

chngcntr

9 Alg. de Horner: Polynomial evaluation

Consider

$$p(x) = \sum_{i=0}^n a_i x^i$$

We wish to compute $p(k)$ for a given $k \in \mathbb{R}$ minimizing the number of operations. Directly computing $a_0 + a_1 k + \dots$ leads to n sums. The i th term requires computing k^i , which means i product operations, for a total of $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ products. The total number of operations is then

$$\Theta = n + n(n+1)/2$$

The associated complexity is $O(n^2)$.

Horner's method consists of re-writing $p(x)$ so that the number of products is reduced. One writes

$$p(x) = a_0 + x b_0$$

where $b_{n-1} = a_n$ and for $0 \leq i < n-1$:

$$b_{i-1} = a_i + x b_i$$

Let $p(x) = 3 + 5x - 4x^2 + 0x^3 + 6x^4$, giving $n = 4$. Then $b_3 = 6$ and

$$b_2 = a_3 + x b_3 = 6x,$$

$$b_1 = a_2 + x b_2 = -4 + x(6x),$$

$$b_0 = a_1 + x b_1 = 5 + x(-4 + x(6x))$$

This finally gives

$$p(x) = 3 + x b_0 = 3 + x(5 + x(-4 + x(6x)))$$

Here, one must perform n sums again but only n products. Thus, there are $\Theta = n + n = 2n$ operations, giving a complexity of $O(n)$ (in the operation space). See the algorithm below:

```

input  $n; a_i, i = 0, \dots, n; x$ 
 $b_{n-1} \leftarrow a_n$ 
for  $i = n - 2$  to  $i = 0$ 
     $b_i = a_{i+1} + x * b_{i+1}$ 
od
 $y \leftarrow a_0 + x * b_0$ 
return  $y$ 

```

It is easy to see in this code that the **for** loop performs $n - 1$ iterations, in each of which a single sum and a single product are computed. The n th sum and n th product are performed in the computation of y , the final result.

A more polished version includes the last computation (the one in the assignment of y) within the loop and makes no use of indexes:

```

input  $n; a_i, i = 0, \dots, n; x$ 
 $b \leftarrow a_n$ 
for  $i = n - 2$  to  $i = -1$ 
     $b = a_{i+1} + x * b$ 
od
return  $b$ 

```

In Python,

```

def horner(coefs, x):
    n = len(coefs)-1
    b = coefs[n]

    for i in reversed(range(-1, n-1)):
        b = coefs[i+1] + x*b

    return b

```

It is trivial to adapt the code so that it returns the coefficients b_0, \dots, b_{n-1} and not the final result, if needed.

10 Error

Let r, \bar{r} be two real numbers s.t. the latter is an approximation of the first. We define the **error** of the approximation to be $r - \hat{r}$, and

$$\Delta r = |r - \bar{r}|, \quad \delta r = \frac{\Delta r}{|r|}$$

With r unknown the strategy is to work with a known bound of r .

11 Non-linear equations

The general problem is to find members of the set \mathcal{R}_f of roots of $f \in \mathbb{R} \rightarrow \mathbb{R}$. The numerical strategy is to iteratively approximate some $r \in \mathcal{R}_f$ until some pre-established threshold in the error of approximation is met.

More formally, the numerical strategy produces a sequence $\{x_k\}_{k \in \mathbb{N}}$ which satisfies

- $\lim_{k \rightarrow \infty} \{x_k\} = r$ for some $r \in \mathcal{R}_f$
- Either $e(x_k) < e(x_{k-1})$ or, more strongly, $\lim_{k \rightarrow \infty} e(x_k) = 0$, where $e(x_k)$ is some appropriate measure of the error of approximation.

11.1 Bisection

A very simple procedure: if a root exists in $[a, b]$, it iteratively shrinks $[a, b]$ in halves (keeping the halves which contain the root) until the interval is of sufficiently small length.

Theorem 1 (Intermediate value). If f is continuous in $[a, b]$ and $f(a)f(b) < 0$, then $\exists r \in \mathcal{R}_f$ s.t. $r \in [a, b]$.

Assume f is continuous. A root exists in $[a, b]$ if $f(a)f(b) < 0$ (**Theorem 1**). If that is the case, the midpoint $(a + b)/2$ is taken as the approximation x_0 . It is also trivial to observe that x_0 is *at most* at a distance of $(b - a)/2$ from the real root, so $e_0 = |x_0 - r| \leq (b - a)/2$.

If $f(x_0) = 0$ the procedure must end because a root was found. Otherwise, suffices to find which half of the interval contains a root computing $f(a)f(c)$ and, if needed, $f(c)f(b)$.

