5 UNIT - Stochastic (Random) Process

We know that the random variable is a rule for assigning $x(s)$ to each and every outcome of x of an experiment.

Now we define a random process as a rule for assigning a real valued function of time $x(t, s)$ to every outcome 's', one function will be assigned. This family of functions is called a random process or stochastic process.

The random process often called as a stochastic process is defined as a collection of functions of time with a probability measure associated.

It is usually denoted by $x(t, s)$.

Interpretations

The random process is a family of functions $x(t, s)$ where t and s are variables. If s is fixed the random process is a function of time. It is called single time function.

For example:-

Consider your classroom. Take sample point as "raju", call "raju" today in microphone, you get one wave function. After one minute, again call "raju". you get another wave function. Similarly for different timings, you have different wave functions corresponding to single sample point is known as sample function.

→ If time is fixed, the random process is a function of s only and hence the random process will represent a random variable at time 't'.

time - 't'

Example-2: At some time $t = 10$ call all the students one by one then we get different wave functions corresponding to different sample points is a random variable.

→ If both t and s are fixed then $x(s, t)$ is just a number.

Types of random process

→ **Continuous random process:**

A continuous random process is one in which the random variable 'x' is continuous and 't' can have any value between t_1 and t_2. The physical examples for continuous random process are dissolving of sugar crystals in coffee. Thermal noice generated in a network, fluctuations in air, temperature and air pressure etc.

→ **Discrete random process**

If the random variable 'x' can assume only certain specified values, while 't' is continuous it is called a discrete random process.

For example:- The voltage available at the one end of switch because of random opening and closing of switch. It is a discrete random process.

→ Deterministic random process:

If the future values of any sample function can be predicted from a knowledge of the past values, then the random process is called "Deterministic Random Process".

For example:-

Consider a random process $x(t) = A\cos(\omega t + \theta)$ which consists of a family of pure sine waves and it is completely specified in terms of the random variables A and θ. Hence, it is deterministic random process.

→ Non deterministic random process:

If the future values of a sample function cannot be predicted from the know of the past values. The random process is called non deterministic random process.

For Ex:- In the case of dissolving of sugar crystals in coffee, it consists of a family of functions that cannot be described in terms of a finite number of parameters.

The future sample future cannot be determined from the past sample functions. so it is a non-deterministic random process.

→ Equal random process:

If two random processes $X(t)$ and $Y(t)$ are equal everywhere if their respective samples $X(t, S_i)$ and $Y(t, S_i)$ are identical for every S_i.

Classification of random process

Classification can be done in another way also. Stochastic process is a function of sample points and time. The sample points may have discrete or continuous values. Similarly the experiments may be defined as discrete or continuous time intervals. Thus a stochastic process may be classified into four types

1) If both x and t are continuous the stochastic process is called as continuous stochastic process.
2) If x is continuous and t is discrete, the stochastic process is called as a discrete stochastic sequence.
3) If x is discrete and t is continuous, the stochastic process is called as a discrete stochastic process.
4) If both x and t are discrete, then the stochastic process is called as a discrete stochastic sequence.

Statesphase

The values assumed by the random variable are called as states.

The set of all possible values of an individual random variable X_n of a stochastic process $\{X_n, n \geq 1\}$ is known as its statesphase.

It is denoted by I.

The statesphase is said to be discrete if it contains a finite or countable infinite coins.

Otherwise it is called Continuous.

For Example :-

1) Suppose a fair die is rolled, let X_n denote the number of sixes appearing in first 'n' throws of die. Then the set of possible values of X_n is the finite set of non negative integers.

Here the statesphase of X_n is discrete.

Markov process

If the future behaviour $X(t_{n+1})$ of the random process depends only on the present value.

$$X(t_{n+1}) = X_{n+1} / X(t_n) = X_n$$

but

not on past values $X_{n-1}, X_{n-2}, X_{n-3} \ldots X_1, X_0$ then the process is said to be a markov process.

The markovian property

$$P\{X(t_{n+1}) = X_{n+1} \mid X(t_n) = X_n, X(t_{n-1}) = X_{n-1}, \ldots\ldots, X(t_1) = X_1, X(t_0) = X_0\}$$

$$= P\{X(t_{n+1}) = X_{n+1} / X(t_n) = X_n\}$$

The joint property is called condition probability is markovian property.

The values $X_0, X_1, \ldots X_n, X_{n+1}$ are called the states of a process.

Depending on the nature of the values of t and $X(t)$. Markov process can be classified as follows, if t and $X(t)$ both are continuous then the process is known as continuous parameter markov process.

If t is continuous, $X(t)$ discrete then the process is known as continuous parameter markov chain.

If t is discrete and $X(t)$ continuous then the process is discrete parameter markov process.

If both t and $X(t)$ are discrete, then the process is discrete parameter markov chain.

Markov chain

A markov process $X(t)$ taking only discrete values for continuous arch discrete time t is called a markov chain.

A sequence of states $\{X_n\}$ is defined as a markov chain if each X_n is a random variable and $P\{X_{n+1} = a_{n+1} / X_n = a_n, X_{n-1} = a_{n-1}, \ldots X_1 = a_1, X_0 = a_0\}$ where $a_0, a_1, a_2, a_3 \ldots a_{n+1}$ are called states of markov chain.

Transition probability

The probability of moving from one state to another or remaining in the same state during a single time period is called the Transition probability.

$$P^{(n-(n-1))}_{x_{n-1} x_n} = P\{X(t_n) = x_n / X(t_{n-1}) = x_{n-1}\}$$

is called the transition probability.

$$P_{ij}^{\overset{\rightarrow \text{No. of steps}}{(m-n)}} = P\{X_m = j / X_n = i\}$$

$\hookrightarrow$ from which state to which one we have to move.

This represent the conditional probability of the system which is in state x_n at time t_n provided that it was previously in state x_{n-1} at time t_{n-1}. Sometimes this probability is known as one step transition probability.

Note:-

$$P_{ij}^{(m-n)} = P\{X_m = j / X_n = i\}$$

Transition probability matrix

The transition probabilities can be arranged in a matrix form and such a matrix is called a one step transition probability matrix, and is denoted by

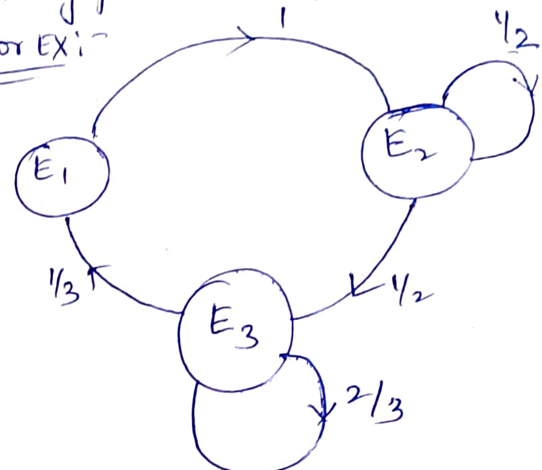
$$P = \begin{array}{c c} & \text{To} \\ & \begin{array}{cccccc} 1 & 2 & \cdots & & & m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ \text{from} \\ \vdots \\ m \end{array} & \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots & P_{1m} \\ P_{21} & P_{22} & P_{23} & \cdots & P_{2m} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ P_{m1} & P_{m2} & \cdots & & P_{mm} \end{bmatrix} \end{array}$$

The matrix P is a square matrix whose elements are non-negative and sum of elements of each row is unity. In general any matrix P whose elements are non-negative and sum of the elements in each row is unity is called a transition probability matrix.

