CHAPTER - 4

DISCRETE PROBABILITY DISTRIBUTIONS

(Binomial, Geometric and Poisson Distributions)

4.1 INTRODUCTION

We have already learnt frequency distributions which are based on the actual observations or experiment. In this chapter, we shall discuss theoretical distributions in which variates are distributed as per some definite law which can be expressed mathematically. That is, theoretical distributions (or Probability distrbutions) are mathematical models. So, it is possible to deduce mathematically what the frequency distribution of certain population should be. Such distributions are called Theoretical distributions or Frequency Distributions. Hence a theoratical distribution is the frequency distribution of a certain event in which frequencies are obtained by mathematical computation.

The value of the random variables and the corresponding probabilities are arranged such a way that one can suit a mathematical function for the probabilities in terms of the value of the random variable. A good model requires sufficiently large number of observations.

Just, as in the case of actual frequency distribution, the various statistical measures such as mean, standard deviation etc., are computed, so also such statistical measures may easily be calculated from a theoretical distribution and they give a very close approximation to such statistical measures as calculated from the actual frequency distribution.

There are two types of theoretical distributions (or Probability distributions)

(*i*) **Discrete Theoretical Distributions**

(*a*) Binomial distribution (*b*) Poisson distribution

(*c*) Rectangular distribution (*d*) Negative Binomial distribution

(*e*) Geometric distribution

(*ii*) **Continuous Theoretical Distributions**

(*a*) Normal distribution (*b*) Student's t-distribution

(*c*) Chi-Square distribution (*d*) F-distribution

In this chapter, we shall study only the three distributions namely, Binomial, Geometric and Poisson distributions. Before we discuss these distributions, we first give a brief introduction of uniform and Bernoulli's distributions.

4.2 DISCRETE UNIFORM DISTRIBUTION

A random variable X has a discrete uniform distribution if and only if its probability distribution is given by

$$p(x) = \frac{1}{k}, \text{ for } x = x_1, x_2, \ldots, x_k.$$

The random variable X is then called discrete uniform random variable.

For example, the distributions

x	0	1
$p(x)$	$\frac{1}{2}$	$\frac{1}{2}$

and

x	0	1	2	3
$p(x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

are discrete uniform distributions.

4.3 BERNOULLI'S DISTRIBUTION

A random variable X which takes two values 0 and 1 with probability q and p respectively *i.e.*, $P(X = 0) = q$ and $P(X = 1) = p$, $q = 1 - p$ is called a *Bernoulli's discrete random variable* and is said to have a *Bernoulli's distribution*. The probability function of Bernoulli's distribution can be written as

$$p(x) = p^x q^{1-x} = p^x (1-p)^{1-x}, x = 0, 1.$$

Note : (*i*) Mean of the Bernoulli's discrete random variable X is

$$\mu = E(X) = \Sigma x_i \cdot p(x_i) = 0 \times q + 1 \times p = p$$

(*ii*) Variance of X is

$$V(X) = E(X^2) - [E(X)]^2 = \Sigma x_i^2 \cdot p(x_i) - \mu^2 = (0^2 \times q + 1^2 \times p) - p^2$$
$$= p - p^2 = p(1-p) = pq$$

Hence the standard deviation is $\sigma = \sqrt{pq}$.

Definition : Suppose, associated with a random trial there is an event called 'success' and the complementary event called 'failure'. Let the probability for success be p and probability for failure be q. Suppose the random trials are prepared n times under identical conditions. These are called Bernoullian trials.

