Categorical Reconstruction Theory

Or: ~What Tony did in his PhD

Based on joint work with Sebastian Halbig and Matti Stroiński

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The classical story

R-Mod

The classical story

R-Mod

Commutative $\mathbb{k}$-algebra

The classical story

R-algebra

B-Mod

R-Mod

The classical story

$$
\begin{tikzcd}
B\text{-Mod} \arrow[d, shift left=3ex] \\
R\text{-Mod} \arrow[u, shift left=3ex] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

The classical story

$$\begin{tikzcd} B\text{-Mod} \arrow[d, shift left] \\ R\text{-Mod} \arrow[u, shift left] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

Monoidal, rigid, . . .

Tannaka duality

$$\begin{tikzcd} B\text{-Mod} \arrow[d, shift left, ""{name=D}] \\ R\text{-Mod} \arrow[u, shift left, ""{name=U}] \arrow[phantom, from=U, to=D, "\dashv"] \end{tikzcd}$$

Tannaka duality

$$\begin{tikzcd} B\text{-Mod} \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\ R\text{-Mod} \arrow[u, shift left=2] \end{tikzcd}$$

Which additional structures or properties on B correspond to additional structures or properties of B-Mod, that are compatible with the forgetful functor?

Tannaka duality

$$\begin{tikzcd} B\text{-Mod} \arrow[d, shift left=2ex] \\ R\text{-Mod} \arrow[u, shift left=2ex] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

Monoidal, rigid

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Tannaka duality

Bialgebra, Hopf algebra

$$\begin{tikzcd} B\text{-Mod} \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\ R\text{-Mod} \arrow[u, shift left=2] \end{tikzcd}$$

Monoidal, rigid

Which additional structures or properties on B correspond to additional structures or properties of B-Mod, that are compatible with the forgetful functor?

Monadic Tannaka duality

$$\begin{tikzcd} \mathscr{C}^T \arrow[d, shift left=2ex] \\ \mathscr{C} \arrow[u, shift left=2ex] \arrow[u, phantom, "\dashv"] \arrow[loop below, "T"] \end{tikzcd}$$

Which additional structures or properties on T correspond to additional structures or properties of $\mathscr{C}^T$, that are compatible with the forgetful functor?

Monadic Tannaka duality

$(x \in \mathscr{C}, \nabla_x : Tx \longrightarrow x)$

$$\begin{tikzcd} \mathscr{C}^T \arrow[d, shift left=2] \\ \mathscr{C} \arrow[u, shift left=2] \arrow[u, phantom, "\dashv"] \arrow[loop below, "T"] \end{tikzcd}$$

Which additional structures or properties on T correspond to additional structures or properties of $\mathscr{C}^T$, that are compatible with the forgetful functor?

Monadic Tannaka duality

Monoidal, closed, rigid

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Bimonad, Hopf monad

Which additional structures or properties on T correspond to additional structures or properties of $\mathscr{C}^T$, that are compatible with the forgetful functor?

Tannaka duality for module categories

Definition

If F is an oplax monoidal functor on a monoidal category $\mathscr{C}$ and $\mathscr{M}$ is a $\mathscr{C}$-module category, then an F**-coaction** on $G\colon \mathscr{M} \longrightarrow \mathscr{M}$ is a natural transformation

$$G_2\colon G(- \triangleright =) \Longrightarrow F(-) \triangleright G(=)$$

satisfying equations.

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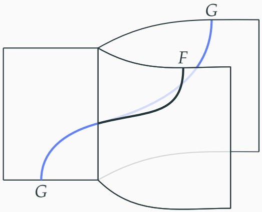

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Theorem (Kelly, Halbig–Z)

Given

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp_{\otimes}"] & \mathscr{D} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

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then F-coactions on G are in bijection with U-actions on V.