Transition Diagrams

Transition Diagram shows the transition probabilities that can occur in any particular situation.

For Ex:-



$$\begin{array}{c c c} & & \begin{array}{ccc} E_1 & E_2 & E_3 \end{array} & \\ \text{from} & \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \end{bmatrix} & \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \end{array}$$

The arrows from each state indicate the possible states to which a process can move from the given state.

The matrix of transition probabilities which corresponds to above diagram is mentioned below. the transition diagram.

A zero element in the above matrix indicates that transition is not possible. The joint probability of a markov chain is given by

$$P\{X_{n+1} = a_{n+1}, X_n = a_n, X_{n-1} = a_{n-1} \ldots X_1 = a_1, X_0 = a_0\}$$

that is the joint probability is the product of conditional probabilities and the initial probabilities

$$P\{X_{n+1} = a_{n+1} / X_n = a_n\}$$
$$= P\{X_n = a_0 / X_{n-1} = a_{n-1}\} \ldots$$
$$P\{X_1 = a_1 / X_0 = a_0\} P(X_0 = a_0)$$

These conditional probability is the product of conditional probabilities and the initial probabilities.
These conditional probabilities called as single step Transition probabilities.

Types of markov chain

Homogeneous markov chain :-

A markov chain is said to be Homogeneous if the transition probabilities are independent of the occurance of transition.

i.e., $P_{ij}(m-1, m) = P_{ij}(n-1, n)$

Non Homogeneous Markov chain: If $P_{ij}^{(n)}$ is dependent on n. Then the markov chain is called to be Non-Homogeneous markov chain.

Finite markov chain: A markov chain with finite no. of steps is said to be a Finite markov chain.

Infinite markov chain: A markov chain with infinite no. of steps is known as infinite markov chain.

Order of markov chain:-

Markov Chains are classified by their order.

→ The case in which probability occurance of each state depends only upon the immediate preceding state. It is said to be first order markov chain.

→ In second order markov chain, it is assumed that the probability of occurance in forth coming period depends upon the state in the last two periods.

→ Similarly in the third order markov chains, it is assumed that probability of a state in the forth coming period depends upon the states in the last three periods.

Questions (sample):-

* Write characteristics of markov chain?

* Write any five applications of markov chain?

Stochastic matrix

A stochastic matrix is a square matrix with non-negative elements and unit-row sums.

Sum of elements in each row = 1

If p and q are stochastic matrices then p×q is also a stochastic matrix and p^n is also a stochastic matrix for all positive integers of n.

1) Which of the following matrices are stochastic?

i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Ans:- The given matrix is not a square matrix. so it is not a stochastic matrix.

ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans:- It is a stochastic matrix [square matrix and row sum is 1]

iii) $\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

Ans:- The given matrix is not a stochastic matrix as 2nd row sum is not equal to 1.

iv) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Ans :- The given matrix is stochastic matrix.

v) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

Ans :- 2nd row consists of negative value. So, it is not a stochastic matrix.

vi) $\begin{bmatrix} 0 & 2 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Ans :- row sum $\neq 1$.
Not a stochastic matrix.

vii) $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$

Ans :- Not a square matrix, so not a stochastic matrix.

viii) $\begin{bmatrix} \frac{15}{16} & \frac{1}{16} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$

Ans :- 2nd row sum $\neq 1$; Not a stochastic matrix

ix) $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Ans :- The given matrix is a stochastic matrix

2) Find x, y, z if

(i) $\begin{bmatrix} 0 & x & \frac{1}{3} \\ 0 & 0 & y \\ \frac{1}{3} & \frac{1}{4} & z \end{bmatrix}$

$$x + \frac{1}{3} = 1$$

$$x = 1 - \frac{1}{3}$$

$$\boxed{x = \frac{2}{3}}$$

$$\boxed{y = 1}$$

$$z + \frac{1}{4} + \frac{1}{3} = 1$$

$$z = 1 - \left[\frac{1}{4} + \frac{1}{3}\right]$$

$$z = 1 - \left[\frac{3+4}{12}\right]$$

$$z = 1 - \frac{7}{12} = \frac{5}{12}$$

$$\boxed{z = \frac{5}{12}}$$

(ii) $\begin{bmatrix} 0 & 0.2 & x \\ x & 0.1 & y \\ 0.1 & 0.2 & z \end{bmatrix}$ are transition probability matrices.

$$0.2 + x = 1$$

$$\boxed{x = 0.8}$$

$$0.8 + 0.1 + y = 1$$

$$\boxed{y = 0.1}$$

$$0.1 + 0.2 + z = 1$$

$$\boxed{z = 0.7}$$

3) The alumni office of a college finds on review that 80% of its alumni who contribute to the annual fund 1 year will also contribute next year and 30% of those who do not contribute 1 year will contribute next year? Write the transition matrix?

Ans:

$$\begin{array}{l|cc} & \multicolumn{2}{c}{\text{To}} \\ & C & NC \\ \hline \text{contribute (C)} & 0.8 & 0.2 \\ \text{Not Contribute (NC)} & 0.3 & 0.7 \end{array}$$

(rows: From)

4) 3 Universities A, B, C are admitting students it is given that 80% of the children of A went to A and the rest went to B. 40% of children of B went to B and rest split evenly to A and C. The children of C 70% went to C, 20% went to A, 10% went to B. Form the markov chain and transition probability matrix.

Ans:

$$\begin{array}{cc} & \text{To} \\ \text{From} & \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.8 & 0.2 & 0 \\ B & 0.3 & 0.4 & 0.3 \\ C & 0.2 & 0.1 & 0.7 \end{array} \end{array}$$

Regular

A stochastic matrix P is said to be Regular, if all entries of some power (P^m) are positive.

A stochastic matrix P is not regular if one occurs in the principal main diagonal.

1) Which of the following stochastic matrices are

a) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

Ans: ∴ Since the principal diagonal has one, it is not a regular matrix.

b) $B = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$

Ans:

$B^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{bmatrix}$

$B^3 = \begin{bmatrix} 0.5 & 0 & 0.5 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$B^4 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

Note that B^4 has all +ve entries.

