Fast lognormal realizations for multi-probe experiments

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1 Notation

χ is comoving radial density. We use units $c = 1$ throughout.

2 Algorithm

- Generate a Gaussian realization of the linear density field δ_G and the newtonian potential ϕ at $z = 0$. For this we use FFTW in a box able to hold a sufficiently large sphere. In what follows, let dx be the comoving resolution used for this.

- Interpolate cartesian grid into spherical shells. These are generated with a width $dr = dx$. The pixels are defined using a CEA scheme (cylindrical equal area), with constant intervals in ϕ and $\mu \equiv \cos\theta$.

  Let N_μ be the number of divisions used in μ. Firstly, we will use $2 \times N_\mu$ divisions in ϕ. Secondly, the largest transverse scale covered by each pixel is given by

  $$\Delta d_{\rm max} = \chi \arccos(1 - 2/N_\mu), \tag{1}$$

  where χ is the comoving distance to the shell. We determine N_μ by trying to achieve $\Delta d_{\rm max} = dx$, which implies $N_\mu = {\rm ceiling}(2/[1 - \cos(dx/\chi)])$.

- While doing the interpolation we compute the relevant quantities at $z = 0$ that will be used later. These are the radial velocity, v_r, proportional to the radial gradient of ϕ, and the transverse tidal field, given by the angular second derivatives of ϕ. These two objects are needed for the computation of RSDs and lensing. We compute these quantities by differentiating through finite differences in the Cartesian grid, rotating and interpolating into the spherical shells (in this order).

- These fields are then evolved in the lightcone in different ways (see details below). Galaxies are Poisson-sampled according to the evolved, bias and lognormalized value of the density field, and are assigned redshift and shape distortions according to the evolved velocity and lightcone-integrated transverse tidal field.

3 Relevant equations

$$\phi_{\bf k}(z=0) = -\frac{3}{2}\Omega_M H_0^2 \frac{\delta_{\bf k}(z=0)}{k^2}, \quad v_r(z=0) = -\frac{2\,f_0}{3\,H_0\Omega_M}\hat{\bf n}\cdot\nabla\phi(z=0) \tag{2}$$

$$\delta(z) = D(z)\,\delta(z=0), \quad v_r(z) = \frac{D(z)f(z)H(z)}{f_0 H_0} v_r(z=0), \quad \phi(z) = (1+z)D(z)\phi(z=0) \tag{3}$$

4 Sources

At each pixel, we compute the physical galaxy density for each galaxy type i as

$$n_i(\chi, \hat{\mathbf{n}}) = \bar{n}_i(\chi) \, \exp\left[D(\chi)b_i(\chi)\left(\delta_G(\chi, \hat{\mathbf{n}}) - D(\chi)b_i(\chi)\sigma_G^2/2\right)\right], \tag{4}$$

where $\bar{n}_i$ is the mean number density of galaxies, and σ_G^2 is the variance of the density field at $z = 0$. $\bar{n}$ is related to $dn/(d\Omega dz)$ as $dn/(d\Omega dz) = \bar{n}\,\chi^2/H$.

Then, at each pixel we sample a number of galaxies from a Poisson distribution with mean $\lambda \equiv V_{\rm pix} n_i$, where $V_{\rm pix}$ is the comoving volume of each spherical voxel. We then place the resulting number of particles inside each voxel at random within it. Each galaxy is given a cosmological redshift corresponding z_c corresponding to its comoving distance by inverting

$$\chi = \int_0^{z_c} \frac{dz}{H(z)}. \tag{5}$$

In addition, each galaxy is given a redshift distortion $\Delta z = v_r$ according to the value of the comoving velocity field in their corresponding voxel.

5 Intensity mapping

The brightness temperature for a line-emitting species a in a voxel i is

$$T_a(\nu, \hat{\mathbf{n}}) = \bar{T}_a(\nu)\left[1 + \Delta_i^a(\chi\hat{\mathbf{n}})\right]. \tag{6}$$

where the mean brightness temperature is

$$\bar{T}_a(z) = \frac{3\hbar A_{21} x_2 c^2}{32\pi G k_B m_a \nu_{21}^2} \frac{H_0^2 \Omega_a(z)(1+z)^2}{H(z)} \tag{7}$$

and Δ_i^a is the redshift-space overdensity of the line-emitting species smoothed over the voxel. Here x_2 is the fraction of atoms in the excited state, Ω_a is the fractional density of the species, ν_{21} is the rest-frame frequency of the transition and A_{21} is the corresponding Einstein coefficient for emission.

