1 Enumerable sets

Definition 1 *An infinite set A is countable or enumerable if there is a bijection $f : \mathbb{N} \to A$.*

Theorem 1 *Any infinite subset of an enumerable set is enumerable.*

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If $\overrightarrow{a} \in A^{\mathbb{N}}$ is such that every $a \in A$ is in $\overrightarrow{a}$, we say $\overrightarrow{a}$ is an enumeration of A.

Definition 2 *If M, N are sets, then $M \sim N$ if and only if there is a bijection $f : M \to N$.*

It is easy to see that $\sim$ is an equivalence relation.

Theorem 2 *$M \sim N$ if and only if $|M| = |N|$.*

If $M_1 \subset M, N_1 \subset N$, it may or may not be possible to put M into 1-1 correspondence with N_1, or N into 1-1 correspondence with M_1. If $M \sim N_1$ for some $N_1 \subseteq N$, but $N \nsim M_1$ for all $M_1 \subseteq M$, we have $|M| < |N|$. If $M \sim N_1 N$ and $N \sim M_1 \subset M$, we must have $|M| = |N|$.

Theorem 3 *If $M sim N_1 \subset N$, and $N \sim M_1 \subset M$, then $M \sim N$, and then $|M| = |N|$.*

Proof 1 *Complete, it is pretty. Page 22.*

Theorem 4 *If $M \subseteq N$, then $|M| \leq |N|$.*

Definition 3 *The cardinal number of the set of natural numbers, and hence of every enumerable set, is denoted $\aleph_0$.*

Theorem 5 *Every infinite set M has an enumerably infinite subset.*

Proof 2 *Let $a_0, a_1, a_2, \ldots \in M$ be distinct elements. Observe that any finite sequence of elements $a'_1, \ldots, a'_n \in M$ is such that $M - \left\{a'_1, \ldots, a'_n\right\}$ is still infinite. But there is a one to one correspondence between $a_0, a_1, a_2, \ldots$ and* $0, 1, 2, \ldots$. ■

Theorem 6 *If M has infinite cardinality, $\aleph_0 \leq |M|$.*

Proof 3 *M has an enumerably infinite subset M_1, and $M_1 \subseteq M$ implies $|M_1| \leq |M|$. But $|M_1| = \aleph_0$. Then $\aleph_0 \leq |M|$.*

Theorem 7 *For any infinite set M, there is a subset $M_1 \subseteq M$ such that $M \sim M_1$.*

Proof 4 *Letting* $P := M - \{a_0, a_1, a_2, \ldots\}$ *for distinct* $a_0, a_1, a_2, \ldots \in M$*, we note that* $M = P + \{a_0, a_1, a_2, \ldots\}$*. But then* M *is equivalent to its proper subset* $M - \{a_0\} = P + \{a_1, a_2 a_3, \ldots\}$.

This theorem is behind the apparent parodox (observed by Galileo) of the natural numbers being in perfect correspondence to some of its subsets.

Theorem 8 *If* M *is an infinite set, its cardinality is unchanged by the introduction or removal of a finite or enumerably infinite set of elements.*

Proof 5 *Complete and understand, page. 24*

2 Peano arithmetic

We axiomatize the following:

- 0 is a natural number.
- If n is a natural number, its successor n' is a natural number.
- The only natural numbers are those given by the previous clauses.

These three clauses ensure the distinctness of all natural numbers, which can be separated into two propositions:

- For any n, m natural, $n = m$ if and only if $n' = m'$.
- $n' \neq 0$ for all natural n.

These were the five axioms which Peano used to characterize natural numbers. Observe that this inductive definition of natural numbers already determines the usual order $<$, where $a < b$ if a is generated before b by successive applications of the unary operator $'$ starting at zero.

Definition 4 *A system S of objects is a non-empty set, class or domain D of objects among which certain relationships are established.*

For example, the sequence $0, 1, 2, \ldots$ of natural numbers constitutes a system of type $(D, 0,')$ where D is a set, 0 a member of it, and $'$ a unary operation. A *representation* (or model) of an abstract system further specifies what the objects in the system are. For example, D may be taken to be the positive integers, in which case the interpretation of the abstract 0 is 1, or the even natural numbers, in which case the interpretation of $'$ is the +2 operation.

Definition 5 *Two systems $(D_1, 0_1,'_1), (D_2, 0_2,'_2)$ are (simply) isomorphic if there is a bijection $f : D_1 \to D_2$ that preserves the relationships.*

More precisely, if $f : D_1 \to 2$ is said bijection, we require $f(0_1) = 0_2$, and whenever $f(m_1) = m_2$, then $f(m'_1{}_1) = m'_2{}_2$.

In a system $S = (D, 0,')$, the symbols $', D, 0$ are called *primitive* or *undefined* notions, insofar as they are not defined prior to the introduction of the axioms. The only information about them is their type, i.e. we know D is a set, 0 a constant, and $'$ a unary operator.

3 Primitive recursive functions

A function $\varphi : \omega \to \omega$ is primitive recursive if it is built using one of the following clauses:

- $\varphi(x) = x'$
- $\varphi(\overrightarrow{x}) = q$, with $q \in \omega$.
- $\varphi(\overrightarrow{x}) = x_i$
- $\varphi(\overrightarrow{x}) = \psi\left(\chi_1(\overrightarrow{x}), \ldots, \chi_m(\overrightarrow{x})\right)$
- $$\varphi(t, \overrightarrow{x}) = \begin{cases} \psi(\overrightarrow{x}) & t = 0 \\ \chi(t-1, \varphi(t-1, \overrightarrow{x}), \overrightarrow{x}) & t > 0 \end{cases}$$

where $\psi, \chi_1, \ldots, \chi_m, \chi$ are primitive recursive. Each of the forms described by the bullet points above is called a *schemata*. One could take the first three bullets to be axioms and the others to be inference rules. If any of the first three forms is satisfied by φ, then φ is called an *initial function*.

If φ is of the recursive form which involves ψ, χ, or $\chi_1, \ldots, \chi_m$, we say φ is an *immediate dependant* of these functions.

Definition 6 *A function φ is primitive recursive if there is a finite sequence $\varphi_1, \ldots, \varphi_k$ of functions, which we call the primitive recursive description of φ, such that each φ_i is either an initial function, or an immediate dependent of preceding functions of the sequence, and $\varphi_k = \varphi$.*

Example 1 *Let $\varphi = \lambda x\,[x + 2]$. Clearly,*

$$\varphi = Suc\,(Suc(x))$$

which entails Suc, Suc, φ is the primitive recursive description of φ.

Example 2 *Consider*

$$\varphi(x, z, y) = \zeta\,(x, \eta\,(y, \theta(x))\,, 2)$$

Then

$$\varphi = \lambda xzy\left[\zeta \circ \left[p^3_1\right], \eta \circ \left[p^3_3, \theta \circ \left[p^3_1\right]\right], C^3_2\right]$$