∴ B is Regular.

c) $C = \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0 & 0.5 & 0.5 \\ 0.6 & 0.4 & 0 \end{bmatrix}$

Ans: $C^2 = \begin{bmatrix} 0.5625 & 0.3125 & 0.125 \\ 0.3 & 0.45 & 0.25 \\ 0.45 & 0.35 & 0.20 \end{bmatrix}$

C^2 has all +ve entries.

∴ C is Regular.

d) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

Ans:- In the first two rows zeroes are not eliminated, while finding higher powers.

∴ Matrix A is not a regular matrix.

e) $C = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Ans:- ∴ Since the principal diagonal has one.

Not a regular matrix.

f) $B = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & 1 \end{bmatrix}$

Ans:-

∴ Not a stochastic matrix,

Not a regular matrix.

g) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Ans:-

∴ Since the principal diagonal has one, it is not a regular matrix.

h) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$

∴ Not a stochastic matrix,

Not a regular matrix.

2) consider the markov chain

$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$. Then find $P_{01}^{(2)}$ and $P(X_2 = 1, X_0 = 0)$ with $P(X_0 = i) = \frac{1}{3}$ $i = 0,1,2$

Sol:-

$$\boxed{\text{From}}\ P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \boxed{\text{To}} \\ \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

$$\boxed{\text{From}}\ P^2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 5/8 & 5/16 & 1/10 \\ 5/16 & 1/2 & 3/16 \\ 3/16 & 9/16 & 1/4 \end{bmatrix} \end{array}$$

$P_{01}^{(2)} = \frac{5}{16}$

(ii) $P(X_2 = 1, X_0 = 0)$

$= P(X_2 = 1 / X_0 = 0) \cdot P(X_0 = 0)$

$= P_{01}^{(2-0)} \cdot \frac{1}{3}$

$= \frac{5}{16} \cdot \frac{1}{3}$

3) $\{X_n, n \geq 0\}$ be a markov chain with 3 states $\{0,1,2\}$ and with transition probability matrix

$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}$ and the initial distribution $P(X_0 = i) = \frac{1}{3}$ for $i = 0,1,2$ then find probability of $P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$

Sol:-
Given that

$$P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$$

To (columns), From (rows)

consider $P\{X_3=1, X_2=2, X_1=1, X_0=2\}$

$= P\{X_3=1/X_2=2\} \cdot P\{X_2=2/X_1=1\} \cdot P\{X_1=1/X_0=2\} \cdot P(X_0=2)$

$= P_{21}^{(3-2)} P_{12}^{(2-1)} P_{21}^{(1-0)} P(X_0=2)$

$= P_{21} P_{12} P_{21} P(X_0=2)$

$= \frac{3}{4} \frac{1}{4} \frac{3}{4} \frac{1}{3}$

$= \frac{3}{64} = 0.046$

4) The transition probability of a markov chain $\{X_n\} = n = 1,2,3 \ldots$ is having 3 states.

$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$ and the

initial distribution is $P^{(0)} = \begin{array}{ccc} 1 & 2 & 3 \\ (0.7, & 0.2, & 0.1) \end{array}$

then find

(i) $P(X_2=3)$

(ii) $P\{X_3=2, X_2=3, X_1=3, X_0=2\}$

Sol:-

(i) $P(X_2=3) = P\{X_2=3/X_0=1\} \cdot P(X_0=1)$
$+ P\{X_2=3/X_0=2\} \cdot P(X_0=2)$
$+ P\{X_2=3/X_0=3\} \cdot P(X_0=3)$

$= P_{13}^{(2-0)} P(X_0=1) + P_{23}^{(2)} P(X_0=2)$
$+ P_{33}^{(2)} P(X_0=3)$

$$P^2 = \begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \hline \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix} \end{array}$$

To (columns), From (rows)

$= (0.26)(0.7) + (0.34)(0.2)$
$+ (0.29)(0.1)$
$= 0.279$

(ii) $P\{X_3=2, X_2=3, X_1=3, X_0=2\}$
$= P\{X_3=2/X_2=3\} \cdot P\{X_2=3/X_1=3\} \cdot$
$P\{X_1=3/X_0=2\} P(X_0=2)$
$= P_{32}^{(1)} P_{33}^{(1)} P_{23}^{(1)} P(X_0=2)$
$= (0.4)(0.3)(0.2)(0.2)$
$= \frac{3}{625}$

5) A Raining process is considered as a two state markov chain. If it rains it is considered in state 0 and it does not rain, the chain is in the state of 1. The transition probability of the markov chain is defined by $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$.

Find the probability that, it will rain for 3 days from today. Assuming that it is raining today. Assume that the mutual probability of state 0 and state 1 has 0.4 and 0.6 respectively.

Sol:- Given,

$$P = \begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ \hline \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

To (columns), From (rows)

$$\begin{array}{cc} 0 & 1 \\ (0.4, & 0.6) \end{array}$$

$$P\{X_3=0 / X_0=0\} = P_{00}^{(3)}$$

$$P^2 = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{bmatrix}$$

$$\therefore P_{00}^{3} = 0.376$$

5) Suppose that the probability of a dry day (state 0) follows a rainy day (state 1) is 1/3 and probability of a rainy day follows a dry day is 1/2. Then find if march 1st is dry day then what is the probability that march 3rd is also dry day. If march 1st is dry day then what is the probability than march 5th is also dry day.

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

(i) P (if march 1^{st} is a dry day then march 3rd is also a dry day).

$$= P(X_2=0 / X_0=0)$$

$$= P_{00}^{(2-0)} = P_{00}^{2}$$

$$P^2 = \begin{bmatrix} 0.4166 & 0.5833 \\ 0.3888 & 0.6111 \end{bmatrix}$$

$$P_{00}^{2} = 0.4166$$

$$1^{st} \xrightarrow[1\,step]{} 2^{nd} \xrightarrow[1\,step]{} 3^{rd}$$

** Sample Question

Define stochastic regular matrix

(ii) P (if march 1^{st} is a dry day then march 5th is also a dry day)

$$= P\{X_4=0 / X_0=0\}$$

$$= P_{00}^{4}$$

$$P^4 = \begin{bmatrix} 0.4004 & 0.5995 \\ 0.3996 & 0.6003 \end{bmatrix}$$

$$P_{00}^{4} = 0.4004$$

*** 1 long Ans

Classification of states

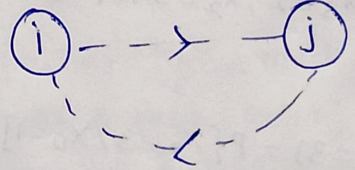
accessible

we say that state j is accessible from state i if there exists a path from i to j.

