5. Stochastic Process

9/12

Stochastic process / Random process : Random process is a rule for assigning a function of time to every point in sample space

i.e $x : S \rightarrow f(t)$ x is mapping to function of time

State: The values assigned by random variables is called states

State space : The set of all possible values of an individual random variable is called state space.

I) Classification of Random / stochastic process

t / $x(t)$	Continuous	discrete
Continuous	continuous random process	Continuous random Sequence
discrete	discrete random process	discrete random Sequence

1) If $x(t)$ is continuous & t is continuous then random process is called continous random process

2) If $x(t)$ is continuous & t is discrete then it is called continuous random sequence

3) If $x(t)$ is discrete & t is continuous then it is called discrete random process

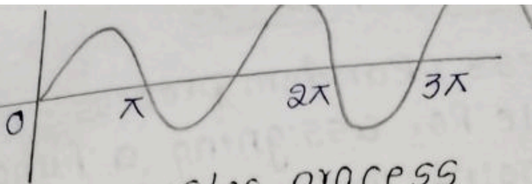
4) If $x(t)$ is discrete & t is discrete then it is called discrete random sequence.

II) → Stochastic process can also be classified as deterministic random process & non-deterministic random process.

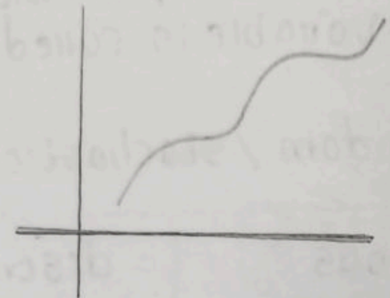
Deterministic random process

The future values of the function can be predicted from the knowledge of past values then random process is called deterministic random process

Ex:



Non deterministic process
If the future values cannot be preditcted based on the knowledge from past then that random process is called non deterministic random process



III) Discrete random process & Continuous random process

Discrete random process : If random process takes values only at some specified interval of time is called discrete random process

Continuous random process : If random process takes continuous values at all time intervals then that random process is called continuous random process

Markov Process

A Stochastic process is called Markov process if the future values of the random process depends only on present state but not on past (state (values) it is

$P\{X_n = t_n / X_{n-1} = t_{n-1}, X_{n-2} = t_{n-2} - - -\}$

i.e probability of $X(t_n+1) <= X(t_n)$

Markov Chain :

A Sequence of states X_n is a markov chain if each X_n is a random variable and if

$P\{X_{n+1} = x_{n+1} / X_n = x_n, x_{n-1} = x_{n-1} - - -\}$

Where the sequence $\{x_0, x_1 - - - x_n\}$ are called states of markov chain

Classification of Markov process

Depending on nature of x_i the markov process is classified as

	Continuous	discrete
Continuous		
discrete		

Transition probability: The probability of moving from one state to an another state or to remain in the same state during a single time period is called Transition probability. Mathematically probability is denoted as

$$P_{x_{(n-1)}, x_n} \{ P[(x(t_n) = x_n / x(t_{n-1}) = x_{n-1} \}$$

is called trasition probability.

This represents the conditional probability of system which is now in x_{n-1} state at t_{n-1} & future is x_n state at some time t_n

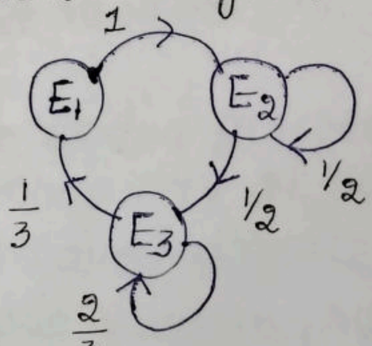
Diagramatic representation of transition probabilities

The transition probabilities can also be represented by 2 types of diagram.

Transition diagram: Transition diagram shows the transition probabilities / shifts that can occur in any particular situation. The arrows from each state indicates the possible state to which a process can move from given state

$$P = \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix} \end{array}$$

Transition diagram

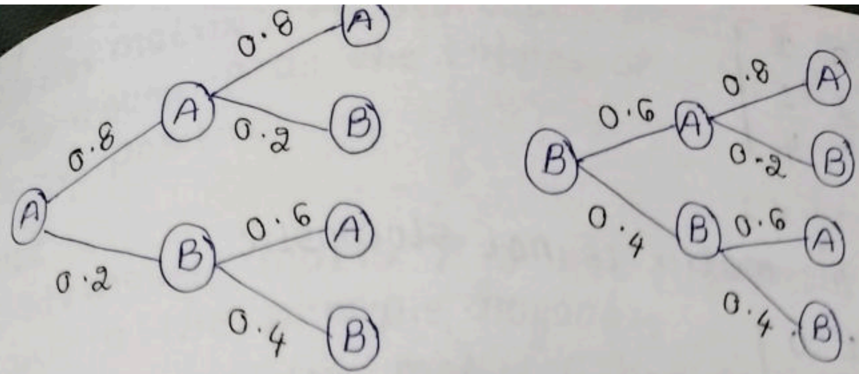


12/12 probability tree diagram

As the name implies this diagram emphasizes the probability & their moment from one step to another along with all possible branches or path that may connect the outcomes over a period of time.

Ex: 2 manufactures A & B are competing with each other in a restricted market over the year A's Customers have exhibited a high degree of loyalty as measured by the fact that constumer is using A's product 80 % of time also former customers purchasing the product from B switched to A 60% of time

Construct a tree diagram



Transition probability matrix

The transition probabilities P_{ij} will be arranged in an matrix form with transition probabilites as

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & \cdots & \cdots \\ \vdots & & & \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

where the first subscript stands for row (present state) and second subscript stands for column (future state). The symbol P_{ij} will be used for the probability of transition from state i to state j.

Properties of transition probability matrix

1) It is a square matrix
2) All entries are 0 & 1 inclusive
3) Sum of the entries in any row must be 1

Stochastic matrix: A stochastic matrix is a square matrix with non -ve elements & unit rows sums.

1) Check which of following matrices are stochastic

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\because$ given matrix is not square matrix, it is not stochastic

2) $\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

$\frac{1}{3}+\frac{1}{4} \neq 1$

$\therefore$ The matrix is not Stochastic

3) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$-1+0 \neq 1$ & -1 is $-$ve

$\therefore$ The matrix is not Stochastic

($\because$ matrix Contains -ve probabilities)

4) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The matrix is Stochastic

5) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

matrix is Stochastic

5) $\begin{bmatrix} 0 & 2 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Row Sum $\neq 1$

Not Stochastic

6) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$

Not Square matrix

Not Stochastic

7) $\begin{bmatrix} \frac{15}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$

$\frac{2}{3}+\frac{4}{3} \neq 1$

8) $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Stochastic matrix

Find value of x, y, z if $\begin{bmatrix} 0 & x & \frac{1}{3} \\ 0 & 0 & y \\ \frac{1}{3} & \frac{1}{4} & z \end{bmatrix}$ is a transition probability matrix

$0+x+\frac{1}{3}=1$

$x = 1-\frac{1}{3} = \frac{2}{3}$

$0+0+y=1$

$y=1$

$\frac{1}{3}+\frac{1}{4}+z=1$

$z = 1-\frac{7}{12} = \frac{5}{12}$

Regular matrix: A Stochastic matrix 'P' is said to be regular if all the entries of some power P^m are positive

Note:
A stochastic matrix P is not regular if one (1) occurs in the principle diagonal

→ If a transition matrix P has some zero entries and P^2 also contains zero entries in the identical places in both P^k & P^{k-1} for any k then P is not regular.

