Lax Module Functors, Reconstruction, and Hopf Algebras

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Our investigation starts with Kelly's classic notion of **doctrinal adjunctions**:

> For monoidal categories $\mathscr{C}$ and $\mathscr{D}$, and an adjunction $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$, there is a bijection between oplax monoidal structures on F and lax monoidal structures on U.

This result provides a kind of **Tannaka reconstruction** for bimonads, as given by Moerdijk.

> There is a bijective correspondence between opmonoidal structures of a monad T on a monoidal category $\mathscr{C}$, and monoidal structures on $\mathscr{C}^T$ such that the forgetful functor is strong monoidal.

This correspondence lifts to the setting of **oplax $\mathscr{C}$-module monads**: monads T on $\mathscr{M}$, for a (left) $\mathscr{C}$-module category $\mathscr{M}$, that are equipped with a natural **action** morphism, resembling that of an opmonoidal comultiplication:

$$T_{\mathsf{a}}\colon T(- \triangleright =) \Longrightarrow - \triangleright T(=).$$

> **Theorem (Halbig–Z)**
> There is a bijective correspondence between oplax $\mathscr{C}$-module structures of a monad T on $\mathscr{M}$, and $\mathscr{C}$-module structures on $\mathscr{M}^T$ such that the forgetful functor U^T is a strict $\mathscr{C}$-module functor.

In contrast to these results stands **Deligne reconstruction**, where one does not require a forgetful functor—at the cost of only recovering the algebraic object of interest up to **Morita equivalence**.

Furthermore, the monads we consider are naturally **lax** module functors. In that case, one obtains a $\mathscr{C}$-module structure on the **Kleisli category** $\mathscr{M}_T$ of T. Under mild additional assumptions, this induces a unique $\mathscr{C}$-module structure on $\mathscr{M}^T$.

> **Proposition (Stroiński–Z)**
> Given a left $\mathscr{C}$-module structure on $\mathscr{M}_T$, there is, up to isomorphism at most one left $\mathscr{C}$-module structure on $\mathscr{M}_T$, such that the canonical inclusion $\iota\colon \mathscr{M}_T \longrightarrow \mathscr{M}^T$ is a strong $\mathscr{C}$-module functor.

Let us now concentrate on the representation theoretic case. For a field $\mathbb{k}$, suppose that $\mathscr{C}$ is a $\mathbb{k}$-linear abelian monoidal category, and that $\mathscr{M}$ is a $\mathbb{k}$-linear abelian left $\mathscr{C}$-module category.

Deligne reconstruction works for nice module categories over nice abelian bases

Paper, Poster, References

An important ingredient in our study of the module structure of $\mathscr{M}^T$ are **internal** projective objects: objects $X \in \mathscr{M}$ such that acting with any projective in $\mathscr{C}$ is a projective in $\mathscr{M}$. If X is **closed**—the adjunction

$$- \triangleright X\colon \mathscr{C} \rightleftarrows \mathscr{M} :\lfloor X, - \rfloor$$

exists—this guarantees the right adjoint to be an exact functor.

If X satisfies the additional condition of being a **$\mathscr{C}$-generator**, then $\lfloor X, - \rfloor$ even reflects zero objects; in particular, all of the preconditions of **Beck's monadicity theorem** for abelian categories hold:

> An adjunction between abelian categories is monadic if and only if the right adjoint is right exact and reflects zero objects.

Naturally, one could instead talk about internal **injective** objects and $\mathscr{C}$-cogenerators. This involves studying the adjunction

$$\lceil X, - \rceil\colon \mathscr{M} \rightleftarrows \mathscr{C} :- \triangleright X.$$

Putting all of these pieces together, we obtain a Deligne-type reconstruction result.

> **Theorem (Stroiński–Z)**
> If $\mathscr{C}$ has enough projectives, then all $\mathscr{C}$-module categories with enough projectives that have a closed $\mathscr{C}$-projective $\mathscr{C}$-generator are of the form $\mathscr{C}^{\lfloor X, - \triangleright X \rfloor}$.

This theorem in particular does not need a rigidity assumption. If this is added, the statement reduces from the monadic to the algebraic case.

> If $\mathscr{C}$ has enough projectives and is rigid, then for all $\mathscr{C}$-module categories $\mathscr{M}$ with enough projectives that have a closed $\mathscr{C}$-projective $\mathscr{C}$-generator, there exists an algebra object $A \in \mathscr{C}$ with $\mathrm{mod}_{\mathscr{C}}(A) \simeq \mathscr{M}$.

A category having enough injectives may instead be replaced by considering its **ind-completion**. Hopf algebraically, this yields a variant of a result by Ostrik.

> **Corollary (Stroiński–Z)**
> Every finite abelian ${}^{H}(\mathsf{vect}_{\mathbb{k}})$-module category $\mathscr{M}$, with $- \triangleright M$ exact for all $M \in \mathscr{M}$, is equivalent to $\mathsf{comod}_{{}^{H}(\mathsf{vect}_{\mathbb{k}})}(C)$, for an H-comodule algebra C.

We also obtain a version of the fundamental theorem of Hopf modules for the case of **Hopf trimodules**. The statement is akin to the quasi-bialgebraic case, as proven by Hausser–Nill and Saracco.

> **Proposition (Stroiński–Z)**
> A bialgebra B admits a twisted antipode if and only if the natural arrow
> $$B \otimes -\colon {}^{B}(\mathsf{Vect}_{\mathbb{k}}) \longrightarrow {}^{B}_{B}(\mathsf{Vect}_{\mathbb{k}})^{B}$$
> is an equivalence.

Lastly, the **fusion operators** of a bimonad in the sense of Bruguières–Lack–Virelizier also fit into this framework—they can be seen as coherence morphisms for a natural module action.

> **Proposition (Stroiński–Z)**
> Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} :U$ be an opmonoidal adjunction. The bimonad $T \overset{\text{def}}{=} UF$ on $\mathscr{C}$ is Hopf if and only if the coherence cells for the natural oplax $\mathscr{C}^T$-module monad structure on T are isomorphisms.