$P_{ij}^{(m)} > 0$ and $P_{ji}^{(m)} > 0$

Communicate

Two states i and j communicate with each other if there exists a path from i to j and j to i.



irreducible markov chain

If every state can be reached from any other state or if all states of a markov chain are communicate with each other then the markov chain is irreducible markov chain.

Finite markov chain

If the no. of states in markov chain are finite, it is a finite markov chain, otherwise infinite.

Not-Null persistant or positive recurrent

A finite irreducible markov chain is called not null persistant or positive recurrent.

Return state

If a state can be return after travelling finite no. of paths then the state is called return state or a state i of markov chain is called a return state if $P_{ii}^{(n)} > 0$

periodic and aperiodic

The period of a return state is defined as gcd $\{n : P_{ii}^{(n)} > 0\}$ and is denoted by d_i

$d_i = \gcd\{n : P_{ii}^{(n)} > 0\}$

If $d_i > 1$ then state i is said to be periodic

If $d_i = 1$ then state i is aperiodic.

ergodic

A positive recurrent and aperiodic state is called ergodic.

A markov chain all of whose states are ergodic is said to be an ergodic chain.

absorbing state

A state i is said to be an absorbing state if $P_{ii} = 1$

** A markov chain is said to be an absorbing markov chain if the below mentioned conditions are satisfied:-

1) The chain has atleast one absorbing state.

2) It is possible to go from every non absorbing state to atleast one absorbing state in one or more steps.

Questions :-

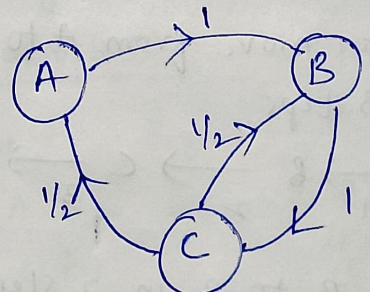
1) 3 boys A, B and C are throwing a ball to each other. 'A' always throws the ball to 'B' and 'B' always throws the ball to 'C' but C is just as likely to throw the ball to 'B' as to 'A'. Show that the process is markovian. Find the transition matrix and classify the states. Do all the states are Ergodic

Sol:- In this problem, the future state is depending on the just present state that is present on which hand ball is there

$\therefore$ The process is markovian.

Transition probability matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$



1) Accessible and Communicate

From the diagram, we can move from any state to another state.

For Example, we can move from

$A \to B$ in 1 step i.e., $A \xrightarrow{1} B$

$A \to C$ in 2 steps i.e., $A \xrightarrow{1} B \xrightarrow{2} C$

$A \to A$ in 3 steps

i.e., $A \xrightarrow{1} B \xrightarrow{2} C \xrightarrow{3} A$

A, B, C are communicate each other.

2) Irreducible

Since 3 states are communicate each other.

∴ The chain is irreducible.

3) Finite markov chain

The chain has only 3 states

⇒ It is a finite chain.

4) Not Null persistant / Positive recurrent

∵ The chain is finite and irreducible

∴ The chain is not null persistant.

5) Return

We can move from A to A in 3 steps

$A \xrightarrow{1} B \xrightarrow{2} C \xrightarrow{3} A$

From B to B in 2 steps

$B \to C \to B$

From C to C in 2 steps

$C \to B \to C$

∴ The three states A, B, C are return states.

6) periodic / aperiodic

State A

We can return to A in 3 steps, 5 steps, 7 steps ...

$(A \to B \to C \to B \to C \to A)$

$d_A = \gcd\{3, 5, 7, \ldots\} = 1$

∴ state A is aperiodic

State B

We can return to B in 2 steps, 3 steps ...

$d_B = \gcd\{2, 3, \ldots\} = 1$

∴ state B is aperiodic

State C

We can return to C in 2 steps, 3 steps ...

$d_B = \gcd\{2, 3, \ldots\} = 1$

∴ state C is aperiodic.

7) Ergodic

The 3 states are positive recurrent and aperiodic.

∴ The states are ergodic states and the chain is ergodic chain.

8) absorbing

$P_{AA} \neq 1$, $P_{BB} \neq 1$, $P_{CC} \neq 1$

∴ A, B, C are non-absorbing states.

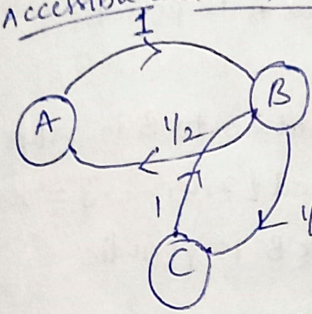
Hence the chain is not an absorbing chain.

2) Find the nature of states of the markov chain with Transition probability matrix

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

1) Accessible and communicate

sol:-



A, B, C are not communicate with each other.

2) Irreducible

since 3 states are not communicate with each other.

∴ The chain is reducible.

3) Finite markov chain.

The chain has only 3 states

⇒ It is a finite chain.

4) NotNull persistant / positive recurrent

∵ The chain is finite and reducible

∴ The chain is null persistant

5) Return

We can move from A to A in 2 steps

$A \xrightarrow[1]{} B \xrightarrow[2]{} A$

From B to B in 2 steps

$B \xrightarrow[1]{} A \xrightarrow[2]{} B$

From C to C in 2 steps

$C \longrightarrow B \longrightarrow C$

∴ The three states A, B, C are return states.

6) periodic / aperiodic

state A

we can return to A in 2 steps, 4 steps, 6 steps ...

$(A \to B \to C \to B \to A)$

$d_A = \gcd\{2, 4, 6 \ldots\} = 2$

∴ state A is periodic

State B

we can return to B in 2 steps

$d_B = \gcd\{2, 2\} = 2$

∴ state B is periodic

state C

we can return to C in 2 steps

$d_C = 2$

∴ state C is periodic.

7) Ergodic

The 3 states are not positive recurrent and periodic.

∴ The states are not ergodic states and the chain is not ergodic chain.

8) absorbing

$P_{AA} \neq 1$, $P_{BB} \neq 1$, $P_{CC} \neq 1$

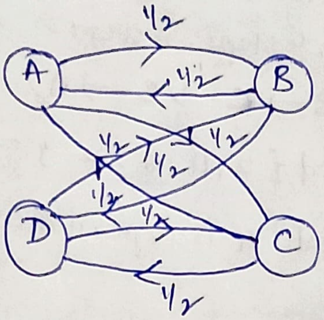
∴ A, B, C are non-absorbing states.

Hence the chain is not an absorbing chain.

5) Find the nature of states of the markov chain with Transition probability Matrix (P) =

$$\begin{array}{c|c} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Sol:- 1) Accessible and communicate



A, B, C communicate with each other.

2) Irreducible

since 3 states are not communicate each other.

∴ The chain is reducible.

3) Finite markov chain

The chain has only 3 states

=) It is a finite chain.

4) Not Null persistant / positive recurrent

∵ The chain is finite and reducible

∴ The chain is null persistant

5) Return

we can move from A to A in 2 steps, 4 steps ...