Bernoulli's Theorem :

If the probability of the occurrence of an event (success) in a single trial is p, then the probability that it will occur exactly r times out of n independent Bernoullian trials is

$${}^nC_r\, p^r q^{n-r} \text{ where } p + q = 1$$

Proof : In a series of n independent trials, the number of r successes and $(n - r)$ failures can happen in ${}^nC_r$ mutually exclusive and exhaustive ways, since that is equal to the number of combinations of n things taken r at a time. Now consider a particular set of r successes and $(n - r)$ failures. Let the probability of success in a single trial be p. Then the probability of its failure is $q\ (= 1 - p)$ and that of success in all the r trials is p^r, by the theorem of compound probability. Also, by the same theorem, the probability of failure in all the $(n - r)$ trials is q^{n-r}. Hence by the multiplication theorem, the probability of getting r successes and $(n - r)$ failures in the particular set considered is equal to $p^r q^{n-r}$. Thus the probability of r successes in n trials with probability p in each trial is ${}^nC_r\, p^r q^{n-r}$.

Note : 1. By Binomial Theorem,

$$(q + p)^n = q^n + {}^nC_1\, q^{n-1} p + {}^nC_2\, q^{n-2} p^2 + \ldots + {}^nC_r\, q^{n-r} p^r + \ldots + p^n.$$

We notice that the probabilities of getting 0, 1, 2, ..., n successes are given respectively by the successive terms of the above binomial expansion. The probability of exactly r successes and $n - r$ failures is ${}^nC_r\, p^r q^{n-r}$ and it is the $(r + 1)^{th}$ term in the above binomial expansion.

2. The probability of getting at least one success in a series of n trials is

$$= {}^nC_1\, q^{n-1} p + {}^nC_2\, q^{n-2} p^2 + \ldots + p^n$$
$$= (q + p)^n - q^n = 1 - q^n$$

3. The probability of no success in n *Bernoullian* trials is given by q^n, and that of all successes by p^n.

BINOMIAL DISTRIBUTION

4.4 BINOMIAL DISTRIBUTION

Binomial distribution was discovered by James Bernoulli in the year 1700 and it is a discrete probability distribution.

Let us visualise a conceptual or practical situation where a trial or an experiment results in only two outcomes, say 'success' and 'Failure'. Further, the result of one trial does not influence the result of next trial, and the probability of success at each trial is the same from trial to trial.

Some of such situations are:

1. Tossing a coin - Head or Tail
2. Birth of a baby - Girl or Boy
3. Auditing a Bill - contains an error or not
4. An advertisement on TV - Recalled by viewer or not.

The conditions for the applicability of a Binomial distribution are as follows:

(*i*) There are n independent trials

(*ii*) Each trial has only two possible outcomes

(*iii*) The probabilities of two outcomes remain constant.

The Binomial distribution can be described with the help of a function and is derived as follows:

Let the number of trials be n. The trials be independent *i.e.*, the success or failure at one trial doesnot affect the outcome of the other trials. Thus the probability of success remains the same from trial to trial. Also let 'p' be the probability of success and 'q' be the probability of failure. Then we have $p + q = 1$ or $q = 1 - p$.

One could be interested in finding the probability of getting r successes, which implies getting $(n - r)$ failures.

One of the ways in which one can get r successes and $(n-r)$ failure is to get r continuous successes first and $(n-r)$ successive failures.

The probability of getting such a sequence of r successes and $(n-r)$ failures is,

$= p^r q^{n-r}$ (using Multiplication Theorem of Probability)

However, the number of ways in which one can get r successes and $n-r$ failures is ${}^nC_r$ and the probability for each of these ways is $p^r q^{n-r}$.

Thus, if x is the random variable representing the number of successes, the probability of getting r successes and $n-r$ failures, in n trials, is given by the probability function

$$P(x = r) = {}^nC_r\, p^r q^{n-r},\ r = 0, 1, 2, ..., n$$

This probability function is popularly known as the Binomial Distribution.

The probabilities of 0, 1, 2, ..., r, ..., n successes are therefore given by

$$q^n,\ {}^nC_1\, p\, q^{n-1},\ {}^nC_2\, p^2 q^{n-2}, ...,\ {}^nC_r\, p^r q^{n-r}, ...,\ p^n.$$

The probability of the number of successes so obtained is called the *Binomial Probability Distribution*, because these probabilities are the successive terms in the expansion of the binomial $(q + p)^n$. This distribution contains two independent constants namely n and p (or q). They are called parameters of the Binomial Distribution. Sometimes, n is also known as the degree of the distribution.

