + \overrightarrow{b}$, is a mapping $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$. As usual, and in a graphical sense, the sum corresponds to the application of the parallelogram rule.

Parallelogram rule. Make $\overrightarrow{a}$ and $\overrightarrow{b}$ coincide in their point of application. From the tip of $\overrightarrow{a}$, draw a copy of $\overrightarrow{b}$, and from the tip of $\overrightarrow{b}$ a copy of $\overrightarrow{a}$. The corner of the thus generated parallelogram is the tip of $\overrightarrow{a} + \overrightarrow{b}$.

Alternatively, from the tip of $\overrightarrow{a}$ write $\overrightarrow{b}$. Then $\overrightarrow{a} + \overrightarrow{b}$ is the vector which goes from the point of application of $\overrightarrow{a}$ to the tip of $\overrightarrow{b}$.

The sum of vectors is commutative, associative, and distributive with respect to scalar product.

If $\overrightarrow{A}$ is a vector, we use A_x and A_y to denote the projection of the vector over the axis x or y, respectively. Using A_x and A_y one forms a rectangular triangle with sides A_x, A_y and a hypotenuse of length $\left|\overrightarrow{A}\right|$.

Let θ be the angle formed by $\overrightarrow{A}$ with the x-axis. Then, using trigonometry,

$$\cos\theta = \frac{A_x}{\left|\overrightarrow{A}\right|}, \qquad \sin\theta = \frac{A_y}{\left|\overrightarrow{A}\right|}$$

from which one can find A_x, A_y assuming one knows θ. From this follows that $\left|\overrightarrow{A}\right|$ and θ fully determine all the information about the vector, insofar as the allow us to determine A_x, A_y. Conversely, knowing A_x and A_y is also sufficient to determine $\overrightarrow{A}$, insofar as

$$\left|\overrightarrow{A}\right| = \sqrt{A_x^2 + A_y^2}, \qquad \frac{A_y}{A_x} = \frac{\left|\overrightarrow{A}\right| \sin\theta}{\left|\overrightarrow{A}\right| \cos\theta} = \tan\theta \Rightarrow \theta = \arctan\left(\frac{A_y}{A_x}\right)$$

As convention, we use $\hat{i}$ to denote the versor (vector of length 1) with direction parallel to the x-axis, and $\hat{j}$ the versor with direction parallel to the y-axis.

Notice that, for any vector $\overrightarrow{A}$, A_x is $\hat{i}$ times A_x, and A_y is $\hat{j}$ times A_y, which means

$$\overrightarrow{A} = A_x\hat{i} + A_y\hat{j}$$

When writing $\overrightarrow{A}$ in this way, we say we write it in term of its components x, y. In terms of linear algebra, it's not hard to see that we are simply expressing that $\hat{i}, \hat{j}$ form a basis of $\mathbb{R}^2$. Thus, it is equivalent to write

$$A_x = \left|\overrightarrow{A}\right| \cos\theta, \qquad A_y = \left|\overrightarrow{A}\right| \sin\theta$$

and

$$\overrightarrow{A} = \left|\overrightarrow{A}\right| \left(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}\right)$$

From this follows as well that

$$\begin{aligned}\overrightarrow{A} + \overrightarrow{B} &= \left(A_x\hat{i} + A_y\hat{j}\right) + (B_x\hat{i} + B_y\hat{j}) \\ &= \hat{i}\,(A_x + B_x) + \hat{j}\,\left(A_y + B_y\right)\end{aligned}$$

which means the sum of vectors has as components the sum of the components.

The scalar product of two vectors, $\overrightarrow{A} \cdot \overrightarrow{B}$, is a scalar defined as

$$\overrightarrow{A} \cdot \overrightarrow{B} = \left|\overrightarrow{A}\right| \left|\overrightarrow{B}\right| \cos\theta$$

where θ is the angle formed by the two vectors. The scalar product is positive if $\cos\theta$ is positive, which occurs for $0 < \theta \leq 90$. It is negative if $\cos\theta$ is negative, i.e. if $90 < \theta \leq 180$. Clearly, $\overrightarrow{A} \cdot \overrightarrow{B} = 0 \iff \theta = 90$.

In general, from the definition follows that

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_x B_x + A_y B_y$$

The vectorial product $\overrightarrow{A} \times \overrightarrow{B}$ is a vector perpendicular to the plane formed by $\overrightarrow{A}$ and $\overrightarrow{B}$. Its module is $\left|\overrightarrow{A}\right| \left|\overrightarrow{B}\right| \sin\theta$, and its direction is given by what's called the right-hand rule.

3.1 Excercises

(2) Sean los vectores $\overrightarrow{A} = 2\hat{\imath} + 3\hat{\jmath}$ $\overrightarrow{B} = 4\hat{\imath} - 2\hat{\jmath}$ y $\overrightarrow{C} = -\hat{\imath} + \hat{\jmath}$. Determinar la magnitud y el ángulo (representación polar) de los vectores resultantes $\overrightarrow{D} = \overrightarrow{A} + \overrightarrow{B} + \overrightarrow{C}$ y $\overrightarrow{E} = \overrightarrow{A} + \overrightarrow{B} - \overrightarrow{C}$. Resolver analítica y gráficamente.

(Analytical solution.) We'll use A_x, A_y to denote the components of the vector $\overrightarrow{A}$, and same for all other vectors. We know the components of $\overrightarrow{D}$ are

$$D_x = A_x + B_x + C_x = 2 + 4 - 1 = 5, \qquad D_y = 3 - 2 + 1 = 2$$

from which readily follows that $|D| = \sqrt{5^2 + 2^2} = \sqrt{29} \approx 5.385$. Similarly,

$$E_x = 2 + 4 + 1 = 7, \qquad E_y = 3 - 2 - 1 = 0$$

from which follows that $|E| = \sqrt{7^2} = 7$.

Now, we must recall that

$$\theta_{\overrightarrow{Z}} = \arctan\left(\frac{Z_y}{Z_x}\right)$$

for any $\overrightarrow{Z}$.

We need not memorize this: it is trigonometrically clear that $Z_x = \cos\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$ and $Z_y = \sin\theta_{\overrightarrow{Z}} \left|\overrightarrow{Z}\right|$, and therefore

$$\frac{Z_y}{Z_x} = \tan\theta$$

And $\arctan$ is the inverse of $\tan$. Anyhow, for $\overrightarrow{E}$ and $\overrightarrow{D}$ we have

$$\theta_{\overrightarrow{E}} = \arctan\left(\frac{E_y}{E_x}\right) = \arctan(0) = 0$$

$$\theta_{\overrightarrow{D}} = \arctan\left(\frac{D_y}{D_x}\right) = \arctan\left(\frac{2}{5}\right) \approx 0.38$$

(3) Can two vectors of different magnitud be combined and yield zero? What about three?

The zero vector is the only vector with magnitude zero. Let $\overrightarrow{A}, \overrightarrow{B}$ arbitrary vectors. Then

$$\left|\overrightarrow{A} + \overrightarrow{B}\right| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}$$

which is zero if and only if

$$(A_x + B_x)^2 + (A_y + B_y)^2 = 0$$

This only holds if $A_x + B_x = A_y + B_y = 0$. But

$$A_x + B_x = 0 \Rightarrow A_x = -B_x, \qquad A_y + B_y = 0 \Rightarrow A_y = -B_y$$

But then

$$|A| = \sqrt{A_x^2 + A_y^2} = \sqrt{(-B_x)^2 + (-B_y)^2} = \sqrt{B_x^2 + B_y^2} = |B|$$

$\therefore \ \left|\overrightarrow{A} + \overrightarrow{B}\right| = 0 \iff \left|\overrightarrow{A}\right| = \left|\overrightarrow{B}\right|$.

It is simple to see that three vectors of different magnitude can add to zero.

Assume $A + B + C = 2\hat{i} + \hat{j}$ and $A = 6\hat{i} - 3\hat{j}$, $B = 2\hat{i} + 5\hat{j}$. Find the components of C. Solve analytically and graphically.

We know

$$6 + 2 + C_x = 2, \qquad -3 + 5 + C_y = 1$$

from which follows that $C_x = -6, C_y = -1$.

(5) A and B have a magnitud of $3m, 4m$ respectively. The angle between them is $\theta = 30$ degrees. Find their scalar product.

Their scalar product is

$$(|B| \cos\theta) |A|$$

Recall that

$$\text{Angle in degrees} = \text{Angle in radians} \cdot \frac{180}{\pi}$$

Thus, thirty degrees equates to $30\frac{\pi}{180} \approx 0.523$ radians. Then the scalar product is

$$4\cos(0.523) \times 3 \approx 10.395$$

(6) Find the angle between $A = 4\hat{i} + 3\hat{j}$ and $B = 6\hat{i} - 3\hat{j}$.

Recall that

$$A \cdot B = |A|\,|B| \cos\theta$$

where θ is the angle between the vectors. This readily entails that

$$\frac{A \cdot B}{|A|\,|B|} = \cos\theta$$

or equivalently that

$$\theta = \arccos\left(\frac{A \cdot B}{|A|\,|B|}\right)$$

Now, $A \cdot B = 4 \times 6 + 3 \times -3 = 24 - 9 = 15$ and $|A|\,|B| = 5 \cdot 6.708 = 33.541$.

Therefore,

$$\theta = \arccos\left(\frac{15}{33.541}\right) = \arccos(0.447) = 1.107$$

(7) Let $\overrightarrow{v} = \left(\frac{1}{3}, \frac{2}{3}\right)$ be the vector of components. Find the components of the vector of module 5 whose direction and orientation (sentido) are those of the given vector.

Assume $\overrightarrow{x} = (x_1, x_2)$ is of magnitude 5. Any vector whose direction and orientation are the same than those of $\vec{v}$ is "a stretching" of $\vec{v}$. In other words, for $\vec{x}$ to satisfy the requirements, we must have

$$\vec{x} = \lambda \vec{y} \tag{1}$$

for some $\lambda \in \mathbb{R}$. (Furthermore, $\lambda > 0$ since otherwise orientation is not preserved.)

Now, from equation (1) follows that

$$\|\vec{x}\| = \lambda \|\vec{y}\| \tag{2}$$

since the magnitude of a scaled vector is the scaled magnitude of the vector. Equation (2) simplifies to

$$\|\vec{x}\| = \lambda \sqrt{1/9 + 4/9} = \frac{\lambda\sqrt{5}}{3} \tag{3}$$

From this readily follows that $\frac{3}{\sqrt{5}}\|\vec{x}\| = \lambda$. But it is a hypothesis that $\|\vec{x}\| = 5$. Therefore,

$$\lambda = \frac{3}{\sqrt{5}} \cdot 5 = \frac{15}{\sqrt{5}} \tag{4}$$

In other words,

$$\vec{x} = \frac{15}{\sqrt{5}} \vec{v} \tag{5}$$

which is ugly but can be simplified.

(8) Write the expression of the vector product $\vec{c} = \vec{u} \times \vec{v}$ in the following cases:

1. $\vec{u}, \vec{v}$ are coplanar. Provide a graphical interpretation.
2. $\vec{u} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{v} = -3\hat{i} + \hat{j} + 2\hat{k}$. Find the module of the resulting vector $\vec{c}$ in two different ways.

(1) Two vectors are coplanar if there is a plane which contains them both. Since the vector product $\vec{u} \times \vec{v}$ is a vector orthogonal to both $\vec{u}$ and $\vec{v}$

(12) Un avión vuela 200 km hacia el NE en una dirección que forma un ángulo de 30 hacia el este de la dirección norte. En ese punto cambia su dirección de vuelo hacia el NO. En esta dirección vuela 60 km formando un ángulo de 45 con la direccióon norte.

(a) Calcular la máxima distancia hacia el este del punto de partida a la que llegó el avión.

(b) Calcular la máxima distancia hacia el norte del punto de partida a la que llegó el avión.

(c) Calcular la distancia a la que se encuentra el avión del punto de partida al cabo de su recorrido.

(d) Determinar vectorialmente el camino que debería hacer para volver al punto de partida. Resolver gráfica y analíticamente.

Sea $\vec{A}$ el vector que describe el primer recorrido, $\vec{B}$ el vector que describe el segundo recorrido. Al final del problema, el avión se encuentra en la posición indicada por $\vec{A} + \vec{B}$.

Como $\vec{A}$ describe un movimiento con un ángulo de $\theta = 60$ grados $(90 - 30)$ respecto al eje y (norte), y una magnitud de 200km, podemos determinarlo recordando que

$$A_x = \left|\vec{A}\right| \cos\theta, \qquad A_y = \left|\vec{A}\right| \sin\theta$$

En radianes, $\theta = \frac{\pi}{180} \times 60 = 1.047$

$$A_x = 200 \times \cos(1.047) = 100.034, \qquad A_y = 22 \times \sin(1.047) = 173.185$$

En conclusión, $\vec{A} = 100.034\hat{\imath} + 173.185\hat{\jmath}$. Mismo razonamiento nos da que el ángulo del segundo vector es de 130 grados, lo cual en radianes nos da $\alpha = \pi/180 \times 130 = 2.268$. Por ende,

$$B_x = 60\cos(2.268) = -38.524, \qquad B_y = 60\sin(2.268) = 45.998$$

Es decir que $\vec{B} = -38.524\hat{\imath} + 45.998\hat{\jmath}$. De esto se sigue que $\vec{C} = \vec{A} + \vec{B} = (61.51, 219.183)$.

(a) Claramente, es la coordenada x del vector $\vec{A}$, 100.034.

(b) Claramente, es la coordenada y del vector $\vec{C}$: 218.183.

(c) Claramente, es la magnitud de $\vec{C}$, es decir $\|\vec{C}\| = \sqrt{61.51^2 + 219.183^2} = 227.65$.

(d) El camino para volver es dado por $\vec{C} \times (-1)$.

4 Cynematics

4.1 Unidimensional movement

The study of movement requires two variables: position (x, in units of length) and time (t, in seconds). We begin our study with unidimensional movement, i.e. movement which occurs along a single axis.

Experimentally, a way to study unidimensional movement could consist in taking a sequence of photographs (from the same position and angle) of the moving object at times $t_1, \ldots, t_n$. Some coordinate system must be imposed upon the space along which the object moves, e.g. setting an axis with origin at the initial position of the object, the same direction as the movement of the object, and some appropriate units. The photographs would then provide a sequence of positions $x_1, \ldots, x_n$.

Clearly, $\{t_n\}, \{x_n\}$ could be understood as defining a discrete function $\varphi(n)$, which on its turn might be interpolated to obtain a continuous approximation $\phi(t)$. To the limit, the continuous approximation converges to what we call a movement function.

Movement function. A movement function $x(t)$ is a continuous, smooth function.

Examples. $x(t) = c$ (reposo), $x(t) = at + b$ (MRU), $x(t) = at^2 + bt + c$ (MRUV).

4.2 Coincidence, displacement, temporal intervals

If A, B are objects with movement functions $x_A(t), x_B(t)$, we say A, B coincide (se encuentran) when $x_A(t) = x_B(t)$.

We define displacement (desplazamiento) (relative to positions x_1, x_2) as $\Delta x = x_2 - x_1$. Notice that Δx is not the same as distance: if one travels from A to B and then to B from A, the distance traveled is to times the distance from A to B, but $\Delta x = 0$.

We also define a temporal interval, relative to two times t_1, t_2, as $\Delta t = t_2 - t_1$, where $t_2 > t_1$.

4.3 Velocity

We define *median velocity* (velocidad media) as

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{1}$$

where $x_2 = x(t_2), x_1 = x(t_1)$. Clearly, $\overline{v}$ is the slope of the line which intersects $x(t)$ at points t_1, t_2. The sign of $\overline{v}$ then determines the direction (sentido) of movement. The unit of $\overline{v}$ is then L/T (length over time), for instance kilometers per hour. Median velocity indicates the rate of change of distance in time.

Clearly, an object in reposo has a median velocity of zero. An object with movement function $x(t) = at + b$ (MRU) has median velocity a. The case of interest is an object with a quadratic movement function (MRUV).

If $x(t) = at^2 + bt + c$, let m the midpoint of the quadratic expression and take $t_1 = m - c, t_2 = m + c$ with $c > 0$. Clearly, the median velocity from t_1 to m is negative, that from m to t_1 is positive, and that from t_1 to t_2 is zero. This is sufficient to suggest that median velocity does not clearly express the nature of the movement.

For that reason, the length Δ of the interval $[t_1, t_2]$ might be reduced in the limit to zero, so that we get an accurate notion of the instantaneous (or close to instantaneous) change of direction. Needless to say, the limit converges to the slope of the line tangent to $(t_1, x(t_1))$, i.e. the derivative of $x(t)$ at t_1. Thus, we obtain the definition of instantaneous velocity, usually called simply velocity:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = v'(t) \tag{2}$$

Again, $[v(t)] = \frac{L}{T}$. Quite clearly, $v(t) = \overline{v}$ for constant and linear functions, but for the quadratic function $x(t)$ we have

$$x(t) = at^2 + bt + c, \qquad v(t) = 2at + b$$

5 De adelante hacia atrás

Recordemos que si $x(t)$ es función de movimiento, $v(t) = \frac{dx}{dt}$ es la velocidad, y $a(t) = \frac{d^2x}{dt^2}$ es la aceleración. Naturalmente, esto significa que

$$x(t) = \int v(t')dt' + D, \qquad v(t) = \int a(t')dt' + C$$

donde D, C son constantes de movimiento que dependerán de las condiciones iniciales.

(†) **Derivada y puntos de inflexión** Un punto de inflexión de f continua y dos veces derivable en $[a, b]$ es un valor $x_0 \in [a, b]$ t.q. $f^{(2)}(x_0) = 0$. Los puntos de inflexión representan un cambio de comportamiento en f, en particular transiciones de cóncava a convexa y viceversa.

Intuitivamente, y exceptuando casos límite (como f'' constante), si $f''(x_0) = 0$, entonces alrededor de x_0 hay un cambio de signo en f'', lo cual quiere decir que en el entorno alrededor de x_0 la función original pasa de crecer a decrecer, o de decrecer a crecer.

6 Movimiento bidimensional (cinemática 2D)

Se modela con una curva en el plano cartesiano. La curva (el dibujo del movimiento) se denomina trayectoria. La trayectoria *no* es una función, obviamente (e.g. una circunferencia es una trayectoria posible).

La descripción de la posición del objeto en cada instante de tiempo t se descsribe con vectores. En el plano cartesiano, decimos que $\vec{r} = \hat{x}i + y\hat{j}$ es un vector posición si la punta de $\vec{r}$ se corresponde con la posición del objeto (en un tiempo dado).

Sea θ el ángulo formado por $\vec{r}$ y el eje x, de manera tal que $x = |\vec{r}|\cos\theta, y = |\vec{r}|\sin\theta$. Es decir, $\vec{r} = |\vec{r}|\cos\theta\hat{i} + |\vec{r}|\sin\theta\hat{j}$.

Ahora pensemos el objeto en movimiento, y que registramos a lo largo del tiempo t las posiciones $x(t), y(t)$. Claramente, $x(t), y(t)$ son funciones de movimiento unidimensionales. Por lo tanto, podemos definir el vector posición de manera general como

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} \tag{1}$$

Ahroa consideremos la trayectoria T (conjunto de puntos en el eje cartesiano) del objeto. Sean $P_1, P_2 \in T$ dos puntos en el plano que pertenecen a la trayectoria. Definimos el desplazamiento del objeto como

$$\Delta\vec{r} = \vec{r_2}\vec{r_1} \tag{2}$$

donde $\vec{r_1}, \vec{r_2}$ son los vectores con puntas en P_1, P_2. De esto se sigue que

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) \tag{3}$$

para dos instantes de tiempo t_1, t_2. Además, es claro que $\Delta\vec{r}$ es el vector que conecta las dos puntas, desde P_1 hasta P_2, y que $\vec{r_2} = \vec{r_1} + \Delta\vec{r}$.

