CHAPTER - 10

STOCHASTIC PROCESSES AND MARKOV CHAINS

10.1 INTRODUCTION

In general, in the previous chapters, we have been concerned with the static aspect of statistical theory. Now we shall deal with the dynamic aspect of statistics or statistics of change. In some situations in science and technology we will be interested in studying "Processes", that is, phenomena that take place with change in time. The theory of probability studied did not have either general procedures or elaborate schemes for solving problems that arise in the study of such phenomena. Hence it is necessary to develop a general theory of **random processes** to study random variables dependent on one or several discretely or continuously varying parameters. This leads to a new concept of indeterminism in dynamic studies. This is referred to as "dynamic indeterminism". Many phenomena occuring in physical and life sciences are studied now not only as random phenomena but also as those changing with time or space.

10.2 DEFINITION

Mathematically, a **stochastic process** is a set of random variables $\{x_t\}$ or $\{x(t)\}$ depending on some real parameter like time t. These are also known as **random processes** or **random functions**.

We will now discuss some examples.

e.g. 1. A queueing system that we have studied, is a random process or stochastic process. The number of people joining in a queue during a time interval, the number of people served from the queue in a particular interval are examples of random variables or stochastic variables.

e.g. 2. Consider the experiment of throwing an unbiased die. Suppose that X_n is the outcome of n^{th} throw, $n \geq 1$. Then $\{X_n / n \geq 1\}$ is a family of random variables, such that for a distinct value of $n\,(=1, 2, 3, ...)$, we get a distinct random variable X_n. Thus $\{X_n / n \geq 1\}$ constitutes a stochastic process. This process is known as **"Bernoulli Process".**

e.g.3. In the same experiment, suppose that X_n is the number of sixes in the first n throws. For a distinct value of $n = 1, 2, ...$, we get a distinct binomial variate X_n. $\{X_n / n \geq 1\}$ which gives a family of random variables. Thus it is a stochastic process and X_n is a stochastic variable.

e.g.4. Again in the same experiment, consider X_n as the maximum number shown in the first n throws. We can see that $\{X_n / n \geq 1\}$ constitutes a stochastic process.

e.g. 5. Consider a random event occuring in time, like, number of telephone calls received at a switch board. Consider an interval $(0,t)$ of duration of t units. Suppose X_t is the random variable which represents the number of incoming calls in that interval. The number of calls within a fixed interval of specific duration is a random variable X_t and the family $\{X_t, t \in T\}, T \in (0, \alpha)$ constitutes a stochastic process.

e.g. 6. Turblent Fluid Flow : In the turblent fluid flow the velocity components u, v, w of the fluid are random variables depending on the space coordinates x, y, z and the time t.

e.g. 7. Movement of molecules of a gas or liquid : In a gas or liquid at random instants the molecule collides with other molecules. Thus its velocity and position are altered. Thus the state of the molecule is subjected to random changes at every instant of time.

e.g. 8. A random walk model : A particle moves in a straight line in steps of unit length. At each stage it can move one step to the right with the probability p or one step to the left with the probability q where $p+q=1$. If the particle starts from origin, its position after n movements is a random variable, which depends on discrete parameter n.

e.g. 9. Communication Process : The amplitude of the signals to be transmitted and amplitude of the noise produced in the channel depending on time are both random variables.

e.g. 10. Gamblers Ruin Problem (Classical Ruin Problem) : Suppose a Gambler wins or looses a unit (Rupee or dollor or pound) with probabilities p and q respectively. Let his initial capital be z and his opponents initial capital be $(a - z)$ so that the combined capital is a. The game continues until the gambler's capital is either reduced to zero or increased to a, that is, until one of the two players is ruined. We are interested in the probability of the gambler's ruin and the probability distribution of the duration of the game. We can notice that the capital of the gambler after n stages is a random stochastic variable.

10.3 RESULT : TO FIND THE PROBABILITY OF GAMBLER RUIN

Let q_z denote the probability of ultimate ruin of the gambler. After the first trial, the gambler's capital is $(z+1)$ if he wins (with probability p) or it is $(z-1)$ if he loses (with probability q).

The probabilities of his ruin, after one trial, will be q_{z+1} and q_{z-1} respectively

so that $q_z = pq_{z+1} + qq_{z-1}$, $1 < z < a-1$ ... (1)

for $z=1$, the first trial may lead to ruin and we may have

$q_1 = pq_2 + q$, $z=1$ ... (2)

Similarly for $z = a-1$, the first trial may result in victory, we may have

$q_{a-1} = qq_{a-2}$, $z = a-1$ ... (3)

To unify our equations we define

$q_0 = 1, q_a = 0$ ... (4)

All these three equations are included in

$$q_z = pq_{z+1} + qq_{z-1}\,,\ 1 \le z \le a-1 \qquad \dots (5)$$

Case I : (Biased case) Suppose $p \neq q$ (i.e.) $p \neq \frac{1}{2}$. Then (5) is a linear difference equation.

We rewrite the equation as $q_z = pq_{z+1} + (1-p)q_{z-1} \Rightarrow pq_{z+1} - qz + (1-p)q_{z-1} = 0$

The characteristic equation is $p\rho^2 - \rho + (1-p) = 0$

The roots of this equation are

$$\rho = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm \sqrt{1-4p+4p^2}}{2p} = \frac{1 \pm \sqrt{(1-2p)^2}}{2p} = \frac{1 \pm (1-2p)}{2p}$$

$$\Rightarrow \rho = \frac{1+(1-2p)}{2p} = \frac{1-p}{p} \text{ and } \rho = \frac{1-(1-2p)}{2p} = \frac{2p}{2p} = 1$$

$$\therefore \rho = \frac{1-p}{p} \text{ or } 1$$

$$\text{or } \rho = \frac{q}{p} \text{ or } \rho = 1$$

Its general solution is, $q_z = A + B(q/p)^z \qquad \dots (6)$

Using boundary conditions from (4)

$$q_0 = A + B = 1 \Rightarrow A = 1 - B$$

$$q_a = 0 = A + B\left(\frac{q}{p}\right)^a = 1 - B + B\left(\frac{q}{p}\right)^a = 1 + B\left[\left(\frac{q}{p}\right)^a - 1\right]$$

$$\therefore B = \frac{1}{1-\left(\frac{q}{p}\right)^a}$$

From this, we get $A = 1 - B = \dfrac{-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^a}$

From (6), $q_z = A + B\left(\dfrac{q}{p}\right)^z \qquad \dots (7)$

$$q_z = \frac{(q/p)^a - (q/p)^z}{(q/p)^a - 1} = \frac{(q/p)^z[(q/p)^{a-z} - 1]}{(q/p)^a - 1}$$

The probability p_z of the opponent's ruin can be obtained from this by writing $a-z$ for z and interchanging p and q so that

$$p_z = \frac{(p/q)^{a-z}[(p/q)^z - 1]}{(p/q)^a - 1} = \frac{1-(p/q)^{-z}}{1-(q/p)^a} = \frac{(q/p)^z - 1}{(q/p)^a - 1} \quad \dots (8)$$

From (8) & 9, $p_z + q_z = 1$ ... (9), so that the game has to terminate (and there is no draw).

Case II : (UnBiased case)

If $p = q = 1/2$, (5) becomes $2q_z = q_{z+1} + q_{z-1}$ ... (10)

Its characteristc equation is $\rho^2 - 2\rho + 1 = 0$ which give that $\rho = 1$ as a double root.

General solution is $q_z = \mathrm{A}z(1)^z + \mathrm{B}(1)^z = \mathrm{A}z + \mathrm{B}$

Using conditions in (4), we get $q_0 = 1 = \mathrm{B}, q_a = 0 = \mathrm{A}a + \mathrm{B} = \mathrm{A}a + 1$

$$\therefore \mathrm{A} = -\frac{1}{a}$$

Substituting we get $q_z = \frac{-1}{a}(z) + 1$

$$\Rightarrow q_z = 1 - z/a = \frac{a-z}{a} \text{ is a solution for (10)} \quad \dots (11)$$

In this case writing $(a-z)$ for z, we get $p_z = z/a$ ... (12)

so that $p_z + q_z = 1$ is true here.

We can reformulate our results as follows: "Let a gambler with an initial capital z play against an infinitely rich adversary who is always willing to play, although the gambler has the privilege of stopping at his pleasure. The gambler adopts the strategy of playing until he either loses his capital or increases it to a (with a net gain $(a-z)$). Then q_z is the probability of his losing, and $(1-q_z)$ the probability of his winning".

Note 1. When $p = q = \frac{1}{2}$, we have $q_z = \frac{a-z}{a}$ and $p_z = \frac{z}{a}$. Suppose the capital of the opponent is comparably greater than the capital of the player *i.e.* $a-z$ is infinitely greater than z which implies that a is infinitely greater than z, then $p_z \to 0$. Thus, we can say that ruin of the opponent is practically impossible, when the players are of equal skill and capital of opponent is comparitively greater than the capital of the player .

2. When $p \neq q$ and $p > q$ and $(a-z) \sim \infty$ (i.e. $a \sim \infty$), we find $q_z = \dfrac{\left(\dfrac{q}{p}\right)^z \left[\left(\dfrac{q}{p}\right)^{a-z} - 1\right]}{\left(\dfrac{q}{p}\right)^a - 1} \sim \left(\dfrac{a}{p}\right)^z$ and $p_z = 1 - \left(\dfrac{q}{p}\right)^z$

Thus "a skillful player with even a slight capital can haves less chance of ruin than a player with larger capital, but less skillful."

This problem can also be discussed with diverse variations like

(*a*) Stakes are doubled with no change in initial capilat.

(*b*) $p \neq q$ (biased)

(*c*) Gambler's ruin with draw

(*d*) Gambler's ruin with correlation.

All these are above the comprehension of a normal student at this level. Those who are keen to learn are advised to look into advanced texts on statistics.

10.4 RANDOM WALK MODELS

We have introduced earlier a random walk model in eg. 8 on page 488.

Suppose a particle moves in a straight line in steps of unit length. At each stage it can move one step to the right with probability p or one step to the left with the probability q, where $p+q=1$. It is natural that if the particle starts from origin its position after n movements in a random variable which depends on the discrete parameter n.

The Random walk models described above is called **symmetric** if $p = q = \frac{1}{2}$. If $p > \frac{1}{2}$ then we say that there is a **drift towards the right**, if $q > \frac{1}{2}$ we say that there is a **drift towards the left**. The walk is said to be **unrestricted** if the particle can go to any point on the line.

In two dimensional random walk the particle moves in unit steps in one of the four directions parallel to x - axis and y - axis.

For a particle starting at the origin the possible positions are all points of the plane with integral valued coordinates. Each position has four neighbours. Similarly in three dimensions each position has six neighbours. The random walk is defined by specifying the corresponding four or six probabilities. Here for simplicity we will consider only the symmetric case where all directions have the same probability.