Definition : A random variable X has a Binomial distribution if it assumes only non-negative values and its probability density function is given by

$$P(X = r) = p(r) = \begin{cases} {}^nC_r p^r q^{n-r};\ r = 0,1,2,....,n;\ q = 1-p \\ 0, \text{otherwise} \end{cases}$$

[JNTU (K) Dec. 2013 (Set No. 1), (H) Dec. 2014]

Note: 1. An alternate notation for binomial distribution is as follows :

$$P(X = x) = p(x) = b(x; n,p) = \begin{cases} {}^nC_x\, p^x q^{n-x};\ x = 0,1,2,....,n \\ 0, \text{otherwise} \end{cases}$$

2. The Binomial distribution function is given by

$$F_X(x) = P(X \le x) = \sum_{r=0}^{n} {}^nC_r p^r q^{n-r}$$

Examples of Binomial Distribution :

(*i*) The number of defective bolts in a box containing 'n' bolts.

(*ii*) The number of machines lying idle in a factory having 'n' machines.

(*iii*) The number of post-graduates in a group of 'n' men.

(*iv*) The number of oil wells yielding natural gas in a group of 'n' wells test drilled.

Conditions of Binomial Distribution :

The Binomial Distribution holds under the following conditions :

1. Trials are repeated under identical conditions for a fixed number of times, say n times.
2. There are only two possible outcomes, *e.g.* success or failure for each trial.
3. The probability of success in each trial remains constant and does not change from trial to trial.
4. The trials are independent *i.e.*, the probability of an event in any trial is not affected by the results of any other trial.

4.5 CONSTANTS OF BINOMIAL DISTRIBUTION

1. Mean of the Binomial Distribution :

[JNTU 2000, 2002, (K) Dec. 2013, Mar. 2014 (Set No. 4), (H) Dec. 2014]

The Binomial probability distribution is given by

$$p(r) = {}^nC_r\, p^r q^{n-r},\ r = 0, 1, 2, ..., n \text{ and } q = 1 - p$$

Mean of X, $\mu = E(X) = \sum_{r=0}^{n} r\, p(r)$

$$= 0 \times q^n + 1 \times {}^nC_1\, p\, q^{n-1} + 2 \times {}^nC_2\, p^2 q^{n-2} + ... + n\,.p^n$$

$$= np\, q^{n-1} + 2\,.\,\frac{n(n-1)}{2!}\, p^2 q^{n-2} + 3\,.\,\frac{n(n-1)(n-2)}{3!}\, p^3 q^{n-3} + ... + np^n$$

$$= np\left[q^{n-1} + (n-1)p\,q^{n-2} + \frac{(n-1)(n-2)}{2!}p^2q^{n-3} + ... + p^{n-1}\right]$$

$$= np\,(q+p)^{n-1} \quad \text{[using Binomial Theorem]}$$

$$= np\,(1) = np \quad [\because p + q = 1]$$

Hence the Arithmetic mean of the Binomial distribution = np.

2. Variance of the Binomial Distribution :

[JNTU 2000, 2002, (K) Dec. 2013, Mar. 2014 (Set No. 4), (H) Dec. 2014]

