Discrete mathematics II : Final exam proofs

SLP

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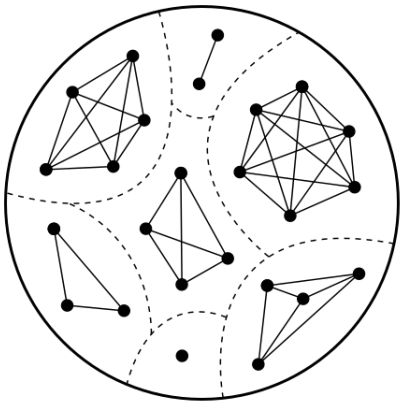

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1 Read me

Most proofs in this document come directly from lectures. A few from PDFs from previous years. In most cases I assume the reader is familiar with the relevant definitions—e.g. in proving a problem is NP-complete I do not waste time reminding him what NP-completeness is—.

This document may contain errors. It will probably contain typing errors, though I have strived to correct all of them. It will perhaps, though hopefully not, contain conceptual errors. In short, these are the notes of a student, not of an expert. Study them with a watchful and critical eye. Should you find an error, let the author know or send a pull request correcting it. All corrections are welcome.

2 Baby Brooks

Notational note. I use $\mathcal{G}$ to denote the Greedy algorithm.

We wish to prove that if $G = (V, E)$ is connected and non-regular, then $\chi(G) \leq \Delta$.

Let $x_0 \in V$ be s.t. $d(x_0) = \delta$. Since G is connected, running BFS from x_0 adds all vertices to the BFS tree. Let O^{-1} be the ordering of the vertices s.t. z is the ith vertex if it was the ith one to be added by BFS. Trivially, x_0 is the first vertex in O^{-1}. Let O be the reverse order, with x_0 last. We will prove $\mathcal{G}$ colors G with at most Δ colors if it uses the ordering O.

Observe that, in the DFS run, every $x \neq x_0$ is inserted by a neighbor that was already in the tree. In other words, in the O^{-1} order, every vertex has a neighbor that precedes him in the order. Consequently, in O^{-1}, every $x \neq x_0$ has a neighbor that succeeds him in the order.

It follows that the worst case scenario for the coloring of $x \neq x_0$ is that it has $d(x) - 1$ preceding neighbors. $\therefore \mathcal{G}$ eliminates at most $d(x) - 1 \leq \Delta - 1$ colors. Then x can be colored with a color in $\{1, \ldots, \Delta\}$.

When $\mathcal{G}$ reaches x_0 it eliminates at most $d(x_0) = \delta$ colors. Since G is non-regular $\delta < \Delta$. $\therefore$ There is at least one color for x_0 in $\{1, 2, \ldots, \Delta\}$.

3 Max flow, min cut

Let f a flow over a network $\mathcal{N}$. We want to prove two things: *(1)* $v(f) \leq Cap(S)$ for any cut S and *(2)* f is maximal iff there is a cut S s.t. $v(f) = Cap(S)$.

(1) We know $v(f) = f(S, \overline{S}) - f(\overline{S}, S)$. Since $f(A, B)$ is a sum over f and $0 \leq f(\overrightarrow{ab}) \leq c(\overrightarrow{ab})$ for any $\overrightarrow{ab} \in E$,

$$v(f) = f(S, \overline{S}) - f(\overline{S}, S) \leq f(S, \overline{S})$$

The same logic implies $f(S, \overline{S}) \leq c(S, \overline{S}) = Cap(S)$. Then $v(f) \leq f(S, \overline{S}) \leq Cap(S)$. ■

(2: ⇐) Assume there is a cut S s.t. $v(f) = Cap(S)$. Let g an arbitrary flow. Then $v(g) \leq Cap(S) = v(f)$. Then f is maximal. Furthermore, it is trivial by definition of Cap that $Cap(T) \geq v(f)$ for any cut T. Then $Cap(T) \geq Cap(S) \Rightarrow S$ is minimal.

(2 : ⇒) Assume f is maximal. Let

$$S = \{s\} \cup \{x \in V : \exists f\text{-camino aumentante entre } s \text{ y } x\}$$

S is a cut because, if $t \in S$, there is an augmenting path $s \ldots t$ and the flow can be augmented, which contradicts that f is maximal.

Recall that $v(f) = f(S, \overline{S}) - f(\overline{S}, S)$. The first term in the difference is

$$f(S, \overline{S}) = \sum_{x \in S, z \notin S, \overrightarrow{xz} \in E} f(\overrightarrow{xz})$$

Let $\overrightarrow{xz} \in E$ a side in the range of the sum above. Then there is an augmenting path $s \ldots x$ and there is no augmenting path $s \ldots z$. But $\overrightarrow{xz} \in E$ and $s \ldots x \ldots z$ is a path. Since it cannot be an augmenting path, we must have $f(\overrightarrow{xz}) = c(\overrightarrow{xz})$. Then $f(\overrightarrow{xz}) = c(\overrightarrow{xz})$ for all $x \in S$, $z \notin S, \overrightarrow{xz} \in E$. Therefore

$$f(S, \overline{S}) = \sum_{\ldots} f(\overrightarrow{xz}) = \sum_{\ldots} c(\overrightarrow{xz}) = Cap(S)$$

