

Logics

FAMAF - UNC

SLP

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1 Functions

A function $f : A \mapsto B$ is a set of tuples $\{(a, b) : a \in A \text{ and } b \in B\}$. The domain \mathcal{D}_f and image I_f of a function have the usual definitions. The kernel of a function is

$$\ker(f) = \{(a, b) \in \mathcal{D}_f^2 : f(a) = f(b)\}$$

From this follows that a function f is injective—that it maps to each element in \mathcal{D}_f a distinct element in the range—iff $\ker(f) = \{(a, b) \in \mathcal{D}_f^2 : a = b\}$.

Given $F : A \mapsto B$ and $S \subseteq A$, we will use $F(S)$ to denote $\{F(a) : a \in S\}$.

2 Equivalence relations

Definition 1 Given a set A , a binary relation over A is a subset of A^2 .

Observe that \emptyset is a binary relationship over any set A . We use $A \propto B$ to say " A is a binary relation over B ". The notation aRb is a shorthand for $(a, b) \in R$.

Observe that $R \propto A$ and $A \subseteq B$ implies $R \propto B$. Many properties of the \propto relation follow from the properties of the \subseteq relation. The properties that a binary relation R may follow are the following, given any $R \propto A$:

- \propto is reflexive: aRa for any $a \in A$.
- \propto is transitive: aRb and bRc implies aRc for any $a, b, c \in A$.
- \propto is symmetric: $aRb \Rightarrow bRa$ for any $a, b \in A$.
- \propto is anti-symmetric: aRb and bRa implies $a = b$ for any $a, b \in A$.

Whether and which of these properties hold depends on the sets in question.

Example. Consider $R = \{(x, y) \in \mathbb{N}^2 : x \leq y\}$. Then $R \propto \mathbb{N}$ and $R \propto \omega$. However, R is reflexive with respect to \mathbb{N} but not with respect to ω , because $(0, 0) \notin R$.

Definition 2 An equivalence relation over A is a binary relation $R \propto A$ s.t. R is reflexive, transitive and symmetric with respect to A .

We write $R \ddot{\propto} A$ to say R is an equivalence relation over A .

Problem 1 Determine true or false for the following statements.

(1) Given X a set, then $R = \emptyset$ is a binary relation over X that is transitive, symmetric and anti-symmetric with respect to X .

We know $\emptyset \propto X$ for any X . Recall that xRx is a shorthand for $(x, x) \in R$ where R is a binary relation. In particular, $(x, x) \notin \emptyset$ for any $x \in X$, so \emptyset is not reflexive. The same applies to all other properties. The statement is false.

(2) If $R \propto X$ and R is not anti-symmetric with respect to X , then R is symmetric with respect to X .

The statement is false. Consider $R = \{(1, 2), (2, 1), (5, 3)\}$ where $R \propto \omega$. Evidently R is not anti-symmetric over ω , because $1R2$ and $2R1$ and yet $2 \neq 1$. However, it is also not symmetric, because $5R3$ and $\neg(3R5)$.

(3) If A a set then $A^2 \propto A$.

Trivially true, since $A^2 \subseteq A^2$.

(4) If $R = \{(x, y) \in \mathbb{N}^2 : x = y\}$ then $R \ddot{\sim} \omega$.

By definition xRx holds. Evidently, $xRy \Rightarrow yRx$ so it is symmetric. Furthermore, $xRy \wedge yRz \Rightarrow xRz$. The statement is true.

(5) If $R \ddot{\sim} B$ and $A \subseteq B$ then $R \ddot{\sim} A$.

We need not even impose the constraint of an *equivalence* relation since the statement is false for any binary relation. In fact, $R \subseteq B^2$ and $A \subseteq B$ does not imply $R \subseteq A^2$. For example, $R = \{(1, 2), (2, 3), (3, 4)\} \subseteq \omega^2$ and $A = \{1, 2\} \subseteq \omega$. However, $R \not\subseteq A^2$. Since the statement is false for all binary relations, and equivalence relations are a form of binary relation, the statement is false.

Definition 3 *The equivalence class of $a \in A$ with respect to equivalence relation $R \ddot{\sim} A$ is*

$$[a]_R = \{b \in A : aRb\}$$

.

We sometimes write simply $[a]$ if the equivalence relation R is understood by the context. We may also write a/R to denote the equivalence class $[a]_R$.

Example. Let $R = \{(x, y) \in \mathbb{Z}^2 : x \text{ has the same parity than } y\}$. Then $[2]$ denotes the set of all numbers that have the same parity than 2; this is, all even numbers.

If $R = \{(x, y) \in \mathbb{Z}^2 : 5 \mid x - y\}$ then $[0] = \{5t : t \in \mathbb{Z}\}$.

Problem 2 *If $R \ddot{\sim} A$ and $a \in A$ then $a \in [a]$.*

True because R is reflexive: $aRa \Rightarrow a \in [a]$ by definition.

Problem 3 *If $R \ddot{\sim} A$ and $a, b \in A$, then $aRb \iff [a] = [b]$.*

Assume aRb . Then, for any $x \in [b]$, transitivity tells us aRx . And because $aRb \Rightarrow bRa$ we have, via the same argument, that for any $y \in [a]$ bRy . Of course,

$$\langle \forall x : x \in A : x \in B \rangle \wedge \langle \forall y : y \in B : y \in A \rangle \Rightarrow A = B$$

So $[a] = [b]$. ■

If we assume $[a] = [b]$ then of course $aRx \iff bRx$. By symmetry we have xRa and then by transitivity $bRx \wedge xRa \Rightarrow bRa \Rightarrow aRb$. ■

Problem 4 Let $R \dot{\sim} A$ and $a, b \in A$. Then $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

Assume $[a] \cap [b] \neq \emptyset$ and $[a] \neq [b]$, which is the negation of the statement we want to prove. Since $[a] \neq [b]$ we cannot have aRb , due to what was proven in the previous exercise. However, since $[a] \cap [b] \neq \emptyset$ there is some $z \in A$ s.t. aRz and bRz . However, $bRz \Rightarrow zRb$ and then aRb by transitivity. This is a contradiction. Then the statement is true.

Definition 4 We use A/R to denote $\{[a] : a \in A\}$ and call this set the quotient of A by R .

In other words, given $R \dot{\sim} A$, the quotient of A by R is the set of all equivalence classes. For example, if $R = \{(x, y) \in \mathbb{R}^2 : x = y\}$ then $\mathbb{R}/R = \{\{x\} : x \in \mathbb{R}\}$.

Definition 5 If $R \dot{\sim} A$, we define $\pi_R : A \mapsto A/R$ defined as $\pi_R(a) = a/R$ for every $a \in A$. We call this function the **canonic projection** with respect to R .