La velocidad media en el intervalo de tiempo $[t_1, t_2]$ se define entonces como

$$\overline{v}_{[t_1,t_2]} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \tag{4}$$

lo cual es claramente un vector que contendrá las velocidades medias en las direcciones x e y. El vector velocidad entonces se define como uno esperaría:

$$
\begin{aligned}
\vec{v}(t) &= \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} \\
&= \frac{d\vec{r}}{dt} \\
&= \frac{d}{dx}\left(x(t)\hat{i} + y(t)\hat{j}\right) \\
&= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}
\end{aligned} \tag{5}
$$

Por lo tanto,

$$
\vec{v}(t) = v_x(t)\hat{i} + v_y(t)\hat{j} \tag{6}
$$

con v_x, v_y las velocidades correspondientes a las funciones de movimiento $x(t), y(t)$.

(†) **Interpretación gráfica de** $\vec{v}$. Gráficamente, el vector velocidad $\vec{v}(t)$ se representa como sigue. Imagine la trayectoria T y un vector posición $\vec{r}$ que conecta con $P \in T$. Entonces $\vec{t}$ será paralelo a la recta tangente a la trayectoria T en el punto P. Esto *no* es una derivada, porque la trayectoria T no es una función. En otras palabras, el vector velocidad es tangente a la trayectoria.

Definimos entonces el versor

$$
\hat{v} = \frac{\vec{v}}{|\vec{v}|} \tag{7}
$$

que nos da la dirección tangencial de la velocidad.

Así como tenemos un vector velocidad, tenemos el vector aceleración

$$
\vec{a}(t) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} \tag{8}
$$

de acuerdo al mismo razonamiento límite que nos dio el resultado (6). El vector aceleración también puede descomponerse en sus direcciones tangencial y normal (respecto a la trayectoria). La aceleración tangencial será la dirección de la velocidad:

$$
\text{aceleración tangencial} \rightarrow \hat{v} = \frac{\hat{v}}{|\vec{v}|}, \qquad \text{aceleración normal} \rightarrow \hat{n} = (\perp \vec{v}) \tag{9}
$$

donde $\perp\hat{v}$ es el vector tal que $\perp\hat{v} \cdot \hat{v} = 0$ (perpendicular).

(†) **Algunas observaciones**. Hablemos en un intervalo de tiempo $[t, t + \Delta t]$.

(a) Cuando $\vec{v}$ cambia de módulo y de sentido, pero no de dirección, la aceleración es puramente tangencial, es decir está en la dirección de la velocidad. (Son paralelos.)

(b) Cuando $\vec{v}$ cambia de dirección y de sentido, pero no de módulo (i.e. el vector apunta hacia otro lado pero tiene el mismo largo), se cumple que la aceleración es perpendicular a la velocidad. Es decir, es puramente normal.

(c) Si $\vec{v}$ cambia de módulo, de sentido y de dirección, el vector $\vec{a}$ tendrá una componente tangencial y normal.

La aceleración puramente tangencial resulta en un movimiento unidimensional, una línea recta. El caso (b) corresponde a un movimiento bidimensional.

6.1 Tiro de proyectil

El tiro de proyectil es un movimiento bidimensional bajo la acción única de la gravedad. Si imaginamos la parábola de un proyectil, surgen preguntas cómo cuál es su altura máxima, su destino final, etc. Asumamos que se ha impuesto un sistema de coordenadas.

Observemos que el objeto *no* tiene aceleración en el eje x, sólo velocidad. En el eje y el proyectil es afectado por la gravedad. Por ende,

$$\vec{a} = o\hat{i} - 9.8\frac{m}{s^2}\hat{j}$$

Luego, en $t = 0$,

$$\vec{v_0} = v_{0x}\hat{i} + v_{0y}\hat{j} = \vec{v}(t = 0)$$

Asumamos que por convención, $\vec{r}(t = 0) = 0$. Por ende,

$$\vec{r}(t = 0) = x_0\hat{i} + y_0\hat{j}$$

Entonces,

$$\vec{v}(t) = v_{0x}\hat{i} - (9.8\frac{m}{s^2}t + v_{0y})\hat{j}$$

Integrando una vez más,

$$\vec{r}(t) = (v_{0x}t + x_0)\hat{i} + \left(-9.8\frac{m}{s^2}\frac{t^2}{2} + v_{0y}t + y_0\right)\hat{j}$$

Pero $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$. Por ende,

$$x(t) = v_{0x}t + x_0, \qquad y(t) = -9.8\frac{m}{s^2}\frac{t^2}{2} + v_{0y}t + y_0$$

Vemos entonces que $x(t)$ es lineal, y su pendiente es la velocidad inicial (en x) dada por v_{0x}. $y(t)$, por otro lado, es cuadrática con máximo. Dicho máximo se corresponde con el punto más alto. Es decir, el punto más alto ocurre en el tiempo t_m tal que

$$v_y(t_m) = 0 \implies t_m = \frac{v_{0y}}{9.8\frac{m}{s^2}}$$

El tiempo de vuelo está dado por el tiempo en que la altura se hace cero, es decir el t_{vuelo} tal que $y(t_{\text{vuelo}}) = 0$. Es la raíz máxima de $y(t)$. La trayectoria será despejando t de x:

$$t = \frac{x - x_0}{v_{0x}}$$

La trayectoria también será una parábola, en este caso particular. Y como en cada punto la aceleración es la aceleración negativa de la gravedad, los vectores aceleración a lo largo del tiempo serán siempre ”flechas rectas hacia abajo”.

7 Descripción vectorial del movimiento

Usamos $\vec{r}(t)$ para denotar al vector posición, $\vec{v}(t)$ para denotar al vector velocidad, y $\vec{a}(t)$ para denotar al vector aceleración. Si $x(t), y(t)$ son las funciones de movimiento en los ejes x, y, entonces

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, \qquad \vec{v}(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}, \qquad \vec{a}(t) = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} \tag{1}$$

y

$$|\vec{r}| = \sqrt{x^2 + y^2} \tag{2}$$

Dados dos instantes $t_1 < t_2$, el desplazamiento del un objeto en movimiento con vector posición $\vec{r}(t)$ es el vector

$$\Delta\vec{r} := \vec{r}(t_2) - \vec{r}(t_1)$$

Notar que este es el vector que ”parte” de la punta de $\vec{r}(t_1)$ y llega a la punta de $\vec{r}(t_2)$. Es decir, marca desde dónde y hasta dónde se movió el cuerpo en el tiempo $[t_1, t_2]$.

La velocidad media del cuerpo en el intervalo $[t_1, t_2]$ es el vector velocidad media:

$$\overline{\vec{v}(t_1, t_2)} := \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}$$

o equivalentemente

$$\overline{\vec{v}(t, \Delta t)} := \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \tag{3}$$

Puesto que $\overline{\vec{v}}$ se obitene multiplicando $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$ por un escalar positivo, ambos vectores tendrán misma dirección y sentido. Es decir, la velocidad media es paralela y equidireccional al desplazamiento.

Respecto al vector velocidad, en un punto dado de la trayectoria, no es difícil ver que ”ocupa” un segmento de la recta tangente a la trayectoria en dicho punto.

With regards to the acceleration vector, we note that

$$\vec{a} = \vec{a}_t + \vec{a}_n$$

where $\vec{a}_t$ is the *tangential acceleration* and $\vec{a}_n$ the normal acceleration. The tangential acceleration operates along the directoin of the velocity, affecting the speed of the object. Normal acceleration is perpendicular to velocity, and it afects the direction of the object. Objects moving in a straight line have a null normal acceleration, since their direction doesn't change. Objects moving in curvilinear fashion always have non-null normal acceleration (and hence non-null acceleration).

If the velocity is constant, such as in uniform circular motion, $\vec{a}_t = 0$ and therefore $\vec{a}$ is purely orthogonal to the velocity. If velocity is not constant but changing, then $\vec{a}$ has a component along the velocity and it is not perpendicular to it.

8 Dinámica Newtoniana

There in this world are four fundamental forces: gravitational, electromagnetic, strong nuclear, and weak nuclear. The first two are the ones we experience in our daily lives. The strong and weak nuclear forces operate at the atomic level, and are not relevant to our discussion.

The gravitational force is an attractive force which operates between any two masses. The electromagnetic force operates between any two charged particles, and can be attractive or repulsive. We will discuss now the gravitational force.

Formally, we define $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ to be a force. The unit of force is the Newton (N), which is defined as the force required to accelerate a mass of $1kg$ at a rate of $1m/s^2$.

Newton discovered fundamental laws of motion, which we summarize here:

Newton's First Law (Law of Inertia). An object at rest remains at rest, and an object in motion continues in motion with constant velocity, unless acted upon by a net external force.

$$\vec{F}_T = 0 \Rightarrow \vec{v} = \text{constant} \tag{1}$$

Newton's Second Law. The net force acting on an object is equal to the time rate of change of its momentum. If the mass of the object is constant, this is equivalent to saying that the net force acting on an object is equal to the product of its mass and its acceleration:

$$\vec{F}_\text{T} = m\vec{a} \tag{2}$$

Newton's Third Law. If object A exerts a force $\vec{F}$ on object B, then B exerts a force $-\vec{F}$ on A. In other words, forces always come in pairs, equal in magnitude and opposite in direction.

We know

$$\vec{v} = \int \vec{a}(t)\ dt + \vec{C}, \qquad \vec{r} = \int \vec{v}(t)\ dt$$

Now, note that

$$\vec{F}_T = F_{Tx}\hat{\imath} + F_{Ty}\hat{\jmath}$$

and since $\vec{F}_T = m\vec{a}$

$$\vec{a} = \frac{\vec{F}_T}{m} = \frac{F_{Tx}\hat{i} + F_{Ty}\hat{j}}{m} = a_x\hat{i} + a_y\hat{j}$$

Then

$$\vec{a} = \frac{\vec{F}}{m} = \frac{d^2\vec{r}}{dt^2} \tag{3}$$

Furthermore,

8.1 Tipos de fuerzas

There are many types of forces, but we will discuss only a few. The MRU (movimiento rectilíneo uniforme) is a movement with no forces acting on the object. The MRUV (movimiento rectilíneo uniformemente variado) is a movement with a constant force acting on the object.

$$\sum \vec{F} = 0 \Rightarrow \vec{v} = \text{constant}, \vec{a} = 0 \qquad \text{(MRU)}$$

$$\vec{F} = \text{constant} \Rightarrow \vec{a} = \text{constant} = \vec{F}/m$$

The gravitatory force is an example of a constant force producing MRUV.

Another force is denote by tension (tensión). It is the force exerted by a rope, string, cable, etc. It is always directed along the rope, and it can be attractive or repulsive. The cable is assumed to be massless and inextensible. It transmits the same force throughout its length.

Another force is the contact force (fuerza de contacto). It is the force exerted by a surface on an object in contact with it. It is always perpendicular to the surface. It is sometimes called the normal force. It is a rule that

$$\vec{a} = 0 \implies \vec{N} + \vec{P} = 0$$

Another force is the elastic force (fuerza elástica). It is the force exerted by a spring when it is stretched or compressed. It is given by Hooke's law:

$$\vec{F}_e = -k\Delta\vec{x} = -k(x - \ell_0)\hat{i}$$

where k is the spring constant, and $\Delta\vec{x}$ is the displacement vector of the spring from its equilibrium position. The negative sign indicates that the force exerted by the spring is in the opposite direction to the displacement.

Another force is the centripetal force (fuerza centrípeta). It is the force that keeps an object moving in a circular path. It is always directed towards the center of the circle.

Another force is the gravitational force (fuerza gravitatoria). It is the force exerted between two masses without contact. It is given by Newton's law of universal gravitation:

$$\vec{F}_g = -G\frac{m_1 m_2}{r^2}\hat{r}$$

where G is the gravitational constant, m_1, m_2 are the masses of the two objects, r is the distance between them, and $\hat{r}$ is the unit vector pointing from one mass to the other. The negative sign indicates that the force is attractive. The force exerted by both objects is equal in magnitude and opposite in direction, in accordance with Newton's third law.

8.2 Excercises

(1) Consider

$$x(t) = 1\left[\frac{m}{s^2}\right]t^2 - 3\left[\frac{m}{s}\right]t$$

the movement function of a body travelling in a straight line, with x in meters and t in seconds.

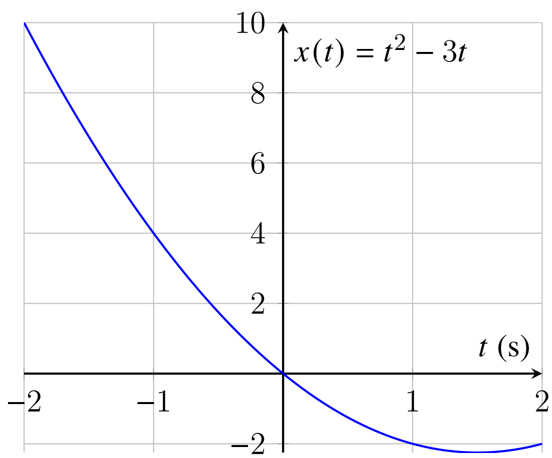
(a) Plot $x(t)$

(b) Determine analytically the median velocity in $[-1, 5]$, $[-1, 8]$, $[-1, 0.9]$, $[-1, 0.99]$, $[-1, 0.999]$.

(c) Let $\Delta t_n = t_n - t_0$ with $\{t_n\} = \{-1, 5, 4, 1, -0.5, -0.8, -0.9, -0.99, -0.999\}$ all measured in seconds. To what value does the median velocity of the object converge as t_n decreases in the interval $[-1, -1 + \Delta t_n]$? What is the geometrical interpretation of this result?

(d) Find the equation for the line tangent to $x(t)$ at $t = -1s$.

(a) Notice that since t is in seconds, $\frac{mt^2}{s^2}$ correctly expresses a quantity in meters, and so does $\frac{mt}{s}$. So we will from now on write simply $x(t) = t^2 - 3t$, understanding that it is a mapping from time in seconds to meters.



(b) The median velocity of an object in the time interval $[t_a, t_b]$ was given by

$$\frac{\Delta x}{\Delta t} = \frac{x(t_b) - x(t_a)}{t_b - t_a} \tag{1}$$

So excercise (b) is as simple as plugging in the corresponding values into equation (1) and I skip it.

(c) Let t be an arbitrary value. Then by definition of $\frac{dx}{dt}$,

$$\lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx}{dt}$$

the derivative of $x(t)$ at time t. In particular, the limit whose convergence we are asked to study is nothing but the limit above with $t = -1$:

$$\lim_{\Delta t \to 0} \frac{x(-1 + \Delta t) - x(-1)}{\Delta t} = x'(-1)$$

So suffices to observe that $x'(t) = 2t - 3$ and $x'(-1) = -5$. In conclusion, the object at time $t = -1$ travels at an instantaneous velocity of -5 meters per second.

(d) The line $\ell(t) = at + b$ tangent to $x(t)$ at $t = -1s$ has slope $a = -5$ and crosses through the point $(-1, 4)$. So we must have $-5(-1) + b = 4 \iff b = 4 - 5 = -1$. So the line is $\ell(t) = -5t - 1$.

(4) Answer the questions.

(a) Can an object have null velocity and yet possess acceleration?

Let $x(t)$ the describe the movement of the object and $v(t) = x'(t)$ its velocity, both as a function of time. Assume for an arbitrary t_0 that $v(t) = 0$. It is very much possible that $v'(t_0) \neq 0$.

Consider, for instance, that $v(t)$ is linear and non-constant, making $v'(t) = a$ a non-null constant. Then there exists a unique root r s.t. $v(r) = 0$, but independently of this fact $v'(r) = a \neq 0$.

Physically, it should be clear that if a non-moving object could not possess acceleration, then it would be impossible for it to pass from a still to a moving state. So, at least at the intutition level, this *reductio ad absurdum* suffices.

(b) Can a moving object have a null displacement in a given interval and yet non-null velocity?

Naturally. Take as example an object moving in circles at a constant, non-null velocity v, and assume it travels a full circle in t seconds. Then all of the intervals in $\{[t_0, t_0 + tk] : k \in \mathbb{N}\}$ are such that they give null displacements. Yet the object *is* moving.

(c) Can an object have an east-bound velocity of while its acceleration is west-bound?

Informally, this is quite clearly the case, insofar as any positively-moving object whose velocity decreases must have a negative acceleration.

(d) Consider an object moving on a straight line, with the east being the positive direction, under a velocity of $v(t) = 20\text{ms}^{-1} - 2\text{ms}^{-2}t$. For $t = 0\text{s}, t = 1\text{s}$, what is the situation?

Its velocity is clearly positive in both cases (20 and 18), evidently decreasing, which points out the fact that its acceleration is negative (-2).

(e) A ball is thrown vertically. What do the *signs* of velocity and acceleration look like as the object ascends, and what does that mean? And when the object descends? What happens at the highest point?

Clearly, its velocity is positive during the ascending phase, and negative during the descending phase. At the highest point, the velocity will be exactly zero.

Conversely, acceleration is always negative due to the force excercised by gravity on the ball.

It is the fact that acceleration is constantly negative what causes the ball not only to lose velocity as it goes up until it begins to fall again, but to then fall more and more rapidly as time goes by.

(5) A particle moves through the x-axis with movement function $x(t) = 3 + 17t - 5t^2$, with x in meters and t in seconds.

(a) What is the position of the particle at times $\{1, 2, 3\}$?

(b) At what point in time does the particle return to the origin?

(c) Find $v(t)$ and determine the instantaneous velocity at times $\{1, 2, 3\}$. When is the velocity null? What is the particles velocity when it crosses the origin?

(d) Plot $x(t), v(t), a(t)$.

(a) Trivial, simply compute $x(1), x(2), x(3)$.

(b) See that

$$x(t) = 0 \iff t = \frac{17}{10} \pm \frac{\sqrt{17^2 + 4 \times 3 \times 5}}{10}$$

which gives approximate solutions $t_1 \approx -0.168, t_2 \approx 3.568$. It makes no sense to speak of negative time and we keep only the positive solution $t = 3.568$. Thus, the particle returns to the origin after approximately 3.568 seconds.

(c) The first derivative of $x(t)$ is

$$v(t) = -10t + 17$$

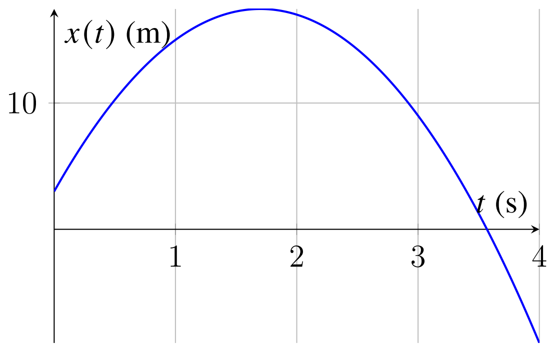
The instantaneous velocity at times 1, 2, 3 are $v(1) = 7, v(2) = -3, v(3) = -13$.

$$17 = 10t \iff t = \frac{17}{10} = 1.7$$

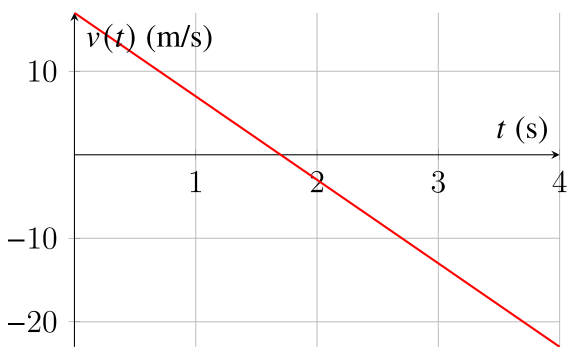
The particle crosses the origin at approximately time 3.568 and its velocity is approximately $v(3.568) = -18.682$.