Variance, $V(X) = E(X^2) - [E(X)]^2$

$$= \sum_{r=0}^{n} r^2 p(r) - \mu^2 = \sum_{r=0}^{n} [r(r-1)+r].p(r) - \mu^2$$

$$= \sum_{r=0}^{n} r(r-1)\,{}^nC_r\, p^r q^{n-r} + \sum_{r=0}^{n} r\, p(r) - \mu^2$$

$$= \left[2.{}^nC_2 p^2 q^{n-2} + 3.2.{}^nC_3 p^3 q^{n-3} + ... + n(n-1)\,p^n\right] + \mu - \mu^2$$

$$= \left[2.\frac{n(n-1)}{2!}p^2q^{n-2} + 6.\frac{n(n-1)(n-2)}{3!}p^3q^{n-3} + ... + n(n-1)\,p^n\right] + \mu - \mu^2$$

$$= n(n-1)p^2\left[q^{n-2} + (n-2)p\,q^{n-3} + \frac{(n-2)(n-3)}{2!}p^2q^{n-4} + ... + p^{n-2}\right] + \mu - \mu^2$$

$$\therefore V(X) = n(n-1)p^2(q+p)^{n-2} + \mu - \mu^2 \quad \text{(by Binomial Theorem)}$$
$$= n(n-1)p^2(1)^{n-2} + \mu - \mu^2 \quad (\text{since } q+p=1)$$
$$= n(n-1)p^2 + np - (np)^2 \quad (\because \mu = np)$$
$$= n^2p^2 - np^2 + np - n^2p^2 = -np^2 + np = np(1-p) = npq$$

$\therefore$ Variance of the Binomial Distribution $= npq$.

Hence the Standard Deviation of the Binomial Distribution $= \sqrt{npq}$.

3. **Mode of the Binomial Distribution :**

Mode of the binomial distribution is the value of x at which $p(x)$ has maximum value.

$$\text{Mode} = \begin{cases} \text{integral part of } (n+1)p, \text{ if } (n+1)p \text{ is not an integer} \\ (n+1)p \text{ and } (n+1)p-1, \text{ if } (n+1)p \text{ is an integer} \end{cases}$$

Recurrence Relation for the Binomial Distribution :

We know that $P(r) = {}^nC_r p^r q^{n-r}$ ... (1)

$\therefore P(r+1) = {}^nC_{r+1} p^{r+1} q^{n-r-1}$ ... (2)

(2) ÷ (1) gives

$$\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r} \cdot \frac{p^{r+1}q^{n-r-1}}{p^r q^{n-r}} = \frac{n-r}{r+1} \cdot \frac{p}{q}$$

or $P(r+1) = \dfrac{(n-r)\,p}{(r+1)\,q} \cdot P(r)$

4.6 BINOMIAL FREQUENCY DISTRIBUTION

If n independent trials constitute one experiment and this experiment is repeated N times, then the frequency of r successes is $N.\ {}^nC_r\ p^r\ q^{n-r}$. Since the probabilities of 0, 1, 2, ..., r, ..., n successes in n trials are given by the terms of the binomial expansion of $(q+p)^n$, therefore in N sets of n trials the theoretical frequencies of 0, 1, 2, ..., r, ..., n successes will be given by the terms of expansion of $N(q+p)^n$.The possible number of successes and their frequencies is called a Binomial Frequency Distribution.

SOLVED EXAMPLES

Example 1 : A fair coin is tossed six times. Find the probability of getting four heads.

Solution : p = Probability of getting a head = $1/2$

q = Probability of not getting head = $1/2$

and $n = 6, r = 4$

We know that $P(r) = {}^nC_r\ p^r\ q^{n-r}$

$$\therefore P(4) = {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{6-4} = \frac{6!}{4!\,2!} \cdot \left(\frac{1}{2}\right)^6 = \frac{6\times 5}{2} \cdot \frac{1}{2^6} = \frac{15}{64} = 0.2344\,.$$

Example 2 : Determine the probability of getting the sum 6 exactly 3 times in 7 throws with a pair of fair dice.

Solution : In a single throw of a pair of fair dice, a sum of 6 can occur in 5 ways : (1, 5), (5, 1), (2, 4), (4, 2) and (3, 3) out of $6 \times 6 = 36$ ways.