Theorem 1 If $R \dot{\sim} A$, then $\ker(\pi_R) = R$. This entails that π_R is injective iff $R = \{(x, y) \in A^2 : x = y\}$.

Problem 5 Let $R = \{(x, y) \in \mathbb{Z}^2 : 5 \mid x - y\}$. Find \mathbb{Z}/R .

Observe that $(5, 0), (6, 1), (7, 2), (8, 3), (9, 4) \in R$. From that point onward (and from $(5, 0)$ downward) we deal with the same equivalence class.

More formally, $[5] = \{5t : t \in \mathbb{Z}\}$, $[6] = \{1, 6, 11, \dots\} = \{5(t+1) : t \in \mathbb{Z}\}$. In general, if $A(t) = \{5t : t \in \mathbb{Z}\}$, then

$$\{A(0), A(1), \dots, A(4)\} = \mathbb{Z}/R$$

Observe that this can be generalized. If $R = \{(x, y) : z \mid x - y\}$ for some fixed $z \in \mathbb{N}$, then

$$\{\{zt : t \in \mathbb{Z}\}, \{z(t+1) : t \in \mathbb{Z}\}, \dots, \{z(t+z-1) : t \in \mathbb{Z}\}\} = \mathbb{Z}/R$$

always with z elements.

Problem 6 Let $R = \{(x, y) \in \mathbb{N}^2 : x, y \leq 6\} \cup \{(x, y) \in \mathbb{N}^2 : x > 6 \wedge y > 6\}$. Prove that R is an equivalence relation over \mathbb{N} and find \mathbb{Z}/R . How many elements does it have?

(1) Let $(a, b) \in R$. We have two possible cases. If (a, b) is s.t. $a, b \leq 6$, then if bRc for some $c \in \mathbb{N}$ we must have $c \leq 6$. This implies $(a, c) \in R$, which means the relation is transitive. A similar argument shows transitivity applies to the case $a, b > 6$. It is very simple to show that the relation is reflexive. To show it is symmetric, simply observe that $(a, b) \in R$ implies either $a, b \leq 6$ or $a, b > 6$ which implies $(b, a) \in R$.

(2) Evidently, $6R5, 6R4, 6R3, \dots$, and $7R8, 7R9, 7R10, \dots$. Thus, the equivalence relation R over \mathbb{Z} has a quotient space

$$\mathbb{Z}/R = \{\{z \in \mathbb{Z} : z \leq 6\}, \{z \in \mathbb{Z} : z > 6\}\} = \{6/R, 7/R\}$$

Problem 7 Give true or false for the following statements.

(1) If R an equivalence relation over $A \neq \emptyset$, then $|A/R| = 1 \iff R = A \times A$.

(\Leftarrow) It is easy to see that $R = A \times A$ is by definition the equivalence relation where any $a \in A$ is equivalent to any $b \in A$. So $|R/A| = 1$.

(\Rightarrow) Let $R = A \times A$. Assume $|A/R| \neq 1$. Since $A \neq \emptyset, A \times A \neq \emptyset$ and $|A/R| > 0$. So we must have $|A/R| > 1$. This implies there is some $a, b \in A$ s.t. $\neg(aRb)$ (otherwise a unique equivalence class would exist). But then $(a, b) \notin A^2$, which contradicts the definition of Cartesian product. Then if $R = A \times A, |A/R| = 1$.

In conclusion, the statement is true.

(2) If $R \ddot{\sim} A$ then $A/R = \{a/R : a \in A\}$.

False. By definition: $A/R = \{a/R : a \in A\} \neq \{\{a/R\} : a \in A\}$

(3) Let $R \ddot{\sim} A$ with $A = \{1, 2, 3, 4, 5\}$. Then $|\{i/R : i \in A\}| = 5$.

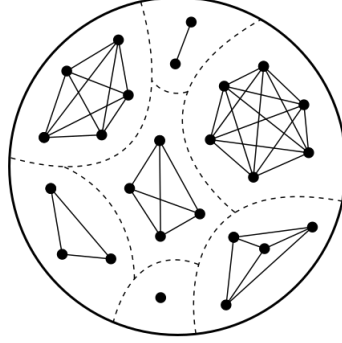
False. It depends on R , which is unspecified. E.g. we have shown that if $R = A^2$ then $|A/R| = 1$.

(4) $A/\{(x, y) \in A^2 : x = y\} = A$.

False, but it is made look like it is true. By definition of $R = \{(x, y) \in A^2 : x = y\}$ we have $x, y \in A \wedge x \neq y \Rightarrow \neg(xRy)$. So $a \in A$ belongs to a single equivalence class a/R . Then $A/R = \{\{a\} : a \in A\} \neq A$.

(5) Let $R \ddot{\sim} A$ and $C \subseteq A, C \neq \emptyset$. Assume xRy for any $x, y \in C$. Then $C \in A/R$.

Figure 1: Graph of a quotient space with 7 equivalent classes. Any two connected vertices denote equivalent elements of a set.



The statement is false. Observe that

$$c/R = C \cup \{x \in A : x \notin C \wedge cRx\}$$

If the second set is non-empty then $C \notin A/R$.

Counter example. Let $A = \{1, 2, 3, 4, 5\}$ and $C = \{1, 2\}$, satisfying the constraints of the problem. If $(1, 3) \in R$ and we assume no non-reflexive relations other than $(1, 2), (1, 3)$ exist, then $A/R = \{\{1, 2, 3\}\} \not\supseteq C$.

Problem 8 Let $R \ddot{\subset} A$. Prove (1) that $\ker(\pi_R) = R$ and (2) π_R is injective iff $R = \{(x, y) \in A^2 : x = y\}$.

(1) By definition $\pi_R(a) = a/R$ which entails that $\ker \pi_R = \{(a, b) : a/R = b/R\}$. Of course $a/R = b/R \iff aRb$. Then $\ker(\pi_R) = \{(a, b) : aRb\} = \{(a, b) : (a, b) \in R\} = R$.

(2) (\Rightarrow) Assume π_R is injective. Then no two elements in the domain map to the same element. Then $\pi_R(a) \neq \pi_R(b)$ for all $a, b \in A, a \neq b$, which entails $a/R \neq b/R$ for all $a, b \in A, a \neq b$. Then each element is only equivalent to itself. Then $R = \{(a, b) \in A^2 : a = b\}$.

(\Leftarrow) Assume $R = \{(a, b) \in A^2 : a = b\}$. Then $\neg(aRb)$ for any $a, b \in A, a \neq b$. Then $\pi_R(a) \neq \pi_R(b)$ for all $a, b \in A, a \neq b$. Then π_R is injective.