$A \xrightarrow[1]{} B \xrightarrow[2]{} D \xrightarrow[3]{} C \xrightarrow[4]{} A$

From B to B in 2 steps, 4 steps ...

$B \xrightarrow[1]{} A \xrightarrow[2]{} C \xrightarrow[3]{} D \xrightarrow[4]{} B$

From C to C in 2 steps, 4 steps ...

$C \xrightarrow[1]{} D \xrightarrow[2]{} B \xrightarrow[3]{} A \xrightarrow[4]{} C$

From D to D in 2 steps, 4 steps ...

$D \xrightarrow[1]{} C \xrightarrow[2]{} A \xrightarrow[3]{} B \xrightarrow[4]{} D$

∴ The 4 states are return states.

6) periodic / aperiodic

state A

we can return to A in 2 steps, 4 steps ...

$d_A = \gcd\{2, 4, \ldots\} = 2$

∴ state A is periodic

state B

we can return to B in 2 steps, 4 steps ...

$d_B = \gcd\{2, 4, \ldots\} = 2$

∴ state B is periodic

state C

we can return to C in 2 steps, 4 steps ...

$d_C = \gcd\{2, 4, \ldots\} = 2$

∴ state C is periodic

state D

we can return to D in 2 steps, 4 steps ...

$d_D = \gcd\{2, 4, \ldots\} = 2$

∴ state D is periodic.

7) Ergodic

The 3 states are positive recurrent and periodic.

∴ The states are not ergodic states and the chain is not ergodic chain

8) absorbing

$P_{AA} \neq 1$, $P_{BB} \neq 1$, $P_{CC} \neq 1$, $P_{DD} \neq 1$

∴ A, B, C, D are non absorbing states.

Hence the chain is not an absorbing chain.

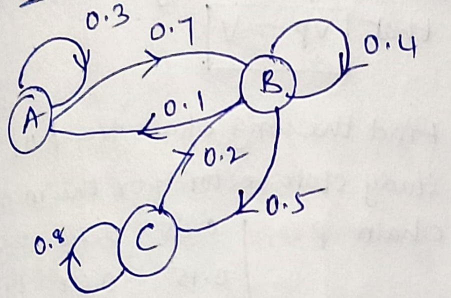
4) The transition probability matrix of a markov chain is

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & 0.3 & 0.7 & 0 \\ B & 0.1 & 0.4 & 0.5 \\ C & 0 & 0.2 & 0.8 \end{array}$$

is irreducible?

Sol:-

Accessible and communicate



Since 3 states are not communicate with each other..

$\therefore$ The chain is reducible.

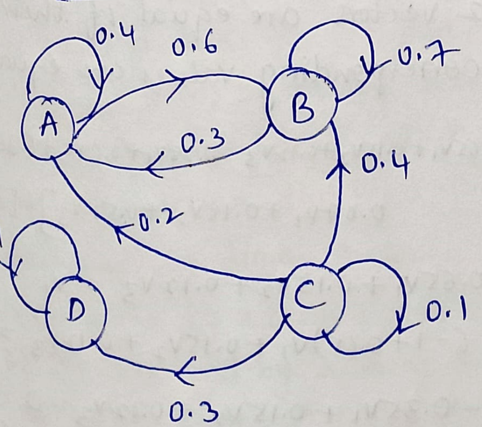
5) The transition probability matrix of a markov chain is

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0.4 & 0.6 & 0 & 0 \\ B & 0.3 & 0.7 & 0 & 0 \\ C & 0.2 & 0.4 & 0.1 & 0.3 \\ D & 0 & 0 & 0 & 1 \end{array}$$

is this matrix irreducible?

Sol:-

Accessible and communicate



Since 4 states are not communicate with each other.

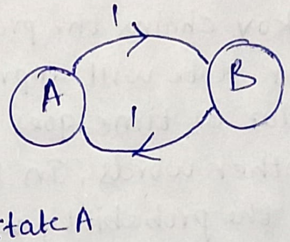
$\therefore$ The chain is not irreducible.

6) Find periodic and aperiodic in the following matrix

i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Sol:-

i) $$\begin{array}{c|cc} & A & B \\ \hline A & 0 & 1 \\ B & 1 & 0 \end{array}$$



State A

we can return to A in 2 steps

$d_A = 2$

$\therefore$ state A is periodic

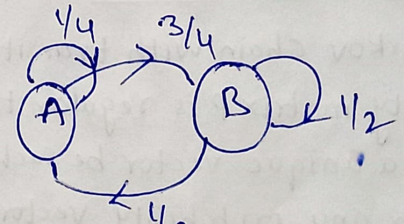
State B

we can return to B in 2 steps

$d_B = 2$

$\therefore$ state B is periodic

ii) $$\begin{array}{c|cc} & A & B \\ \hline A & \frac{1}{4} & \frac{3}{4} \\ B & \frac{1}{2} & \frac{1}{2} \end{array}$$



State A

we can return to A in 1 step, 2 steps.

$d_A = \gcd\{2, 1\} = 1$

$\therefore$ state A is aperiodic

State B

We can return to B in 1 step, 2 steps

$d_B = \gcd\{1, 2\} = 1$

$\therefore$ state B is aperiodic.

Study-state condition

Stable probability

In many markov chains the probability for a particular state will approach a limiting value as time goes to infinity. In other words, In the far future, the probability won't be changing much from one transition to the next transition. These limiting probabilities are called the state probabilities.

** Study-state condition

If a system is such that each state has probability = its stable probability, the probability will persist for all the time. Then the system is said to be in study-state condition.

Study state vector (or) equilibrium vector of a markov chain

If a markov chain with transition probability matrix P is regular then there is a unique vector be such that for any probability vector v such that vP^n.

$$vP^n \simeq v$$

vector v is called the equilibrium vector or the fixed vector or study-state vector of the markov chain. This is also called long-range trend of markov chain.

Probability vector

A probability vector is a matrix of only one row having non-negative entries with sum of the entries = 1

Result:-

If a markov chain with transition matrix P is regular then there exists a probability vector v such that $\boxed{vP = v}$

1) Find the long chain trend or study state vector for the markov chain $P = \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix}$

Sol:- Since all entries are positive, therefore P is a regular matrix.