The procedure to generate intensity maps is:

- We cycle over each voxel in the spherical shells for which we have stored the value of the density and velocity fields.
- We compute the overdensity in the voxel using a log-normal model:

$$1 + \delta_i^a = \exp\left[D(\chi)b_a(\chi)\left(\delta_G(\chi, \hat{\mathbf{n}}) - D(\chi)b_a(\chi)\sigma_G^2/2\right)\right]. \tag{8}$$

- We sub-sample the voxel in $N_{\rm sub}$ random points. Each point is assigned a brightness temperature

$$T_{a,{\rm sub}} = \bar{T}_a(\nu)\,(1 + \delta_i^a)\frac{v_{\rm vox}}{N_{\rm sub}}, \tag{9}$$

  where $v_{\rm vox}$ is the comoving volume of the voxel, as well as a redshift displacement given by v_r, the value of the lightcone-evolved radial velocity field in the voxel.
- We compute the frequency channel corresponding to each point from their redshift (computed as the sum of its cosmological redshift and the RSD term), as well as the pixel index in that frequency channel corresponding to the angular coordinates of the point. We then add the brightness temperature of this point computed in the previous step to the total brightness temperature in the pixel.
- Each intensity mapping pixel is finally divided by the total comoving volume covered by the pixel.

6 Shear

We compute the shear tensor as

$$\hat{T} \equiv \begin{pmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{yy} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{zz} \end{pmatrix}, \quad \hat{R} \equiv \begin{pmatrix} \sin\theta\,\cos\phi & \sin\theta\,\sin\phi & \cos\theta \\ \cos\theta\,\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}, \tag{10}$$

$$\hat{t} \equiv \hat{R}\cdot\hat{T}\cdot\hat{R}^T, \quad \hat{\tau} \equiv \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}, \tag{11}$$

$$\hat{\Gamma}(\chi,\hat{\mathbf{n}}) \equiv \int_0^\chi d\chi'\,\hat{\tau}(\chi'\hat{\mathbf{n}})\,\chi'\left(1-\frac{\chi'}{\chi}\right) = \int_0^\chi d\chi'\,\hat{\tau}(\chi'\hat{\mathbf{n}})\,\chi' - \frac{1}{\chi}\int_0^\chi d\chi'\,\hat{\tau}(\chi'\hat{\mathbf{n}})\,\chi'^2. \tag{12}$$

$$\hat{\Gamma}(\chi_i) = \hat{I}_i - \frac{1}{\chi_{i+1/2}}\hat{J}_i, \quad \hat{I}_i = \hat{I}_{i-1} + \hat{\tau}(\chi_i)\frac{\chi^2_{i+1/2}-\chi^2_{i-1/2}}{2}, \quad \hat{J}_i = \hat{J}_{i-1} + \hat{\tau}(\chi_i)\frac{\chi^3_{i+1/2}-\chi^3_{i-1/2}}{3} \tag{13}$$

7 CMB lensing

For $\kappa \equiv \mathrm{Tr}(\hat{\Gamma})/2$ we compute, from the simulation, $\kappa(\chi_{\rm max})$, where $\chi_{\rm max}$ is the maximum radial distance that fits in the box. In order to get CMB lensing we need $\kappa(\chi_{\rm LSS}) = \tilde{\kappa}_{\rm max} + \Delta\kappa$, where

$$\tilde{\kappa}_{\rm max} \equiv \int_0^{\chi_{\rm max}} d\chi\delta(\chi)\frac{\chi}{2}\left(1-\frac{\chi}{\chi_{\rm LSS}}\right) \tag{14}$$

$$\Delta\kappa \equiv \int_{\chi_{\rm max}}^{\chi_{\rm LSS}} d\chi\delta(\chi)\frac{\chi}{2}\left(1-\frac{\chi}{\chi_{\rm LSS}}\right). \tag{15}$$