Tannaka duality for module categories

Corollary (Halbig–Z)

$$\begin{tikzcd} \mathscr{C} \arrow[d, dotted, "\mathrm{act}"'] \\ \mathscr{M} \end{tikzcd}$$

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$$\begin{tikzcd} \mathscr{M} \arrow[loop left] \arrow[r, shift left=2, "G"] \arrow[r, phantom, "\perp"] & \mathscr{N} \arrow[l, shift left=2, "V"] \arrow[loop right] \\ \mathscr{C} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp_\otimes"] & \mathscr{D} \arrow[l, shift left=2, "U"] \end{tikzcd}$$

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Corollary (Halbig–Z)

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\begin{tikzcd}
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\mathscr{M} \arrow[loop below, "T \text{ oplax module}"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathscr{M}^T \arrow[l, shift left]
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\mathscr{C} \arrow[d, dotted, "\mathrm{act}"'] \arrow[drr, dotted, bend left, "\mathrm{act}"] & & \\
\mathscr{M} \arrow["T"', loop below] \arrow[rr, shift left] \arrow[rr, phantom, "\perp"] & & \mathscr{M}^T \arrow[ll, shift left]
\end{tikzcd}
$$

T oplax module

Theorem (Kelly, Halbig–Z)

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\mathscr{C} \arrow[r, shift left, "\mathrm{Id}_{\mathscr{C}}"] \arrow[r, phantom, "\perp_{\otimes}"] & \mathscr{C} \arrow[l, shift left, "\mathrm{Id}_{\mathscr{C}}"]
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Tannaka duality for module categories

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Theorem (Kelly, Halbig–Z)

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$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left=2, "F^T"] \arrow[r, phantom, "\perp"] & \mathcal{M}^T \arrow[l, shift left=2, "U^T"] \\ \mathcal{C} \arrow[r, shift left=2, "\mathrm{Id}_{\mathcal{C}}"] \arrow[r, phantom, "\perp_{\otimes}"] & \mathcal{C} \arrow[l, shift left=2, "\mathrm{Id}_{\mathcal{C}}"] \end{tikzcd}$$

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Duality in monoidal categories

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\end{tikzcd}
$$

Rigidity and closedness of a monoidal category may be recognised by structures on the underlying monad, but what about other kinds of dualities?

Duality in monoidal categories

Theorem (Halbig–Z)

Rigid monoidal

Tensor representable

Grothendieck–Verdier

Closed monoidal

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Duality in monoidal categories

$\mathrm{ev}^{\ell}_x\colon {}^{\vee}x \otimes x \longrightarrow 1,\ \mathrm{coev}^{\ell}_x\colon 1 \longrightarrow x \otimes {}^{\vee}x,\ \mathrm{ev}^r$, and coev^r

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Rigid monoidal

Tensor representable

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Closed monoidal

$- \otimes x \dashv [x, -]_\ell$ and $x \otimes - \dashv [x, -]_r$

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Duality in monoidal categories

Th... (Hallai... Z)

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Rigid monoidal

Tensor representable

Grothendieck–Verdier

Dualising object d, and antiequivalence D, such that Dx represents $\mathscr{C}(- \otimes x, d)$.

Closed monoidal

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Duality in monoidal categories

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Rigid monoidal

Duals Lx and Rx, such that

$- \otimes x \dashv - \otimes Lx$ and $x \otimes - \dashv Rx \otimes -$

Tensor representable

recognised by structures on the

Dualising object d, and antiequivalence D,

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Grothendieck–Verdier

Closed monoidal

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Duality in monoidal categories

Theorem (Halbig–Z)

Rigid monoidal

$\subsetneq$

Tensor representable

$\subsetneq$

Grothendieck–Verdier

$\subsetneq$

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Duality in monoidal categories

Theorem (Halbig–Z)

Hopf algebras, Hopf monads

Rigid monoidal

$\subsetneq$

Tensor representable

$\subsetneq$

Grothendieck–Verdier

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Duality in monoidal categories

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Hopf algebras, Hopf monads

$$\begin{array}{c} \textit{Rigid monoidal} \\ \rotatebox{90}{$\subsetneq$} \\ \textit{Tensor representable} \\ \rotatebox{90}{$\subsetneq$} \\ \textit{Grothendieck–Verdier} \\ \rotatebox{90}{$\subsetneq$} \\ \textit{Closed monoidal} \end{array}$$

Hopf monads, Gabi monads (WIP)

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Duality in monoidal categories

Theorem (Halbig–Z)

Hopf algebras, Hopf monads

Rigid monoidal

$\subsetneq$

Tensor representable

$\subsetneq$

Grothendieck–Verdier

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Closed monoidal

???

