Category Theory and All That

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Tony Zorman

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What I like:

What I like:

- Duality.

What I like:

- Duality.
- Abstraction.

What I like:

- Duality.
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- Unification.

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Duality

$\mathrm{vect}_{\mathbb{C}}$

Duality

$\mathrm{rep}(G)$

$\mathrm{vect}_{\mathbb{C}}$

Duality

(Compact) group

$\mathsf{rep}(G)$

$\mathsf{vect}_{\mathbb{C}}$

Duality

$$\begin{tikzcd} \mathsf{rep}(G) \arrow[d, "U"] \\ \mathsf{vect}_{\mathbb{C}} \end{tikzcd}$$

Duality

$$\begin{array}{c} \mathsf{rep}(G) \\ \Big\downarrow{\scriptstyle U} \\ \mathsf{vect}_{\mathbb{C}} \end{array} \qquad \boxed{\mathrm{End}_{\otimes}(U) \cong G}$$

Duality/Abstraction

R-Mod

Duality/Abstraction

R-Mod

Commutative $\Bbbk$-algebra

Duality/Abstraction

R-algebra

B-Mod

R-Mod

Duality/Abstraction

$$
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R\text{-Mod}
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«adjectives$_1$»

Duality/Abstraction

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«adjectives$_1$»

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Duality/Abstraction

«adjectives$_2$»

$$\begin{tikzcd} B\text{-Mod} \arrow[d, "U"] \\ R\text{-Mod} \end{tikzcd}$$

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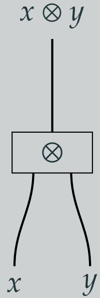

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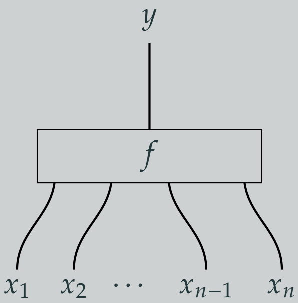

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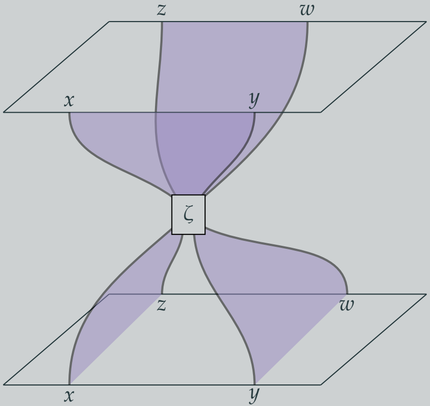

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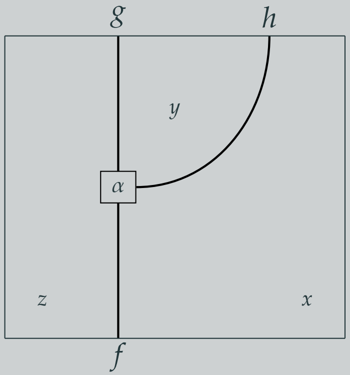

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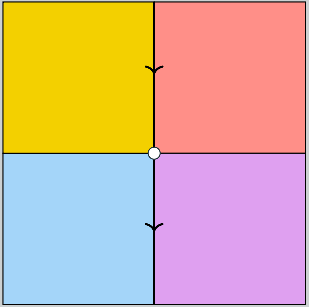

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Lemma (Yoneda lemma)

For every pseudofunctor $F : \mathbb{B}^{\mathrm{op}} \longrightarrow \mathbb{C}\mathsf{at}$ *we have*

$$\mathsf{Str}(よ, F) \xrightarrow{\sim} F.$$

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For every $\mathscr{M} \in \mathscr{C}\text{-}\mathsf{Mod}$ *we have*

$$_{\mathscr{C}}\mathsf{StrMod}(\mathscr{C}, \mathscr{M}) \xrightarrow{\sim} \mathscr{M}.$$

