We shall prove 3-color is NP complete. In order to do this, we will prove 3-SAT $\leq_\rho$ 3-Color. In other words, given an instance of 3-SAT of the form

$$B = \bigwedge i = 1^m (l_{i1} \vee l_{i2} \vee l_{i3})$$

with each literal l_{ij} a case of the variables $x_1, \ldots, x_n$, we shall provide an effective procedure that constructs a special graph $\mathcal{G}$ s.t. $\mathcal{G}$ is 3-colorable iff B is satisfiable.

(1 : Building $\mathcal{G}$) We shall define $\mathcal{G}$ by parts; namely,

1. Two special vertices s and t that are connected.
2. n triangles, each connecting the vertices in $\{t, v_i, w_i : 1 \leq i \leq n\}$
3. m triangles formed by the vertices $\{b_{i1}, b_{i2}, b_{i3} : 1 \leq i \leq m\}$
4. A tip u_{ij} each connected to b_{ij} and s.

Now let us define

$$\psi(l_{ij}) = \begin{cases} v_k & l_{ij} = x_k \\ w_k & l_{ij} = \overline{x_k} \end{cases}$$

Then we also include in $\mathcal{G}$ the sides $\big\{u_{ij}\ \psi(l_{ij} : 1 \leq i \leq m, 1 \leq j \leq 3)\big\}$. In other words, we connect each tip u_{ij} to either v_k or w_k, depending on what the literal l_{ij} is.

This completes the construction of $\mathcal{G}$. Now we shall prove $\mathcal{G}$ is 3-colorable iff B is satisfiable.

(2 : Proving $\Rightarrow$) Assume $\mathcal{G}$ has a proper coloring of three colors or less. Since $\mathcal{G}$ contains triangles, it must be a coloring of exactly three colors. We shall define

$$\overrightarrow{b_k} = \begin{cases} 1 & c(v_k) = c(s) \\ 0 & c(v_k) \neq c(s) \end{cases}$$

and prove that $B(\overrightarrow{b}) = 1$. Proving this equates to proving there is at least one j in $\{1, 2, 3\}$ s.t. $l_{ij}(\overrightarrow{b}) = 1$ for any arbitrary i. To prove this, we shall take u_{ij} and analyze what is color entails about the truth assignment.

The triangle $\{b_{i1}, b_{i2}, b_{i3}\}$ must contain $c(t)$ at some b_{ij_0} fixed. Take u_{ij_0}. Note that $c(s) \neq c(u_{ij_0}) \neq c(t)$. And since $\psi(u_{ij_0})$ cannot have the color of t, it must be the case that $c\left(\psi\left(u_{ij_0}\right)\right) = c(s)$. Now consider these cases.

Case 1. If $\psi(u_{ij_0}) = v_k$, it follows that $l_{ij} = x_k$.j Then $c(v_k) = c(s) \Rightarrow \overrightarrow{b_k} = 1 \Rightarrow l_{ij}(\overrightarrow{b}) = 1$. $\therefore$ $B_i(\overrightarrow{b}) = 1$.

Case 2. If $\psi(u_{ij_0}) = w_k$ then $l_{ij} = \overline{x_k}$. Since $c(w_k) = c(s)$ in this case, $c(v_k) \neq c(s)$ and $\overrightarrow{b}_k = 0$. Then $l_{ij}(\overrightarrow{b}) = 1$. $\therefore$ $B_i(\overrightarrow{b}) = 1$.

In both cases, for an arbitrary i, the coloring of $\mathcal{G}$ allows us to define an assignment ve^3 that makes $B_i(\overrightarrow{b}) = 1$. Of course, this assignment is s.t. $B(\overrightarrow{b}) = 1$. ■

(3 : Proving $\Leftarrow$) Assume B is satisfiable by a boolean vector $\overrightarrow{b}$. Then for any given i in $[1, m]$ we have $B_i(\overrightarrow{b}) = 1$. Then $l_{ij_0}(\overrightarrow{b}) = 1$ for (at least) a fixed j_0, $1 \le j_0 \le 3$.

Let $C = \{0, 1, 2\}$ a set of colors and define $c(s) = 0, c(t) = 1$. Let

$$c(v_k) = \begin{cases} c(s) & \overrightarrow{b}_k = 1 \\ 2 & \overrightarrow{b_k} = 0 \end{cases} \qquad c(w_k) = \begin{cases} 2 & \overrightarrow{b_k} = 1 \\ c(s) & \overrightarrow{b_k} = 0 \end{cases}$$

Clearly, $\{s, t\}$ is properly colored and $\{t, v_i, w_i\}$ is properly colored. All that is left is to color the triangles with tips.

Let

$$c(u_{ij}) = \begin{cases} 2 & j = j_0 \\ c(t) & j \neq j_0 \end{cases}$$

Of course, each $\{u_{ij}, s\}$ is properly colored. But what about $\{u_{ij}, \psi(l_{ij})\}$? Well, there are two cases to consider.

If $j = j_0$, $c(u_{ij}) = 2$ and $l_{ij}(\overrightarrow{b}) = 1$. If $\psi(l_{ij}) = v_k$, this means $x_k(\overrightarrow{b}) = 1 \Rightarrow \overrightarrow{b_k} = 1$. Then v_k is colored with $c(s) \neq c(u_{ij})$ and the coloring is correct. If $\psi(l_{ij}) = w_k$, entailing that $l_{ij} = \overline{x_k}$, then $\overrightarrow{b}_k = 0$ necessarily, in which case $c(w_k) = c(s) \neq c(u_{ij})$.

If $j \neq j_0$, then $c(u_{ij}) = c(t)$. But $\psi(l_{ij}) \in \{v_k, w_k\}$ never takes the color of t, and the coloring is correct.

All that is left is to color the triangle $\{b_{i1}, b_{i2}, b_{i3}\}$. But this is trivial. Simply let $c(b_{ij_0}) = c(s)$, ensuring that $\{b_{ij_0}, u_{ij_0}\}$ are properly colored, and color the remaining two vertices with $c(t)$ and 2 in any order.

We have used $\overrightarrow{b}$ to define a 3-coloring of $\mathcal{G}$. ■