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1 Taylor

Let $f \in C^n[a,b]$ and assume $f^{(n+1)}$ exists in (a,b). Then for any $c, x \in [a,b]$ there is some ζ between c and x s.t.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)(x-c)^k}{k!} + E_n(x) \tag{1}$$

where

$$E_n(x) = \frac{f^{(n+1)}(\zeta)(x-c)^{n+1}}{(n+1)!}$$

Equation (1) is called the Taylor expansion of f around c.

Observation. The famous *mean value theorem* is simply the case $n = 0$ of Taylor's expansion: if $f \in C[a,b]$ and f' exists on (a,b), then for $x, c \in [a,b]$

$$f(x) = f(c) + f'(\zeta)(x-c)$$

where ζ is between c and x. Take $x = b, c = a$ and the theorem appears:

$$f(b) - f(a) = f'(\zeta)(b-a)$$

We typically extend the Taylor approximation of f around a point r, where $r = x + h$ is an approximation some value of interest x. This is useful because said approximation gives

$$f(r) = f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \ldots + \frac{f^{(n)}(x)}{n!}h^n + E_n(h)$$

In other words, this strategy allows us to extend $f(r)$ in terms of x and h, the approximation and its error. Usually, r, h are unknown but h can be bounded.

2 Alg. de Horner: Polynomial evaluation

Consider

$$p(x) = \sum_{i=0}^{n} a_i x^i$$

We wish to compute $p(k)$ for a given $k \in \mathbb{R}$ minimizing the number of operations. Directly computing $a_0 + a_1 k_1 + \ldots$ leads to n sums. The ith term requires computing k^i, which means i product operations, for a totall of $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ products. The total number of operations is then

$$\Theta = n + n(n+1)/2$$

The associated complexity is $O(n^2)$.

Horner's method consists of re-writing $p(x)$ so that the number of products is reduced. One writes

$$p(x) = a_0 + x b_0$$

where $b_{n-1} = a_n$ and for $0 \leq i < n-1$:

$$b_{i-1} = a_i + x b_i$$

Let $p(x) = 3 + 5x - 4x^2 + 0x^3 + 6x^4$, giving $n = 4$. Then $b_3 = 6$ and

$$\begin{aligned} b_2 &= a_3 + x b_3 = 6x, & b_1 &= a_2 + x b_2 = -4 + x(6x), \\ b_0 &= a_1 + x b_1 = 5 + x(-4 + x(6x)) \end{aligned}$$

This finally gives

$$p(x) = 3 + x b_0 = 3 + x(5 + x(-4 + x(6x)))$$

Here, one must perform n sums again but only n products. Thus, there are $\Theta = n + n = 2n$ operations, giving a complexity of $O(n)$ (in the operation space). See the algorithm below:

$$
\begin{array}{l}
\textbf{input } n; a_i, i = 0, \ldots, n; x \\
b_{n-1} \leftarrow a_n \\
\textbf{for } i = n-2 \textbf{ to } i = 0 \\
\qquad b_i = a_{i+1} + x * b_{i+1} \\
\textbf{od} \\
y \leftarrow a_0 + x * b_0 \\
\textbf{return } y
\end{array}
$$

It is easy to see in this code that the **for** loop performs $n-1$ iterations, in each of which a single sum and a single product are computed. The nth sum and nth product are performed in the computation of y, the final result.

A more polished version includes the last computatoin (the one in the assignment of y) within the loop and makes no use of indexes:

$$
\begin{array}{l}
\textbf{input } n; a_i, i = 0, \ldots, n; x \\
b \leftarrow a_n \\
\textbf{for } i = n-2 \textbf{ to } i = -1 \\
\qquad b = a_{i+1} + x * b \\
\textbf{od} \\
\textbf{return } b
\end{array}
$$

In Python,

```
def horner(coefs, x):
  n = len(coefs)-1
  b = coefs[n]

  for i in reversed(range(-1, n-1)):
    b = coefs[i+1] + x*b

  return b
```

It is trivial to adapt the code so that it returns the coefficients $b_0, \ldots, b_{n-1}$ and not the final result, if needed.

3 Error

Let $r, \overline{r}$ be two real numbers s.t. the latter is an approximation of the first. We define the **error** of the approximation to be $r - \hat{r}$, and

$$\Delta r = |r - \overline{r}|\,, \qquad \delta r = \frac{\Delta r}{|r|}$$

With r unknown the strategy is to work with a known bound of r.

4 Non-linear equations

The general problem is to find members of the set $\mathcal{R}_f$ of roots of $f \in \mathbb{R} \to \mathbb{R}$. The numerical strategy is to iteratively approximate some $r \in \mathcal{R}_f$ until some pre-established threshold in the error of approximation is met.

More formally, the numerical strategy produces a sequence $\{x_k\}_{k\in\mathbb{N}}$ which satisfies

- $\lim_{k\to\infty} \{x_k\} = r$ for some $r \in \mathcal{R}_f$
- Either $e(x_k) < e(x_{k-1})$ or, more strongly, $\lim_{k\to\infty} e(x_k) = 0$, where $e(x_k)$ is some appropriate measure of the error of approximation.

4.1 Bisection

A very simple procedure: if a root exists in $[a, b]$, it iteratively shrinks $[a, b]$ in halves (keeping the halves which contain the root) until the interval is of sufficiently small length or the root is found.

Theorem 1 (Intermediate value)**.** If f is continuous in $[a, b]$ and $f(a)f(b) < 0$, then $\exists r \in \mathcal{R}_f$ s.t. $r \in [a, b]$.

Assume f is continuous. A root exists in $[a, b]$ if $f(a)f(b) < 0$ (**Theorem 1**). If that is the case, the midpoint $(a + b)/2$ is taken as the approximation x_0. It is also trivial to observe that x_0 is *at most* at a distance of $(b - a)/2$ from the real root, so $e_0 = |x_0 - r| \leq (b - a)/2$.

If $f(x_0) = 0$ the procedure must end because a root was found. Otherwise, sufficies to find which half of the interval contains a root computing $f(a)f(c)$ and, if needed, $f(c)f(b)$.

The iterations may stop after reaching a maximum number of steps, when $|f(c)|$ is sufficiently close to zero, or when the error bound $|e_k| \leq (b_k - a_k)/2$ (where $[a_k, b_k]$ is the interval of this iteration) is sufficiently small.

(!) The algorithm not always converges. Take $f(x) = 1/x$. Clearly, it has no root. Yet setting $a = -1, b = 1$ in the initial iteration falsely passes the test. (The problem obviously is that f is not continuous in $[-1, 1]$.) If one sets

Input : a, b, δ, M, f

Output : Tupla de la forma: $(r, \text{cota de error})$

$f_a \leftarrow f(a)$

$f_b \leftarrow f(b)$

if $f_a * f_b > 0$

 return ?

fi

for $i = 1$ **to** $i = M$ **do**

 $c \leftarrow a + (b - a)/2$

 $f_c \leftarrow f(c)$

 if $f_c = 0$ **then**

  return $(c, 0)$

 fi

 $\epsilon = \dfrac{b - a}{2}$

 if $\epsilon < \delta$ **then**

  break

 fi

 if $f_a * f_c < 0$ **then**

  $b \leftarrow c$

  $f_b = f(b)$

 else

  $a \leftarrow c$

  $f_a = f(a)$

 fi

od

return (c, ϵ)

```
def bisection(f : callable, a : float, b : float, delta : float, M : int):

  s, e = f(a), f(b) # function values at (s)tart, (e)nd of interval

  if s*e > 0:
    raise ValueError("Interval [a, b] contains no root.")

  for i in range(M):

    c = a + (b-a)/2
    m = f(c) # value of f at (m)idpoint

    if m == 0:
      return c, 0

    e = (b-a)/2
    if e < delta:
      return c, e

    if s*m < 0:
      b = c
      e = f(b)
    else:
      a = c
      s = f(a)

  return c, e
```

Theorem 2. If $\{[a_i, b_i]\}_{i=0}^{\infty}$ are the intervals generated by the bisection method on iterations $i = 0, 1, \ldots$, then:

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ is a member of $\mathcal{R}_f$.
2. If $c_n = \frac{1}{2}(a_n + b_n), r = \lim_{n\to\infty} c_n$, then $|r - c_n| \le \frac{1}{2^{n+1}}(b_0 - a_0)$

Proof. (1) It is clear that $a_i \le a_{i+1}$ and $b_i \ge b_{i+1}$, since the interval on each iteration shrinks in one direction.

$\therefore$ a_n, b_n are monotonous.

But clearly a_n is bounded by b_0 and b_n is bounded by a_0.

$\therefore$ a_n, b_n are monotonous and bounded.

$\therefore$ Their limits exist.

It is also clear that the interval shrinks to half its size on each iteration:

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}), \qquad n \ge 1 \tag{1}$$

By recurrence on (1),

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0), \qquad n \ge 0 \tag{2}$$

Then

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} \frac{1}{2^n}(b_0 - a_0) = 0 \tag{3}$$

$\therefore$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Since the limit of a_n, b_n exists and f is by assumption continuous, the composition limit theorem applies and:

$$\begin{aligned}
&\lim_{n\to\infty} (f(a_n) \cdot f(b_n)) \\
&= \lim_{n\to\infty} f(a_n) \cdot \lim_{n\to\infty} f(b_n) && \{\text{Product of limits}\} \\
&= f\left(\lim_{n\to\infty} a_n\right) \cdot f\left(\lim_{n\to\infty} b_n\right) && \{\text{Composition limit theorem}\} \\
&= [f(r)]^2 && \left\{r = \lim_{n\to\infty} a_n\right\}
\end{aligned} \tag{4}$$

The invariant of the algorithm is $f(a_n)f(b_n) < 0$. But due to the last result,

$$\lim_{n\to\infty} f(a_n)f(b_n) \le 0 \iff [f(r)]^2 \le 0 \iff f(r) = 0$$

$\therefore$ $r = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ is a root.

(2) Follows directly from result (2)

$$
\begin{aligned}
|r - c_n| &= \left| r - \frac{1}{2}(b_n - a_n) \right| \\
&\leq \left| \frac{1}{2}(b_n - a_n) \right| \\
&= \left| \frac{1}{2^{n+1}}(b_0 - a_0) \right| && \{\text{Result (2)}\}
\end{aligned}
$$

4.2 Newton's method

Assume $r \in \mathcal{R}_f$ and $r = x + h$, with x an approximation of r and h its error. Assume f'' exists and is continuous in some I around x s.t. $r \in I$. What we explained on Taylor expansions around a point gives:

$$0 = f(r) = f(x+h) = f(x) + f'(x)h + O(h^2)$$

If x is sufficiently close to r, h is small and h^2 even smaller, so that $O(h^2)$ is unconsiderable:

$$0 \approx f(x) + hf'(x)$$

Therefore,

$$h \approx -\frac{f(x)}{f'(x)} \tag{1}$$

From this follows that $r = x + h$ is approximated by

$$r \approx x - \frac{f(x)}{f'(x)}$$

Since the approximation in (5) truncated the terms of $O(h^2)$ complexity, this new approximation is closer to r than x originally was. In other words, $x - f(x)/f'(x)$ is a better approximation to r than x itself.

Thus, if x_0 is an original approximation, we can define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

to produce a sequence of approximations. This is the fundamental idea of Newton's method.

Input: x_0, M, δ, ϵ;
$v \leftarrow f(x_0)$
if $|v| < \epsilon$ **then return** x_0 **fi**
for $k = 1$ **to** $k = M$ **do**
    $x_1 \leftarrow x_0 - \dfrac{v}{f'(x_0)}$
    $v \leftarrow f(x_1)$
    if $|x_1 - x_0| < \delta \vee v < \epsilon$ **then**
        return x_1
    fi
    $x_0 \leftarrow x_1$
od
return x_0

The predicate $|x_1 - x_0| < \delta$ checks whether our algorithm is adjusting x in a negligible degree. If that is the case, we should stop.

Theorem 3. If f'' continuous around $r \in \mathcal{R}_f$ and $f'(r) \neq 0$, then there is some $\delta > 0$ s.t. if $|r - x_0| \leq \delta$, then:

- $|r - x_n| \leq \delta$ for all $n \geq 1$.
- $\{x_n\}$ converges to r
- The convergence is quadratic, i.e. there is a constant $c(\delta)$ and a natural N s.t. $|r - x_{n+1}| \leq c\,|r - x_n|^2$ for all $n \geq N$.

Proof. Let $e_n = r - x_n$ be the error in the nth approximation. Assume f'' is continuous and $f(r) = 0$, $f'(r) \neq 0$. Then

$$\begin{aligned} e_{n+1} &= r - x_{n+1} \\ &= r - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) \\ &= r - x_n + \frac{f(x_n)}{f'(x_n)} \\ &= \frac{e_n f'(x_n) + f(x_n)}{f'(x_n)} \end{aligned} \tag{3}$$

Thus, the error at any given iteration is a function of the error at the previous iteration. Now consider the expansion of $f(r)$ as

$$f(r) = f(x_n - e_n) = f(x_n) + e_n f'(x_n) + \frac{e_n^2 f''(\zeta_n)}{2} \tag{4}$$

for ζ_n between x_n and r. This equation gives

$$e_n f'(x_n) + f(x_n) = -\frac{1}{2} f''(\zeta_n) e_n^2 \tag{5}$$

The expression in (5) is the numerator in (3), whereby we obtain via substitution:

$$e_{n+1} = -\frac{1}{2} \frac{f''(\zeta_n) e_n^2}{f'(x_n)} \tag{6}$$

Equation (6) ensures that the error scales quadratically. Now we wish to bound the error expression in (6). To bound e_{n+1}, we take $\delta > 0$ to define a neighbourhood of length δ around r. For any x in this neighbourhood, (6) reaches its maximum when the numerator is maximized and the denominator is minimized:

$$c(\delta) = \frac{1}{2} \frac{\max_{|x-r| \leq \delta} |f''(x)|}{\min_{|x-r| \leq \delta |f'(x)|}}$$

In other words, $c(\delta)$ is the maximum value which e_{n+1} can take if ζ_n, x_n are assumed to belong to the neighbourhood. Now we make two assumptions:

1. x_0 belongs to the neighbourhood, i.e. $|x_0 - r| \leq \delta$
2. δ is sufficiently small so that $\varrho := \delta c(\delta) < 1$.

Note that, since ζ_0 is between x_0 and r, assumption (1) ensures that ζ_0 is also in the neighbourhood, i.e. $|r - \zeta_0| \leq \delta$. Then we have:

$$|e_0| = \frac{1}{2} |f''(\zeta_0)/f'(x_0)| \leq c(\delta)$$

Then:

$$\begin{aligned}
|x_1 - r| &= |e_1| \\
&= \left| e_0^2 \cdot \frac{1}{2} f''(\zeta_0)/f'(x_0) \right| \\
&\leq |e_0^2| c(\delta) && \left\{ \frac{1}{2} f''(\zeta_0)/f'(x_0) \leq c(\delta) \right\} \\
&\leq |e_0| \delta c(\delta) && \{ |e_0| \leq \delta \} \\
&= |e_0| \, \varrho && \{ \varrho = \delta c(\delta) \} \\
&< |e_0| && \{ \varrho < 1 \} \\
&\leq \delta
\end{aligned}$$

$\therefore |e_1| < |e_0| \leq \delta$, which means the error decreases. This argument may be repeated inductively, giving:

$$
\begin{aligned}
|e_1| &\leq \varrho\, |e_0| \\
|e_2| &\leq \varrho\, |e_1| \leq \varrho^2\, |e_0| \\
|e_3| &\leq \varrho\, |e_2| \leq \varrho^3\, |e_0| \\
&\vdots
\end{aligned}
$$

In general, $|e_n| \leq \varrho^n\, |e_0|$. And since $0 \leq \varrho < 1$, we have $\varrho^n \to 0$ when $n \to \infty$, entailing that $|e_n| \to 0$ when $n \to \infty$.

