Given a first order type τ, we use S^τ to denote the set of τ-sentences, i.e. formulas with no free variables. Let $T = (\Sigma, \tau)$ a theory. Then we can define

$$\varphi \dashv\vdash_T \psi \iff T \vdash (\varphi \leftrightarrow \psi)$$

or equivalently

$$\dashv\vdash_T = \{(\varphi, \psi) \in S^\tau : T \vdash (\varphi \leftrightarrow \psi)\}$$

It is easy to see through natural deduction that $\dashv\vdash_T$ is an equivalence relation.

We say that a sentence $\varphi \in S^\tau$ is refutable when $(\Sigma, \tau) \vdash \neg\varphi$. It is also easy to prove via natural deduction that the set of theorems and the set of refutable sentences form two distinct equivalent classes under $\dashv\vdash$. In other words, any pair of theorems imply each other, and any pair of refutable sentences impliy each other.

Let $[\varphi]$ denote the equivalent class of φ with respect to $\dashv\vdash_T$. Define **s** as

$$[\varphi] \ \mathbf{s} \ [\psi] = [(\varphi \vee \psi)]$$

In short, let the supremum of the equivalent classes of two formulas be the class of their disjunction. In particular, the supremum of (the classes of) two theorems is the class of theorems, and the supremum of (the classes of) a theorem and a refutable sentence still is the class of theorems. Only the supremum of (the classes of) two refutable sentences is the class of refutable sentences.

Define the corresponding operations for the infimum and complement of equivalent classes:

$$\begin{aligned} &[\varphi] \ \mathbf{i} \ [\psi] = [(\varphi \wedge \psi)] \\ &([\varphi])^c = [\neg\varphi] \end{aligned}$$

With 0^T as the set of theorems in T, 1^T as the set of refutable sentences in T,

$$\mathcal{B} = \left(S^\tau/\dashv\vdash_T, \mathbf{s}, \mathbf{i},^{T}, 0^T, 1^T\right)$$

is a Boolean algebra. This algebra is called *Lindenbaum*'s algebra (under theory T). Observe that, by virtue of Dedekind's theorem, this algebra has an associated partial order. This order is described as follows:

$$[\varphi] \leq [\psi] \iff T \vdash (\varphi \to \psi)$$

A Lindenbaum's algebra is used to prove Godel's completeness theorem.

Further description

Let $\psi \in S^{\tau}$ s.t. $T \nvdash \psi$. It is easy to see that natural deduction allows us to conclude,