Some observations:

- K_n can span every $G \in \mathcal{G}_n$.
- For any $G \in \mathcal{G}_{n,m}$, there are $M = \binom{n}{2} - m$ edges that must be removed to span it from a K_n.
- The order in which the edges are removed does not matter.
- The space of prunable edges $\mathcal{E}$ is not constant, since an edge may become a bridge and disappear from $\mathcal{E}$.
- Following the previous statement: $\mathcal{E}$ initializes as $\Lambda(n)$ but loses an element per generated bridge.

We know $|\Lambda(n)| = \binom{n}{2}$. There are

$$\binom{\frac{n(n-1)}{2}}{\frac{n(n-1)}{2} - m} = \binom{\frac{n(n-1)}{2}}{m} =: C_{n,m}$$

graphs in $\mathcal{G}_{n,m}$, but some of them are disconnected. The question is: how many of them are disconnected.

Let $\mathcal{A}$ be the class of all graphs. We wish to produce a generating function for $\mathcal{A}$; this is, a series s.t. its kth coefficient is the number of graphs with n vertices, m edges. We know this quantity due to the derivation above, and all that is left is to expand it into a series for each n, m.

The mixed exponential generating function for $\mathcal{A}$ is then

$$A(x) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{\frac{n(n-1)}{2}}{m} y^m \right) \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} (1+y)^{\frac{n(n-1)}{2}} \frac{x^n}{n!}$$

$$= 1 + \sum_{n=1}^{\infty} (1+y)^{n(n-1)/2} \frac{x^n}{n!}$$

Now, every graph in $\mathcal{A}$ is a set of connected graphs. In other words, if we define C the class of connected graphs, the relationship between these two clases is the set-of relation. This means

$$A(x) = \exp C(x)$$

But we know $A(x)$, so we can find $C(x)$ by taking $\ln A(x)$:

$$C(x) = \ln \left[1 + \sum_{n=1}^{\infty} (1+y)^{n(n-1)/2} \frac{x^n}{n!} \right]$$

Here, we recall that

$$\log(1+u) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{u^k}{k}$$

which entails

$$C(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\sum_{n=1}^{\infty} (1+y)^{n(n-1)/2} \frac{x^n}{n!} \right]^k$$

Thus, $C(x)$ produces an enumeration of all connected graphs of n vertices, and we can arrive at the expression for all connected graphs of N vertices and M edges:

$$N! y^M x^N \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k} \left[\sum_{n=1}^{N} (1+y)^{n(n-1)/2} \frac{x^n}{n!} \right]^k$$

For example, for $M = N - 1$ across $N = 2, 3, \ldots$, this effectively produces the sequence

$$1, 1, 3, 16, 125, 1296, \ldots$$

which matches the number of trees indicated by the Prufer sequence.

We know $\mathcal{G}_{n,m} = \binom{\binom{n}{2}}{m}$. Assuming n is fixed, this gives us the generating function

$$\begin{aligned} A(z) &= \sum_{k\geq 0}^{\binom{n}{2}} \binom{\binom{n}{2}}{m} \frac{z^k}{k!} \\ &= (1+z)^{\binom{n}{2}} \frac{1}{k!} \end{aligned}$$

for the size of $\mathcal{G}_{n,k}$ across values of k. Of course, this generating function counts connected and non-connected graphs. But any graph that is not connected is a set of connected graphs. Which entails that if $C_{n,m}$ is the set of connected graphs with n vertices, m edges, then it induces a generating function

$$\begin{aligned} B(z) &= \ln A(z) \\ &= \ln \left[(1+z)^{\binom{n}{2}} \right] \\ &= \binom{n}{2} \ln(1+z) \\ &= \binom{n}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \\ &= \frac{n(n-1)}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \end{aligned}$$

For $n = 3$ this gives:

$$3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = 3 \left[1 - \frac{x^2}{2} + \frac{x^3}{3} + \dots \right]$$

Any $G \in C_{n,m}$ corresponds univocally to a set of edges s.t. removing those edges from a K_n produces G. This readily entails that, if we let $\mathcal{E}_{n,m} \subseteq \Lambda(n)$ be the class of edges which, if removed from a K_n, produce a graph in $C_{n,m}$,

$$|\mathcal{E}_{n,m}| = \mathbb{G}(n, m)$$

Furthermore, for any $W \in \mathcal{E}_{n,m}$ it is the case that $|W| = \binom{n}{2} - m$.