Thus p = Probability of occurrence of 6 in one throw $= \dfrac{5}{36}$

$$q = 1 - p = 1 - \frac{5}{36} = \frac{31}{36}$$

n = Number of trials = 7

$\therefore$ Probability of getting 6 exactly thrice in 7 throws

$$= {}^7C_3 p^3 q^{7-3} = {}^7C_3 \left(\frac{5}{36}\right)^3 \left(\frac{31}{36}\right)^4 = \frac{35(125)(31)^4}{(36)^7} = 0.0516 \text{ (nearly)}.$$

Example 3 : A die is thrown 6 times. If getting an even number is a success, find the probabilities of (*i*) at least one success (*ii*) ≤ 3 successes (*iii*) 4 successes.

Solution : In a single throw of a die, an even number can occur in 3 ways out of 6 ways. Thus

p = Probability of occurrence of an even number in one throw $= \dfrac{3}{6} = \dfrac{1}{2}$

n = Number of trials = 6

(*i*) $P(r \geq 1) = 1 - P(r = 0)$

$$= 1 - \left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64} = 0.9844$$

(*ii*) $P(r \leq 3) = P(r=0) + P(r=1) + P(r=2) + P(r=3)$

$$= \left(\frac{1}{2}\right)^6 ({}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3) = \frac{21}{32} = 0.6562$$

(*iii*) $P(r=4) = {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^2 = \frac{15}{64} = 0.2344$

Example 4 : Ten coins are thrown simultaneously. Find the probability of getting at least (*i*) seven heads **[JNTU 1999, 2007, 2008S, (K) May 2013 (Set No. 2),(H) III yr. Nov. 2015]**

(*ii*) six heads **[JNTU (A) Dec, 2009, (K) May 2013 (Set No. 2)]**

(*iii*) one head **[JNTU (H) III yr. Nov. 2015]**

Solution : p = Probability of getting a head = $1/2$

q = Probability of not getting a head = $1/2$

The probability of getting r heads in a throw of 10 coins is

$$P(X = r) = p(r) = {}^{10}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r} \ ; r = 0, 1, 2, \ldots, 10$$

(*i*) Probability of getting at least seven heads is given by

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= {}^{10}C_7\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{10-r}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 + {}^{10}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 + {}^{10}C_9\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right) + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}\left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right] = \frac{1}{2^{10}}(120 + 45 + 10 + 1)$$

$$= \frac{176}{1024} = 0.1719$$

(*ii*) This is left as an exercise to the student.

(*iii*) P (at least 1 head) $= p(r \geq 1) = 1 - p(r = 0)$

$$= 1 - {}^{10}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{2^{10}}$$

Example 5 : Two dice are thrown five times. Find the probability of getting 7 as sum (*i*) at least once (*ii*) exactly two times *iii*) $P(1 < X < 5)$. **[JNTU (H) Nov. 2009, May 2017]**

Solution : p = The probability of getting a sum 7 in a single throw of a pair of dice

$$= \frac{6}{36} = \frac{1}{6}$$

q = The probability of not getting a sum 7 in a throw of a pair of dice $= 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

Number of trials, $n = 5$.

(*i*) P (at least once) $= P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^5C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 = 1 - \left(\frac{5}{6}\right)^5$$

(*ii*) P (exactly two times) $= P(X = 2) = {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4$

$$= 10 \cdot \frac{5^4}{6^6} = \frac{2}{6}\cdot\left(\frac{5}{6}\right)^5 = \frac{1}{3}\cdot\left(\frac{5}{6}\right)^5$$

(*iii*) $P(1 < X < 5) = P(X = 2) + P(X = 3) + P(X = 4)$

$$= {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 + {}^5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 + {}^5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2$$

$$= 10\cdot\frac{5^4}{6^6} + 10\cdot\frac{5^3}{6^6} + 5\cdot\frac{5^2}{6^6} = \frac{5^3}{6^6}(50 + 10 + 1) = 61\left(\frac{5^3}{6^6}\right)$$

