1 Equivalence relations

Definition 1 *Given a set A, a binary relation over A is a subset of A^2.*

Observe that $\emptyset$ is a binary relationship over any set A. We use $A \propto B$ to say "A is a binary relation over B". The notation aRb is a shorthand for $(a, b) \in R$.

Observe that $R \propto A$ and $A \subseteq B$ implies $R \propto B$. Many properties of the $\propto$ relation follow from the properties of the $\subseteq$ relation. The properties that a binary relation R *may* follow are the following, given any $R \propto A$:

- $\propto$ is reflexive: aRa for any $a \in A$.
- $\propto$ is transitive: aRb and bRc implies aRc for any $a, b, c \in A$.
- $\propto$ is symmetric: $aRb \Rightarrow bRa$ for any $a, b \in A$.
- $\propto$ is anti-symmetric: aRb and bRa implies $a = b$ for any $a, b \in A$.

Whether and which of these properties hold depends on the sets in question.

Example. Consider $R = \left\{(x, y) \in \mathbb{N}^2 : x \leq y\right\}$. Then $R \propto \mathbb{N}$ and $R \propto \omega$. However, R is reflexive with respect to $\mathbb{N}$ but not with respect to ω, because $(0, 0) \notin R$.

Definition 2 *An equivalence relation over A is a binary relation $R \propto A$ s.t. R is reflexive, transitive and symmetric with respect to A.*

We write $R \ddot{\propto} A$ to say R is an equivalence relation over A.

Problem 1 *Determine true or false for the following statements.*

(1) Given X a set, then $R = \emptyset$ is a binary relation over X that is transitive, symmetric and anti-symmetric with respect to X.

We know $\emptyset \propto X$ for any X. Recall that xRx is a shorthand for $(x, x) \in R$ where R is a binary relation. In particular, $(x, x) \notin \emptyset$ for any $x \in X$, so $\emptyset$ is not reflexive. The same applies to all other properties. The statement is false.

(2) If $R \propto X$ and R is not anti-symmetric with respect to X, then R is symmetric with respect to X.

The statement is false. Consider $R = \{(1, 2), (2, 1), (5, 3)\}$ where $R \propto \omega$. Evidently R is not anti-symmetric over ω, because $1R2$ and $2R1$ and yet $2 \neq 1$. However, it is also not symmetric, because $5R3$ and $\neg(3R5)$.

(3) If A a set then $A^2 \propto A$.

Trivially true, since $A^2 \subseteq A^2$.

(4) If $R = \left\{(x, y) \in \mathbb{N}^2 : x = y\right\}$ then $R \ddot{\propto} \omega$.

By definition xRx holds. Evidently, $xRy \Rightarrow yRx$ so it is symmetric. Furthermore, $xRy \wedge yRz \Rightarrow xRz$. The statement is true.

(5) If $R \ddot{\propto} B$ and $A \subseteq B$ then $R \ddot{\propto} A$.

We need not even impose the constraint of an *equivalence* relation since the statement is false for any binary relation. In fact, $R \subseteq B^2$ and $A \subseteq B$ does not imply $R \subseteq A^2$. For example, $R = \{(1, 2), (2, 3), (3, 4)\} \subseteq \omega^2$ and $A = \{1, 2\} \subseteq \omega$. However, $R \not\ddot{\propto} A$. Since the statement is false for all binary relations, and equivalence relations are a form of binary relation, the statement is false.

Definition 3 *The equivalence class of $a \in A$ with respect to equivalence relation $R \ddot{\propto} A$ is*

$$[a]_R = \{b \in A : aRb\}$$

.

We sometimes write simply $[a]$ if the equivalence relation R is understood by the context.

Example. Let $R = \left\{(x, y) \in \mathbb{Z}^2 : x \text{ has the same parity than } y\right\}$. Then $[2]$ denotes the set of all numbers that have the same parity than 2; this is, all even numbers.

If $R = \left\{(x, y) \in \mathbb{Z}^2 : 5 \mid x - y\right\}$ then $[0] = \{5t : t \in \mathbb{Z}\}$.