Category theory

$$
\begin{tikzcd}
(a \bullet b) \circ (c \bullet d) \circ (x \bullet y) \arrow[r, "\mathrm{id}\circ(\eta_c\bullet\eta_d)\circ(\eta_x\bullet\eta_y)"] \arrow[d, "(\eta_a\bullet\eta_b)\circ(\eta_c\bullet\eta_d)\circ\mathrm{id}"'] & (a \bullet b) \circ (Tc \bullet Td) \circ (Tx \bullet Ty) \arrow[r, "\mathrm{id}\circ\xi_{Tc,Td,Tx,Ty}"] \arrow[dd, "(\eta_a\bullet\eta_b)\circ\mathrm{id}"] & (a \bullet b) \circ ((Tc \circ Tx) \bullet (Td \circ Ty)) \arrow[r, "(\eta_a\bullet\eta_b)\circ(\eta_{Tc\circ Tx}\bullet\eta_{Td\circ Ty})"] \arrow[dd, "(\eta_a\bullet\eta_b)\circ\mathrm{id}"'] & (Ta \bullet Tb) \circ (T(Tc \circ Tx) \bullet T(Td \circ Ty)) \arrow[ddd, "\xi_{Ta,Tb,T(Tc\circ Tx),T(Td\circ Ty)}"] \\
(Ta \bullet Tb) \circ (Tc \bullet Td) \circ (x \bullet y) \arrow[d, "\xi_{Ta,Tb,Tc,Td}\circ\mathrm{id}"'] \arrow[dr, "\mathrm{id}\circ(\eta_x\bullet\eta_y)"] & & & \\
((Ta \circ Tc) \bullet (Tb \circ Td)) \circ (x \bullet y) \arrow[dr, "\mathrm{id}\circ(\eta_x\bullet\eta_y)"'] \arrow[dd, "(\eta_{Ta\circ Tc}\bullet\eta_{Tb\circ Td})\circ(\eta_x\bullet\eta_y)"'] & (Ta \bullet Tb) \circ (Tc \bullet Td) \circ (Tx \bullet Ty) \arrow[r, "\mathrm{id}\circ\xi_{Tc,Td,Tx,Ty}"'] \arrow[d, "\xi_{Ta,Tb,Tc,Td}\circ\mathrm{id}"] & (Ta \bullet Tb) \circ ((Tc \circ Tx) \bullet (Td \circ Ty)) \arrow[uur, "\mathrm{id}\circ(\eta_{Tc\circ Tx}\bullet\eta_{Td\circ Ty})"'] \arrow[ddd, "\xi_{Ta,Tb,Tc\circ Tx,Td\circ Ty}"'] & \\
& ((Ta \circ Tc) \bullet (Tb \circ Td)) \circ (Tx \bullet Ty) \arrow[dl, "(\eta_{Ta\circ Tc}\bullet\eta_{Tb\circ Td})\circ\mathrm{id}"] \arrow[dd, "\xi_{Ta\circ Tc,Tb\circ Td,Tx,Ty}"] & & (Ta \circ T(Tc \circ Tx)) \bullet (Tb \circ T(Td \circ Ty)) \arrow[dd, "(Ta\circ T^{\circ}_{2,Tc,Tx})\bullet(Tb\circ T^{\circ}_{2,Td,Ty})"] \\
(T(Ta \circ Tc) \bullet T(Tb \circ Td)) \circ (Tx \bullet Ty) \arrow[d, "\xi_{T(Ta\circ Tc),T(Tb\circ Td),Tx,Ty}"'] & & & \\
(T(Ta \circ Tc) \circ Tx) \bullet (T(Tb \circ Td) \circ Ty) \arrow[d, "(T^{\circ}_{2,Ta,Tc}\circ Tx)\bullet(T^{\circ}_{2,Tb,Td}\circ Ty)"'] & (Ta \circ Tc \circ Tx) \bullet (Tb \circ Td \circ Ty) \arrow[l, "(\eta_{Ta\circ Tc}\circ\mathrm{id})\bullet(\eta_{Tb\circ Td}\circ\mathrm{id})"] \arrow[r, "\mathrm{id}"] & (Ta \circ Tc \circ Tx) \bullet (Tb \circ Td \circ Ty) \arrow[uur, "(\mathrm{id}\circ\eta_{Tc\circ Tx})\bullet(\mathrm{id}\circ\eta_{Td\circ Ty})"] \arrow[r, "(\mathrm{id}\circ\eta_{Tc}\circ\eta_{Tx})\bullet(\mathrm{id}\circ\eta_{Td}\circ\eta_{Ty})"] \arrow[dll, "(\eta_{Ta}\circ\eta_{Tc}\circ\mathrm{id})\bullet(\eta_{Tb}\circ\eta_{Td}\circ\mathrm{id})"] \arrow[dr, "\mathrm{id}"] & (Ta \circ T^2c \circ T^2x) \bullet (Tb \circ T^2d \circ T^2y) \arrow[d, "(Ta\circ\mu_c\circ\mu_x)\bullet(Tb\circ\mu_d\circ\mu_y)"] \\
(T^2a \circ T^2c \circ Tx) \bullet (T^2b \circ T^2d \circ Ty) \arrow[rrr, "(\mu_a\circ\mu_c\circ Tx)\bullet(\mu_b\circ\mu_d\circ Ty)"'] & & & (Ta \circ Tc \circ Tx) \bullet (Tb \circ Td \circ Ty)
\end{tikzcd}
$$

Thanks!

tony-zorman.com/zmp25.pdf