(d) The acceleration is constant: $a(t) = -10$.

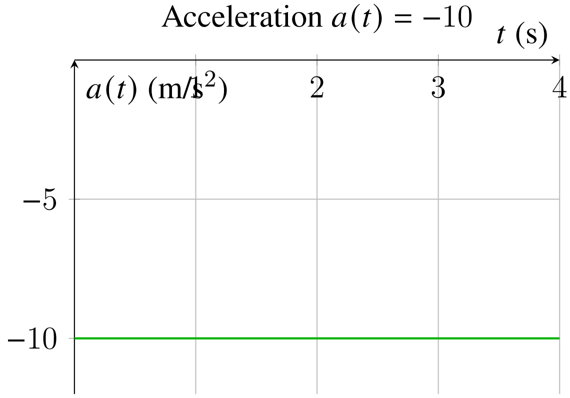
Position $x(t) = 3 + 17t - 5t^2$



Velocity $v(t) = 17 - 10t$



Acceleration $a(t) = -10$



(**6**) (a) Determine the instantaneous acceleration of the object plotted in the exercise for $t = 3, t = 11$.

(b) Compute the distanace traveled by the object in the time intervals $[0, 5]$ and $[0, 9]$ and $[0, 15]$.

(c) Knowing that $x(t = 6) = 0$, find the position of the object at $t = 0$.

(d) Give an expression for the objects position for all t.

(e) Plot $x(t), v(t), a(t)$.

The velocity of the object is constantly 20m/s from time $t = 0$ to time $t = 6$. Then it linearly increases until it reaches 44m/s at $t = 9$, from whereon it linearly decreases until it reaches 0ms at time $t = 15$.

The linear expression $\ell_1(t) = a_1 t + b_1$ which satisfies $\ell(6) = 20, \ell(9) = 44$ is such that

$$6a_1 + b_1 = 20, \qquad 9a_1 + b_1 = 44$$

The associated system of equations yields that $b_1 = 20 - 6a_1$, from which follows that $9a_1 + (20 - 6a_1) = 44$, entailing

$$a_1 = \frac{24}{3} = 8$$

From this readily follows that $b_1 = 20 - 6 \times 8 = -28$.

Similarly, the linear expression $\ell_2(t) = a_2 t + b_2$ which gives a line s.t. $\ell_2(9) = 44, \ell_2(15) = 0$ must satisfy

$$9a_2 + b_2 = 44, \qquad 15a_2 + b_2 = 0$$

Then $b_2 = -15a_2$ and $9a_2 - 15a_2 = 44$, entailing $a_2 = -\frac{22}{3}$. From this follows that $b_2 = 110$ via simple calculations. Thus,

$$v(t) = \begin{cases} 20 & 0 \leq t \leq 6 \\ 8t - 28 & 6 < t \leq 9 \\ -\frac{22}{3}t + 110 & 9 < t \leq 15 \end{cases}$$

It should be intuitive to graps that the distance travelled $d(a, b)$ in the interval $[a, b]$ is

$$d(a,b) = \int_a^b |v(t)|\ dt$$

If $v(t)$ is in meters per second, and t is in seconds, the total number of meters travelled in a time interval $[a, b]$ is the summation of the meters per second travelled in every instant! This will involve the anti-derivative of $v(t)$, i.e. the movement function $x(t)$, which we might as well compute at once.

$$\begin{aligned}
x(t) &= \int v(t)\ dt \\
&= \begin{cases} 20t + C_1 & 0 \le t \le 6 \\ 8\frac{t^2}{2} - 28t + C_2 & 6 < t \le 9 \\ -\frac{22}{3}\frac{t^2}{2} + 110t + C_3 & 9 < t \le 15 \end{cases} \\
&= \begin{cases} 20t + C_1 & 0 \le t \le 6 \\ 4t^2 - 28t + C_2 & 6 < t \le 9 \\ -\frac{22}{6}t^2 + 110t + C_3 & 9 < t \le 15 \end{cases}
\end{aligned}$$

The constants C_1, C_2, C_3 must satisfy the restriction of continuity and of preserving the necessary values. In particular, we need $20(0) + C_1 = 20$. Since we know $x(6) = 0$, we need $120 + C_1 = 0$, i.e. $C_1 = -120$. We also need

$$4(6^2) - 28(6) + C_2 = 0$$

to ensure continuity, so

$$C_2 = 24$$

Then we can know what $x(9)$ is and deduce C_3, which ends up being -597.

$$\therefore\ x(t) = \begin{cases} 20t - 120 & 0 \le t \le 6 \\ 4t^2 - 28t + 24 & 6 < t \le 9 \\ -\frac{22}{6}t^2 + 100t - 597 & 9 < t \le 15 \end{cases}$$

In any case, we could have computed the distance travelled without $x(t)$ (I computed $x(t)$ because it's part of the exercise):

$$
\begin{aligned}
d(0,5) &= \int_0^5 v(t)\ dt \\
&= 20 \times t]_0^5 \\
&= 20 \times (5) \\
&= 100
\end{aligned}
$$

$$
\begin{aligned}
d(0,9) &= \int_0^6 v(t)\ dt + \int_6^9 v(t)\ dt \\
&= 20 \times t\Big]_0^6 + (4t^2 - 28t)\Big]_6^9 \\
&= 120 + [(4 \times 81 - 28 \times 9) - (4 \times 38 - 28 \times 6)] \\
&= 120 + 96 \\
&= 216
\end{aligned}
$$

etc.

(7) A car and a truck leave at the same instant, the car initially being a certain distance behind the truck. The latter has a constant acceleration of $1.2m/s^2$, while the car accelerates at $1.8m/s^2$. The car reaches the truck when the latter has covered 45 meters.

(a) How much time does it take for the car to reach the truck?

(b) What is the initial distance between both vehicles?

(c) What is the velocity of each in the moment the cross paths?

(d) Plot $x(t), v(t), a(t)$.

(a) Since both vehicles have constant accelerations, they have linear velocities and therefore quadratic movement functions. They will meet when the parabolas corresponding to these functions intersect.

Let $x_1(t)$ denote the movement function of the car, $x_2(t)$ that of the truck. We then wish to find the solutions to $x_1(t) = x_2(t)$. Now,

$$v_1(t) = \int a_1(t) = \int 1.8 \, dt = 1.8t + C_1 \tag{2}$$

$$v_2(t) = \int a_2(t) = \int 1.2 \, dt = 1.2t + C_2 \tag{3}$$

are the velocities of the car (v_1) and the truck (v_2). Since at $t = 0$ the velocities of both vehicles is zero (they start from rest), it is necessary that $C_1 = C_2 = 0$.

We know that the car reaches the truck when the latter has covered 45 meters, so the question is what is the time t_0 when the distance covered by the truck is that one? In other words, we need to find t_0 such that

$$\begin{aligned}
&\int_0^{t_0} v_2(t) \, dt = 45 \\
\iff &\int_0^{t_0} 1.2t \, dt = 45 \\
\iff &\left[0.6t^2\right]_0^{t_0} = 45 \\
\iff &0.6t_0^2 = 45 \\
\iff &t_0 = \sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3}
\end{aligned}$$

Thus, the vehicles meet at time $t_0 = 5\sqrt{3}$.

(b) The initial distance between both vehicles is given by $|x_1(0) - x_2(0)|$. From the velocities v_1, v_2 we can determine that

$$x_1(t) = 0.9t^2 + C_1', \qquad x_2(t) = 0.6t^2 + C_2' \tag{4}$$

Let us fix our coordinate system so that the starting position of the truck corresponds to the origin. Then $C_2' = 0$. Knowing that both vehicles coincide at time $t_0 = 5\sqrt{3}$, we also know $X_1(t_0) = x_2(t_0)$, i.e.

$$0.9(25 \times 3) + C_1' = 0.6(25 \times 3) \tag{5}$$

which entails $67.5 + C_1' = 45$, from which follows that $C_1' = -22.5$. Thus, the original distance of both vehicles is 22.5m.

(c) This consists simply of computing $v_1(t_0), v_2(t_0)$. Trivial.

(d) Meh.

(8) A car travels parallel to a train rail. The car stops at a red light in the exact instant when a train passes with a constant velocity of 12m/s. The car remains at halt for 6s and then continues with a constant acceleration of $2m/s^2$.

(a) Determine the time it takes for the car to reach the train, with $t = 0$ being the instant in which the car halted.

(b) Determine the distance traveled by the car from the red light until it reached the train.

(c) Determine the car's velocity at the instant it reaches the train.

(a) Let $a_1(t)$ be the acceleration of the car, defined as

$$a_1(t) = \begin{cases} 0 & 0 \leq t < 6 \\ 2 & t \geq 6 \end{cases} \tag{6}$$

Let $v_2(t) = 12m/s$ be the constant velocity of the train. Let the point of halt be the origin of our coordinate system, so that at time $t = 0$ (when the car halted) both the train and the car are at position zero. Observe then that it follows that $x_2(t) = 12t$ (in meters) via integration of $v_2(t)$ and the necessary condition of the constant of integration being zero.

Integration of equation (6) gives

$$v_1(t) = \begin{cases} C_1 & 0 \leq t < 6 \\ 2t + C_2 & t \geq 6 \end{cases} \tag{7}$$

where the constants of integration must satisfy two constraints: (a) $v_1(0) = 0$ and v_1 must be continuous. From this follows that $C_1 = 0$ and that $2(6) + C_2 = 0$, i.e. $C_2 = -12$. Therefore,

$$v_1(t) = \begin{cases} 0 & 0 \leq t < 6 \\ 2t - 12 & t \geq 6 \end{cases} \tag{8}$$

Via integration of v_1,

$$x_1(t) = \begin{cases} C_1' & 0 \leq t < 6 \\ t^2 - 12t + C_2' & t \geq 6 \end{cases} \tag{9}$$

Again, C_1' must of course be zero, and $x_1(6)$ must also be zero, meaning that $C_2 = -36+12(6) = 36$. Therefore,

$$x_1(t) = \begin{cases} 0 & 0 \leq t < 6 \\ t^2 - 12t + 36 & t \geq 6 \end{cases} \tag{10}$$

The car reaches the train at the time $t_0 > 6$ which satisfies $x_1(t_0) = x_2(t_0)$, so we solve

$$t^2 - 12t + 36 = 12t \iff t^2 - 24t + 36 = 0$$

which has solutions

$$\frac{24}{2} \pm \frac{\sqrt{24^2 - 4 \times 36}}{2} = 12 \pm \frac{\sqrt{432}}{2} \approx 12 \pm 10.392$$

Keeping only the positive solution, we have that $t_0 \approx 22.392$.

(b) The distance traveled by the car from the red light until it reached the train is the distance traveled from $t = 0$ to $t = t_0$, i.e.

$$\int_0^{t_0} v_1(t)\ dt = |x_1(t_0) - x_1(t_0)| = x_1(t_0) \approx 268.697$$

where the equality above holds only because velocity is always positive (i.e. the car moves only in one direction).

(c) Simply computing $v_1(t_0)$ gives the answer.

(9) A ball is thrown vertically and upwards from the floor with initial velocity v_0. Write the equations for the movement of the ball and plot graphically the vectors $\vec{y}(t), \vec{v}(t), \vec{a}(t)$. Identify the conditions for the instant of maximum height and the instant it reaches the floor.

The move is strictly vertical, so $\vec{r}(t) = 0\hat{\imath} + y(t)\hat{\jmath}$ and we need only determine the unidimensional vertical movement function $y(t)$. Now, the ball is affected only by gravitiy, i.e. it is subjected to a constant acceleration of $\vec{a}(t) = 0\hat{\imath} - 9.8\frac{m}{s^2}\hat{\jmath}$. From this we can derive the vertical velocity:

$$v_y(t) = -9.8 \int dt = -9.8t + C$$

The constant of integration must satisfy the initial velocity being v_0, so we must have

$$v_y(t) = v_0 - 9.8t$$

From this follows that

$$r_y(t) = \int v_0 - 9.8t \; dt = v_0 t - \frac{9.8}{2}t^2 + C'$$

If we assume the position on the floor (vertically) is zero, we must have $C' = 0$, and

$$r_y(t) = v_0 t - 4.9t^2$$

In summary,

$$\vec{r}(t) = 0\hat{\imath} - (v_0 t - 4.9t^2)\hat{\jmath}, \qquad \vec{v}(t) = 0\hat{\imath} - 9.8t\hat{\jmath}, \qquad \vec{a}(t) = 0\hat{\imath} - 9.8\hat{\jmath}$$

Maximum hight will occur at time $t \neq 0$ when the vertical velocity of the ball is exactly zero. So, we solve

$$v_0 - 9.8t = 0 \iff \frac{v_0}{9.8} = t$$

It will reach the floor at time $t \neq 0$ when the vertical position of the ball is zero, so we solve

$$v_0 t - 4.9t^2 = 0 \iff t(v_0 - 4.9t) = 0$$

The root $t = 0$ is not a solution that interests us, so we only care about the root that solves $v_0 - 4.9t = 0$, i.e. $t = \frac{v_0}{4.9}$.

(10) A rock is thrown vertically and upwards. On its path, it crosses point A with velocity v and point B, which is 3m higher than A, at velocity $v/2$. Determine v an the maximum height reached by the rock above point B.

Let $\Delta y = y_B - y_A = 3\text{m}$ the distance between A and B. We know the rock has a constant acceleration $-g$ due to gravity. This tells us that velocity is linear and movement is quadratic. In general,

$$a(t) = -g, \qquad v(t) = v_0 - gt, \qquad x(t) = y_0 + v_0 t - g\frac{t^2}{2}$$

Now, given an arbitrary velocity v occurring when the object is at position y,

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = a$$

assuming a constant acceleration a. However, since the position y is a function of time, the chain rule gives

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$$

From this follows $a = v\frac{dv}{dy}$ or equivalently $a\ dy = v\ dv$. Integrating from initial positions and velocities y_0, v_0;

$$\int_{y_0}^{y} a\ dy = \int_{v_0}^{v} v\ dv$$
$$\Rightarrow\ v^2 - v_0^2 = 2a(y - y_0)$$

In other words, for any given velocity v occurring at position y, the relationship above holds. In particular, if we treat A as our initial position,

$$\frac{v^2}{4} - v^2 = 2a(y_B - y_A)$$

Since $y_B - y_A = 3\text{m}$,

$$\frac{v^2}{4} - v^2 = 6a$$

Now, the acceleration in our problem is simply $-g$, and then the equation above gives solutions for v given by

$$\begin{aligned}
&\frac{-3v^2}{4} = -6g \\
\iff &v^2 = 8g \\
\iff &v = \pm\sqrt{8g}
\end{aligned}$$

Now, since velocities can only be positive, we keep $v = \sqrt{8g}$ as the only solution.

Now the question is, what is the maximum height? Well, in the path from B to the top, velocity drops from $\frac{v}{2}$ to zero. So again, if we take as reference B,

$$0 - \left(\frac{v}{2}\right)^2 = 2a(y_{\text{top}} - y_B)$$

Let $h = y_{\text{top}} - y_B$. Then

$$h = -\frac{v^2}{8a}$$

But $a = -g$ and $v = \sqrt{8g}$, giving

$$h = \frac{8g}{8g} = 1$$

So the maximum height above B is 1 meter.

(11) The movement in the plane of a particle is given by $x(t) = at^2$, $y(t) = bt^3$, with $a = 3\frac{m}{s^2}$, $b = 2\frac{m}{s^3}$.

(a) Find the trajectory of the particle. Plot it.

(b) Compute the acceleration at $t = 12$s.

(c) What is the angle formed by the velocity vectors and the acceleration at that instant?

(d) Determine the instant t_1 where acceleration is parallel to the line $y = x$, and the instant t_2 in which velocity is parallel to said line.

(e) Determine the median velocity in the interval (t_1, t_2).

(a) The trajectory of the particle is

$$\begin{aligned} S &= \{(x(t), y(t)) : t \in \mathbb{R}\} \\ &= \left\{(3t^2, 2t^3) : t \in \mathbb{R}\right\} \end{aligned}$$

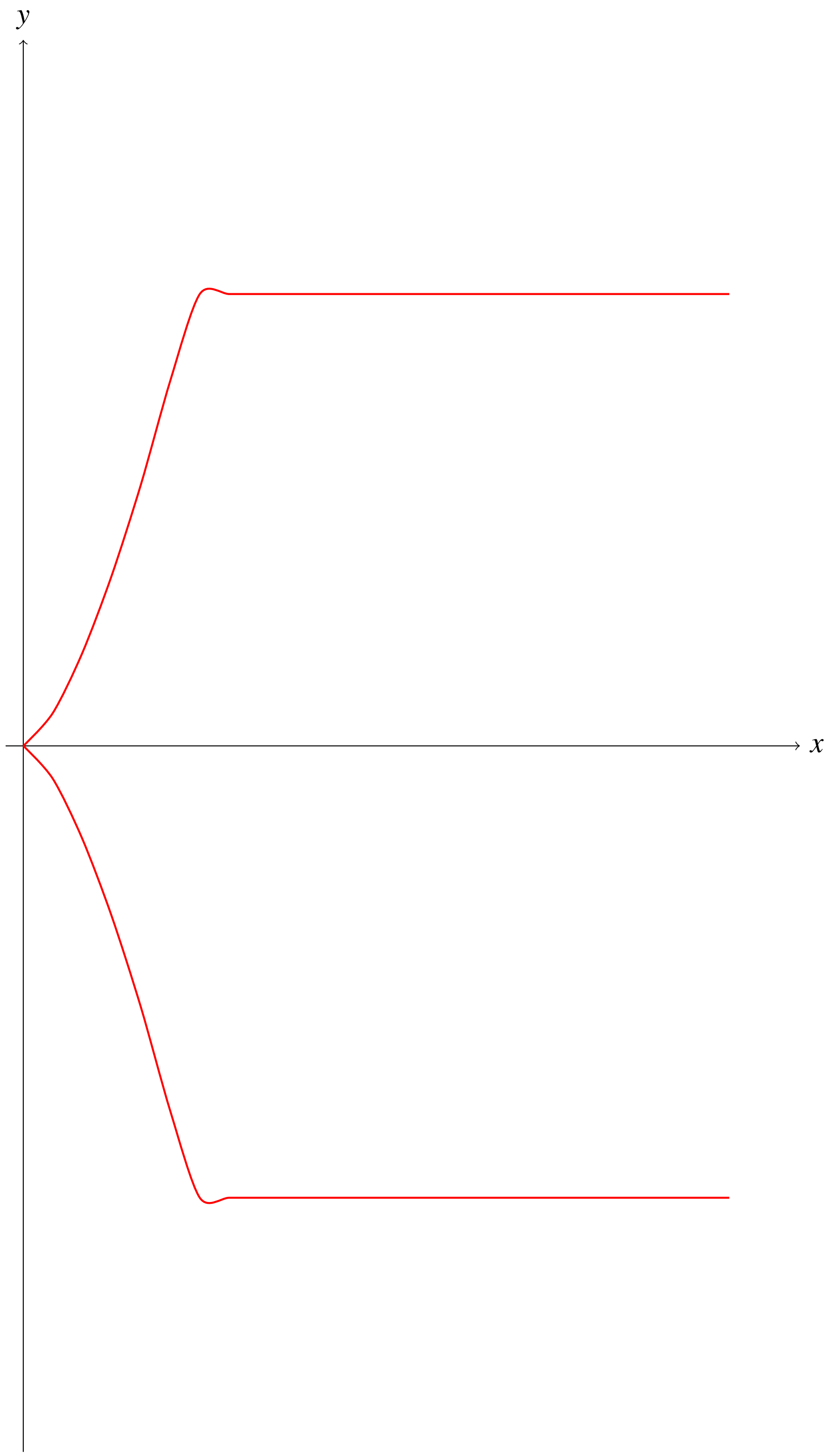
Since $x = 3t^2, t^2 = \frac{x}{3}$. And since $y = 2t^3$, we have $y^2 = 4t^6 = 4(t^2)^3 = 4\left(\frac{x}{3}\right)^3 = \frac{4}{27}x^3$. In summary,

$$y^2 = \frac{4}{27}x^3$$

or equivalently

$$y = \pm\frac{2}{3}\sqrt{x^3}$$

This entirely defines S.