Theorem 4. If f'' is continuous in $\mathbb{R}$, and if f is increasing, convex, and has a root, then said root is unique and Newton's method converges to it from any starting point.

Recall that f is convex if $f''(x) > 0$ for all x. Graphically, it is convex if the line connecting two arbitrary points of f lies above the curve of f between those two points.

4.3 Secant method

In Netwon's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The function of interest is f. We cannot escape computing $f(x_n)$, but it would be desirable to avoid the computation of $f'(x_n)$, which may potentially be expensive. Since

$$f'(x) = \lim_{h \to x} \frac{f(x) - f(h)}{x - h}$$

it is natural to suggest

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \tag{1}$$

Graphically, this means we are not using the line tangent to the point $(x_n, f(x_n))$ but the line secant to the points $(x_n, f(x_n)$ and $(x_{n-1}, f(x_{n-1}))$. The point x_{n+1} is then the value of x where this secant line has a root.

4.4 Fixed point iteration

The key observation is this: if $r \in \mathcal{R}_f$, then $g(x) = x - kf(x)$ has r as fixed point, for any $k \in \mathbb{R}$. Inversely, if g has a fixed point in r, then $r \in \mathcal{R}_f$.

Theorem 5. (1) Let $g \in C[a, b]$ and assume $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there is a fixed point of g in $[a, b]$.

(2) If, on top of previous conditions, g is differentiable in (a, b) and there is some $k < 1$ s.t. $|g'(x)| \leq k$ for all $x \in (a, b)$, then the fixed point referred in (1) is unique.

Theorem 6 (Mean value theorem)**.** Let $f : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b) with $a < b$. Then there is some $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The interpretation is simple: consider the line secant to f on a, b. The theorem ensures that there is some point c s.t. the line tangent to c is parallelt to said secant (equal slopes).

Proof. (1) If a or b are fixed points the proof is done so assume otherwise. Since $g(x) \in [a, b]$, we have $g(a) > a$ and $g(b) < b$.

Take $\varphi(x) = g(x) - x$, which is continuous and defined in $[a, b]$. Then

$$\varphi(a) = g(a) - a > 0, \qquad \varphi(b) = g(b) - b < 0$$

Then $\varphi(a)\varphi(b) < 0$. Then, by the intermediate value theorem, φ has a root in (a, b). In otherwords, there is at least one p s.t.

$$\varphi(p) = g(p) - p = 0$$

$\therefore g(p) = p$ is a fixed point of g.

(2) Assume two distinct fixed points p, q exist in $[a, b]$. The mean value theorem ensures the existence of some ζ between p, q (and thus in $[a, b]$) s.t.t

$$g'(\zeta) = \frac{g(a) - g(b)}{a - b} \iff g'(\zeta)(a - b) = g(a) - g(b) \tag{1}$$

By hypothesis, $|g'(x)| \leq k < 1$. Since p, q are assumed to be fixed points, equation (1) gives:

$$\begin{aligned} |p - q| &= |g(p) - g(q)| \\ &= |g'(\zeta)|\,|p - q| \\ &\leq k\,|p - q| < |p - q| \end{aligned}$$

But this is absurd. The contradiction arises from assuming p, q to be distinct. Therefore, the fixed point is unique.

The fixed point algorithm begins with an approximation p_0. Then,

$$p_n = g(p_{n-1})$$

If g continuous and the sequence converges, then it converges to a fixed point, since:

$$p := \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p)$$

Input: p, M, δ
$p_{\text{previous}} = p$
for $i = 1$ **to** $i = M$ **do**
    $p \leftarrow g(p)$
    if $\left|p - p_{\text{previous}}\right| < \delta$ **then**
        return p
    fi
    $p_{\text{previous}} = p$
od
return p

Theorem 7. Let $g \in C[a, b]$ be a self-map of $[a, b]$ differentiable in (a, b). Assume there is a constant $0 < k < 1$ s.t. $|g'(x)| \leq k$ for all $x \in (a, b)$.

For all $p_0 \in [a, b]$, the sequence $p_n = g(p_{n-1})$ converges to the unique f ixed point p in (a, b).

Proof. The mean value theorem ensures that

$$\begin{aligned} |p_n - p| &= |g(p_{n-1}) - g(p)| \\ &= |g'(\zeta_n)||(p_{n-1} - p)| \\ &\leq k\,|p_{n-1} - p| \end{aligned}$$

with $\zeta_n \in (a, b)$. More succintly, with $e_n := p_n - p$,

$$|e_n| \leq k\,|e_{n-1}| \leq k\,|e_{n-2}| \leq \ldots \leq k\,|e_0|$$

By recurrence,

$$|e_n| \leq k^n \, |e_0|$$

Since $0 < k < 1$, $k^n \to 0$ when $n \to \infty$, which entails $|e_n| \to 0$ when $n \to \infty$. It follows that $\{p_n\} \to p$ when $n \to \infty$.

Now let us consider the error of this method. Take $p_n = p + e_n$ and consider the Taylor expanssion of g around p evaluated at $p_n = p + e_n$:

$$g(p_n) = g(p + e_n) = \sum_{i=1}^{m-1} \frac{g^{(i)}(p)}{i!} e_n^i + \frac{f^{(m)}(\zeta_n)}{(n+1)!} e_n^m \tag{2}$$

See that in (2), n corresponds to the iteration we are dealing with, and thus ζ_n and e_n depend on it. On the contrary, m is the degree to which we expand the series of g around p evaluated at p_n. We also assume that ζ_n lies between p_n and p.

By definition, $g(p_n) = p_{n+1}$ so (2) is nothing but an expression for this value. Assume $g^{(k)}(p) = 0$ for $k = 1, 2, \ldots, m-1$, but $g^{(m)}(p) \neq 0$. Then

$$\begin{aligned} e_{n+1} &= p_{n+1} - p \\ &= g(p_n) - g(p) \\ &= \frac{g^{(m)}\,(\zeta_n)}{m!} e_n^m \end{aligned}$$

More succintly,

$$e_{n+1} = \frac{g^{(m)}\,(\zeta_n)}{m!} e_n^m$$

Then

$$\lim_{n \to \infty} \left| \frac{e_{n+1}}{e_n^m} \right| = \frac{|g^m(p)|}{m!}$$

which is a constant. In conclusion, if the derivatives of g are null in p up to the order $m - 1$, the method as an order of convergence of at least m. Three results follow from this fact.

4.5 Exercises

(1) Let $f(x) = (x+2)(x+1)^2x(x-1)^3(x-2)$. To which root does the biscection method converge on the following intervals?

$$[-1.5, 2.5], \qquad [-0.5, 2.4], \qquad [-0.5, 3], \qquad [-3, -0.5]$$

(a) The midpoint of $I_0 = [-1.5, 2.5]$ is $c_0 := (2.5 - 1.5)/2 = 1/2$. Since $f(a)f(c) < 0$, we have $I_1 = [-1.5, 0.5]$. The midpoint of I_1 is $c_1 = -0.5$, so I_2 will be $[-0.5, 0.5]$. The only root in this interval is $r = 0$, so the algorithm converges to it.

(b) The midpoint of $I_0 = [-0.5, 2.4]$ is $c := (2.4 - 0.5)/2 = 0.95$. Then $I_1 = [-1.5, 0.95]$. Same logic gives $c_1 = -0.725$ and then $I_2 = [-0.725, 0.95]$. The only root here is zero again.

(c, d) Same.

(2) We wish to find a root of f in $[a, b]$ using bisection method and ensuring that the error is not greater than $\epsilon \in \mathbb{R}^+$.

(a) Estimate the number of iterations sufficient to meet the criterion.

(b) What is the number of iterations for $a = 0, b = 1, \epsilon = 10^{-5}$?

Let $e_n = x_n - r$. It is trivial to note that $|e_n| \le \frac{b_n - a_n}{2}$. Furthermore, the length of I_1 is half the length of I_0, that of I_2 is half that of I_1, etc. In other words,

$$|e_0| \le \frac{b-a}{2}, \qquad |e_1| \le \frac{b-a}{2^2}, \qquad |e_2| \le \frac{b-a}{2^3}, \ldots$$

In general,

$$|e_n| \le \frac{b-a}{2^{n+1}}$$

Imposing

$$|e_n| \le \frac{b-a}{2^{n+1}} \le \epsilon$$

we satisfy our criterion, but we wish to express this bound in terms of n. Now, clearly,

$$\begin{aligned}
& \frac{b-a}{2^{n+1}} \le \epsilon \\
\iff & \frac{b-a}{\epsilon} \le 2^{n+1} \\
\iff & \log_2\left(\frac{b-a}{\epsilon}\right) - 1 \le n \\
\Longleftarrow & \log_2\left(\frac{b-a}{\epsilon}\right) \le n \\
\iff & \frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln 2} \le n
\end{aligned}$$

which is our final answer.

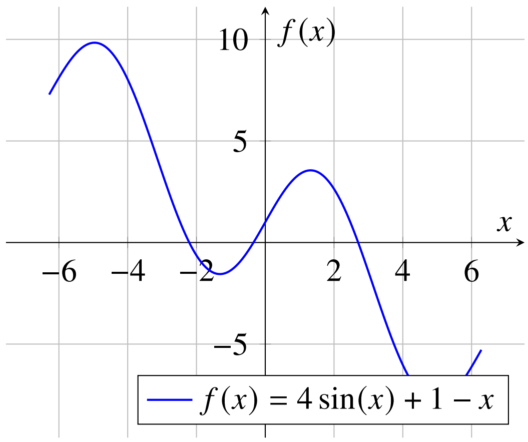
(b) For $a = 0, b = 1, \epsilon = 10^{-5}$, we need

$$n \geq \frac{\ln\left(\frac{1}{10^{-5}}\right)}{\ln 2} \approx 16.609$$

so $n = 17$ would suffice.

(3) Determine graphically some root of $f(x) = 4\sin x + 1 - x$ and perform three iterations of the bisection method to approximate. How many steps are needed to ensure an error less than 10^{-3}?

Let us unveil the full power of LaTex:



I'm too lazy to perform the steps of the algorithm. The number of steps needed again are given by

$$n \geq \frac{\ln\left(\frac{4-2}{10^{-3}}\right)}{\ln 2} \approx 10.96$$

so taking $n = 11$ suffices.

(4) Let $a > 0$. Computing $\sqrt{a}$ is equivalent to finding the root of $f(x) = x^2 - a$.

(a) Show that Newton's sequence for this case is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) Prove that f or any $x_0 > 0$, the approximations $\{x_n\}$ satisfy $x_n \geq \sqrt{a}$ for $n \geq 1$.

(c) Prove $\{x_n\}$ is sdecreasing.

(d) Conclude that the sequence converges to $\sqrt{a}$

(a) In Newton's algorithm,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Clearly,

$$f'(x) = \frac{d}{dx}(x^2 - a) = 2x$$

Therefore,

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - a}{2x_n} \\ &= x_n - \frac{1}{2}\left(x_n - \frac{a}{x_n}\right) \\ &= \frac{1}{2}x_n + \frac{1}{2}\frac{a}{x_n} \\ &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \qquad \blacksquare \end{aligned}$$

(b) Let $x_0 > 0$. Recall that, among all Pythagorean means, the arithmetic mean is the greatest, asuming positively-valued vectors. In particular, it is greater or equal to the geometric mean:

$$\frac{1}{N}\sum_{i=1}^{n} y_i \geq \sqrt[n]{\prod_{i=1}^{n} y_i}$$

for any set of points $y_1, \ldots, y_n$ all positive. In particular,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \sqrt{x_n \frac{a}{x_n}} = \sqrt{a} \qquad \blacksquare$$

(c)

$$\begin{aligned}
&\frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \leq x_n \\
\iff &x_n + \frac{a}{x_n} \leq 2x_n \\
\iff &\frac{a}{x_n} \leq x_n \\
\iff &a \leq x_n^2 \\
\iff &\sqrt{a} \leq x_n
\end{aligned}$$

which is true due to point (b).

(d) Let $e_n = x_n - \sqrt{a}$. We have shown $\{x_n\}$ to be decreasing and bounded below by $\sqrt{a}$. Therefore, it converges to a limit L (with L the infimum of $\{x_n\}$). Then

$$\lim_{n\to\infty} x_n = \frac{1}{2}\lim_{n\to\infty}\left(x_{n-1} + \frac{a}{x_{n-1}}\right) = \frac{1}{2}L + \frac{a}{2L}$$

This induces the equation

$$\begin{aligned}
L = \frac{L}{2} + \frac{a}{2L} \iff &\frac{L}{2} = \frac{a}{2L} \\
\iff &L^2 = a \\
\iff &L = \sqrt{a} \qquad \blacksquare
\end{aligned}$$

(5) Propose an iteration formula to approximate $\frac{1}{\sqrt{a}}$, with $a > 0$, using Newton's method. Decide the number of iterations needed so that the relative error in the approximation is less than 10^{-4} when starting from $x_0 = 1$ and taking $a = 5$.

Error: $e_n = r - x_n$, quadratitc, i.e. $|r - x_{n+1}| \leq c|r - x_n|^2$.

(a. Iteration formula) Let $a > 0$ and assume we wish to approximate $1/\sqrt{a}$. Let $\varphi = \frac{1}{a}$, so that $\frac{1}{\sqrt{a}} = \sqrt{\varphi}$. We see that we can express the problem of finding the reciprocal of a root in terms of a simple root.

We know from the previous exercise that the iteration formula for $\sqrt{\varphi}$ is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\varphi}{x_n}\right)$$

Now take $x_0 = 1$ and $a = 5$, so that $\varphi = \frac{1}{5}$. The relative error of approximation on iteration n is

$$e_n = \frac{\left|x_n - \frac{1}{\sqrt{5}}\right|}{\sqrt{5}}$$

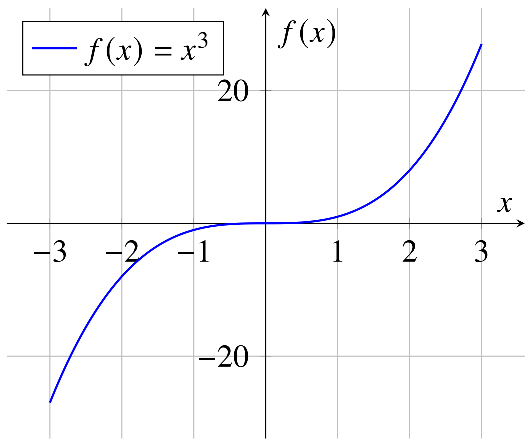
Brute-forcing allows us to see that x_0, x_1, x_2, x_3 do not meet the criterion, but

$$x_4 = 0.4472137791286728 \text{ (jaja)}$$

has $e_4 < 10^{-4}$.

(6) Propose an iteration formula for $\sqrt[3]{R}$ where $R > 0$. Plot the function to see where the procedure converges.

Observe that finding $\sqrt[3]{R}$ is equivalent to finding a root of $f(x) = x^3 - R$.