10.5 SPECIFICATION OF STOCHASTIC PROCESSES

State space : The values assumed by the random variable are called **states**. The set of all possible values of an individual random variable X_n of a stochastic process $\{X_n, n \geq 1\}$ is known as its **state space**. It is denoted by I. The state space is said to be **discrete** if it contains a finite or countable infinity of points, otherwise it is called ***continuous***.

e.g. 11. Suppose a fair die is rolled. Let X_n denote the total number of sixes appearing in the first n throws of a die. Then the set of possible values of X_n is the finite set of non-negative integers $0,1,2,.....,n$. Here the state X_n is discrete.

e.g. 12. Consider $X_n = Z_1 + Z_2 + ... + Z_n$ where Z_i is a continuous random variable assuming values in $[0,\infty)$. Then the set of possible values of X_n is in the interval $[0,\infty)$. Then the state space of X_n is continuous.

In the above two examples, we have considered the parameter n of X_n as non-negative integer $(n \geq 0)$. We considered the state of system at distinct "*time*" points $n = 0,1,2,......$ only. Here the word *"time"* is used in a wider sense.

We can again think of a family of random variables $\{X_t, t \in T\}$ such that the state of system is known at every instant over a finite or infinite interval. Then the system is defined for a continuous range of times. We say that we have a family of random variables in continuous time. A Stochastic process in *continuous time* can have either a disrete or a continuous state space.

e.g. 13. Suppose that X_t gives the number of outgoing calls at a switch board in an interval $(0,t)$. Here the state space of X_t is discrete, though X_t is defined for a continuous range of time. Thus this is a process in continuous time having discrete state space.

e.g. 14. Suppose X_t represents the minimum temparature at a particular place in $(0,t)$, then the set of possible values of X_t is continuous. This is a system in continuous time having a continuous state space.

10.6 RELATIONSHIP

The relationship among the numbers of a family $\{X_n\}$ is of importance. The nature of the dependence varies. We will describe some Stochastic Process according to the nature of dependence relationship existing among the members of the family.

Definition : Dependent Stochastic Processes

In some cases the members of the family $\{X_n\}$ are mutually dependent. They are called **Dependent Stochastic Processes**. The Bernoulli's process discribed earlier is an example of Dependent Stochastic Processes.

Definition : Stochastic Process with Independent Increments

If for all $t_1, t_2, ..., t_n, t_1 < t_2 < ..... < t_n$, the random variables $X(t_2) - X(t_1), X(t_3) - X(t_2),$ ...., $X(t_n) - X(t_{n-1})$ are independent, then $\{X(t), t \in T\}$ is said to be a **Stochastic Process with Independent Increments.**

10.7 MULTI-DIMENSIONAL STOCHASTIC PROCESS

Until now we have assumed that the values assumed by the random variable X_n are one-dimensional. But the process $\{X_n, n \geq 1\}$ may be multi-dimensional.

e.g.9. Consider $X_{(t)} = \{X_{1(t)}, X_{2(t)}\}$ where X_1 represents maximum and X_2 represents minimum temparature at a place in the interval of time $(0,1)$. This is a two dimensional stochastic proess in continuous time having continuous state space. Similarly we can have multi-dimensional process.

10.8 CLASSIFICATION OF STOCHASTIC PROCESSES

From the above discussion it is clear that a Stochastic process is a function of sample points and time. The sample points may have discrete or continuous values. Similarly the experiments may be defined as discrete of continuous time intervals. Thus a Stochastic process may be classified into 4 types.

1. If both x and t are continuous the Stochastic process is called as "**Continuous Stochastic Process**".
2. If x is continuous and t is discrete, the stochastic process is called as a **Discrete stochastic** process.
3. If x is discrete and t is continuous, the stochastic process is called as a **Discrete Stochostic Process.**
4. If both x and t are discrete, then the Stochastic process is called a **Discrete Stochastic Process.**

We put these in tabular form

$x(t)$ \ t	Continuous	Discrete
Continuous	Continuous Stochastic Process	Discrete Stochastic Sequence
Discrete	Discrete Stochastic Process	Discrete Stochastic Sequence

We can classify Stochastic process in another way also. It can be classified as deterministic Stochastic Process and non-determininstic Stochastic process.

Def. A random process is called a **Deterministic Stochastic Process** if all the future values can be predicted from past observations.

A stochastic process is called **Non-Deterministic Stochastic Process** if future values of any sample function cannot be predicted from past observations.

Two random variables $X(t)$ and $Y(t)$ are **equal everywhere** if their respective sample spaces are identical for every t.

In case of continuous time both the symbols $\{X_t, t \in T\}$ or $\{X(t), t \in T\}$ where T is finite or infinite interval is used. The parameter t is usually interpretted as time, though it may represent distance, length, thickness and so on.

10.9 PROBABILISTIC STRUCTURE

We defined a random process in two different ways. One, as a collection of different time functions and another as a collection of different random variables defined at different time instants. We will study the probabilistic structure of a random variable using these two difinitions.

By second definition a random process becomes a random variable, when time is fixed. A random variable is characterised by a probability density function and it is possible to calculate different statistical properties like mean, variance and other moments.

10.10 PROBABILITY DISTRIBUTION AND DENSITY FUNCTION

To each stochastic variable we can define the Probability function as

$F_x(x_1 : t_1) = P[x(t_1) \le x_1]$ for any real number x_1.

This is called the first order distribution function of the random variable $x(t_1)$.

The **first order probability density function** $f_x(x_1;t_1)$ is defined as the derivative of first order probability distribution function.

$$f_x(x_1;t_1) = \frac{d}{dx_1} F_x(x_1 : t_1)$$

Similarly, we can define **second - order joint probability distribution function** as

$$F_x(x_1, x_2 : t_1, t_2) = P\{x(t_1) \le x_1, x(t_2) \le x_2\}$$

and the **second order joint probability density function** as

$$f_x(x_1, x_2 : t_1, t_2) = \frac{\partial^2}{\partial x_1\, \partial x_2}[F_x(x_1 x_2 : t_1 t_2)]$$

We can extend this idea to n random variables.

Thus we can define nth order joint probability distribution function as

$$F_x(x_1, x_2, \ldots\ldots\ldots, x_n : t_1, t_2, \ldots\ldots\ldots, t_n) = P\{x(t_1) \le x_1, x(t_2) \le x_2, \ldots\ldots, x(t_n) < x_n\}$$

n^{th} order probability function is defined as

$$f_x(x_1, x_2, \ldots\ldots, x_n : t_1, t_2, \ldots\ldots, t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots\ldots\ldots \partial x_n} F_x(x_1, x_2, \ldots\ldots, x_n : t_1, t_2, \ldots\ldots, t_n)$$

10.11 STATIONARITY

Stationarity of a random process explains the time invariance of certain properties. For a stationary process, the distribution function or certain expected values do not change with time.

Contradictarily, for a non-stationary random process, any of its density functions or probabilty functions or any of its movements depends on the precise value of time.

For stationary process, $f_x(x_1, t_1) = f_x(x_1, t_1 + \delta)$ should be true for any t, and a time shift of δ.

As $f_x(x_1;t_1)$ is independent of t_1, the mean of each random variable is same. Hence the ...of the process is a constant.

Thus, $E[x(t)] = \bar{X} = \text{constant}$

Def. A random process is said to be **stationary of order two** if for all t_1, t_2 and δ its ...nd order density functions satisfy the condition, $f_x(x_1, x_2 : t_1, t_2) = f_x(x_1, x_2 : t_{1+\delta}, t_{2+\delta})$

n^{th} order stationary :

A process is **stationary to order *n***, if for *n* random variables of the process considered ...es $t_1, t_2, ..., t_n$ their nth order joint density function is invariant with time origin shift.

$$f_x(x_1, x_2, ..., x_n : t_1, t_2, ..., t_n) = f(x_1, x_2, ..., x_n : t_1 + \delta, t_2 + \delta, ..., t_n + \delta)$$

Definition : Evolutionary process

A process which is not stationary is said to be **evolutionary.**

...12 TIME AVERAGES

Previously we defined statistical averages of the stochostic process, by viewing that as ...llection of random variables indexed in time. Now, we shall consider the stochastic process ...collection of time (sample) functions and define the time averages for the random process.

Def. The **time average of a stochastic process** is defined as

$$A[x(t)] = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

We can define time mean, variance for a sample function also. If $x(t)$ represents a ...ple function, then mean of the sample function (time function) is

$$\bar{x} = A[x(t)] = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

The time auto correlation of a sample function is defined as

$$R_{xx}(\tau) = A[x(t) \,.\, x(t+\tau)] = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, dt$$

The time averages like $\bar{x}$ of a sample function are constant. Similarly, the mean of other ...ple functions will each be constant. These constants again constitute a random variable. ...can define the statistical average of these random variables.

For stationary process, which is not time dependent. $E[\bar{x}] = \bar{X}$,

$E[R_{xx}(\tau)] = R_{xx}(\tau)$

If the process has zero variance then $\bar{x} = \bar{X}$ and $R_{xx}(\tau) = R_{XX}(\tau)$.

10.13 MARKOV PROCESS

A Stochastic process is said to be a "Markov Process" if given the value of $X(t)$, the value of $X(v), v > t$ does not depend on the values of $X(u)$ for $u < t$. **The future behaviour of the Markov process depends only on the present value and not on the past values.**

Def. A Stochastic process $X(t)$ is said to be Markovian if

$$P[X(t_{n+1}) \le x_{n+1} / X(t_n) = x_n, X(t_{n-1} = x_{n-1}, ..., X(t_0) = t_0]$$
$$= P[X(t_{n+1}) \le X(t_n)] = x_n] \text{ where } t_0 \le t_1 \le t_2 ...... \le t_n < t_{n+1}$$

$X_0, X_1, X_2, ..., X_n$ are called the states of the process.

10.14 MARKOV CHAIN

A sequence of states $[X_n]$ is a Markov chain if each X_n is a random variable and if

$$P[X_{n+1} = x_{n+1} / X_n = x_n, X_{n-1} = x_{n-1}, ...X_0 = x_0]$$
$$= P[X_n = n / X_n = x_n]$$

10.15 MARKOV CHAINS

Consider a system which can be in any one of a finite number of states $E_1, E_2, ..., E_n$. We also assume that the probability of the system being in a given state at the next trial depends only on its present state and not upon the states it may have been in earlier times. If at any time, the system is in a state E_i, the probability of it being in the state E_j at the next trail is p_{ij}. This process is known as **Markov chain** and this property of the process is called **Markov property**.

In this process the out come E_j is no longer associated with a fixed probability but to every pair (E_i, E_j) there corresponds a conditional probability p_{ij}; given that E_i has occured at some trail, the probability of E_j at the next trial is p_{ij}. In addition to the p_{ij} we must be given the probability a_j of the outcome E_j at the initial trial. For p_{ij} to have the meaning attributed to them, the probabilities of sample sequences corresponding to two, three or four trials must be defined by $P\{(E_i, E_j)\} = a_i p_{ij}$, $P\{(E_i, E_j, E_k)\} = a_i p_{ij} p_{jk}$,

$P\{(E_i, E_j, E_k, E_r)\} = a_i\, p_{ij}\, p_{jk}\, p_{kr}$, and generally

$$P\{(E_{i0}, E_{i1}, ..., E_{in})\} = a_{i0}\, p_{i0i1}\, p_{i1i2} .... p_{i_{n-2}i_{n-1}}\, p_{i_{n-1}}\, p_{i_n} \quad ... (1)$$

We now formally give the definition of a Markov chain.