1) Check which of following stochastic matrices are regular

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

It is not regular since 1 is present in principal diagonal

$$\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is not regular
∵ 1 is present in diagonal position

$$\begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad P^2 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.375 & 0.375 & 0.25 \end{bmatrix}$$

∵ zeroes are occuring in identical places the matrix cannot be regular

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

∴ Not regular

$$\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.5 & [illegible] & [illegible] \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix} \qquad P^5 = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.125 & 0.5 & 0.375 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

$\therefore$ matrix is regular

$$\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.5625 & 0.3125 & 0.125 \\ 0.3 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.2 \end{bmatrix}$$

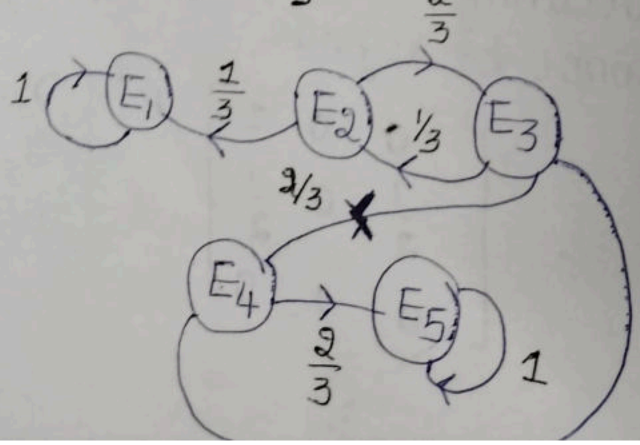
$\therefore$ matrix is regular

$$\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

not regular   not regular   not regular

Q: Consider the transition probability matrix find its graph

$$\begin{array}{c|c} & \begin{matrix} E_1 & E_2 & E_3 & E_4 & E_5 \end{matrix} \\ \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



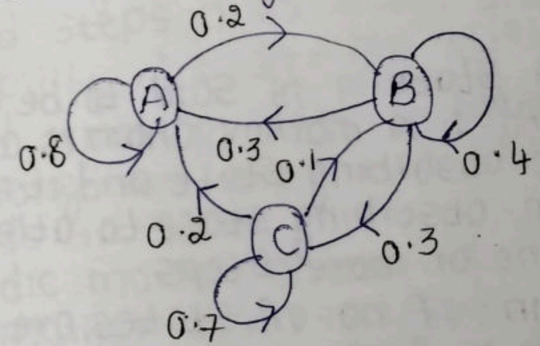
3 unive... students

Q. It is given that 80% of the Children of A went to A & rest went to B. 40% of Children of B went to B and rest split equally b/w A & C. Of the children of C 70% went to C and 20% went to A & 10% went to B. Form transition matrix & markov chain

Transition probability matrix

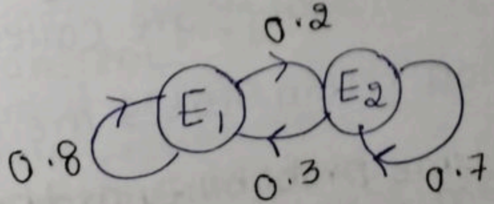
$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

Transition diagram (or) Markov Chain



The Alumin of a College finds on review that 80% of its Alumin who Contribute to annual fund one year will also Contribute next year & 30% of those do not contribute one year will contribute next. Write transition matrix & state transition diagram

$$\begin{array}{c} \\ E_1 \text{ Cont} \\ E_2 \text{ not Cont} \end{array} \begin{array}{c} \begin{array}{cc} E_1 \text{ Count} & E_2 \text{ not contr} \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$



Classification of States of Markov Chains:

1) accessible: A state j is accessible from another state i if $P_{ij} > 0$ for some $n > 0$

2) Communicate: Two accessible states i and j are said to communicate. State i and j with itself for all $i \geq 0$. If state i communicates with j and state j communicates with state k, then state i communicates with state k. Two states that communicate are in the same class. A state i is called an essential state if it communicates with every state it leads to $i \to j, j \to i$, $i \leftrightarrow j$

3) irreducible markov chain and irreducible matrix

irreducible markov chain and for all i and j, then every If $P_{ij} > 0$ for some n and for all i and j, then every state can be reached from every other state, such a chain is said to be irreducible. The transition probability matrix of an irreducible chain is an irreducible matrix. Otherwise the chain is said to be reducible or non-irreducible

4) Absorbing state: A state i is said to be an absorbing state if $P_{ii}^{m} = 1$. A markov chain is absorbing if it has atleast one absorbing state and it is possible to go from every non-absorbing state to atleast one absorbing state in one or more steps

4a) Finite marchochain: If no. of states are finite then the markov chain is finite markov chain

4b) Infinite markovchain: If no. of states are infinite then the markov chain is infinite markov chain.

4c) non-null persistant: Finite irreducible markov chain is non-null persistant

5) return state: A state i of a markov chain is called a return state if $P_{ij} > 0$ for some $n \geq 1$ (A state can be returned after a no. of steps it is called return state).

6) Period, periodic, aperiodic: The period of a return state is defined as greatest common divisor (GCD) of all m such that $P_{ii}^{m} > 0$ i.e $d_i = GCD\{m : P_{ii}^{m} > 0\}$ no. of steps in which the state can be restricted is period. A state i is said to be periodic with period d_i if $d_i > 1$ and aperiodic if $d_i = 1$

5b) recurrent state: return states are called recurrent state

5c) Transient state: non return states are called Trasient states

7) First time return time probability or the recurrence time probability: The probability that the chain starting from state i, for the first time at the n^{th} step by $f_{ii}^{(n)} = n = 1, 2 \ldots$ and is called as first time return time probability. If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$ the return to state i is certain and the state i is said to be persistent

or recurrent. If $F_{ii}<1$ the state i is said to be transient.
If state i communicates with state j is also recurrent.
The parameter $\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called the mean recurrence time of state i. If the mean recurrence time μ_{ii} is finite the state i is called to be non-null persistent or positive persistent and $\mu_{ii} = \infty$ it is null persistent.

8) ergodic state: positive nonnull perisistent to aperiodic recurrent (or positive persistent) and periodic state is called ergodic.

9) Regular Chain: A transition probability matrix is said to be a regular matrix if all entries of $P^{(m)}$ $(m=2,3 \ldots)$ are non-zero positive values. A homogeneous Markov chain is said to be regular chain if its transition probability matrix is a regular matrix.

10) A homogeneous markov chain will have a transition probability matrix that is independent of initial state and steps n as. $n \to \infty$ and is called steady state probability, i.e $\varrho_j = \lim_{n\to\infty} P_{ij}^{(n)}$ and $\sum \varrho_j = 1$ where ϱ_j is called limiting state probability and is interpreted as the long run proportion of time the markov chain spends in state j.

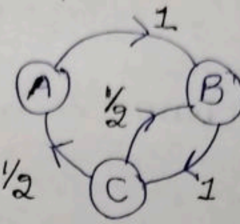
ergodic markov chain: If every state is ergodic then the markov chain is ergodic markov chain.

absorbing markov chain: 1) atleast one absorbing state
2) from non-absorbing state to absorbing state ∃ atleast one path

* accessible $i \to j$ or j is reachable from i, j ∃ a path from i to j
* Communicate two states i and j communicate if $i \to j$ and $j \to i$ i.e $i \leftrightarrow j$
* All states communicate then Markov chain is irreducible
* All states do not communicate then markov chain is reducible
* return state: If a state can be returned after possible no. of paths then the state is return state
period = GCD {no. of paths required to return that path}
aperiodic state if $d_i = 1$
periodic state if $d_i > 1$
* transient: If ∃ atleast one path which cannot be returned to the original state then state is transient or non-return states are transient states
* recurrent: If state is not transient then it is recurrent or return state are recurrent states
* ergodic: positive recurrent + aperiodic state is ergodic
* absorbing states: If probability of state is 1 then the state is absorbing state.