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Theorem (Halbig–Z)

Rigid monoidal

$\subsetneq$

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$\subsetneq$

Closed monoidal

What about dualities that aren't necessarily reflected in the fibre functor? Can a monad create additional structure that wasn't present in its base category?

R- and K-matrices

Theorem

Suppose that H is a bialgebra. Then R-matrices – special invertible elements in $H \otimes H$ – are in bijection with braidings on the category of H-modules.

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Theorem (Bruguières–Virelizier)

Let $\mathscr{C}$ be a monoidal category, and T an opmonoidal monad on $\mathscr{C}$. Then braidings on $\mathscr{C}^T$ are in bijection with R-matrices on T.

R- and K-matrices

Definition

A **duoidal** category is a pseudomonoid in $\mathbb{M}\mathsf{onCat}$.

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$$\zeta\colon (x \bullet y) \circ (a \bullet b) \Longrightarrow (x \circ a) \bullet (y \circ b)$$

satisfying equations.

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Example

Braided monoidal categories, $({}_K\mathcal{M}_K, \otimes_{K\otimes K}, \otimes_K)$, $(\mathrm{End}_\kappa(\mathcal{V}), *, \circ)$.

Theorem (Bruguières–Virelizier)

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Theorem (Z)

Let $\mathscr{C}$ be a preduoidal category, and T a separately opmonoidal monad on $\mathscr{C}$. Then interchanges on $\mathscr{C}^T$ are in bijection with R-matrices on T.

R- and K-matrices

Definition

A **duoidal** category is a pseudomonoid in $\mathbb{M}\mathsf{onCat}$.

$$\zeta\colon (x \bullet y) \circ (a \bullet b) \Longrightarrow (x \circ a) \bullet (y \circ b)$$

satisfying equations.

Example

Braided monoidal categories, $({}_K\mathcal{M}_K, \otimes_{K \otimes K}, \otimes_K)$, $(\mathrm{End}_\kappa(\mathcal{V}), *, \circ)$.

Theorem (Bruguières–Virelizier)

Let $\mathscr{C}$ be a monoidal category, and T an opmonoidal monad on $\mathscr{C}$. Then braidings on $\mathscr{C}^T$ are in bijection with R-matrices on T.

R- and K-matrices

Theorem (WIP)

$$\begin{tikzcd} \mathscr{C} \arrow[loop below, "T\ \mathit{quasi}\Delta"] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathscr{C}^T \arrow[l, shift left] \end{tikzcd}$$

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Duoidal version?

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Reconstruction up to Morita equivalence

Theorem (Stroiński–Z)

Let $\mathscr{C}$ be an abelian monoidal category and $\mathscr{M}$ an abelian $\mathscr{C}$-module category.

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Let $\mathscr{C}$ be an abelian monoidal category and $\mathscr{M}$ an abelian $\mathscr{C}$-module category. Suppose $\ell \in \mathscr{M}$ is closed $(- \triangleright \ell \dashv \lfloor \ell, - \rfloor)$ and "nice".

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$$\mathscr{M} \simeq \mathscr{C}^{\lfloor \ell, - \triangleright \ell \rfloor}.$$

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Furthermore, there is a bijection

$$\{(\mathscr{M}, \ell) \text{ as above}\} / \mathscr{M} \simeq \mathscr{N} \overset{\cong}{\longleftrightarrow} \left\{ \begin{matrix} \text{Right exact lax } \mathscr{C}\text{-module} \\ \text{monads on } \mathscr{C} \end{matrix} \right\} / \, \mathscr{C}^T \simeq \mathscr{C}^S$$

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Module structure?

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Reconstruction up to Morita equivalence

Braided?

Theorem (Stroiński–Z)

Let $\mathscr{C}$ be an abelian monoidal category and $\mathscr{M}$ an abelian $\mathscr{C}$-module category. Suppose $\ell \in \mathscr{M}$ is closed ($- \triangleright \ell \dashv \lfloor \ell, - \rfloor$) and "nice". Then there is an equivalence of $\mathscr{C}$-module categories

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Thanks!

tony-zorman.com/amp25.pdf

References i

[AHLF18] Marcelo Aguiar, Mariana Haim, and Ignacio López Franco. **Monads on higher monoidal categories.** English. In: **Appl. Categ. Struct.** 26.3 (2018), pp. 413–458. ISSN: 0927-2852. DOI: 10.1007/s10485-017-9497-8.