Now consider the second term in the difference:

$$f(\overline{S}, S) = \sum_{w \notin S, x \in S, \overrightarrow{wx} \in E} f(\overrightarrow{wx})$$

Let $\overrightarrow{wx}$ an arbitrary side in the sum above. Again, there must be an augmenting path $s \ldots x$, but not one $s \ldots w$. But $\overrightarrow{wx}$ is a side, and then $s \ldots \overleftarrow{xw}$ is not augmenting only if $f(\overleftarrow{xw}) = 0$. This means $f(\overrightarrow{wx}) = 0$ for all $\overrightarrow{wx}$ in the range of the sum above.

$\therefore v(f) = Cap(S) - 0 = Cap(S)$. ■

4 Edmond-Karp

Primero, recordemos algunas definiciones:

- *Cut :* A cut is a set $S \subseteq V$ s.t. $s \in S, t \notin S$. A theorem establishes that $v(f) \leq CAP(S)$ for any cut S. If $CAP(S) = v(f)$, we say S is minimal.

Furthermore, if $v(f) = CAP(S)$, then f is maximal and S is minimal. And if f is maximal, necessarily there is some S s.t. $v(f) = CAP(S)$.

4.1 Complexity

Edmond-Karp runs BFS ζ times to find the f-augmenting path. At the end of each run, it updates the flow. Each BFS run has complexity $O(\sum_{x \in V} d(x)) = O(2m) = O(m)$. Updating the flow has a complexity $O(n)$ because the f-augmenting path is of length at most n. Then the complexity is $\zeta\ (O(m) + O(n)) = \zeta O(m)$.

ζ is bounded by the number of f-augmenting paths that can be found. This number is $m \times \varphi$, where φ is the number of times a side can become critical. Let us determine φ.

Let $f_0, f_1, f_2, \ldots$ the flows obtained over the iterations of E.K. Let $\overrightarrow{xy}$ a side that becomes critical at iteration k. Then either it saturated being forward, or it emptied being backward. Let us look at each case.

Case 1: It saturated being forward.

(1.1) Assume it saturated being forward. Then a f_k-augmenting path of the form $s \ldots \overrightarrow{xy} \ldots t$ was found. Since we are using E.K., this path is of minimal length, which means $d_k(y) = d_k(x) + 1$.

(1.2) Assume $\overrightarrow{xy}$ becomes critical again at some iteration $j > k$. There are two options: it emptied because $\overrightarrow{yx}$ was used backwardly in the iteration, or it saturated again because, in some iteration $i \in (k, j)$, a fraction of the flow was returned backwardly. In both cases, there is a $i \leq k$ s.t. a f_i-augmenting path of the form $s \ldots \overrightarrow{yx} \ldots t$ was found. Since we are using E.K., this path is of minimal length, and $d_i(x) = d_i(y) + 1$.

(1. 3) Because the length of the augmenting paths never decreases across iterations, $d_j(t) \geq d_i(t) = d_i(x) + b_i(x)$. This means

$$d_j(t) \geq d_i(y) + 1 + b_i(x) \geq d_k(y) + 1 + b_k$$

But $d_k(y) = d_k(x) + 1$. Then

$$d_j(t) \geq d_k(x) + 1 + 1 + b_k(x) = d_k(t) + 2$$

Case 2: It emptied being backwards

(1.1) Assume $\overrightarrow{xy}$ empties. Then the f_k-augmenting path is of the form $s \ldots \overrightarrow{yx} \ldots t$ with $d_k(x) = d_k(y) + 1$.

(1.2) Assume it becomes critical again at iteration j. Then it is either saturated or emptied again. In both cases, some iteration $i \leq j$ must find the f_i-augmenting path $s \ldots \overrightarrow{xy} \ldots t$ and therefore $d_i(y) = d_i(x) + 1$.

(1.3) We know $d_j(t) \geq d_i(t)$. But

$$\begin{aligned}
&= d_i(y) + b_i(y) \\
&= d_i(x) + 1 + b_i(y) \\
&\leq d_k(x) + 1 + b_k(y) \\
&= d_k(y) + 1 + 1 + b_k(y) \\
&= d_k(t) + 2
\end{aligned}$$

Then $d_j(t) \geq d_k(t) + 2$.

Conclusion. Once a side $\overrightarrow{xy}$ becomes critical, the length of the augmenting paths found must increase at least by two before it can become critical again. Then a side can become critical at most $\varphi = \frac{n}{2} = O(n)$ times. Then the complexity of Edmond-Karp is $(\varphi \times m) \times O(m) = O(nm^2)$.

