Reconstruction for Lax Module Monads

Foregoing the fibre functor in *n* easy steps!

Based on joint work with Matti Stroiński: `arXiv:2409.00793`

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Given a monoidal category $\mathscr{C}$, are all left $\mathscr{C}$-module categories equivalent to the modules of an algebra object in $\mathscr{C}$?

Theorem ([Ost03; EGNO15])

Let $\mathscr{C}$ be a finite tensor category and let $\mathscr{M}$ be a finite abelian $\mathscr{C}$-module category, such that the evaluation functor $- \triangleright \ell \colon \mathscr{C} \longrightarrow \mathscr{M}$ is exact, for all $\ell \in \mathscr{M}$. Then there exists an algebra object $A \in \mathscr{C}$ such that there is an equivalence of $\mathscr{C}$-module categories $\mathrm{mod}_{\mathscr{C}}(A) \simeq \mathscr{M}$.

Finiteness assumptions, exactness assumptions, and rigidity assumptions.

Proposition ([DSPS19])

In the absence of rigidity, there are finite abelian $\mathscr{C}$-module categories that cannot be realised as the modules of an algebra object in $\mathscr{C}$.

Given a monoidal category $\mathscr{C}$, are all left $\mathscr{C}$-module categories equivalent to the modules of a monad on $\mathscr{C}$?

The setup

Let $\mathscr{C}$ be a $\Bbbk$-linear monoidal, and $\mathscr{M}, \mathscr{N}$ $\Bbbk$-linear left $\mathscr{C}$-module categories.

$$\otimes\colon \mathscr{C} \otimes_{\Bbbk} \mathscr{C} \longrightarrow \mathscr{C}, \qquad \triangleright\colon \mathscr{C} \otimes_{\Bbbk} \mathscr{M} \longrightarrow \mathscr{M}.$$

such that for all $x, y, z \in \mathscr{C}$ and $m \in \mathscr{M}$, e.g.,

$$(x \otimes y) \otimes z \cong x \otimes (y \otimes z) \qquad \text{and} \qquad (x \otimes y) \triangleright \ell \cong x \triangleright (y \triangleright \ell).$$

A functor $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ is a *lax $\mathscr{C}$-module functor* if there exists an appropriately associative and unital natural transformation

$$F_2\colon - \triangleright F(=) \Longrightarrow F(- \triangleright =).$$

The functor F is *oplax* if F_2 goes the other way, and *strong* if it is invertible.

The Yoneda lemma™

Proposition

There is an equivalence of $\mathscr{C}$-module categories

$$\mathscr{M} \simeq \mathsf{Str}\mathscr{C}\mathsf{Mod}(\mathscr{C}, \mathscr{M}), \qquad \ell \longmapsto - \triangleright \ell, \qquad F1 \longleftarrow F.$$

In particular, $\mathscr{C}^{\mathrm{rev}} \simeq \mathsf{Str}\mathscr{C}\mathsf{Mod}(\mathscr{C}, \mathscr{C})$.

Study cases in which $- \triangleright \ell$ admits a right adjoint. The resulting monad canonically has a lax $\mathscr{C}$-module structure. Then apply Beck's monadicity theorem.

Kelly and Beck

Theorem (Kelly's doctrinal adjunctions, [Kel74])

Given an adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ between monoidal categories, oplax monoidal structures on F are in bijective correspondence with lax monoidal structures on U.

An adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{M} :U$ is called *monadic* if the canonical comparison functor $K\colon \mathscr{M} \longrightarrow \mathscr{C}^{UF}$ is an equivalence.

Theorem (Abelian monadicity, [BZBJ18])

An adjunction is monadic if and only if U is exact and reflects zero objects.

Internal projectives

Let $\mathscr{C}$ and $\mathscr{M}$ be abelian. An object $\ell \in \mathscr{M}$ is *closed* if there is an adjunction

$$- \triangleright \ell \colon \mathscr{C} \rightleftarrows \mathscr{M} : \lfloor \ell, - \rfloor.$$

A closed object is called *$\mathscr{C}$-projective* if $\lfloor \ell, - \rfloor$ is (right) exact and a *$\mathscr{C}$-generator* if it is faithful.

Example

- Every object in a rigid monoidal category $\mathscr{C}$ is $\mathscr{C}$-projective.
- Finite $\mathscr{C}$-module categories over finite tensor categories always admit $\mathscr{C}$-projective $\mathscr{C}$-generators [EGNO15; DSPS19].

Only the Eilenberg–Moore category of an oplax $\mathscr{C}$-module monad has a canonical $\mathscr{C}$-module structure.

Linton coequalisers

Definition

The *Linton coequaliser* of $x \in \mathscr{C}$ and $m \in \mathscr{M}^T$ is:

$$T(x \triangleright Tm) \underset{\mu_{x \triangleright m} \circ T T_{\mathsf{a};x,m}}{\overset{T(x \triangleright \nabla_m)}{\rightrightarrows}} T(x \triangleright m) \longrightarrow\!\!\!\!\rightarrow x \blacktriangleright m.$$

Theorem ([SZ24])

The Eilenberg–Moore category of any right exact lax $\mathscr{C}$-module monad can be equipped with a canonical $\mathscr{C}$-module structure by mean of Linton coequalisers.

