1 Finance with certainty

A financial operation may be understood as a loan. A lender gives a borrower a certain amount of money, under the condition that the borrower shall return it with interest. We assume the interest is previously agreed upon, which means we are dealing with financial *certainty*.

The initial capital es the amount of money being borrowed. The final capital is the amount that is returned. The *plazo* (term) is the time the operation lasts. The interest is the difference between the initial and the final capital. In general, the interest is proportional to the initial capital.

Since the interest depends on the initial capital, generally what is done is setting an *interest rate*. The interest rate is the interest corresponding to each unit of capital per each unit of time.

Example. For example, 5% to 30 days, or 10% monthly, or daily 0.001 (0.1%), or annual 25%.

Example. An initial capital of 10.000 with an interest rate of 5% every 30 days, we have an interest in 30 days of $10.000 \times 0.05 = 500$.

Example. If an interest of 4000 is paid for a loan of 200.000 for 45 days, the interest rate is to 45 days is 2%:

$$4000/200.000 = 0.02$$

If the interest rate is set to an amount of time t less than the term of the loan, we have different systems.

1. *Simple capitalization system*: In the simple capitalization system, the interest is directly proportional to the term:

   $$F = I(1 + it)$$

   where F, I are the final and initial capitals, i is the interest rate, and t is the time expressed in the time unit of the interest.

2. *Composite capitalization system*: Once we reach time t, we operate as if we had removed the money from the loan and re-inserted the amount obtained.

   $$F = C(1 + i)^t, \qquad i = C\left((1 + i)^t - 1\right)$$

   where C is the capital we have at the moment (with accumulated interests) and t is the number of terms that passed.

Example of simple capitalization. An interest rate of 5% to 30 days and an initial capital of 10.000 gives an interest of: 500 in 30 days, 1000 in 60 days, 1500 in 90 days, etc. Notice that each time 30 days pass, we increment our money by $500 = 10.000 \times 0.05$.

Notice that after 30 days we would have 10.500, and after 60 we would have 11.000. But there's a problem: 11.000 is less than what would be obtained with an interest rate of 5% over the 10.500 we had after the first 30 days.

In other words, if the 5% of 10.000 is 500, the 5% of 10.500 is more than 500. So we see that as time goes by, with the simple capitalization system, the growth becomes less and less relative to the capital at hand.

Example of composite capitalization. An interest rate of 5% every 30 days with initial capital of 10.000. Then the cumulative interest will be:

1. 500 in 30 days.
2. 1025 in 60 days $(500 + 0.05 \times 10.500)$
3. 1576 in 90 days $(1025 + 0.05 \times 11.025)$

Problem. Fixed term to 35 days with $I = 100.000$. An automatic renovation is done in the following two vencimientos. The interest on each term are 2.4%, 2.4%, 2.8%. Compute the final capital after 105 days if: (a) total renovation is done, (b) partial renovation is done.

Solution. The first term goes from day 0 to 35, the second from day 35 to 70, and the third from day 70 to 105. The interest rate of each of these terms was specified above.

With total renovation, this would work as follows. On day 35, after the first term, we would have 100.000×1.025. On day 70 we would apply the interest of the term to the total capital so far accrued, so we would have $100.000(1.025)(1.024)$. Lastly, on day 105, we would apply the interest again to the total capital accrued, resulting in $100.000(1.025)(1.024)(1.028) = 107898.88 = F$. with an interest of $F - I = 7898.88$.

With partial renovation, on day 35 we would again have 100.000×1.025. On day 70, we would add to our accrued capital the interest of the term applied to the initial capital: $100.000(1.025 + 0.024)$. Then, on day 105, we would have $100.000(1.025 + 0.024 + 0.028)$. (If this is unclear, apply the distributive law and it will be.)

1.1 Nominal and effective rates

The TNA (Tasa Nominal Anual) is an annual rate that serves as reference to compute rates in different terms through direct proportionality. For instance, if to 30 days we have a TNA of

25%, this means the that the rate to 30 days is

$$I_{30\text{ days}} = TNA \times \frac{\text{rate in days}}{365} = 0.25 \times \frac{30}{365} = 0.0205 = 2.05\%$$

Notice that $30/365$ is the "how much are 30 days to a year".

The TEA (Tasa Efectiva Anual) is a rate equivalent to the effective rate being applied, but corresponding to an annual term. For instance, if for a fixed term to 30 days we have a rate of 2.05%, then the composite capitalization is

$$1 \times (1.0205)^{\frac{365}{30}} = 1.2800$$

which means the TEA is 28%.

1.2 Continuous capitalization

If the TNA is r, the final capital in a year over an initial capital C, with continuous capitalization, is

$$C \times \lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^m = C \times e^r$$

Then

$$TEA = e^t - 1$$

meaning that the final capital es

$$C(t) = C \times e^{rt}$$

When modeling, we need to take r as $r(t)$, i.e. as a rate that may continuously vary in time, which adds extra complexity.

1.3 Discounted value

When comparing two quantities in different time instances, we determine the *discounted value* or *actual value* of a capital. Let C_1 the capital at time t_1, C_2 the capital at time t_2, with $t_1 < t_2$. Then via composite capitalization with rate i

$$C_1 = \frac{C_2}{(1+i)^{t_2 - t_1}}$$

is the discounted value from C_2 at time t_1.