The iterations may stop after reaching a maximum number of steps, when $|f(c)|$ is sufficiently close to zero, or when the error bound $|e_k| \leq (b_k - a_k)/2$ (where $[a_k, b_k]$ is the interval of this iteration) is sufficiently small.

(!) The algorithm not always converges. Take $f(x) = 1/x$. Clearly, it has no root. Yet setting $a = -1, b = 1$ in the initial iteration falsely passes the test. (The problem obviously is that f is not continuous in $[-1, 1]$.) If one sets

Input : a, b, δ, M, f

Output : Tupla de la forma: $(r, \text{cota de error})$

$f_a \leftarrow f(a)$

$f_b \leftarrow f(b)$

if $f_a * f_b > 0$

return ?

fi

for $i = 1$ **to** $i = M$ **do**

$c \leftarrow a + (b - a)/2$

$f_c \leftarrow f(c)$

if $f_c = 0$ **then**

return $(c, 0)$

fi

$\epsilon = \frac{b - a}{2}$

if $\epsilon < \delta$ **then**

break

fi

if $f_a * f_c < 0$ **then**

$b \leftarrow c$

$f_b = f(b)$

else

$a \leftarrow c$

$f_a = f(a)$

fi

od

return (c, ϵ)

```

def bisection(f : callable, a : float, b : float, delta : float, M : int):

    s, e = f(a), f(b) # function values at (s)tart, (e)nd of interval

    if s*e > 0:
        raise ValueError("Interval [a, b] contains no root.")

    for i in range(M):

        c = a + (b-a)/2
        m = f(c) # value of f at (m)idpoint

        if m == 0:
            return c, 0

        e = (b-a)/2
        if e < delta:
            return c, e

        if s*m < 0:
            b = c
            e = f(b)
        else:
            a = c
            s = f(a)

    return c, e

```

Theorem 2. If $\{[a_i, b_i]\}_{i=0}^{\infty}$ are the intervals generated by the bisection method on iterations $i = 0, 1, \dots$, then:

1. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ is a member of \mathcal{R}_f .
2. If $c_n = \frac{1}{2}(a_n + b_n)$, $r = \lim_{n \rightarrow \infty} c_n$, then $|r - c_n| \leq \frac{1}{2^{n+1}}(b_0 - a_0)$

Proof. (1) It is clear that $a_i \leq a_{i+1}$ and $b_i \geq b_{i+1}$, since the interval on each iteration shrinks in one direction.

$\therefore a_n, b_n$ are monotonous.

But clearly a_n is bounded by b_0 and b_n is bounded by a_0 .

$\therefore a_n, b_n$ are monotonous and bounded.

\therefore Their limits exist.

It is also clear that the interval shrinks to half its size on each iteration:

$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}), \quad n \geq 1 \quad (1)$$

By recurrence on (1),

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0), \quad n \geq 0 \quad (2)$$

Then

$$\lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} \frac{1}{2^n}(b_0 - a_0) = 0 \quad (3)$$

$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$.

Since the limit of a_n, b_n exists and f is by assumption continuous, the composition limit theorem applies and:

$$\begin{aligned} & \lim_{n \rightarrow \infty} (f(a_n) \cdot f(b_n)) \\ &= \lim_{n \rightarrow \infty} f(a_n) \cdot \lim_{n \rightarrow \infty} f(b_n) \quad \{\text{Product of limits}\} \\ &= f\left(\lim_{n \rightarrow \infty} a_n\right) \cdot f\left(\lim_{n \rightarrow \infty} b_n\right) \quad \{\text{Composition limit theorem}\} \\ &= [f(r)]^2 \quad \left\{r = \lim_{n \rightarrow \infty} a_n\right\} \end{aligned} \quad (4)$$

The invariant of the algorithm is $f(a_n)f(b_n) < 0$. But due to the last result,

$$\lim_{n \rightarrow \infty} f(a_n)f(b_n) \leq 0 \iff [f(r)]^2 \leq 0 \iff f(r) = 0$$

$\therefore r = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ is a root.

(2) Follows directly from result (2)

$$\begin{aligned}
|r - c_n| &= \left| r - \frac{1}{2}(b_n - a_n) \right| \\
&\leq \left| \frac{1}{2}(b_n - a_n) \right| \\
&= \left| \frac{1}{2^{n+1}}(b_0 - a_0) \right|
\end{aligned}$$

{Result (2)}

11.2 Newton's method

Taylor: repaso. El desarrollo de una f suficientemente diferenciable alrededor de un punto r es

$$f(x) = f(r) + f'(r)(x - r) + \frac{f''(r)}{2!}(x - r)^2 + \dots + \frac{f^{(n)}(r)}{n!}(x - r)^n + R_n(x)$$

donde $R_n(x)$ es el resto.