4.2 Augmenting paths are non-decreasing

Let $A = \{x \in V : d_{k+1}(x) < d_k(x)\}$ and assume $A \neq \emptyset$. Let $x_0 \in A$ be the vertex whose distance $d_{k+1}(x_0)$ from s is minimal; i.e. $d_{k+1}(x_0) \leq d_{k+1}(y)\ \forall y \in A$. Since $x_0 \in A$, $d_{k+1}(x_0) < d_k(x_0) \leq \infty$. Then there exists a $\mathcal{P}_{k+1} = s \dots x_0$ of minimal length. Let z be the predecessor of x_0 in this path.

By definition of d_f, the length of the path is $d_{k+1}(x_0)$. Because the path is of minimal length to x_0, it is of minimal length to any predecessor of x_0 in it, including z. Then $d_{k+1}(z) = d_{k+1}(x_0) - 1$. This implies $z \notin A$.

There are two possible cases: either $\overrightarrow{xz} \in E$ or $\overrightarrow{zx} \in E$.

(Case 1): If $\overrightarrow{zx_0} \in E$, then $d_{k+1}(z) < d_{k+1}(x_0)$. Since $z \notin A$,

$$d_k(z) \leq d_{k+1}(z) < d_{k+1}(x_0) < \infty$$

Since $d_k(z) < \infty$, there is an f_k-augmenting path from s to z. Then, in principle, $s \dots zx$ could be an augmenting path. But if this were the case,

$$d_k(x_0) \leq d_k(z) + 1 \leq d_{k+1}(z) + 1 = d_{k+1}(x)$$

which implies $x_0 \notin A$ ($\perp$). $s \dots zx$ is not f_k-a.p. This can only happen if $f_k(\overrightarrow{zx_0}) = c(\overrightarrow{zx_0})$ (the side is saturated). Since the side is used in a f_{k+1}-a.p. it must be the case that $f_{k+1}(\overrightarrow{zx_0}) < c(\overrightarrow{zx_0})$. This means $\overrightarrow{zx}$ was used backwards in the kth iteration, or rather that $f_k = s \dots \overleftarrow{x_0 z} \dots t$.

Since this is Edmond-Karp, augmenting paths are of minimal length. Then

$$\begin{aligned} d_k(z) &= d_k(x_0) + 1 \\ &> d_{k+1}(x_0) + 1 \\ &= d_{k+1}(z) + 2 \\ &\geq d_k(z) + 2 \end{aligned}$$

which is absurd.

1.3.2 If $\overrightarrow{xz}$ is a side then again

$$d_k(z) \leq d_{k+1}(z) < d_{k+1}(x_0) < \infty$$

Then there is an f_k-a.p. $s \dots z$ and, at least in principle, $s \dots \overrightarrow{zx}$ could also be augmenting. But if this were the case, we would have

$$d_k(x_0) = d_k(z) + 1 \leq d_{k+1}(z) + 1 = d_{k+1}(x_0)$$

which would imply $x_0 \notin A$ ($\perp$). Then $s \dots \overrightarrow{zx_0}$ is not augmenting, which means the side $\overrightarrow{x_0z}$ cannot be used backwards. This can only be true if $f_k(\overrightarrow{x_0z} = 0$. But since the side is used backwards in the f_{k+1}-augmenting path, it must be used forward in this iteration. Which means $s \dots \overrightarrow{x_0z} \dots t$ is augmenting. But then

$$\begin{aligned} d_k(z) &= d_k(x_0) + 1 \\ &> d_{k+1}(x_0) + 1 \\ &= d_{k+1}(z) + 2 \\ &\geq d_k(z) + 2 \end{aligned}$$

which is absurd.

Conclusion. In both cases a contradiction arises. The contradiction comes from assuming $A \neq \emptyset$. Then $A = \emptyset$. ■

5 Complexity of Dinitz

We will prove the complexity of Dinitz is $O(n^2m)$.

Let ψ the complexity of finding a blocking flow and φ the complexity of building an auxiliary network. We know the level t in the auxiliary networks is increasing, which implies there are at most n auxiliary networks. $\therefore$ The complexity of Dinitz is $(\psi + \varphi)\, O(n)$.

Since we use DFS, $\varphi = O(m)$. Let us prove $\psi = O(nm)$. If we can prove this, we will have the overall complexity $(O(nm) + O(m))\, O(n) = O(n^2m)$, which is what we want.

Original algorithm. In this version, the auxiliary network is s.t. all vertices have an exiting edge. This means DFS always reachs t without need to backtrack. Then DFS is not $O(m)$ but $O(N) = O(n)$, where N is the number of levels in the auxiliary network.

Every path deletes at least one edge from the auxiliary network. $\therefore$ There are $O(m)$ paths. $\therefore$ the complexity of finding the paths and updating the flow with them is $O(nm)$. But there is an extra cost associated to the property that all vertices have an exiting edge.

To enforce this invariant, Dinitz used the pruning operation (*podar*). Pruning goes from high- to low-level vertices and deletes the sides that have no exiting edge. Checking if a vertex has an exiting edge is $O(1)$; there are $O(n)$ vertices and one pruning operation per path; there are $O(m)$ paths. $\therefore$ Pruning so far is $O(mn)$.