(b) Clearly,

$$v_x(t) = 2at, \qquad v_y(t) = 3bt^2$$

meaning that

$$a_x(t) = 2a, \qquad a_y(t) = 6bt$$

So the acceleration at time 12 is

$$\vec{a}(12) = 2a\hat{i} + 72b\hat{j} = 6\hat{i} + 144\hat{j}$$

(c) Recall that the angle between two vectors $\vec{u}, \vec{w}$ is

$$\theta = \arccos\left(\frac{u \cdot w}{|u|\,|w|}\right)$$

because the dot product is $u \cdot w = |u|\,|w| \cos\theta$. Now,

$$\vec{v}(12) = 72\hat{i} + 864\hat{j}$$

It is simple to compute:

$$|\vec{v}(12)| = 866.994, \qquad |\vec{a}(12)| = 144.125$$

and

$$\vec{v}(12) \cdot \vec{a}(12) = 6 \cdot 72 + 144 \cdot 864 = 124848$$

Then

$$\theta = \arccos\left(\frac{124848}{866.994 \times 144.125}\right) = \arccos(0.999) = 0.044$$

In degrees, these are $0.044 \times \frac{180}{\pi} = 2.521$.

(d) It is quite simple to reason and see that any vector parallel to $y = x$ is such that its x and y coordinates are the same. So the find t_1 s.t. $\vec{v}(t_1)$ is parallel to $y = x$, we need only find t_1 s.t.

$$\vec{v}_x(t_1) = \vec{v}_y(t_1)$$

But this holds if and only if

$$\begin{aligned} 2at = 3bt^2 &\iff 6t = 6t^2 \\ &\iff t = t^2 \\ &\iff t \in \{0, 1\} \end{aligned}$$

But 0 is a trivial solution, so we keep only 1. Same goes for acceleration:

$$\begin{aligned} a_x(t_2) = a_y(t_2) &\iff 6 = 12t_2 \\ &\iff \frac{1}{2} = t_2 \end{aligned}$$

(e) Median velocity was defined as $\Delta x/\Delta t$. So the median velocity in $(t_1, t_2) = (\frac{1}{2}, 1)$ is

$$\frac{\Delta x}{\Delta t} = \frac{x\,(1) - x\left(\frac{1}{2}\right)}{1 - \frac{1}{2}} = \frac{2.25}{0.5} = 4.5$$

(12) A bullet is shot horizontally from a canon placed on a platform of height 44m. Iis exit velocity is $25m/s$. Assume the terrain is horizontal and perfectly plain.

(a) Write the movement equations.

(b) Draw the vectors $\vec{r}(t), \vec{v}(t), \vec{a}(t)$ at the highest point of the curve and when the ball reaches the ground.

(c) How much time does the ball remain in the air before hitting the ground?

(d) What is its reach, i.e. at what distance from the cannon does it hit the ground?

(e) What is the magnitude of the vertical component of $\vec{v}$ when the bullet reaches the ground?

(f) Repeate (c) for the case when the ball is dropped in free fall from the platform.

(g) Consider now that the exit velocit has vertical direction and posit the movement equations.

Recall that, when a projectile is shot, it has no horizontal acceleration, only horizontal velocity. Its vertical acceleration is given by gravity alone. Thus, it is always the case for a projectile that

$$\vec{a} = 0\hat{i} - g\hat{j}$$

where g is in m/s^2. From this readily follows:

$$\vec{v}(t) = v_{0x}\hat{i} - (gt + v_{0y})\hat{j}, \qquad \vec{r}(t) = (v_{0x}t + x_0)\hat{i} + (-\frac{g}{2}t^2 + v_{0y}t + y_0)\hat{j}$$

Fix $x_0 = 0$. We know, from the conditions of the problem, that $y_0 = 44$, $v_{0x} = 25$ (in meters per second) and $v_{0y} = 0$ (since the ball is shot horizontally, it has no vertical velocity at the start.) Thus,

$$\vec{v}(t) = 25\hat{i} - gt\hat{j}, \qquad \vec{r}(t) = 25t\hat{i} + \left(-\frac{g}{2}t^2 + 44\right)\hat{j}$$

(b) Observe that the y-coordinate of $\vec{r}(t)$, henceforth denoted $\vec{r}_y(t)$, is zero if and only if

$$
\begin{aligned}
& \vec{r}_y(t) = 0 \\
\iff\quad & -\frac{g}{2}t^2 + 44 = 0 \\
\iff\quad & t^2 = \frac{2}{g}44 \\
\iff\quad & t = \pm\sqrt{\frac{88}{g}} \\
\iff\quad & t \approx \sqrt{\frac{88}{9.8}} \\
\iff\quad & t \approx \sqrt{8.97} \\
\iff\quad & t \approx \sqrt{8.97} \\
\iff\quad & t \approx 2.996
\end{aligned}
$$

where we kept only the positive solution because $t \geq 0$. So the object touches the aground after approximately three seconds. This means $\vec{r}_y(t)$ is a quadratic function with roots ± 2.996, whose midpoint is zero (i.e. the curve is symmetric around the y-axis). This is the formal way of saying something obvious: the ball reaches its maximum height at $t = 0$ (since then it falls). Then it is simple to see:

$$
\vec{r}(0) = (0, 44)^\top, \qquad \vec{v}(0) = (25, 0)^\top, \qquad \vec{a}(0) = (0, -g)^\top
$$

It is easy to imagine what these vectors look like when graphed.

(c) From our computation in (b) we already know the ball remains in the air almost three seconds before hitting the ground.

(d) Let $t_0 = 2.996$ the time at which the bullet hits the ground. Observe that $\vec{r}_x(t_0) = 25(t_0) = 75.9$. So the projectile hits the ground at 75.9 meters from its starting position in the x-axis.

(e) Note that $\vec{v}(t_0) = (25, gt_0)^\top \approx (25, -29.3608)^\top$. The magnitude of the vertical component is the second component of the vector given.

9 Movimiento circular

Es un movimiento en el plano cuya trayectoria es una circunferencia, o parte de ella. Es decir, es un movimiento que satisface

$$[x(t) - x_0]^2 + [y(t) - y_0]^2 = R^2 \tag{1}$$

con (x_0, y_0) el centro de la circunferencia y R su radio. Si el sistema de coordinadas tiene origen en el centro de la circunferencia, $x_0 = 0$, $y_0 = 0$ y se obtiene

$$x^2(t) + y^2(t) = R^2 \tag{2}$$

Usaremos esta convención. El vector posición $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ se puede expresar en función del radio, que es constante, y el ángulo $\theta(t)$ que el vector forma con el eje x:

$$\vec{r}(t) = R\left[(\cos\circ\,\theta)(t)\hat{\imath} + (\sin\circ\,\theta)(t)\hat{\jmath}\right] \tag{3}$$

De esto se sigue que $|\vec{r}(t)| = R$ (lo cual es gráficamente obvio). Derivando, obtenemos

$$\vec{v}(t) = R\frac{d\theta}{dt}\left[-\sin\left(\theta(t)\right)\hat{\imath} + \cos\left(\theta(t)\right)\hat{\jmath}\right] \tag{4}$$

La derivada $\frac{d\theta}{dt}$ se denomina *velocidad angular*, y se denota

$$\omega(t) = \frac{d\theta(t)}{dt}$$

So we may write:

$$\vec{v}(t) = R\omega(t)\left[-\sin(\theta(t))\hat{\imath} + \cos(\theta(t))\hat{\jmath}\right] \tag{5}$$

from which follows

$$|\vec{v}(t)| = R\,|\omega(t)| \tag{6}$$

Again, if we write $\gamma(t) = d\omega(t)/dt$,

$$\vec{a}(t) = R\left[\gamma(t)\left(-\sin(\theta(t))\hat{\imath} + \cos(\theta(t))\hat{\jmath}\right) - \omega^2(t)\left(\cos(\theta(t))\hat{\imath} + \sin(\theta(t))\hat{\jmath}\right)\right] \tag{7}$$

It can be proven that $\vec{r}(t)\cdot\vec{v}(t) = 0$, i.e. the position and velocity vectors are always perpendicular. This makes graphical sense but still is quite beautiful. Furthermore, it is also true that with

$$\hat{r}(t) = \cos(\theta(t))\hat{\imath} + \sin(\theta(t))\hat{\jmath}, \qquad \hat{u}(t) = -\sin(\theta(t))\hat{\imath} + \cos(\theta(t))\hat{\jmath}$$

we have

$$\hat{r}(t)\cdot\hat{u}(t) = 0, \qquad |\hat{r}(t)| = 1, \qquad |\hat{u}(t)| = 1 \tag{8}$$

In other words, $\hat{r}(t)$ and $\hat{u}(t)$ are orthonormal unit vectors. If one notes that $\hat{r}(t)$ is simply a normalized version of $\vec{r}(t)$, and that $\hat{u}(t)$ is a normalized version of $\vec{v}(t)$, then it is clear that the previous result is just a ”normalized” version of the previous one.

Cuando descomponemos el vector $\vec{a}$ en aceleración tangencial y normal, en un movimiento circular, siempre sucede que la aceleración normal tiene misma dirección y contrario sentido al vector posición.

10 Dirección de fuerzas

10.1 Fuerza de roce (o fricción)

Involucra contacto paralelo de un cuerpo con una superficie. Hay estático y dinámico. Trataremos el estático primero.

En el roce estático, el cuerpo no se mueve ($\vec{v} = 0$). Digamos que queremos empujar una mesa apoyada en el piso. Si al hacer fuerza empujando, la mesa aún no se mueve, es porque existe otra fuerza (la de rozamiento estático) que equilibra o supera la que estamos haciendo.

Sabemos, por las leyes de Newton, que la suma de las fuerzas es igual a la masa por aceleración:

$$\begin{aligned}\sum \vec{F} &= m\vec{a} \\ &= \vec{N} + \vec{P} + \vec{F}_{\text{ext}} + \vec{F}_{\text{roce estatico}}\end{aligned}$$

Aquí, la fuerza normal $\vec{N}$ equilibra $\vec{P}$, i.e. $\vec{N} + \vec{P} = 0$. Como el objeto no se mueve, la aceleración es cero y por lo tanto $m\vec{a} = 0$. Entonces necesariamente la fuerza externa y la fuerza de roce estático también suman cero (se equilibran).

A su vez, hay un valor máximo que la fuerza de roce estático puede tomar:

$$\left|\vec{F}_{\text{r.e.}}^{\text{max}}\right| = \mu_e \left|\vec{N}\right|$$

donde $\mathcal{M}_e$ se llama coeficiente de roce estático.

Pasemos al caso dinámico, cuando el objeto empieza a moverse. Existe cuando $\vec{v} \neq 0$, tiene una constante $\mathcal{M}_a < \mathcal{M}_e$ que lo acota. Esta fuerza no necesita de otras fuerzas aplicadas paralelas a la superficie: se opone al movimiento y disminuye $\vec{v}$. Es la fricción que ”frena” a un objeto que se mueve/desliza. Siempre se cumple

$$\left|\vec{F}_{\text{r.d.}}\right| = \mu_a \left|\vec{N}\right|$$

10.2 Movimiento oscilatorio armónico

Digamos que tenemos un resorte horizontal (e.g. contra la pared) de constante k y longitud ℓ_0 con una masa m en el extremo.

Ley de Hooke. La fuerza de un resorte de constante k es $\vec{F}_k = -k\Delta\ell$ donde ℓ es la distancia entre el extremo y la posición de equilibrio del extremo.

Las fuerzas involucradas en un resorte que ha sido estirado o movido fuera de su equilibrio son: la fuerza normal $\vec{N}$, la fuerza $\vec{P}$, la fuerza del resorte $vecF_k$. Por ende, la segunda ley de Newton nos da

$$\begin{aligned} & \sum \vec{F} = m\vec{a} \\ \Rightarrow \quad & \vec{N} + \vec{P} + \vec{F}_k = m\vec{a} \end{aligned}$$

Pero la aceleración es una aceleración en x (se mueve horizontalmente). Es decir, $\vec{a} = a_x\hat{\imath} + 0\hat{\jmath} = a_x\hat{\imath}$. Sabemos además que $\vec{N} + \vec{P} = 0$. Por lo tanto,

$$\vec{F}_k = m\vec{a} = ma_x\hat{\imath}$$

Pero $a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$. Por ende

$$\vec{F}_k = m\frac{d^2x}{dt^2}\hat{\imath}$$

o bien, por la ley de Hooke,

$$-k(x - x_0) = m\frac{d^2x}{dt^2}\hat{\imath}$$

donde x es la posición actual de la masa en el extremo del resorte y x_0 su posición original (de equilibrio). Sea $\widetilde{m} = \frac{d^2x}{dt^2}\hat{\imath}$. Entonces

$$a_x = \widetilde{x} + \frac{k}{m}(x - x_0) = 0$$

El problema es que la ecuación

$$v = \int a_x = \int \widetilde{x} + \frac{k}{m}(x(t) - x_0)\ dt$$

no se puede resolver, porque $v(t)$ depende de $x(t)$. Si no sabemos $x(t)$ no podemos continuar.

Sea $\mu = x - x_0$, tal que $\mu^{(1)} = x^{(1)}$ y $\mu^{(2)} = x^{(2)} = \widetilde{x}$. Entonces

$$\widetilde{x} + \frac{k}{m}(x - x_0) = 0$$

puede escribirse como

$$\mu^{(2)} + \frac{k}{m}\mu = 0$$

de lo cual se sigue

$$\mu^{(2)} = -\frac{k}{m}\mu \tag{1}$$

Es decir, necesitamos una función u tal que su segunda derivada sea una constante por sí misma. Podemos ver que

$$\mu(t) = A\sin\left(\omega t + \phi\right), \qquad \mu(t) = A'\sin(\omega t) + B'(\cos\omega t)$$

donde ϕ es un ángulo inicial (fase) satisfacen la ecuación (1). Tomemos

$$\mu(t) = x(t) - x_0 = A\sin(\omega t + \phi)$$

Entonces

$$\mu^{(1)}(t) = \mu^{(1)}(t) = v(t) = \omega A\cos(\omega t + \phi)$$

y

$$\begin{aligned}
\mu^{(2)}(t) &= a(t) \\
&= -\omega^2 A\sin(\omega t + \phi) \\
&= -\omega^2\mu(t) \\
&= -\frac{k}{m}\mu(t)
\end{aligned}$$

Es decir,

$$\omega^2 = \sqrt{\frac{k}{m}}$$

Decimos que $T = \frac{2\pi}{\omega}$ es el período.

Repasemos. Por la segunda ley de newton, teníamos

$$\mu^{(2)}(t) + \frac{k}{m}\mu(t) = 0$$

La solución que hallamos es $\mu(t) = A\sin(\omega t + \phi)$, con $\omega = \sqrt{\frac{k}{m}}$. El período es $T = \frac{2\pi}{\omega}$. ¿Pero quién es A?

A determina la amplitud del movimiento, en unidades de longitud, y se corresponde con el apartamiento máximo, i.e. la máxima distancia entre la masa y su punto de equilibrio x_0. Ahora bien, como

$$v(t) = A\omega\cos(\omega t + \phi)$$

tenemos que $v_{\max} = A\omega$ (cuando el coseno es uno). Además,

$$\mu(0) = A\sin(\phi_0)$$

y a esto le denominamos fase inicial. Describe algo que sucede en el instante $t = 0$.

Los puntos exxtrmeos del movimiento, con distanca A, son donde la velocidad es cero. El punto de equilibrio es el punto donde la velocidad es máxima.

11 Energía

(Cap 7 Serway)

La energía es la capacidad de un sistema para realizar un proceso (cambio). Matemáticamente, es un escalar independiente del tiempo. Un sistema puede ser un sistema de partículas, una masa particular, entre otras cosas. La unidad de la energía es $[E] = [F]\,[d]$, fuerza por distancia, y se denomina Jule o simplemente J. Usaremos conceptos como *trabajo* (W de *work*), energía potencial, energía cinemática, energía mecánica, calor, etc.

11.0.1 El trabajo (invertido/realizado) de una fuerza

Supongamos un sistema simple con un cuerpo puntual, un recorrido unidimensional, con desplazamiento bajo la acción de una fuerza. Sea $\vec{\Delta x} = \Delta x \hat{i} = (x_f - x)\hat{i}$, con x_f la posición final. Sabemos que

$$\vec{F} = F_x \hat{i} + F_y \hat{j} = F \cos\theta \hat{i} + F \sin\theta \hat{j}$$

El **trabajo** invertido por la fuerza en mover el objeto es

$$W := F_x \vec{\Delta x} \tag{1}$$

es decir, es fuerza por distancia. Claramente,

$$W = F \cos\theta \vec{\Delta x} = \vec{F} \cdot \vec{\Delta x}$$

Es decir, W es el component de la fuerza en la dirección del movimiento, producto por el deslpazamiento. (Notar que $\vec{F}\vec{\Delta x}$ es un producto escalar: $\vec{A} \cdot \vec{B} = |A|\,|B| \cos\theta$ con θ el ángulo entre ambos.)

Las unidades del trabajo son unidades de fuerza por unidades de longitud: $[W] = [F]\,[\Delta x]$.

Ahora bien, es posible que $\vec{F}$ varíe con la posición, i.e. que $\vec{F}$ sea una función x. Entonces, podemos calcular diferenciales de trabajo, es decir el cambio en el trabajo de acuerdo a la posición:

$$\frac{dW}{dx} = \vec{F}(x)$$

o bien

$$dw = \vec{F}(x)\ d\vec{x} \tag{2}$$

Integrando (2) obtenemos

$$W = \int_{x_0}^{x_f} \vec{F}(x)\ d\vec{x} \tag{3}$$

Más complejo aún, ¿qué pasa si la trayectoria del objeto es 2D (curva)? También aquí la fuerza varía dependientemente de la trayectoria. Sea $d\vec{s}$ el vector que indica el cambio de posición instantáneo del objeto en su trayectoria. Se deduce:

$$W = \int_{s_0}^{s_f} \vec{F}(s)\, d\vec{s} \tag{4}$$

A la integral de (4) se le llama *integral de línea*.

Si hay muchas fuerzas aplicadas, $\vec{F}_1, \vec{F}_2, \ldots$, el trabajo total será

$$W_{\text{total}} = W_{F_1} + WF_2 + \ldots = \sum_i W_{F_i}$$

Respecto al signo del trabajo, recordemos que $W = \vec{F} \cdot \vec{\Delta x}$. Si $0 \leq \theta < \frac{\pi}{2}$, entonces la fuerza está "tironeando" del cuerpo, pues el ángulo está entre 0 y 45 grados. El trabajo será positivo, y diremos que la fuerza *entrega* trabajo.

