Category Theory and All That

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Tony Zorman

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What I like:

What I like:

- Duality.

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- Duality.
- Abstraction.

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- Unification.

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Duality

$\mathrm{vect}_{\mathbb{C}}$

Duality

$\mathrm{rep}(G)$

$\mathrm{vect}_{\mathbb{C}}$

Duality

(Compact) group

$\mathrm{rep}(G)$

$\mathrm{vect}_{\mathbb{C}}$

Duality

$$\begin{tikzcd} \mathsf{rep}(G) \arrow[d, "U"] \\ \mathsf{vect}_{\mathbb{C}} \end{tikzcd}$$

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$\boxed{\mathrm{Aut}_{\otimes}(U) \cong G}$

Duality/Abstraction

R-Mod

Duality/Abstraction

R-Mod

Commutative $\mathbb{k}$-algebra

Duality/Abstraction

R-algebra

B-Mod

R-Mod

Duality/Abstraction

$$\begin{tikzcd} B\text{-Mod} \arrow[d, "U"] \\ R\text{-Mod} \end{tikzcd}$$

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Duality/Abstraction

«adjectives$_2$»

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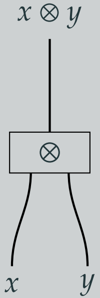

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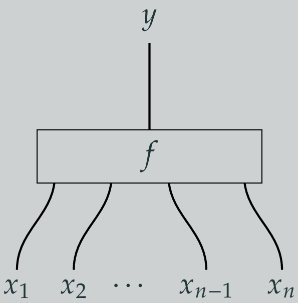

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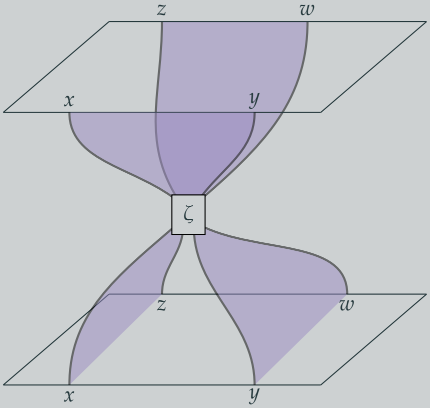

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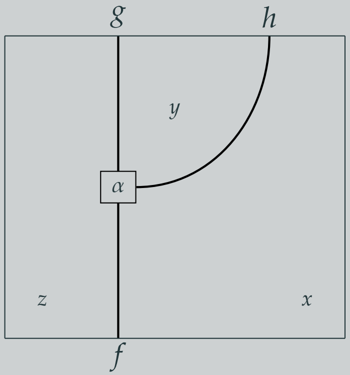

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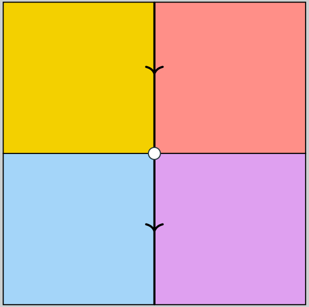

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Lemma (Yoneda lemma)

For every pseudofunctor $F : \mathbb{B}^{\mathrm{op}} \longrightarrow \mathbb{C}\mathsf{at}$ we have

$$\mathsf{Str}(よ, F) \xrightarrow{\sim} F.$$

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For every $\mathscr{M} \in \mathscr{C}$-$\mathsf{Mod}$ we have

$$_{\mathscr{C}}\mathsf{StrMod}(\mathscr{C}, \mathscr{M}) \xrightarrow{\sim} \mathscr{M}.$$