Example 6 : It has been claimed that in 60% of all solar heat installations the utility bill is reduced by at least one-third. Accordingly, what are the probabilities that the utility bill will be reduced by at least one-third in (*i*) four of five installations (*ii*) at least four of five installations. **[JNTU 1999S, (A) Apr. 2012 (Set No. 4)]**

Solution : p = The probability that in the solar heat installations the utility bill is reduced by one-third

$= 60\% = 0.6$

$q = 1 - p = 1 - 0.6 = 0.4$

(*i*) $r = 4, n = 5$

We have $P(X = r) = {}^nC_r\, p^r q^{n-r}$

$$\therefore P(X = 4) = {}^5C_4(0.6)^4(0.4)^{5-4} = 5(0.6)^4(0.4) = 0.2592$$

(*ii*) $P(X \geq 4) = P(X = 4) + P(X = 5) = 0.2592 + {}^5C_5(0.6)^5$ [From (*i*)]

$= 0.2592 + 0.07776 = 0.337$

Example 7 : If 3 of 20 tyres are defective and 4 of them are randomly chosen for inspection, what is the probability that only one of the defective tyre will be included ? **[JNTU 2000S]**

Solution : Let p = Probability of a defective tyre (success) $= \frac{3}{20}$

Then q = Probability of a non-defective tyre

$$= 1 - p = 1 - \frac{3}{20} = \frac{17}{20}$$

Given $n = 4$

The probability that exactly one tyre will be defective

$$= P(r = 1) = {}^4C_1\left(\frac{3}{20}\right)^1\left(\frac{17}{20}\right)^{4-1} = \frac{4.3.17^3}{(20)^4} = 0.3685$$

Example 8 : The incidence of an occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of 6 workers chosen at random, four or more will suffer from the disease ? **[JNTU 2001 S, (K) Nov. 2011 (Set No. 4)]**

Solution : The probability of a worker suffering from disease, $p = 20\% = 0.2$

$\therefore$ The probability that no worker suffering from disease, $q = 1 - p = 0.8$

No. of workers, $n = 6$

The probability that four or more workers will suffer from disease

$$= P(X \geq 4)$$

$$= P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^6C_4(0.2)^4(0.8)^2 + {}^6C_5(0.2)^5(0.8) + {}^6C_6(0.2)^6$$

$$= (0.2)^4[15(0.8)^2 + 6(0.2)(0.8) + (0.2)^2]$$

$$= (0.2)^4\ [9.6 + 0.96 + 0.4] = 0.0175$$

Example 9 : Determine the binomial distribution for which the mean is 4 and variance 3 **[JNTU 1999, (H) Sept. 2017]**

Solution : Given mean of the distribution = 4 *i.e.*, $np = 4$ ... (1)

and variance of the distribution = 3 *i.e.*, $npq = 3$ ... (2)

(2) ÷ (1) gives $q = 3/4$

$$\therefore p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

From (1), $n = \frac{4}{p} = 16$

$\therefore$ The given Binomial Distribution has parameters $n = 16$ and $p = 1/4$.

Hence the Binomial distribution is

$$P(X = x) = p(x) = \begin{cases} {}^{16}C_4 \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{16-r}, & r = 0,1,2,\ldots,n \\ 0, \text{otherwise} \end{cases}$$

Note : To find mode of the distribution, we have $(n+1)\,p = \frac{17}{4} = 4.25$, which is not an integer.

Hence the unique mode of the distribution = integral part of $(n+1)p$ *i.e.*, $4.25 = 4$

Example 10 : If the probability of a defective bolt is $1/8$, find :

(*i*) The mean (*ii*) The variance for the distribution of defective bolts of 640.

[JNTU (H) Nov. 2009 (Set No. 4)]

Solution : We are given

p = The probability of a defective bolt = $1/8$ and $n = 640$.

$\therefore$ Mean of the distribution, $\mu = np = \frac{640}{8} = 80$

Also $q = 1 - p = 1 - \frac{1}{8} = \frac{7}{8}$.