But erasing the vertices has some complexity. This is done at most once per vertex, and it is likely that once we prune one vertex, pruning the following ones is less complex (because we are removing edges). So we will calculate the *average* complexity, not the worst-case complexity, of the problem.

Deleting a vertex x and its edges is $O(d(x))$; over all vertices this gives $\sum O(d(x)) = O(m)$ (Handshaking Lemma). $\therefore \psi = O(nm) + O(nm) + O(m) = O(nm)$.

$\therefore$ The complexity of Dinitz is $(O(nm) + O(m))\, O(n) = O(n^2m)$

Western version. The simplest approach is to give the pseudo-code. We are speaking of finding a blocking flow in the auxiliary network, do not think we are talking of the original network.

```
g := 0
bool flag := true
while flag do
  type path := [s]
  int x := s
    while x ≠ t do
      if Γ+(x) ≠ 0 then
          tomar y ∈ Γ+(x)
          agregar y a path
          x := y                    {Esta línea y la anterior son la parte (A) de avanzar}
      else
        if x ≠ s then               {(R) Retroceder}
          z := elemento anterior a x en path
          borro x de path
          borro zx⃗ de la n.a.
          x := z
        else
          flag := 0
        fi
      fi
    od
    if x = t then                   {(I) Incrementar}
      aumentar flujo g a lo largo de path
      borrar lados saturados
    fi
od
```

Let $\Sigma = \{A, I, R\}$. A Dinitz run (in the Western version) is a word $w \in \Sigma^*$. Consider all over this language of the form $A \dots AX$ with $X \in \Sigma$, $X \neq A$. Each A is $O(1)$; each R is $O(1)$; each I is traversing the path twice, once to update the flow, once to erase the edges $\Rightarrow$ I is $O(n)$.

Each A moves the pivot x from one level to the next. $\therefore$ There are $O(n)$ letters A in the word.

$$\therefore \quad (1)O(A\ldots AR) = O(n) + O(1) + O(n)$$
$$(2)O(A\ldots AI) = O(n)(\#A) + O(n)(\#I) = O(n)$$

Each R deletes an edge and each R deletes *at least* an edge. $\therefore$ There are $O(m)$ words of the form $A\ldots AX$.$\therefore$ The total complexity is $O(nm)$.

6 Codes

6.1 Hamming bound

Theorem 1 (Cota de Hamming) *Sea $C \subseteq \{0,1\}^n$. Sea $\delta = \delta(c)$ y $t = \left\lfloor \frac{\delta-1}{2} \right\rfloor$. Entonces*

$$|C| \leq \frac{2^n}{1 + n + \binom{n}{2} + \ldots + \binom{n}{t}}$$

Definition 1 (Disco) *The disk of radius r around $\alpha \in \{0,1\}^n$ is*

$$D_r(\alpha) = \{\gamma \in \{0,1\}^n : d_H(\alpha, \gamma) \leq r\}$$

Let $A = \bigcup_{v \in C} D_t(v)$. Since C corrects t errors, for any two different words $v, w \in C$,$D_t(v) \cap D_t(w) = \emptyset$. This means A is a disjoint union. $\therefore |A| = \sum_{v \in C} |D_t(v)|$.

Let $S_r(v) = \{w \in \{0,1\}^n : d_H(v,w) = r\}$. Then $D_t(v) = \bigcup_{r=0}^{t} S_r(v)$, and this union is trivially disjoint. Then $|D_t(v)| = \sum_{r=0}^{t} |S_r(v)|$. The question is what is the value of $|S_r(v)|$.

Recall that

$$\begin{aligned} w \in S_r(v) &\iff d_H(v,w) = r \\ &\iff w \text{ differes by } r \text{ bits from } v \end{aligned}$$

In other words, each $w \in S_r(v)$ is fully determined by the bits which makes it different from v, and this set of bits fully determine w. This means there is a bijection $\psi : S_r(v) \to B$ where B is the set of all subsets of r bits from a set of n bits. set of all subsets of r bits from a set of n bits.

Because it is a bijection,

$$|S_r(v)| = |B| = \binom{n}{r}$$

This means

$$
\begin{aligned}
|D_t(v)| &= \sum_{r=0}^{t} |S_r(v)| \\
&= \sum_{r=0}^{t} \binom{n}{r}
\end{aligned}
$$

By definition of A,

$$
\begin{aligned}
|A| &= \sum_{v \in C} \left(\sum_{r=0}^{t} \binom{n}{r} \right) \\
&= |C| \sum_{r=0}^{t} \binom{n}{r} \\
\Rightarrow |C| &= \frac{|A|}{\sum_{r=0}^{t} \binom{n}{r}}
\end{aligned}
$$

Since $A \subseteq \{0,1\}^n \Rightarrow |A| \leq 2^n$,

$$
|C| \leq \frac{2^n}{\sum_{r=0}^{t} \binom{n}{r}}
$$

6.2 $\delta(C) = \min\{j : \exists S \subseteq H_{*n} : |S| = j \wedge S \textbf{ is LD}\}$

Notation. I use H_{*n} to denote the set with the n columns of H. I use $H^{(i)}$ to denote the ith column of H.