But $f(x)$ is simply a vertical displacement of x^3, so $\frac{d}{dx}x^3 = \frac{d}{dx}f(x)$ (which holds algebraically). In particular, the derivative of x^3 approaches 0 as $x \longrightarrow 0$, meaning that Newton's method will fail to converge for intervals of length L around 0 (with L unspecified). The graph suggests that an appropriate value for L is 1.

That said, since $\frac{d}{dx}f(x) = \frac{d}{dx}x^3$ (in other words, since the derivative of the function is independent of R), and $\frac{d}{dx}x^3 = 3x^2$, we propose

$$x_{n+1} = x_n - \frac{x_n^3}{3x_n^2} = x_n - \frac{x_n}{3} = \frac{2x_n}{3}$$

(7) (a) Utilizando el teorema del valor intermedio, demostrar que $g(x) = \arctan(x) - \frac{2x}{1+x^2}$ tiene raíz $\alpha \in [1, \sqrt{3}]$.

(b) Then show that if $\{x_n\}$ is the sequence generated by Newton's method for $f(x) = \arctan(x)$, with $x_0 = \alpha$, it is the case that $x_n = (-1)^n\alpha$.

(a) It is known that $\arctan x$ is continuous in $\mathbb{R}$. Since $1 + x^2 > 0$ for all x, $2x/(1 + x^2)$ is also continuous in $\mathbb{R}$. $\therefore$ g is continuous in $\mathbb{R}$. And it is easy to verify as well that $g(1)g(\sqrt{3}) < 0$.

$\therefore$ By virtue of the intermediate value theorem, there is a root α of g in $[1, \sqrt{3}]$.

(b) Let $g_1(x) = \arctan x$, $g_2(x) = \frac{2x}{1+x^2}$, so that $g = g_1 - g_2$. Since $\alpha > 0$, we have $g_1(\alpha) > 0$, $g_2(\alpha) > 0$. And since $g(\alpha) = 0$ if and only if $g_1(\alpha) - g_2(\alpha) = 0$, we conclude that $g_1(\alpha) = g_2(\alpha)$. In other words,

$$\arctan\alpha = \frac{2\alpha}{1+\alpha^2} \tag{1}$$

Since the derivative of $\arctan x$ is $1/(1 + x^2)$, equation (1) may be expressed as follows:

$$\arctan\alpha = 2\alpha \arctan'(\alpha) \tag{2}$$

This entails that

$$\arctan'\alpha = \frac{\arctan\alpha}{2\alpha} \tag{3}$$

Now take $x_0 = \alpha$ and consider Newton's sequence for $f(x) = \arctan x = g_1(x)$. Clearly,

$$\begin{aligned}
x_1 &= \alpha - \frac{f(\alpha)}{f'(\alpha)} \\
&= \alpha - \arctan\alpha \times \frac{2\alpha}{\arctan\alpha} && \{\text{Eq. } (3)\} \\
&= \alpha - 2\alpha \\
&= -\alpha
\end{aligned}$$

Same logic gives $x_2 = \alpha, x_3 = -\alpha, \ldots$ and the result should be easy to generalize.

(8) Consider for the fixed-point iteration the following functions, whose least positive root we wish to find:

$$\phi(x) = x^3 - x - 1, \qquad \psi(x) = 2x - \tan x, \qquad \varphi(x) = \exp(-x) - \cos x$$

Find an iteration function and an interval which guarantees the method's convergence.

(ϕ) Let us analyze ϕ in order to ascertain where its roots are.

Consider that $\phi'(x) = 3x^2 - 1$, which means ϕ' has roots wherever $3x^2 = 1$, which holds if and only if $x^2 = \frac{1}{3}$, or equivalently $x = \pm\frac{\sqrt{3}}{3}$. Furthermore, $\phi'(x) < 0$ in the region $(-\sqrt{3}/3, \sqrt{3}/3)$ and $\phi'(x) > 0$ elsewhere. In conclusion, ϕ is decreasing in $(-\sqrt{3}/3, \sqrt{3}/3)$ and increasing everywhere else.

Now, observe that $\phi\left(\sqrt{3}/3\right) < 0$. Combined with the fact that ϕ is increasing in $(\sqrt{3}/3, \infty)$, this means there is a root of ϕ in this interval. (Note that ϕ is a polynomial without asymptotic behavior.) Furthermore, $\phi\left(-\sqrt{3}/3\right) < 0$. Again, this means there is no root in $(\infty, \sqrt{3}/3)$.

$\therefore$ ϕ has one and only one root and it belongs to $(\sqrt{3}/3, \infty)$.

Now, suffices to note that $f(1.3)i < 0, f(1.4) > 0$, and the intermediate value theorem ensures that there is a root in $(1.3, 1.4)$. $\therefore$ The only root of ϕ lies within $(1.3, 1.4)$.

Now, we need only propose a function f s.t. r is a fixed-point of f and $f(x) \in (1.3, 1.4)$ for all $x \in (1.3, 1.4)$. Consider that

$$\phi(x) = 0 \iff x^3 = x + 1 \iff x = \sqrt[3]{x+1} \tag{4}$$

So letting $f(x) := \sqrt[3]{x+1}$ ensures that the fixed point of f is the root of ϕ. Furthermore, $f(1.3) \approx 1.32, f(1.4) \approx 1.33$. Now,

$$f'(x) = \frac{1}{\sqrt[3]{(x+1)^2}}$$

Since $f'(x) > 0$ (as is simple to note), we know f is increasing, which means all $f(x) \in (1.32, 1.33)$ for $x \in [1.3, 1.4]$. Furthermore, $f'(x) \in (0, 1)$ and $f'(x)$ is clearly decreasing. This means that in $[1.3, 1.4]$, f' has its maximum at $f'(1.4) \approx 0.573$. In other words, if we let $k = 0.573$, we know $|g'(x)| = g'(x) < k$ for all $x \in [1.3, 1.4]$.

$\therefore$ f is a self-map of $[1.3, 1.4]$, differentiable in $(1.3, 1.4)$, and there is a constant $k \in (0, 1)$ s.t. $|g'(x)| < k$ for all $x \in (1.3, 1.4)$—where incidentally this constant is $g'(1.3)$.]

$\therefore$ By virtue of **Theorem 7**, the fixed-point algorithm will converge to the unique root $r \in (1.3, 1.4)$ if using the iteration function $f(x) = \sqrt[3]{x + 1}$ and the interval $[1.3, 1.4]$.

(ψ) Let $\psi(x) = 2x - \tan x$. A root exists for $\psi(x)$ whenever

$$x = \frac{\tan x}{2} = \frac{2\sin x}{\cos x}$$

So we may define $g(x) := \tan x/2$ guarantying that any fixed point of g is a root of ψ. Now, $\tan 0 = 0$ entails that $g(0) = 0$. Furthermore, $g(\pi/4) = 1/2$. Since $g'(x) = \sec^2(x)/2$ is strictly positive, g is strictly increasing and this means for $x \in [0, \frac{\pi}{4}]$ we have $g(x) \in [0, 1/2] \subseteq [0, \frac{\pi}{4}]$.

$\therefore$ g is a self-map in $[0, \pi/4]$.

$\therefore$ There is a fixed-point of g in $[0, \pi/4]$.

Consider now $g'(x) = \frac{1}{2}\sec^2(x) = \frac{1}{2\cos^2 x}$. This is clearly bounded in $(0, 1]$. To be more precise, it is geometrically obvious that, for all $x \in [0, \pi/4]$, $\sqrt{2}/2 \leq \cos x \leq 1$, which means $1/2 \leq \cos^2 x \leq 1$. In particular, $g'(x)$ reaches its maximum when $\cos^2 x$ reaches its minimum, so $g'(x)$ reaches its maximum at $x = \frac{\pi}{4}$:

$$g'(\pi/4) = \frac{1}{2\cos^2\frac{\pi}{4}} = \frac{1}{2 \cdot 1/2} = 1$$

It follows that there is some constant $k \in (0, 1)$ such that $|g'(x)| \leq k$ for all $x \in (0, \pi/4)$.

$\therefore$ There is a unique fixed point of g in $[0, \pi/4]$.

$\therefore$ There is a unique root of $\psi(x)$ in $[0, \pi/4]$ and the iteration method converges to it using this interval and the iteration function g.

(φ) Consider $\varphi(x) = \exp(-x) - \cos x$. This function is zero if and only if $e^{-x} = \cos x$, which may be expressed as $x = -\ln(\cos x)$. In other words, the roots of φ correspond to the fixed points of $f(x) = -\ln(\cos x)$.

Now, $-1 \le \cos x \le 1$ but ln is defined only in $\mathbb{R}^+$. From this follows that f is defined only when $\cos x > 0$, i.e. in the right-hand half of the unite circle. This corresponds to values of x in $[0, \pi/2)$ or $(3\pi/2, 2\pi]$ (extended by any factor $2\pi k$, $k \in \mathbb{Z}$).

Take $I := [0, \pi/4] \subseteq \mathrm{Dom}(f)$. See that $f(0) = -\ln(1) = 0$ and $f(\pi/4) = -\ln(\sqrt{2}/2) \approx 0.346 < \pi/4$. Furthermore, with $u = \cos x$,

$$\frac{df}{dx} = -\frac{d}{du}\ln(u) \times \frac{d}{dx}\cos x = \frac{\sin x}{\cos x} = \tan x$$

which is strictly positive in $[0, \pi/4]$. This suffices to prove that $f(x) \in [0, \pi/4]$ for all $x \in [0, \pi/4]$.

$\therefore$ f is a self-map of $[0, \pi/4]$.

$\therefore$ There is a fixed point of f in $[0, \pi/4]$.

Now, $\tan x$ is increasing in $[0, \pi/4]$ and, in particular, $\tan 0 = 0, \tan \frac{\pi}{4} = 1$. This suffices to show that $|g'(x)| < 1$ for all $x \in (0, \pi/4)$.

$\therefore$ There is a unique fixed point of f in $[0, \pi/4]$ and the fixed point iteration algorithm converges to it when starting from said interval with f as iteration function.

(10) Let $x_{n+1} = 2^{x_n - 1}$ the formula used to solve $2x = 2^x$. What interval should be chosen to ensure $\{x_n\}$ is convergent? Calculate its limit.

The fixed-point algorithm uses the formula $p_n = g(p_{n-1})$ where g is a function s.t. the fixed points of g are roots of some original function of interest f. In this case, clearly $g(x) = 2^{x-1}$. To ensure convergence, we must find an interval I s.t. g is a self-map of I and g' lies within a unit neighbourhood of 0.

Now, clearly the equation $2x = 2^x$ has solutions $x = 1, x = 2$, and no other. So whatever self-map I we build must contain either 1 or 2. So take $I = [0, 1]$.

Clearly, if $x \in I$, then $-1 \leq x - 1 \leq 0$. This means 2^{x-1} has exponent at least -1, when $g(0) = 2^{-1} = \frac{1}{2}$. Furthermore, 2^{x-1} has exponent at most 0, when $g(1) = 2^0 = 1$. This suffices to show that $g(x) \in I$ for all $x \in I$.

Now,

$$\frac{d}{dx} 2^{x-1} = \frac{d}{du} 2^u \times \frac{d}{dx}(x-1) = 2^u \ln u$$

In short, $g'(x) = 2^{x-1} \ln(2)$. For $x \in [0, 1]$, we have already established that $0 \leq 2^{x-1} \leq 1$. Therefore, $0 \leq g'(x) \leq \ln(2) < 0$ for all $x \in [0, 1]$. In other words, g' lies within a unit-distance of zero when its domain is restricted to I.

$\therefore$ The algorithm converges to the unique solution of $2x = 2^x$ in $[0, 1]$ (which is 1) when starting from said interval with iteration function g.

(11) Suppose $\{x_n\}$ converges to r and that $x_{n+1} = g(x_n)$ where $|g(y) - g(x)| \leq \lambda|y - x|$ for all x, y with $\lambda \in (0, 1)$. Determine the error bound on each iteration as a function of the difference between the last two iteration values. In other words, find C s.t.

$$|x_{n+1} - r| \leq C\,|x_{n+1} - x_n|$$

Recall that $x_{n+1} = g(x_n)$. This means

$$|x_{n+1} - r| = |g(x_n) - r|$$

But r is a fixed-point of g, i.e. $r = g(r)$. Then

$$|g(x_n) - r| = |g(x_n) - g(r)|$$

By assumption, then,

$$\begin{aligned} |x_{n+1} - r| &= |g(x_n) - g(r)| \\ &\leq \lambda\,|x_n - r| \end{aligned}$$

Recall that $|e_n| = |x_n - r| \leq k^n\,|e_{n-1}|$ for some $k \in (0, 1)$. Since the property above holds for any $\lambda \in (0, 1)$, it holds for said k.

Since $|x_n - r| \leq k^n\,|e_{n-1}|$, and $k^n \in (0, 1)$ entails $k^n\,|e_{n-1}| < |e_{n-1}|$, we have $|x_n - r| < |x_{n-1} - r|$. In other words, successive approximations in the sequence become increasingly closer to r. This means

$$|x_{n+1} - r| \leq k\,|x_n - r|$$

Wtf now?

5 Polynomial interpolation

Teorema fundamental del álgebra. Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots.

Theorem 8. Given $x_0, \dots, x_n$, $y_0, \dots, y_n$, there is a unique polynomial p_n of degree $\mathrm{gr}(p_n) \leq n$ s.t. $p_n(x_i) = y_i$ for all i.

Proof. (Existence) If $n = 0$ simple $p_0(x) = y_0$ which is trivial. So take as inductive hypothesis the existence of p_{k-1}, of degree $\leq k-1$, s.t. $p_{k-1}(x_i) = y_i$ for $i = 0, \dots, k-1$. We will construct a polynomial p_k of degree $\leq k$ s.t. $p_k(x_i) = y_i$ for $0 \leq i \leq k$.

Consider

$$p_k(x) = p_{k-1}(x) + c(x - x_0)(x - x_1) \dots (x - x_{k-1})$$

with c yet to be determined. Its degree is $\leq k$ and it obviously interpolates the points $(x_0, y_0), \dots, (x_{k-1}, y_{k-1})$. Now consider the equation:

$$p_k(x_k) = y_k$$

or equivalently

$$p_{k-1}(x_k) + c(x_k - x_0)(x_k - x_1) \dots (x_k - x_{k-1})$$

Solving for c, we find

$$c = \frac{y_k - p_{k-1}(x_k)}{(x_k - x_0) \dots (x_k - x_{k-1})}$$

Notice that c is well defined because each $x_0, \dots, x_n$ is distinct and therefore the denominator is never zero. This proves the polynomial exists.

(Uniqueness) Assume two interpolating polynomoials p_n, q_n exist. Let $h = p_n - q_n$. Clearly, its degree is $\leq n$ and $h(x_i) = 0$ for each $0 \leq i \leq n$. But this means h has $n + 1$ real roots. Then, for the fundamental theorem of algebra, $h(x) = 0$ for all x and then $p_n = q_n$.

5.1 Newton's form

Given $x_0, \dots, x_n$ we define Newton's basis polynomials:

$$\eta_i(x) = \prod_{j=0}^{i-1} (x - x_j), \qquad 0 \leq i \leq n$$

where $\eta_0(x) := 1$. Applying the construction seen in the last proof recurrently, we obtain:

$$p_k(x) = \sum_{i=0}^{k} c_i \eta_i(x) = \sum_{i=0}^{k} c_i \prod_{j=0}^{i-1} (x - x_j)$$

Here, each c_i is obtained as in the last proof.