Si $\frac{\pi}{2} < \theta \leq \pi$, el producto escalar será negativo y por ende lo será el trabajo. Este caso se corresponde por ejemplo con un cuerpo que se mueve hacia la derecha y una fuerza que lo "tira" hacia arriba a la izquierda, u directamente a la izquierda. Decimos entonces que la fuerza *frena* al objeto.

Si $\theta = \frac{\pi}{2}$, es decir el ángulo es de 90 grados, el producto escalar es cero, y el trabajo es cero. La fuerza normal es un ejemplo.

11.1 Trabajo y energía cinética

Imaginemos un cuerpo con movimiento unidimensional en un riel x entre dos posiciones x_0, x_f. En particular, es movido por una fuerza $\vec{F}$ que no es totalmente paralela al riel, sino que tiene cierto ángulo (e.g. 45 grados o $\pi/4$). Asumamos que dicha fuerza es constante.

Dado que $\vec{F} == F_x\hat{i} + F_y\hat{j} = m\vec{a}$, y que $\vec{a} = a\hat{i}$ (pues el movimiento es solo horizontal), ¿cuáles son las ecuaciones de movmiento del objeto? Well,

$$F_x = ma, \qquad F_y + N - mg = 0$$

And since the force is constant, so is the acceleration factor a. It follows that $v = at + v_0, x = \frac{a}{2}t^2 + v_0t + x_0$.

Now,

$$v_f = at_f + v_0 \implies t_f = \frac{v_f - v_0}{a}$$

Furthermore,

$$x_f = x(t_f) = \frac{a}{2}t_f^2 + v_0 t_f + x_0$$

from which follows via algebraic manipulation that

$$x_f - x_0 = \frac{1}{2a}(v_f^2 - v_0^2) \tag{1}$$

Now, equation (1) may be expanded so that

$$\Delta x a = \frac{1}{2}v_f^2 - \frac{1}{2}v_0^2$$

where $\Delta x = x_f - x_0$. But we know $F_x = am$ or rather $a = F_x/m$. So

$$\Delta x F_x/m = \frac{1}{2}v_f^2 - \frac{1}{2}v_0^2 \tag{2}$$

Multiplying both sides by m,

$$\Delta x F_x = \frac{m}{2}v_f^2 - \frac{m}{2}v_0^2 \tag{3}$$

But $\Delta x F_x$ is the work. Now, we define $K = T = E_c$ the kinetic energy as $\frac{1}{2}mv^2$, which is a factor relating mass and velocity. In particular,

$$K_0 = \frac{1}{2}mv_0^2 \ \text{ (Initial kinetic energy) }, \qquad K_f = \frac{1}{2}mv_f^2 \ \text{ (Final k.e.)}$$

So equation (3) is nothing but

$$W = K_f - K_0 = \Delta K \tag{4}$$

In conclusion, the work invested by the forces is nothing but the variation or difference between the kinetic energies. (Note that here W denotes the total work, i.e. the sum of all works from all forces.)

Summary. The work identities are:

$$W = \vec{F}\vec{\Delta x}, \qquad W = \int_{t_0}^{t_f} \vec{F}\, d\vec{s}, \qquad K = \frac{1}{2}m\,|\vec{v}|^2, \qquad W = \Delta K$$

11.2 Trabajo de la fuerza del resorte

Assume we have a resorte R with a mass m at its end. Assume the mass is at x_0, it reaches x_f, and its equilibrium point is $x_e < x_0$. Fix the coordinate system so that in the x-axis, x_e corresponds to zero. Assume the mass is moving positively (to the right of the equilibrium). We want to study W_R, the work invested by the resorte on the mass. But we know that the force of the resorte is not constant, but it depends on the distance from the mass to the origin. From one of the work identities (see summary above),

$$W_R = \int_{x_0}^{x_f} \vec{F}_R(x)\, d\vec{x}$$

We know $\vec{F_R} = -kx\hat{i}$ (Hooke). Since the movement is horizontal, $d\vec{x} = dx\hat{i}$. So

$$W_r = \int_{x_0}^{x_f} (-kx\hat{i})(dx\,\hat{i}) = \int_{x_0}^{x_f} (-kx)\, dx$$

This is nothing but

$$-k\left(\frac{1}{2}x^2\right)\Big]_{x_0}^{x_f}$$

which gives

$$W_R = k\frac{x_0^2}{2} - k\frac{x_f^2}{2} \tag{1}$$

We see then that the work of a resorte depends only of the initial and final positions. Furthermore, if $x_0 = x_f$ then $W_R = 0$.

11.3 Trabajo de un agente externo sobre el resorte

Imagine an external agent (e.g. a hand) slowly stretching the resorte in the positive direction. Then $\vec{F}_{\text{ext}} = -\vec{F}_R$ implies

$$W_{\text{ext}} = -W_R$$

In other words, the external force must oppose that of the resorte.

11.4 Masa unida a un resorte en movimiento oscilatorio armónico

Now, at x_0 the velocity is v_0, at x_f it is v_f. Recalling that $W_R = \Delta K$, and

$$k\frac{x_0^2}{2} - k\frac{x_f^2}{2} = \overbrace{K_f - K_0}^{\Delta K} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$$

We can order this equation so that

$$\frac{k}{2}x_0^2 + \frac{1}{2}mv_0^2 = \frac{k}{2}x_f^2 + \frac{1}{2}mv_f^2$$

In other words, at the initial and the final state some things are the same. There is a conservation. Recalling that $\frac{1}{2}mv_0^2 = K_0$, $\frac{1}{2}mv_f^2 = K_f$, we can define the *potential energy*

$$U = \frac{k}{2}x^2$$

the equation above states that

$$E_0 = U_0 + K_0 = U_f + K_f = E_f \tag{1}$$

where $E = U + K$ is the mechanical energy. Since $E_0 = E_f$, we know the energy is conserved and the variation in energy is zero: $\Delta E = 0$.

Summary.

$$W_R = \frac{k}{2}x_0^2 - \frac{k}{2}x_f^2 = U_0 - U_f = \Delta U \tag{2}$$

$$U = \frac{1}{2}kx^2 + C, \qquad C = 0 \tag{3}$$

$$E = U + K \tag{4}$$

12 Fuerza gravitatoria

Think of a projectile shot with upward (purely vertical) movement. The only force presente comes from the projectile's weight $\vec{P}$. Let y_0 be the height at t_0, y_f the height at time t_f. We know $\vec{P} = -mg\hat{j}$. Now, if $d\vec{y}$ is the displacement differential (instantaneous direction differential), $d\vec{y} = dy\hat{j}$.

Now,

$$\begin{aligned}
W_P &= \int_{y_0}^{y_f} \vec{P}\, d\vec{y} \\
&= \int_{y_0}^{y_f} -mg\, dy \\
&= -mg \int_{y_0}^{y_f} dy \\
&= -mg\, y\Big]_{y_0}^{y_f} \\
&= -mg\left(y_f - y_0\right) \\
&= mgy_0 - mgy_f \\
&= mg\Delta y
\end{aligned}$$

So the work of the force $\vec{P}$ in a projectile traveling upwards depends only on the initial and final positions.

It just so happens that in tiro de proyectil the same happens. In a cycle, i.e. in any projectile shot s.t. $y_0 = f_y$, $W = 0$. This agains suggests a conservation of energy. In particular, if we define the potential gravitatory energy $U_G = mgy + C$, where we can safely impose $C = 0$ (since we can choose any coordinate system we like),

$$W_P = U_0 - U_f = -\Delta U$$

Note that $[U]$ = energy. Now, we may recall that $W_P = \Delta K$, i.e.

$$\begin{aligned} W_P &= K_f - K_0 \\ \Rightarrow \quad U_0 - U_F &= K_f - K_0 \\ \Rightarrow \quad U_0 + K_0 &= U_f + K_f \end{aligned}$$

Since $U + K$ is the energy, we found

$$E_0 = E_f \tag{1}$$

once more. So gravitational energy is also conservative. Also note that since $E = U + K$,

$$E = mgy + \frac{1}{2}m\,|\vec{v}|^2 \tag{2}$$

13 Fuerzas conservativas y no conservativas

If there are only conservative forces involved,

$$W = -\Delta U$$

where U is the potential energy. And since $W = \Delta K$ (which holds in every situation),

$$-\Delta U = \Delta K \iff \Delta K + \Delta U = 0$$

And since we define $E = U + K$, this entails

$$\Delta E = 0$$

i.e. the variation in energy is null (energy is conserved).

But what happens if we have non-conservative forces? Well, the work will be the sum of the work invested by conservative and non-conservative forces,

$$W = W_{\text{con}} + W_{\text{non-con}} = \Delta K$$

And since $W_{\text{con}} = -\Delta U$, as stated above,

$$W = -\Delta U + W_{\text{non-con}} = \Delta K$$

This means

$$W_{\text{non-con}} = \Delta K + \Delta U = \Delta(K + U) = \Delta E$$

In summary,

$$W_{\text{non-con}} = \Delta E \tag{1}$$

Note that from this follows that if $W_{\text{non-con}} = 0$, then $\Delta E = 0$ which is what we already knew.

14 Problems

14.1 Guía 2

(1) A mass A of 100kg lies on a surface with inclination of 30 with respect to the horizontal. There is no friction between the mass and the surface. A rope AB connects the mass to a pulley at the top of the incline, parallel to the surface, and from there to a hanging mass P of unknown weight. Furthermore, there is a second rope AC which is horizontal and connects the mass to a pole C lying on top of place where the inclination begins (i.e. to the left of the mass) and from which an object Q of 10kg hangs.

(a) Find the weight of P knowing that the system is in equilibrium.

(b) Find the reaction of the plain over block A.

(a) Let $\vec{D}$ be the force pulling A down the slope, $\vec{U}$ the force pulling A up the slope. It should be clear that $\vec{D} = \vec{W} + \vec{Q}$, where $\vec{W}$ is the force exerted by the weight of A down the slope and $\vec{Q}$ is the force exerted by the rope tying A to mass Q. Let us fist study $\vec{W}$.

Gravity and weight. Firstly, observe that

$$\vec{W} = -m_A g\hat{j} = -100g\hat{j} \tag{1}$$

The unit vector up the slope $\hat{u}$ is

$$\hat{u} = (\cos 30°, \sin 30°) \tag{2}$$

Thus, the component of gravity down the slope is

$$W_{||} := \vec{W} \cdot \vec{u} = -100g \sin 30° \tag{3}$$

Mass Q. Clearly,

$$Q = \vec{Q} = -m_Q\hat{g}i = -10g\hat{i} \tag{4}$$

Thus, the projection of $\vec{Q}$ onto the slope is

$$Q_{||} = \vec{Q} \cdot \vec{u} = -10g \cos 30° \tag{5}$$

Thus, we obtain the total force being exerted down the slope:

$$\therefore\ D_{\parallel} = -10g\cos 30^\circ - 100g\sin 30^\circ = -5\sqrt{3}g - 50g = -g(5\sqrt{3} + 50) \tag{6}$$

Since the system as at equilibrium, the component of $\vec{U}$ up the slope must exactly counter-act the component of $\vec{D}$ down the slope. So we know

$$\vec{U}_{\parallel} = g\left(5\sqrt{3} + 50\right) \tag{7}$$

But $\vec{U}$ operates only in direction parallel to the slope, from which readily follows that

$$m_P g = g\left(5\sqrt{3} + 50\right) \iff m_P = 5\sqrt{3} + 50 \approx 58.66 \tag{8}$$

Thus, we have deduced that the mass of P is approximately 58.66kg.

(b) Since friction plays no role in our ideal scenario, the reaction of the plain over block A is given purely by the normal vector $\vec{N}$. Since the object lies in equilibrium at the surface, the forces acting perpendicularly to the surface counterbalance each other. This forces are $\vec{W}$, $\vec{Q}$ and $\vec{N}$.

The unit vector in the direction of the normal vector (i.e. perpendicular to the slope) is

$$\hat{p} = (\cos 120^\circ, \sin 120^\circ) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \tag{9}$$

So the components of $\vec{W}$ and $\vec{Q}$ perpendicular to the surface are

$$W_{\perp} = \vec{W} \cdot \vec{p} = (-m_A g)\left(\frac{\sqrt{3}}{2}\right) = -50g\sqrt{3} \tag{10}$$

$$Q_{\perp} = \vec{Q} \cdot \vec{p} = (-m_Q g)\left(-\frac{1}{2}\right) = 5g \tag{11}$$

And since the system is at equilibrium,

$$W_{\perp} + Q_{\perp} + N = 0 \tag{12}$$

with N the magnitude of the normral vector, from which follows

$$N = -W_{\perp} - Q_{\perp} = 50g\sqrt{3} - 5g = g(50\sqrt{3} - 5) \approx 81.60g \tag{13}$$

where $[81.60] = \text{kg}$ and $[g] = \frac{m}{s^2}$. In conclusion,

$$\therefore N \approx (81.60 \times 9.8)N = 800N \tag{14}$$

(2) A body of mass $m = 10\,\mathrm{kg}$ rests on a horizontal surface without friction. A person pulls the block with a rope attached to it in the horizontal direction with a force of magnitude $|\vec{F}| = 20\,\mathrm{N}$. Calculate the acceleration of the block, assuming the mass of the rope is negligible.

Newton's second law states that the total force exerted on an object is equal to its mass times its acceleration, i.e.

$$\vec{F} = m\vec{a}$$

From this readily follows that

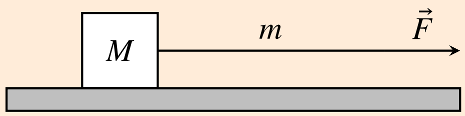
$$\begin{aligned} \left|\vec{F}\right| &= m\,|\vec{a}| \\ \Rightarrow \quad 20N &= 10\mathrm{kg}\,|\vec{a}| \end{aligned}$$

Since the object moves only in the horizontal direction, the y-component of the acceleration is null, and $\vec{a} = a_x\hat{i}$, from which readily follows that $|\vec{a}| = a_x$.

$$\therefore \quad 20N = 10\mathrm{kg}\,a_x$$

$$\therefore a_x = \frac{20N}{10\mathrm{kg}} = 2\frac{m}{s^2}$$

(3) Consider a block of mass M being pulled with force $\vec{F}$ applied through a rope of mass m. Assume friction is negligible and that the rope is non-extensible. (a) Compute the acceleration of the block-rope system. (b) Consider the diagram which has only the rope and verify that $F_{BC}/F = M/(M+m)$. Consider the case $m << M$



(a) We know $\vec{F}/m = \vec{a}$, so if we consider the block-rope system with mass $\phi = M + m$,

$$\vec{a} = \frac{\vec{F}}{\phi} \tag{15}$$

which is all we can say.

(b) The rope of mass m receives two forces: $\vec{F}$, which pulls it to the right, and from Newton's third law $\vec{F}_{BC}$, which is the force exerted by the block on the rope, and which pulls it to the left. So

$$\vec{F} + \vec{F}_{BC} = m\vec{a} \tag{16}$$

But we know

$$a = \frac{F}{M+m} \tag{17}$$

So

$$F + F_{BC} = m\frac{F}{M+m} \tag{18}$$

from which follows that

$$F_{BC} = F - \frac{mF}{M+m} = F\left(1 - \frac{m}{M+m}\right) \tag{19}$$

But

$$1 - \frac{m}{M + m} = \frac{M + m}{M + m} - \frac{m}{M + m} = \frac{M}{M + m} \tag{20}$$

So we have

$$F_{BC} = F\left(\frac{M}{M + m}\right) \tag{21}$$

from which follows that

$$F_{BC}/F = M/(M + m) \tag{22}$$

QED.

(4) An 8kg block A lies on an incline of angle $\alpha = 37°$ without friction. It is joined through a rope and polley without friction to a block B of mass $m_B = 4$kg. Determine $\vec{a}$ the acceleration of the masses and the tension of the ropes when the system is left to evolve freely. Create a diagram of the isolated body.

The mass A moves only up/down the surface. So if we let $\hat{u}$ be the unit vector parallel and equidirectional to the surface,

$$\sum \vec{F}_A = m_A \vec{a}_A = m_A a_A \hat{u} \tag{23}$$

Similarly,

$$\sum \vec{F}_B = m_B \vec{a}_B = m_B a_B \hat{j} \tag{24}$$

This entails that, in magnitude,

$$\left|\sum \vec{F}_A\right| = m_A a_A, \qquad \left|\sum \vec{F}_B\right| = m_B a_B \tag{25}$$

But since the objects move under equal and opposite accelerations, $a_A = -a_B$. So if we define $a := a_A$ as the acceleration of A, equation (25) becomes

$$\left|\sum \vec{F}_A\right| = m_A a, \qquad \left|\sum \vec{F}_B\right| = m_B(-a) \tag{26}$$

Now, let θ be the angle between $\hat{u}$ and $\vec{W}$ the weight vector of A. Then the projection of $\vec{W}$ along the surface is $W_{\|} = -m_A g \cos\theta$. And since this and the tension of the rope are the only forces moving A along the surface, we have

$$\left|\sum \vec{F}_A\right| = W_{\|} + T = -m_A g \cos\theta + T \tag{27}$$

Similar reasoning gives that

$$\left|\sum \vec{F}_B\right| = -m_B g + T \tag{28}$$

So, combining these last two results with equation (26),

$$T - m_A g \cos\theta = m_A a, \qquad T - m_B g = m_B(-a) \tag{29}$$

Solving T in one of these equations and substituting it into the other, with $\theta = 90^\circ - 37^\circ = 53^\circ$, $m_A = 8\mathrm{kg}, m_B = 4\mathrm{kg}$, one readily obtains after a bit of algebra that

$$a \approx -0.665\frac{m}{s^2} \tag{30}$$

(5) Dos bloques de masas m_1 y m_2 están en contacto sobre una mesa sin rozamiento. Una fuerza horizontal $\vec{F}$ se aplica sobre el primer bloque.

(a) Encuentre la aceleración del sistema y la fuerza de contacto entre los dos bloques. Evaluar para el caso que $m_1 = 2\text{kg}, m_2 = 1\text{kg}$ y $\left|\vec{F}\right| = 3N$.

(b) Muestre que si la misma fuerza $\vec{F}$ se aplica en sentido contrario, es decir sobre m_2 en lugar de m_1, la fuerza de contacto será distinta. Explique realizando un diagrama de cuerpo aislado.

(a) Considering the two blocks as a system, the mass of the system is $m_1 + m_2$, so

$$F = (m_1 + m_2)a \Rightarrow a = \frac{F}{m_1 + m_2} \tag{31}$$

If $m_1 = 2\text{kg}, m_2 = 1\text{kg}$ and $\left|\vec{F}\right| = 3N$,we have

$$a = \frac{3N}{3\text{kg}} = 1\frac{m}{s^2} \tag{32}$$

Consider the lone body diagram of m_2. The only force exerted on this body is the contact force F_c, so the magnitude of its acceleration is equal to the magnitude of this force. But its acceleration is equal to the acceleration of the system (would it make sense for the second block to accelerate at a different rate than the system?) In short,

$$F_c = m_2 a = \frac{F m_2}{m_1 + m_2} \tag{33}$$

(b) By the same reasoning as before, if the force were applied leftwards onto m_B, the contact force would be the only thing moving m_A, so that

$$F_C = m_1 a \neq m_2 a \tag{34}$$

at least as long as $m_1 \neq m_2$.