Let $s = \min\{j : \exists S \subseteq H_{*n} : |S| = j \wedge S \text{ is LD}\}$. This implies there are s columns $H^{(j_1)}, \ldots, H^{(j_s)}$ s.t. $\sum x_i H^{(j_i)} = 0$ for $x_1, \ldots, x_s$ not all null.

(1) Let $w := \sum x_i e_{j_i}$ where e_k is the vector with all zeroes except at the kth coordinate. Since not all x_i are zeroes, $w \neq 0$. Now,

$$\begin{aligned} Hw^t &= H\left(x_1 e_{j_1} + \ldots + x_s e_{j_s}\right)^t \\ &= x_1 H e_{j_1}^t + \ldots + x_s H e_{j_s}^t \\ &= \sum x_i H^{(j_i)} && \left\{\text{Because } He_j^t = H^{(j)}\right\} \\ &= 0 \end{aligned}$$

Then $w \in Nu(H) = C$. But $|w| \leq s$ and $w \neq 0$. We know $\delta = \min\{|x| : x \in C, c \neq 0\}$.

$\therefore\ \delta \leq |w| \leq s$.

(2) Let $v \in C$ s.t. $\delta = |v|$. Then there are $i_1, \ldots, i_\delta$ s.t. $v = e_{i_1} + \ldots + e_{i_\delta}$. Since $v \in C, Hv^t = 0$, which using the same logic as before gives $\sum H^{(i_j)} = Hv^t = 0$.

This implies $\left\{H^{(i_1)}, \ldots, H^{(i_\delta)}\right\}$ is LD.

$\therefore\ s \leq \delta$.

(3) Points *(1)* and *(3)* imply $s = \delta$.

7 Matchings

7.1 Konig

We want to prove that any bipartite and regular graph $G = (V, E)$ has a perfect matching. Let X, Y be the two parts of G. For any $W \subseteq V$ let $E_W := \{wu \in E : w \in W\}$.

(1) Let $S \subseteq X$ and $l \in E_S$. It follows that

$$\exists x \in S, y \in Y : l = xy = yx$$

$\therefore y \in \Gamma(x)$. And since $x \in S$ we have $y \in \Gamma(S)$ and $l \in E_{\Gamma(S)}$.

$\therefore E_S \subseteq E_{\Gamma(S)}$ and $|E_S| \leq |E_{\Gamma(S)}|$.

(2) Let us calculate $|E_W|$ when $W \subseteq X$.

Observe that $E_W = \bigcup_{w \in W} \{wv : v \in \Gamma(w)\}$. Furthermore, the union is disjoint, because $wv \in E_W \Rightarrow w \in X \Rightarrow v \in Y$. Then

$$|E_W| = \sum_{w \in W} |\Gamma(w)| = \sum_{w \in W} d(w)$$

Since G is regular, $d(w) = \delta = \Delta$.

$\therefore |E_W| = \Delta |W|$

(3) Using what we established in *(1)*, it follows from *(2)* that

$$|S|\Delta \leq |\Gamma(S)|\Delta \Rightarrow |S| \leq |\Gamma(S)|$$

This holds for any $S \subseteq X$. Then Hall's theorem implies there is a complete matching from X to Y. To prove it is perfect, we must prove $|X| = |Y|$.

But since X, Y are the two parts of G, $E = E_X = E_Y$. Then $|E_X| = |E_Y|$, which implies $|X|\Delta = |Y|\delta \Rightarrow |X| = |Y|$.

Alternatively, since there is a complete matching from X to Y, $|X| \leq |Y|$. But the choice of X over Y was arbitrary, and then the same holds for Y. Then $|X| = |Y|$.

In both caes the matching is perfect.

7.2 Hall

Let $G = (V, E)$ a bipartite graph with parts X and Y, and let $Z \in \{X, Y\}$. We want to prove that there is a complete matching from X to Y iff $\forall S \subseteq Z : |S| \leq |\Gamma(S)|$.

($\Rightarrow$) The proof is trivial, because if such matching exists, it induces an injective function $f : X \to Y$ s.t. $f(x) \in \Gamma(x)$. Since it is an injection, $|f(S)| = |S|$ for any S. Then $f(S) \subseteq \Gamma(S) \Rightarrow |S| \leq |\Gamma(S)|$.

($\Leftarrow$) Assume the Hall condition $|S| \leq |\Gamma(S)|$ holds. Assume that, after running the algorithm to find a maximal matching, an incomplete matching is found. We will build $S \subseteq X$ that violates our assumption (we could use $S \subseteq Y$ without loss of generality).

(1) Let S_0 be the set of rows unmatched and $T_1 = \Gamma(S_0)$. Observe that, by assumption, $S_0 \neq \emptyset$, and all columns in T_1 have a match that is not in S_0. Let S_1 the set of rows matching columns of T_1 and $T_2 = \Gamma(S_1) - T_1$. Generally,

$$S_i = \text{Rows matching with } T_i$$
$$T_{i+1} = \Gamma(S_i) - \bigcup_{j=0}^{j=i} T_j$$

The algorithm stops only when it is revising a row and this row has no available neighbors; this is, it only stops passing from a S_i to a T_{i+1} when $T_{i+1} = \emptyset$. Furthermore, since each column only labels a single row (that of its match), and T_i "creates" S_i, we have $|S_j| = |T_j|$.