Let $f_{n,m} : \mathcal{E}_{n,m} \mapsto C_{n,m}$ denote the bijection $f(W) = \Big(V(K_n), E(K_n) - W\Big)$. We shall prove that *(1)* our algorithm effectively constructs a $W \in \mathcal{E}_{n,m}$ and computes $f(W)$ and *(2)* that any $W \in \mathcal{E}_{n,m}$ has an equal probability of being constructed.

(1) The algorithm iteratively removes edges ensuring that the connectivity invariant is preserved. It is trivial to see that it removes $k := \binom{n}{2} - m$ edges. Let $S = \{e_1, \ldots, e_k\}$ be the set of randomly sampled edges, where e_i was sampled at the ith edge-removing iteration.

It follows that, in the edge-removing iterations, the sampling spaces $E_1, \ldots, E_r$ are

$$\begin{aligned}
E_1 &= \big\{e \in W : W \in \mathcal{E}_{n,m}\big\} \\
E_2 &= \big\{e \in W : W \in \mathcal{E}_{n,m} \wedge \{e_1\} \subseteq W\big\} \\
E_3 &= \big\{e \in W : W \in \mathcal{E}_{n,m} \wedge \{e_1, e_2\} \subseteq W\big\} \\
&\vdots
\end{aligned}$$

Thus, the general form is $E_i = \big\{e \in W : W \in \mathcal{E}_{n,m} \wedge \{e_1, \ldots, e_{i-1}\} \subseteq W\big\}$.

It follows that $S = \{e_1, \ldots, e_k\} \subseteq W$ for some $W \in \mathcal{E}_{n,m}$. But $|S| = |W| = k$. Then $S = W$ and $S \in \mathcal{E}_{n,m}$. And since S is the set of removed edges, the algorithm generates $f(S)$.

(2) Since there is a bijection between $C_{n,m}$ and $\mathcal{E}_{n,m}$, a graph is more probable than others if and only if there is an $S \in \mathcal{E}_{n,m}$ that is more probably constructed than others. This could only be true for two cases: *(1)* An edge or set of edges in S is more likely to be chosen, or *(2)* S contains more elements than other members of $\mathcal{E}_{n,m}$. But *(1)* is impossible if the selection is random, and *(2)* contradicts that $|S| = \binom{n}{2} - m$ for every $S \in \mathcal{E}_{n,m}$.

$\therefore$ The algorithm is correct and is unbiased.

1. Define $E = \mathcal{E}_{n,m}$.
2. Define $S = \emptyset$.
3. Define $G = K_n$.
4. Sample randomly an edge e s.t. $e \in W$ for some $W \in E$.
5. Compute $G = f'_{n,m}(e, G)$.
6. Update $S = S \cup e$ and $E = \{W \in E : S \subseteq W\}$
7. If number of edges is not m go to *(4)*.

When the algorithm finishes, S contains a certain number of selected edges; and not only this, $S \in \mathcal{E}_{n,m}$ (prove this). Thus, the computation equates to $f_{n,m}(S)$.

To prove that $S \in \mathcal{E}_{n,m}$, we must only see that the *(1)* $|S| = \binom{n}{2} - m$ and *(2)* that removing S from K_n does not disconnect the graph. This two statements are obvious by construction of S.

Of course, this is equivalent to defining E and S as before and doing the following process $\binom{n}{2} - m$ times:

1. Sample randomly an edge e s.t. $e \in W$ for some $W \in E$.
2. Update $S = S \cup e$ and $E = \{W \in E : S \subseteq W\}$

and then computing $f_{n,m}(S)$. The only difference lies in that in one algorithm the edges are removed on iterations, and on the other the edges are removed at the end.

That $S \in \mathcal{E}_{n,m}$ is obvious by construction, since all elements in S are sampled from either $\mathcal{E}_{n,m}$ or

So we have a possible recursive procedure:

Thus, the step of our original algorithm which checks if removing the edge disconnects the graph simply checks whether the chosen edge e is such that there is some $W \in E$ s.t. $e \in W$.

The bijection $f_{n,m}$ allows us to ask the question differently. Instead of asking if certain graphs are more likely to be generated, we ask whether certain edge sets S are more likely to be constructed.

This corresponds to asking whether, in the domain of $f_{n,m}$, an edge is in more sets than other edges. If this were the case, then removing this edge from the K_n would result in more possible graphs than removing another edge. But this makes no sense, because after removing any edge from a K_n I can produce the same number of graphs than before; i.e. a unique removal does not shrink E (this should be proven).