1) 3 boys A, B, C are throwing ball to each other. A always throws the ball to B and B always throws the ball to C. But C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix & Classify the states

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$



Since the future state depends upon present state. Hence matrix is markovian

Classification:

Accessibility:

A is reachable from B, B is reachable from C, A is reachable from C & so on.

$\therefore$ All states are accessible from all other states

Communicate:

$A \rightleftarrows B \quad B \rightleftarrows C \quad C \rightleftarrows A$

$\therefore$ All states communicate with each other

irreducible:

$\because$ If all states communicate with each other

$\therefore$ The given markov chain is irreducible

Finite markov chain:

$\because$ There are 3 states (finite no. of states).

The given markov chain is finite markov chain

non-null persistant:

$\because$ the given markov chain is finite & irreducible

$\therefore$ given markov chain is non-null persistant

Return state:

A, B, C are return states

Hence there are recurrent states

period:

Period of A = Gcd of $\{3, 5, 7 \ldots\} = 1$

$\therefore$ A is aperiodic state

Period of B = Gcd of {...}

∴ B is aperiodic state
period of C = Gcd of {2,3} = 1
∴ C is aperiodic state

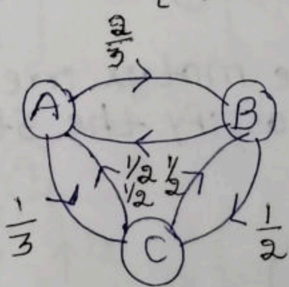
∴ A, B, C are non-null persisent & aperiodic state hence they are ergodic state.

→ ∴ If all states are ergodic then that chain is called ergodic markov chain

Absorbing state:

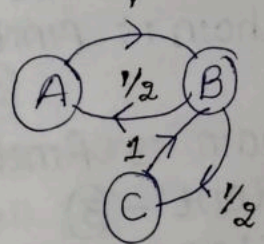
A, B, C are non-absorbing state ($\because p_{ii} \neq 1$)

The 3 state markov chain is given by the transition probability $P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & \frac{2}{3} & \frac{1}{3} \\ B & \frac{1}{2} & 0 & \frac{1}{2} \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$ prove that the chain is irreducible



∴ All states communicate with each other
Hence the markov chain is irreducible markov chain

Find the nature of states of markov chain with transition probability matrix $P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & \frac{1}{2} & 0 & \frac{1}{2} \\ C & 0 & 1 & 0 \end{array}$



→ All states communicate with each other
Hence given markov chain is irreducible markov chain -①

→ ∴ the no. of states are finite the given markov chain is finite markov chain -②

From ① & ② the given markov chain is non-null persistant -③

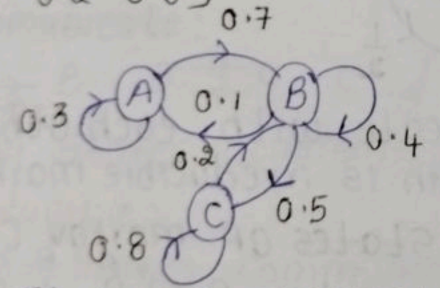
A, B, C are return states
Hence they are recurrent states -④

13/12 period of state A = Gcd {2, 4, 6 ...} = 2
if period > 1, state is periodic
∴ State A is periodic
period of state B = Gcd {2} = 2
∴ State B is periodic
period of state C = Gcd {2, 4 ...} = 2
∴ State C is periodic
∴ A, B, C are periodic states — ⑤
∴ $P_{ii} \neq 1$ for A, B, C
→ ∴ A, B, C are non-absorbing states
∴ Given markov chain is non-absorbing markov chain — ⑦
From ③ & ⑤
A, B, C are not ergodic states — ⑥
∴ The given markov chain is not ergodic

1) The transition probability matrix (TPM) of a markov chain

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix} \end{array}$$

is the matrix irreducible / classify the states



(markov chain)

∵ All states communicate with each other
The given markov chain is irreducible — ①
∵ The no. of states are finite
The given markov chain is finite markov chain — ②
From ① & ②
The given markov chain is finite & irreducible
i.e non-null persistent — ③
A, B, C are return states
Hence A, B, C are called recurrent states — ④
period of state A = Gcd {1, 2, 3, 4, 5} = 1
∴ A is aperiodic state
period of B = Gcd {1, 2, 3 ...} = 1
∴ B is aperiodic state
period of state C = Gcd {1, 2 ...} = 1
∴ C is aperiodic state

∴ A, B, C
From ③ &
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2)
1)

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

i)

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A, B, C
Hence A
∴ period
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period
∴ B
Period

$\therefore$ A, B, C are aperiodic states - (5)

From (3) & (5)

The given markov chain is non-null persisents & all states are aperiodic

Hence it is ergodic markov chain - (6)

$P_{ii} \neq 1$ for states A, B, C

so A, B, C are called non absorbing states

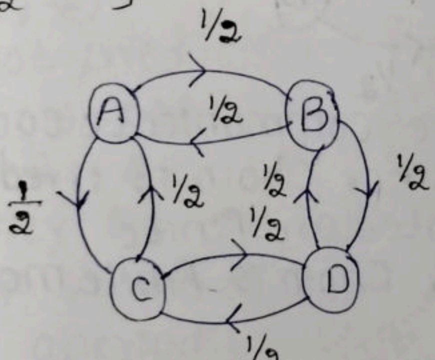
$\therefore$ All states are absorbing states Hence the given markov chain is non absorbing markov chain

classify the following markov chain

i) $\begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$

ii) $\begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$

i)



$\therefore$ All states communicate with each other

$\therefore$ The given markov chain is irreducible - (1)

$\therefore$ The no. of states are finite

$\therefore$ The given markov chain is finite markov chain - (2)

From (1) & (2)

The given markov chain is finite irreducible

The given markov chain is non-null persistent - (3)

i.e non-null persistent

A, B, C are return states

Hence A, B, C are called recurrent states - (4)

$\therefore$ period of state A = Gcd{2, 4 ...} = 2

(period of A) $\therefore$ A is aperiodic state

period of state B = Gcd{2, 4 ...} = 2

$\therefore$ B is aperiodic state

period of state C = Gcd{2, 4 ...} = 2

period of state C = Gcd{2, 4 ...} = 2

period of state C = ... periodic state

... is periodic state

period of State D = Gcd {2, 4 ...} = 2

∴ D is periodic State

∴ A, B, C, D are periodic states —(5)

From (3) & (5)

The given markov chain is non-null persisent

& all states are periodic

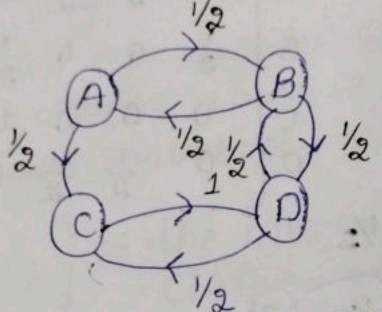
∴ Hence it is not ergodic markov chain —(6)

∴ $P_{ii} \neq 1$ for states A, B, C, D

so A, B, C, D are non-absorbing states

∴ All states are non-absorbing the given markov chain is non-absorbing markov chain

ii)



∴ All states are communicate each other

∴ The given markov chain is irreducible —(1)

∴ The no. of states are finite

The given markov chain is finite markov chain —(2)

From (1) & (2)

The given markov chain is finite irreducible

i.e non-null persistent —(3)

A, B, C, D are return states

Hence A, B, C, D are called recurrent states —(4)

Period of State A = Gcd {2, 4, 6 ...} = 2

∴ A is periodic State

period of State B = Gcd {2, 4, 6 ...} = 2

∴ B is periodic state

period of State C = Gcd {2, 4, 6 ...} = 2

∴ C is periodic state

period of State D = Gcd {2 ...} = 2

∴ D is periodic state

∴ A, B, C, D are periodic states —(5)

