# Reconstruction for Lax Module Monads

## Foregoing the fibre functor in $n$ easy steps!

Based on joint work with Matti Stroiński: `arXiv:2409.00793`

2025-07-17

Tony Zorman

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**Given a monoidal category $\mathscr{C}$, are all left $\mathscr{C}$-module categories equivalent to the modules of an algebra object in $\mathscr{C}$?**

**Theorem ([Ost03; EGNO15])**

*Let $\mathcal{C}$ be a finite tensor category and let $\mathcal{M}$ be a finite abelian $\mathcal{C}$-module category, such that the evaluation functor $- \triangleright \ell \colon \mathcal{C} \longrightarrow \mathcal{M}$ is exact, for all $\ell \in \mathcal{M}$. Then there exists an algebra object $A \in \mathcal{C}$ such that there is an equivalence of $\mathcal{C}$-module categories $\mathrm{mod}_{\mathcal{C}}(A) \simeq \mathcal{M}$.*

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**Proposition ([DSPS19])**

*In the absence of rigidity, there are finite abelian $\mathscr{C}$-module categories that cannot be realised as the modules of an algebra object in $\mathscr{C}$.*

**Given a monoidal category $\mathscr{C}$, are all left $\mathscr{C}$-module categories equivalent to the modules of a monad on $\mathscr{C}$?**

# The setup

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such that for all $x, y, z \in \mathcal{C}$ and $m \in \mathcal{M}$, e.g.,

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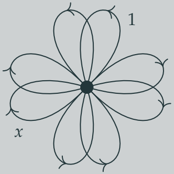
The functor $F$ is *oplax* if $F_2$ goes the other way, and *strong* if it is invertible.

# Module categories as deloopings

The *delooping* $\mathbf{B}\mathscr{C}$ of a monoidal category $\mathscr{C}$ is a 2-category with one object.



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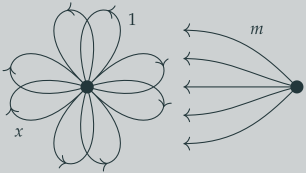

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# To Kelly and Beck

**Theorem (Kelly's doctrinal adjunctions, [Kel74])**

*Given an adjunction* $F \colon \mathscr{C} \rightleftarrows \mathfrak{D} : U$ *between monoidal categories, oplax monoidal structures on* $F$ *are in bijective correspondence with lax monoidal structures on* $U$.

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**Theorem (Beck's monadicity theorem)**

*An adjunction $F\colon \mathscr{C} \rightleftarrows \mathfrak{D} : U$ is monadic if and only if $U$ is conservative, $\mathfrak{D}$ has coequalisers of $U$-split pairs, and $U$ preserves them.*

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**Theorem (Abelian monadicity)**

*An adjunction $F\colon \mathcal{C} \rightleftarrows \mathfrak{D} : U$ is monadic if $U$ is exact and reflects zero objects.*

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**Example**

- Every object in a rigid monoidal category $\mathscr{C}$ is $\mathscr{C}$-projective.
- Finite $\mathscr{C}$-module categories over finite tensor categories always admit $\mathscr{C}$-projective $\mathscr{C}$-generators [EGNO15; DSPS19].

**Only the Eilenberg–Moore category of an oplax $\mathscr{C}$-module monad has a canonical $\mathscr{C}$-module structure.**

# Linton coequalisers

**Definition**

The *Linton coequaliser* of $x \in \mathscr{C}$ and $m \in \mathscr{M}^T$ is:

$$T(x \triangleright Tm) \underset{\mu_{x \triangleright m} \circ T T_{\mathsf{a};x,m}}{\overset{T(x \triangleright \nabla_m)}{\rightrightarrows}} T(x \triangleright m) \longrightarrow\!\!\!\!\to x \blacktriangleright m.$$

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**Theorem ([AHLF18; SZ24])**

*The Eilenberg–Moore category of any right exact lax $\mathscr{C}$-module monad can be equipped with a canonical $\mathscr{C}$-module structure by mean of Linton coequalisers.*

# The reconstruction result

**Theorem ([SZ24])**

*Let $\mathscr{C}$ be an abelian monoidal category, $\mathscr{M}$ an abelian $\mathscr{C}$-module category, and assume that $\ell \in \mathscr{M}$ is a closed $\mathscr{C}$-projective $\mathscr{C}$-generator.*

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$$\mathscr{M} \simeq_{\triangleright} \mathscr{C}^{\lfloor \ell, - \triangleright \ell \rfloor}.$$

*Furthermore, there is a bijection*

$$\{(\mathscr{M}, \ell) \text{ as before}\} \big/ \mathscr{M} \simeq \mathscr{N} \xleftrightarrow{\cong} \left\{ \begin{matrix} \text{Right exact lax } \mathscr{C}\text{-module} \\ \text{monads on } \mathscr{C} \end{matrix} \right\} \Big/ \mathscr{C}^T \simeq \mathscr{C}^S$$

# (Op)lax module functors in action

# Hopf trimodules

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*Let $B$ be a bialgebra, and define $\mathcal{V} := {}^{B}\mathsf{Vect}$. There is a monoidal equivalence*

$$\begin{aligned} {}^{B}_{B}\mathsf{Vect}^{B} &\longrightarrow \mathsf{LexfLax}\mathcal{V}\mathsf{Mod}(\mathcal{V},\mathcal{V}) \\ X &\longmapsto (X \square_B -, \chi) \end{aligned}$$

*between the category of Hopf trimodules, and the category of left exact finitary lax $\mathcal{V}$-module endofunctors on $\mathcal{V}$.*

# The interchanger

For all $M, N \in {}^{B}\mathsf{Vect}$, the arrow

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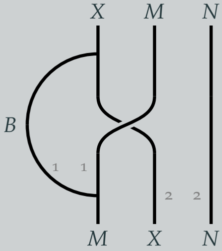

# Deducing a theorem for Hopf trimodules

**Proposition**

*Let $\mathscr{C}$ be a left closed monoidal category such that every lax $\mathscr{C}$-module endofunctor of $\mathscr{C}$ is strong—in other words, that the monoidal embedding*

$$\mathsf{Str}\mathscr{C}\mathsf{Mod}(\mathscr{C},\mathscr{C}) \hookrightarrow \mathsf{Lax}\mathscr{C}\mathsf{Mod}(\mathscr{C},\mathscr{C})$$

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**Corollary ([SZ24])**

*A bialgebra $B$ admits a twisted antipode if and only if the canonical functor $B \otimes_{\Bbbk} -\colon {}^{B}\mathsf{Vect} \longrightarrow {}^{B}_{B}\mathsf{Vect}^{B}$ is an equivalence.*

# Fusion operators for Hopf monads

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*Further, $T_{\mathsf{rf}}$ is an isomorphism if and only if $T_2$ is.*

# Thanks!

`tony-zorman.com/ct2025`

Reconstruction of module categories in the infinite and non-rigid settings.
`arXiv:2409.00793`

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