Approximation of neutrino phase space integral

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To calculate the neutrino density, we want to calculate integral of the kind

$$\rho = \int \frac{\sqrt{q^2+m^2}}{e^{q/kT}+1} d^3q, \tag{1}$$

see e.g. Dodelson, eq. 2.79. Without loss of generality, this is equivalent of caclulating

$$I(m_T) = \int_{r=0}^{r=\infty} \frac{r^2\sqrt{m_t^2+r^2}}{e^r+1} dr \tag{2}$$

There are two well known limits of this integral which one can do analytically, namely the radiation limit and CDM limit:

$$I(m_t=0) = \frac{7}{120}\pi^4 \tag{3}$$

$$I(m_t=\infty) = \frac{3}{2}\zeta(3)m_t \tag{4}$$

where ζ is Riemann ζ function. We can first expand both ends using Taylor expansion. This gives

$$I(m_t=0) = \frac{7}{120}\pi^4 + \frac{1}{24}\pi^2 m_t^2 \tag{5}$$

$$I(m_t=\infty) = \frac{3}{2}\zeta(3)m_t + \frac{45}{4}\zeta(5)m_t^{-1} + \frac{2835}{32}\zeta(7)m_t^{-3} \tag{6}$$

$$\tag{7}$$

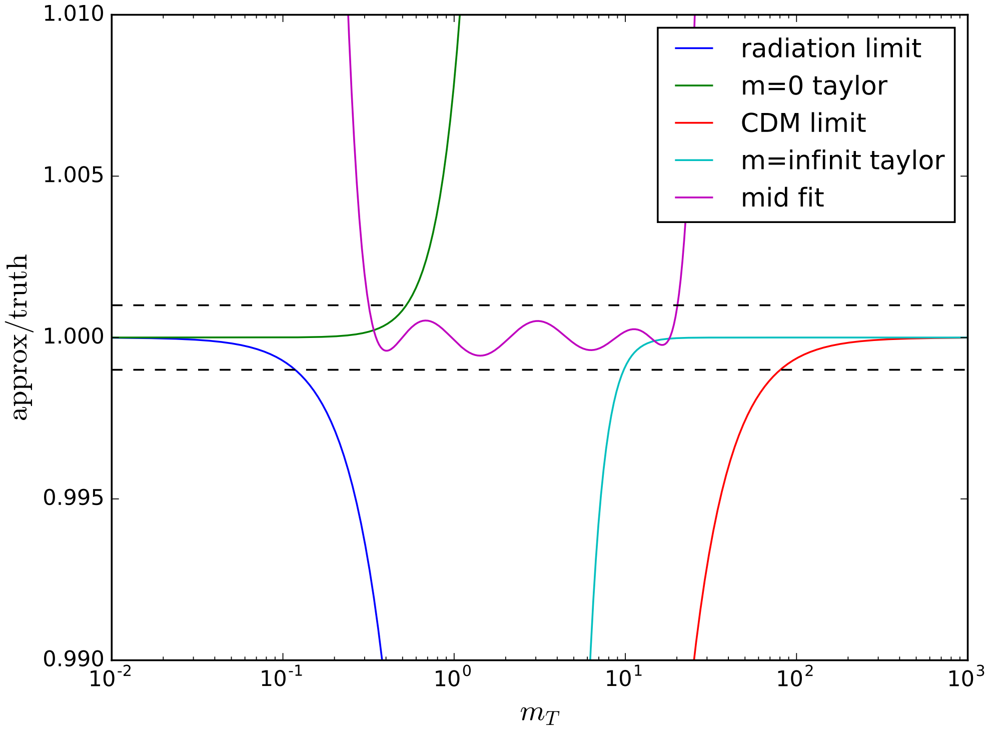

Figure 1: Approximations in three different regimes considered here. Accuracy is better than 10^{-3} everywhere (dashed lines).

In between, I could not find anything better than a simply polynomial fit in the intermediate region in log-log space. There we have

$$I(m_t) \approx \exp\left[\sum \frac{a_i}{i!} \log m_t\right], \tag{8}$$

where $a_{i=0\ldots7}$ =`1.79925723, 0.10857284, 0.09180734, 0.29183885, 0.1584564, -0.77416485, -0.11203872, 1.12435601`. This is shown in Figure 1. Note that accuracy of 10^{-3} in the phase-space integral will produce accuracy in hubble parameter of $O(10^{-5}$ since neutrinos are heavily subdominant when this matters.