(6) A ball of mass $m = 10\text{kg}$ hangs from a rope on the roof of a car. The maximum tension which the rope can take without breaking is $500N$. What is the maximal horizontal acceleration which the car can recah without the rope breaking? Determine the angle between the rope and the vertical for that maximal acceleration.

The forces operating on the ball are its weight, which pulls it downard ($\vec{W} = W\hat{j}$), and the tension of the rope. From Newton's second law,

$$\begin{aligned} \vec{W} + \vec{T} &= m\vec{a} \\ \Rightarrow \vec{T} &= m\vec{a} - \vec{W} \\ &= ma\hat{i} - W\hat{j} \end{aligned}$$

(since acceleration is strictly horizontal). Furtheremore, we know

$$W = -g10\text{kg} = -9.8\frac{m}{s^2} \cdot 10\text{kg} = -98N$$

We are interested in the acceleration which breaks the rope, i.e. which causes a tension superior to $500N$. The tension of the rope is

$$\begin{aligned} \left|\vec{T}\right| &= \sqrt{(ma)^2 + W^2} \\ &= \sqrt{100\text{kg}^2 a^2 + 98^2 N^2} \\ &= \sqrt{100\text{kg}^2 a^2 + 9604N^2} \end{aligned}$$

So to find the acceleration which causes the tension to be $500N$, we solve

$$\begin{aligned} 500N &= \sqrt{100\text{kg}^2 a^2 + 9604N^2} \\ \Longleftrightarrow 500^2 N^2 &= 100\text{kg}^2 a^2 + 9604N^2 \\ \Longleftrightarrow a &= 49.03\frac{m}{s^2} \end{aligned}$$

as it is easy to see algebraically. So, the rope breaks if $a \geq 49.03$, meaning that the rope will not break if the acceleration lies in the open interval $a < 49.03$. This interval has 49.03 as supremum, but has no maximum.

(b) The angle between the tension vector $\vec{T}$ and the horizontal is

$$\arctan\left(\frac{T_y}{T_x}\right)$$

We know $\vec{T} = ma\hat{i} - W\hat{j} = 10\text{kg}a - 98N\hat{j}$. Taking the maximal acceleration $a = 49.03m/s^2$, we have $\vec{T} = 490.3N\hat{i} - 98N\hat{j}$. So the angle between the tension vector and the horizontal is

$$\arctan\left(\frac{98N}{490.3N}\right) = 0.197\text{rad}$$

Now, the angle between $\vec{T}$ and the vertical will be $90° = \frac{\pi}{2}$rad minus the angle between $\vec{T}$ and the horizontal. So the angle between $\vec{T}$ and the vertical is

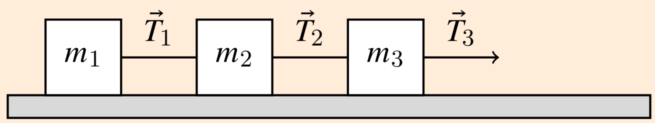
$$\frac{\pi}{2}\text{rad} - 0.197\text{rad} = 1.37379633\text{rad} \tag{35}$$

Problem 7. Three blocks of masses m_1, m_2, and m_3 are connected by strings that sustain tensions $\vec{T}_1$ and $\vec{T}_2$ on a frictionless table. They are pulled by a string attached to m_3 with a tension $|\vec{T}_3| = 60$ N. The strings are massless and inextensible.

(a) Find $\vec{T}_1$ and $\vec{T}_2$ as functions of the block masses.

(b) Evaluate (a) for the particular case $m_1 = 20$ kg, $m_2 = 20$ kg, and $m_3 = 30$ kg.

(c) Repeat (a) and (b) for the case of vertical motion.



(a) As always, let us ask: what are the forces at work on each body? The mass m_1 is affected only by $\vec{T}_1$, so

$$\vec{T}_1 = m_1 a \tag{36}$$

The mass m_2 is affected by $\vec{T}_2$, which pulls it to the right, and $\vec{T}_1$, which pulls it to the left, so

$$\vec{T}_2 - \vec{T}_1 = m_2 a \tag{37}$$

By the same logic,

$$\vec{T}_3 - \vec{T}_2 = m_3 a \tag{38}$$

From (38) we have

$$\vec{T}_3 - m_3 a = \vec{T}_2 \iff 60N - m_3 a = \vec{T}_2 \tag{39}$$

Substituting this into (37), we have

$$60N - m_3 a - \vec{T}_1 = m_2 a \iff 60N - m_3 a - m_2 a = \vec{T}_1 \tag{40}$$

Substituting into (36),

$$60N - m_3 a - m_2 a = m_1 a \iff \frac{60N}{m_1 + m_2 + m_3} = a \tag{41}$$

Now that we have determined the acceleration a as a function of the masses, we have

$$\begin{aligned}
T_1 &= \frac{m_1 60N}{m_1 + m_2 + m_3} \\
T_2 &= m_2 a + T_1 \\
&= \frac{60N m_2}{m_1 + m_2 + m_3} + \frac{m_1 60N}{m_1 + m_2 + m_3} \\
&= 60N \frac{m_1 + m_2}{m_1 + m_2 + m_3}
\end{aligned}$$

(9) A block of mass m slides on the floor while a force of magnitude $|\vec{F}| = 12\ \mathrm{N}$ pulls it at an angle θ with the horizontal. The coefficient of kinetic friction is 0.4. The angle θ can vary between $0°$ and $90°$, and the block always remains on the floor. What is the angle θ that gives the maximum acceleration?

The forces acting on the mass are the normal force, its weight (gravity), $\vec{F}$ and the dynamic friction $\vec{F}_{r.d.}$. Since the object moves only horizontally, $\vec{a} = a\hat{i}$. So by Newton's second law,

$$\vec{W} + \vec{N} + \vec{F} + \vec{F}_{r.d.} = ma\hat{i}$$

from which follows

$$F_{\parallel} + F_{r.d.} = ma \tag{42}$$

where $F_{\parallel}$ is the projection of F parallel to the x-axis. We know $F_{r.d.} = \mu \left|\vec{N}\right|$ where μ is the dynamic frictoin coefficient. So we obtain

$$\left|\vec{F}\right| \cos\theta + \mu_{r.d.} \left|\vec{N}\right| = ma \tag{43}$$

Using the information of the exercise, this gives

$$12N \cos\theta + 0.4 \left|\vec{N}\right| = ma \tag{44}$$

We must know determine the magnitude of $\vec{N}$, the vertical forces at play. These are (a) the force that opposes gravity, of magnitude equal to that of gravity, and the vertical component of $\vec{F}$, which we term $F_{\|}$. So

$$\left|\vec{N}\right| = N = gm_A + 12N \sin\theta \tag{45}$$

In conclusion,

$$12N \cos\theta + \mu \left(gm_A + 12N \sin\theta\right) = ma \tag{46}$$

Rearranging,

$$a = \frac{12N(\cos\theta + \mu\sin\theta) + \mu g m_A}{m} \tag{47}$$

This will of course be maximized when $\varphi(\theta) = \cos\theta + \mu\sin\theta$ is maximized, under the constraint $0 \leq \theta \leq \pi/2$. The derivative of φ is

$$\varphi'(\theta) = -\sin\theta + \mu\cos\theta \tag{48}$$

and this derivative is zero if and only if $\sin\theta = \mu\cos\theta$, which holds if and only if $\frac{\sin\theta}{\cos\theta} = \mu$, or equivalently $\tan\theta = \mu$. But this holds if and only if $\arctan\mu = \theta$. With $\mu = 0.4$, we obtain $\theta \approx 0.380$.

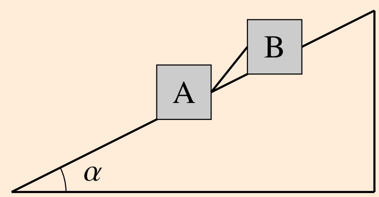
$\therefore$ Acceleration is maximized if $\theta \approx 0.380$rad $\approx 21.77°$.

10. Two masses m_A and m_B slide down an inclined plane, connected by a massless rope, where m_A pulls m_B. The inclination angle is α. The coefficient of kinetic friction between m_A and the plane is μ_A, and between m_B and the plane is μ_B.

(a) Find an expression for the tension $\vec{T}$ in the rope that connects m_A and m_B, expressed in terms of the problem variables.

(b) Find an expression for the acceleration $\vec{a}$ that both bodies have.

(c) How do the answers change if instead m_B pulls m_A?



Object A is affected by gravity, the normal force, the tension of the rope, and friction:

$$\sum \vec{F}_A = \vec{T}_A + \vec{N}_A + \vec{W}_A + \vec{R}_A = m_A\vec{a}$$

Let us decompose these forces into components parallel and perpendicular to the slope. Let $\hat{x}', \hat{j}'$ be the unit vectors parallel and perpendicular to the slope, with $\hat{x}'$ going up the slope and $\hat{j}$ with the same direction as $\vec{N}$.

(Weight.) The angle between $\vec{W}_A$ and $\hat{x}'$ is $\frac{\pi}{2} - \alpha$, so we readily know that

$$W_{A\|} = -m_A g \sin\alpha, \qquad W_{A\perp} = -m_A g \cos\alpha \tag{49}$$

(Normal.) The normal force will be strictly perpendicular to the incline and will counteract gravities force:

$$N_{A\|} = 0, \qquad N_{A\perp} = m_A g \cos\alpha \tag{50}$$

(Friction.) Friction is parallel to the slope and contrary to the direction of movement, so it will be directed up the slope. Thus,

$$R_{A\|} = \mu_A \left|\vec{N}\right| = \mu_A m_A g \cos\alpha, \qquad R_\perp = 0 \tag{51}$$

(Tension.) The tension also has a null perpendicular component, acting only parallel to the surface. For object A, the tension of the rope acts up the slope and towards object B. And from our previous results, it follows that

$$\begin{aligned} & T_{\|} + N_{A\|} + W_{A\|} + R_{A\|} = m_A a \\ \iff \quad & T_{\|} - m_A g \sin\alpha + \mu_A m_A g \cos\alpha = m_A a \\ \iff \quad & T_{\|} = m_A a + m_A g \sin\alpha - \mu_A m_A g \cos\alpha \end{aligned} \tag{52}$$

Note that we do not write $T_{A\|}$ but simply $T_{\|}$, since the tension is equal in absolute value for A and B, i.e. we may define $T_{\|} := T_{A\|}$ and then simply $T_{B\|} = -T_{\|}$.

Now, the same results follow for object B, but of course with its own mass and friction coefficient. So

$$\begin{aligned} & -T_{\|} + N_{B\|} + W_{B\|} + R_{B\|} = m_B a \\ \iff \quad & -T_{\|} - m_B g \sin\alpha + \mu_B m_B g \cos\alpha = m_B a \end{aligned} \tag{53}$$

If we add (52) and (53), we see that T disappears and we obtain an equation with only a unknown:

$$\begin{aligned} & \text{Eq. (52)} + \text{Eq. (53)} \\ \Rightarrow \quad & (N_{A\|} + W_{A\|} + R_{A\|}) + (N_{B\|} + W_{B\|} + R_{B\|}) = a(m_A + m_B) \\ \Rightarrow \quad & (-m_A g \sin\alpha - m_B g \sin\alpha) + (\mu_A m_A g \cos\alpha + \mu_B m_B g \cos\alpha) = a(m_A + m_B) \\ \Rightarrow \quad & (-g \sin\alpha (m_A + m_B)) + (g \cos\alpha (m_A \mu_A + m_B \mu_B)) = a(m_A + m_B) \\ \Rightarrow \quad & -g \sin\alpha + g \cos\alpha \, \frac{m_A \mu_A + m_B \mu_B}{m_A + m_B} = a \end{aligned}$$

Now that we know a, we may substitute it in the simplified form of equation (52) to obtain an expression of $T_{\|} =: T$ in terms of the problem variables:

$$
\begin{aligned}
& T = m_A a + m_A g \sin\alpha - \mu_A m_A g \cos\alpha \\
\iff\quad & T = m_A\left(a + g\sin\alpha - \mu_A g\cos\alpha\right) \\
\iff\quad & T = m_A\left(-g\sin\alpha + \frac{m_A\mu_A + m_B\mu_B}{m_A + m_B} g\cos\alpha + g\sin\alpha - \mu_A g\cos\alpha\right) \\
\iff\quad & T = m_A g\cos\alpha\left(\frac{m_A\mu_A + m_B\mu_B}{m_A + m_B} - \mu_A\right) \\
\iff\quad & T = m_A g\cos\alpha\left(\frac{m_A\mu_A + m_B\mu_B - \mu_A(m_A + m_B)}{m_A + m_B}\right) \\
\iff\quad & T = m_A g\cos\alpha\left(\frac{m_A\mu_A + m_B\mu_B - m_A\mu_A - m_B\mu_A}{m_A + m_B}\right) \\
\iff\quad & T = m_A g\cos\alpha\left(\frac{m_B(\mu_B - \mu_A)}{m_A + m_B}\right) \\
\iff\quad & T = g\cos\alpha\,\frac{m_A m_B(\mu_B - \mu_A)}{m_A + m_B}
\end{aligned}
$$

11. The spring of a laboratory dynamometer has been stretched $11.7\,\text{cm}$ at the maximum scale, which corresponds to $2\,N$.

(a) What is the spring constant of the spring with which the dynamometer has been manufactured?

(b) How much will it stretch when a force $|\vec{F}| = 0.4\,N$ is applied to it?

Sea k la constante del resorte. Sabemos que la fuerza del resorte es

$$\vec{F}_k = -k\Delta\ell \tag{54}$$

donde ℓ es la distancia entre el extremo y la posición de equilibrio. Ahora bien, sabemos además que si $\delta\ell = 11.7\text{cm}$ la fuerza se corresponde con $2N$, es decir que bajo este supuesto

$$N + W + F_k = 2N \tag{55}$$

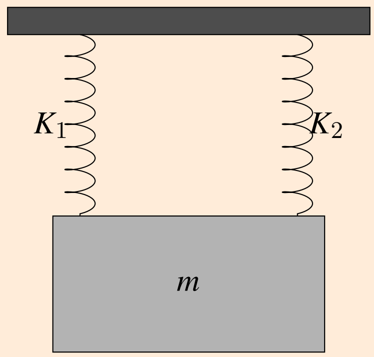
Pero asumimos que $\vec{N} + \vec{W} = 0$, con lo cual $F_k = 2N$, i.e.

$$k(11.7\text{cm}) = 2N \tag{56}$$

Por lo tanto,

$$\begin{aligned} k &= \frac{2N}{11.7\text{cm}} \\ &= \frac{2}{11.7}\frac{\text{kg}\cdot ms^{-2}}{cm} \\ &= \frac{2}{0.117}\frac{N}{m} \end{aligned}$$

Problema 12. Dos resortes de longitudes naturales $L_0 = 0.5$ m pero con diferentes constantes elásticas, $K_1 = 50$ N/m y $K_2 = 100$ N/m, se encuentran colgados del techo. Un cuerpo de masa $m = 2.5$ kg que inicialmente está suspendido de ellos es estirado hacia abajo hasta que la longitud de los resortes se duplica. ¿Cuál es la aceleración $\vec{a}$ que adquiere el cuerpo cuando se deja libre?



The acceleration of the body is strictly vertical, i.e. $\vec{a} = a\hat{j}$ for some $a \in \mathbb{R}$. Both springs will exert a force on the object, and this force will also be strictly vertical. In particular, the forces acting on the body when it is at a distance $\Delta\ell$ from the point of equilibrium are

$$\vec{F}_1 = -K_1\Delta\ell\ \hat{j}, \qquad \vec{F}_2 = -K_2\Delta\ell\ \hat{j}, \qquad \vec{W} = -gm\ \hat{j}$$

It follows that

$$\begin{aligned} &- K_1\Delta\ell - K_2\Delta\ell - gm = ma \\ \Rightarrow &- \Delta\ell(K_1 + K_2) - gm = ma \\ \Rightarrow &- \Delta\ell(150\text{N / m}) - gm = ma \end{aligned}$$

Since in the conditions described the length of the springs is doubled, going from 0.5m to 1m, the mass will be at a distance of -0.5m from its resting position. So we will have

$$
\begin{aligned}
& \frac{1}{2}\mathrm{m}\,(150\mathrm{N/m}) - gm = ma \\
\Rightarrow \quad & 75N - gm = ma \\
\Rightarrow \quad & \frac{75N}{2.5\mathrm{kg}} - g = a \\
\Rightarrow \quad & 30ms^{-2} - 9.8ms^{-2} = a
\end{aligned}
$$

So $a = 20.2ms^{-2}$.

(13) A spring with constant k has an extreme fixed adn the other coinciding with coordinate x_0 when there is nodeformation. A mass m lies at this extreme, is displaced to position x_1, and let go from that position. Assume $k = 8\text{N/m}$, $m = 2\text{kg}, x_0 = 40\text{cm}, x_1 = 55\text{cm}$.

(a) Determine $x(t), v(t), a(t)$ and plot them.

(b) Determine the period, frequency, and extreme coordinates of the movement, as well as the module of the velocity of m, at the point of equilibrium.

Recordemos que

$$a(t) = -\omega^2 x(t), \qquad v(t) = \omega A\cos(\omega t + \phi), \qquad x(t) = x_0 + A\sin(\omega t + \phi)$$

con $\omega = \sqrt{k/m}$ y con $x(t)$ el desplazamiento de la masa, en el tiempo t, respecto a la posición de equilibrio. Dado que $x(0) = x_1, v(0) = 0$, tenemos

$$\omega A\cos(\phi) = 0, \qquad A\sin(\phi) = x_1 - x_0$$

Como $x_1 \neq 0$, tenemos que $A \neq 0, \sin(\phi) \neq 0$, de lo cual se sigue que $\phi \neq 0$. Por ende, en la ecuación $\omega A\cos(\phi) = 0$, dado que $\omega \neq 0$, debe satisfacerse $\cos(\phi) = 0$, es decir $\phi = \frac{\pi}{2}$. Usando este resultado en la ecuación de la derecha, vemos que

$$A\sin\left(\frac{\pi}{2}\right) = x_1 - x_0 \iff A = x_1 - x_0 = 15\text{cm}$$

Tomando

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\text{N}}{2\text{kg m}}} = 2\frac{ms^{-1}}{m} = 2s^{-1}$$

obtenemos

$$x(t) = 40\text{cm} + 15\text{cm}\sin\left(2s^{-1}\cdot t + \frac{\pi}{2}\right)$$

$$a(t) = 2s^{-1}\cdot 15\text{cm}\cdot\cos\left(2s^{-1}\cdot t + \frac{\pi}{2}\right)$$

etc. Simplificar unidades (e.g. $a(t)$ queda todo en cm sobre segundos, etc.)

(b) El periodo es $T = \frac{2\pi}{\omega} = \frac{2\pi}{2s^{-1}} = \pi\mathrm{s}$. La frequencia es la inversa del periodo, i.e. $F = \frac{1}{\pi\mathrm{s}}\frac{1}{\pi}\mathrm{Hz}$.

15 P3

1. Una masa pequeña se coloca en el extremo de una cuerda de largo $L = 132\,\text{cm}$ y se suelta desde el reposo, siendo $\theta_A = 5$ (posición A). Sabiendo que $d = 66\,\text{cm}$, determinar:

(a) El valor de la velocidad en el punto más bajo de la trayectoria (posición B).

(b) El valor de θ_C para la máxima altura que alcanza la masa (posición C).

(c) La tensión de la cuerda en la posición B.