2.1 Partitions and equivalence

A partition \mathcal{P} of a set A is a set s.t. every $P \in \mathcal{P}$ is a subset of A , $P_1 \cap P_2 = \emptyset$ for any $P_1, P_2 \in \mathcal{P}, P_1 \neq P_2$; and $\bigcup_{P \in \mathcal{P}} P = A$.

Given a partition \mathcal{P} of a set A , a valid binary relation is

$$R_{\mathcal{P}} = \{(a, b) : a, b \in S \text{ for some } S \in \mathcal{P}\}$$

Observe that $R_{\mathcal{P}}$ is an equivalence relation. First of all, $aR_{\mathcal{P}}a$ because a is always in the same partition than a . Furthermore, if $aR_{\mathcal{P}}b$ and $bR_{\mathcal{P}}c$ then a and c are in the same partition. Lastly, if a is in the same partition than b , then b is in the same partition than a (symmetry).

Furthermore, if $R \ddot{\sim} A$ is an arbitrary equivalence relation, then A/R is a partition of A . To each element $a \in A$ corresponds some a/R that *at least* contains a ; from this follows trivially that $\bigcup_{a \in A} a/R = A$. Furthermore, if $a/R \neq b/R$ for some $a, b \in A$, then $a/R \cap b/R = \emptyset$ —otherwise, some element $c \in A$ equivalent to a and b should exist, but this would contradict the hypothesis that a and b are not equivalent. That $a/R \subseteq A$ for every $a \in A$ follows trivially from the definition of equivalence class.

Theorem 2 *Let A an arbitrary set, \mathcal{P}_A the set of all partitions of A and \mathcal{R}_A the set of all binary equivalence relations over A . Then*

$$\begin{array}{ll} \mathcal{P}_A \mapsto \mathcal{R}_A & \mathcal{R}_A \mapsto \mathcal{P}_A \\ \mathcal{P} \mapsto R_{\mathcal{P}} & R \mapsto A/R \end{array}$$

are bijections one the inverse of the other.

Problem 9 *Say true, false or imprecise the following statements.*

(1) If \mathcal{P} a partition of X and $x \in X$, then $x/\mathcal{P} \in \mathcal{P}$.

Imprecise. \mathcal{P} is a partition, not a binary relation, and thus the expression x/\mathcal{P} is undefined.

(2) $\mathcal{P} = \{1, 3/2, 4/5, 6\}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$.

Imprecise. The expression $3/2, 4/5$, etc. are undefined.

(3) If \mathcal{P} a partition of X , then $\mathcal{P} \cap X = \emptyset$.

The statement is true. The set \mathcal{P} contains *sets* of elements of X ; the set X contains elements of X . Therefore, each $P \in \mathcal{P}$ is of a different type than each $x \in X$.

(4) If $R \dot{\sim} A$, then $A \cap A/R = \emptyset$.

We know A/R is a partition of A , and in the previous problem we have already stated that $A \cap \mathcal{P} = \emptyset$ for any partition \mathcal{P} of A . So the statement is true.

(5) If $R \dot{\sim} A$ and there is a bijection between A and A/R , then $R = \{(x, y) \in A^2 : x = y\}$.

The statement is false. Consider $A = \mathbb{N}$ and R the equivalence relation s.t. A/R is the partition

$$\{\{1\}, \{2, 3\}, \{4, 5, 6\}, \{7, 8, 9, 10\}, \dots\}$$

Then $F(1) = \{1\}, F(2) = \{2, 3\}, F(3) = \{4, 5, 6\}, \dots$ is a bijection.

It is interesting to study the finite case, however. If $A = \{a_1, \dots, a_n\}$ a finite set, and F is bijective, we must have

$$F(a_1) = X_1, \dots, F(a_n) = X_n$$

with $X_i \neq X_j$ for $i, j \in [1, n]$. In other words, $|A/R| = |A|$, which implies A/R is a partition of A into singleton sets. And because every element must be equivalent to itself, $A/R = \{\{a_1\}, \dots, \{a_n\}\} \Rightarrow R = \{(x, y) \in A^2 : x = y\}$.

2.2 Functions with domain A/R

In general, defining $f : A/R \mapsto B$ leads to ambiguity. For example, if we define $f(a/R) = f([a]) = a^2$ and R is the relationship "has the same parity", then the fact that $[2] = [4]$ would lead us to expect $f([2]) = 4 = f([4]) = 16$.

Notwithstanding, one of the fundamental ideas of modern algebra relates to a function of precisely this form:

Theorem 3 *If $f : A \mapsto B$ is onto, then $\bar{f}(a/\ker f) = f(a)$ defines a bijection $\bar{f} : A/\ker f \mapsto B$.*

Proof. (Is a function) Observe that $\bar{f}(a/\ker f) = f(a)$ is uniquely determined for any $a \in A$.

(Injective) Let $a_1, a_2 \in A$ arbitrary elements with $a_1/\ker f \neq a_2/\ker f$. Assume $\bar{f}(a_1) = \bar{f}(a_2)$. Then $f(a_1) = f(a_2)$, which entails $(a_1, a_2) \in \ker f$, which contradicts the assumption. Then \bar{f} is injective.

(*Surjective*) Let $b \in B$ an arbitrary element. Since f is surjective, $b = f(a)$ for some $a \in A$. From this follows $b = \bar{f}(a/\ker f)$.

Since \bar{f} is injective and surjective, \bar{f} is a bijection.

The theorem above guarantees, for any surjective f , the existence of a mapping from the quotient space $A/\ker f$ onto I_f .

Problem 10 Say true, false or imprecise for the following statements.

(1) Let $R = \{(x, y) \in \mathbb{Z}^2 : 2 \mid x - y\}$. The equation $f(n/R) = \frac{1}{n^2+1}$ correctly defines a function.

False. Observe that

$$\mathbb{Z}/R = \{\{z \in \mathbb{Z} : z \text{ is even}\}, \{z \in \mathbb{Z} : z \text{ is odd}\}\}$$

We would then expect $f(0/R) = f(2/R) \iff 1 = \frac{1}{5}$. (\perp)

(2) If $R \propto A$ then $f : A/R \mapsto A$ defined as $f(a/R) = a$ is onto.

Imprecise because f is not necessarily a function and hence we cannot say it is onto.

3 Partial orders

Definition 6 If $R \propto A$ is reflexive, transitive and anti-symmetric, then it is a partial order.

We use \leq to denote the binary relation that is a partial order. Because we define \leq as a binary relation, we must emphasize that \leq denotes a set of 2-uples. Furthermore, $<$ denotes $\{(a, b) \in \leq : a \leq b \wedge a \neq b\}$.

