I shall prove 3-SAT is NP complete by proving SAT $\leq_\rho$ 3-SAT. In order to do this, we are to provide an algorithm or effective procedure $\mathcal{A}$ that is capable of constructing an instance of 3-SAT from an instance of SAT in polynomial time, and construct it in such a way that one if satisfiable if and only if the other one is satisfiable.

Let

$$B = \bigwedge_{i=1}^{m} \{l_{i1} \vee \ldots \vee l_{ir_i}\}$$

be an instance of SAT, where r_i is the number of literals in the ith term of B. We shall define the following effective procedure $\mathcal{P}$: For any arbitrary B_i, we shall construct E_i as follows:

- If $r_i = 1$,

$$E_i = (l_{i1} \vee y_1 \vee y_2) \wedge \{l_{i1} \vee \overline{y_1} \vee y_2\} \wedge (l_{i1} \vee y_1 \vee \overline{y_2}) \wedge (l_{i1} \vee \overline{y_1} \vee \overline{y_2})$$

- If $r_i = 2$,

$$E_i = (l_{i1} \vee l_{i2} \vee y_1) \wedge (l_{i1} \vee l_{i2} \vee \overline{y_1})$$

- If $r_i = 3$, $E_i = B_i$.
- If $r_i \geq 4$, then

$$\begin{aligned} E_i =& (l_{i1} \vee l_{i2} \vee y_1) \wedge (\overline{y_1} \vee y_2 \vee l_{i3}) \wedge (\overline{y_2} \vee y_3 \vee l_{i4}) \\ & \wedge \ldots \\ & \wedge (\overline{y_{(r-4)}} \vee y_{(r_i-3)} \vee l_{i(r_i-2)}) \wedge (\overline{y_{(r_i-3)}} \wedge l_{i(r_i-1)} \wedge l_{ir_i}) \end{aligned}$$

where each y_j are new variables. We shall prove $E = \bigwedge_{i=1}^{m'} E_i$ is satisfiable iff B is satisfiable.

($\Rightarrow$) Assume $(\overrightarrow{b}, \overrightarrow{u})$ is an assignment for the x, y variables respectively s.t. $E(\overrightarrow{b}, \overrightarrow{u}) = 1$. Then, for any arbitrary E_i, there is at least some literal that evaluates to one under this assignment. Let us consider by cases.

(1: $r_i = 1$). Assume $B_i(\overrightarrow{b}) = 0$. Then $l_{i1}(\overrightarrow{b}) = 0$. But since $E_i(\overrightarrow{b}, \overrightarrow{u}) = 1$, we must have

$$(y_1 \vee y_2) \wedge \ldots \wedge (\overline{y_1} \vee \overline{y_2})(\overrightarrow{u}) = 1$$

But it is trivial to see that any of the possible assignments makes some terms true and others false simultaneously, which is a contradiction. So $B_i(\overrightarrow{b}) = 1$.

(2 : $r_i = 2$). Assume $B_i(\overrightarrow{b}) = 0$. Since $E_i(\overrightarrow{b}, \overrightarrow{u}) = 1$ we must have

$$[y_1 \wedge y_2](\overrightarrow{u}) = 1 \Rightarrow \perp$$

Then $B_i(\overrightarrow{b}) = 1$.

(3 : $r_i = 3$). Trivial.

(4: $r_i \geq 4$). Assume $B(\overrightarrow{b}) = 0$. Then E_i is an expression of the form $y_1 \wedge (\overline{y_1} \vee y_2) \wedge (\overline{y_2} \vee y_3) \wedge \ldots \wedge \overline{y_{r_i-2}}$. Necessarily, y_1 must be true, which entails y_2 must be true, which inductively entails y_k is true for any k. But then the last term is false. $(\perp)$

In all possible cases, $B_i(\overrightarrow{b}) = 1$ for any i. Then $B(\overrightarrow{b}) = 1$.

$(\Leftarrow)$ Assume $\overrightarrow{b}$ is an assignment s.t. $B(\overrightarrow{b}) = 1$. Take an arbitrary B_i. Since it is true under $\overrightarrow{b}$, there is at least one fixed j_0 s.t. $l_{ij_0}(\overrightarrow{b}) = 1$. Let us define the assignment $\overrightarrow{u}$ as follows:

$$\begin{aligned} u_1 = u_2 = \ldots = u_{j_0-2} = 1 \\ u_{j_0-1} = u_{j_0} = \ldots = u_k = 0 \end{aligned}$$

for all k variables $y_1, \ldots, y_k$. We shall prove this assignment makes $E(\overrightarrow{b}, \overrightarrow{u}) = 1$. For this to occur, suffices that $E_i(\overrightarrow{b}, \overrightarrow{u}) = 1$. There are four possible cases.

(1: $r_i = 1$). In this case, each term in the series of conjuctions will either be anterior to the appearence of l_{ij_0}, posterior to it, or will be the term with l_{ij_0}. If it is the term with l_{ij_0} it will be true by assumption. If it is anterior it will contain at least one y_k, and by definition this will be true. If it is posterior it will contain at least one $\overline{y}_{ik}$ and it will be true.

(2 : $r_i = 2$). By def. of E_i, both terms contain all l_{ij}, so both terms will contain l_{ij_0} and will be true.

(3 : $r_i = 3$). Trivial.

(4: $r_i \geq 4$). Observe that E_i will be of the form

$$
\begin{aligned}
E_i &= (l_{i1} \vee l_{i2} \vee y_1) && y_1(\overrightarrow{u}) = 1 \text{ by def} \\
&= (\overline{y_1} \vee y_2 lor l_{i3}) && \text{True for the same reason} \\
&\vdots \\
&= (\overline{y_{(j_0-3)}} \vee y_{j_0-2} \vee l_{i(j_0-1)}) \\
&= (\overline{y_{(j_0-2)}} \vee y_{(j_0-1)} \vee l_{ij_0}) \\
&= (\overline{y_{(j_0-1)}} \vee y_{j_0} \vee l_{i(j_0+1)}) \\
&\vdots \\
&= (\overline{y_{r_i-2}} \vee l_{i(r_i-1)} \vee l_{ir})
\end{aligned}
$$

all true.