[BLV11] Alain Bruguières, Steve Lack, and Alexis Virelizier. **Hopf monads on monoidal categories.** In: **Advances in Mathematics** 227.2 (2011), pp. 745–800. ISSN: 0001-8708. DOI: 10.1016/j.aim.2011.02.008.

[CSZ25] Kevin Coulembier, Mateusz Stroiński, and Tony Zorman. **Simple algebras and exact module categories.** In: **arXiv e-prints** (2025). arXiv: 2501.06629 [math.RT].

[Day77] B. J. Day. **Note on monoidal monads.** English. In: **J. Aust. Math. Soc., Ser. A** 23 (1977), pp. 292–311. ISSN: 0263-6115.

[Del02] P. Deligne. **Catégories tensorielles. (Tensor categories).** French. In: **Mosc. Math. J.** 2.2 (2002), pp. 227–248. ISSN: 1609-3321. URL: www.ams.org/distribution/mmj/vol2-2-2002/abst2-2-2002.html.

[DSPS19] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. **The balanced tensor product of module categories.** English. In: **Kyoto J. Math.** 59.1 (2019), pp. 167–179. ISSN: 2156-2261. DOI: 10.1215/21562261-2018-0006.

References ii

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. **Tensor categories.** Vol. 205. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015, pp. xvi+343. ISBN: 978-1-4704-2024-6. DOI: 10.1090/surv/205.

[HN99] Frank Hausser and Florian Nill. **Integral Theory for Quasi-Hopf Algebras.** In: **arXiv e-prints** (1999). arXiv: math/9904164 [math.QA].

[HZ23] Sebastian Halbig and Tony Zorman. **Duality in Monoidal Categories.** In: **arXiv e-prints** (2023). arXiv: 2301.03545 [math.CT].

[HZ24a] Sebastian Halbig and Tony Zorman. **Diagrammatics for Comodule Monads.** In: **Appl. Categ. Struct.** 32 (2024). Id/No 27, p. 17. ISSN: 0927-2852. DOI: 10.1017/CBO9781139542333.

[HZ24b] Sebastian Halbig and Tony Zorman. **Pivotality, twisted centres, and the anti-double of a Hopf monad.** English. In: **Theory Appl. Categ.** 41 (2024), pp. 86–149. ISSN: 1201-561X.

[Kel74] G. M. Kelly. **Doctrinal adjunction.** English. Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 257-280 (1974). 1974.

[Lin69] F. E. J. Linton. **Coequalizers in categories of algebras.** English. Semin. Triples categor. Homology Theory, ETH 1966/67, Lect. Notes Math. 80, 75-90 (1969). 1969.

References iii

[Moe02] Ieke Moerdijk. **Monads on tensor categories.** In: **J. Pure Appl. Algebra** 168.2-3 (2002). Category theory 1999 (Coimbra), pp. 189–208. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(01)00096-2.

[Ost03] Viktor Ostrik. **Module categories, weak Hopf algebras and modular invariants.** English. In: **Transform. Groups** 8.2 (2003), pp. 177–206. ISSN: 1083-4362. DOI: 10.1007/s00031-003-0515-6.

[Ost04] Victor Ostrik. **Tensor categories (after P. Deligne).** In: **arXiv e-prints** (2004). arXiv: math/0401347 [math.CT].

[Sar17] Paolo Saracco. **On the structure theorem for quasi-Hopf bimodules.** English. In: **Appl. Categ. Struct.** 25.1 (2017), pp. 3–28. ISSN: 0927-2852. DOI: 10.1007/s10485-015-9408-9.

[SZ24] Mateusz Stroiński and Tony Zorman. **Reconstruction of module categories in the infinite and non-rigid settings.** In: **arXiv e-prints** (2024). arXiv: 2409.00793 [math.QA].

[Zor25a] Tony Zorman. **Categorical Reconstruction Theory.** PhD thesis. TU Dresden, 2025. URL: https://nbn-resolving.org/urn:nbn:de:bsz:14-qucosa2-979696.

[Zor25b] Tony Zorman. **Duoidal R-Matrices.** In: **arXiv e-prints** (2025). arXiv: 2503.03445 [math.CT].