Definition 7 Let \leq be a partial order over A . If $a < b$ and there is no z s.t. $a < z$ and $z < b$, then we write $a < b$ and read "b covers a" or "a is covered by b".

Observe that $<$ is itself the binary relation

$$\{(a, b) \in A^2 : a < b \wedge \neg(\exists z \in A : a < z \wedge z < b)\}$$

Definition 8 We say \leq is a total order over A if it is a partial order s.t. $x \leq y$ or $y \leq x$ for any $x, y \in A$.

Partially or totally ordered sets are pairs (P, \leq) where \leq is a partial or total order (respectively) over P .

3.1 Maximum, minimum, maximal, minimal

Given a poset (P, \leq) , x is a maximum if $a \leq x$ for all $a \in P$. The definition of a minimum is analogous.

Theorem 4 If (P, \leq) a poset, then (P, \leq) has at most one maximum.

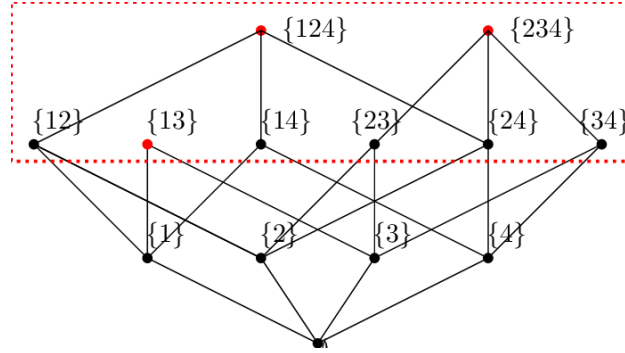
Proof. Assume (P, \leq) is a poset with two distinct maximums x, y . By definition then $x \leq y$ and $y \leq x$. By anti-symmetry we have $x = y$, which is a contradiction.

Given a poset (P, \leq) , we use 1 to denote its maximum and 0 to denote its minimum, if they exist.

A maximal element of a poset (P, \leq) is any $a \in P$ s.t. there is no $b \in P$ s.t. $a < b$. In other words, a maximal element is an element that has no successor in the order. Similarly, $a \in P$ is minimal if there is no $b \in P$ s.t. $b < a$. In other words, a minimal element is one that has no predecessor.

Problem 11 True or false: If (P, \leq) a poset and $a \in P$ is not a maximum, then $a < b$ for some $b \in B$.

False. Consider any poset (P, \leq) that has $n > 1$ maximals m_1, \dots, m_n . Then, for any $i, j = 1, \dots, n$, m_i is not a maximum (because $m_j \not\leq m_i$) but $m_i \not< b$ for all $b \in B$. For an example of a poset with $n = 3$ maximals, see the graph below.



Problem 12 True or false: If (P, \leq) a poset without maximal elements, then P is infinite.

False, but only for a special case. If $P \neq \emptyset$, then it is true that for any $a_1 \in P$ there is some a_2 s.t. $a_1 < a_2$, and this extends to infinity: $a_1 < a_2 < \dots$. However, if $P = \emptyset$, then the only binary relation over \emptyset is $\emptyset^2 = \emptyset$, which gives the poset (\emptyset, \emptyset) . This poset is not only a partial order but a total order; it contains no maximal elements, and yet it is not infinite.

3.2 Supremum and infimum

Let (P, \leq) a poset and $S \subseteq P$. We say $a \in P$ is an upper bound of S in (P, \leq) when $b \leq a$ for all $b \in S$.

Note. $\emptyset \subseteq P$, so what's the deal? Well, every element in \emptyset (which is no element at all) is lesser than any $a \in P$. In other words, every element in P is an upper bound of \emptyset .

Note 2. For any given $S \subseteq P$, many upper bounds may exist (see the previous note).

An element $a \in P$ is called the *supremum* of S in (P, \leq) when two properties hold:

- a is an upper bound of S in (P, \leq)

- For any $b \in P$, if b is an upper bound of S in (P, \leq) , then $a \leq b$.

In other words, a is a supremum if it is the lesser upper bound. It is always unique.

Example. Let (\mathbb{N}, \leq) denote the usual order over \mathbb{N} and $S = \{1, 2, 3\}$. Any natural $n \geq 3$ is an upper bound of S in (\mathbb{N}, \leq) . However, 3 is the only supremum of S .

The definitions of the lower bound and the infimum are analogous. A lower bound of $S \subseteq P$ in (P, \leq) is any $a \in P$ s.t. $a \leq b$ for all $b \in S$. The infimum is the greatest lower bound, or the lower bound a satisfying that any lower bound a' is s.t. $a' \leq a$.

Problem 13 Prove that if a, a' are supremums of S in (P, \leq) , then $a = a'$.

By definition, a, a' are the least upper bounds of S . If $a < a'$ then a' is no longer the least upper bound and hence $a' \leq a$. The same reasoning gives $a \leq a'$. Then, by anti-symmetry, $a = a'$.

The previous problem shows that we can speak of *the* supremum of $S \subseteq P$ for any poset (P, \leq) .

Problem 14 Let (P, \leq) a poset. (1) If $a \leq b$ then $\sup\{a, b\} = b$. (2) Find $\{a \in P : a \text{ is upper bound of } \emptyset \text{ in } (P, \leq)\}$. (3) If the supremum of \emptyset in (P, \leq) exists, it is a minimum element of (P, \leq) .

(1) The statement is trivially true.

(2) Assume $P \neq \emptyset$. Since $\emptyset \subseteq P$ it is correct to speak of the upper bound of \emptyset in (P, \leq) . However, any element $a \in P$ is an upper bound of \emptyset in (P, \leq) . The reason is that to prove $a \in P$ is *not* an upper bound of \emptyset , we should find some $x \in \emptyset$ s.t. $x \not\leq a$ —in other words, because the definition of upper bound involves a universal quantifier, its negation involves an existential, a counter-example. And since \emptyset has no elements, there is no such counter-example. In conclusion,

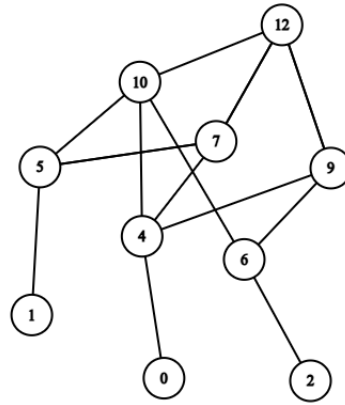
$$\{a \in P : a \text{ is upper bound of } \emptyset \text{ in } (P, \leq)\} = P$$

However, if $P = \emptyset$ (and therefore $\leq = \emptyset = \emptyset^2$), there is no upper bound of \emptyset in (\emptyset, \emptyset) .