Example. Consider the points $(1, 2)$, $(2, 5)$, $(3, 3)$. We begin with $p_0(x) = 2$, the first interpolation which interpolates $(1, 2)$.

Now,

$$\begin{aligned} p_1(x) &= p_0(x) + c_0(x - x_0) \\ &= 2 + c_0(x - 1) \end{aligned}$$

Now, we pose the equation:

$$\begin{aligned} p_1(x_1) &\iff y_1 \equiv p_1(2) = 5 \\ &\iff 2 + c_0(2 - 1) = 5 \\ &\iff c_0 = 3 \end{aligned}$$

Then $p_1(x) = 2 + 3(x - 1) = 3x - 1$. Now we repeat:

$$\begin{aligned} p_2(x) &= p_1(x) + c_1(x - x_0)(x - x_1) \\ &= (3x - 1) + c_1(x - 1)(x - 2) \end{aligned}$$

We pose the equation:

$$\begin{aligned} & (3 \cdot 3 - 1) + c_1(3 - 1)(3 - 2) = 3 \\ \iff & 8 + 2c_1 = 3 \\ \iff & 2c_1 = -5 \\ \iff & c_1 = -\frac{5}{2} \end{aligned}$$

from which we have

$$p_2(x) = (3x - 1) - \frac{5}{2}(x - 1)(x - 2)$$

In Newton's form,

$$\begin{aligned} p_2(x) &= 2 + 3(x - 1) - \frac{5}{2}(x - 1)(x - 2) \\ &= \sum_{i=0}^{2} c_i \eta_i(x) \end{aligned}$$

with $c_0 = 2, c_1 = 3, c_2 = -\frac{5}{2}$.

5.2 Lagrange's form

Given $(x_0, y_0), \ldots, (x_n, y_n)$, we define Lagrange's basic polynyomials:

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}, \qquad i = 0, \ldots, n$$

Note that $\mathrm{gr}(\ell_i) = n$ for all i and $\ell_i(x_j) = \delta_{ij}$ for all i, j. In other words, $\ell_i(x)$ is nothing but a polynomial that becomes null at each x_j except at x_i, where it is one.

The Lagrange form of the interpolating polynomial es

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x)$$

Prove that $\sum_{i=0}^{n} \ell_i(x) = 1$.

Observe that the polynomial $\sum_{i=0}^{n} \ell_i(x)$ has roots x_0

Let $\gamma(x) = \sum_{i=0}^{n} \ell_i(x) - 1$. Any root of this polynomial is a value where the sum of each $\ell_i(x)$ is 1. Since $\ell_i(x_j) = \delta_{ij}$, it is the case that each x_j is a root of γ. Therefore, γ has at least $n + 1$ roots, but its degree is n. By the fundamental theorem of algebra, $\gamma(x) = 0$. But then $\sum_{i=0}^{n} \ell_i(x) = 1$ necessarily. ■

It should be clear from the fact that $\ell_i(x_j) = \delta_{ij}$ that Lagrange's form is a valid interpolation. We don't really care what its value is beyond the arguments $x_0, \ldots, x_n$.

5.3 Error of interpolation

Theorem 9 (Error of interpolation)**.** Let $f \in C^{n+1}[a, b]$ and p a polynomial of degree $\leq n$ which interpolates f on $n + 1$ distinct points within $[a, b]$. Then for each $x \in [a, b]$, there is a number $\zeta = \zeta_x \in (a, b)$ s.t.

$$f(x) - p(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}\eta_{n+1}(x)$$

or equivalently

$$f(x) - p(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}\prod_{j=0}^{n}(x - x_i)$$

5.4 Divided differences

In Newton's form, we use $f[x_0, \ldots, x_k]$ to denote c_k. In other words, we re-write

$$\begin{aligned} p_k(x) &= \sum_{i=0}^{k} f[x_0, \ldots, x_i]\eta_i(x) \\ &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_1)(x - x_2) + \ldots \end{aligned}$$

If $k = 0$, $f[x_0] = f(x_0)$; if $k = 1$,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

where f is the function being interpolated.

Theorem 10. Given $x_0, \ldots, x_n$,

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

This allows us to construct a table for the so called divided differences. For instance, with $n = 3$:

$$\begin{array}{c|c|c|c|c}
x_0 & f[x_0] & f[x_0, x_1] & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] \\
x_1 & f[x_1] & f[x_1, x_2] & f[x_1, x_2, x_3] & \\
x_2 & f[x_2] & f[x_2, x_3] & & \\
x_3 & f[x_3] & & &
\end{array}$$

Example. Assume $f(3) = 1$, $f(1) = -3$, $f(5) = 2$, $f(6) = 4$ where these arguments are $x_0, \ldots, x_3$. We begin the table thus:

$$\begin{array}{c|c|c|c|c}
3 & 1 & f[x_0, x_1] & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] \\
1 & -3 & f[x_1, x_2] & f[x_1, x_2, x_3] & \\
5 & 2 & f[x_2, x_3] & & \\
6 & 4 & & &
\end{array}$$

This is information sufficient to compute $f[x_2, x_3]$, which is

$$f[x_2, x_3] = \frac{f(x_2) - f(x_3)}{x_2 - x_3} = \frac{2-4}{5-6} = 2$$

giving

$$\begin{array}{c|c|c|c|c}
3 & 1 & f[x_0, x_1] & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] \\
1 & -3 & f[x_1, x_2] & f[x_1, x_2, x_3] & \\
5 & 2 & 2 & & \\
6 & 4 & & &
\end{array}$$

Same logic gives $f[x_1, x_2] = 5/4$ and $f[x_0, x_1] = 2$:

$$\begin{array}{c|c|c|c|c}
3 & 1 & 2 & f[x_0, x_1, x_2] & f[x_0, x_1, x_2, x_3] \\
1 & -3 & 5/4 & f[x_1, x_2, x_3] & \\
5 & 2 & 2 & & \\
6 & 4 & & &
\end{array}$$

Now, $f[x_0, x_1, x_2] = (f[x_1, x_2] - f[x_0, x_1])/(x_2 - x_0)$. Using the table, this gives $(5/4 - 2)/(2) = -3/8$. Similarly, $f[x_1, x_2, x_3] = (f[x_2, x_3] - f[x_1, x_2])/(x_3 - x_1) = (2 - 5/4)/(5) = 3/20$.

3	1	2	$-3/8$	$f[x_0, x_1, x_2, x_3]$
1	-3	$5/4$	$3/20$	
5	2	2		
6	4			

The last value is computed in similar fashion.

Theorem 11 (Error of interpolation with divided differences)**.** Let p with degree $\leq n$ an interpolator of f on nodes $x_0, \ldots, x_n$. If $t \neq x_i$ for all i is a real number, then

$$f(t) - p(t) = f[x_0, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j)$$

($\star$) **Observation**. Assume $p(x)$ is of degree 1 (linear). Then $f(x) - p(x) = \frac{f^{(2)}(\zeta_x)}{2!}(x - x_0)(x - x_1)$ for some ζ_x in $[x_0, x_1]$, in accordance with the **Error of interpolation** theorem. See that $\varphi(x) = (x - x_0)(x - x_1)$ is a quadratic expression with roots x_0, x_1 and minimum at $x_m = (x_0 + x_1)/2$. And since $\varphi(x_m)$ is negative, $\varphi(x) \geq \varphi(x_m) \Rightarrow |\varphi(x)| \leq |\varphi(x_m)|$. In consequence,

$$|\varphi(x)| \leq |(x - x_0)(x - x_1)| = \frac{|x_1 - x_0|^2}{4}$$

In consequence,

$$|f(x) - p(x)| \leq \frac{f^{(2)}(\zeta_x)}{8} |x_1 - x_0|^2$$

If we choose M the maximum of $f^{(2)}(x)$ in $[x_0, x_1]$, then we have

$$|f(x) - p(x)| \leq \frac{f^{(2)}(\zeta_x)}{8} |x_1 - x_0|^2 \leq \frac{M}{8} |x_1 - x_0|^2$$

In consequence,

$$|f(x) - p(x)| \leq \frac{\max_x f^{(2)}_{|[x_0, x_1]}(x)}{8} |x_1 - x_0|^2$$

5.5 Hermite interpolation

Hermite interpolation consists of interpolating a function f *and its derivative* in certain nodes $x_0, \ldots, x_n$. For instance, if two points are given, we wish

$$p(x_i) = f(x_i), \qquad p'(x_i) = f'(x_i), \qquad \text{for } i = 0, 1$$

See that this gives four conditions, which means it is reasonable to seek a solution in Π_3 the space of all polynomials of degree ≤ 3. (An element in Π_3 has four coefficients.) But instead of writing $p(x)$ in terms of the coefficients for $1, x, x^2, x^3$, we shall write

$$p(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^2(x - x_1)$$

which gives

$$p'(x) = b + 2c(x - x_0) + 2d(x - x_0)(x - x_1) + d(x - x_0)^2$$

Writing the polynomial this way allows us to express the four conditions as follows:

$$\begin{aligned} a &= f(x_0) \\ b &= f'(x_0) \\ f(x_1) &= a + bh + ch^2 \\ f'(x_1) &= b + 2ch + dh^2 \end{aligned}$$

where $h = x_1 - x_0$. This approach readily gives a and b, c can be determined from the third equation, and d from the fourth equation.

Now, observe that from the third equation,

$$\begin{aligned} c = \frac{f(x_1) - a - bh}{h^2} &= \frac{f(x_1) - f(x_0)}{h^2} - \frac{f'(x_0)h}{h^2} \\ &= \frac{f(x_1) - f(x_0)}{(x_1 - x_0)^2} - \frac{f'(x_0)}{(x_1 - x_0)} \end{aligned}$$

5.6 Splines

A spline is an interval-based polynomial approximation. We say $S(x)$ defined on $[x_0, x_n]$ is a spline of degree k if

1. S is polynomial of degree $\leq k$ on each sub-interval $[x_i, x_{i+1})$ para $i = 0, \ldots, n-1$;
2. The derivaitves of $S^{(i)}$ are continuous $[x_0, x_n]$ for $i = 0, \ldots, k-1$.

A linear spline is a spline of the form:

$$S(x) = \begin{cases} S_2(x) = a_0 x + b_0 & x \in [x_0, x_1) \\ S_1(x) = a_1 x + b_1 & x \in [x_1, x_2) \\ \vdots & \\ S_{n-1}(x) = a_{n-1} x + b_{n-1} & x \in [x_{n-1}, x_n) \end{cases}$$

where each a_i, b_i is to be determined. This gives $2n$ conditions. Clearly, for a fixed i,

$$\begin{aligned} a_i x_i + b_i &= S_i(x_i) = f(x_i) \\ a_i x_{i+1} + b_i &= \lim_{x \to x_{i+1}} S_i(x) = S_{i+1}(x_{i+1}) = f(x_{i+1}) \end{aligned}$$

Subtracting the first equation in the second one,

$$a_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \qquad b_i = f(x_i) - a_i x_i$$

The error of approximation in a linear spline can be determined if we assume each $x_0, \ldots, x_n$ to be equidistant. In other words, assume f is two times continuously differentiable in $[a, b]$ and $x_k = a + kh$ for $h = (b-a)/n$ the length of each sub-interval. Then on each interval we have a degree 1 polynomial, which means the error of interpolation for each $x \in [a, b]$ satisfies

$$|e(x)| < \frac{M}{8} h^2$$

where $|f''(x)| \leq M$ for all $x \in [x_0, x_n]$. (See **Observation** marked with $\star$.)

5.6.1 Cubic splines

5.7 Exercises

(1) Construct the Lagrange and Newton interpolating polynomials for $f(x) = 1/x$ taking $x_0 = 2, x_1 = 2.5, x_2 = 4$. Compare them and give their degrees. Graph them. Analize the results (?).

(Newton) Newton's interpolating polynomial has the form

$$\varphi(x) = \sum_{i=0}^{n} a_i \eta_i(x)$$

where each a_i is to be determined and $\eta_i = \prod_{j=0}^{i-1}(x - x_j)$. We first do a brute construction, then a construction using divided differences.

(Newton, brute) Take $\varphi_0(x) = \frac{1}{2}$, a polynomial interpolating f at x_0. Now let $\varphi_1(x) = \varphi_0(x) + c(x - x_0)$ and solve $\varphi_1(x_1) = f(x_1)$:

$$\begin{aligned}
&\frac{1}{2} + c(2.5 - 2) = f(2.5) \\
\iff &c = 2 \times \left(\frac{2}{5} - \frac{1}{2}\right) \\
\iff &c = \frac{4}{5} - \frac{5}{5} \\
\iff &c = -\frac{1}{5}
\end{aligned}$$

$\therefore\ \varphi_1(x) = \frac{1}{2} - \frac{1}{5}(x - 2)$.

Now we let $\varphi_2(x) = \frac{1}{2} - \frac{1}{5}(x - 2) + c(x - 2)(x - 2.5)$ and solve for c in $\varphi_2(4) = f(4)$:

$$\begin{aligned}
&\frac{1}{2} - \frac{1}{5} \times 2 + 2 \times \frac{3}{2} c = \frac{1}{4} \\
\iff &\frac{1}{2} - \frac{2}{5} + 3c = \frac{1}{4} \\
\iff &c = \frac{1}{3}\left(\frac{10}{40} + \frac{16}{40} - \frac{20}{40}\right) \\
\iff &c = \frac{1}{3}\left(\frac{3}{20}\right) \\
\iff &c = \frac{1}{20}
\end{aligned}$$

So finally we have the following polynomial in Newton's form:

$$\varphi(x) = \frac{1}{2} - \frac{1}{5}(x-2) + \frac{1}{20}(x-2.5)(x-2)$$

which is of degree 2.

(Newton, divided diffs.) The table of divided differences to interpolate f on x_0, x_1, x_2 is

$$\begin{array}{c|c|c|c} x_0 & f[x_0] & f[x_0,x_1] & f[x_0,x_1,x_2] \\ x_1 & f[x_1] & f[x_1,x_2] & \\ x_2 & f[x_2] & & \end{array} \quad \Rightarrow \quad \begin{array}{c|c|c|c} 2 & 1/2 & f[x_0,x_1] & f[x_0,x_1,x_2] \\ 2.5 & 2/5 & f[x_1,x_2] & \\ 4 & 1/4 & & \end{array}$$

Now, $f[x_1, x_2] = (f[x_2] - f[x_1])/(x_2 - x_1) = (1/4 - 2/5)/(1.5) = -1/10$:

$$\begin{array}{c|c|c|c} 2 & 1/2 & f[x_0,x_1] & f[x_0,x_1,x_2] \\ 2.5 & 2/5 & -1/10 & \\ 4 & 1/4 & & \end{array}$$

Now, same rule gives $f[x_0, x_1] = -1/5$:

$$\begin{array}{c|c|c|c} 2 & 1/2 & -1/5 & f[x_0,x_1,x_2] \\ 2.5 & 2/5 & -1/10 & \\ 4 & 1/4 & & \end{array}$$

Lastly, $f[x_0, x_1, x_2] = (f[x_1, x_2] - f[x_0, x_1])/(x_2 - x_0) = 1/20$:

$$\begin{array}{c|c|c|c} 2 & 1/2 & -1/5 & 1/20 \\ 2.5 & 2/5 & -1/10 & \\ 4 & 1/4 & & \end{array}$$

Then,

$$\begin{aligned} \varphi(x) &= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\ &= \frac{1}{2} - \frac{1}{5}(x-2) + \frac{1}{20}(x-2.5)(x-2) \end{aligned}$$

which is is what we had obtained before.