Usualmente, queremos tomar $r = x + h$, donde x es una aproximación de r y h el error de aproximación. Entonces es provechoso expandir $f(r)$ alrededor de su estimación x :

$$f(r) = f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots + \frac{f^{(n)}(x)}{n!}h^n + R_n(h)$$

Esto es **recontra** útil porque nos dice cuánto se diferencia $f(r)$ de nuestra aproximación $f(x)$ (pues expresa $f(r)$ como $f(x)$ más algo).

Usualmente r, h son desconocidos pero h puede acotarse.

El resto R_n del teorema puede expresarse como sigue:

$$f(r) = f(x + h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots + \frac{f^{(n)}(x)}{n!}h^n + \frac{f^{(n+1)}(\zeta)}{(n+1)!}h^{n+1}$$

para algún $\zeta \in (x, h)$. Esta forma de expresar el error de aproximación con el polinomio de Taylor se usará mucho.

Assume $r \in \mathcal{R}_f$ and $r = x + h$, with x an approximation of r and h its error. Assume f'' exists and is continuous in some I around x s.t. $r \in I$. What we explained on Taylor expansions around a point gives:

$$0 = f(r) = f(x + h) = f(x) + f'(x)h + O(h^2)$$

If x is sufficiently close to r , h is small and h^2 even smaller, so that $O(h^2)$ is unconsiderable:

$$0 \approx f(x) + hf'(x)$$

Therefore,

$$h \approx -\frac{f(x)}{f'(x)} \tag{1}$$

From this follows that $r = x + h$ is approximated by

$$r \approx x - \frac{f(x)}{f'(x)}$$

Since the approximation in (5) truncated the terms of $O(h^2)$ complexity, this new approximation is closer to r than x originally was. In other words, $x - f(x)/f'(x)$ is a better approximation to r than x itself.

Thus, if x_0 is an original approximation, we can define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (2)$$

to produce a sequence of approximations. This is the fundamental idea of Newton's method.

```

Input:  $x_0, M, \delta, \epsilon$ ;
 $v \leftarrow f(x_0)$ 
if  $|v| < \epsilon$  then return  $x_0$  fi
for  $k = 1$  to  $k = M$  do
     $x_1 \leftarrow x_0 - \frac{v}{f'(x_0)}$ 
     $v \leftarrow f(x_1)$ 
    if  $|x_1 - x_0| < \delta \vee v < \epsilon$  then
        return  $x_1$ 
    fi
     $x_0 \leftarrow x_1$ 
od
return  $x_0$ 

```

The predicate $|x_1 - x_0| < \delta$ checks whether our algorithm is adjusting x in a negligible degree. If that is the case, we should stop.

Theorem 3. If f'' continuous around $r \in \mathcal{R}_f$ and $f'(r) \neq 0$, then there is some $\delta > 0$ s.t. if $|r - x_0| \leq \delta$, then:

- $|r - x_n| \leq \delta$ for all $n \geq 1$.
- $\{x_n\}$ converges to r
- The convergence is quadratic, i.e. there is a constant $c(\delta)$ and a natural N s.t. $|r - x_{n+1}| \leq c |r - x_n|^2$ for all $n \geq N$.

Proof. Let $e_n = r - x_n$ be the error in the n th approximation. Assume f'' is continuous and $f(r) = 0$, $f'(r) \neq 0$. Then

$$\begin{aligned}
e_{n+1} &= r - x_{n+1} \\
&= r - \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) \\
&= r - x_n + \frac{f(x_n)}{f'(x_n)} \\
&= \frac{e_n f'(x_n) + f(x_n)}{f'(x_n)}
\end{aligned} \tag{3}$$

Thus, the error at any given iteration is a function of the error at the previous iteration. Now consider the expansion of $f(r)$ as

$$f(r) = f(x_n - e_n) = f(x_n) + e_n f'(x_n) + \frac{e_n^2 f''(\zeta_n)}{2} \tag{4}$$

for ζ_n between x_n and r . This equation gives

$$e_n f'(x_n) + f(x_n) = -\frac{1}{2} f''(\zeta_n) e_n^2 \tag{5}$$

The expression in (5) is the numerator in (3), whereby we obtain via substitution:

$$e_{n+1} = -\frac{1}{2} \frac{f''(\zeta_n) e_n^2}{f'(x_n)} \tag{6}$$

Equation (6) ensures that the error scales quadratically. Now we wish to bound the error expression in (6). To bound e_{n+1} , we take $\delta > 0$ to define a neighbourhood of length δ around r . For any x in this neighbourhood, (6) reaches its maximum when the numerator is maximized and the denominator is minimized:

$$c(\delta) = \frac{1}{2} \frac{\max_{|x-r| \leq \delta} |f''(x)|}{\min_{|x-r| \leq \delta} |f'(x)|}$$