(3) The statement is false. The hypothesis gives $P \neq \emptyset$, and we know any $a \in P$ is an upper-bound of \emptyset . If we assume \emptyset has a supremum over (P, \leq) , then it is the least upper-bound. In other words, it is some $m \in P$ s.t. $m \leq a$ for any $a \in P$. By definition, m is a minimum.

Problem 15 Give a finite poset with three elements x_1, x_2, x_3 s.t. (1) $\{x_1, x_2, x_3\}$ is an anti-chain, meaning that $x_i \not\leq x_j$ when $i \neq j$; (2) $\sup\{x_i, x_j\}$ doesn't exist for any $i \neq j$; (3) $\sup\{x_1, x_2, x_3\}$ exists.

A poset that satisfies this can be any that has the following Hasse diagram:



Here, 0, 1, 2 are x_1, x_2, x_3 . The supremum on any pair of them does not exist because each $\{x_i, x_j\}$ has two upper bounds that are not ordered with respect to one another. For example, the two smallest upper bounds of $\{1, 0\}$ are 10, 7. But $10 \not\leq 7$ and $7 \not\leq 10$. However, $\sup\{0, 1, 2\} = 12$.

Problem 16 If (P, \leq) a poset and $a = \sup(S)$ then $a = \sup(S \cup \{a\})$.

The statement is true. Our hypothesis is that $x \leq a$ for any $x \in S$, and $a \leq b$ for any upper-bound b of S . This evidently still holds for $S \cup \{a\}$, because $a \leq a$.

Problem 17 Let (P, \leq) a poset and $a \in P$. Then a is a maximum of (P, \leq) iff $a = \sup(P)$.

(\Rightarrow) Assume a is a maximum of (P, \leq) . Then $x \leq a$ for all $x \in P$. Then a is an upper-bound of P . Furthermore, if there were some $u \in P$ s.t. u is an upper bound and $u < a$, then by definition u would not be an upper-bound of P because $a \not\leq u$. Then a is the least upper bound of P . ■

(\Leftarrow) Assume a is the supremum of P . Then $x \leq a$ for all $x \in P$. The definition of a supremum of $S \subseteq P$ over (P, \leq) requires that the supremum be an element of P . Then $a \in P$. Then by definition a is the maximum of P .

Note. The problem reveals a property; namely, that if $S \subseteq P$ and $\sup(S)$ over (P, \leq) satisfies $\sup(S) \in S$, then this supremum is the maximum of (S, \leq) . Alternatively, this can be stated as follows: *The maximum of a poset (P, \leq) , if it exists, is the supremum m of P over (P, \leq) whenever $m \in P$.*

Problem 18 Give true, false or imprecise.

(1) If (P, \leq) a poset and $S \subseteq P$, then $a = \sup(S)$ in (P, \leq) iff $a \in S$ and $b \leq a$, for all $b \in S$.

False. It is not necessary that $\sup(S) \in S$. Consider the last graph we gave, where $\sup\{0, 1, 2\} = 12$ is not in $\{0, 1, 2\}$.

(2) Let (P, \leq) a poset and $S \subseteq P$ and $a \in P$ an upper bound of S . If a is not the supremum of S , then there is some upper bound b of S s.t. $b < a$.

The statement is false. If a is an upper bound of S but it is not the supremum, it could very well be the case that another upper bound b exists, with $a \not\leq b$ and $a \not\geq b$.

For an example, go at the last graph we showed; imagine the maximum (i.e. 12) does not exist. Then consider that 10 is an upper bound of $\{0, 1\}$ but not a supremum, and yet there is no upper bound b of $\{0, 1\}$ s.t. $10 < b$.

Problem 19 Let $P = \{0\} \cup \{x \in \mathbb{R} : 1 < x \leq 2\}$. Let

$$\leq = \{(x, y) \in P^2 : x \leq y\}$$

Let $S = \{x \in \mathbb{Q} : 1 < x \leq 2\}$. Does S have an infimum over (P, \leq) ?

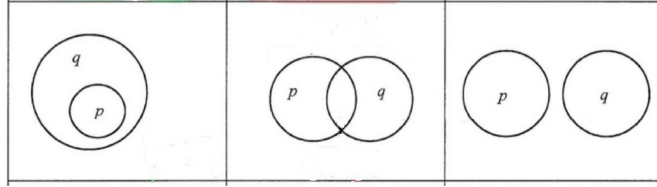
The order is the usual order, but over $P = \{0\} \cup (1, 2]$. The set S (and in fact P as well) has only one lower bound over (P, \leq) ; namely, 0. Observe that 1 is not a lower bound because $1 \notin P$, and there is no such thing as the "first rational number". Since 0 is the *only* lower bound it is also the greatest lower bound.

Problem 20 Say true or false. Let

$$\mathcal{D}((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$$

Let $P = \{\emptyset\} \cup \{\mathcal{D}((x_0, y_0), r) : x_0, y_0 \in \mathbb{R}, r > 0\}$. In the poset (P, \subseteq) , there is always $\inf\{D_1, D_2\}$, for any $D_1, D_2 \in P$.

$\mathcal{D}((x_0, y_0), r)$ is the set of points within a circumference with center (x_0, y_0) and radius r . So P is the set of all disks, including \emptyset . Two disks may be related in one and only one of the ways schematized by the following Venn diagrams:



Formally, for $D_1, D_2 \in P$, the image depicts the following exhaustive and mutually exclusive cases:

- $D_1 \subseteq D_2$,
- $D_1 \cap D_2 \neq \emptyset$ but $D_1 \not\subseteq D_2$
- $D_1 \cap D_2 = \emptyset$.

It is easy to prove that in the first and third cases, there is an infimum. However, consider the case $D_1 \cap D_2 \neq \emptyset$ with $D_1 \not\subseteq D_2$. Let D_3 a disk s.t. $D_3 \subseteq D_1 \cap D_2$ —this is, D_3 is an arbitrary, non-empty lower bound of $\{D_1, D_2\}$. Then, given any arbitrary $(z_1, z_2) \notin D_3$ that lies in $D_1 \cap D_2$, we can define $D_z = \mathcal{D}((z_1, z_2), \epsilon)$, with $\epsilon > 0$ a quantity sufficiently small to guarantee $D_z \cap D_3 = \emptyset$ and $D_z \subseteq D_1 \cap D_2$. It is evident that D_z is a lower bound of $\{D_1, D_2\}$; but since $D_z \not\subseteq D_3$ we cannot say D_3 is the greatest lower bound.

The argument above holds for any lower bound $D_3 \subseteq D_1 \cap D_2$. In general terms, we have shown that, in the case $D_1 \cap D_2 \neq \emptyset$, $D_1 \not\subseteq D_2$, for any lower bound D_3 of $\{D_1, D_2\}$, we can find a lower bound D_z that is not a subset of D_3 . Therefore no greater lower bound exists and there is no infimum. Thus, the statement is false.