(Lagrange) Lagrange's polynomial has the form

$$\phi(x) = \sum_{i=0}^{n} y_i \ell_i(x)$$

where

$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Then,

$$\begin{aligned}
\ell_0(x) &= \left(\frac{x - 2.5}{2 - 2.5}\right)\left(\frac{x - 4}{2 - 4}\right) \\
&= \left(-\frac{1}{2}(x - 2.5)\right)(-2(x - 4)) \\
&= (x - 2.5)(x - 4)
\end{aligned}$$

$$\begin{aligned}
\ell_1(x) &= \left(\frac{x - 2}{2.5 - 2}\right)\left(\frac{x - 4}{2.5 - 4}\right) \\
&= -\frac{3}{4}(x - 2.5)(x - 4)
\end{aligned}$$

$$\begin{aligned}
\ell_2(x) &= \left(\frac{x - 2}{4 - 2}\right)\left(\frac{x - 2.5}{4 - 2.5}\right) \\
&= 3(x - 2)(x - 4)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi(x) &= \frac{1}{2}\ell_0(x) + \frac{2}{5}\ell_1(x) + \frac{1}{4}\ell_2(x) \\
&= \frac{1}{2}(x - 2.5)(x - 4) - \frac{3}{10}(x - 2.5)(x - 4) + \frac{3}{4}(x - 2)(x - 4)
\end{aligned}$$

(2) Prove: If f polynomial of degree $\leq n$ then the polynomial of degree $\leq n$ that interpolates f at $x_0, \dots, x_n$ is f itself.

It is a theorem that there exists a polynomial of degree $\leq n$ that interpolates f in the points $x_0, \dots, x_n$, and that this polynomial is unique. f is a polynomial of degree $\leq n$ that interpolates f at $x_0, \dots, x_n$. This concludes the proof.

Given $x_0, \ldots, x_n$, prove the following properties of Lagrange's basic polynomials $\ell_k(x)$:

1. Their sum is 1.
2. Their linear combination with coefficients $x_0, \ldots, x_n$ is x.
3. Their linear combinations with coefficients $x_0^m, \ldots, x_n^m$, with $m \leq n$, is x^m.

(1) This was already proven in a previous section.

(2) See that

$$\sum_{k=0}^{n} x_k \ell_k(x) = x \iff \sum_{k=0}^{n} x_k \ell_k(x) - x = 0 \tag{1}$$

Let $\phi(x) = \sum x_k \ell_k(x) - x$. Since $\ell_k(x_j) = \delta_{kj}$, $\varphi(x_j) = x_j - x_j = 0$, meaning that each x_j is root of ϕ. This means ϕ is a polynomial of degree $\leq n$ with $n+1$ roots. Then by virtue of the fundamental theorem of algebra, it is necessarily the case that $\phi(x) = 0$. Then the RHS of (1) holds, which concludes the proof.

(3) Assume $m \leq n$. See that:

$$\sum_{k=0}^{n} x_k^m \ell_k(x) = x^m \iff \sum_{k=0}^{n} x_k^m \ell_k(x) - x^m = 0 \tag{2}$$

Let $\phi(x) = \sum x_k^m \ell_k(x) - x^m$. The polynomial has degree $\leq n$ due to the fact that $m \leq n$. Once more, $\phi(x_j) = x_j^m - x_j^m = 0$. Etc.

(6) Let $f : [0,5] \to \mathbb{R}$, $f(x) = 2^x$. Let P_n a polynomial of degree at most n that interpolates f at $n+1$ distinct points in $[0,5]$. Prove that for any x in said interval,

$$|P(x) - f(x)| \leq \frac{32 \times 5^{n+1}}{(n+1)!}$$

Recall that for $x \in [0,5]$,

$$P(x) - f(x) = \frac{f^{(n+1)}(\zeta_x)}{(n+1)!}\eta_{n+1}(x)$$

for some $\zeta_x \in [0,5]$. Now, $\frac{d}{dx}2^x = \ln 2 \times 2^x$, whose derivative is $\ln^2 2 \times 2^x$, whose derivative is $\ln^3 2 \times 2^x$, etc. $\therefore$ $f^{(n+1)}(x) = \ln^{n+1} 2 \times 2^x = (n+1)\ln 2 \times 2^x$.

$$\begin{aligned}
\therefore\ P(x) - f(x) &= \frac{\ln^{n+1}(2) \times 2^{\zeta_x}}{(n+1)!}\prod_{j=0}^{n}(x - x_j) \\
&= \frac{\ln^{n+1}(2) \times 2^{\zeta_x}}{n!}\prod_{j=0}^{n}(x - x_j)
\end{aligned}$$

Since $0 < \ln(2) < 1$, we know $0 < \ln^{n+1}(2) < 1$, and therefore

$$\frac{\ln^{n+1}(2) \times 2^{\zeta_x}}{(n+1)!}\prod_{j=0}^{n}(x - x_j) < \frac{2^{\zeta_x}}{(n+1)!}\prod_{j=0}^{n}(x - x_j)$$

Necessasrily, $2^{\zeta_x} \leq 5$ and

$$\prod_{j=0}^{n}(5 - x_j) \leq \prod_{j=0}^{n} 5 = 5^{n+1}$$

From this follows that

$$P(x) - f(x) \leq \frac{2^5 \times 5^{n+1}}{(n+1)!}$$

Since the RHS is positive, taking absolute value on both sides gives us the desired result.

(7) Prove that when interpolating $\cosh(x)$ with a polynomial $p(x)$ of degree ≤ 22 in $[-1,1]$, the error is $\leq 5 \times 10^{-16}$.

A polynomial of degree $n = 22$ has 23 coefficients, corresponding to the need of determining 23 nodes $x_0, \ldots, x_{23}$. So we let $n = 23$. It is known that $\frac{d}{dx}\cosh x = \sinh x$, whose derivative is once more $\cosh x$. So, $\cosh^{(n+1)} = \cosh^{(24)} = \cosh$. In consequence,

$$p(x) - \cosh(x) = \frac{\cosh(\zeta_x)}{(n+1)!}\prod_{j=0}^{n}(x - x_j)$$

for some $\zeta_x \in (-1,1)$. The graph of $\cos h$ is symmetric (the function is even). Therefore, it achieves its maximum (restricted to $[-1,1]$) at $\cosh(1) = \cosh(-1) = (e^1 + e^{-1})/2 = (e^2+1)/2e$. So

$$\left|\frac{\cosh(\zeta_x)}{(n+1)!}\right|\left|\prod_{j=0}^{n}(x - x_j)\right| \leq \frac{e^2+1}{2e(n+1)!}\left|\prod_{j=0}^{n}(x - x_j)\right|$$

Now, since $x \in [-1,1]$, it is obvious that the maximum value which the factorial (and its absolute value) can take is 1. So

$$\frac{e^2+1}{2e(n+1)!}\left|\prod_{j=0}^{n}(x - x_j)\right| \leq \frac{e^2+1}{2e(n+1)!}$$

Now, with a calculator one can see that $24! > 10^{16}$. Furthermore, $(e^2+1)/2e < 5$. So,

$$\frac{e^2+1}{2e(n+1)!} < \frac{5}{10^{16}} = 5 \times 10^{-16} \qquad \blacksquare$$

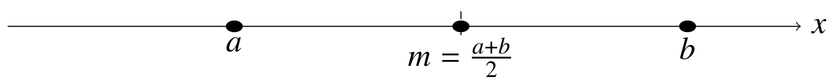
(8) (a) Let $a < b$, m the midpoint between them, $p = m - h$ and $q = m + h$ for $0 \leq h \leq (b-a)/2$. Prove that for all $x \in [a, b]$,

$$|(x-p)(x-q)| \leq \frac{(b-a)^2}{4}$$

(b) Let $x_i = a + i(\frac{b-a}{n})$ for $i = 0, \ldots, n$ equidistant points in $[a, b]$. Prove that for all $x \in [a, b]$,

$$|(x-x_0)\ldots(x-x_n)| \leq \frac{(b-a)^{n+1}}{2^{n+1}}$$

(a) See the graph below for reference.



Define $f(x)$ as the quadratic function with roots a, b and upward tails: $f(x) = (x-a)(x-b)$. We know that if $x_0 = p, x_1 = q$, then a linear interpolation of f at nodes x_0, x_1 on the interval $[a, b]$ satisfies:

$$|f(x) - p(x)| = \left|\frac{f''(\zeta_x)}{2!}\right| |(x-x_0)(x-x_1)| \leq \frac{M}{8}|x_1 - x_0|^2$$

with M the maximum of $|f''(x)|$ in $[a, b]$ and some $\zeta_x \in (a, b)$. Now,

$$|x_1 - x_0|^2 = |(m+h) - (m-h)|^2 = 4h^2$$

Therefore,

$$\begin{aligned}
&\left|\frac{f''(\zeta_x)}{2}\right| |(x-p)(x-q)| \leq \frac{4h^2 \times M}{8} \\
\Rightarrow &\left|\frac{f''(\zeta_x)}{2}\right| |(x-p)(x-q)| \leq \frac{M}{2}h^2 \\
\Rightarrow &|f''(\zeta_x)|\, |(x-p)(x-q)| \leq Mh^2
\end{aligned}$$

Here's the trick: since f is quadratic, f'' is constant and therefore $|f''(\zeta_x)| = |M|$. So dividing by $|M|$ on both sides we get

$$|(x-p)(x-q)| \leq h^2$$

Finally, we know $0 \leq h \leq (b-a)/2$, so

$$|(x-p)(x-q)| \leq h^2 \leq \left(\frac{b-a}{2}\right)^2 = \frac{(b-a)^2}{4}$$

which is what we wanted to show.

(b) Let $x_0, \ldots, x_n$ s.t. $x_i = a + i\frac{(b-a)}{n}$ for $0 \leq i \leq n$. We wish to show that for all $x \in [a, b]$,

$$|\eta_{n+1}(x)| \leq \frac{(b-a)^{n+1}}{2^{n+1}}$$

Take x_i, x_{i+1} and let f be the quadratic function with roots x_i, x_{i+1} and upward tails. Using the exact same reasoning of (a), we know

$$|(x-x_i)(x-x_{i+1})| \leq \frac{(x_{i+1}-x_i)^2}{2^2} \tag{3}$$

Since the points are equidistant, $x_{i+1} - x_i = \frac{b-a}{n}$, as is easy to prove algebraically, so equation (2) is equivalent to:

$$|(x-x_i)(x-x_{i+1})| \leq \left(\frac{b-a}{2n}\right)^2 \tag{4}$$

See that exercise (a) satisfied this formula, where had nodes x_0, x_1 and therefore $n = 1$. In other words, exercise (a) was the base case for an inductive proof and we can now assume that the statement holds for $n = k-1$ for some $k > 2$. Our inductive case consists of having points $x_0, \ldots, x_k$, where we wish to prove

$$|(x-x_0)\ldots(x-x_{k-1})| \leq \frac{(b-a)^k}{2^k} \Rightarrow |(x-x_0)\ldots(x-x_{k-1})(x-x_k)| \leq \frac{(b-a)^{k+1}}{2^{k+1}}$$

So assume

$$|\eta_k(x)| \leq \frac{(b-a)^k}{2^k}$$

Now take

$$|\eta_k(x)|\,|(x - x_{k+1})| \leq \frac{(b-a)^k}{2^k}\,|(x - x_{k+1})|$$

Since $x \in [a, b]$ and $x_{k+1} = b$ is the last node, $|x - x_{k+1}| \leq b - a$ (i.e. the maximum distance that x can take from b is when x is exactly a). So

$$|\eta_k(x)|\,|(x - x_{k+1})| \leq \frac{(b-a)^k}{2^k}\,|(x - x_{k+1})| \leq \frac{(b-a)^k}{2^k}(b-a)$$

Something's off. But should be along this lines.

(9) (a) Let $f(x) = \cos x\pi$. Find a polynomial of degree ≤ 3 that verifies:

$$p(-1) = f(-1), \qquad p(0) = f(0), \qquad p(1) = f(1), \qquad p'(1) = f'(1)$$

(b) Find a polynomial of degree ≤ 4 that verifies previous conditions and the added condition $p''(1) = f''(1)$.

(a) Let $x_0 = -1, x_2 = 0, x_3 = 1$. To keep track of which nodes have double (or more) conditions, let $z_0 = x_0, z_1 = x_1, z_2 = x_2, z_3 = x_2$ (since x_2 has two conditions, one for p and one for p'.) Then, our table of divided differences is

$$\begin{array}{c|c|c|c|c}
z_0 & f[z_0] & f[z_0, z_1] & f[z_0, z_1, z_2] & f[z_0, z_1, z_2, z_3] \\
z_1 & f[z_1] & f[z_1, z_2] & f[z_1, z_2, z_3] & \\
z_2 & f[z_2] & f[z_2, z_3] & & \\
z_3 & f[z_3] & & &
\end{array}$$

So now we simply compute the first column.

$$\begin{array}{c|c|c|c|c}
-1 & -1 & f[z_0, z_1] & f[z_0, z_1, z_2] & f[z_0, z_1, z_2, z_3] \\
0 & 1 & f[z_1, z_2] & f[z_1, z_2, z_3] & \\
1 & -1 & f[z_2, z_3] & & \\
1 & -1 & & &
\end{array}$$

Amazing. So now we compute $f[z_2, z_3] = f[x_2, x_2]$. Since this is a repeated node, by definition it corresponds to $f'(x_2) = f'(0)$. So, we see that $f'(0) = -\sin(0\pi)\pi = 0$. Similarly,

$$f[z_1, z_2] = \frac{f[z_2] - f[z_1]}{z_2 - z_1} = \frac{-2}{1} = -2$$

and

$$f[z_0, z_1] = \frac{f[z_1] - f[z_0]}{z_1 - z_0} = 2$$

So,

$$\begin{array}{r|r|r|c|c}
-1 & -1 & 2 & f[z_0, z_1, z_2] & f[z_0, z_1, z_2, z_3] \\
0 & 1 & -2 & f[z_1, z_2, z_3] & \\
1 & -1 & 0 & & \\
1 & -1 & & &
\end{array}$$

Now,

$$f[z_1, z_2, z_3] = \frac{f[z_2, z_3] - f[z_1, z_2]}{z_3 - z_1} = \frac{2}{1} = 2$$

$$f[z_0, z_1, z_2] = \frac{f[z_1, z_2] - f[z_0, z_1]}{z_2 - z_0} = -\frac{2}{2} = -2$$

$$\begin{array}{r|r|r|r|c}
-1 & -1 & 2 & -2 & f[z_0, z_1, z_2, z_3] \\
0 & 1 & -2 & 2 & \\
1 & -1 & 0 & & \\
1 & -1 & & &
\end{array}$$

At last,

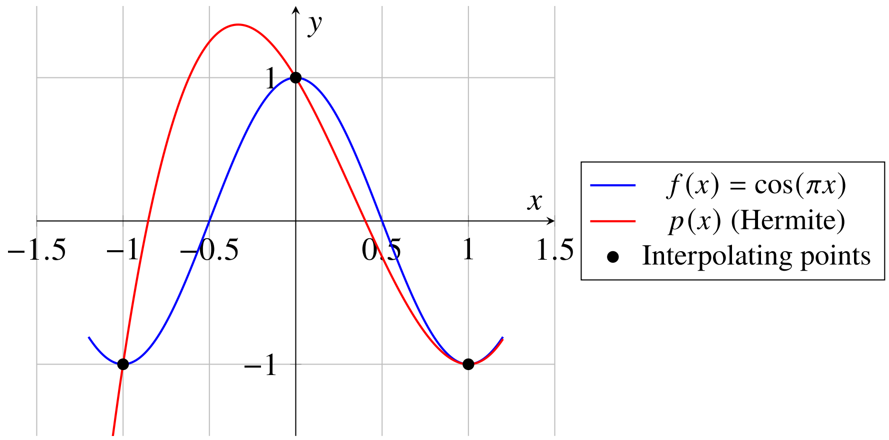
$$f[z_0, \ldots, z_3] = \frac{f[z_1, z_2, z_3] - f[z_0, z_1, z_2]}{z_3 - z_0} = 2$$

The interpolating polynomial is built using Newton's form—remember that $f[x_0], f[x_0, x_1], \ldots$ etc are the coefficients $c_0, c_1, \ldots$ in Newton's form $\sum_{i=0}^{n} c_i \eta_i(x)$. So,

$$p(x) = f[z_0] + f[z_0, z_1](x - z_0) + f[z_0, z_1, z_2](x - z_0)(x - z_1) + f[z_0, \ldots, z_3](x - z_0)(x - z_1)(x - z_2)$$

Simplifying,

$$p(x) = -1 + 2(x + 1) - 2(x + 1)x + 2(x + 1)x(x - 1)$$



y
x
1
−1
−1.5
−1
−0.5
0.5
1
1.5
$f(x) = \cos(\pi x)$
$p(x)$ (Hermite)
Interpolating points

(10) We wish to approximate $f(x) = \sqrt{x}$ with an error of at most 5×10^{-8} using a linear spline and quadratic interpolation every three nodes.