The reconstruction result

Theorem ([SZ24])

Let $\mathscr{C}$ be an abelian monoidal category, $\mathscr{M}$ an abelian $\mathscr{C}$-module category, and assume that $\ell \in \mathscr{M}$ is a closed $\mathscr{C}$-projective $\mathscr{C}$-generator. Then there is an equivalence of $\mathscr{C}$-module categories

$$\mathscr{M} \simeq \mathscr{C}^{\lfloor \ell, - \triangleright \ell \rfloor}.$$

Furthermore, there is a bijection

$$\left\{(\mathscr{M}, \ell) \text{ as before}\right\} \big/ \mathscr{M} \simeq \mathscr{N} \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{Right exact lax } \mathscr{C}\text{-module} \\ \text{monads on } \mathscr{C} \end{matrix} \right\} / \; \mathscr{C}^T \simeq \mathscr{C}^S$$

$$(\mathscr{M}, \ell) \longmapsto \lfloor \ell, - \triangleright \ell \rfloor$$

$$(\mathscr{C}^T, T1) \longleftarrow T$$

Back to Hopf algebras

Let H be a Hopf algebra and $\mathscr{C} := {}^H\mathsf{vect} \implies \mathsf{Ind}(\mathscr{C}) \simeq {}^H\mathsf{Vect}$.

Let $\mathscr{M}$ be an abelian $\mathscr{C}$-module category such that $\mathsf{Ind}(\mathscr{M})$ admits a coclosed $\mathsf{Ind}(\mathscr{C})$-injective $\mathsf{Ind}(\mathscr{C})$-cogenerator.

Then there exists an H-comodule coalgebra C such that $\mathsf{Ind}(\mathscr{M}) \simeq \mathsf{Comod}_H C$ as $\mathsf{Ind}(\mathscr{C})$-module categories.

This restricts to a $\mathscr{C}$-module equivalence $\mathscr{M} \simeq \mathsf{comod}_H C$.

(Op)lax module functors in action

Hopf trimodules

Theorem ([SZ24])

Let B be a bialgebra, and define $\mathcal{V} := {}^{B}\mathsf{Vect}$. There is a monoidal equivalence

$$\begin{aligned} {}^{B}_{B}\mathsf{Vect}^{B} &\longrightarrow \mathsf{LexfLax}\mathcal{V}\mathsf{Mod}(\mathcal{V},\mathcal{V}) \\ X &\longmapsto (X \,\square_B -, \chi) \end{aligned}$$

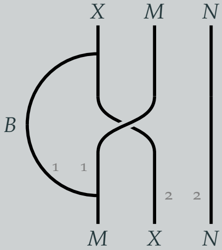
between the category of Hopf trimodules, and the category of left exact finitary lax $\mathcal{V}$-module endofunctors on $\mathcal{V}$.

The interchanger

For all $M, N \in {}^B\mathsf{Vect}$, the arrow

$$\chi_{M,N} \colon M \otimes_{\Bbbk} (X \,\square_B\, N) \longrightarrow X \,\square_B\, (M \otimes_{\Bbbk} N)$$

is defined by



Deducing a theorem for Hopf trimodules

Proposition

Let $\mathscr{C}$ be a left closed monoidal category such that every lax $\mathscr{C}$-module endofunctor of $\mathscr{C}$ is strong—in other words, that the monoidal embedding

$$\mathsf{Str}\mathscr{C}\mathsf{Mod}(\mathscr{C},\mathscr{C}) \hookrightarrow \mathsf{Lax}\mathscr{C}\mathsf{Mod}(\mathscr{C},\mathscr{C})$$

is an equivalence. Then $\mathscr{C}$ is left rigid.

Corollary ([SZ24])

A bialgebra B admits a twisted antipode if and only if the canonical functor $B \otimes_{\Bbbk} - \colon {}^{B}\mathsf{Vect} \longrightarrow {}^{B}_{B}\mathsf{Vect}^{B}$ is an equivalence.

Fusion operators for Hopf monads

Proposition ([SZ24])

Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} \colon U$ be an oplax monoidal adjunction. The strong monoidal structure of U turns $\mathscr{C}$ into a $\mathscr{D}$-module category via $- \triangleright = \;:=\; U(-) \otimes =$. The bimonad $T := UF$ on $\mathscr{C}$ becomes an oplax $\mathscr{D}$-module monad. In particular, the right fusion operator is the "free part" of the coherence morphism:

$$T_{2;F,\mathrm{Id}} = T_{\mathsf{rf}}.$$

Further, T_{rf} is an isomorphism if and only if T_2 is.

Thanks!

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Reconstruction of module categories in the infinite and non-rigid settings.
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