Let $v = [v_1 \ v_2 \ v_3]$ be the long-range trend or study state vector then $v = vP$

$$[v_1 \ v_2 \ v_3] \begin{bmatrix} 0.65 & 0.28 & 0.07 \\ 0.15 & 0.67 & 0.18 \\ 0.12 & 0.36 & 0.52 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

2 vectors are equal if their corresponding values are equal

$$[0.65v_1 + 0.15v_2 + 0.12v_3 \quad 0.28v_1 + 0.67v_2 + 0.36v_3 \quad 0.07v_1 + 0.18v_2 + 0.52v_3] = [v_1 \ v_2 \ v_3]$$

$$0.65v_1 + 0.15v_2 + 0.12v_3 = v_1$$

$$(-1 + 0.65)v_1 + 0.15v_2 + 0.12v_3 = 0$$

$$-0.35v_1 + 0.15v_2 + 0.12v_3 = 0 \quad \text{(1)}$$

$0.28V_1 + 0.67V_2 + 0.36V_3 = V_2$

$0.28V_1 + (-1 + 0.67)V_2 + 0.36V_3$

$0.28V_1 - 0.33V_2 + 0.36V_3 = 0$ ——— (2)

$0.07V_1 + 0.18V_2 + 0.52V_3 = V_3$

$0.07V_1 + 0.18V_2 + (-1+0.52)V_3 = 0$

$0.07V_1 + 0.18V_2 - 0.48V_3 = 0$ ——— (3)

(2) + (3) = −(1) ⇒

−[(1) + (2)] = (3)

eq. (1), (2), (3) are linearly dependent.

Take only (1), (2) and we know that $\boxed{V_1 + V_2 + V_3 = 1}$ ——— (4)

on solving (1), (2), (4)

$V_1 = \frac{104}{363}$ $V_2 = \frac{532}{1089}$

$V_3 = \frac{245}{1089}$

$$V = \begin{bmatrix} \frac{104}{363} & \frac{532}{1089} & \frac{245}{1089} \end{bmatrix}$$

1) The weather in a certain spot is classified as fair, cloudy (without rain) or rainly. A fair day is followed by a fair day 60% of the time and by a cloudy day 25% of time. A cloudy day is followed by a cloudy day 35% of the time and by a rainy day 25% of the time. A rainy day is followed by a cloudy day 40% of time and by a rainy day 25% of the time. Initial probabilities are 0.3, 0.3 and 0.4.

(i) Find the probability that there will be a rainy day after 3 days.

(ii) What portion of the days is expected to be fair or cloudy in the long run.
$\hookrightarrow n \to \infty$

$$P = \begin{array}{c|ccc} & F & C & R \\ \hline F & 0.6 & 0.25 & 0.15 \\ C & 0.4 & 0.35 & 0.25 \\ R & 0.35 & 0.4 & 0.25 \end{array}$$

Initial probabilities $\Pi = [0.3\ 0.3\ 0.4]$

the 2nd day probabilities

$$\Pi P = \underset{1\times3}{[0.3\ 0.3\ 0.4]} \underset{3\times3}{\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix}}$$

$= [0.44\ 0.34\ 0.22]$

For 3rd day, $\Pi P^2 = (\Pi P)(P)$

$$= \underset{1\times3}{[0.44\ 0.34\ 0.22]} \underset{3\times3}{\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix}}$$

$= [0.477\ 0.317\ 0.206]$

After 3 days $= \Pi P^3 = (\Pi P^2)P$

$$= [0.477\ 0.317\ 0.206] \begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix}$$

$= [0.4851\ 0.3126\ 0.2023]$

∴ The probability that there will be a rainy day after 3 days is 0.2023

Fair day prob. ⟶ 0.4851

cloudy day prob. ⟶ 0.3126

ii) After long time proportion of the days expected to be fair, cloudy or rainy day is obtained by using the study-state vector.

Sol:- Let $[v_1\ v_2\ v_3]$ be the study-state vector then $VP = V$

$$\Rightarrow [v_1\ v_2\ v_3]\begin{bmatrix} 0.6 & 0.25 & 0.15 \\ 0.4 & 0.35 & 0.25 \\ 0.35 & 0.4 & 0.25 \end{bmatrix} = [v_1\ v_2\ v_3]$$

$\Rightarrow 0.6v_1 + 0.4v_2 + 0.35v_3 = v_1$

$-0.4v_1 + 0.4v_2 + 0.35v_3 = 0$ ——— (1)

$\Rightarrow 0.25v_1 + 0.35v_2 + 0.4v_3 = v_2$

$0.25v_1 - 0.65v_2 + 0.4v_3 = 0$ ——— (2)

$\Rightarrow 0.15v_1 + 0.25v_2 + 0.25v_3 = v_3$

$0.15v_1 + 0.25v_2 - 0.75v_3 = 0$ ——— (3)

(1) + (2) = −(3)

∴ eq. (1), (2), (3) are linearly dependent

consider (1), (2) and $v_1 + v_2 + v_3 = 1$

$$\left.\begin{aligned} -0.4v_1 + 0.4v_2 + 0.35v_3 &= 0 \\ 0.25v_1 - 0.65v_2 + 0.4v_3 &= 0 \\ v_1 + v_2 + v_3 &= 1 \end{aligned}\right\}$$

$\Rightarrow v_1 = 0.33 \quad v_2 = 0.33 \quad v_3 = 0.33$

2) If the transition probability matrix of market shares of 3 brands A, B, C is

$$P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}$$

and the initial market shares are 50%, 25%, 25%. Find

(i) The market share in 2nd and 3rd periods.

(ii) Find the limiting probabilities or study-state vector.

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix} \end{array}$$

Initial probabilities $\Pi = [0.5\ 0.25\ 0.25]$

the 2nd period probabilities

$$\Pi P = \underset{1\times 3}{[0.5\ 0.25\ 0.25]} \underset{3\times 3}{\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}}$$

$= [0.4875\ 0.2375\ 0.275]$

For 3rd period, $\Pi P^2 = (\Pi P)(P)$

$$= \underset{1\times 3}{[0.4875\ 0.2375\ 0.275]} \underset{3\times 3}{\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix}}$$

$= [0.4812\ 0.2387\ 0.28]$

(ii) let $[V_1\ V_2\ V_3]$ be the study-state vector then $VP = V$

$$\Rightarrow [V_1\ V_2\ V_3]\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.35 & 0.25 & 0.4 \end{bmatrix} = [V_1\ V_2\ V_3]$$

$\Rightarrow 0.4V_1 + 0.8V_2 + 0.35V_3 = V_1$

$= -0.6V_1 + 0.8V_2 + 0.35V_3 = 0$ ——— (1)

$\Rightarrow 0.3V_1 + 0.1V_2 + 0.25V_3 = V_2$

$= 0.3V_1 - 0.9V_2 + 0.25V_3 = 0$ ——— (2)

$\Rightarrow 0.3V_1 + 0.1V_2 + 0.4V_3 = V_3$

$= 0.3V_1 + 0.1V_2 - 0.6V_3 = 0$ ——— (3)

(1) + (2) = −(3)

∴ eq. (1), (2), (3) are linearly dependent

consider (1), (2) and $V_1 + V_2 + V_3 = 1$

$$\left.\begin{aligned} -0.6V_1 + 0.8V_2 + 0.35V_3 &= 0 \\ 0.3V_1 - 0.9V_2 + 0.25V_3 &= 0 \\ =1V_1 + 1V_2 + 1V_3 &= 1 \end{aligned}\right\}$$

$V_1 = \frac{103}{214}$ $V_2 = \frac{51}{214}$ $V_3 = \frac{30}{107}$

$$\left[\frac{103}{214}\ \ \frac{51}{214}\ \ \frac{30}{107}\right] = [V_1\ V_2\ V_3]$$

Definition of limiting probabilities

If $\{X_n / n \geq 0\}$ is a markov chain whose all states are positive recurrent and aperiodic, then the limiting probability $\lim_{n\to\infty} P_{ij}^{(n)} = V_j$ is independent of initial state i and is depending only on reaching state or destination state j.