10.16 DEFINITION

[JNTU (H) Sup. 2011 (Set No. 3)]

A sequence of trials with possible outcomes $E_1, E_2 ...$ is called a **Markov chain** if the probabilities of sample sequences are defined by (1) in terms of a probability distribution

...for E_i at the initial (or zero-th) trial and fixed conditional probabilities p_{ij} of ...given that E_i has occured at the preceding trial. The possible outcomes E_j are usually ...red to as possible states of the system. p_{ij} is called the probability of a transition from E_i ... E_j.

...nother Definition: The stochastic process $\{x_n, n = 0,1,2,.....\}$ is Called a Markov chain ...r $i, j, i_1, i_2, ...i_{n-1} \in \mathrm{N}$, $\mathrm{P}\{x_n = j / x_{n-1} = i, x_{n-2} = i_1, ...x_0 = i_{n-1}\}$

$$\mathrm{P}\{x_n = j / \mathrm{X}_{n-1} = i\} = p_{ij}$$

...never the first number is defined.

The outcomes E_j are called **states** of the Markov chain. If X_n has the outcome E_j *i.e.* ...$= j$, the process is said to be at the state E_j or simply at the state j at the n^{th} trial. To ...e j there is no longer a fixed probability $\mathrm{P}(\mathrm{X}_n = j)$ but to a pair of states (i, j) at the two ...cessive trials (say n^{th} and $(n+1)^{th}$ trials) there is conditional probability p_{ij}. This is the ...bability of transition from the state at the n^{th} trial to the state j at $(n+1)^{th}$ trial.

The transition probability may or may not be independent of n. If the transition probability ...ndependent of n, then Markov chain is said to be "***homogeneous***" (or to have ***stationary ...nsition probabilities***). If it is dependent on n, then the chain is said to be ***non-homogeneous.*** ...e transition probability p_{ij} referes to the states (i, j) at two successive trails. The transition ...one step and p_{ij} is called ***one-step or unit step transition probability.***

In more general case, we are concerned with the pair of states (i, j) at two non-successive ...als, say state i at n^{th} trial and state j at $(n+m)^{th}$ trial. The corresponding transition probability ...then called m-step transition probability and is denoted by $p_{ij}^{(m)}$

i.e. $p_{ij}^{m} = \mathrm{P}(\mathrm{X}_{n+m} = k / \mathrm{X}_n = j)$ (and this is beyond the scope of the present book)

...17 TRANSITION PROBABILITY MATRIX

The transition probabilities p_{ij} will be arranged in a matrix of transition probabilities

$$\mathrm{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots \\ p_{21} & p_{22} & p_{23} & \cdots \\ p_{31} & p_{32} & p_{33} & \cdots \\ \cdots & \cdots & \cdots & .. \\ \cdots & \cdots & \cdots & .. \end{bmatrix}$$

where the first subscript stands for row, the second for column. Clearly P is a square matrix with non-negative elements and unit row sums. This matrix is called **transition probability matrix (tpm)**.

The symbol p_{ij} will be used for the probability of transition from state i to state j in one generation.

Properties of transition probability Matrix

A transition probability matrix has several features.

1. It is a square matrix, since all possible states must be used both as rows and columns.

2. All entries are between 0 and 1, inclusive; this is because all the entries represent probabilities.

3. The sum of the entries in any row must be 1, since the numbers in the row give the probability of changing from the state at the left to one of the right.

If P^2 represents the matrix product P.P, then P^2 gives probabilities of a transition from one state to another in two repetitions of an experiment. In general P^k gives the probabilities of a transition from one state to another in k repetitions of an experiment.

Definition : A **Stochastic Matrix** is a square matrix with non-negative elements and unit row sums.

10.18 ORDER OF A MARKOV CHAIN

Definition: A Markov chain $\{X_n\}$ is said to be of order $s\ (=1,2,....)$, if for all n

$$P\,(X_n = k/X_{n-1} = j, X_{n-2} = j_1, ..., X_{n-s} = j_{s-1}\}$$

$$= P\,\{X_n = k/X_{n-1} = j, ..., X_{n-s} = j_{s-1}\}$$

A Markov chain $\{X_n\}$ is said to be of order one (or simply a Markov chain) if

$$P\{X_n = k/X_{n-1} = j, X_{n-2} = j_1, ...\}$$

$$= P\{X_n = k/X_{n-1} = j\} = p_{jk}$$

unless stated otherwise, we mean by Markov chain, A chain of order one.

A chain is said to be of order zero if $p_{ij} = p_j\ \forall i$. This implies the independence of X_n and X_{n-1}.

A Markov chain $\{X_n/n \geq 0\}$ with k states, when k is finite is said to be a ***Finite Markov Chain***. The transition matrix in this case is a square matrix with k rows and columns.

If the possible values of X_n are ......, $-2,-1,0,1,2,.....$ then the Markov chain is said to be **denumerably infinite.**

10.19 Theorem: If P and Q are stochastic matrices then product PQ is also a stochastic ...x. Thus P^n is a stochastic matrix for all positive integer values of n.

10.20 Definition: A stochastic matrix P is said to be **regular** if all the entries of some ...r P^m are positive.

10.21 Theorem: A stochastic matrix P is not regular if 1 occures in the principle main ...onal.

SOLVED EXAMPLES

Example 1 : Which of the following matrices are stochastic

(i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 \\ 1/3 & 1/4 \end{bmatrix}$ (iv) $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

(vi) $\begin{bmatrix} 0 & 2 \\ 1/4 & 1/4 \end{bmatrix}$ **[JNTU (H) May 2011 (Set No. 4)]**

Solution : (i) is not a square matrix.

$\therefore$ it is not stochastic.

(ii) The matrix is a square matrix with non-negative entries and sum of the elements in ...ch row is equal to 1.

$\therefore$ The matrix is stochastic.

(iii) The matrix is a square matrix but sum in each row is not equal to 1. So it is not ...chastic.

(iv) It is a stochastic matrix.

(v) The matrix is not stochastic, because it contains negative elements.

(vi) The matrix is square matrix but sum in each row is not equal to 1. It is not a ...chastic matrix.

Example 2 : Test the following matrices are stochastic or not.

(a) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{15}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **[JNTU (H) June 2012 (Set No.4)]**

Solution : (a) Given matrix is not a square matrix.

$\therefore$ It is not a stochastic matrix.

(b) The matrix is a square matrix with non-negative entries.

But sum of elements in each row is not equal to 1.

$\therefore$ The matrix is not stochastic.

(c) The given matrix is square matrix with non-negative entries and sum of the elements in each row is equal to 1. Thus it is a stochastic matrix.

Example 3 : Find the value of x, y, z if $\begin{bmatrix} 0 & x & 1/3 \\ 0 & 0 & y \\ 1/3 & 1/4 & z \end{bmatrix}$ is a transition probability matrix.

[JNTU(H) Dec. 2019 (R15)]

Solution : Sum of the elements in each row is equal to 1.

$$\Rightarrow x+\frac{1}{3}=1 \Rightarrow x=\frac{2}{3}$$

and $0+y=1 \Rightarrow y=-1$

$$\frac{1}{3}+\frac{1}{4}+z=1 \Rightarrow z=1-\frac{1}{3}-\frac{1}{4}=\frac{12-4-3}{12}=\frac{5}{12}$$

Example 4 : Let M_n denote the sequence of sample means from a random process X_n:

$$M_n=\frac{X_1+X_2+....+X_n}{n}$$

Is M_n a Markov process ?

Solution : $M_n=\frac{1}{n}\sum_{i=1}^{n} X_i=\frac{1}{n}[X_n+(n-1)M_{n-1}]$

$$=\frac{1}{n}X_n+\left(1-\frac{1}{n}\right)M_{n-1}$$

Clearly if M_{n-1} is given then M_n depends only on X_n and is independent of $M_{n-2}, M_{n-3}, ...$

Therefore, M_n is a Markov process.

Example 5 : Which of the Stochastic matrices are regular.

(*i*) $A=\begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$ (*ii*) $C=\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$ (*iii*) $A=\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$

(*iv*) $B=\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ (*v*) $B=\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **[JNTU (H) Dec. 2011 (Set No. 1)]**

Solution : (*i*) Not regular since 1 lies on the main diagonal

(*ii*) $C^2=C\cdot C=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$, $C^3=C\cdot C^2=\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$

$$C^4=C^3\cdot C=\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}, \quad C^5=\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 1/2 & 3/8 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Since all the entries of some power of C are positive, C is regular stochastic matrix.

(iii) $$A^2 = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix} = \begin{bmatrix} 0.5625 & 0.3125 & 0.125 \\ 0.3 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.2 \end{bmatrix}$$

Since all the entries in A^2 are positive, A is regular.

(iv) $$B^2 = B \cdot B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 3/8 & 3/8 & 1/4 \end{bmatrix}; \quad B^3 = B^2 \cdot B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 7/16 & 7/16 & 1/8 \end{bmatrix};$$

$$B^4 = B^3 \cdot B = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 15/32 & 7/32 & 1/16 \end{bmatrix}$$

Since entries b_{13}, b_{23} are zero, for all powers of B, B is not regular.

(v) We can caluculate $B^2 = \begin{bmatrix} 0.25 & 0 & 0.75 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$

$$B^3 = \begin{bmatrix} 0.125 & 0 & 0.875 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad B^4 = \begin{bmatrix} 0.0625 & 0 & 0.9375 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Further powers of B will still give the same zero entries, so no power of matrix B contains possible entries. Then B is not regular.

(or) B is not regular since 1 occurs in principal diagonal.

Note : If a transition matrix P has some zero entries and P^2 also contains zero entries, we may wonder how far shall we compute P^k to be certain that the matrix is not regular. The answer is that if zeros occur in the identical places in both P^k and P^{k+1} for any k, they will appear in those places for higher powers of P, so P is not regular.

Example 6 : Which of the following matrices are regular

(a) $\begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **[JNTU (H) May 2012 (Set No.2)]**

Solution : In all the matrices the principal diagonal contains 1. Thus they are not regular.

Example 7 : When there are only two possible states E_1 and E_2 the matrix of transition probabilities is necessarily of the form $P = \begin{bmatrix} 1-p & p \\ \alpha & 1-\alpha \end{bmatrix}$

Such a chain could be realised by the following conceptual experiment. A particle moves along the x-axis in such a way that its absolute speed remains constant but the direction of the motion can be reversed. The system is said to be in state E_1 if the particle moves in the positive direction and in state E_2 if the motion is to the left. Then p is the probability of a reversal when the particle moves to the right and α the probability of a reversal when it moves to the left.