Define $S = \bigcup S_i, T = \bigcup T_i$, and note that all the S_i are disjoint and all the T_i are disjoint. Then

$$\begin{aligned} |S| &= \sum |S_i| \\ &= |S_0| + \sum |T_i| \\ &= |S_0| + |T| \end{aligned}$$

$\therefore$ $|S| > |T|$ (since $S_0 \neq \emptyset$).

We must only prove now that $T = \Gamma(S)$.

(1) T are the labeled columns, and each column is labeled by a row in S. Each row only labels its neighbors. This implies $T \subset \Gamma(S)$.

(2) Assume $y \in \Gamma(S)$ and $y \notin T$. Then y was not labeled. But since $y \in \Gamma(S)$ there is an $x \in S$ s.t. $y \in \Gamma(x)$. Then each time the algorithm passes through x it should label y, which contradicts the fact that y is not labeled. Then $y \in T$. Then $\Gamma(S) \subseteq T$.

$\therefore \Gamma(S) = T$ and $|S| > |\Gamma(S)|$. But this contradicts the hypothesis that the Hall condition holds. The contradiction comes from assuming there wasn't a complete matching. $\therefore$ There is a complete matching. ■

8 P-NP

8.1 2-Color is polynomial

We must give a polynomial algorithm that decides whether an arbitrary $G = (V, E)$ is $\chi(G) = 2$.

$n\_colored := 0$
while $j < n$ **do**
    $x := v \in V, v$ not colored
    $C(x) := 1$
    $n\_colored := n\_colored + 1$
    $Q =$ queue with only x
    while $Q \neq \emptyset$ **do**
        $p := pop!(Q)$
        for $w \in \Gamma(p)$ **do**
            if w is not colored **do**
                $push!(w, Q)$
                $C(w) = 3 - C(p)$
                $n\_colored = n\_colored + 1$
for $\{v, w\} \in E$ **do**
    if $C(v) = C(w)$ **return** $False$ **else return** $True$

Complexity. The inner **while** traverses the vertices in the connected component of x, which means the outer **while** is executed once per connected component. Furthermore, inside the inner **while** we loop through $\Gamma(p)$. Then the complexity of the inner **while** is

$$O\left[\sum_{p \in C(x)} d(p)\right] = O\left[2 \times \text{edges in } C(x)\right] = O(\#\text{edges in } C(x))$$

where $C(x)$ is the connected component of x. The **for** loop is $O(m)$ of course. $\therefore$ the algorithm is polynomial.

Correctness. It is trivial to note that if the algorithm returns $True$ then G is 2-color. Now, assume the algorithm returned $False$. Then there is some $\{v, w\} \in E$ s.t. $C(v) = C(w)$. These belong to the same connected component.

Let x be the root of this connected component from which the queue was built. Without loss of generality, assume v entered the queue first. Then, when v became the first element in the queue, w must have already been in the queue. Otherwise, because they are neighbors, v would have added w with the color $3 - C(v)$, which contradicts the hypothesis. Then there is a chain of inclusion in the queue

$$x = v_r \rightarrow v_{r-1} \rightarrow \ldots \rightarrow v_1 \rightarrow v_0 = v$$

$$x = w_t \rightarrow w_{t-1} \rightarrow \ldots \rightarrow w_1 \rightarrow w_0 = w$$

Since v came first, we must have $r \leq t$, but since w is the queue when v is its first element, $t \leq r + 1$. Now, since the color depends of the parity of r and t, $C(v) = C(w) \Rightarrow t \equiv r \mod 2$. Then $t = r$.

Let k the first index s.t. $v_k = w_k$; then we have a path $v \rightarrow v_1 \rightarrow v_k = w_k \rightarrow w_{k-1} \rightarrow w_{k-2} \ldots w$ with $wk + 1$ vertices. But since v, w are an edge, the graph contains C_{2k+1}. $\therefore \chi(G) \leq 3$.

8.2 3SAT es NP-Completo

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Let $B = B_1 \wedge \ldots B_m$ an instance of SAT with variables $x_1, \ldots, x_n$. We build an instance of 3-SAT by transforming each B_i into an E_i as follows:

Complete.

$$E_i = (e_1 \vee e_2 \vee y_1) \wedge (\overline{y_1} \vee y_2 \vee e_3) \wedge (\overline{y_2} \vee y_3 \vee e_4) \vee \ldots (\overline{y_{k-3}} \vee e_{k-1} \vee e_k)$$

We want to prove

$$\exists \overrightarrow{b} : B(\overrightarrow{b}) = 1 \iff \exists \overrightarrow{\alpha} : \tilde{B}(\overrightarrow{b}, \overrightarrow{\alpha}) = 1$$

(⇐) Asume $B(\overrightarrow{b}) = 0$. Then $D_i(\overrightarrow{b}) = 0$ for some i. Let $e_1, \ldots, e_k$ be the literals in D_i.

