This entry provides a succint overview of what Gödel's completeness theorem is about. My purpose is to write an overview sufficiently simple for a reader with little experience in logic to comprehend. This is *not* a technical writing: no proofs are given and fundamental concepts (e.g. first order type, first order theory, etc.) are only schematically described. For a more advanced covering of these subjects, visit the *Logics* document in the *Studies* section of this site.

Gödel's incompleteness theorem is a foundational piece of mathematical logic. Mathematical logic concerns itself with the mathematical description of mathematics - it is mathematics describing itself.

Mathematicians prove things. A proof is to the mathematician what a note is to the musician. As the musician combines notes to form chords, and these are in turn organized in harmonic progressions, and from this organization the beauty of music emerges, the mathematicians too combine their proofs, use them to derive new ones, and from these workings of the mind truth, or at least a glimpse of it, arises. But what is a proof?

The question "what is a proof?" has many possible answers, but to the hammer everything is a nail, and thus the mathematician, in almost primitive and brute fashion, attempts to answer it mathematically. He searches for a mathematical description of a proof, one mathematical enough so that we may in turn prove things about it.

Perhaps the most widely known mathematical model of a proof is *natural deduction*, developed by Gentzen. Suffices to mention that natural deduction is a rigorous proof system, with rules so clear that a computer could follow. Given axioms $\varphi_1, \ldots, \varphi_n$, natural deduction systematizes the operations which our intuitive understanding of deduction considers valid to perform with these axioms.

It should be emphasized that a proof system, insofar as it provides only rules to manipulate symbols, is a syntactic object. In a sense, if we were to define a language $\mathcal{L}$ with the usual logical symbols ($\wedge, \vee, \forall, \neg$, etc.), with its words satisfying the syntax of logics (e.g. $\varphi \wedge \phi$ is in $\mathcal{L}$, but $\varphi\wedge$ is not), a proof system in its whole could be considered as a function $f : \mathcal{L}^+ \to \mathcal{L}^+$ - this is, a mapping from valid logical sentences to other valid logical sentences. There is no such thing as *meaning* and not a glimpse of *truth values*. Only grammar.

In particular, in the mathematician's daily life, the alphabet of logic is complemented with other symbols which describe constants, functions and relations.

These symbols are dependent of an interpretation: the letter π may refer to the ratio of a circle's circumference to its diameter, but it could also be used to define a function $\pi : A \rightarrow B$, or whatever else we wanted. The symbols which complement those of logic in a particular context are provided in what is termed a *first order type*.

A *first order type* τ contains three sets of symbols: *constants*, *functions* and *relations*, and each of the symbols in these sets possesses some arity. First order types, in combination with logical sentences and symbols in $\mathcal{L}$, provide a complete alphabet and syntax for the generation of sentences which can be interpreted mathematically. > *Example*. Assume the set of constants in τ is
$a, aa, aaa, aaaa, \dots,$ > the set of relationships is
$\leq$, and the other sets are empty. Assume the arity of $\leq$ is 2. Then we > could produce the τ-dependent sentence "$aa \leq aaa \wedge \varphi \neq \phi$".

The symbols in a first order type τ are imbued with meaning through what is called a *model*. A *model* of τ is made up of a set (e.g. $\mathbb{R}$) and an interpretation of each symbol in τ.

> *Example*. Assume the set of constants in τ is
$a, aa, aaa, aaaa, \dots,$ > the set of relationships is
$\leq$, and the other sets are empty. Assume the arity of $\leq$ is 2. Then a model with set $A = \mathbb{N}$ that interprets $\overbrace{a \dots a}^{n \text{ times}}$ as n, and that interprets $\leq$ as
$(a, b) : a \mid b$, essentially corresponds to the poset $(\mathbb{N}, |)$.

Given a model, we need to rigorously define when a sentence φ will be true or not. Sentences usually contain variable names, and thus their truth value does not depend only on how we interpret the symbols in φ (i.e. on the model), but also on what values we assign to the variables. We will not discuss how assignments of values to variables work, and we will assume there is a rigorous way to evaluate a logical sentence under a model given an assignment. In other words, we will assume we have defined a rigorous and non-ambiguous way to declare "φ is true/false when the variables $x_1, \dots, x_n$ take on the values $a_1, \dots, a_n$, under the interpretation given by model M of the symbols in φ".

We thus arrive at the concept of a theory. A theory is a set of *general* sentences depending on a first order type τ. By *general* I mean that these sentences do not contain the name of constants. The sentences in a theory are pure in the sense that, under whatever model of τ, they will never speak of specific elements, but will make general statements about what is true or false.

> *Example*. Assume the of constants in τ is
$a, aa, aaa, aaaa, \dots$ > , the set of relationships is
$\leq$, and the other sets are > empty. Assume the arity of $\leq$ is 2. Then a theory would never contain a sentence > of the form $aa \leq aaa$, because that speaks of

particular elements, but it could > contain a sentence of the form: $\forall x, y : (x \leq y \wedge y \leq x) \implies x = y$.

We will use (Σ, τ) to refer to a theory whose axioms (sentences) are in Σ and which depends on the first order type τ.

We are almost ready to describe the Gödel's completeness theorem. As a last preliminary, we must only point out that the theory we have developed allows for two related, yet different kinds of entailment. One is *syntactic entailment*, meaning the production of symbols within a proof system, and the other is *semantic entailment*, meaning the determination that a sentence is true as a consequence of the truth value of other sentences.

If a proof system allows us to deduce φ from the statements in a theory (Σ, τ), we say $(\Sigma, \tau) \vdash \varphi$. Recall that proof systems are entirely syntactic: $(\Sigma, \tau) \vdash \varphi$ equates to saying "the grammar of our proof system (whatever it may be) allows us to derive φ from the symbols in Σ".

On the other hand, if under all possible models (interpretations) of a first order type, the truth of the axioms in Σ entails that φ is true, we write $(\Sigma, \tau) \vDash \varphi$. This is a semantic entailment: we are not speaking of symbol manipulation, but about truth evaluation. We are saying: "under any interpretation (semantics) of the symbols of our language, φ logically follows from the axioms in Σ".

The question which Gödel's theorem answers is a troubling one. Under a given theory (Σ, τ), many statements will conceivably be true. But are they always provable through a rigorous prove system? In other words, if something is true, can we guarantee that we can prove it, where "prove it" has a rigorous and systematic definition under a system of proof?

It is conceivable that, given a system of proof such as *natural deduction*, tomorrow, or in a century, a mathematician may think of a clever deduction rule which our system does not incorporate. Such rule could potentially allow us to prove new things, things we could not prove before. In short, it is conceivable that our proof system is incomplete.

What Gödel proved is that this imagination, though conceivable, is not the case. The theorem states, in a short and brief line, that *whatever is true under a theory can be proven through a system of proof*. In other words, that

$$(\Sigma, \tau) \vDash \varphi \implies (\Sigma, \tau) \vdash \varphi$$

for any theory (Σ, τ) and any logical sentence φ. The theorem thus ensures a tight and tranquilizing link between *semantics* and *syntax*, assuring us that whatever is semantically true under a theory is provable under such theory.