The strategy to compute these is:

- We compute $\tilde{\kappa}_{\rm max}$ from $\kappa(\chi_{\rm max})$ using the same strategy used for shear, but taking care to divide by $\chi_{\rm LSS}$ instead of $\chi_{\rm max}$.
- We compute $\Delta\kappa$ as a Gaussian realization constrained to have the right correlation with $\tilde{\kappa}_{\rm max}$. Do do this we start by rewriting the previous equation as

  $$\tilde{\kappa}_{\rm max} \equiv \int d\chi w_1(\chi)\delta(\chi\hat{\mathbf{n}}), \quad \Delta\kappa \equiv \int d\chi w_2(\chi)\delta(\chi\hat{\mathbf{n}}), \tag{16}$$

  $$w_1(\chi) \equiv \frac{\chi}{2}\left(1-\frac{\chi}{\chi_{\rm LSS}}\right)\Theta(\chi,0,\chi_{\rm max}), \quad w_2(\chi) \equiv \frac{\chi}{2}\left(1-\frac{\chi}{\chi_{\rm LSS}}\right)\Theta(\chi,\chi_{\rm max},\chi_{\rm LSS}). \tag{17}$$

  where $\Theta(x,x_0,x_f)$ is a top-hat function.

  The covariance matrix of the two terms is therefore

  $$\langle|\tilde{\kappa}_{{\rm max},\ell m}|^2\rangle = \frac{2}{\pi}\int_0^\infty dk\,k^2\,P(k)\,w^2_{1,\ell}(k), \quad \langle|\Delta\kappa_{\ell m}|^2\rangle = \frac{2}{\pi}\int_0^\infty dk\,k^2\,P(k)\,w^2_{2,\ell}(k) \tag{18}$$

  $$\langle{\rm Re}(\tilde{\kappa}_{{\rm max},\ell m}\Delta\kappa_{\ell m})\rangle = \frac{2}{\pi}\int_0^\infty dk\,k^2\,P(k)\,w_{1,\ell}(k)\,w_{2,\ell}(k), \quad w_{i,\ell}(k) \equiv \int_0^\infty d\chi\,w_i(\chi)\,j_\ell(k\chi) \tag{19}$$

  Thus we generate a realization of $\Delta\kappa$ at each multipole order as a Gaussian number with distribution $\mathcal{N}(\mu,\sigma)$, where

  $$\mu = \frac{\langle{\rm Re}(\tilde{\kappa}_{{\rm max},\ell m}\Delta\kappa_{\ell m})\rangle}{\langle|\tilde{\kappa}_{{\rm max},\ell m}|^2\rangle}\tilde{\kappa}_{{\rm max},\ell m}, \quad \sigma = \langle|\Delta\kappa_{\ell m}|^2\rangle - \frac{\langle{\rm Re}(\tilde{\kappa}_{{\rm max},\ell m}\Delta\kappa_{\ell m})\rangle^2}{\langle|\tilde{\kappa}_{{\rm max},\ell m}|^2\rangle} \tag{20}$$

8 Full-sky expressions

The angular power spectrum between two contributions is:

$$C_\ell^{ij} = 4\pi\int_0^\infty \frac{dk}{k}\,\mathcal{P}_\Phi(k)\Delta^i_\ell(k)\Delta^j_\ell(k). \tag{21}$$

The expressions for density, RSD, magnification, lensing convergence and CMB lensing are:

$$\Delta_\ell^D(k) = \int dz p_z(z) b(z) T_\delta(k,z) j_\ell(k\chi(z)) \tag{22}$$

$$\Delta_\ell^{RSD}(k) = \int dz \frac{(1+z)p_z(z)}{H(z)} T_\theta(k,z) j_\ell''(k\chi(z)) \tag{23}$$

$$\Delta_\ell^M(k) = -\ell(\ell+1) \int \frac{dz}{H(z)} W^M(z) T_{\phi+\psi}(k,z) j_\ell(k\chi(z)), \tag{24}$$

$$\Delta_\ell^L(k) = -\frac{\ell(\ell+1)}{2} \int \frac{dz}{H(z)} W^L(z) T_{\phi+\psi}(k,z) j_\ell(k\chi(z)), \tag{25}$$