3.3 Poset homomorphism

Let $(P, \leq), (Q, \leq')$ two posets. A function $F : P \mapsto Q$ is called a homomorphism from (P, \leq) to (Q, \leq') iff

$$\forall x, y \in P : x \leq y \Rightarrow F(x) \leq' F(y)$$

We say F is an isomorphism of (P, \leq) in (Q, \leq') if F is a bijective homomorphism and F^{-1} is a homomorphism from (Q, \leq') in (P, \leq) .

Note. Not all bijective homomorphism satisfy the last property. For example,

$$\begin{aligned} P &= (\{1, 2\}, \{(1, 1), (2, 2)\}) \\ Q &= (\{1, 2\}, \{(1, 2), (2, 2), (1, 2)\}) \end{aligned}$$

Then $F : \{1, 2\} \mapsto \{1, 2\}$ with $F(1) = 1, F(2) = 2$ is a bijective homomorphism. However, F^{-1} is not a homomorphism because $1 \leq' 2$ and $F^{-1}(1) = 1, F^{-1}(2) = 2, 1 \not\leq 2$.

The following theorem states that an isomorphism preserves all the properties of interest.

Theorem 5 *Let $(P, \leq), (Q, \leq')$ two posets. Assume F is an isomorphism from (P, \leq) to (Q, \leq') . Then $x \leq y$ iff $F(x) \leq' F(y)$. Furthermore, if x is a maximum, a minimum, a maximal or a minimal of (P, \leq) , then $F(x)$ is that same thing of (Q, \leq') . Moreover, for any $x, y, z \in P$, $z = \sup \{x, y\}$ if and only if $F(z) = \sup \{F(x), F(y)\}$, and the same applies to the infimum. Lastly, $x < y$ if and only if $F(x) <' F(y)$.*

Problem 21 *Prove that if $(P, \leq), (Q, \leq')$ posets with an isomorphism F , then for all $x, y \in P$ we have $x < y \iff F(x) <' F(y)$.*

(\Rightarrow) Assume $x < y$. Then $F(x) \leq' F(y)$. Assume $F(x) = F(y)$. Then $F^{-1}(F(x)) = F^{-1}(F(y))$, which contradicts the assumption. Then $F(x) <' F(y)$.

(\Leftarrow) Assume $F(x) <' F(y)$. Then we have $x \leq y$ (because F^{-1} is an homomorphism). If $x = y$ and $F(x) <' F(y)$, we have $F(y)$ covers $F(x)$ but y does not cover x (\perp). Then $x < y$.

Problem 22 Now prove x is a maximum iff $F(x)$ is a maximum.

(\Rightarrow) Assume $x \in P$ is a maximum of (P, \leq) . Then $\forall y \in P : y \leq x$. Then $\forall y \in P : F(y) \leq' F(x)$. Then $F(x)$ is a maximum of (Q, \leq') .

(\Leftarrow) Assume $F(x)$ is a maximum of (Q, \leq') with $x \in P$. Then $\forall y \in P : F(y) \leq' F(x)$. Then $\forall y \in P : F^{-1}(F(y)) \leq F^{-1}(F(x))$ or rather $\forall y \in P : y \leq x$.

Problem 23 Now prove $x < y \iff F(x) < F(y)$.

Assume $x < y$ for $x, y \in P$. Then $y \leq x$ and for all $z \in P$ s.t. $y \leq z$ we have $x \leq z$. The first fact gives $F(y) \leq' F(x)$. The second fact gives $F(x) \leq F(z)$ for all $z \in P$ s.t. $y \leq z$. Then $F(x) <' F(y)$. The other side of the implication is left to the reader.

Problem 24 Give true, false or imprecise for the following statements.

(1) If (P, \leq) , (P, \leq') are finite and isomorphic, then $\leq = \leq'$.

True. Observe that $x \leq y \iff x \leq' y$ which by definition entails $(x, y) \in \leq \iff (x, y) \in \leq'$.

(2) If (P, \leq) a poset s.t. every $F : P \mapsto P$ is homomorphic from (P, \leq) in (P, \leq) , then $|P| = 1$.

False. Assume $P = \emptyset$. There is only one function $F : P \rightarrow P$, namely $\emptyset^2 = \emptyset$. This function is a homomorphism because no counter-example can be found to the defining properties of a homomorphism in the empty set. So $P = \emptyset$ satisfies the properties but $|P| \neq 1$.

3.4 Lattices

A poset (P, \leq) is called a lattice if for any $x, y \in P$, $\sup \{x, y\}$ and $\inf \{x, y\}$ exist. Informally, this means that any pair of elements in P is related to some common successor and some common predecessor in P . We use (L, \leq) to denote a lattice.

Problem 25 Prove that $(\mathbb{N}, |)$ is a lattice. Does it have maximum and minimum?

We skip the proof that $(\mathbb{N}, |)$ is a poset. Let $n_1, n_2 \in \mathbb{N}$ two arbitrary numbers. Because the set $\mathcal{D}(n_1, n_2) = \{d \in \mathbb{N} : d | n_1, d | n_2\}$ is a finite set over the natural numbers, it has a maximum. Of course, from a lattice perspective, $\mathcal{D}(n_1, n_2)$ is the set of lower bounds of $\{n_1, n_2\}$. Then $\inf \{n_1, n_2\} = \max \mathcal{D}(n_1, n_2)$ is guaranteed to exist. The proof that $\sup \{n_1, n_2\}$ exists is similar.

Because $1 | n$ for any $n \in \mathbb{N}$, 1 is a minimum. However, there is no natural $m \in \mathbb{N}$ s.t. $n | m$ for every n , so the set lacks a maximum.

Problem 26 Show that if (P, \leq) is a total order then it is lattice.

Assume (P, \leq) is a total order. If $\dots \leq p_0 \leq p_1 \leq p_2 < \dots$ is the (potentially infinite) order of P , then for any $i, k \in \omega$, $\sup \{p_i, p_{i+k}\} = p_{i+k}$ and $\inf \{p_i, p_{i+k}\} = p_i$. Then (P, \leq) is a lattice.

Problem 27 If (P, \leq) a lattice then $\sup(S)$ exists for any $S \subseteq P$?

The statement is false. (\mathbb{N}, \leq) with \leq the usual order is a total order and therefore a lattice, and $\sup(\mathbb{N})$ does not exist.

Problem 28 True or false: If (P, \leq) a lattice and $S \subseteq P$, then $(S, \leq \cap S^2)$ is a lattice.