From (3) & (5)

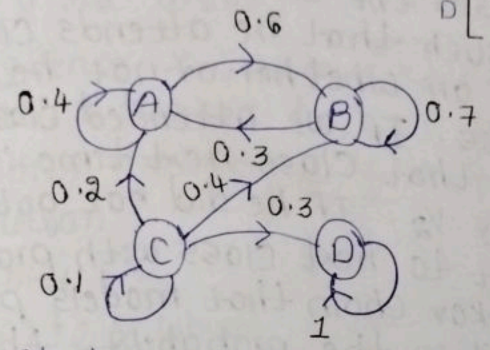
The given markov chain is non-null persistent & periodic

∴ Hence it is not ergodic markov chain

$P_{ii} \neq 1$

∴ The given markov chain is non absorbing markov chain

Find the nature of States of markov Chain with TPM $P =$ $\begin{matrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{matrix}$ $\begin{array}{c|c} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$



∵ All States are not communicating each other
∴ The given markov Chain is not irreducible - (1)
∵ The no. of States are finite
The given markov Chain is finite markov Chain - (2)
From (1) & (2)
It is not irreducible & finite markov Chain
Hence it is not non-null Persistent - (3)
(period of) All States are return States
Hence they are recurrent States - (4)
period of $A = \text{Gcd}\{1, 2, 3\} = 1$
∴ A is aperiodic State
period of $B = \text{Gcd}\{1, 2, 3\} = 1$
∴ B is aperiodic State
period of $C = \text{Gcd}\{1\} = 1$
∴ C is aperiodic State
period of $D = \text{Gcd}\{1\} = 1$
∴ D is aperiodic State
∴ A, B, C, D are aperiodic States - (5)
From (3) & (5)
The given markov chain is not non-null persistent and aperiodic
∴ Hence it is not ergodic markov chain - (6)
$p_{ii} = 1$ for State D
So A, B, C, D are absorbing States
∴ The given markov Chain is absorbing markov Chain

15/12 1) Peter takes the course basic stochastic process this quarter on Tue, Thurs & Friday. The classes starts at 10 am. Peter is used to work until late in the night and consequently he sometimes misses the class. His attendance behaviour is such that he attends classes depending only on whether or not he went to the latest class. If we attended class one day then he goes to that class next time it meets with probability 1/2. If he did not got to 1 class then he will got to next class with probability 3/4. Discuss the markov chain that models peter's attendance. What is the probability that he will attend the class on Thur if he went to class on Friday.

Let zero (0) be case of attending class.
Let 1 be case of not attending class
The transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

∴ If peter goes to college on friday he will attend clg on Thursday with a probability of

$$P\{X_2 = 0 / X_0 = 0\} = P_{00}^2 = \frac{5}{8}$$

$$P^2 = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix} \end{array}$$

2) A raining process is considered as a 2 state markov chain. If it rains it is considered to be in state 0. and if it does not rain it is considered to be in state 1. The transition probability of the markov chain is defined by $\begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$ find probability that it will rain for 3 days from today. Assuming that the mutual probabilities of state 0 (or) state 1 are 0.4 & 0.6 respectively

The probability that it will rain for 3 days from today $= P\{X_3 = 0 / X_0 = 0\} = P_{00}^{3} = 0.376$

$$P^3 = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix} \end{array}$$

Let sequence $X_n, n > 0$ $\{X_n / n > 0\}$ be a markov chain with 3 states $\{0, 1, 2\}$ and with transition probability matrix $\begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ and the initial distribution probability $P\{X_0 = i\} = \frac{1}{3}$ where $i = 0, 1, 2$ then find probability of $P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} \end{array}$$

$P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$ (is joint probability where X_3 depends on X_2, X_2 depends on X_1, X_1 depends on X_0 & X_0 is independent)

* every join probability can be converted into conditional probability

$= P\{X_3 = 1 / X_2 = 2\} \cap P\{X_2 = 2 / X_1 = 1\} \cap P\{X_1 = 1 / X_0 = 2\} \cap P(X_0 = 2)$

$= P_{21}^{(3-2)} \cdot P_{12}^{(2-1)} \cdot P_{21}^{(1-0)} \cdot P(X_0 = 2)$

$= P_{21}^{1} \cdot P_{12}^{1} \cdot P_{21}^{1} \cdot \frac{1}{3}$ $\quad (\because P(X_0 = i) = \frac{1}{3}$ where $i = 0, 1, 2)$

$= \frac{3}{4} \times \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3}$

$= \frac{3}{64}$

4) Consider the markov chain $P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ find transition probability matrix transition probability and probability of $P_{01}^{(2)}$ and probability of $P(X_2=1, X_0=0)$ with $P\{X_0=i\} = \frac{1}{3}$, where $i=0,1,2$

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$$

$$P^2 = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{1}{2} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} & \frac{1}{4} \end{bmatrix} \end{array} \Rightarrow P_{01}^{(2)} = \frac{5}{16}$$

$$\Rightarrow P(X_2=1, X_0=0) = P\{X_2=1/X_0=0\} \cap P(X_0=0)$$

$$= P_{01}^{(2-0)} \cdot P(X_0=0)$$

$$= P_{01}^{2} \cdot P(X_0=0)$$

$$= \frac{5}{16} \times \frac{1}{3} = \frac{5}{48}$$

16/12 An urn initially contains 5 black & 5 white balls. The following experiment is repeated indefinetely. A ball is drawn from the urn. If the ball is white it is put back in the urn otherwise it is left out. Let X_n be the no. of balls remaining in urn. after n draws from the urn. Is X_n a markov process if so find the appropriate transition probability matrix

b) Find the 1 step TPM P for X_n

c) Find the 2 step transition probability P^2 by matrix multiplication. What happens to X_n if x tends to ∞ Use your answer to guess the limit of P_n as $n \to \infty$

The number x_n of black balls in the urn completely specifies the probability of outcomes of a trial which is dependent only on present but not past

$\therefore x_n$ is markov process

$$P(x_n=0/x_{n-1}=0) = \frac{5c_1}{5c_1} = 1$$

$$P(x_n=1/x_{n-1}=0) = P(x_n=2/x_{n-1}=0) = \dots$$

$$P(x_n=5/x_{n-1}=0) = 0$$

$$P(x_n=0/x_{n-1}=1) = P(x_n=2/x_{n-1}=\,)$$

$$P(x_n=0/x_{n-1}=1) = \frac{1c_1}{6c_1} = \frac{1}{6}$$

$$P(x_n=1/x_{n-1}=1) = \frac{5c_1}{6c_1} = \frac{5}{6}$$

$$P(x_n=2/x_{n-1}=1) = P(x_n=3/x_{n-1}=1) =$$

$$P(x_n=4/x_{n-1}=1) = P(x_n=5/x_{n-1}=1) = 0$$

$$P(x_n=0/x_{n-1}=2) = P(x_n=3/x_{n-1}=2) =$$

$$P(x_n=4/x_{n-1}=2) = P(x_n=5/x_{n-1}=2) = 0$$

$$P(x_n=1/x_{n-1}=2) = \frac{2c_1}{7c_1} = \frac{2}{7}$$

$$P(x_n=2/x_{n-1}=2) = \frac{5c_1}{7c_1} = \frac{5}{7}$$

$$P(x_n=0/x_{n-1}=3) = P(x_n=1/x_{n-1}=3) = P(x_n=4/x_{n-1}=3)$$

$$= P(x_n=5/x_{n-1}=3) = 0$$

$$P(x_n=2/x_{n-1}=3) = \frac{3c_1}{8c_1} = \frac{3}{8}$$

$$P(x_n=3/x_{n-1}=3) = \frac{5c_1}{8c_1} = \frac{5}{8}$$

$$P(x_n=0/x_{n-1}=4) = P(x_n=1/x_{n-1}=4) = P(x_n=2/x_{n-1}=4)$$

$$= P(x_n=5/x_{n-1}=4) = 0$$

$$P(x_n=3/x_{n-1}=4) = \frac{4c_1}{9c_1} = \frac{4}{9}$$

$$p(x_n=4/x_{n-1}=4) = \frac{5c_1}{9c_1} = \frac{5}{9}$$

$$p(x_n=0/x_{n-1}=5) = P(x_n=1/x_{n-1}=5) = P(x_n=2/x_{n-1}=5)$$

$$= p(x_n=3/x_{n-1}=5) = 0$$

$$* \; p(x_n=4/x_{n-1}=5) = \frac{5c_1}{10c_1} = \frac{5}{10}$$

$$p(x_n=5/x_{n-1}=5) = \frac{5c_1}{10c_1} = \frac{5}{10}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{5}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{10} & \frac{5}{10} \end{bmatrix}$$