1) Consider a markov chain with states phase $\{1, 2, 3\}$ and transition probability matrix $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

then which of the following probabilities are true?

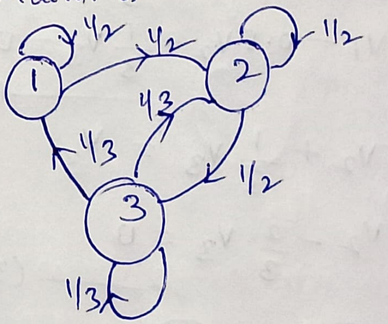
(1) $\lim_{n\to\infty} P_{11}^{(n)} = \frac{2}{9} = V_1$

(2) $\lim_{n\to\infty} P_{21}^{(n)} = 0 = V_1$

(3) $\lim_{n\to\infty} P_{32}^{(n)} = \frac{1}{3} = V_2$

(4) $\lim_{n\to\infty} P_{13}^{(n)} = \frac{1}{3} = V_3$

sol: The transition probability diagram of the given transition probability Matrix is



1) we can move from any state to another state from the graph

∴ The chain is irreducible.

2) The states 1, 2, 3 are return states.

3) The given chain is finite chain because it has 3 states.

∴ The chain is '+ve' recurrent (finite & irreducible)

periodic or aperiodic

state 1 can return in 1 step and 3 steps, - - -

$d_1 = \gcd\{1, 3 \ldots\} = 1$

$d_1 = 1 \Rightarrow$ state 1 is aperiodic

States 2, 3

States 2, 3 can return in 1 step, 2 steps, 3 steps ----.

$d_2 = \gcd\{1, 2, 3 - - -\} = 1$

$d_2 = 1$

$\Rightarrow$ states 2, 3 are aperiodic

Let $V = [v_1 \ v_2 \ v_3]$ be the study-state vector then $VP = V$

$$[v_1 \ v_2 \ v_3]\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} = [v_1 \ v_2 \ v_3]$$

$\Rightarrow 0.5v_1 + \frac{1}{3}v_3 = v_1$

$= -0.5v_1 + \frac{1}{3}v_3 = 0$ ——— (1)

$\Rightarrow 0.5v_1 + 0.5v_2 + \frac{1}{3}v_3 = v_2$

$= 0.5v_1 - 0.5v_2 + \frac{1}{3}v_3 = 0$ ——— (2)

$\Rightarrow 0.5v_2 + \frac{1}{3}v_3 = v_3$

$= 0.5v_2 - \frac{2}{3}v_3 = 0$ ——— (3)

(1) + (2) = −(3)

∴ eq. (1), (2), (3) are linearly dependent

consider (1), (2) and $v_1 + v_2 + v_3 = 1$

$$\left.\begin{aligned} -0.5v_1 + \tfrac{1}{3}v_3 &= 0 \\ 0.5v_1 - 0.5v_2 + \tfrac{1}{3}v_3 &= 0 \\ v_1 + v_2 + v_3 &= 1 \end{aligned}\right\}$$

$v_1 = \frac{2}{9}$ $\quad v_2 = \frac{4}{9}$ $\quad v_3 = \frac{1}{3}$

$[v_1 \ v_2 \ v_3] = \left[\frac{2}{9} \ \frac{4}{9} \ \frac{1}{3}\right]$

(1) $\lim_{n\to\infty} P_{11}^{(n)} = \frac{2}{9} = v_1$ True

(2) $\lim_{n\to\infty} P_{21}^{(n)} = 0 \neq v_1$ false

(3) $\lim_{n\to\infty} P_{32}^{(n)} = \frac{1}{3} \neq v_2$ false

(4) $\lim_{n\to\infty} P_{13}^{(n)} = \frac{1}{3} = v_3$ True

2) consider a markov chain with states phase $\{1, 2, 3\}$ and transition probability matrix $P = \begin{bmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$

then which of the following probabilities are true?

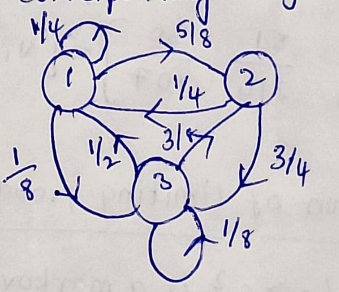
(1) $\lim_{n\to\infty} P_{12}^{(n)} = 0$

(2) $\lim_{n\to\infty} P_{12}^{(n)} = \lim_{n\to\infty} P_{21}^{(n)}$

(3) $\lim_{n\to\infty} P_{22}^{(n)} = \frac{1}{8}$

(4) $\lim_{n\to\infty} P_{21}^{(n)} = \frac{1}{3}$

Sol:- The transition diagram corresponding to given matrix is



1) We can move from any state to another state in finite no. of steps.

The given chain is irreducible

It has only 3 states

The chain is finite

2) Every irreducible and finite chain is positive recurrent.

$\therefore$ The chain is positive current.

All 3 states are return states.

3) periodic or aperiodic

state 1 : we can return state 1 in one step, 2 steps, 3 steps and so on.

$\therefore d_1 = \gcd\{1, 2, 3, \ldots\} = 1$

$\therefore$ state1 is aperiodic.

state 2 : we can return state 2 in 2 steps and so on.

$d_2 = \gcd\{2, 3, \ldots\} = 1$

$\therefore$ state2 is aperiodic.

state 3 : we can return state3 in 1 step, 2 steps, 3 steps and so on.

$d_3 = \gcd\{1, 2, 3, \ldots\} = 1$

$\therefore$ state 3 is aperiodic.