Example 8 : Suppose that a coin with probability p for a head is tossed indefinitely. Let X_n, the outcome of the n^{th} trail be k, where $k(=0,1,2,....,n)$ denotes that there is a run of k successes (i.e.) the length of the uninterupted block of heads is k. $\{X_n\}$ constitutes a Markov chain, with unit step transition probabilities.

$$p_{jK} = P\{X_n = k/X_{n-1} = j\} = p, k = j+1$$

$$= q, k = 0$$

$$= 0, \text{ otherwise}$$

Then the transition matrix is given by

$$\text{States of } \downarrow X_{n-1} \quad \begin{array}{c c} & \begin{array}{cccccccc} 0 & 1 & 2 & .... & k & k+1 & ... & \end{array} \\ \begin{array}{c} 1 \\ 2 \\ . \\ k \\ . \end{array} & \begin{bmatrix} q & p & 0 & .... & 0 & 0 & .... \\ q & 0 & p & .... & 0 & 0 & .... \\ q & 0 & 0 & .... & 0 & 0 & .... \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \end{bmatrix} \end{array}$$

Example 9 : Let $\{X_n, n > 0\}$ be a Markov chain with three states $\{0,1,2\}$ and with transition matrix $\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$ and the initial distribution $P\{X_0 = i\} = 1/3, i = 0,1,2$ then

we have $P\{X_1 = 1/X_0 = 2\} = 3/4$; $P\{X_2 = 2/X_1 = 1\} = 1/4$

$$P\{X_2 = 2, X_1 = 1, X_0 = 2\} = P\{X_0 = 2/X_1 = 1\}.P\{X_1 = 1/X_0 = 2\} = \frac{1}{4}.\frac{3}{4} = \frac{3}{16}$$

$$P\{X_2=2, X_1=1, X_0=2\} = P\{X_2=2, X_1=1 / X_0=2\}.P\{X_0=2\} = \frac{3}{16}.\frac{1}{3} = \frac{1}{16}$$

$$P\{X_3=1, X_2=2, X_1=1, X_0=2\}$$

$$= P\{X_3=1 / X_2=2, X_1=1, X_0=2\}.P\{X_2=2, X_1=1, X_0=2\}$$

$$= P\{X_3=1 / X_2=2\}.\frac{1}{16} = \frac{3}{4}.\frac{1}{16} = \frac{3}{64}$$

Example 10 : Discuss about system S that is in the states A_1, A_2, A_3; transition from to state occurs in accordance with the scheme of a homogeneous Markov chain; the ition probabilities are given by the matrix

$$\pi_1 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Solution : We see that if the system was in the state A_1, then after a change of the state ne step it will remain in the same state with a probability of 1/2, and it will pass to state A_2 a probability of 1/6, and to state A_3 with a probability of 1/3. But if the system was in the A_2, then after the transition it can (with equal probability) find itself only in states A_1 d A_3; it cannot pass from state A_2 into A_2. The last row of the matrix shows us that m the state A_3 the system can pass to any one of the possible states with one and the same bability 1/3.

Example 11 : The three state Markov chain is given by the transition probability matrix

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

. Prove that the chain is irreducible. **[JNTU (H) June 2013 (Set No.3)]**

Solution : In this Markov chain, all the states communicate with each other.

Suppose we consider the states as 0, 1, 2.

It is possible to go from state 0 to state 1 with a probability of 1/2.

Again it is possible to go from state 1 to state 2 with a probability of 1/3.

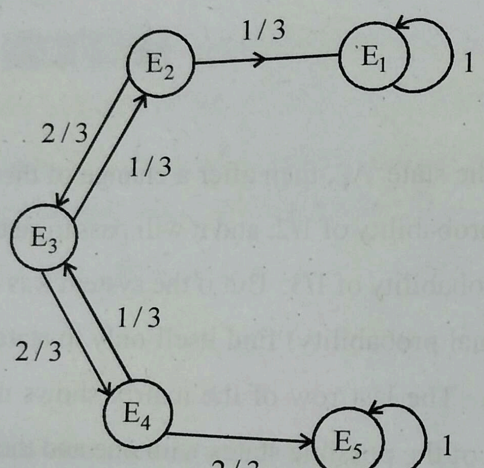
Thus it is possible to go from state 0 to state 2.

So, the chain is irreducible and all the states are recurrent.

Example 12 : Consider the transition probability matrix, find its graph.

$$\begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{array} \begin{array}{c} \begin{array}{ccccc} E_1 & E_2 & E_3 & E_4 & E_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solution : Its graph is shown below.

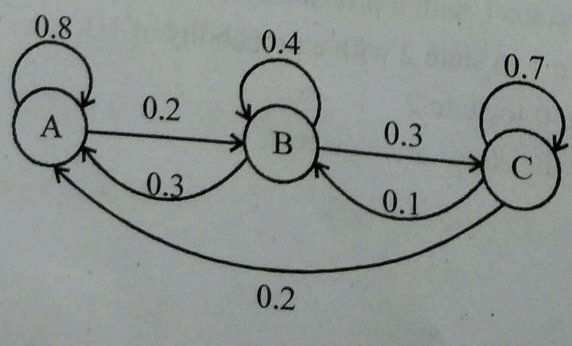


Example 13 : Three universities A, B, C are admitting students. It is given that 80 percent of the children of A went to A and the rest went to B. 40 percent of the children of B went to B and rest split evenly between A and C of the children of C seventy percent went to C and 20 percent went to A and 10 percent to B. Form the Markov chain and transition matrix.

Solution :

$$\text{Transition matrix is } P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix} \end{array}$$

Markov Chain is



HIGHER TRANSITION PROBABILITIES

...8 CHAPMAN - KOLMOGROV EQUATION

We have considered unit step or one-step transition probabilities, the probability of X_n ... that of X_{n-1}. We have considered the probability of the outcome at the n^{th} step or trial ... the outcome at the previous step, p_{ij} gives the probability of unit step transition from the ... i at a trial to the state j at the next trial.

Also m step transition probability is given by

$$P\{X_{m+n} = j/X_n = i\} = p_{ij}^{(m)} \quad \cdots (1)$$

$p_{ij}^{(m)}$ gives the probability that from the state i at the n^{th} trial, the state j is reached at ...$+n)$ trial in m steps.

The one step transition probabilities $p_{ij}^{(1)}$ are denoted simply by p_{ij}

Consider $p_{ij}^{(2)} = P\{X_{n+2} = j/X_n = i\}$ ...(2)

The state j can be reached from the state i in two steps through some intermediate stage r.

$$P\{X_{n+2} = j, X_{n+1} = r/X_n = i\}$$

$$= P\{X_{n+2} = j/X_{n+1} = r, X_n = i\} . P\{X_{n+1} = r/X_n = i\}$$

$$= p_{rj}^{(1)} \, p_{ir}^{(1)} = p_{ir} \, p_{rj}$$

Since these intermediate states $r, r = 1, 2, ...$ are mutually exclusive, we have

$$p_{ij}^{(2)} = P\{X_{n+2} = j/X_n = i\} = \sum_r P\{X_{n+2} = j, X_{n+1} = r/X_n = i\}$$

$$= \sum p_{ir} \, p_{rj} \quad \ldots (3) \quad \text{Summing over all intermediate states.}$$

By induction, we have

$$p_{ij}^{(m+1)} = P\{X_{n+m+1} = j/X_n = i\}$$

$$= \sum_r P\{X_{n+m+1} = j/X_{n+m} = r\} . P\{X_{n+m} = r/X_n = i\}$$

$$= \sum_r p_{rj} \, p_{ir}^{(m)}$$

Similarly we get

$$p_{ij}^{(m+1)} = \sum_r p_{ir} \, p_{rj}^{(m)}$$

In general we can have

$$p_{ij}^{(m+n)} = \sum_r p_{rj}^{(n)}\, p_{ir}^{(m)} = \sum_r p_{ir}^{(n)}\, p_{rj}^{(m)} \qquad \text{... (4)}$$

This equation is a special case of **Chapman - Kolmogrov equation**, which is satisfied by the transition probabilities of a Markov chain. (This can be seen in Advanced texts on statistics)

Ex. 1. Verify Chapman-Kolmgrov equation with the following example, write the transition probability matrix given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0 & 0.2 & 0 & 0.8 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution : We have
$$P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{bmatrix} 0.2 & 0 & 0.8 & 0 \\ 0 & 0.36 & 0 & 0.64 \\ 0.4 & 0 & 0.96 & 0 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \end{array}$$

In the matrix P^2, consider the value of $p_{11}^{(2)} = 0.36$ we are considering the probability of starting from state 1 and reaching the state 1. It is possible through any of the intermediate stages $k\ (k = 0,1,2,3)$.

State 1 to 1 can be reached is

$p_{10}, p_{01} \rightarrow$ here the intermediate state is 0

$p_{11}, p_{11} \rightarrow$ here the intermediate state is 1

$p_{12}, p_{21} \rightarrow$ here the intermediate state is 2

$p_{13}, p_{31} \rightarrow$ here the intermediate state is 3

$$\therefore\ p_{11}^{(2)} = p_{10}p_{01} + p_{11}\cdot p_{11} + p_{12}\cdot p_{21} + p_{12}\cdot p_{31}$$
$$= (0.2\times 1) + (0\times 0) + (0.8\times 0.2) + (0\times 0) = 0.36$$

Thus Chapman - Kolmogrov equations are satisfied.

...29 RESULTS IN TERMS OF TRANSITION MATRICES

Let $P = (p_{ij})$ denote the transition matrix of the unit step transitions and $P^{(m)} = \left(p_{ij}^{(m)}\right)$ ...ote the transition matrix of m - step transitions. We can see that elements of $P^{(2)}$ are the ...ents of the matrix P^2.

Thus $P^{(2)} = P.P = P^2$

Similarly $P^{(m+1)} = P^{(m)}.P = P.P^{(m)}$ and

$P^{(m+n)} = P^{(m)}.P^{(n)} = P^{(n)}.P^{(m)}$

SOLVED EXAMPLES

Example 1 : An urn initially contains five black balls and five white balls. The following ...periment is repeated indefinitely. A ball is drawn from the urn; if the ball is white it is put ...ck in the urn, otherwise it is left out. Let X_n be the number of black balls remaining in the ...n after n draws from the urn.

Is X_n a Markov process ? If so,

(a) Find the apropriate transition probabilities.

(b) Find the one - step transiton probability matrix P for X_n.

(c) Find the two-step transition probability matrix P^2 by matrix multiplication.

(d) What happens to X_n as n approaches infinity ? Use your answer to guess the limit ...P^n as $n \to \infty$.

Solution : The number X_n of black balls in the urn completely specifies the probability of ...e outcomes of a trial; therefore X_n is independent of its past values and X_n is a Markov ...ocess.

$$P[X_n = 4 \mid X_{n-1} = 5] = \frac{5}{10} = 1 - P[X_n = 5 \mid X_{n-1} = 5]$$

$$P[X_n = 3 \mid X_{n-1} = 4] = \frac{4}{9} = 1 - P[X_n = 4 \mid X_{n-1} = 4]$$

$$P[X_n = 2 \mid X_{n-1} = 3] = \frac{3}{8} = 1 - P[X_n = 3 \mid X_{n-1} = 3]$$

$$P[X_n = 1 \mid X_{n-1} = 2] = \frac{2}{7} = 1 - P[X_n = 2 \mid X_{n-1} = 2]$$

$$P[X_n = 0 \mid X_{n-1} = 1] = \frac{1}{6} = 1 - P[X_n = 1 \mid X_{n-1} = 1]$$

$$P[X_n = 0 \mid X_{n-1} = 0] = 1$$

All the transition probability are independent of time.