Two step transition probability matrix

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{11}{36} & \frac{25}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{21} & \frac{65}{144} & \frac{25}{49} & 0 & 0 & 0 \\ 0 & \frac{3}{28} & \frac{225}{448} & \frac{25}{64} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{95}{192} & \frac{25}{81} & 0 \\ 0 & 0 & 0 & \frac{2}{9} & \frac{19}{36} & \frac{1}{4} \end{bmatrix}$$

The urn contains only white balls as $n \to \infty$

$$P^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$= P(X_n=2/X_{n-1}=5)$

A fair die is tossed repeatedly if X_n denotes the maximum of the numbers occuring in the first n numbers. Find TPM P of the markov chain X_n & $P(X_2=6)$ find also P^2 and $P(X_2=6)$

State space = {1, 2, 3, 4, 5, 6}

X_n = The max of numbers occuring in first trials

$P(X_n=1/X_{n-1}=1) = \frac{1}{6} = P(X_n=2/X_{n-1}=1)$

$= P(X_n=3/X_{n-1}=1) = P(X_n=4/X_{n-1}=1) = P(X_n=5/X_{n-1}=1)$

$= P(X_n=6/X_{n-1}=1)$

$P(X_n=1/X_{n-1}=2) = 0$

$P(X_n=2/X_{n-1}=2) = \frac{2}{6}$

$P(X_n=3/X_{n-1}=2) = P(X_n=4/X_{n-1}=2) = P(X_n=5/X_{n-1}=2)$

$= P(X_n=6/X_{n-1}=2) = \frac{1}{6}$

$P(X_n=1/X_{n-1}=3) = P(X_n=2/X_{n-1}=3) = 0$

$P(X_n=3/X_{n-1}=3) = \frac{3}{6} = \frac{1}{2}$

$P(X_n=4/X_{n-1}=3) = P(X_n=5/X_{n-1}=3) = P(X_n=6/X_{n-1}=3) = \frac{1}{6}$

$P(X_n=1/X_{n-1}=4) = P(X_n=2/X_{n-1}=4) = P(X_n=3/X_{n-1}=4) = 0$

$P(X_n=4/X_{n-1}=4) = \frac{4}{6}$

$P(X_n=5/X_{n-1}=4) = P(X_n=6/X_{n-1}=4) = \frac{1}{6}$

$P(X_n=1/X_{n-1}=5) = P(X_n=2/X_{n-1}=5) = P(X_n=3/X_{n-1}=5)$

$P(X_n=4/X_{n-1}=5) = 0$

$P(X_n=5/X_{n-1}=5) = \frac{5}{6}$

$P(X_n=6/X_{n-1}=5) = \frac{1}{6}$

$P(X_n=1/X_{n-1}=6) = P(X_n=2/X_{n-1}=6) = P(X_n=3/X_{n-1}=6)$

$= P(X_n=4/X_{n-1}=6) = P(X_n=5/X_{n-1}=6) = 0$

$P(X_n=6/X_{n-1}=6) = \frac{6}{6} = 1$

$\rightarrow \infty$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1/2 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \end{bmatrix} \end{array}$$

Steady state probability

In many markov chain the probability for a particular state will approach a limiting value as time goes to infinity. In other words in the far future the probability won't be changing from one transition to next transition. These limiting values are called Steady State probabilities or stable probabilities

Steady state condition

If a system is such that each state has probability equal to its stable probability. The probability will persist for all time. Then the system is said to be in Steady State condition.

Steady state vector / equillibrium vector of a markov chain

If a markov chain with transition matrix P is regular then there exists a unique vector V such that for any probability vector v & for large values of n. $vP^n \simeq V$, Vector V is called the equillibrium vector / fixed vector / steady state vector of markov chain. This is also called long range trend vector of markov chain.

19/12 Probability vector: A probability vector is a matrix having only one row whose row sum is 1.

Note:

If a markov chain with transition matrix P is regular then there exists a probability vector V such that $VP = PV$ where V gives the long range trend of markov chain

1. Find the long range trend or study vector for markov chain with transition matrix

$$\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

Let $P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$

$\because$ All entries of matrix P are positive

$\therefore$ P is regular

probability vector v such that

$$vP = v$$

$$[v_1 \ v_2 \ v_3] \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

$$[0.65v_1 + 0.15v_2 + 0.12v_3 \quad 0.28v_1 + 0.67v_2 + 0.36v_3 \quad 0.07v_1 + 0.18v_2 + 0.52v_3] = [v_1, v_2, v_3]$$

$$0.65v_1 + 0.15v_2 + 0.12v_3 = v_1$$

$$0.28v_1 + 0.67v_2 + 0.36v_3 = v_2$$

$$0.07v_1 + 0.18v_2 + 0.52v_3 = v_3$$

$$-0.35v_1 + 0.15v_2 + 0.12v_3 = 0 \quad -(1)$$

$$0.28v_1 - 0.33v_2 + 0.36v_3 = 0 \quad -(2)$$

$$0.07v_1 + 0.18v_2 - 0.48v_3 = 0 \quad -(3)$$

$$v_1 + v_2 + v_3 = 1 \quad -(4) \quad (\because v \text{ is probability vector})$$

(1) (2) (3) are linearly dependent

Solving (1), (2), (4)

$$v_1 = \frac{104}{363}, \quad v_2 = \frac{532}{1089}, \quad v_3 = \frac{245}{1089}$$

probability vector / long range trend vector

$$V = [v_1 \ v_2 \ v_3]$$

$$V = \left[\frac{104}{363}, \frac{532}{1089}, \frac{245}{1089}\right]$$

2. Find the equillibrium vector or steady state vector for transition matrix $P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$

Let $P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$

$\because$ In P all entries are positive

P is a regular matrix

Let V be equillibrium vector

$\therefore \quad VP = V$

$$[v_1 \; v_2 \; v_3] \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = [v_1 \; v_2 \; v_3]$$

$$[0.5v_1 + 0.1v_2 + 0.2v_3 \quad 0.2v_1 + 0.4v_2 + 0.2v_3 \quad 0.3v_1 + 0.5v_2 + 0.6v_3]$$

$$= [v_1 \; v_2 \; v_3]$$

$$0.5v_1 + 0.1v_2 + 0.2v_3 = v_1$$

$$0.2v_1 + 0.4v_2 + 0.2v_3 = v_2$$

$$0.3v_1 + 0.5v_2 + 0.6v_3 = v_3$$

$$-0.5v_1 + 0.1v_2 + 0.2v_3 = 0 \quad -(1)$$

$$0.2v_1 - 0.6v_2 + 0.2v_3 = 0 \quad -(2)$$

$$0.3v_1 + 0.5v_2 - 0.4v_3 = 0 \quad -(3)$$

$v_1 + v_2 + v_3 = 1 \quad -(4)$ ($\because$ V is probability vector)