Problem 2 *If $R \ddot{\propto} A$ and $a \in A$ then $a \in [a]$.*

True because R is reflexive: $aRa \Rightarrow a \in [a]$ by definition.

Problem 3 *If $R \ddot{\propto} A$ and $a, b \in A$, then $aRb \iff [a] = [b]$.*

Assume aRb. Then, for any $x \in [b]$, transitivity tells us aRx. And because $aRb \Rightarrow bRa$ we have, via the same argument, that for any $y \in [a]$ bRy. Of course,

$$\langle \forall x : x \in A : x \in B \rangle \wedge \langle \forall y : y \in B : y \in A \rangle \Rightarrow A = B$$

So $[a] = [b]$. ■

If we assume $[a] = [b]$ then of course $aRx \iff bRx$. By symmetry we have xRa and then by transitivity $bRx \wedge xRa \Rightarrow bRa \Rightarrow aRb$. ■

Problem 4 *Let $R \ddot{\propto} A$ and $a, b \in A$. Then $[a] \cap [b] = \emptyset$ or $[a] = [b]$.*

Assume $[a] \cap [b] \neq \emptyset$ and $[a] \neq [b]$, which is the negation of the statement we want to prove. Since $[a] \neq [b]$ we cannot have aRb, due to what was proven in the previous exercise. However, since $[a] \cap [b] \neq \emptyset$ there is some $z \in A$ s.t. aRz and bRz. However, $bRz \Rightarrow zRb$ and then aRb by transitivity. This is a contradiction. Then the statement is true.

Definition 4 *We use A/R to denote $\{[a] : a \in A\}$ and call this set the quotient of A by R.*

In other words, given $R \ddot{\propto} A$, the quotient of A by R is the set of all equivalence classes. For example, if $R = \left\{(x, y) \in \mathbb{R}^2 : x = y\right\}$ then $\mathbb{R}/R = \{\{x\} : x \in \mathbb{R}\}$.

Problem 5 *Let $R = \left\{(x, y) \in \mathbb{Z}^2 : 5 \mid x - y\right\}$. Find $\mathbb{Z}/R$.*

Observe that $(5, 0), (6, 1), (7, 2), (8, 3), (9, 4) \in R$. From that point onward (and from $(5, 0)$ downward) we deal with the same equivalence class.

More formally, $[5] = \{5t : t \in \mathbb{Z}\}$, $[6] = \{1, 6, 11, \ldots\} = \{5(t + 1) : t \in \mathbb{Z}\}$. In general, if $A(t) = \{5t : t \in \mathbb{Z}\}$, then

$$\{A(0), A(1), \ldots, A(4)\} = \mathbb{Z}/R$$

Observe that this can be generalized. If $R = \{(x, y) : z \mid x - y\}$ for some fixed $z \in \mathbb{N}$, then

$$\{\{zt : t \in \mathbb{Z}\}, \{z(t + 1) : t \in \mathbb{Z}\}, \ldots, \{z(t + z - 1) : t \in \mathbb{Z}\}\} = \mathbb{Z}/R$$

always with z elements.

1.1 Partitions and equivalence

Given a partition $\mathcal{P}$ of a set A, a valid binary relation is

$$R_{\mathcal{P}} = \{(a, b) : a, b \in S \text{ for some } S \in \mathcal{P}\}$$

This is in fact an equivalence relation.

Theorem 1 *Let A an arbitrary set, $\mathscr{P}_A$ the set of all partitions of A and $\mathscr{R}_A$ the set of all binary equivalence relations over A. Then*

$$\begin{aligned} \mathscr{P}_A &\mapsto \mathscr{R}_A \\ \mathcal{P} &\mapsto R_{\mathcal{P}} \end{aligned} \qquad\qquad \begin{aligned} \mathscr{R}_A &\mapsto \mathscr{P}_A \\ R &\mapsto A/R \end{aligned}$$

are bijections one the inverse of the other.