Determine the least number of nodes n of the form $x_i = 1 + \frac{i}{n}$, with $i = 0, \ldots, n$, and interval length h, so that the error bound is met.

(Linear spline) See that the desired approximation falls within $[1, 2]$. For a linear spline, the error of approximation obeys

$$|e(x)| < \frac{M}{8} h^2, \qquad x \in [1, 2] \text{ and } M = \max |f''|$$

where we can think of h as a function of n, i.e. $h(n) = \frac{1}{n}$ for $n \in \mathbb{N}$.

$$\begin{aligned} & \frac{M}{8} h(n)^2 \leq 5 \times 10^{-8} \\ \iff & \frac{M}{(n)^2} \leq 40 \times 10^{-8} \\ \iff & \frac{10^8 M}{40} \leq n^2 \\ \iff & \sqrt{\frac{10^8 M}{40}} \leq n \end{aligned}$$

Now, suffices to see that

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f''(x) = -\frac{1}{4x^{3/2}}$$

Clearly then $|f''(x)|$ is decreasing and its maximum in $[1, 2]$ occurs at $x = 1$:

$$M = |f''(1))| = \frac{1}{4}$$

So, we require

$$
\begin{aligned}
&\sqrt{\frac{10^8}{40 \times 4}} \leq n \\
\iff &\frac{10^4}{\sqrt{160}} \leq n \\
\iff &\frac{10.000}{\sqrt{16 \times 10}} \leq n \\
\iff &\frac{10.000}{4\sqrt{10}} \leq n \\
\iff &\frac{2500}{\sqrt{10}} \leq n \\
\iff &790.569 \leq n
\end{aligned}
$$

So fixing $n = 791$ suffices.

(Quadratic interpolation) Assume we group $x_0, \ldots, x_n$ into x_0, x_1, x_2, then x_3, x_4, x_5, etc. Let $\overrightarrow{x}_i$ denote the ith grouping of three nodes. We are of course assuming that there are $n + 1 = 3k$ nodes with $k \in \mathbb{N}$. The function $h(n)$ which specifies the distance between nodes will be specified later.

Assume each $\overrightarrow{x}_i$ is used to fit a quadratic polynomial $q_i(x) = a_i x^2 + b_i x + c_i$. The error of interpolation will then be specific to each interval. In other words, for any x belonging to the interval specified by $\overrightarrow{x}_i$,

$$
|e(x)| = \left| \frac{f^{(3)}(\zeta_x)}{3!} \prod_{\widetilde{x} \in \overrightarrow{x}_i} (x - \widetilde{x}) \right|
$$

for some ζ_x in the interval of interest. Taking $M = \max \left| f^{(3)}(x) \right|$, with $f^{(3)}(x) = \frac{3}{8x^{\frac{5}{2}}}$, we obtain $M = 3/8$. Now the question becomes whether we can bound the factorial expression. See that

$$
\varphi(x) = (x - x_0)(x - x_1)(x - x_2)
$$

is a cubic function with distinct roots. (We could have chosen any successive values for x_0, x_1, x_2.) Consider φ as restricted to the interval $[x_0, x_2]$. We know that it will have three roots, a maximum x_M in $[x_0, x_1]$ and a minimum x_m at $[x_1, x_2]$. Since the roots are equidistant (root symmetry), φ is symmetric around the mid-root x_1 and $\varphi(x_m) = \varphi(x_M)$. But where is the critical point?

($\star$) Let us consider a centered cubic polynomial, without loss of generality. Let $\phi(x) = (x - a)x(x - b)$, with mid-root zero. Assuming root symmetry, $a = -b$. So letting $r = b$ we have

$$\phi(x) = (x-r)x(x+r) = x^3 - xr^2$$

Its derivative $\phi'(x) = 3x^2 - r^2$ is zero if and only if $x = \pm\frac{r}{\sqrt{3}}$. We have already established that these critical points are equal in their absolute values. Now it only suffices to see that

$$\phi\left(\frac{r}{\sqrt{3}}\right) = -\frac{2r^3}{3\sqrt{3}} = -\frac{(c-a)^3}{12\sqrt{3}}$$

(because $r = (c-a)/2$). Therefore,

$$\max|\phi(x)| = \frac{(c-a)^3}{12\sqrt{3}}$$

From $(\star)$ readily follows that

$$\left|\prod_{\widetilde{x}\in\overrightarrow{x}_i}(x-\widetilde{x})\right| \leq \frac{h^3}{12\sqrt{3}}$$

where h is the distance between the last node in a grouping and the first (i.e. $x_2 - x_0 = x_5 - x_3 = \ldots$ etc.) In consequence,

$$|e(x)| = \left|\frac{f^{(3)}(\zeta_x)}{3!}\prod_{\widetilde{x}\in\overrightarrow{x}_i}(x-\widetilde{x})\right|$$
$$\Rightarrow |e(x)| \leq 3/8\frac{h^3}{12\sqrt{3}}$$

Here, h is a function of n. If $n = 2$ (three points), then $h = 2/3$. If $n = 5$ (six points), then each sub-interval is of length $1/6$ and $h = 2/6$. In general, if $n = 3k$, then $h = \frac{2}{n}$. So we have

$$|e(x)| \leq \frac{3}{8\times\sqrt{3}}\frac{2}{n} = \frac{3}{4\sqrt{3}n}$$

So now all that is needed is to bound the RHS expression to 5×10^{-8}:

$$\frac{3}{4\sqrt{3}n} \leq 5 \times 10^{-8} \iff \dots$$

bla bla. This is the simplest part of the problem so I skip it.

(10) Let $f(x) = \cos x$. Determine the step-length h and minimum number of nodes $n + 1$ needed to approximate $f(x)$ via linear spline in $[0, 2\pi]$ with an error less than or equal to 5×10^{-7}.

We know

$$|e(x)| \leq \frac{M}{8} h(n)^2$$

where $h(n) = \frac{2\pi}{n}$ anda $M = \max |f''(x)|$. Since $f''(x) = -\cos x$, its maximum in $[0, 2\pi]$ is 1. From this follows that

$$|e(x)| \leq \frac{4\pi^2}{8n^2} = \frac{1}{2} \left(\frac{\pi}{n}\right)^2$$

and all that is left is bounding said expression to be less than or equal to 5×10^{-7}. If one does the math, we obtain that said condition holds iff $n =\geq \pi \times 10^3 \approx 3141.59$, which means letting $n = 3142$ suffices. So the number of nodes needed is $n + 1 = 3143$. The step length h results $\frac{2\pi}{3142} \approx 0.001999 \approx 0.002$.

(12) (a) Determine α, β, γ such that

$$S(x) = \begin{cases} \alpha x^3 + \gamma x & 0 \leq x \leq 1 \\ -\alpha x^3 + \beta x^2 - 5\alpha x + 1 & 1 \leq x \leq 2 \end{cases}$$

is a cubic spline.

(b) With the determined values, decide if S interpolates $f(x) = 2^x + 1/2x^2 - 1/2x - 1$ in $[0, 2]$ with nodes $\{0, 1, 2\}$.

(c) Graph f and S in $[0, 2]$.

(a) $S(x)$ must satisfy the interpolation constraint and the continuity constraint. Note that $S(x)$ are polynomials with continuous first and second derivatives in $[0, 2]$. Furthermore,

$$S'(x) = \begin{cases} 3\alpha x^2 + \gamma & 0 \leq x \leq 1 \\ -3\alpha x^2 + 2\beta x - 5\alpha & 1 \leq x \leq 2 \end{cases}, \qquad S''(x) = \begin{cases} 6\alpha x & 0 \leq x \leq 1 \\ -6\alpha x + 2\beta & 1 \leq x \leq 2 \end{cases}$$

For all f in $\{S, S', S''\}$ we wish that

$$\lim_{x \to 1(\leftarrow)} f(x) = \lim_{x \to 1(\rightarrow)} f(x)$$

$(a.a)$ Beginning with $S''(x)$, we impose

$$6\alpha = -6\alpha + 2\beta \Rightarrow 12\alpha = 2\beta \Rightarrow \beta = 6\alpha$$

$(a.b)$ Now taking $S'(x)$, we impose

$$3\alpha + \gamma = -3\alpha + 2\beta - 5\alpha$$

Substituting with $\beta = 6\alpha$, we find

$$\gamma = -6\alpha + 2(6\alpha) - 5\alpha \Rightarrow \gamma = \alpha$$

$(a.c)$ Now taking $S(x)$, we impose

$$\alpha + \gamma = -\alpha + \beta - 5\alpha + 1$$

Substituting with $\gamma = \alpha, \beta = 6\alpha$, we find

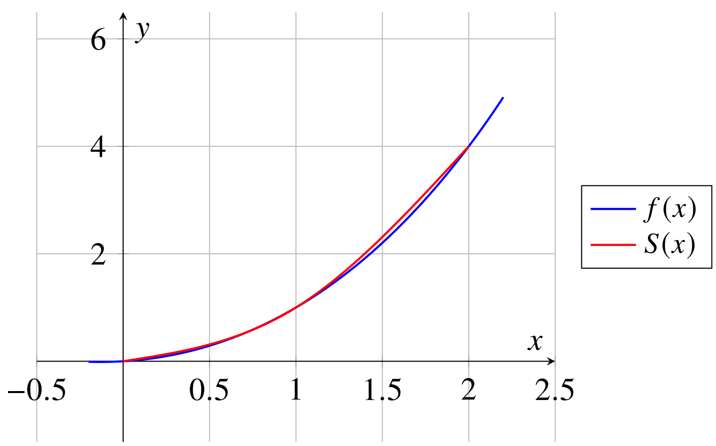
$$2\alpha = 1 \Rightarrow \alpha = \frac{1}{2}$$

So we finally obtain

$$\alpha = \frac{1}{2}, \beta = 3, \gamma = \frac{1}{2}$$

(b) It is simple to see that the spline does interpolate f simply by evaluating S and f on $0, 1, 2$.

(c)



6 Function approximation

6.1 Least squares method

Assume $(x_1, y_1), \ldots (x_m, y_m)$ are points from a function f we wish to approximate. A reasonable approach is to find a linear function with coefficients a_0, a_1 that minimize

$$E = \sum_{i=1}^{m} (y_i - (a_1 x_i + a_0))^2$$

Doing the standard things (take partial derivative, equate it to zero, etc.) we obtain the following normal equations to minimize E:

$$a_0 = \frac{\left(\sum x_i^2\right)\left(\sum y_i\right) - \left(\sum x_i y_i\right)\left(\sum x_i\right)}{m \sum x_i^2 - \sum x_i}, \qquad a_1 = \frac{m \sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{m \sum x_i^2 - \sum x_i}$$

where all sums range from $i = 1$ to m.

The general case is when f is approximated with a polynomial of degree $\leq n$, with $n < m - 1$. Now, we need to determine the coefficients $a_0, \dots, a_n$ that minimize

$$\sum_{i=1}^{m} (y_i - p_n(x_i))^2$$

Again, doing the standard procedure, the following system of equation emerges for variables $a_0, \dots, a_n$:

$$\begin{bmatrix} \sum x_i^0 & \sum x_i^1 & \sum x_i^2 & \dots & \sum x_i^n \\ \sum x_i^1 & \sum x_i^2 & \sum x_i^3 & \dots & \sum x_i^{n+1} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \sum x_i^2 & \sum x_i^{n+1} & \sum x_i^{n+2} & \dots & \sum x_i^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sum y_i x_i^0 \\ \sum y_i x_i^1 \\ \vdots \\ \sum y_i x_i^n \end{bmatrix}$$

6.2 Non-polynomial models

It is possible to propose a non-polynomial model for f, e.g. $\hat{f}(x) = be^{ax}$ or whatever. This produces a non-linear system of equations which cannot be solved simply. However, taking the logarithm of the proposed model allows for a linear approximation. For example, if $\hat{f}(x) = be^{ax}$ then

$$\ln \hat{f}(x) = \ln b + ax$$

which is a linear model that can be fitted using least squares. However, it is important to recall that the values $y_1, \dots, y_m$ must be scaled with $\ln y_1, \dots, \ln y_m$ to fit the logarithmic model.

6.3 Error of approximation

Assume $f \in C[a, b]$ and $P_n(x)$ of degree $\leq n$ an approximation via least squares. Then

$$E = E(a_0, \dots, a_n) = \int_a^b (f(x) - P_n(x))^2 \ dx = \int_a^b \left[f(x) \ sum_{k=0}^n a_k x^k \right]^2 \ dx$$

In general, we wish to find $a_0, \dots, a_n$ such that E is minimal. In general, the partial derivative of E with respect to a_j is zero if and only if

$$\sum_{k=0}^{n} \left(a_k \int_a^b x^{k+j} \ dx \right) = \int_a^b x^j f(x) \ dx, \qquad j = 0, \dots, n$$

This readily provides a $(n+1) \times (n+1)$ system of equations which is to be solved to minimize the error:

$$\begin{bmatrix} \int_a^b dx & \int_a^b x \ dx & \dots & \int_a^b x^n \\ \int_a^b x \ dx & \int_a^b x^2 \ dx & \dots & \int_a^b x^{n+1} \\ \vdots & \vdots & \dots & \vdots \\ \int_a^b x^n & \int_a^b x^{n+1} & \dots & \int_a^b x^{2n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \int_a^b x^0 f(x) \ dx \\ \int_a^b x f(x) \ dx \\ \vdots \\ \int_a^b x^n f(x) \ dx \end{bmatrix}$$

Notice that, in LHS matrix, the coefficient c_{jk} is

$$\int_a^b x^{j+k} \ dx = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1}$$

which allows for a more direct computation of the system. The LHS matrix is called **Hilbert matrix** and it is famously ill-conditioned—i.e. small changes in the coefficients greatly affect the final result.