(1) (2) (3) are linearly dependent

(1), (2) (4)

$$v_1 = \frac{1}{4} \quad v_2 = \frac{1}{4} \quad v_3 = \frac{1}{2}$$

$\therefore$ probability vector/long rangetrend

$$V = [v_1 \; v_2 \; v_3] = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right]$$

3) Find study state vector for

$$P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$$

Let $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$

$\therefore$ In P all entries are positive

P is a regular matrix

Let v be probability vector

$$vP = v$$

$$[v_1 \ \ v_2] \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix} = [v_1 \ \ v_2]$$

$$[0.25v_1 + 0.5v_2 \quad 0.75v_1 + 0.5v_2] = [v_1 \ \ v_2]$$

$$0.25v_1 + 0.5v_2 = v_1$$

$$0.75v_1 + 0.5v_2 = v_2$$

$$-0.75 + 0.5v_2 = 0 \quad -(1)$$

$$0.75v_1 - 0.5v_2 = 0 \quad -(2)$$

$$v_1 + v_2 = 1 \quad -(3)$$

(1) (2) are linearly independent

Solving (1) & (4):

we get

$$v_1 = \frac{2}{5} \qquad v_2 = \frac{3}{5}$$

probability vector / long range trend

$$v = [v_1, v_2]$$

$$v = \left[\frac{2}{5}, \frac{3}{5}\right]$$

4) A housewife buys 3 types of cereals A, B, C She never buys same Cereals in successive weeks however if she buys Cereal A, the next week she buys B however if she buys B or C The next week it is 3 times as likely to buy A as other cereals. In the long run how often does she buys each of three cereals

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 0 \\ 3x & 0 & x \\ 3x & x & 0 \end{array}\right] \end{array}$$

$\because$ P is a probability matrix
row sum = 1

$\therefore 3x + 0 + x = 1$

$4x = 1$

$x = \frac{1}{4}$

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \left[\begin{array}{ccc} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array}\right] \end{array}$$

$$P^2 = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{16} & \frac{3}{16} & 0 \\ \frac{3}{16} & \frac{3}{4} & \frac{1}{16} \end{bmatrix} \qquad P^3 = \begin{bmatrix} \frac{3}{16} & \frac{13}{16} & 0 \\ \frac{39}{64} & \frac{3}{16} & \frac{13}{64} \\ \frac{39}{64} & \frac{13}{64} & \frac{3}{16} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} \frac{39}{64} & \frac{3}{16} & \frac{13}{64} \\ \frac{75}{256} & \frac{169}{256} & \frac{3}{64} \\ \frac{75}{256} & \frac{21}{32} & \frac{13}{256} \end{bmatrix}$$

$\therefore$ P is regulart

Let $v_1\ v_2\ v_3$ be long range trend vector.

$\therefore VP = V$

$$[v_1\ v_2\ v_3] \left\{ \begin{array}{ccc} \frac{39}{64} & \frac{3}{16} & \frac{13}{64} \\ \frac{75}{256} & \frac{169}{256} & \frac{3}{64} \end{array} \right\} \times \begin{bmatrix} 0 & 1 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} & 0 \end{bmatrix} [v_1\ v_2\ v_3]$$

$$\begin{bmatrix} 0 \\ \frac{13}{64} \\ \frac{3}{16} \end{bmatrix}$$

ctor·

$[v_1\ v_2\ v_3]$

$$\left[0v_1 + \tfrac{3}{4}v_2 + \tfrac{3}{4}v_3 \quad v_1 + 0v_2 + \tfrac{1}{4}v_3 \quad 0v_1 + \tfrac{1}{4}v_2 + 0v_3\right] = [v_1\ v_2\ v_3]$$

$$0v_1 + \frac{3}{4}v_2 + \frac{3}{4}v_3 = v_1$$

$$v_1 + 0v_2 + \frac{1}{4}v_3 = v_2$$

$$0v_1 + \frac{1}{4}v_2 + 0v_3 = v_3$$

$$-v_1 + \frac{3}{4}v_2 + \frac{3}{4}v_3 = 0 \quad -(1)$$

$$v_1 - v_2 + \frac{1}{4}v_3 = 0 \quad -(2)$$

$$0v_1 + \frac{1}{4}v_2 - v_3 = 0 \quad -(3)$$

$v_1 + v_2 + v_3 = 1$ — (4) ($\because v$ is probability vector)

(1) (2) (3) are linearly dependent

Solving (1) (2) (4)

$$v_1 = \frac{3}{7} \quad v_3 = \frac{4}{35} \quad v_2 = \frac{16}{35}$$

probability of buying Cereal $A = \frac{3}{7}$

$B = \frac{16}{35}$

$C = \frac{4}{35}$

20/12

5) The whether in certain spot is classified as fair, cloudy or rainy. A fair day is followed by a fair day 60% of the time and by a cloudy day 25% of time. A cloudy day is followed by a cloudy day 35% of time & by a rainy day 25% of time. A rainy day is followed by a cloudy day 40% of time & rainy day 25% of time. Find probability Initial probabilities 0.3, 0.3, 0.4. Find probability that it will be a rainy day after 3 days. What portion of the day is expected to be fair, cloudy or rainy in long run.

$$P = \begin{array}{c} \\ F \\ C \\ R \end{array} \begin{array}{c} \begin{array}{ccc} F & C & R \end{array} \\ \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix} \end{array}$$

$\because$ In P all entries are positive

P is a regular matrix

Let v be probability vector

$$VP = V$$

$$[v_1 \; v_2 \; v_3] \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix} = [v_1 \; v_2 \; v_3]$$

$$[0.6v_1 + 0.4v_2 + 0.35v_3 \quad 0.25v_1 + 0.35v_2 + 0.4v_3 \quad 0.15v_1 + 0.25v_2 + 0.25v_3] = [v_1 \; v_2 \; v_3]$$

$$0.6v_1 + 0.4v_2 + 0.35v_3 = v_1$$

$$0.25v_1 + 0.35v_2 + 0.4v_3 = v_2$$

$$0.15v_1 + 0.25v_2 + 0.25v_3 = v_3$$

$$0 = -0.4v_1 + 0.4v_2 + 0.35v_3 \quad — (1)$$

$$0 = 0.25v_1 - 0.65v_2 + 0.4v_3 \quad — (2) \qquad v_1 + v_2 + v_3 = 1 \quad — (4)$$

$$0 = 0.15v_1 + 0.25v_2 - 0.75v_3 \quad — (3)$$

(1) (2) (3) are linearly dependent

(1) (2) (4) solving

$$v_1 = \frac{155}{318} \qquad v_2 = \frac{33}{106} \qquad v_3 = \frac{32}{159}$$

$$[v_1 \; v_2 \; v_3] = V = \left[\frac{155}{318}, \frac{33}{106}, \frac{32}{159}\right]$$

$\therefore$ probability of having fair day on in long run is $\frac{155}{318}$

probability of having cloudy day in long run is $\frac{33}{106}$

probability of having rainy day in long run is $\frac{32}{159}$

$v^{(0)} = [0.3, 0.3, 0.4]$

$$\boxed{v^{(k+1)} = v^{(k)} P} \quad \text{--- (1)}$$

put $k=0$ in (1)

$$v^{1} = v^{(0)}P = [0.3, 0.3, 0.4]\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix}$$

$$= [0.44, 0.34, 0.22]$$

Transition probability after 1 day.