En el sistema de coordenadas que elegí,

$$A = L\cos\theta\,\hat{i} + L\sin\theta\,\hat{j}, \qquad B = 132\text{cm}\,\hat{i}$$

La distancia vertical en el espacio físico se corresponde con la distancia horizontal en mi sistema de coordenadas, y se obtiene que la distancia vertical (física) entre A y B es

$$\Delta h = B_x - A_x \approx 0.51\text{cm} \tag{1}$$

Recordemos que la energía kinética (en un momento dado) y la energía potencial son:

$$K = \frac{1}{2}mv^2, \qquad U = mg\Delta h$$

donde Δh es la altura del objeto y $E = U + K$.

Tomemos como momento inicial el instante en que el objeto se suelta desde el reposo, y como momento final el instante en que el objeto ocupa el punto B. Pues la velocidad en el instante cero es cero,

$$E_0 = K_0 + U_0 = 0 + mg\Delta h = mg\Delta h$$

En otras palabras, en el instante cero, la energía se corresponde con la energía potencial, pues el objeto está en reposo con altura Δh. Ahora bien, de manera análoga,

$$E_f = K_f + U_f = \frac{1}{2}mv_f^2 + 0 = \frac{1}{2}mv_f^2$$

donde v_f es la velocidad en el instante en que el cuerpo ocupa el punto B. En otras palabras, en este caso, la energía se corresponde enteramente con la energía kinética. Por preservación de la energía, $E_f = E_0$, y por lo tanto obtenemos:

$$mg\Delta h = \frac{1}{2}mv_f^2 \tag{2}$$

lo cual implica que $v_f = \sqrt{2g\Delta h}$. Aproximando $g \approx 9.8\frac{m}{s^2}$ y $\Delta h \approx 0.51\text{cm}$, obtenemos

$$\begin{aligned} v_f &= \sqrt{2 \cdot 9.8\frac{\text{m}}{\text{s}^2} \cdot 0.0051\text{m}} \\ &\approx 0.316\text{m/s} \end{aligned}$$

(b) En el instante t_C en que el péndulo se detiene alcanzando el punto C, toda la energía es potencial. Pero por preservación de la energía, $E_C = E_f$. Como $E_c = U_C$ (la energía es estrictamente potencial), y $E_f = K_f$ (la energía es estrictamente kinética), obtenemos $E_C = K_f$, es decir

$$mg\Delta h' = \frac{1}{2}mv_f^2 \tag{3}$$

donde $\Delta h' = B_x - C_x$ es la distancia vertical (en el espacio físico) entre C y B. Ahora bien, de la ecuación anterior se deduce

$$\Delta h' = \frac{1}{2g}v_f^2 \tag{4}$$

donde conocemos v_f^2 por el punto (a). Ahora bien, notemos que en nuestro sistema de coordenadas, la posición en el eje x del punto C se corresponde con

$$C_x = \cos\theta_C\,|C| + d \tag{5}$$

con $|C| = L - d$. Luego

$$C_x = \cos\theta_C(L - d) + d \tag{6}$$

Por lo tanto, combinando (6) y (4), obtenemos

$$
\begin{aligned}
\frac{1}{2g}v_f^2 &= B_x - \cos\theta_C(L-d) - d \\
\Rightarrow \quad -\frac{\frac{1}{2g}v_f^2 - B_x + d}{L-d} &= \cos\theta_C
\end{aligned}
$$

Es fácil ver que

$$\frac{1}{2g}v_f^2 \approx \frac{0.316^2}{19.6}\frac{\frac{m^2}{s^2}}{\frac{m}{s^2}} = 0.005\mathrm{m} = 0.5\mathrm{cm}$$

y que $-B_x + d = (-132+66)\mathrm{cm} = -66\mathrm{cm}, L - d = 66\mathrm{cm}.$ Luego obtenemos

$$
\begin{aligned}
-\frac{0.5\mathrm{cm} - 66\mathrm{cm}}{66\mathrm{cm}} &= \cos\theta_C \\
\Rightarrow \quad \frac{65.5}{66}\mathrm{cm} &= \cos\theta_C \\
\Rightarrow \quad 0.992\mathrm{cm} &= \cos\theta_C
\end{aligned}
$$

Como $\cos^{-1}(0.992) = 0.126\mathrm{rad}$, obtenemos que $\theta_C = 0.126\mathrm{rad}$.

(c) Se nos pide la tensión de la cuerda. Recordemos que en un movimiento pendular como éste, la velocidad angular en el punto de equilibrio es cero. Recordemos que la aceleración centrípeta es

$$a_c = \frac{v^2}{L} \tag{7}$$

y por ende la segunda ley de Newton nos dice

$$T - W = ma_c \tag{8}$$

donde T es la tensión de la cuerda (vertical y positiva hacia arriba), W es la gravedad (vertical y negativa hacia abajo). Por lo tanto,

$$T - W = m\frac{v^2}{L} \iff T = m\frac{0.316^2\mathrm{m}^2/\mathrm{s}^2}{L} + mg \tag{9}$$

Como L está en centímetros, $\frac{L}{100}$m se corresponderá con la longitud de la soga en metros, y el término izquierdo quedará en metros sobre segundos cuadrados. Se obtiene entonces

$$T \approx m\frac{0.099}{1.32}\frac{\mathrm{m}}{s^2} + m9.8\frac{\mathrm{m}}{s^2} = m\left(9.875\mathrm{m/s^2}\right) \tag{10}$$

2. Un bloque de $20\,kg$ es empujado sobre una superficie horizontal, por medio de una fuerza $\vec{F}$ que forma un ángulo θ con ésta (ver figura). Durante el movimiento la fuerza aumenta de acuerdo con la relación $|\vec{F}(x)| = 6x\,N$.

(a) Calcule el trabajo realizado por esta fuerza mientras el cuerpo se desplazó en línea recta desde $x = 10\,m$ hasta $x = 20\,m$.

(b) Calcule la energía cinética del cuerpo en la posición final, asumiendo que se parte del reposo. Considere dos casos:

(i) $\mu_d = 0$

(ii) $\mu_d = 0.05$

(a) Recordemos que el trabajo realizado por una fuerza $\vec{F}$ al desplazar un cuerpo con un desplazamiento $\Delta\vec{r}$ es el producto punto de ambos vectores:

$$W = |F| \cdot |\Delta\vec{r}| \cdot \cos\theta \tag{11}$$

con θ el ángulo entre ambos vectores. En el caso de un movimiento horizontal, como el problema, $\Delta\vec{r} = \Delta x \cdot \hat{i}$ y por lo tanto escribimos simplemente $W = F \cdot \Delta x \cdot \cos\theta$. Notar que si la fuerza y el desplazamiento tienen la misma dirección, $\cos\theta = 1$ y $W = F\Delta x$, y si la fuerza y el desplazamiento tienen dirección opuesta, $\cos\theta = -1$ y $W = -F\Delta x$ (es decir, la fuerza hace un trabajo negativo).

En nuestro problema, la magnitud $\vec{F}$ es variable respecto a la posición x. Por lo tanto, debemos integrar la expresión para la fuerza en cada una de las posiciones tomadas por el cuerpo:

$$W_F = \int_{x=10\mathrm{m}}^{x=20\mathrm{m}} \left|\vec{F}(x)\right| \cos\theta\; dx = 6\mathrm{N}\cos\theta \int_{x=10\mathrm{m}}^{x=20\mathrm{m}} x\; dx = 900\cos\theta\ \mathrm{J}$$

(Notar, si esto no es claro, que dx es un desplazamiento infinitesimalmente pequeño. Es decir que la expresión siendo integrada sigue siendo exactamente de la forma de la ecuación 11: magnitud de la fuerza por magnitud del desplazamiento por ángulo entre ambos, i.e. producto punto).

(b) Recordemos que $W = \Delta K$. A su vez, $W = W_F + W_R$ con W_F el trabajo de la fuerza $\vec{F}$ y W_R el trabajo de la fuerza de rozamiento $\vec{R}$. En términos generales, asumiendo un coeficiente de rozamiento dinámico μ_d arbitrario,

$$W_R = \left|\vec{R}\right| \Delta x \cos\varphi = \mu_d \left|\vec{N}\right| \Delta x \cdot \cos\varphi$$

donde φ el ángulo entre la dirección del movimiento y la fuerza de rozamiento. Pero la fuerza de rozamiento se opone al movimiento, i.e. su ángulo es 180 grados y $\cos\varphi = \cos 180° = -1$. La fuerza normal tiene magnitud equivalente a la magnitud de la gravedad. Luego

$$W_R = -\mu_d mg \cdot \Delta x = -\mu_d 1960\text{J}$$

Por ende,

$$W = W_R + W_F = 900\cos\theta\text{J} - \mu_d 1960\text{J}$$

Entonces

$$\Delta K = 900\cos\theta \text{ J} - \mu_d 1960 \text{ J}$$

Pero la velocidad inicial es cero, así que $\Delta K = K_f$ la energía kinética final. Con lo cual

$$K_f = (900\cos\theta - \mu_d 1960) \text{ J}$$

Sustituyendo con $\mu_d = 0, \mu_d = 0.05$ se obtienen las respuestas.

(3) A 1kg mass is left to slide down a plane with a $\theta = 30°$ angle with respect to the horizontal and from a height of 1m.

(a) Find the velocity of the block when it reaches the floor, assuming no friction.

(b) Same but assuming $\mu_d = 0.3$.

(c) Compare the value of (a) with the value obtained if the block is dropped in free fall from the same height.

(d) For (b), compute the loss of energy.

(a) Recall that $K = \frac{1}{2}mv^2$ and that $W = \Delta K$. Since the projection of the weight vector along the incline is $P_{\|} = mg\sin\theta$,

$$W = P_{\|}\Delta x' \tag{12}$$

where $\Delta x'$ is the change of position in terms parallel to the incline. The incline is the magnitude of a vector $\vec{h}$ with y-coordinate 1m and angle θ, i.e. $\Delta x' = \left|\vec{h}\right|$. Here, we recall

$$h_x = \left|\vec{h}\right|\cos\theta, \qquad h_y = \left|\vec{h}\right|\sin\theta$$

From the LHS and the fact that $h_y = 1\text{m}$,

$$\Delta x' = |h| = \frac{1\text{m}}{\sin\theta}$$

In short, substituting in (13) with (14),

$$W = mg\sin\theta\cdot\frac{1\text{m}}{\sin\theta} = mg\text{m} = 1\text{kg}\cdot 9.8\frac{\text{m}}{\text{s}^2}\cdot 1\text{m} = 9.8\text{N}\cdot 1\text{m} = 9.8\text{J} \tag{13}$$

The velocity at the initial instant is zero, and at the final instant is unknown, but since $W = \Delta K$ we have

$$\begin{aligned}
& W = K_f - K_0 \\
\iff\quad & 9.8\text{J} = \frac{1}{2}mv_f^2 \\
\iff\quad & \frac{19.6\text{J}}{1\text{kg}} = v_f^2 \\
\iff\quad & 19.6\frac{\text{m}^2}{\text{s}^2} = v_f^2 \\
\iff\quad & \sqrt{19.6\frac{\text{m}^2}{\text{s}^2}} = v_f \\
\iff\quad & 4.42718872424\frac{\text{m}}{\text{s}} = v_f
\end{aligned}$$

(b) If we assume there is friction with $\mu_d = 0.3$, then $\vec{R}$ will oppose the sliding movement with magnitude $\mu_d\left|\vec{N}\right|$. The magnitude of the normal force balances the perpendicular component of the weight, i.e. $\left|\vec{N}\right| = P_\perp = mg\cos\theta$. So now the total work is

$$
\begin{aligned}
W &= (P_{\parallel} - R)\Delta x' \\
&= (mg\sin\theta - \mu_d\ mg\cos\theta)\ \frac{1\text{m}}{\sin\theta} \\
&= mg\ 1\text{m} - \mu_d\ mg\frac{\cos\theta}{\sin\theta}\ 1\text{m} \\
&= 1\text{m}\ mg\ (1 - \mu_d\cot\theta) \\
&= 9.8\text{J}\left(1 - 0.3\cdot\sqrt{3}\right) \\
&= 9.8\text{J}\cdot 0.480384758 \\
&= 4.70777063\text{J}
\end{aligned}
$$

Having found the work, we again use that $W = K_f$ in this scenario to obtain

$$
4.70777063\text{J} = \frac{1}{2}mv_f^2 \iff \sqrt{9.41554126\frac{\text{m}^2}{s^2}} = v_f \tag{14}
$$

which gives $v_f = 3.06847539668\text{m/s}$.

(c) If the object is let to fall freely from one meter, we have

$$
W = mg\ 1\text{m}
$$

which entails

$$
v_f = \sqrt{2\cdot g\ 1\text{m}} = 4.42718872424\frac{\text{m}}{\text{s}}
$$

So interestingly, at the final instant, the velocity of the object if left to slide coincides with the velocity of the object is left to fall.

(4) A mass of 1kg compresses a spring of constant $k = 2$N/m over a horizontal surface without friction. The spring is compressed 0.3m with respect to its equilibrium position. In a given moment, it is released. The mass is not tied to the spring.

(a) Explain what happens when the spring is released.

(b) Compute the work done by the spring.

(c) Compute the final velocity which the object raches after being released.

(d) At the moment when the body is released, it makes contact with a surface with friction and $\mu_d = 0.2$. Explain what happens, and the work done by the force of friction when the mass reaches half the velocity which it carried at the moment of release.

(c) Compute the distance the body travels before stopping.

(†) I misunderstood the statement of the problem and used y instead of x coords. I'm lazy and will not change it. Just think of y as x.

(a) When the spring is released, the potential energy of the spring becomes kinetic energy, and there's a transfer of energy from the spring unto the mass.

The mass is pushed away in horizontal and positive direction with the force transfered by the spring, becoming immediately affected by gravity.

(b) The work done by the spring will correspond to

$$W_R = \int_{x_0}^{x_f} |\vec{R}(x)| \, dx$$

where $y_0 = y_e - 0.3$m and y_f is the point at which the mass stops making contact with the spring. Theoretically, y_f will correspond to the point at which the spring stops exerting force unto the mass, i.e. when $\left|\vec{R}\right| = 0$. Since, by virtue of Hooke's law, $\left|\vec{R}\right| = -k\Delta y$, it will be zero exactly at y_e.

Since the movement of the spring is strictly vertical, $d\vec{y} = dy$. The magnitude of the force at an arbitrary point y is $-k\Delta y$, and if we set the origin of our coordinate system precisely at the equilibrium y_e then $\Delta y = y$. From this follows that

$$W_R = \int_{y_0}^{y_f} \vec{R}(y) \, dy = \int_{y_0}^{y_f} -ky \ \ dy = -k \left. \frac{y^2}{2} \right]_{y_0}^{y_f} = \frac{k}{2}(y_0^2 - y_f^2)$$

So,

$$W_R = \frac{2\text{N}}{2\text{m}}\left(-0.3^2\text{m}^2\right) = 0.09\text{Nm} = 0.09\text{J} \tag{15}$$

(c) We are asked to compute v_f the final velocity reached by the mass after being released. Recall that

$$W = \Delta K, \qquad K_f = \frac{1}{2}mv_f^2$$

Knowing that $K_0 = 0$, we obtain $W = K_f$, i.e.

$$0.09\text{J} = \frac{1}{2}\ 1\text{kg}\ v_f^2$$

from which follows

$$\sqrt{\frac{2 \cdot 0.09\text{J}}{\text{kg}}} = v_f$$

Solving, $v_f = 0.42426406871\frac{m}{s}$.

(d) Upon making contact with the frictioning surface, friction will exert a force contrary to the direction of movement, reducing the acceleration of the object.

Let y_* be the point at which the object has lost half of its velocity, i.e. the point at which $v_* = \frac{1}{2}v_f$. (We still use v_f as the velocity of the object when it loses contact with the spring.) The work of friction, like any other work, obeys

$$
\begin{aligned}
W_F &= \Delta K \\
&= K_* - K_f \\
&= \frac{1}{2} m v_*^2 - \frac{1}{2} m v_f^2 \\
&= \frac{1}{2} m \left(\frac{1}{2} v_f\right)^2 - \frac{1}{2} m v_f^2 \\
&= \frac{1}{8} m v_f^2 - \frac{1}{2} m v_f^2 \\
&= m v_f^2 \left(\frac{1}{8} - \frac{1}{2}\right) \\
&= m v_f^2 \left(-\frac{3}{8}\right) \\
&= 1\text{kg} \cdot 0.18 \frac{\text{m}^2}{\text{s}^2} \cdot -\frac{3}{8} \\
&= \left(0.18 \cdot -\frac{3}{8}\right) \mathbf{J} \\
&= -0.0675 \mathbf{J}
\end{aligned}
$$

(e) When the object stops, its velocity is zero. When the object leaves the spring, its velocity is v_f, as computed before. Therefore, $W = \Delta K = K_s - K_f = -K_f$, with W the work of all the forces involved in moving the object across its path.

That said, $W = |F| \Delta y$, with F the magnitud of all forces involved in moving the object across its path. From this follows that $-K_f = |F| \Delta y$, i.e. $\Delta y = -\frac{K_f}{|F|}$. So we have obtained an expression for the distance travelled.

Now, once the object leaves the spring, the only force which contributes to its movement is the friction of the surface, which acts to stop it. So in fact $F = R$ the magnitud of friction. We know this magnitud is $-\mu_D \left|\vec{N}\right|$, and here $\left|\vec{N}\right| = mg$, counteracting gravity.

$$
\begin{aligned}
\Delta y &= -\frac{K_f}{\mu_D m g} \\
&= \frac{\frac{1}{2} m v_f^2}{(0.2) \cdot m \cdot 9.8\text{m/s}^2} \\
&= \frac{5}{2} \cdot \frac{1}{9.8\text{m/s}^2} \cdot 0.18(\text{m/s})^2 \\
&= 0.04591836734 \cdot \frac{\text{m}^2}{\text{s}^2} \cdot \frac{\text{s}^2}{\text{m}} \\
&= 0.04591836734 \text{ m} \\
&= 4.591836734 \text{ cm}
\end{aligned}
$$

(5) A mass of $m = 5\text{kg}$ is gently layed upon a spring of constant $k = 2\text{N/m}$, which is vertically placed over a horizontal surface.

(a) Describe what happens.

(b) Compute the work done by the spring when the body reaches equilibrium.

(c) Assume the body falls freely from a height of $h = 1\text{m}$ before hitting the spring. Compute how much the spring compresses from its equilibrium position.

(a) When the mass is gently placed upon the object, the force of gravity acts downard pressing the spring. When the spring is displaced from equilibrium, it exerts an upward, contrary force which seeks to restore equilibrium.

(b) The body will reachequilibrium when the magnitude of the forces acting upon it becomes zero. The two forces acting upon the object are gravity and the spring's force. In other words, the body will reach equilibrium if and when

$$-mg = k\Delta y \iff -\frac{5\text{kg} \cdot 9.8\text{m/s}^2}{2\text{N/m}} = \Delta y \iff \Delta y = -24.5\text{m}$$

$\therefore$ The object reaches a state of equilibrium when displaced 24.5 meters below the spring's resting point.

The work exerted by the spring upon the objece is non-constant and therefore

$$\begin{aligned}
W_R &= \int_{y=0}^{y=\Delta y} F_R(y)\ dy \\
&= \int_{y=0}^{y=\Delta y} -ky\ dy \\
&= -k\ \frac{y^2}{2}\Big]_0^{\Delta y} \\
&= -k\left(\frac{(-24.5\text{m})^2}{2}\right) \\
&= -2\frac{\text{N}}{\text{m}}(300.125\text{m}^2) \\
&= -600.25\text{J}
\end{aligned}$$

(c) What is the energy of our system in the initial setting? The mass will begin to fall immediately, but at the initial instant it stands with zero velocity and only potential energy $U_m^2 = mg\Delta h$. Since the spring is at equilibrium, its energy is zero.