In other words, $c(\delta)$ is the maximum value which e_{n+1} can take if ζ_n, x_n are assumed to belong to the neighbourhood. Now we make two assumptions:

1. x_0 belongs to the neighbourhood, i.e. $|x_0 - r| \leq \delta$
2. δ is sufficiently small so that $\varrho := \delta c(\delta) < 1$.

Note that, since ζ_0 is between x_0 and r , assumption (1) ensures that ζ_0 is also in the neighbourhood, i.e. $|r - \zeta_0| \leq \delta$. Then we have:

$$|e_0| = \frac{1}{2} |f''(\zeta_0)/f'(x_0)| \leq c(\delta)$$

Then:

$$\begin{aligned}
|x_1 - r| &= |e_1| \\
&= \left| e_0^2 \cdot \frac{1}{2} f''(\zeta_0)/f'(x_0) \right| \\
&\leq |e_0|^2 c(\delta) && \left\{ \frac{1}{2} f''(\zeta_0)/f'(x_0) \leq c(\delta) \right\} \\
&\leq |e_0| \delta c(\delta) && \{|e_0| \leq \delta\} \\
&= |e_0| \varrho && \{\varrho = \delta c(\delta)\} \\
&< |e_0| && \{\varrho < 1\} \\
&\leq \delta
\end{aligned}$$

$\therefore |e_1| < |e_0| \leq \delta$, which means the error decreases. This argument may be repeated inductively, giving:

$$\begin{aligned}
|e_1| &\leq \varrho |e_0| \\
|e_2| &\leq \varrho |e_1| \leq \varrho^2 |e_0| \\
|e_3| &\leq \varrho |e_2| \leq \varrho^3 |e_0| \\
&\vdots
\end{aligned}$$

In general, $|e_n| \leq \varrho^n |e_0|$. And since $0 \leq \varrho < 1$, we have $\varrho^n \rightarrow 0$ when $n \rightarrow \infty$, entailing that $|e_n| \rightarrow 0$ when $n \rightarrow \infty$.

Theorem 4. If f'' is continuous in \mathbb{R} , and if f is increasing, convex, and has a root, then said root is unique and Newton's method converges to it from any starting point.

Recall that f is convex if $f''(x) > 0$ for all x . Graphically, it is convex if the line connecting two arbitrary points of f lies above the curve of f between those two points.

11.3 Secant method

In Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The function of interest is f . We cannot escape computing $f(x_n)$, but it would be desirable to avoid the computation of $f'(x_n)$, which may potentially be expensive. Since

$$f'(x) = \lim_{h \rightarrow x} \frac{f(x) - f(h)}{x - h}$$

it is natural to suggest

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \quad (1)$$

Graphically, this means we are not using the line tangent to the point $(x_n, f(x_n))$ but the line secant to the points $(x_n, f(x_n))$ and $(x_{n-1}, f(x_{n-1}))$. The point x_{n+1} is then the value of x where this secant line has a root.

11.4 Fixed point iteration

The key observation is this: if $r \in \mathcal{R}_f$, then $g(x) = x - kf(x)$ has r as fixed point, for any $k \in \mathbb{R}$. Inversely, if g has a fixed point in r , then $r \in \mathcal{R}_f$.

Theorem 5. (1) Let $g \in C[a, b]$ and assume $g(x) \in [a, b]$ for all $x \in [a, b]$. Then there is a fixed point of g in $[a, b]$.

(2) If, on top of previous conditions, g is differentiable in (a, b) and there is some $k < 1$ s.t. $|g'(x)| \leq k$ for all $x \in (a, b)$, then the fixed point referred in (1) is unique.

Theorem 6 (Mean value theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ continuous and differentiable on (a, b) with $a < b$. Then there is some $c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The interpretation is simple: consider the line secant to f on a, b . The theorem ensures that there is some point c s.t. the line tangent to c is parallel to said secant (equal slopes).

Proof. (1) If a or b are fixed points the proof is done so assume otherwise. Since $g(x) \in [a, b]$, we have $g(a) > a$ and $g(b) < b$.