6.4 A few polynomial theorems

Recall that vectors $\phi_1, \dots, \phi_n$ are linearly independent if $\sum_{i=0}^n c_i \phi_i = 0$ entails that $c_0 = \dots = c_n = 0$. In particular, a function space $\mathcal{F}$ is a vector space such that $\Pi = \bigcup_{i=0}^{\infty} \Pi_i \subseteq \mathcal{F}$.

Theorem 12 (Polynomials of different degrees are linearly independent)**.** Let $\phi_0, \dots, \phi_n$ such that $\phi_j \in \Pi_j$. Then $\{\phi_i\}_{i=0}^n$ is linearly independent in any interval $[a, b]$.

Proof. Assume $c_0, \dots, c_n$ are such that $P(x) = c_0\phi_0(x) + \dots + c_n\phi_n(x) = 0$ for any $x \in [a, b]$. Since $P(x)$ is null for all x in $[a, b]$, each exponentiation x^k must be zero. Since $c_n\phi_n(x)$ is the only term which includes x^n, we must have $c_n = 0$, which readily entails $P(x) = c_0\phi_0(x) + \dots + c_{n-1}\phi_{n-1}(x)$. By recurrence, $c_0 = \dots = c_n = 0$. ■

The previous theorem is highly general, since any polynomial of degree n is by definition a linear combination of $n+1$ polynomials of degree $0, 1, \ldots, n$. For instance, $1+3x+2x^2$ can be considered a linear combination of the polynomials $1, x, x^2$ with coefficients $1, 3, 2$, or of the polynomials $1, 3x, 4x^2$ with coefficients $1, 1, 1/2$, etc.

Theorem 13. Let $\{\phi_0, \ldots, \phi_n\}$ be a set of linearly independent polynomials for any interval $[a, b]$, each of degree $\leq n$. Then any polynomial of degree $\leq n$ can be expressed as a linear combination of $\{\phi_0, \ldots, \phi_n\}$.

6.5 Weighted function approximation: Diagonalizing Hilbert matrix

A weight function $\omega(x)$ on interval I is a function such that:

1. $\omega(x) > 0$ for all $x \in I$.
2. $\omega(x) \neq 0$ for all x in any arbitrary sub-interval of I.

Condition (2) means that ω cannot be constantly zero in any sub-interval, i.e. any zero mapping of ω must be a point. Weight function allow one to give more or less importance to approximations in different regions of an interval of interest.

Now assume we have a linearly independent set of functions $\{\phi_0, \ldots, \phi_n\}$ in $[a, b]$, a weight function ω defined on $[a, b]$, and f continuous in $[a, b]$ which we wish to approximate. Then there is a set of coefficients $a_0, \ldots, a_n$ of

$$P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

that minimize the following error function:

$$E = E(a_0, \ldots, a_n) = \int_a^b \omega(x) \left[f(x) - P(x) \right]^2 \, dx$$

This is the same as we did before: we are approximating f via a polynomial by finding the coefficients that minimize the error of approximation. But now we are (a) including a weight function and (b) defining $P(x)$ as a linear combination of independent polynomials.

Once more, taking the derivative of E with respect to an arbitrary a_j, we find that said derivative is zero if and only if

$$\sum_{k=0}^{n} a_k \int_a^b \omega(x)\phi_k(x)\phi_j(x)\,dx = \int_a^b \omega(x)f(x)0\phi_j(x)\,dx, \qquad j = 0, \ldots, n$$

which readily provides a system of equations. The system is complex, but it can be greatly simplified if we could choose $\{\phi_0, \ldots, \phi_n\}$ such that

$$\int_a^b \omega(x)\phi_k(x)\phi_j(x)\,dx = \alpha_j \delta_{jk}$$

for some $\alpha_j > 0$. In that case, the equation would simplify to

$$a_j \int_a^b \omega(x)\phi_j^2(x)\,dx = \int_a^b \omega(x)f(x)\phi_j(x)\,dx, \qquad j = 0, \ldots, n$$

By assumption, this would yield

$$a_j\alpha_j = \int_a^b \omega(x)f(x)\phi_j(x)\,dx, \qquad j = 0, \ldots, n$$

or equivalently

$$a_j = \frac{1}{\alpha_j}\int_a^b \omega(x)f(x)\phi_j(x)\,dx, \qquad j = 0, \ldots, n$$

In short, a relatively simple system of equations emerges if $\phi_0, \ldots, \phi_n$ are intelligently chosen. But how to choose them intelligently?

[Ortogonal set] A set $\{\phi_0, \ldots, \phi_n\}$ of functoins defined in $[a, b]$ is ortogonal with respect to a weight function ω iff

$$\int_a^b \omega(x)\phi_k(x)\phi_j(x)\,dx = \alpha_j \delta_{jk}, \qquad j = 0, \ldots, n$$

for $\alpha_j > 0$. If $\alpha_j = 1$ for all $j = 0, \ldots, n$ then we say the set is ortonormal.

We can formalize what was said so far as follows:

Theorem 14 (Ortogonal polynomial approximation theorem)**.** If $\{\phi_0, \ldots, \phi_k\}$ defined in $[a, b]$ is ortogonal with respect to ω defined in $[a, b]$, then the least squared approximation of f weighted by ω is

$$P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

with

$$a_k = \frac{1}{\alpha_k} \int_a^b \omega(x) f(x) \phi_k(x)\, dx$$

where $\alpha_k = \int_a^b \omega(x) \phi_k^2(x)\, dx$.

Theorem 15 (Ortogonal set generation)**.** The following is an ortogonal set of functions defined in $[a, b]$ with respect to a weight function ω:

$$\phi_0(x) = 1, \qquad \phi_1(x) = x - B_1, \qquad x \in [a, b]$$

where

$$B_1 = \frac{\int_a^b x\omega(x)\phi_0^2(x)\, dx}{\int_a^b \omega(x)\phi_0^2(x)\, dx}$$

For $k \geq 2$,

$$\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k \phi_{k-2}(x), \qquad x \in [a, b]$$

with

$$B_k = \frac{\int_a^b x\omega(x)\phi_{k-1}^2(x)\, dx}{\int_a^b \omega(x)\phi_{k-1}^2(x)}, \qquad C_k = \frac{\int_a^b x\omega(x)\phi_{k-1}(x)\phi_{k-2}(x)\, dx}{\int_a^b \omega(x)\phi_{k-2}^2(x)\, dx}$$

In general, we will use a few ortogonal sets which are already known, built recurrently from the previous theorem:

1. **Legendre polynomials**:

$$\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{3}, \phi_3(x) = x^3 - \frac{3}{5}x, \phi_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{25}, \dots$$

2. More examples in the textbook.

6.6 Exercises

(1) Approximate $f(x)$ with a polynomial of degree 1 and $g(x)$ with a polynomial of degree two, where:

x	0	1	2	3	4	5	6	7	8	9
$\hat{f}(x)$	-0.1	1.1	1.9	3.2	3.8	5.0	6.0	7.3	8.1	8.9

and

x	-1	0	1	3	6
$\hat{g}(x)$	6.1	2.8	2.2	6	26.9

(f) We know $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ is the least squared approximation of f if and only if $a_0, \ldots, a_n$ are solutions to the following system of equations:

$$\begin{bmatrix} \sum_{i=0}^m x_i^0 & \sum_{i=0}^m x_i^1 & \ldots & \sum_{i=0}^m x_i^m \mid \sum_{i=0}^m x_i^0 y_i \\ \sum_{i=0}^m x_i^1 & \sum_{i=0}^m x_i^2 & \ldots & \sum_{i=0}^{m+1} x_i^m \mid \sum_{i=0}^m x_i^1 y_i \\ \vdots & \vdots & \ldots & \vdots \\ \sum_{i=0}^m x_i^m & \sum_{i=0}^m x_i^{m+1} & \ldots & \sum_{i=0}^m x_i^{2m} \mid \sum_{i=0}^m x_i^m y_i \end{bmatrix}$$

where m is the number of points $(x_i, \hat{f}(x_i))$. Since we wish to approximate f with a polynomial of degree 1, this gives a 2×2 matrix with coefficients

$$c_{11} = \sum_{i=0}^{9} (1) = 10, \qquad c_{11} = \sum_{i=0}^{9} x = 45,$$
$$c_{21} = \sum_{i=0}^{9} x_i = 45, \qquad c_{22} = \sum_{i=0}^{n} x_i^2 = 285$$

The solution vector, on the other hand, is:

$$w_1 = \sum_{i=0}^{9} y_i = 45.2, \qquad w_2 = \sum_{i=0}^{9} x_i y_i = 286.7$$

So, suffices to solve the system

$$\begin{bmatrix} 10 & 45 & 45.2 \\ 45 & 285 & 286.7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4.5 & 4.52 \\ 1 & 6.33 & 6.371 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4.5 & 4.52 \\ 0 & 1.83 & 1.85 \end{bmatrix}$$

So $a_1 = \frac{1.85}{1.83} a_2 = 1.01 a_2$, bla bla. Linear system, cursillo de ingreso.

(g) Not that different from (f): calculate the matrix, which will be 3×3 now (degree two polynomial), solve the resulting system. Calculating the matrix is computing sums (stupid), solving the system is solving a linear system (stupid).

(2) Prove that with $n + 1$ distinct points, the best polynomial approximation (in the sense of least squares) of degree n coincides with the interpolating polynomial.

Let $p_n(x)$ be the interpolating polynomial of f on points $x_0, \ldots, x_n$. Let $\ell_n(x)$ be the least squares approximation of degree n. Observe that

$$\sum_{i=0}^{n} (y_i - p_n(x_i))^2 = 0$$

Since $\ell_n(x)$ minimizes the error among all polynomials of degree n, and taking p_n we have an error of zero, we must have

$$\sum_{i=0}^{n} (y_i - \ell_n(x))^2 = 0$$

which holds if and only if $\ell_n(x)$ is an interpolating polynomial. But the interpolating polynomial is unique. $\therefore$ $p_n(x) = \ell_n(x)$.

(3) Find the polynomial of degree 0 that best approximates $f : [a, b] \to \mathbb{R}$ on $x_1, \ldots, x_n$ in $[a, b]$.

We wish to find a constant polynomial $p_0(x) = c$ such that

$$S = \sum_{i=1}^{n} [y_i - p_0(x_i)]^2 = \sum_{i=1}^{n} [y_i - c]^2$$

is minimized. Taking

$$\begin{aligned}
\frac{\partial S}{\partial c} &= \sum_{i=1}^{n} \frac{\partial}{\partial c}(y_i - c)^2 \\
&= \sum_{i=1}^{n} \frac{\partial u^2}{\partial u} \frac{\partial}{\partial c}(y_i - c) \\
&= \sum_{i=1}^{n} 2(y_i - c)(-1) \\
&= 2 \sum_{i=1}^{n} (c - y_i) \\
&= 2 \left(cn - \sum_{i=1}^{n} y_i \right)
\end{aligned}$$

The expression above is zero if and only if

$$c = \frac{1}{n} \sum_{i=1}^{n} y_i$$

Therefore, the constant polynomial which best approximates f (in the sense of least squares) is the one whose value is the mean of $y_1, \ldots, y_n$.

(4) Approximate the following data with a model of the form $f(x) \sim ae^{bx}$.

x	-1	0	1	2
y	8.1	3	1.1	0.5

If $f(x) = ae^{bx}$ then $g(x) = \ln(f(x)) = \ln a + bx$. Now take the logarithm of the sampled image of f:

x	-1	0	1	2
y	2.09	1.09	0.09	-0.69

We can fit the linear model $g(x) = \ln a + bx$ to this data using the standard procedure. The 2×2 matrix associated to the system whose solutions are $\ln a, b$ is given by:

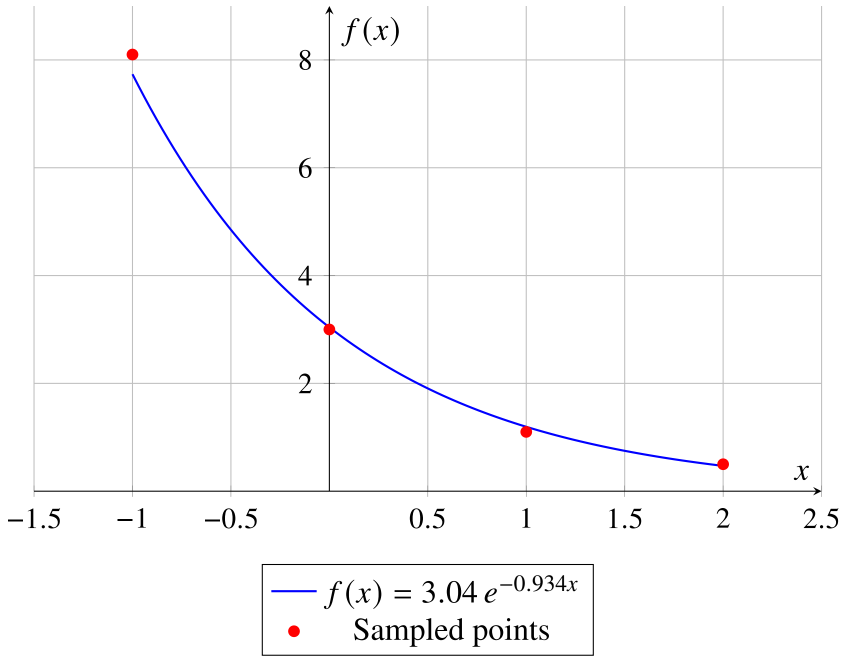
$$\sum_{i=0}^{3} x^0 = 4, \qquad \sum_{i=0}^{3} x_i 2, \qquad \sum_{i=0}^{3} x_i^2 = 6,$$
$$\sum_{i=0}^{3} x_i^0 y_i = 2.58, \qquad \sum_{i=0}^{3} x_i y_i = -3.38$$

In other words, the system is

$$\begin{bmatrix} 4 & 2 & 2.58 \\ 2 & 6 & -3.38 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1.112 \\ 0 & 1 & -0.934 \end{bmatrix}$$

From this follows:

$$\ln a = 1.112 \text{ (or } a = e^{1.112} \approx 3.04\text{)}, \qquad b = -0.934$$



$f(x)$
8
6
4
2
x
-1.5
-1
-0.5
0.5
1
1.5
2
2.5
$f(x) = 3.04\,e^{-0.934x}$
Sampled points

(5) Same, but with $f(x) \sim -e^{ax^2+bx+c}$ and values

x	-1	0	1	2
y	-1.1	-0.4	-0.9	-0.5

An obvious problem is that $\mathcal{D}(\ln) = \mathbb{R}^+$. So take

$$\phi(x) = \ln(-f(x)) = ax^2 + bx + c$$

so that the transformation we need to apply to the data is $\ln \circ \psi$ with ψ the negation function:

x	-1	0	1	2
y	0.09	-0.91	-0.1	-0.69

Since $n = 2$, we need to compute a 3×3 matrix with coefficients

$$\mathbf{A} = \begin{bmatrix} \sum x_i^0 & \sum x_i^1 & \sum x_i^2 \\ \sum x_i^1 & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix}$$

to create the system

$$\mathbf{A}\begin{bmatrix} a & b & c \end{bmatrix}^\top = \begin{bmatrix} \sum y_i & \sum x_i y_i & \sum x_i^2 y_i \end{bmatrix}^\top$$