Note. Recall that the potential energy of a spring is $U_R = \frac{k}{2}x^2$ with x its distance from equilibrium.

In short,

$$E_0 = mg\Delta h$$

At the final state, the mass has compressed the spring to a maximum distance from equilibrium, standing still. Since it stands still, its velocity is zero and therefore so is its kinetic energy. Its potential energy will be $U_m^f = mg\Delta h'$ with $\Delta h'$ its distance from equilibrium in the final state. On the other hand, the spring will now be compressed and still, thereby containing potential energy $U_R^f = \frac{k}{2}\Delta h'$. So

$$E_f = mg\Delta h' + \frac{k}{2}(\Delta h')^2$$

Due to preservation of energy, we have

$$mg\Delta h = mg\Delta h' + \frac{k}{2}(\Delta h')^2$$

Now, if the origin of our system is at the equilibrium position, $\Delta h = 1m$ and $\Delta h' := x$ is unknown, producing the equation

$$\begin{aligned} mg = mgx + \frac{k}{2}x^2 \iff& \quad \frac{k}{2}x^2 + mgx - mg = 0 \\ \iff& \quad \frac{k}{2\phi}x^2 + x - 1 = 0 \end{aligned}$$

with $\phi := mg = 49\text{N}$. So we found a quadratic equation with solutions

$$x = \frac{\phi}{k}\left(-1 \pm \sqrt{1 + \frac{2k}{\phi}}\right)$$

Now suffices to see that

$$\frac{\phi}{k} = \frac{49}{2} = 24.5$$

from which follows that $2k/\phi = 2 \cdot \left(\frac{1}{24.5}\right) = 0.08163265306$. So we obtain

$$x = 24.5\left(-1 + \sqrt{1.08163265306}\right) = 24.5\,(0.04001569846) = 0.98038461227$$

Since x is by definition in units of meters, we have found that $x = \Delta h' \approx 0.98\mathrm{m}$. So the spring is compressed approximately 0.98 meters below its equilibrium point by the free falling mass.

(6) Un embalaje de masa $m = 250$kg está colgado de un cable de largo $L = 10$m. Se lo mueve hacia un lado apartándolo de la vertical una longitud $l = 1$m y se lo sostiene allí.

(a) ¿Cuál es la fuerza necesaria para mantener el embalaje en esa posición?

(b) ¿Se hace trabajo para sostenerlo allí?

(c) ¿Se hizo trabajo para moverlo de lado? ¿Cuánto?

(d) La tensión del cable, ¿efectúa algún trabajo?

(a) Sea F_l la fuerza necesaria para mantener el embalaje una longitud $l = 1$m a la izquierda. El embalaje es afectado por la gravedad (en su componente angular) y la tensión del cable, que son fuerzas paralelas y opuestas en dirección.

Notemos que al desplazar el objeto a la izquierda, su movimiento describe un arco y la soga termina formando un ángulo θ respecto a la vertical, satisfaciendo:

$$L \sin\theta = l \iff \sin\theta = \frac{l}{L} \iff \theta = \arcsin\left(\frac{1}{10}\right)$$

La tensión del cable apunta hacia el origen del cable satisfaciendo

$$T_x = T\sin\theta, \qquad T_y = T\cos\theta$$

Como el objeto está en equilibrio, el componente horizontal de la tensión debe contrarrestar el componente horizontal de la fuerza F_l, y el componente vertical debe contrarrestar la gravedad. Es decir, $T_x = F_l, T_y = mg$. Pero

$$T_y = mg \Rightarrow mg = T\cos\theta \Rightarrow T = \frac{mg}{\cos\theta}$$

de lo cual se sigue que

$$F_l = T_x = T\sin\theta = \frac{mg}{\cos\theta}\sin\theta = mg\tan\theta$$

$\therefore$ La magnitud de la fuerza que mantiene al objeto en la posición es $mg\tan\theta$, y como dicha fuerza solo tiene un componente horizontal, esto la define completamente.

(b) No. El trabajo de cualquier fuerza involucrada será la magnitud de dicha fuerza por el desplazamiento del cuerpo. Pero estamos asumiendo que el cuerpo está sostenido en un punto fijo, i.e. su desplazamiento es cero.

(c) El trabajo realizado para moverlo es dado por $W = \Delta K = \Delta U$, puesto que en el momento inicial y final la energía kinética es cero. La energía potencial del objeto es dada por $mg\Delta h$ con Δh la altura alcanzada por la masa (asumiendo que en el reposo la altura es cero).

Del mismo modo que $L\sin\theta$ describe la distancia horizontal que recorre el objeto respecto al punto en que el cable está colgado, $L\cos\theta$ describe la distanca vertical (respecto a dicho punto). Se sigue que

$$
\begin{aligned}
W &= \Delta U \\
&= U_f - U_0 \\
&= mg(L - L\cos\theta) - 0 \\
&= mgL(1 - \cos\theta) \\
&= mgL(1 - \cos\left(\arcsin\frac{1}{10}\right)) \\
&= mg \cdot 0.0501256289\text{m} \\
&= 250\text{kg} \cdot 9.8\text{m/s}^2 \cdot 0.0501256289\text{m} \\
&= 122.807790805\text{J}
\end{aligned}
$$

(7) Un carro de montaña rusa sin fricción comienza en un punto A con velocidad v_0. Asuma que el carro puede ser considerado puntual y que siempre se mantiene en la vía.

(a) Calcule la energía total inicial del sistema.

(b) ¿Con qué velocidades llegará a B y a C?

(c) Calcule la desaceleración cnostante que debe aplicarse en D para que se detenga en E.

(a) Notemos que la única fuerza involucrada en el sistema es la gravedad $\vec{G}$. De esto se sigue que la energía del sistema es

$$E = K + U = \frac{1}{2}mv_0^2 + mgh$$

(b) El trabajo realizado por la gravedad en el intervalo $x \in [x_A, x_B]$ es $mg\Delta y = mg(h - h) = 0$. Se sigue que $\Delta K = 0$, i.e. $K_A = K_B$. Pero entonces se sigue fácilmente que $v_A = v_B$. Es decir, la velocidad al llegar a B será la misma velocidad con la que se empezó en A.

Respectot a C, tenemos que el trabajo de la gravedad es $mg(h - h')$, con h' la altura del punto C. Se sigue que

$$\begin{aligned} & mg(h - h') = \frac{1}{2}mv_C^2 - \frac{1}{2}mv_A^2 \\ \iff \quad & g\Delta h + \frac{1}{2}v_A^2 = \frac{1}{2}v_C^2 \\ \iff \quad & \sqrt{2g\Delta h + v_A^2} = v_C \end{aligned}$$

con $\Delta h = h - h'$.

(c) Asumamos que se detiene en E. Entonces la energía en E es cero (no hay altura, i.e. no hay energía potencial gravitatoria; no hay movimiento, i.e. no hay energía cinética). La energía en D, según el mismo razonamiento, es estrictamente cinética: $E_D = K_D = \frac{1}{2}mv_D^2$.

Como el objeto se detuvo, se aplicó una fuerza F desaceleradora a partir de D, y el trabajo de dicha fuerza es $W_F = F \cdot L$ con L el desplazamiento vertical entre D y E. Pero por el teorema del trabajo y la energía, obtenemos entonces

$$F \cdot L = K_E - K_D = -K_D = -\frac{1}{2}mv_D^2$$

Por lo tanto,

$$F = -\frac{mv_D^2}{2L}$$

Pero $F = ma$ la aceleración producida por F (y lo que deseamos conocer). Por ende,

$$ma = -\frac{mv_D^2}{2L} \iff a = -\frac{v_D^2}{2L}$$

Pero v_D es desconocido. Sin embargo, podemos expresar v_D^2 en términos de v_0 si notamos que

$$W = K_A - K_D \iff -mgh = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_D^2 \iff v_D^2 = 2gh + v_0^2$$

Por lo tanto,

$$a = -\frac{2gh + v_0^2}{2L} = -\frac{gh}{L} - \frac{v_0^2}{2L}$$

(8)

Sistema de coordenadas: origen en B, eje x crece hacia la derecha, eje y crece hacia abajo.

(a) La masa se mueve de B a C sin recibir fuerza alguna más que la fricción $\vec{R}$. Se satisface que $v_C = 0$. La energía del sistema una vez que el objeto alcanza la región horizontal del recorrido es dada por la energía kinética del movimiento y la energía del rozamiento. En particular, sabemos que

$$W = K_C - K_B = -K_B = -\frac{1}{2}mv_B^2$$

Pero sabemos que la fricción hace el siguiente trabajo negativo:

$$W = -\left|\vec{R}\right| \cdot d$$

y $\left|\vec{R}\right| = |N|\,\mu_d = \mu_d mg$. Por ende, $W = -\mu_d mg\ d$ y obtenemos

$$-\mu_d mgd = -\frac{mv_B^2}{2} \iff \mu_d = \frac{v_B^2}{2gd}$$

Tomando $v_B = 3.6$m/s, $g = 9.8$m/s^2, $d = 2.7$m, obtenemos

$$\mu_d = 0.24489795918 \approx 0.245$$

(b) El trabajo realizado por la gravedad contra la fricción es $W_G = \left|\vec{G}\right| \Delta y = mgy$, donde y es la altura del punto A en nuestro sistema de coordenadas. Como A está a la altura del centro de la circunferencia, sabemos que $y = R = 1.5$m. Por lo tanto, el trabajo realizado por la gravedad es $mg \cdot 1.5$m.

El trabajo neto del sistema es

$$W = W_R + W_G$$

y $W = \Delta K = \frac{1}{2}mv_B^2$. Por ende,

$$W_G = \frac{1}{2}mv_B^2 - W_R$$

Por lo tanto,

$$\begin{aligned}
&\frac{1}{2}mv_B^2 = W_R + W_G \iff \\
\iff \quad & W_R = \frac{1}{2}mv_B^2 - W_G \\
\iff \quad & W_R = \frac{1}{2}mv_B^2 - mg \cdot 1.5\text{m} \\
\iff \quad & W_R = 1\text{kg}\left(6.48\frac{\text{m}^2}{\text{s}^2} - 14.7\frac{\text{m}^2}{\text{s}^2}\right) \\
\iff & W_R = -8.22\text{J}
\end{aligned}$$

Como el trabajo de la fricción es de -8.22J, el trabajo que se realiza para contrarrestarla es de esta misma magnitud.

16 Electroestática

16.1 Electric charges and Coulomb's law

The concept of electric force, very much like gravitational force, is rooted only in observation. It is the case in nature that certain objects interact with each other at a distance. Gravitational force describes one such kind of interaction—particularly one that is always attractive, conservational, proportional to the masses, etc. Electric force describes another such kind of interaction, one that is sometimes attractive and sometimes repulsive, that is independent of the masses involved.

We introduce the concept of *charge* to account for two observed phenomena. Firstly, that electrostatic interactions can be repulsive or attractive, where we say that identical charges attract and opposite charges repel each other. Secondly, that the strength of electrostatic interactions varies depending on the interacting objects.

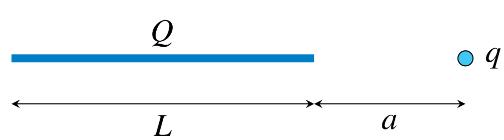
Coulomb's law describes the electric force between two charges:

$$\vec{F}_E = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r} \qquad \text{(Coloumb's law)}$$

where r is the distance between the two charges, $\hat{r}$ is a unitary vector in the direction of the force, and ϵ_0 is a constant termed the *permittivity of free space* or *electric constant*:

$$\epsilon_0 = 8.854187817 \times 10^{12} \qquad \text{(Permittivity of free space)}$$

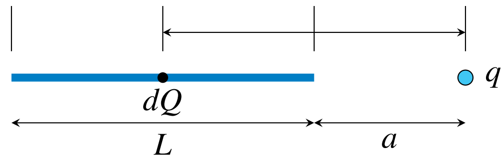
The force exerted by multiple charges on another charge is additive. To understand this, consider the diagram below, where a series of charges $q_0, \ldots, q_n$ are arranged in a line of distance L and a charge q is at a distance a from the end of the line. We use Q to denote the total charge $\sum q_i$.



In such a situation, we use *charge density* to denote Q/L. Let x be the distance from an arbitrary point in the segment and the charge q. Then

$$dQ = \frac{Q}{L}\,dx$$

is the amount of charge in a segment of the line charge of length dx. Since we can make dx arbitrarily small, dQ can be taken to represent a point charge, so that we may use Coloumb's law.



$$d\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{dQ \cdot q}{x^2}\hat{r}$$

Hence, since the force of multiple charges is additive,

$$\begin{aligned}\vec{F} = \int_a^{a+L} dF &= \int_a^{a+L} \frac{1}{4\pi\epsilon_0}\frac{dQ \cdot q}{x^2}\hat{r} \\ &\frac{\hat{r}}{4\pi\epsilon_0}\int_a^{a+L}\frac{q \cdot Q/L}{x^2}\,dx\end{aligned}$$

Note that $\hat{r}$ is constant: it always points either horizontally towards q or horizontally "against" q (depending on whether Q and q are identically charged or not). It is only for this reason that it can be factored out of the integral. Now, solving the integral, we obtain

$$\begin{aligned}\vec{F} = \frac{\hat{r}}{4\pi\epsilon_0}\int_a^{a+L}\frac{q \cdot Q/L}{x^2}\,dx &= \frac{\hat{r}q \cdot Q}{L4\pi\epsilon_0}\int_a^{a+L}\frac{1}{x^2}\,dx \\ &= \hat{r} \cdot \frac{1}{4\pi\epsilon_0}\frac{qQ}{a(a+L)}\end{aligned}$$

16.2 Electric fields

Coloumb's law describes forces acting between two distant charges. However, we can model the situation differently. Since any charged object Q will exert electrostatic force to other nearby charged objects, we can think of Q as creating a *field* around it. Nearby objects are within this field at a certain distance from Q, and the force in the field (due to Coloumb's law) will be inversely proportional to the square of said distance.

Let q by a charge nearby Q, or within the field of Q. Then

$$\vec{E} = \frac{\vec{F}}{q} \tag{Electric field}$$

Note that $\vec{E}$ is nothing but the *normalized* force which Q exerts to q. We can think of this as: the force which Q would exert on an object of charge 1+. Note that the dimensions the electric field are newtons over coulombs.

If we have multiple charges scattered in space, the electric field generated by all of them is the sum of their individual fields:

$$\vec{E} = \frac{1}{4\pi\epsilon 0} \sum_i \frac{q_i}{r^2} \hat{r}_i$$

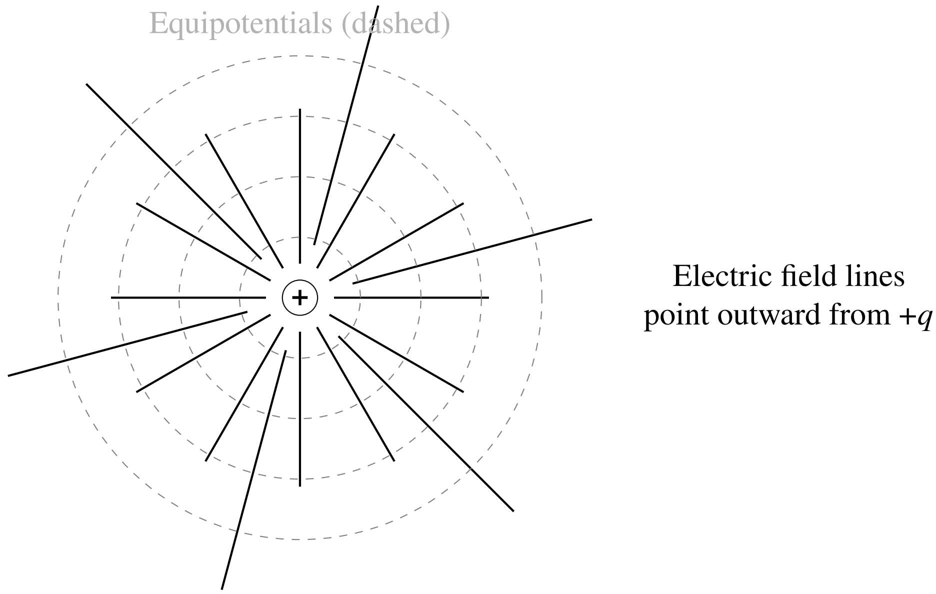
Note that this is a vector sum. If the charges are smeared out in a continuous distribution, the summation evolves into an integral:

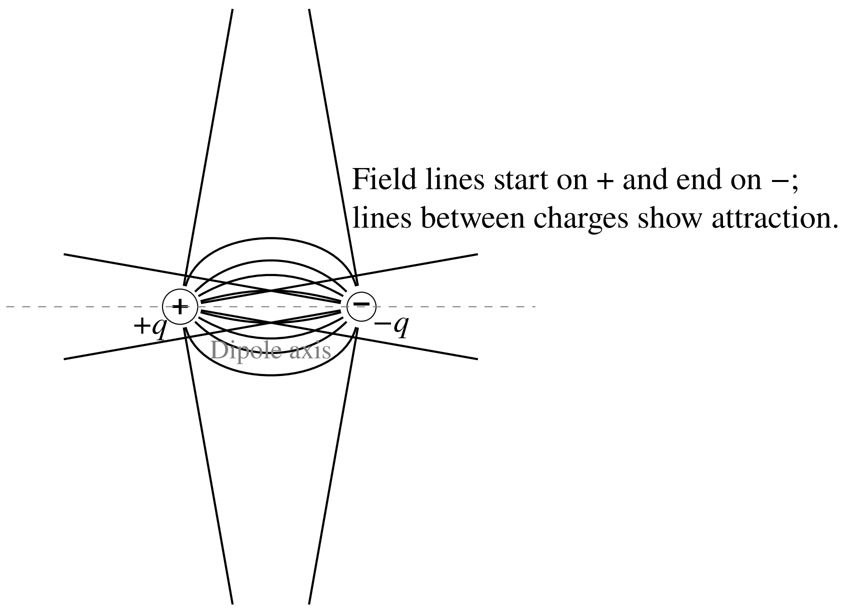
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r^2} \hat{r}$$

with r the distance between each dq and the location of interest. If said distribution is uniform, and particularly linear, then the charge on each segment is $\lambda = Q/L$ with L the length of the segment in question. If it is a surface, $\sigma = Q/A$ with A the area is the charge on each point. If it is a 3D distribution, $\varrho = Q/V$ is the charge per volume. So:

- For linear distributions of charge, $dq = \lambda$ dl
- For surface distributions, $dq = \sigma$ ds
- For volumetric distributions, $dq = \varrho$ dv

where dl, ds and dv are the infinitesimal regions of a line, a surface or a volume, respectively.





16.3 Electric potential and electric potential energy

Electric potential energy is the negative of the work needed to move a charge *against* an electric field and into some reference position. Electric potential depends on the distance of the movement, the magnitude of the charge, and the strengt of the electric field:

- Weak electric field, long distance → medium electric potential
- Strong electric field, long distance → High electric potential

- Weak electric field, short distance → Low electric potential
- Weak electric field, high distance → Medium electric potential

(Of course the items above are only schematic.)

Electric potential, also called **voltage**, is the *difference* in potential energy per unite charge between two locations in an electric field. Why per unit charge? Well, we have already established that