$$v^{(1)} = [0.44, 0.34, 0.22]$$

Transition probability after 2 days

$$v^{(2)} = v^{(1)} P = [0.44, 0.34, 0.22]\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix}$$

$$= [0.477, 0.317, 0.206]$$

Transition probability after 3 days

$$v^{(3)} = v^{(2)} P = [0.477, 0.317, 0.206]\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix}$$

$$= [\underset{\text{fair}}{0.4851}, \underset{\text{cloudy}}{0.3126}, \underset{\text{rainy}}{0.2023}]$$

$\therefore$ After 3 days the probability of having rainy $= 0.2023$

Limiting probability

Let sequence x_n such that $n \geq 0$ $\{x_n / n \geq 0\}$ be a markov chain whose all states are recurrent and aperiodic then limiting probability i.e

$$\lim_{n\to\infty} P_{ij}^{(n)} = V_j$$

1) If Transition probability matrix of 3 brands A, B & C is $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}$ and the initial market shares are 15%, 25%, 25%. market shares in

Find the 2nd & 3rd periods

Find limiting probabilities

$$P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}$$

$50\% = 0.5 \quad 25\% = 0.25 \quad 25\% = 0.25$

$\because$ All entries of P are positive

$\therefore$ P is regular — (1)

we know $V^{(k+1)} = V^{(k)} P$ — (2)

put $k=0$ in (2)

$$V^{(1)} = V^{(0)} P = [0.5 \quad 0.25 \quad 0.25] \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}$$

$$= [0.4875 \quad 0.2375 \quad 0.275]$$

$$V^{(2)} = V^{(1)} P = [0.4875 \quad 0.2375 \quad 0.275] \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}$$

$$= [0.48125 \quad 0.2387 \quad 0.28]$$

Market shares in 2^{nd} period are

48.75%, 23.75%, 27.5

$V^{(3)} = V^{(2)} P =$ Market shares in 3^{rd} period are 48.125%, 23.87, 28%.

Let $V = [v_1 \ v_2 \ v_3]$ be the long range trend vector

$\therefore VP = V$

$$[v_1 \quad v_2 \quad v_3] \times \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix} = [v_1 \quad v_2 \quad v_3]$$

$0.4v_1 + 0.8v_2 + 0.35v_3 = v_1$

$0.3v_1 + 0.1v_2 + 0.1v_3 = v_2$

$0.35v_1 + 0.25v_2 + 0.4v_3 = v_3$

$-0.6v_1 + 0.8v_2 + 0.35v_3 = 0$ — (1)

$0.3v_1 - 0.9v_2 + 0.1v_3 = 0$ — (2)

$0.35v_1 + 0.25v_2 - 0.6v_3 = 0$ — (3)

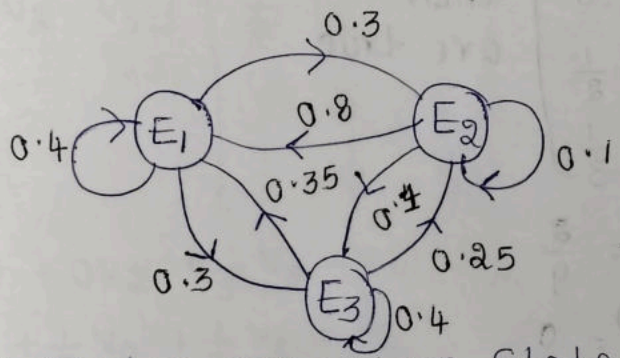
$0.4V_1 + 0.8V_2 + 0.35V_3 = V_1$
$0.3V_1 + 0.1V_2 + 0.25V_3 = V_2$
$0.3V_1 + 0.1V_2 + 0.4V_3 = V_3$

$-0.6V_1 + 0.8V_2 + 0.35V_3 = 0 \quad -(1)$
$0.3V_1 - 0.9V_2 + 0.25V_3 = 0 \quad -(2)$
$0.3V_1 + 0.1V_2 - 0.6V_3 = 0 \quad -(3)$
$V_1 + V_2 + V_3 = 1 \quad -(4)$

$(2) + (3) = -(1)$

(1) (2) (3) are linearly dependent
solving (2) (3) and (4)

$V_1 = \frac{103}{214}, \quad V_2 = \frac{51}{214}, \quad V_3 = \frac{30}{107}$



∵ All States are return State
∴ It is recurrent markov Chain
period of E_1 = Gcd {1, 2, 3 . . .} = 1
∴ E_1 State is aperiodic state
period of E_2 = Gcd {1, 2, 3 - - -} = 1
∴ E_2 State is aperiodic State
period of E_3 = Gcd {1, 2, 3 . . .} = 1
∴ E_3 State is aperiodic state
∴ Matrix/marchov Chain is recurrent & has aperiodic States

$$\lim_{n\to\infty} P_{ij}^{(n)} = V_j$$

$$\lim_{n\to\infty} P_{11}^{(n)} = V_1 = 0.4813$$

$$\lim_{n\to\infty} P_{12}^{(n)} = V_2 = 0.238$$

$$\lim_{n\to\infty} P_{13}^{(n)} = V_3 = 0.28$$

$\lim\limits_{n\to\infty} P_{21}^{(n)} = v_1 = 0.4813$

$\lim\limits_{n\to\infty} P_{22}^{(n)} = v_2 = 0.238$

$\lim\limits_{n\to\infty} P_{23}^{(n)} = v_3 = 0.28$

$\lim\limits_{n\to\infty} P_{31}^{(n)} = v_1 = 0.4813$

$\lim\limits_{n\to\infty} P_{32}^{(n)} = v_2 = 0.238$

$\lim\limits_{n\to\infty} P_{33}^{(n)} = v_3 = 0.28$

2) Consider a markov Chain with State Space $\{1,2,3\}$ and Transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

then which of the following are true

1) $\lim\limits_{n\to\infty} P_{11}^{(n)} = \frac{2}{9}$

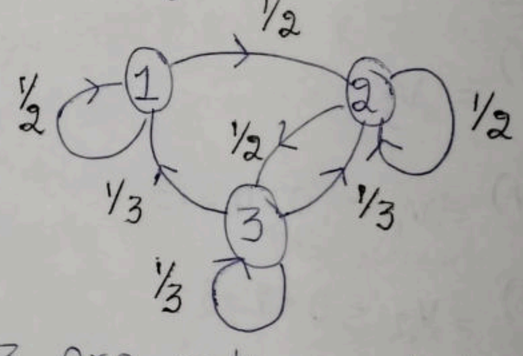
2) $\lim\limits_{n\to\infty} P_{21}^{(n)} = 0$

3) $\lim\limits_{n\to\infty} P_{32}^{(n)} = \frac{1}{3}$

4) $\lim\limits_{n\to\infty} P_{13}^{(n)} = \frac{1}{3}$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad P^2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{5}{12} & \frac{5}{12} \\ \frac{5}{18} & \frac{4}{9} & \frac{5}{18} \end{bmatrix}$$

$\therefore$ P is regular matrix



1, 2, 3 are return states

$\therefore$ It is recurrent markov Chain period of 1 = Gcd {1, 3 ...} = 1

∴ 1 is aperiodic state

period of 2 = Gcd {1, 2 ..} = 1

∴ 2 is aperiodic state

period of 3 = Gcd {1, 2 ...} = 1

∴ 3 is aperiodic state

∴ markov chain is recurrent & aperoidic

∴ P is regular & given markov chain is recurrent and each state is aperiodic state

let long rang trend vector be

$V = [v_1 \ v_2 \ v_3]$

∴ $VP = V$

$$[v_1 \ v_2 \ v_3] \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

$\frac{1}{2}v_1 + 0v_2 + \frac{1}{3}v_3 = v_1$

$\frac{1}{2}v_1 + \frac{1}{2}v_2 + \frac{1}{3}v_3 = v_2$

$(\frac{1}{3}v_1 + \frac{1}{3}v_2 + \frac{1}{3}v_3 = v_3)$ $\quad 0v_1 + \frac{1}{2}v_2 + \frac{1}{3}v_3 = v_3$