If $k = 3$ a contradiction ensues trivially. If $k = 2$, then $D_i = e_1 \vee e_2$ and then $E_i = (e_1 \vee e_2 \vee y_1) \wedge (e_1 \vee e_2 \vee \overline{y_1})$. Since $D_i = 0$, $e_1 \vee e_2 = 0$ and therefore $e_1 = e_2 = 0$ From this follows $E_i = y_1 \wedge \overline{y_1} = 1$. $(\perp)$

If $k = 1$ then $e_1 = 0$ and therefore $E_i = (y_1 \vee y_2) \wedge (y_1 \vee \overline{y_2}) \wedge (\overline{y_1} \vee y_2) \wedge (\overline{y_1} \vee \overline{y_2}) = 0$. But by assumption $E_i = 1 (\perp)$.

If $k \geq 4$ we must observe that, since $D_i(\overrightarrow{b}) = 0$, we have $e_1 = e_2 = \ldots = e_k = 0$. Then this literals are neutral elements in the disjunctions and can be ignored. Since $E_i(\overrightarrow{b}, \overrightarrow{\alpha}) = 1$, its first term is true; in other words, $e_1 \vee e_2 \vee y_1 = 1 \Rightarrow y_1 = 1$. In all the following cases (except the last), $E_i = \overline{y_{i-1}} \vee y_i$ must be true; this is, $y_i \Rightarrow y_{i+1}$ is true. But the last term is $\overline{y_{k-3}}$, which cannot be true because y_1 and $y_1 \Rightarrow y_2 \Rightarrow \ldots \Rightarrow y_{k-3}$. $(\perp)$

(⇒) Assume $B(\overrightarrow{b}) = 1$. For $k = 1, k = 2$, define $y_i = 0$ for all i. $\therefore D_i(\overrightarrow{b}) = 1 \Rightarrow E_i(\overrightarrow{b}, \overrightarrow{\alpha}) = 1$. For $k = 3$ the result is trivial. Let us consider the case $k \geq 4$.

Since $D_i(\overrightarrow{b}) = 1$ is a true disjunction, at least one e_r is true under $\overrightarrow{b}$. Define the following assignment:

$$y_1 = y_2 = \ldots = y_{r-2} = 1$$
$$y_i = 0 \text{ para todos los demás } i$$

Then

$$
\begin{aligned}
E(\overrightarrow{b}, \overrightarrow{\alpha}) = (e_1 \vee e_2 \vee y_1) \qquad & \{\text{True because } y_1 = 1\} \\
\wedge\, (\overline{y_1} \vee y_2 \vee e_3) \qquad & \{\text{True because } y_2 = 1\} \\
\vdots & \\
\wedge (\overline{y_{r-3}} \vee y_{r-2} \vee e_{r-1}) \qquad & \{\text{True because } y_{r-2} = 1\} \\
\wedge (\overline{y_{r-2}} \vee y_{r-1} \vee e_r) \qquad & \{\text{True because } e_r = 1\} \\
\wedge (\overline{y_{r-1}} \vee y_r \vee e_{r+1}) \qquad & \{\text{True because } y_{r-1} = 0\} \\
\vdots & \\
\wedge (\overline{y_{k-3}} \vee e_{k-1} \vee e_k) \qquad & \{\text{True because } y_{k-3} = 0\}
\end{aligned}
$$

$\therefore$ Our assignment makes $\tilde{B}$ true.

8.3 3-Color es NP-Completo

Sabemos que 3-Color $\in NP$. La idea es ver que $3 - SAT \leq_p 3 - COLOR$. Debemos crear un grafo $G = (V, E \cup F)$ tal que B es satisfactible si y solo si $\chi(G) \leq 3$.

Let $B = D_1 \wedge \ldots \wedge B_m$ with variables $x_1, \ldots, x_n$ and each $B_i = (l_{i1} \vee l_{i2} \vee l_{i3})$. Let $\mathcal{G} = (V, E)$ a graph formed as follows:

- A nucleus t from which n triangles with sides $\{t, v_1, w_1\}, \ldots, \{t, v_n, w_n\}$ form.
- m claws, each with a triangle $\{b_{i1}, b_{12}, b_{13}\}$, and such that from each b_{ij} sprouts a tip u_{ij}.
- A source s connected to t and to every tip u_{ij}.

Now, let $\psi : \{l_{11}, \ldots, l_{m3}\} \to V$ a function that maps a literal to v_k if the literal is x_k, and to w_k if the literal is $\overline{x_k}$. In other words,

$$\psi(l_{ij}) := \begin{cases} v_k & l_{ij} = x_k \\ w_k & l_{ij} = \overline{x_k} \end{cases}$$

We will use ψ to create our graph $G = (V, E \cup F)$ by letting

$$F = \left\{u_{ij}\psi(l_{ij}) : 1 \leq i \leq m, 1 \leq j \leq 3\right\}$$

In other words, we connect each u_{ij} to either v_k or w_k, depending on whether $l_{ij} = x_k$ or $\overline{x_k}$. Now that we have defined G, we must only prove that B is satisfiable iff G is 3-colorable.