Now,

$$\sum_{i=0}^{3} x^0 = 4, \qquad \sum_{i=0}^{3} x_i = 2, \qquad \sum_{i=0}^{3} x_i^2 = 6, \qquad \sum_{i=0}^{3} x_i^3 = 8, \qquad \sum_{i=0}^{3} x_i^4 = 18$$
$$\sum_{i=0}^{3} x_i^0 y_i = -1.61, \qquad \sum_{i=0}^{3} x_i y_i = -1.57, \qquad \sum_{i=0}^{3} x_i^2 y_i = -2.77$$

So the system can be expressed as:

$$\begin{bmatrix} 4 & 2 & 6 & -1.61 \\ 2 & 6 & 8 & -1.57 \\ 6 & 8 & 18 & -2.77 \end{bmatrix}$$

which solves to

$$a = -0.4285, \qquad b = -0.2555, \qquad c = 0.1025$$

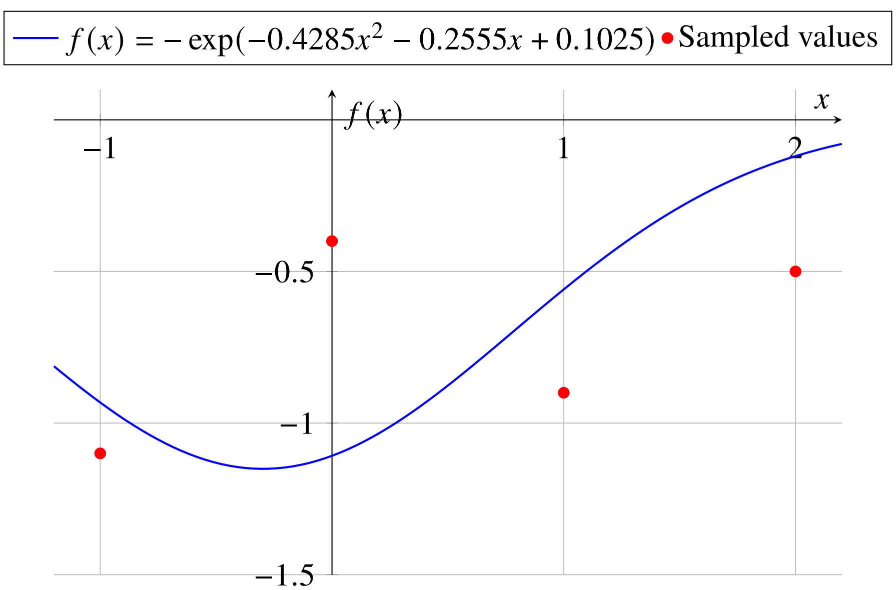
Now, since

$$\phi(x) = \ln(-f(x)) = ax^2 + bx + c$$

we find

$$f(x) = -\exp\left(ax^2 + bx + c\right) = -e^{-0.4285x^2 - 0.2555x + 0.1025}$$

to be the least square fit of our sampled values with model f.



(8) Approximate the data with $f(x) \sim a\cos x + b\sin x$.

x	0	1	2	3	4	5	6	7	8	9	10
y	1.8	3.5	2.1	-1.0	-3.3	-2.7	0.9	3.3	2.8	-0.1	-3.0

There is no way to transform f into a polynomial model (that I kwow of). So let's brute-force this:

$$\begin{aligned} E = E(a,b) &= \sum_{i=0}^{n} (y_i - f(x_i))^2 \\ &= \sum_{i=0}^{n} (y_i - (a\cos x_i + b\sin x_i))^2 \\ &= \sum_{i=0}^{n} (y_i - a\cos x_i - b\sin x_i)^2 \end{aligned}$$

Now, it is easy to see that

$$\frac{\partial E}{\partial a} = 2\sum_{i=0}^{n}(a\cos x_i + b\sin x_i - y_i)\cos x_i, \qquad \frac{\partial E}{\partial b} = 2\sum_{i=0}^{n}(a\cos x_i + b\sin x_i - y_i)\sin x_i$$

Equating both expressions to zero, one obtains the system:

$$\begin{aligned} a\sum_{i=0}^{n}\cos^2 x_i + b\sum_{i=0}^{n}\sin x_i\cos x_i &= \sum_{i=0}^{n} y_i\cos x_i \\ a\sum_{i=0}^{n}\cos x_i\sin x_i + b\sum_{i=0}^{n}\sin^2 x_i &= \sum_{i=0}^{n} y_i\sin x_i \end{aligned}$$

In other words, we readily obtain the system

$$\begin{bmatrix} \sum_{i=0}^{n}\cos^2 x_i & \sum_{i=0}^{n}\sin x_i\cos x_i & \sum_{i=0}^{n} y_i\cos x_i \\ \sum_{i=0}^{n}\cos x_i\sin x_i & \sum_{i=0}^{n}\sin^2 x_i & \sum_{i=0}^{n} y_i\sin x_i \end{bmatrix}$$

Suffices to evaluate these sums using a calculator and solving the system to obtain the coefficients a, b which minimize the squared distances from the model to the set of data points.

(9) Consider Legendre's polynomials $\mathcal{P} := \{P_0, P_1, P_2\}$ on $[-1, 1]$ given by

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = x^2 - 1/3$$

Verify that the set $\mathcal{P}$ is ortogonal.

The set $\mathcal{P}$ is ortogonal (in $[a, b]$) if and only if there is a weight function ω such that

$$\int_a^b \omega(x) P_k(x) P_j(x)\, dx = \alpha_j \delta_{jk}, \qquad j = 0, 1, 2$$

for some $\{\alpha_0, \alpha_1, \alpha_2\}$ all positive. In particular, if $\alpha_0 = \alpha_1 = \alpha_2 = 1$, the set is ortonormal. Now, consider that $\Omega(x) = 1$ is a weight function. It is easy to verify that

$$\int_{-1}^1 \Omega(x) P_j^2(x)\, dx > 0$$

for $j = 0, 1, 2$. Now, $\int_{-1}^1 \Omega(x) P_0(x) P_1(x)\, dx = \int_{-1}^1 x\, dx = 0$. Similarly,

$$\int_{-1}^1 \Omega(x) P_0(x) P_2(x)\, dx \int_{-1}^0 x^2 - 1/3\, dx = \frac{2}{3} - \frac{2}{3} = 0$$

Etc. This excercise is silly as it amounts to solving integrals.

(10) Determine the linear and quadratic approximations of $f(x) = e^x$ in the least squares sense using Legendre's polynomials in $[-1, 1]$.

Let $\Phi = \left\{1, x, x^2 - 1/3\right\}$, and use ϕ_0, ϕ_1, ϕ_2 to refer to the elements in this set in the order I wrote them. We know Φ to be ortogonal.

(Linear approximation) There is a set of coefficients a_0, a_1 of

$$P(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

which minimizes

$$\int_{-1}^{1} [f(x) - P(x)]^2 \, dx$$

Since Φ is ortonormal, we have

$$a_j = \frac{1}{\alpha_j} \int_a^b f(x)\phi_j(x) \, dx, \qquad j = 0, 1$$

Recall that

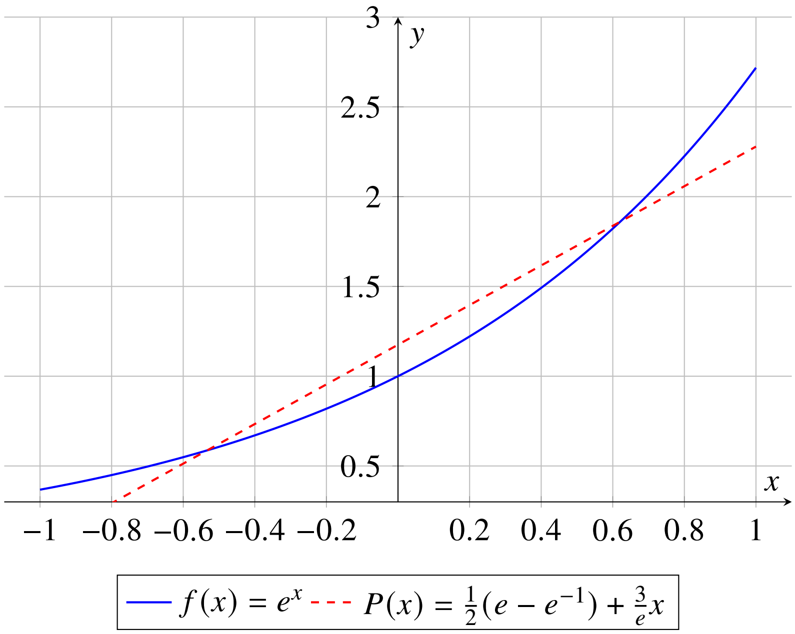
$$\alpha_0 = \int_{-1}^{1} \phi_0^2(x) = 2, \qquad \alpha_1 = \int_{-1}^{1} \phi_1^2(x) \, dx = \frac{2}{3}, \qquad \alpha_2 = \int_{-1}^{2} \phi_2^2(x) \, dx = \frac{8}{45}$$

This readily gives the system of equations:

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^{1} e^x \, dx = \frac{1}{2}(e - e^{-1}) \\ a_1 &= \frac{3}{2} \int_{-1}^{1} x e^x \, dx = \frac{3}{2} \left[e^{-1} + e^{-1}\right] = \frac{3 \cdot 2}{2e} = \frac{3}{e} \end{aligned}$$

Hence, the desired polynomial approximation of $f(x) = e^x$ on $[-1, 1]$ is:

$$P(x) = \frac{1}{2}(e - e^{-1}) + \frac{3}{e} \cdot x$$

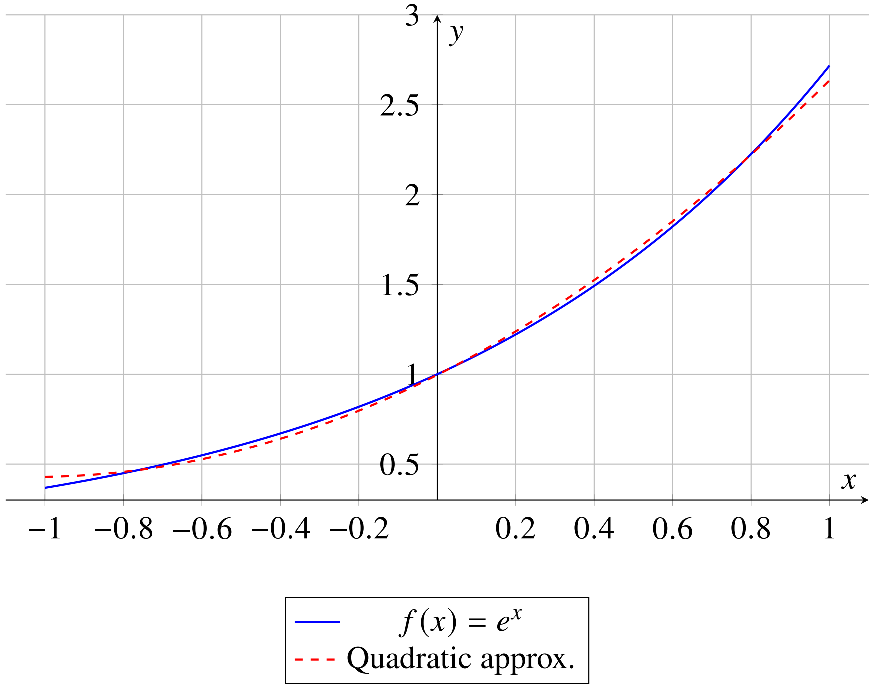


(Quadratic approximation) For a quadratic approximation, we now compute as well

$$a_2 = \frac{45}{8} \int_{-1}^{1} e^x (x^2 - 1/3)\, dx = \frac{15(2e^2 - 14)}{8e}$$

So, the approximation is

$$P(x) = \frac{1}{2}(e - e^{-1}) + \frac{3}{e} \cdot x + \frac{15(2e^2 - 14)}{8e} \left(x^2 - \frac{1}{3}\right)$$


3
2.5
2
1.5
1
0.5
y
x
−1 −0.8 −0.6 −0.4 −0.2 0.2 0.4 0.6 0.8 1
$f(x) = e^x$
Quadratic approx.

7 Numerical integration

We'll estimate $\int_a^b f(x)\ dx$ via $\sum_{i=0}^n a_i f(x_i)$, where $(x_0, y_0), \ldots, (x_n, y_n)$ are known values of f (which itself might be unknown) in the interval $[a, b]$.

More generally, take $\{x_i\}_{i=0}^n$ in $[a, b]$ and P_n the interpolating polynomial of f in those nodes in Lagrange's form. Then we have

$$P_n(x) = \sum_{i=0}^n f(x_i)\ell_i(x), \qquad e_n(x) = \frac{f^{(n+1)(\zeta_x)}}{(n+1)!}\eta_{n+1}(x)$$

for $\zeta_x \in (x_0, x_n)$. Using the fact that $f(x) = P_n(x) + e_n(x)$,

$$\begin{aligned}
\int_a^b f(x)\ dx &= \int_a^b P_n(x)\ dx + \int_a^b e_n(x)\ dx \\
&= \int_a^b \sum_{i=0}^n f(x_i)\ell_i(x)\ dx + \int_a^b \frac{f^{(n+1)(\zeta_x)}}{(n+1)!}\eta_{n+1}(x)\ dx \\
&= \sum_{i=0}^n a_i f(x_i) + \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\zeta_x) eta_{n+1}(x)\ dx
\end{aligned}$$

from $\zeta_x \in (x_0, x_n)$ and $a_i = \int_a^b \ell_i(x)\ dx$, $i = 0, \ldots, n$.

$$\therefore \int_a^b f(x)\ dx \approx \sum_{i=0}^n a_i f(x_i) \tag{1}$$

with an error

$$E_n(f) = \frac{1}{(n+1)!}\int_a^b f^{(n+1)}(\zeta_x)\eta_{n+1}(x)\ dx \tag{2}$$

7.1 Trapeze rule

Trapeze's rule is the name for the case $n = 1$ (linear integration). By convention, $x_0 = a, x_1 = b, h = b - a$. The interpolating polynomial es

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

with error

$$e_1(x) = \frac{f''(\zeta_x)}{2!}(x - x_0)(x - x_1)$$

Simplifying equation (1) for this case,

$$\begin{aligned}\int_a^b f(x)\, dx &\approx f(x_0)\int_a^b \frac{x - x_1}{x_0 - x_1}\, dx + f(x_1)\int_a^b \frac{x - x_0}{x_1 - x_0}\, dx \\ &= \frac{h}{2}\left(f(a) + f(b)\right)\end{aligned}$$

To simplify the expression of the error, the following theorem is used.

Theorem 16 (Pretty theorem)**.** Assume $f \in C[a, b]$ and g integrable and either always positive or always negative in $[a, b]$. Then there is some $c \in (a, b)$ such that

$$\int_a^b f(x)g(x)\, dx = f(c)\int_a^b g(x)\, dx$$

If $g(x) \equiv 1$ then $\int_a^b f(x)\, dx = f(c)(b - a)$, entailing that

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\, dx$$

It is easy to see that $(x - x_0)(x - x_1)$ satisfies the conditions of the theorem, meaning that there is some ζ (independent of x) such that

$$\begin{aligned}\int_a^b (\zeta_x)(x - a)(x - b)\, dx &= f''(\zeta)\int_a^b (x - a)(x - b)\, dx \\ &= f''(\zeta)\left[\frac{x^3}{3} - \frac{a + b}{2}x^2 + abx\right]_a^b\end{aligned}$$