$$\Delta_\ell^C(k) = \frac{\ell(\ell+1)}{2} \int_0^{\chi_*} d\chi \frac{\chi_* - \chi}{\chi\chi_*} T_{\phi+\psi}(k,z) j_\ell(k\chi), \tag{26}$$

where

$$W^M(z) \equiv \int_z^\infty dz' p_z(z') \frac{2-5s(z')}{2} \frac{\chi(z')-\chi(z)}{\chi(z')} \tag{27}$$

$$W^L(z) \equiv \int_z^\infty dz' p_z(z') \frac{\chi(z')-\chi(z)}{\chi(z')} \tag{28}$$

9 Limber approximation

The Limber approximation is

$$j_\ell(x) \simeq \sqrt{\frac{\pi}{2\ell+1}}\, \delta\left(\ell + \frac{1}{2} - x\right). \tag{29}$$

Thus for each k and ℓ we can define a radial distance $\chi_\ell \equiv (\ell + 1/2)/k$

10 Expressions in the Limber approximation

The expressions above can be written as follows in the Limber approximation. First, the power spectrum can be rewritten as

$$C_\ell^{ij} = \frac{2}{2\ell+1} \int_0^\infty dk\, P_\delta\,(k, z=0)\, \tilde{\Delta}_\ell^i(k) \tilde{\Delta}_\ell^j(k). \tag{30}$$

where

$$\tilde{\Delta}_\ell^D(k) = p_z(\chi_\ell)\, b(\chi_\ell)\, D(\chi_\ell)\, H(\chi_\ell) \tag{31}$$

$$\tilde{\Delta}_\ell^{RSD}(k) = \frac{1+8\ell}{(2\ell+1)^2}\, p_z(\chi_\ell)\, f(\chi_\ell)\, D(\chi_\ell)\, H(\chi_\ell) - \frac{4}{2\ell+3}\sqrt{\frac{2\ell+1}{2\ell+3}} p_z(\chi_{\ell+1})\, f(\chi_{\ell+1})\, D(\chi_{\ell+1})\, H(\chi_{\ell+1}) \longrightarrow 0 \tag{32}$$

$$\tilde{\Delta}_\ell^{ISW}(k) = \frac{3\Omega_{M,0}H_0^2}{k^2} H(\chi_\ell) g(\chi_\ell)\,[1 - f(\chi_\ell)] \tag{33}$$

$$\tilde{\Delta}_\ell^M(k) = 3\Omega_{M,0}H_0^2 \frac{\ell(\ell+1)}{k^2} \frac{D(\chi_\ell)}{a(\chi_\ell)\chi_\ell} W^M(\chi_\ell) \longrightarrow 3\Omega_{M,0}H_0^2 \frac{\chi_\ell D(\chi_\ell)}{a(\chi_\ell)} W^M(\chi_\ell) \tag{34}$$

$$\tilde{\Delta}_\ell^L(k) = \frac{3}{2}\Omega_{M,0}H_0^2 \sqrt{\frac{(\ell+2)!}{(\ell-2)}} \frac{1}{k^2} \frac{D(\chi_\ell)}{a(\chi_\ell)\chi_\ell} W^M(\chi_\ell) \longrightarrow \frac{3}{2}\Omega_{M,0}H_0^2 \frac{\chi_\ell D(\chi_\ell)}{a(\chi_\ell)} W^M(\chi_\ell) \tag{35}$$

$$\tilde{\Delta}_\ell^C(k) = \frac{3}{2}\Omega_{M,0}H_0^2 \frac{\ell(\ell+1)}{k^2} \frac{D(\chi_\ell)}{a(\chi_\ell)\chi_\ell} \frac{\chi_* - \chi_\ell}{\chi_*} \Theta(\chi_\ell - \chi_*) \longrightarrow \frac{3}{2}\Omega_{M,0}H_0^2 \frac{\chi_\ell D(\chi_\ell)}{a(\chi_\ell)} \frac{\chi_* - \chi_\ell}{\chi_*} \Theta(\chi_\ell - \chi_*) \tag{36}$$