False. Consider as a counter example $(\{1, 2, 3, 6\}, |)$. It is evident that this is a lattice, and here

$$| = \{(1, 2), (1, 3), (1, 6), (2, 6), (3, 6)\}$$

Now consider $(\{1, 2, 3\}, \{(1, 2), (1, 3)\})$. This is obviously not a lattice.

Problem 29 True or false: If (P, \leq) a lattice and $S \subseteq P$ non-empty and s.t. $(S, \leq \cap S^2)$ a lattice, then for any $a, b \in S$, $\inf \{a, b\}$ in (P, \leq) coincides with $\inf \{a, b\}$ in $(S, \leq \cap S^2)$.

Should be true. COMPLETE.

Problem 30 Let $P \subseteq \mathcal{P}(\mathbb{N})$ and assume (P, \leq) a lattice with

$$\leq = \{(A, B) \in P \times P : A \subseteq B\}$$

Is $\inf \{A, B\} = A \cap |_{P^2} B$?

Since (P, \leq) a lattice we know the infimum of any pair of elements always exist. Let $A, B \in P$ and assume $\inf \{A, B\} = I$. Then, by definition, $I \subseteq A$ and $I \subseteq B$. Furthermore, for any $I' \in P$ s.t. $I' \subseteq A$ and $I' \subseteq B$ we have $I' \subseteq I$. It follows that for every $x \in A \cap B$ we have $x \in I$. Then $I = A \cap B$. And since we have imposed the condition $A, B \in P$, the restriction of the intersection to P^2 satisfies what we have shown. The statement is true.

Problem 31 *If (P, \leq) a lattice and m is a maximal element of (P, \leq) , then m is a maximum of (P, \leq) . Is this true if (P, \leq) is not a lattice?*

The statement is true. Assume m is not a maximum. Then either there is some $m' \in P$ s.t. $m \leq m', m \neq m'$, or there is some $x \in P$ s.t. $x \not\leq m$. If the first case holds then m is not maximal (\perp). If the second case holds then $\sup \{x, m\}$ does not exist and (P, \leq) is not a lattice (\perp). Then m is a maximum. ■

3.5 Binary operations

Given a set A , a binary operation over A is a function $f : A^2 \rightarrow A$ s.t. $\mathcal{D}_f = A$. A lattice has by definition two binary operations: \inf and \sup . We will write $a \vee b$ and $a \wedge b$ to denote the supremum and infimum of $\{a, b\} \subseteq P$, respectively.

Some properties with their proofs: Assume $x, y \in (L, \leq)$ a lattice.

$$(1) x \leq x \vee y$$

Proof. $x \leq x \vee y$ by definition of supremum, because $x \vee y$ is the least $z \in L$ s.t. $x \leq z, y \leq z$.

$$(2) x \wedge y \leq x$$

Proof. The proof is similar to the previous case.

$$(3) x \vee x = x$$

Proof. $\sup \{x, x\} = \sup \{x\}$ and of course x is the lesser element in L s.t. $x \leq x$.

$$(4) x \wedge x = x$$

Proof. Similar to the previous case.

$$(5) x \vee y = y \vee x$$

Proof. Trivial; left to the reader.

$$(6) x \wedge y = y \wedge x$$

Theorem 6 Let (L, \leq) a lattice. For any $x, y \in L$, we have $x \leq y \iff x \vee y = y$. Furthermore, $x \leq y \iff x \wedge y = x$.

Theorem 7 (Absorption laws) Let (L, \leq) a lattice and $x, y, z \in L$. Then (1) $x \vee (x \wedge y) = x$ and (2) $x \wedge (x \vee y) = x$.

Theorem 8 (Order preservation) If $x \leq z$ and $y \leq w$, then $x \circ y \leq z \circ w$, with $\circ \in \{\vee, \wedge\}$.

Some proving tips.

- If you want to prove $x \vee y \leq z$, it suffices to show $x \leq z$ and $y \leq z$.

Justification. Assume $x \leq z, y \leq z$. Then z is an upper bound of $\{x, y\}$. Since $x \vee y$ is the least upper bound, $x \vee y \leq z$.

- If you want to prove $z \leq x \wedge y$, it suffices to show $z \leq x$ and $z \leq y$.

Justification. If $z \leq x, z \leq y$, then z is a lower bound of $\{x, y\}$. Then, because $x \wedge y$ is the least lower bound of this set, $z \leq x \wedge y$.

Theorem 9 (Associativity) For any $x, y, z \in L$ with (L, \leq) a lattice, $(x \vee y) \vee z = x \vee (y \vee z)$, and the same holds for \wedge .

Proof. (1) Firstly, we will prove $(x \vee y) \vee z \leq x \vee (y \vee z)$. To do this, we will prove the expression to the right is an upper-bound of the terms in the expressions to the left.

(1.1) It follows directly from the definition of supremum that $x \leq x \vee (y \vee z)$. Furthermore, let $\varphi = y \vee z$, so that by definition $y \leq \varphi$. Since $\varphi \leq x \vee \varphi$ we have $y \leq x \vee \varphi$ by transitivity. In other words, $y \leq x \vee (y \vee z)$. Then $x \vee (y \vee z)$ is an upper bound of $\{x, y\}$. Then $x \vee y \leq x \vee (y \vee z)$.

(1.2) That $z \leq x \vee (y \vee z)$ is clear from the fact that $z \leq y \vee z$ and $y \vee z \leq x \vee (y \vee z)$ (apply transitivity).

From (1.1, 1.2) follows that $x \vee (y \vee z)$ is an upper bound of $\{x \vee y, z\}$. Then $(x \vee y) \vee z \leq x \vee (y \vee z)$. ■

(2) In a similar way, we can prove that $x \vee (y \vee z) \leq (x \vee y) \vee z$. Since $\varphi \leq \psi$ and $\psi \leq \varphi$ imply $\varphi = \psi$ for any $\varphi, \psi \in L$, this concludes the proof.

Theorem 10 *If (L, \leq) a lattice and $x, y, z \in L$, then $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$.*

Proof. (1) Observe that $(x \wedge y) \vee (x \wedge z) \leq x$. The reason is that $x \wedge y \leq x$ trivially, $x \wedge z \leq x$ trivially, and therefore x is an upper bound of $\{x \wedge y, x \wedge z\}$. Then the supremum of this set is necessarily less than or equal to x .

(2) Observe that $(x \wedge y) \vee (x \wedge z) \leq y \vee z$. The reason is that $x \wedge y \leq y \leq y \vee z$ and $x \wedge z \leq z \leq y \vee z$. Then $y \vee z$ is an upper bound of $\{x \wedge y, x \wedge z\}$, and then the supremum of this set is less than or equal to $y \vee z$.