X_n

$$(b)\quad P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ X_{n-1}\ 4 \\ 5 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{5}{7} & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{10} & \frac{5}{10} \end{bmatrix}$$

(c) The two-step transition probability matrix P^2 by matrix multiplication is

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{11}{36} & \frac{25}{36} & 0 & 0 & 0 & 0 \\ \frac{1}{21} & \frac{65}{144} & \frac{25}{49} & 0 & 0 & 0 \\ 0 & \frac{3}{28} & \frac{225}{448} & \frac{25}{64} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{95}{192} & \frac{25}{81} & 0 \\ 0 & 0 & 0 & \frac{2}{9} & \frac{19}{36} & \frac{1}{4} \end{bmatrix}$$

(d) As $n \to \infty$ eventually all black balls are removed. Thus

$$P^n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2 : Peter takes the course Basic Stochastic Processes this quarter on Tuesday. Thursday, and Friday. The classes start at 10:00 am. Peter is used to work until late in the night and consequently, he sometimes misses the class. His attendance behaviour is such that he attends class depending only on whether or not he went to the latest class. If he attended class one day, then he will go to class next time it meets with probability 1/2. If he did not go to one class, then he will go to the next class with probability 3/4. Describe the Markov chain that models Peter's attendance. What is the probability that he will attend class on Thursday, if he went to class on Friday ?

Solution : Let $X_n = 0$ if Peter goes to the $n-th$ class meeting and $X_n = 1$ if he skips it. The process $\{X_n, n \geq 1\}$ is a Markov Chain with state space $I = \{0,1\}$ and one-step transition probability matrix $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$

p {Peter will attend on Thursday | he went on Friday} $p_{11}^{(2)} = \frac{5}{8}$ where $p_{11}^{(2)}$ is taken from the two - step transition matrix

$$P^2 = \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \\ \frac{9}{16} & \frac{7}{16} \end{bmatrix}$$

Example 3 : The alumni office of a college finds, on review, that 80% of its alumni who contribute to the annual fund one year will also contribute next year and 30% of those who do not contribute one year will contribute next. Write the transition matrix.

Solution : Consider the state corresponding to alumnus giving a donation in any one year as state 1 and state corresponding alumnus not giving a donation in that year as state 2.

The transition probability matrix is given by
$$P = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.8 & 0.2 \\ 2 & 0.3 & 0.7 \end{array}$$

Example 4 : A raining process is considered as a two state Markov chain. If it rains, it is considered to be in state 0 and it does not rain, the chain is in the sate of 1. The transition probability of the Markov chain is defined by $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$. Find the probability that it will rain for 3 days from today assuming that it is raining today. Assume that the mutual probabilities of state 0 or state 1 as 0.4 and 0.6 respectively. **[JNTU (H) Apr., June 2012 (Set No. 1)]**

Solution : The 1 step transition probability of the matrix is given by $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$

$$p(2) = P^2 = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}$$

$$p(3) = P^3 = \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$$

The probability that it will rain on third day, given that it will rain today is 0.376.

Example 5 : Consider Markov chain

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 3/4 & 1/4 & 0 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 0 & 3/4 & 1/4 \end{array}$$

. Find $P_{01}^{(2)}$ and $P(X_2 = 1, X = 0)$

Solution : The two step transition matrix is given by

$$P^2 = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 5/8 & 5/16 & 1/16 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{bmatrix} \end{array}$$

Hence $P_{01}^{(2)} = P\{X_{n+2} = 1/X_n = 0\} = \dfrac{5}{16}$ for $n \geq 0$

Thus $P\{X_2 = 1/X_n = 0\} = 5/6$ and

$$P\{X_2 = 1, X_0 = 0\}$$

$$= P(X_2 = 1, X = 0\} = P\{X_2 = 1/X_0 = 0\}.P\{X_0 = 0\}$$

$$= \frac{5}{16}.\frac{1}{3} = \frac{5}{48}$$

Example 6 : Suppose that the probability of a dry day (state 0) follows a rain day (state 1) is 1/3, and probability of a rain day follows a rain day is 1/2. Then we have a two state Markov chain such that $p_{10} = 1/3$ are $p_{01} = 1/2$ and transition probability matrix (tpm) is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

Solution : From this $P^2 = \begin{bmatrix} 5/12 & 7/12 \\ 7/18 & 11/18 \end{bmatrix}$ and $P^4 = \begin{bmatrix} 173/432 & 259/432 \\ 259/432 & 389/432 \end{bmatrix}$

From this we can conclude that if March 1st is a dry day then the probability that March 3rd is a dry day is 5/12 are March 5 is a dry day is 173/432.

Example 7 : The transition probability matrix of a Markov chain $\{X_n\}; n = 1, 2, 3, ...$ having three states 1,2 and 3 is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $P^{(0)} = (0.7, 0.2, 0.1)$. **[JNTU (H) Dec. 2013, (A) Nov. 2015, Nov. 2019]**

Find (*i*) $P\{X_2 = 3\}$, (*ii*) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$

Solution : We have $P(X_0 = 1) = 0.7; P(X_0 = 2) = 0.2, P(X_0 = 3) = 0.1$

$$\text{Also, } p^{(2)} = P^2 = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & \quad 2 & \quad 3 \end{array} \\ \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} \end{array}$$

(i) $P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3/X_0 = i\} \times P\{X_0 = i\}$

$$= p_{13}^{(2)} . P\{(X_0 = 1) + p_{23}^{(2)} . P(X_0 = 2) + p_{33}^{(2)} P(X_0 = 3)\}$$

$$= 0.26 \times 0.7 + 0.34 + 0.2 + 0.29 \times 0.1$$

$$= 0.182 + 0.068 + 0.029 = 0.279$$

(ii) $P\{X_1 = 3/X_0 = 2\} = P_{23} = 0.2$ ... (1)

$$P\{X_1 = 3, X_0 = 2\} = P\{X_1 = 3/X_0 = 2\} \times P\{X_0 = 2\}$$

$$= 0.2 \times 0.2 = 0.04 \text{ using (1)} \quad \text{... (2)}$$

$$P(X_2 = 3, X_1 = 3, X_0 = 2\} = P\{X_2 = 3/X_1 = 3, X_0 = 2\} \times P\{X_1 = 3, X_0 = 2\}$$

$$= P\{X_2 = 3/X_1 = 3\} \times P\{X_1 = 3, X_0 = 2\} \text{ by Markov property}$$

$$= 0.3 \times 0.04 \text{ by equation (2)}$$

$$= 0.012$$

$$P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2/X_2 = 3, X_1 = 3, X_0 = 2\} \times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2/X_2 = 3\} \times P\{X_2 = 3, X_1 = 3, X_0 = 2\} \text{ by Markov property}$$

$$= 0.4 \times 0.012 \text{ by equation (3)}$$

$$= 0.0048$$

Example 8 : A fair die is tossed repeatedly. If X_n denotes the maximum of the numbers occuring in the first n tosses, find the transition probability matrix P of the Markov chain $\{X_n\}$. Find also P^2 and $P(X_2 = 6)$. **[JNTU (H) Sup May 2011 (Set No. 1), Dec. 2019]**

Solution : State space $= \{1, 2, 3, 4, 5, 6\}$

Let $X_n =$ the maximum of the numbers occuring in the first n trails = 3, say

Then $X_{n+1} = 3$, if the $(n+1)^{th}$ trail results is 1, 2 or 3

$= 4$, if the $(n+1)^{th}$ trail results is 4

$= 5$, if the $(n+1)^{th}$ trail results is 5

$= 6$, if the $(n+1)^{th}$ trail results is 6

$$\therefore \ P\{X_{n+1} = 3/X_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$$

$$P\{X_{n+1} = i/X_n = 3\} = \frac{1}{6}, \text{ when } i = 4,5,6.$$

$\therefore$ The transition probability matrix of the chain is

$$P = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^2 = \frac{1}{36} \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

Initial state probability distribution is

$$P^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right) \text{ (Since all the values 1,2, ..., 6 equally likely)}$$

$$P\{x_2 = 6\} = \sum_{i=1}^{6} P\{X_2 = 6/X_0 = i\} \times P\{X_0 = i\}$$

$$= \frac{1}{6} \times \sum_{i=1}^{6} P_{i6}^{(2)}$$

$$= \frac{1}{6} \times \frac{1}{36} \times (11+11+11+11+11+36) = \frac{91}{216}$$

Example 9 : Consider a communication system which transmits the two digits 0 and 1 ough several stages. Let $\{X_n, n \geq 1\}$ be the digit leaving the nth stage of the system and X_0 e the digit entering the first stage (leaving the 0th stage). At each stage there is a constant obability q that the digit which enters will be transmitted unchanged (*i.e.* the digit will remain changed when it leaves), and probability p otherwise (*i.e.* the digit changes when it leaves), $p+q=1$.

Solution : Here $\{X_n, n \geq 0\}$ is a homogeneous two - state Markov chain with unit - step nsition matrix.

$$P = \begin{pmatrix} q & p \\ p & q \end{pmatrix}$$

We can prove by Mathematical Induction that

$$P^m = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(q-p)^m & \frac{1}{2}-\frac{1}{2}(q-p)^m \\ \frac{1}{2}-\frac{1}{2}(q-p)^m & \frac{1}{2}+\frac{1}{2}(q-p)^m \end{pmatrix}$$

Here $p_{00}^{(m)} = p_{11}^{(m)} = \frac{1}{2}+\frac{1}{2}(q-p)^m$ and

$p_{01}^{(m)} = p_{10}^{(m)} = \frac{1}{2}-\frac{1}{2}(q-p)^m$ and

as $m \to \infty$ $\text{Lt } p_{00}^{(m)} = \text{Lt } p_{01}^{(m)} = \text{Lt } p_{10}^{(m)} = \text{Lt } p_{11}^{(m)} \to \frac{1}{2}$

Suppose that the initial distribution is given by

$$P\{X_n = 0\} = a \text{ and } P\{X_0 = 1\} = b = 1-a$$

then we have $P\{X_m = 0, X_0 = 0\} = P\{X_m = 0/X_0 = 0\}P\{X_0 = 0\} = a\, p_{00}^{(m)}$

and $P\{X_m = 0, X_0 = 1\} = bp_{10}^{(m)}$

The probability that the digit entering the first stage is 0 given that the digit leaving the th stage is 0 can be evaluated by applying Baye's rule.