$-\frac{1}{2}v_1 + 0v_2 + \frac{1}{3}v_3 = 0$ — (1)

$\frac{1}{2}v_1 - \frac{1}{2}v_2 + \frac{1}{3}v_3 = 0$ — (2)

$0v_1 + \frac{1}{2}v_2 - \frac{2}{3}v_3 = 0$ — (3)

$v_1 + v_2 + v_3 = v$ — (4)

(1) (2) (3) are linearly dependent

Solving (1) (2) (4)

$v_1 = 0.222 = \frac{2}{9}$ $\quad v_2 = 0.444 = \frac{4}{9}$ $\quad v_3 = 0.333 = \frac{1}{3}$

1) $\lim_{n\to\infty} P_{11}^{(n)} = \frac{2}{9}$ (True)

2) $\lim_{n\to\infty} P_{21}^{(n)} = 0$ (False)

3) $\lim_{n\to\infty} P_{32}^{(n)} = \frac{1}{3}$ (False)

4) $\lim_{n\to\infty} P_{13}^{(n)} = \frac{1}{3}$ (True)

3) Consider a markov chain with state space $S = \{1, 2, 3\}$ and TPM $P =$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \end{array}$$

then which of following are true

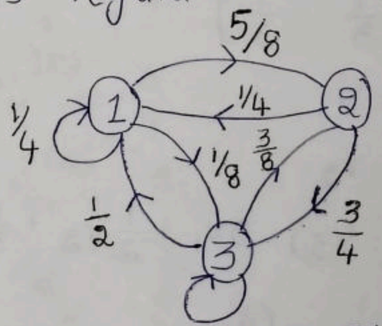
1) $\lim_{n\to\infty} P_{12}^{(n)} = 0$

2) $\lim_{n\to\infty} P_{12}^{(n)} = \lim_{n\to\infty} P_{21}^{(n)}$

3) $\lim_{n\to\infty} P_{22}^{(n)} = \frac{1}{8}$ 4) $\lim_{n\to\infty} P_{21}^{(n)} = \frac{1}{3}$

$$P = \begin{bmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \quad P^2 = \begin{bmatrix} \frac{9}{32} & \frac{13}{64} & \frac{33}{64} \\ \frac{7}{16} & \frac{7}{16} & \frac{1}{8} \\ \frac{9}{32} & \frac{23}{64} & \frac{23}{64} \end{bmatrix}$$

$\therefore$ P is regular matrix



$\therefore$ 1, 2, 3 are return states

$\therefore$ It is recurrent markov chain

period of 1 = Gcd{1, 2 . . .} = 1

$\therefore$ 1 is aperiodic state

period of 2 = Gcd{2, 3 . . .} = 1

$\therefore$ 2 is aperiodic state

period of 3 = Gcd{1, 2 . . .} = 1

$\therefore$ 3 is aperiodic state

$\therefore$ markov chain is aperiodic

$\therefore$ P is regular & given markov chain is recurrent & aperoidic

let long range trend vector

$V = [v_1 \ v_2 \ v_3]$

$V = VP$

$$[v_1 \quad v_2 \quad v_3] = \begin{bmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} [v_1 \quad v_2 \quad v_3]$$

$$\frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 = v_1$$

$$\frac{5}{8}v_1 + 0v_2 + \frac{3}{8}v_3 = v_2$$

$$\frac{1}{8}v_1 + \frac{3}{4}v_2 + \frac{1}{8}v_3 = v_3$$

$$-\frac{3}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 = 0 \quad -(1)$$

$$\frac{5}{8}v_1 - v_2 + \frac{3}{8}v_3 = 0 \quad -(2)$$

$$\frac{1}{8}v_1 + \frac{3}{4}v_2 - \frac{7}{8}v_3 = 0 \quad -(3)$$

$$v_1 + v_2 + v_3 = v \quad -(4)$$

(1) (2) (3) are linearly dependent

Solving (1) (2) & (4)

$v_1 = \frac{1}{3} \quad v_2 = \frac{1}{3} \quad v_3 = \frac{1}{3}$

1) $\lim_{n\to\infty} P_{12}^{(n)} = 0$ (False)

2) $\lim_{n\to\infty} P_{12}^{(n)} = \lim_{n\to\infty} P_{21}^{(n)}$ = True

3) $\lim_{n\to\infty} P_{22}^{(n)} = \frac{1}{8}$ (False) 4) $\lim_{n\to\infty} P_{21}^{(n)} = \frac{1}{3}$ (True)

Absorbing State

Absorbing state of an markov chain is an absorbing state i of an state if $P_{ii} = 1$

Absorbing markov chain

A markov chain is an absorbing markov chain if following 2 conditions are satisfied

1) The chain as atleast 1 absorbing state
2) It should be possible to go from any non absorbing state to an absorbing state

Note:
1) The 2nd condition does not mean that it is possible to go from any non-absorbing to any absorbing but it is possible to go to same absorbing state
2) In a markov chain, the state which is not absorbing is called transient

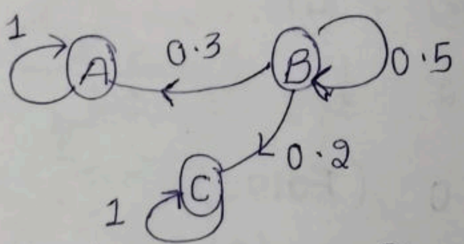
1) Identify all absorbing states in markov chain having following matrices. Decide whether markov chain is absorbing or not

a) $\begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

a) $P_{11} = 1$ & $P_{33} = 1$

$\therefore$ 1 & 3 states are absorbing states

$\therefore$ 2 is non absorbing state



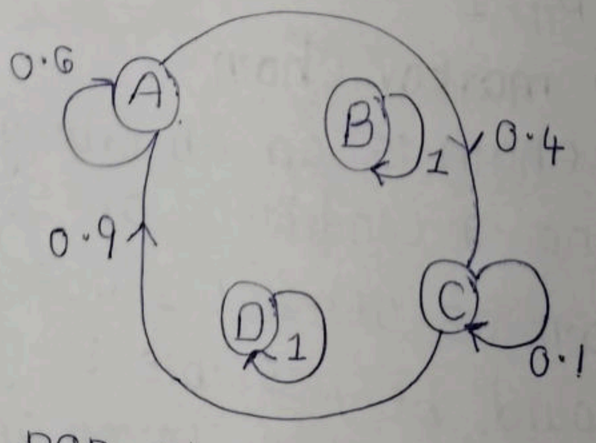
It is possible to go from non-absorbing state to absorbing state.

$\therefore$ It is absorbing markov chain

b) $P_{22} = 1$ & $P_{44} = 1$

$\therefore$ 2 & 4 are absorbing states

1 & 3 are non absorbing states



$\therefore$ It is non absorbing chain

1) markov ... will enter ...
2) The p... closer ...
& The l...
3) changing ... state
A gamb... at a ti... He stop...
a) Wha... related ...
b) Wha... his mo...
c) Wha... more t...

properties of absorbing Chains

1) Regardless of original State of an absorbing markov Chain in a finite no. of Steps. The Chain will enter an absorbing State & Stay in that State

2) The powers of transition matrix get Closer & Closer to Some particular matrix

3) The long range trend depends on initial State Changing the initial State Can Change the final State

1) A gambler has 2 rupees, he bets 1 rupee at a time & wins ₹1 with a probability of $\frac{1}{2}$. He stops playing if he looses ₹2 or wins ₹4

a) What is transition probability matrix of related markov chain

b) What is the probability that he has lost his money at the end of 5 place.

c) What is probability that the game last more than 7 place