(3) Results (1) and (2) entail $(x \wedge y) \vee (x \wedge z)$ is a lower bound of $\{x, y \vee z\}$. Then $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$.

Using the same tricks we can prove $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$, which completes the proof. ■

4 Lattices as algebras

We have treated lattices as a special kind of poset. However, a lattice can be modeled as a special kind of algebra. In general, a lattice is any 3-uple (L, \vee, \wedge) with L a set and \vee, \wedge binary relations over L that satisfy the following properties:

For any $x, y, z \in L$:

- $x \vee x = x \wedge x$
- $x \vee y = y \vee x$ (Commutativity)
- $x \wedge y = y \wedge x$ (Commutativity)
- $(x \vee y) \vee z = x \vee (y \vee z)$ (Associativity)
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ (Associativity)
- $x \vee (x \wedge y) = x$
- $x \wedge (x \vee y) = x$

Viewed in this way, if (L, \leq) a lattice *in the poset* sense, then we have (L, \vee, \wedge) a lattice *in the algebraic sense* where \vee, \wedge denote the supremum and infimum operators. More formally,

Theorem 11 (Dedekind) *If (L, \vee, \wedge) a lattice, the binary relation $x \leq y \iff x \vee y = y$ is a partial order over L s.t. $\sup \{x, y\} = x \vee y$, $\inf \{x, y\} = x \wedge y$, for any $x, y \in L$.*

We call \leq the order associated to (L, \vee, \wedge) and (L, \leq) the poset associated to (L, \vee, \wedge) .

4.1 Distributive lattice

A lattice (L, \vee, \wedge) is said to be distributive when, for any $x, y, z \in L$, we have $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. It can be proven that if this property holds (distributivity of \wedge over \vee), its complementary property holds (distributivity of \vee over \wedge).

4.2 Sub-lattices and sub-universes

If (L, \wedge, \vee) , (L', \wedge', \vee') are lattices, we say the first is a sub-lattice of the other iff

- $L \subseteq L'$
- $\vee = \vee' \upharpoonright_{L \times L}$ and $\wedge = \wedge' \upharpoonright_{L \times L}$

We say $S \subseteq L$ is a sub-universe of (L, \vee, \wedge) if $S \neq \emptyset$ and S is closed under \vee, \wedge .

Note. The concepts of sub-lattice and sub-universe are similar but not identical. A sub-universe of (L, \vee, \wedge) is a *set*; a sub-lattice of (L, \vee, \wedge) is a lattice. It is true that if S is a sub-universe, then $(S, \vee \upharpoonright_{S \times S}, \wedge \upharpoonright_{S \times S})$ is a sub-lattice, and that every sub-lattice is obtained in this manner. In other words, there is a bijection between sub-lattices and sub-universes.

4.3 Lattice homomorphisms and isomorphisms

Let (L, \vee, \wedge) , (L', \vee', \wedge') be lattices. A function $F : L \mapsto L'$ is a lattice homomorphism from (L, \vee, \wedge) in (L', \vee', \wedge') iff

$$F(x \circ y) = F(x) \circ' F(y)$$

with \circ either \vee or \wedge . A homomorphism is called an isomorphism when it is bijective and its inverse is a homomorphism as well. We write $(L, \wedge, \vee) \simeq (L', \wedge', \vee')$ to say that two lattices are isomorphic.

Theorem 12 *If F is a bijective homomorphism between two lattices, then it is an isomorphism.*

Theorem 13 *Let F an homomorphism from (L, \vee, \wedge) in (L', \vee', \wedge') . Then I_F is a sub-universe of (L', \vee', \wedge') , and in consequence F is an homomorphism from (L, \vee, \wedge) in $(I_F, \vee' \upharpoonright_{I_F \times I_F}, \wedge' \upharpoonright_{I_F \times I_F})$.*

Theorem 14 *Let (L, \vee, \wedge) and (L', \vee', \wedge') lattices with associated posets $(L, \leq), (L', \leq')$. Then F is an isomorphism of (L, \vee, \wedge) in (L', \vee', \wedge') iff F is an isomorphism from (L, \leq) to (L', \leq') .*

4.4 Lattice congruence

A congruence over a lattice (L, \vee, \wedge) is an equivalence relation $\theta \dot{\sim} L$ s.t.

$$x_1 \theta x_2 \text{ and } y_1 \theta y_2 \Rightarrow (x_1 \vee y_1) \theta (x_2 \vee y_2) \text{ and } (x_1 \wedge y_1) \theta (x_2 \wedge y_2)$$

This condition allows us to define two binary operations $\hat{\vee}, \hat{\wedge}$ over L/θ as follows:

$$\begin{aligned} x/\theta \hat{\vee} y/\theta &= (x \vee y)/\theta \\ x/\theta \hat{\wedge} y/\theta &= (x \wedge y)/\theta \end{aligned}$$

Examples. (1) Consider the lattice $(\{1, 2, 3, 4, 5, 6\}, \max, \min)$. Let θ be the equivalence relation given by the partition $\{\{1, 2\}, \{3\}, \{4, 5\}\}$. Then θ is a congruence. For example,

$$\left. \begin{array}{l} 1 \theta 2 \\ 4 \theta 5 \end{array} \right\} \Rightarrow (1 \max 4) \theta (2 \max 5)$$

which holds, because $4 \theta 5$. The same can be verified for the min operation. Of course, we have that $\{1, 2\} \widehat{\max} \{4, 5\} =$

$(1 \max 4)/\theta = 4/\theta = \{4, 5\}$. In a sense, we are taking the supremum and infimum operations to a broader dimension, not applying to individual elements but to sets of equivalent elements.

Note. Observe that a congruence is an equivalence relation that is preserved in the supremum and infimum operations of a lattice. In the previous example, because it is a congruence, we expect that because 1 is equivalent to 2, and 4 is equivalent to 5, that the supremum of 1 and 4 matches the supremum of 2 and 5.

Theorem 15 *If (L, \vee, \wedge) a lattice and θ a congruence relation of this lattice, then $(L/\theta, \widetilde{\vee}, \widetilde{\wedge})$ is a lattice.*

We use $\widetilde{\leq}$ to denote the partial order associated to the lattice $(L/\theta, \widetilde{\vee}, \widetilde{\wedge})$.

Theorem 16 *If (L, \vee, \wedge) a lattice and θ a congruence over this lattice, then*

$$x/\theta \widetilde{\leq} y/\theta \iff y\theta(x \vee y)$$

for any $x, y \in L$.

Theorem 17 *If $F : (L, \wedge, \vee) \mapsto (L', \wedge', \vee')$ an homomorphism, then $\ker(F)$ is a congruence over (L, \wedge, \vee) .*