We have

$P\{X_0 = 0/X_m = 0\}$

$$= \frac{P\{X_m = 0/X_0 = 0\}P\{X_0 = 0\}}{P\{X_m = 0/X_0 = 0\}P\{X_0 = 0\} + P\{X_m = 0/X_0 = 1\}P\{X_0 = 1\}}$$

$$= \frac{a\, p_{00}^{(m)}}{a\, p_{00}^{(m)} + b\, p_{10}^{(m)}}$$

$$= \frac{a\left\{\frac{1}{2}+\frac{1}{2}(q-p)^m\right\}}{a\left\{\frac{1}{2}+\frac{1}{2}(q-p)^m\right\}+b\left\{\frac{1}{2}-\frac{1}{2}(q-p)^m\right\}}$$

$$= \frac{a\{1+(q-p)^m\}}{1+(a-b)(q-p)^m}$$

10.30 CLASSIFICATION OF STATES AND CHAINS

[JNTU (H) June 2012 (Set No.3)]

The states of a Markov chain $\{X_n, n \geq 0\}$ can be classified in a distinctive manner according to some fundamental properties of the system.

1. If the probability $P_{ij}^{(n)}$ is non-zero for some $n \geq 1$, then we say that the state j can be reached from the state i. We say that state j is **accessible** from i.
2. Two accessible states i and j are said to **communicate.** It is clear that i communicates with itself for all $i \geq 0$. If the state i communicates with state j and state j communicates with state k, then the state i communicates with state k. Two states that communicate are in the same class. A state is called an essential state if it communicates with every state it leads to.
3. If every state can be reached from any state (in any number of transitions) then the chain is said to be ***irreducible.*** Then the transition matrix is ***irreducible.*** Otherwise, the chain is said to be ***reducable*** or ***non-reducible.*** **[JNTU (H) May 2013]**
4. A state i is said to be an **absorbing state** if and only if $p_{ii} = 1$. A Markov chain is absorbing if it has at least one absorbing state and it is possible to go from every non-absorbing state to atleast one absorbing state in one or more steps.

   [JNTU (H) Nov. 2015]
5. A state i of a Markov chain is called a **return state** if $p_{ii}^{(n)} > 0$ for some $n \geq 1$.
6. A state is said to be ***periodic*** with period $t\,(>1)$ if the return to the state is possible only in $t, 2t, 3t, \ldots\ldots$ steps, where t is the greatest integer with this property. In this case $p_{ii}^{(n)} = 0$, unless n is an integral multiple of t.

   The state i is said to be **aperiodic** (or non-periodic) of no such $t\,(>1)$ exists.

Alternative Definition

7. The period of a return state is defined as the greatest common divisor (GCD) of all m such that $p_{ii}^{(m)} > 0$ i.e., $d_i = \text{GCD}\,\{m;\, p_{ii}^{(m)} > 0\}$

   A state i is said to be periodic with period d_i, if $d_i > 1$ and aperiodic if $d_i = 1$.

8. The probability that the chain starting from state i returns to state i, for the first time at the n^{th} step is denoted by $f_{ii}^{(n)}, n = 1,2,3,.....$ and is called as **first time return time probability** or **the recurrence time probability.**

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$ the return to state i is certain, and the state i is said to be ***persistent***

recurrent. If $F_{ii} < 1$, (the return to the state i is uncertain) the state i is said to be ***transient***.

state i communicates with state j and if state j is recurrent, then state i is also recurrent .

e parameter $\mu_{ii} = \sum_{n=1}^{\infty} n\, f_{ii}^{(n)}$ is called the ***mean recurrence time*** of the state i. If the mean

currence time μ_{ii} is finite, the state i is said to be ***non-null persistent*** or ***positive persistent***

d if $\mu_{ii} = \infty$, it is ***null-perisistent.***

9. A positive recurrent (or positive persistent) and aperiodic state is called ***ergodic***. A Markov chain all of whose states are ergodic is set to be a ***ergodic chain***. **[JNTU (H) May 2013]**

10. A tpm, P is said to be a **regular matrix** if all entries of $P^m (m = 2,3,....)$ are non-zero positive values. A homogenous Markov chain is said to be **regular chain** if its tpm is a regular matrix. **[JNTU (H) Nov. 2015]**

A tpm P is said to be a **stochastic matrix** if the elements of each of the rows are non-negative and the sum of elements in each rows is equal to 1.

11. A homogeneous Markov chain will have a tpm that is independent of initial state i and steps n as $n \to \infty$ and is called **steady state probability,** *i.e.* $q_j = \underset{n\to\infty}{Lt}\, P_{ij}^{(n)}$ and $\sum_{\forall j} q_j = 1$ where q_j is called limiting state probability and is interpreted as the **long-run proportion** of time the Markov chains spends in state j.

.31 MARKOV ANALYSIS

The Markov analysis is a method used to forecast the value of a variable whose predicted ue is influenced only by its current state, not by any prior activity. In essence, it predicts a dom variable based solely upon the current circumstances surrounding the variable.

The technique is named after Russian mathematician Andrei Andreyevich Markov, who eered the study of stochastic processes, which are processes that involve the operation of nce. He first used this method to predict the movements of gas particles trapped in a tainer. Markov analysis is often used for predicting behaviors and decisions within large ups of people.

Understanding Markov Analysis : The Markov analysis process involves defining the elihood of a future action given the current state of a variable. Once the probabilities of re actions at each state are determined, a decision tree can be drawn. Then, then the elihood of a result can be calculated, given the current state of a variable. Markov analysis several applications in the business world. It is often used to predict the number of defective pieces that will come off an assembly line, given the operating status of the machines on the line.

Advantages of Markov Analysis : The primary benefits of Markov analysis are simplicity and out of sample forecasting accuracy. Simple models, such as those used for Markov analysis, are often better at making predictions than more complicated models. This result is well-known in econometrics.

Disadvantages of Markov Analysis : Markov analysis is not very useful for explaining events, and it cannot be the true model of the underlying situation in most cases. Yes, it is relatively easy to estimate conditional probabilities based on the current state. However, that often tells one little about why something happened.

SOLVED EXAMPLES

Example 1 : Three boys A, B and C are throwing a ball to each other. A always throws the ball to B and B always throws the ball to C; but C is just as likely to throw the ball to B as to A. Show that the process is Markovian. Find the transition matrix and classify the states. Do all the states are ergodic ? **[JNTU (H) Nov. 2010, Dec. 2011 (Set No. 1), Dec. 2019]**

Solution : The transition probability matrix of the process $\{X_n\}$ is given as,

$$\begin{array}{c} \rightarrow \\ \text{States of } X_n \end{array}$$

$$\downarrow \begin{array}{c}\text{States of } X_{n-1} \\ P\end{array} = \begin{array}{c} \\ A \\ B \\ C \end{array}\!\!\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

States of X_n depend only on states of X_{n-1} but not on states of $X_{n-2}, X_{n-3}, \ldots$ or earlier states.

$\therefore$ $\{X_n\}$ is a Markov chain.

$$\text{Now } P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}, P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$P_{11}^{(3)} > 0, P_{13}^{(2)} > 0, P_{21}^{(2)} > 0, P_{22}^{(2)} > 0, P_{33}^{(2)} > 0$ and all other $P_{ij}^{(1)} > 0$

$\therefore$ The chain is irreducible. We can see that

$$P^4 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}; P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}; P^6 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 3/8 & 1/2 \\ 1/8 & 3/8 & 3/8 \end{bmatrix} \text{ and so on.}$$

We note that $p_{ii}^{(2)}, p_{ii}^{(3)}, p_{ii}^{(5)}, p_{ii}^{(6)}$ etc., are > 0 for $i = 2,3$

G. C. D. of $2,3,5,6,.....= 1$.

$\therefore$ The states 2,3 (*i.e.* B and C) are periodic with period 1 *i.e.* aperiodic.

We note that $p_{11}^{(3)}, p_{11}^{(5)}, p_{11}^{(6)}$ etc.. are > 0 and G. C. D. of $3,5,6,...... = 1$

$\therefore$ The state 1 (*i.e.* state A) is periodic with period 1, *i.e.* a periodic.

Since the chain is finite and irreducible, all its states are non-null persistant.

Moreover all the states are **ergodic.**

Example 2 : Find the nature of states of the Markov chain with tpm.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Solution : $$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P^3 = P^2 . P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = P$$

$P^4 = P^3 . P = P^2$

Similarly $P^6 = P^4 . P^2 = P^2 . P^2 . P^2 = P^2$

$$P^5 = P^3 . P^2 = P . P^2 = P$$

In general, $P^{2n} = P^2$ and $P^{2n+1} = P^{2n} . P = P^3 = P$

Also, we observe $\quad p_{00}^{(2)} > 0, p_{02}^{(2)} > 0, p_{11}^{(2)} > 0, p_{20}^{(2)} > 0, p_{22}^{(2)} > 0$

$$p_{01}^{(1)} > 0, p_{10}^{(1)} > 0, p_{12}^{(1)} > 0, p_{21}^{(1)} > 0$$

Hence, the Markov chain is irreducible. Also $p_{ii}^{(2)} > 0, p_{ii}^{(4)} > 0, p_{ii}^{(6)} > 0$ and so on for ry i. Thus $p_{ii}^{(2n)} > 0, p_{ii}^{(2n+1)} = 0$ for each i. Therefore, all the states of the Markov chain periodic with period 2.

Since the Markov chain is finite and irreducible, all its states are non-null persistent. All s are not **ergodic.**

Example 3 : Check whether the following Markov chain is regular and ergodic.

$$(i)\ P=\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (ii)\ \begin{bmatrix} 1 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$ **[JNTU (H) Dec. 2011 (Set No.3)]**

Solution : We check regularity of $P(n)$ matrices for different n. We observe that, when n is odd $P(n)$ is the original matrix. If n is even

$$p(n)=\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$\therefore$ For any value of n, we get matrices with zero only.

$\therefore$ The chain is not regular.

It is possible to go from state 0 to state 1 or state 2 or from state 1 to state 0 or state 3 and from state 2 to state 0 or state 1.

$\therefore$ All the states are ergodic.

$$(ii)\ P^2=\begin{bmatrix} 3/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/4 & 1/4 & 3/4 \end{bmatrix};\quad P^3=\begin{bmatrix} 2 & 1 & 1 & 3/4 \\ 1 & 3/8 & 3/8 & 3/4 \\ 1 & 3/8 & 3/8 & 3/4 \\ 3/4 & 1/2 & 1/2 & 5/8 \end{bmatrix}$$

P^2, P^3 are matrices without zero elements.

We can prove that $P^m (m = 2,3,....)$ has all the entries with possible non-zero values.

$\therefore$ P is a regular matrix.

Example 4 : The transition probability matrix of a Markov chain is given by

$$\begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.1 & 0.4 & 0.5 \\ 0 & 0.2 & 0.8 \end{bmatrix}.$$ Is this matrix irreducible ?

[JNTU (H) Dec. 2011 (Set No. 3), Nov. 2015, Dec. 2019]

Solution : Consider the three states as 0, 1, 2.

In this chain we go from state 0 to state 1 with a probability of 0.7 and from state 1 to 2 with a probability 0.5.

Thus it is possible to go from state 0 to state 2.

$\therefore$ The chain is irreducible and all the states are recurrent.