Let $V = [v_1\ v_2\ v_3]$ be the study-state vector then $vp = v$

$$[v_1\ v_2\ v_3]\begin{bmatrix} \frac{1}{4} & \frac{5}{8} & \frac{1}{8} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} = [v_1\ v_2\ v_3]$$

$\Rightarrow \frac{1}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 = v_1$

$= -\frac{3}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 = 0$ ——— (1)

$\Rightarrow \frac{5}{8}v_1 + \frac{3}{8}v_3 = v_2$

$= \frac{5}{8}v_1 - v_2 + \frac{3}{8}v_3 = 0$ ——— (2)

$\Rightarrow \frac{1}{8}v_1 + \frac{3}{4}v_2 + \frac{1}{8}v_3 = v_3$

$= \frac{1}{8}v_1 + \frac{3}{4}v_2 - \frac{7}{8}v_3 = 0$ ——— (3)

(1) + (2) = −(3)

$\therefore$ eq. (1), (2), (3) are linearly dependent

consider (1), (2) and $v_1 + v_2 + v_3 = 1$

$-\frac{3}{4}v_1 + \frac{1}{4}v_2 + \frac{1}{2}v_3 = 0$ ——— (1)

$\frac{5}{8}v_1 - v_2 + \frac{3}{8}v_3 = 0$ ——— (2)

$v_1 + v_2 + v_3 = 1$ ——— (4)

solving (1), (2), (4)

we get,

$v_1 = \frac{1}{3} \quad v_2 = \frac{1}{3} \quad v_3 = \frac{1}{3}$

$[v_1\ v_2\ v_3] = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$

(1) $\lim_{n\to\infty} P_{12}^{(n)} = 0 \neq v_2$ False

(2) $\lim_{n\to\infty} P_{12}^{(n)} = \lim_{n\to\infty} P_{21}^{(n)}$ True

$v_2 = v_1$

(3) $\lim_{n\to\infty} P_{22}^{(n)} = \frac{1}{8} \neq v_2$ False

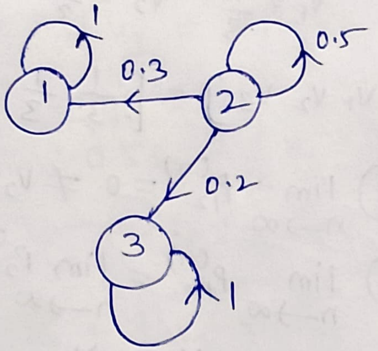
(4) $\lim_{n\to\infty} P_{21}^{(n)} = \frac{1}{3} = v_1$ True

3) Identify all absorbing states in markov chain having the following matrices. Decide whether the markov chain is absorbing or not?

①
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \\ 2 & 0.3 & 0.5 & 0.2 \\ 3 & 0 & 0 & 1 \end{array}$$

②
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & 0.6 & 0 & 0.4 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 0.9 & 0 & 0.1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

① Ans:

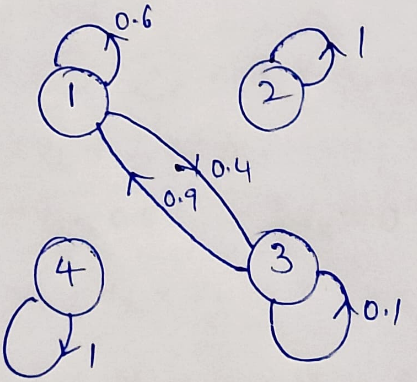


$P_{11} = 1 \quad P_{33} = 1$

It is possible to go from state 2 (Non absorbing state) to state 1, 3.

Markov chain is absorbing.

② Ans:



It is not possible to go from non-absorbing states 1, 3 to absorbing states 2, 4.

∴ The given chain is not an absorbing chain.

For an Absorbing chain

we know that $P_{ii} = 1$ then state i is an absorbing state.

A markov chain is an absorbing if the following 2 conditions are satisfied.

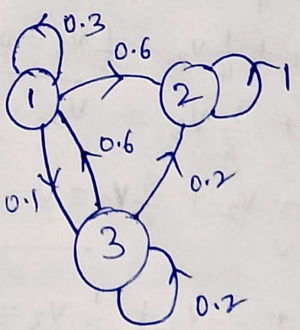
- The chain has atleast one absorbing state.
- If it is possible to go from any non-absorbing state to an absorbing state $P_{ii} = 1$ (Not necessarily in more than 1 step).

4) Identify all absorbing states in markov chain having the following matrices. Decide whether the markov chain is absorbing or not?

①
$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0.3 & 0.6 & 0.1 \\ 2 & 0 & 1 & 0 \\ 3 & 0.6 & 0.2 & 0.2 \end{array}$$

②
$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1/2 & 0 & 1/2 & 0 & 0 \\ 3 & 0 & 1/2 & 0 & 1/2 & 0 \\ 4 & 0 & 0 & 1/2 & 0 & 1/2 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{array}$$

①

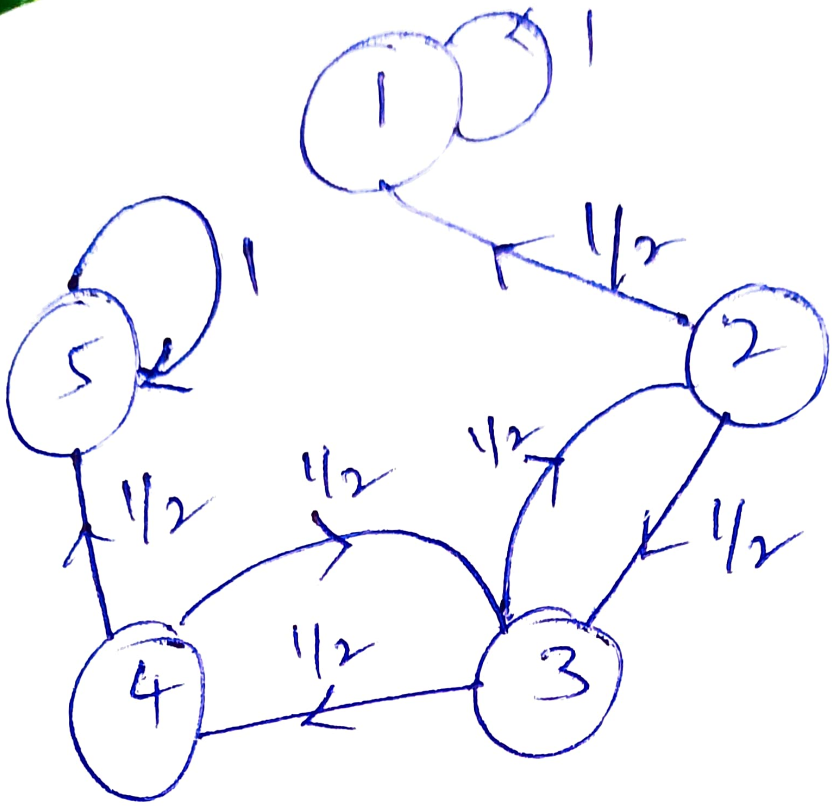


$P_{22} = 1$

It is possible to go from states 1, 3 (Non absorbing state) to state 2.

Markov chain is absorbing.

(2)



$P_{11} = 1 \quad P_{55} = 1$

It is possible to go from Non absorbing states to absorbing states.

Markov chain is absorbing.