Proof of ($\Leftarrow$) *:* Assume $\chi(G) \leq 3$. Since G has triangles, $\chi(G) = 3$. Let $\overrightarrow{b}_k = [c(v_k) = c(s)]$.

We must prove $B_i(\overrightarrow{b}) = 1$ for all $1 \leq i \leq m$. Take an arbitrary B_i.

> *(1)* The triangle $\{b_{i1}, b_{i2}, b_{i3}\}$ must have $c(b_{ij}) = c(t)$ for some j. Then, since $\left\{b_{ij}, u_{ij}\right\}$ is a side, $c(u_{ij}) \neq c(t)$. And since $\left\{u_{ij}, s\right\}$ is a side, $c(u_{ij}) \neq c(s)$. $\therefore u_{ij}$ was colored with the third color.
>
> *(2)* Since $\left\{u_{ij}, \psi(l_{ij})\right\}$ is a side, and $\left\{\psi(l_{ij}), t\right\}$ is a side, $c(\psi(l_{ij})) = c(s)$.

We have established that $c(\psi(l_{ij})) = c(s)$. By definition, we have two cases.

> *(1)* If $\psi(l_{ij}) = v_k$, $l_{ij} = x_k$ and $c(v_k) = c(s)$, which means $\overrightarrow{b}_k = 1$. Then $l_{ij}(\overrightarrow{b}) = 1$. Then $D_i(\overrightarrow{b}) = 1$.
>
> *(2)* If $\psi(l_{ij}) = w_k$, then $l_{ij} = \overline{x_k}$. Since $\{v_k, w_k\}$ is a side, these vertices have different colors. Then $c(v_k) \neq c(s)$. Then $\overrightarrow{b}_k = 0$. Then $l_{ij}(\overrightarrow{b}) = \overline{x_k}(\overrightarrow{b}) = 1$. Then $D_i(\overrightarrow{b}) = 1$.

In both cases, $D_i(\overrightarrow{b}) = 1$. This holds for any $i = 1, 2, \ldots, m$. Then $B(\overrightarrow{b}) = 1$. ■

Proof of $(\Rightarrow)$ *:* Assume there is some $\overrightarrow{b}$ s.t. $B(\overrightarrow{b}) = 1$. Let $C = \{C_s, C_t, C\}$ a set of three colors, and let $c(s) = C_s$, $c(t) = C_t$, and

$$c(v_k) = \begin{cases} C_s & b_k = 1 \\ C & b_k = 0 \end{cases}, \qquad c(w_k) = \begin{cases} C & b_k = 1 \\ C_s & b_k = 0 \end{cases}$$

Clearly, we are ensuring $c(v_k) \neq c(w_k) \neq c(t)$, so the triangles $\{t, v_i, w_i\}$ are all properly colored. And of course, $\{t, s\}$ is also properly colored. All we have to do is look at the claws.

First, since $\exists \overrightarrow{b} : B(\overrightarrow{b}) = 1$, then $D_i(\overrightarrow{b}) = 1$ for all i. This means, in an arbitrary D_i, there is at least one l_{ij} s.t. $l_{ij}(\overrightarrow{b}) = 1$. Let's color the tips of the claw as follows:

$$c(u_{ir}) = \begin{cases} C & r = j \\ C_t & r \neq j \end{cases}$$

Clearly, $\{u_{ir}, s\}$ is properly colored. What about $\{u_{ir}, \psi(l_{ir})\}$? There are two cases:

> *(Case $r \neq j$) :* In this case, $c(u_{ir}) = C_t$, and since $\psi(l_{ir})$ is either a v or a w, $c(\psi(l_{ir})) \neq C_t$.
>
> *(Case $r = j$) :* Here, $c(u_{ir}) = C$. If $l_{ij} = x_k, \psi(l_{ij}) = v_k$. By definition of l_{ij}, $l_{ij}(\overrightarrow{b}) = 1$. Then $c(\psi(l_{ij})) = c(v_k) = C_s$.

So, in both cases we are properly coloring $\{u_{ir}, \psi(l_{ir})\}$.

Now that we have colored the tips u_{ir}, we only have to color the triangles in the claw, $\{b_{i1}, b_{i2}, b_{i3}\}$. Let $c(b_{ij}) = C_t$ and the other two be colored in any of the possible ways. Clearly, the triangle is properly colored. Furthermore,

- $\{b_{ij}, u_{ij}\}$ is properly colored, because $c(b_{ij}) = C_t, c(u_{ij}) = C$.
- $\{b_{ir}, u_{ir}\}$ with $r \neq j$ is properly colored, because $c(u_{ir}) = C_t$ and $c(b_{ir}) \neq C_t$.

We have given a proper coloring of all vertices using $\overrightarrow{b}$.

8.4 Trisexual marriage