Example 5 : Consider the Markov chain with transition probabilty matrix

$$\begin{bmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ Is this matrix irreduciable. **[JNTU (H) Apr. 2012 (Set No. 4)]**

Solution : All the states do not communicate with each other. The chain is not irreducible ergodic.

We observe that from state 3 no other state is accessible and $p_{33} = 1$.

$\therefore$ The state 3 is an absorbing state.

Example 6 : Find periodic and aperiodic states in each of the following transition probability matrices.

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Solution : (*a*) Here starting in state 0 (or 1); we return back to state 0 (or 1) after an even number of transitions. So states are with period 2.

(*b*) Both the states are aperiodic as period of each state is 1. Starting in state 0(or 1), it has a probability to return to the same state in one transition.

3.32 STEADY STATE CONDITION

Stable Probability : In many Markov chains, the probability for a particular state will approach a limiting value as time goes to infinity. In other words, in the far future, the probability won't be changing much from one transition to the next transition. These limiting values are called the stable probabilitites.

Steady State Condition : If a system is such that each state has probability equal to its stable probability, the probability will persist for all the time. Then the system is said to be in steady state condition.

3.33 STEADY STATE VECTOR (OR) EQUILIBRIUM VECTOR OF A MARKOV CHAIN

If a Markov chain with transition matrix P is regular, then there is a unique vector V such that, for any probability vector V and for large values of n, $V.P^n \simeq V$. Vector V is called the **equilibrium vector** or the **fixed vector** or **steady state vector** of the Markov chain. This is also called **long range trend of the markov Chain**.

Probability Vector : A probability vector is a matrix of only one row, having non-negative entries, with the sum of the entries equal to 1.

Result. If a Markov chain with transition matrix P is regular, then there exists a probability vector V such that

$$VP = V$$

Sol. By the definition of equilibrium vector, we have

$$V.P^n \simeq V$$

Multiplying with P on both sides,

$$V.P^nP \simeq VP$$

$$\Rightarrow \quad V.P^{n+1} \simeq VP \quad \ldots (1)$$

Since $V.P^n \simeq V$ for large volume of n, it is also true that $V.P^{n+1} \simeq V$ for large values of n.

Thus we have $V.P^{n+1} \simeq V \quad \ldots (2)$

From (1) and (2), we get $VP = V$

This vector V gives the **long-range trend** of the Markov chain.

SOLVED EXAMPLES

Example 1 : Find the long range trend or steady state vector for the Markov chain with transition matrix $\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$

<u>**Solution :**</u> All the entries in the matrix are positive. So, this is a regular matrix.

$$\text{Let } P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$$

Let V the probability vector $[v_1 \; v_2 \; v_3]$. We want to find V such that $VP = V$

$$\Rightarrow [v_1 \;\; v_2 \;\; v_3]P = [v_1 \;\; v_2 \;\; v_3]$$

$$\Rightarrow [v_1 \;\; v_2 \;\; v_3]\begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix} = [v_1 \;\; v_2 \;\; v_3]$$

$$\Rightarrow [0.65v_1 + 0.15v_2 + 0.12v_3 \quad 0.28v_1 + 0.67v_2 + 0.36v_3 \quad 0.07v_1 + 0.18v_2 + 0.52v_3]$$

$$= [v_1 \;\; v_2 \;\; v_3]$$

Equating the corresponding entries

$$0.65v_1 + 0.15v_2 + 0.12v_3 = v_1$$

$$0.28v_1 + 0.67v_2 + 0.36v_3 = v_2$$

$$0.07v_1 + 0.18v_2 + 0.52v_3 = v_3$$

$$\Rightarrow -0.35v_1 + 0.15v_2 + 0.12v_3 = 0 \qquad \dots (1)$$

$$0.28v_1 - 0.33v_2 + 0.36v_3 = 0 \qquad \dots (2)$$

$$0.07v_1 + 0.18v_2 - 0.48v_3 = 0 \qquad \dots (3)$$

We observe that (3) is the sum of (1) and (2) multiplied by -1.
So we will not consider this equation.

Since V is a probability vector we have $v_1 + v_2 + v_3 = 1$ ... (4)

Writing in the matrix form

$$\begin{bmatrix} -0.35 & 0.15 & 0.12 \\ 0.28 & -0.33 & 0.36 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving using Gauss-Jordan Method

$$\begin{bmatrix} -0.35 & 0.15 & 0.12 \\ 1.33 & 0.12 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{104}{363} \\ \frac{532}{1089} \\ \frac{245}{1089} \end{bmatrix}$$

Thus we get $v_1 = \frac{104}{363}, v_2 = \frac{532}{1089}, v_3 = \frac{245}{1089}$

Steady State Vector (or) Equilibrium Vector,

$$V = \begin{bmatrix} \frac{104}{363} & \frac{532}{1089} & \frac{245}{1089} \end{bmatrix} = \begin{bmatrix} 0.2865 & 0.4885 & 0.2250 \end{bmatrix}$$

Example 2 : Find the equilibrium vector (or) steady state vector for the transition matrix.

$$P = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}$$

Solution : We observe that all the entries in the matrix are positive.

So, this is a regular matrix. Let V be the probability vector.

Let $V = [v_1 \quad v_2 \quad v_3]$. We want to find V such that $VP = V$

$$\Rightarrow [v_1 \quad v_2 \quad v_3] \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.2 & 0.2 & 0.6 \end{bmatrix} = [v_1 \quad v_2 \quad v_3]$$

$$\Rightarrow \quad 0.5v_1 + 0.1v_2 + 0.2v_3 = v_1$$

$$0.2v_1 + 0.4v_2 + 0.2v_3 = v_2$$

$$0.3v_1 + 0.5v_2 + 0.6v_3 = v_3$$

$$\Rightarrow \quad -0.5v_1 + 0.1v_2 + 0.2v_3 = 0 \quad \dots (1)$$

$$0.2v_1 - 0.6v_2 + 0.2v_3 = 0 \quad \dots (2)$$

$$0.3v_1 + 0.5v_2 - 0.4v_3 = 0 \quad \dots (3)$$

(3) is the sum of (1) & (2) multiplied by -1.

$\therefore$ We will not consider (3)

Also we have $v_1 + v_2 + v_3 = 1 \quad \dots (4)$

Writing in the matrix form, we get

$$\begin{bmatrix} -0.5 & 0.1 & 0.2 \\ 0.2 & -0.6 & 0.2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Using Gauss-Jordan method,

$$R_1 \leftrightarrow R_3 \text{ gives } \begin{bmatrix} 1 & 1 & 1 \\ 0.2 & -0.6 & 0.2 \\ -0.5 & 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} R_2 - 0.2R_1 \\ R_3 + 0.5R_1 \end{matrix} \text{ gives } \begin{bmatrix} 1 & 1 & 1 \\ 0 & -0.8 & 0 \\ 0 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.2 \\ 0.5 \end{bmatrix}$$

$$\frac{R_2}{-0.8} \text{ gives } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.25 \\ 0.5 \end{bmatrix}$$

$$R_2 - R_1 \text{ gives } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \\ 0.5 \end{bmatrix}$$

$$R_3 - 0.6R_2 \text{ gives } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \\ 0.35 \end{bmatrix}$$

$$\frac{R_3}{0.7} \text{ gives } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \\ 0.5 \end{bmatrix}$$

$$R_1 - R_3 \text{ gives } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix}$$

$\therefore v_1 = 0.25, v_2 = 0.25, v_3 = 0.5$

Thus $V = \begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix}$ is the Steady State Vector or Equilibrium Vector.

Example 3 : Find the steady state vector for $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$

Solution : We take, $VP = V$

$\Rightarrow V(P - I) = 0 \qquad \ldots (1)$

Given $P = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$

$\therefore P - I = \begin{bmatrix} -0.75 & 0.75 \\ 0.5 & -0.5 \end{bmatrix}$

From (1) V must be a 2 rowed vector .

Let $V = [v_1 \quad v_2]$

From (1) $[v_1 \quad v_2]\begin{bmatrix} -0.75 & 0.75 \\ 0.5 & -0.5 \end{bmatrix} = [0 \quad 0]$

$\Rightarrow -(0.75)v_1 + (0.5)v_2 = 0$ ... (2)

$(0.75)v_1 - (0.5)v_2 = 0$ ... (3)

Since V is a equilibrium vector we must have

$v_1 + v_2 = 1$ ... (4)

Solving, $(-0.75)v_1 + (0.5)v_2 = 0$

$(0.75)v_1 + (0.75)v_2 = 0.75$ .... $(4) \times (0.75)$

$v_2 = 0.75 \Rightarrow v_1 = 0.25$

$V = [0.25 \quad 0.75]$

Example 4 : A house wife buys 3 kinds of cereals A, B and C. She never buys the same cereals in successive weeks. If she buys Cereal A, the next week she buys B. However if she buys B or C, the next week it is 3 times as likely to buy A as other cereals. In the long run, how often does she buy each of the three cereals.

Solution : The transition probability matrix of the purchasing of the house wife can be

written as
$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} \end{array}$$

Let $V = \{v_0, v_1, v_2\}$ be the steady state probability distribution of the Markov Chain.

Using $VP = V$, we have

$$[v_0 \quad v_1 \quad v_2]\begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & 1/4 \\ 3/4 & 1/4 & 0 \end{bmatrix} = [v_0 \quad v_1 \quad v_2]$$

$\frac{3}{4}v_1 + \frac{1}{4}v_2 = v_0$ ... (1)

$v_0 + \frac{1}{4}v_2 = v_1$ ... (2)

$\frac{1}{4}v_1 = v_2 \Rightarrow v_1 = 4v_2$ ... (3)

Also $v_0 + v_1 + v_2 = 1$ ... (4)

From (2) & (3), $v_0 + \frac{1}{4}v_2 = 4v_2$

$\Rightarrow v_0 = \frac{15}{4}v_2$

Substituting in (4), $\frac{15}{4}v_2 + 4v_2 + v_2 = 1 \Rightarrow v_2 = \frac{4}{35}$

$\Rightarrow v_0 = \frac{15}{35}; \Rightarrow v_1 = \frac{16}{35}; \Rightarrow v_2 = \frac{4}{35}$

$\therefore$ In the long run, probability of buying cereal $A = \frac{15}{35}$

probability of buying cereal $B = \frac{16}{35}$

probability of buying cereal $C = \frac{4}{35}$.

Example 5 : The weather in certain spot is classified as fair, cloudy (without rain) or rainy. A fair day is followed by a fair day 60% of the time and by a cloudy day 25% of the time. A cloudy day is followed by a cloudy day 35% of the time and by a rainy day 25% of the time. A rainy day is followed by a cloudy day 40% of the time and by a rainy day 25% of the time. Initial probabilities are 0.3, 0.3 and 0.4. Find the probability that there will be rainy day after 3 days. What portion of the days is expected to be fair, cloudy or rainy in the long run.