Category theory

$$
\begin{tikzcd}
(a \bullet b) \circ (c \bullet d) \circ (x \bullet y) \arrow[r, "\mathrm{id} \circ (\eta_c \bullet \eta_d) \circ (\eta_x \bullet \eta_y)"] \arrow[d, "(\eta_a \bullet \eta_b) \circ (\eta_c \bullet \eta_d) \circ \mathrm{id}"'] & (a \bullet b) \circ (Tc \bullet Td) \circ (Tx \bullet Ty) \arrow[r, "\mathrm{id} \circ \xi_{Tc,Td,Tx,Ty}"] \arrow[dd, "(\eta_a \bullet \eta_b) \circ \mathrm{id}"] & (a \bullet b) \circ ((Tc \circ Tx) \bullet (Td \circ Ty)) \arrow[r, "(\eta_a \bullet \eta_b) \circ (\eta_{Tc \circ Tx} \bullet \eta_{Td \circ Ty})"] \arrow[dd, "(\eta_a \bullet \eta_b) \circ \mathrm{id}"'] & (Ta \bullet Tb) \circ (T(Tc \circ Tx) \bullet T(Td \circ Ty)) \arrow[ddd, "\xi_{Ta,Tb,T(Tc \circ Tx),T(Td \circ Ty)}"] \\
(Ta \bullet Tb) \circ (Tc \bullet Td) \circ (x \bullet y) \arrow[d, "\xi_{Ta,Tb,Tc,Td} \circ \mathrm{id}"'] \arrow[dr, "\mathrm{id} \circ (\eta_x \bullet \eta_y)"] & & & \\
((Ta \circ Tc) \bullet (Tb \circ Td)) \circ (x \bullet y) \arrow[dr, "\mathrm{id} \circ (\eta_x \bullet \eta_y)"'] \arrow[dd, "(\eta_{Ta \circ Tc} \bullet \eta_{Tb \circ Td}) \circ (\eta_x \bullet \eta_y)"'] & (Ta \bullet Tb) \circ (Tc \bullet Td) \circ (Tx \bullet Ty) \arrow[r, "\mathrm{id} \circ \xi_{Tc,Td,Tx,Ty}"'] \arrow[d, "\xi_{Ta,Tb,Tc,Td} \circ \mathrm{id}"] & (Ta \bullet Tb) \circ ((Tc \circ Tx) \bullet (Td \circ Ty)) \arrow[uur, "\mathrm{id} \circ (\eta_{Tc \circ Tx} \bullet \eta_{Td \circ Ty})"'] \arrow[ddd, "\xi_{Ta,Tb,Tc \circ Tx,Td \circ Ty}"'] & \\
& ((Ta \circ Tc) \bullet (Tb \circ Td)) \circ (Tx \bullet Ty) \arrow[dl, "(\eta_{Ta \circ Tc} \bullet \eta_{Tb \circ Td}) \circ \mathrm{id}"] \arrow[dd, "\xi_{Ta \circ Tc,Tb \circ Td,Tx,Ty}"] & & (Ta \circ T(Tc \circ Tx)) \bullet (Tb \circ T(Td \circ Ty)) \arrow[dd, "(Ta \circ T^\circ_{2,Tc,Tx}) \bullet (Tb \circ T^\circ_{2,Td,Ty})"] \\
(T(Ta \circ Tc) \bullet T(Tb \circ Td)) \circ (Tx \bullet Ty) \arrow[d, "\xi_{T(Ta \circ Tc),T(Tb \circ Td),Tx,Ty}"'] & & & \\
(T(Ta \circ Tc) \circ Tx) \bullet (T(Tb \circ Td) \circ Ty) \arrow[d, "(T^\circ_{2,Ta,Tc} \circ Tx) \bullet (T^\circ_{2,Tb,Td} \circ Ty)"'] & (Ta \circ Tc \circ Tx) \bullet (Tb \circ Td \circ Ty) \arrow[l, "(\eta_{Ta \circ Tc} \circ \mathrm{id}) \bullet (\eta_{Tb \circ Td} \circ \mathrm{id})"] \arrow[r, "\mathrm{id}"] & (Ta \circ Tc \circ Tx) \bullet (Tb \circ Td \circ Ty) \arrow[r, "(\mathrm{id} \circ \eta_{Tc} \circ \eta_{Tx}) \bullet (\mathrm{id} \circ \eta_{Td} \circ \eta_{Ty})"] \arrow[uur, "(\mathrm{id} \circ \eta_{Tc \circ Tx}) \bullet (\mathrm{id} \circ \eta_{Td \circ Ty})"] \arrow[dll, "(\eta_{Ta} \circ \eta_{Tc} \circ \mathrm{id}) \bullet (\eta_{Tb} \circ \eta_{Td} \circ \mathrm{id})"] \arrow[dr, "\mathrm{id}"] & (Ta \circ T^2c \circ T^2x) \bullet (Tb \circ T^2d \circ T^2y) \arrow[d, "(Ta \circ \mu_c \circ \mu_x) \bullet (Tb \circ \mu_d \circ \mu_y)"] \\
(T^2a \circ T^2c \circ Tx) \bullet (T^2b \circ T^2d \circ Ty) \arrow[rrr, "(\mu_a \circ \mu_c \circ Tx) \bullet (\mu_b \circ \mu_d \circ Ty)"'] & & & (Ta \circ Tc \circ Tx) \bullet (Tb \circ Td \circ Ty)
\end{tikzcd}
$$

Thanks!

tony-zorman.com/zmp25.pdf