Take $\varphi(x) = g(x) - x$, which is continuous and defined in $[a, b]$. Then

$$\varphi(a) = g(a) - a > 0, \quad \varphi(b) = g(b) - b < 0$$

Then $\varphi(a)\varphi(b) < 0$. Then, by the intermediate value theorem, φ has a root in (a, b) . In otherwords, there is at least one p s.t.

$$\varphi(p) = g(p) - p = 0$$

$\therefore g(p) = p$ is a fixed point of g .

(2) Assume two distinct fixed points p, q exist in $[a, b]$. The mean value theorem ensures the existence of some ζ between p, q (and thus in $[a, b]$) s.t.

$$g'(\zeta) = \frac{g(a) - g(b)}{a - b} \iff g'(\zeta)(a - b) = g(a) - g(b) \quad (1)$$

By hypothesis, $|g'(x)| \leq k < 1$. Since p, q are assumed to be fixed points, equation (1) gives:

$$\begin{aligned} |p - q| &= |g(p) - g(q)| \\ &= |g'(\zeta)| |p - q| \\ &\leq k |p - q| < |p - q| \end{aligned}$$

But this is absurd. The contradiction arises from assuming p, q to be distinct. Therefore, the fixed point is unique.

The fixed point algorithm begins with an approximation p_0 . Then,

$$p_n = g(p_{n-1})$$

If g continuous and the sequence converges, then it converges to a fixed point, since:

$$p := \lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} g(p_{n-1}) = g\left(\lim_{n \rightarrow \infty} p_{n-1}\right) = g(p)$$

```

Input:  $p, M, \delta$ 
 $p_{\text{previous}} = p$ 
for  $i = 1$  to  $i = M$  do
     $p \leftarrow g(p)$ 
    if  $|p - p_{\text{previous}}| < \delta$  then
        return  $p$ 
    fi
     $p_{\text{previous}} = p$ 
od
return  $p$ 

```

Theorem 7. Let $g \in C[a, b]$ be a self-map of $[a, b]$ differentiable in (a, b) . Assume there is a constant $0 < k < 1$ s.t. $|g'(x)| \leq k$ for all $x \in (a, b)$.

For all $p_0 \in [a, b]$, the sequence $p_n = g(p_{n-1})$ converges to the unique fixed point p in (a, b) .

Proof. The mean value theorem ensures that

$$\begin{aligned} |p_n - p| &= |g(p_{n-1}) - g(p)| \\ &= |g'(\zeta_n)| |p_{n-1} - p| \\ &\leq k |p_{n-1} - p| \end{aligned}$$

with $\zeta_n \in (a, b)$. More succinctly, with $e_n := p_n - p$,

$$|e_n| \leq k |e_{n-1}| \leq k |e_{n-2}| \leq \dots \leq k |e_0|$$

By recurrence,

$$|e_n| \leq k^n |e_0|$$

Since $0 < k < 1$, $k^n \rightarrow 0$ when $n \rightarrow \infty$, which entails $|e_n| \rightarrow 0$ when $n \rightarrow \infty$. It follows that $\{p_n\} \rightarrow p$ when $n \rightarrow \infty$.

Now let us consider the error of this method. Take $p_n = p + e_n$ and consider the Taylor expansion of g around p evaluated at $p_n = p + e_n$:

$$g(p_n) = g(p + e_n) = \sum_{i=1}^{m-1} \frac{g^{(i)}(p)}{i!} e_n^i + \frac{g^{(m)}(\zeta_n)}{(n+1)!} e_n^m \quad (2)$$

See that in (2), n corresponds to the iteration we are dealing with, and thus ζ_n and e_n depend on it. On the contrary, m is the degree to which we expand the series of g around p evaluated at p_n . We also assume that ζ_n lies between p_n and p .

By definition, $g(p_n) = p_{n+1}$ so (2) is nothing but an expression for this value. Assume $g^{(k)}(p) = 0$ for $k = 1, 2, \dots, m-1$, but $g^{(m)}(p) \neq 0$. Then

$$\begin{aligned} e_{n+1} &= p_{n+1} - p \\ &= g(p_n) - g(p) \\ &= \frac{g^{(m)}(\zeta_n)}{m!} e_n^m \end{aligned}$$

More succinctly,

$$e_{n+1} = \frac{g^{(m)}(\zeta_n)}{m!} e_n^m$$

Then

$$\lim_{n \rightarrow \infty} \left| \frac{e_{n+1}}{e_n^m} \right| = \frac{|g^{(m)}(p)|}{m!}$$

which is a constant. In conclusion, if the derivatives of g are null in p up to the order $m-1$, the method has an order of convergence of at least m . Three results follow from this fact.