[JNTU (H) Dec. 2019 (R15)]

Solution : We write TPM as

$$P = \begin{array}{c} \\ \text{Fair} \\ \text{Cloudy} \\ \text{Rainy} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Fair} & \text{Cloudy} & \text{Rainy} \end{array} \\ \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.40 & 0.25 \end{bmatrix} \end{array}$$

Intial probabilities are given by $\pi = [0.3 \quad 0.3 \quad 0.4]$ (say)

$$\text{TPM for day 2} = \pi p = [0.3 \quad 0.3 \quad 0.4]\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.40 & 0.25 \end{bmatrix}$$

$$= [0.44 \quad 0.34 \quad 0.22]$$

$$\text{TPM for day 3} = \pi p^2 = \begin{bmatrix}0.44 & 0.34 & 0.22\end{bmatrix}\begin{bmatrix}0.6 & 0.25 & 0.15\\0.4 & 0.35 & 0.25\\0.35 & 0.40 & 0.25\end{bmatrix}$$

$$=\begin{bmatrix}0.477 & 0.317 & 0.206\end{bmatrix}$$

$$\text{After day 3} = \text{Day 4} = \pi p^3 = \begin{bmatrix}0.477 & 0.317 & 0.206\end{bmatrix}\begin{bmatrix}0.6 & 0.25 & 0.15\\0.4 & 0.35 & 0.25\\0.35 & 0.40 & 0.25\end{bmatrix}$$

$$=\begin{bmatrix}0.4851 & 0.3126 & 0.2023\end{bmatrix}$$

After long time, proportion of the days expected to be fair, cloudy or rainy is obtained by using the limiting probabilities.

That $\pi_i, i = 0,1,2$ are obtained by solving the set of equations

$$\pi_0 = 0.6\pi_0 + 0.25\pi_1 + 0.15\pi_2 \quad \ldots (1)$$

$$\pi_1 = 0.4\pi_0 + 0.35\pi_1 + 0.25\pi_2 \quad \ldots (2)$$

$$\pi_2 = 0.35\pi_0 + 0.40\pi_1 + 0.25\pi_2 \quad \ldots (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad \ldots (4)$$

On solving (1), (2), (3) and (4) we get

$$\pi_0 = 0.33, \pi_1 = 0.33, \pi_2 = 0.33$$

Example 6 : If the transition probability matrix market shares of three brands A, B and C is $\begin{bmatrix}0.4 & 0.3 & 0.3\\0.8 & 0.1 & 0.1\\0.35 & 0.25 & 0.4\end{bmatrix}$ and the initial market share are 50%, 25% and 25%, find

(*a*) The market share in second and third periods

(*b*) The limiting probabilities

[JNTU (H) Dec. 2019 (R15)]

Solution : The TPM of the market shares is given by

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix}0.4 & 0.3 & 0.3\\0.8 & 0.1 & 0.1\\0.35 & 0.25 & 0.4\end{bmatrix}\end{array}$$

The initial market shares are $\pi = \begin{bmatrix}0.50 & 0.25 & 0.25\end{bmatrix}$ (say)

(*a*) To find the Market shares in second period

$$\pi P = [0.50 \quad 0.25 \quad 0.25]\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.25 \end{bmatrix}$$

$$= [0.4875 \quad 0.2375 \quad 0.2750] = S_2 \text{ (say)}$$

3rd period $= \pi P^2 = S_2(P)$

$$= [0.4875 \quad 0.2375 \quad 0.2750]\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.25 \end{bmatrix}$$

$$= [0.4813 \quad 0.2387 \quad 0.2800]$$

$\therefore$ The market shares in 2nd period are 48.7%, 23.75 %, 27.50% and in the 3rd period ...48.13%, 23.87%, 28%

...SORBING MARKOV CHAINS

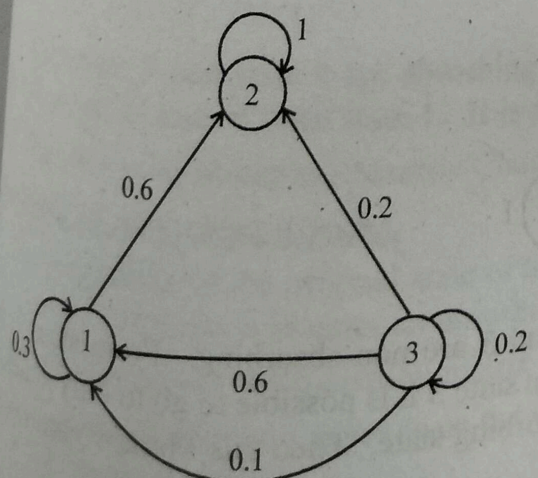
Not all Markov chains are regular. We now discuss one type of Markov chain that do not ...lve transition matrices that are regular. It is called absorbing Markov Chain.

e.g. : Suppose a Markov chain has transition matrix.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0 & 1 & 0 \\ 0.6 & 0.2 & 0.2 \end{bmatrix} \end{array}$$

Here, we observe that p_{12} the probability of going from state 1 to state 2, is 0.6 and p_{22} ...probability of staying in state 2 is 1.

Thus, once state 2 is entered, it is impossible to leave. For this reason state 2 is called an ...sorbing state. The diagram shows that it is not possible to leave state 2.



Generalizing this idea, we define the following.

Absorbing State : State i of a Markov Chain is an absorbing state if $p_{ii} = 1$

We now define an absorbing Markov chain.

Absorbing Markov Chain

A Markov chain is an absorbing chain if the following two conditions are satisfied.

1. The chain has itleast one absrobing state.
2. If is possible to go from any non-absorbing state to an absorbing state (not necessarily in more than one step)

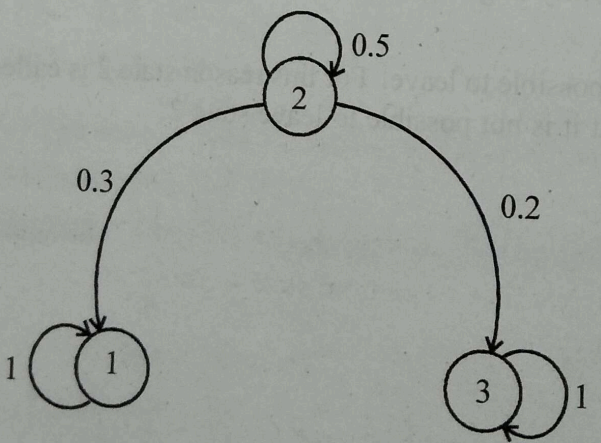
Note that the second condition does not mean that it is possible to go from any non-absorbing state to any absorbing state, but it is possible to go to some absorbing state.

In an absorbing Markov chain, a state which is not absorbing is called **transient.**

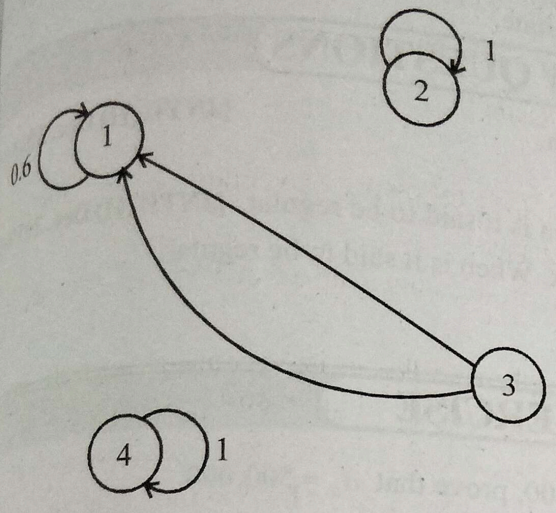
Ex. 1. Identify all absorbing states in the Markov chains having the following matrices. Decide whether the Markov chain is absorbing

$$(a)\ \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad (b)\ P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 1 & 0 & 0 \\ 0.9 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Solution : We have $p_{11} = 1$ and $p_{33} = 1$. Thus state 1 and state 3 are absorbing states. The only non-absorbing state is 2. It is possible to go to absorbing state 1 from state 2 with a probability of 0.3. There is a probability of 0.2 for going from state 2 to obsorbing state 3. So, it is possible to go from the non-absorbing state to an absorbing state. This Markov chain is absorbing. The transition diagram is

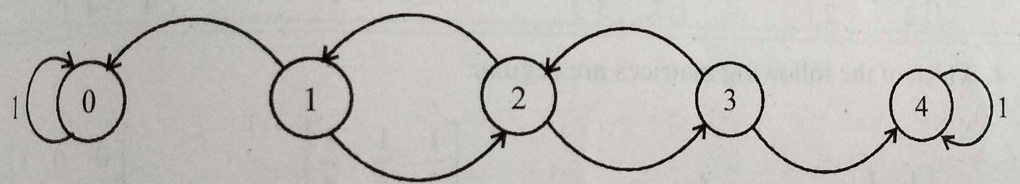


(b) State 2 and 4 are absorbing. States 1 and 3 are non-absorbing. From state 1, it is possible to go to only to states 1 or 3; from state 3 it is possible to go to only states 1 or 3. Neither non-absorbing leads to an absorbing state. Then this Markov chain is non-absorbing



e.g.1. **Drunkard's Walk :** A man walks along a four block stretch of park avenue. If he at corner 1,2 or 3, then he walks to the left or right with equal probability. He continuous until reaches corner 4 which is a bar or corner 0, which is his home. If he reaches either home bar he stays there.

The transition diagram is



$$TPM = P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Here state 0 and state 4 are absorbing. State 1, state 2, state 3 are non-absorbing. It is possible to go to state 0 from state 1. It is possible to go from state 4 from state 3.

Thus this is an absorbing Markov Chain.

Properties of Absorbing Chains.

1. Regardless of the original state of an absorbing Markov Chain, in a finite number of steps the chain will enter an absorbing state and then stay in that state.
2. The powers of the transition matrix get closes and closes to some particular matrix.
3. The long term trend depends on the initial state. Changing the initial state can change the final result.

This property distinguishes absorbing Markov chains from regular Markov Chains, where the final result is independent of the initial state.

REVIEW QUESTIONS

1. Define an Absorbing Markov Chain. [JNTU (H) Dec. 2019 (R15)]
2. Define Markov Chain.
3. What is a Stochostic Matrix ? When is it said to be regular. [JNTU (H) Dec. 2019 (R18)]
4. Define transition Probability matrix. When is it said to be regular.
5. Define Markov Process.
6. Define Probability Vector.

EXERCISE

1. If $p = 1/2$, $q = 1/2$, $z = 500$, $a = 1500$, prove that $d_z = 500,000$
2. If $p = 1/2$, $q = 1/2$, $z = 1$, $a = 1000$ prove that $d_z = 999$
3. Which of the following matrices are stochastic

(*a*) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$ (*b*) $\begin{bmatrix} \frac{15}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$ (*c*) $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (*d*) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

4. Which of the following matrices are regular

(*a*) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ (*b*) $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (*c*) $C = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$ (*d*) $D = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$

ANSWERS

3. (*a*) not stochastic (*b*) not stochastic (*c*) stochastic (*d*) not stochastic
4. (*a*) not regulat (*b*) not regular (*c*) not regular (*d*) regular