Hence variance of the distribution = $npq = (np)q = \mu q = 80 \times \frac{7}{8} = 70$

Example 11 : In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails. **[JNTU 2003, 2004 (Set No. 4)]**

Solution : The probability of getting a head, $p = 1/2$

$\therefore$ The probability of getting a tail, $q = 1/2$.

Here $n = 12$

The probability of getting 8 heads and 4 tails in 12 trails

$$= P(X = 8) = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4$$

$$= \frac{12!}{8!\,4!} \cdot \left(\frac{1}{2}\right)^{12} = \frac{495}{(2)^{12}}$$

$\therefore$ The expected number of such cases in 256 sets

$$= 256 \times P(X = 8) = 2^8 \times \frac{495}{2^{12}} = \frac{495}{16} = 30.9375 \sim 31$$

Example 12 : If the probability of success is $1/20$, how many trials are necessary in order that the probability of at least one success is just greater than $1/2$.

Solution : Here $p = \frac{1}{20}$. So $q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

Let n be the required number of trails so that the probability of at least one success is $1 - q^n = 1 - \left(\frac{19}{20}\right)^n$ $\quad [\because p(r \geq 1) = 1 - p(r = 0) = 1 - {}^nC_0 p^0 q^{n-0} = 1 - q^n]$

[Also Refer Note (2) on page 135]

Suppose $1 - \left(\frac{19}{20}\right)^n > \frac{1}{2}$ *i.e.*, $\left(\frac{19}{20}\right)^n < \frac{1}{2}$

Take $\left(\frac{19}{20}\right)^n = \frac{1}{2} \Rightarrow n \log\left(\frac{19}{20}\right) = \log\left(\frac{1}{2}\right)$

$\Rightarrow n = \frac{-0.6931}{-0.0513} = 13.51$

We have to take $n > 13.51$ $\quad \therefore n = 14$

Example 13 : Six dice are thrown 729 times. How many times do you except at least three dice to show a 5 or 6 ?

Solution : p = Probability of occurrence of 5 or 6 in one throw = $\frac{2}{6} = \frac{1}{3}$

$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$ and $n = 6$

The probability of getting at least three dice to show a 5 or 6 = $P(X \geq 3)$

$$= P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + {}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + {}^6C_6\left(\frac{1}{3}\right)^6$$

$$= \frac{1}{(3)^6}[160 + 60 + 12 + 1] = \frac{233}{729}$$

$\therefore$ The expected number of such cases in 729 times = $729\left(\frac{233}{729}\right) = 233$

Example 14 : Two dice are thrown 120 times. Find the average number of times in which, the number on the first die exceeds the number on the second die?

[JNTU (K) 2009 (Set No. 2)]

Solution : If a pair of dice is tossed then S = {1,2,3,4,5,6} X {1,2,3,4,5,6} and $n(S) = 36$. Let X be the random variable. The favourable cases are (2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5).

$\therefore p$ = probability of occurrence with the number on first dice exceeds the number on the second dice in one throw

$$= \frac{15}{36} = \frac{5}{12}$$

Hence $E(X)$ = mean = $np = 120\left(\frac{5}{12}\right) = 50$.

Example 15 : If the probability that a man aged 60 will live to be 70 is 0.65, what is the probability that out of 10 men, now 60, at least 7 will live to be 70 ?

Solution : p = The probability that a man aged 60 will live to be 70 = 0.65

$\therefore q = 1 - p = 1 - 0.65 = 0.35$

n = No. of men = 10

$\therefore$ Required probability $= P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$= {}^{10}C_7(0.65)^7(0.35)^3 + {}^{10}C_8(0.65)^8(0.35)^2 + {}^{10}C_9(0.65)^9(0.35) + (0.65)^{10}$

$= (0.65)^7[120.(0.35)^3 + 45(0.35)^2(0.65) + 10\times(0.35)(0.65)^2 + (0.65)^3]$

$= (0.65)^7\,[5.145 + 3.583 + 1.479 + 0.275]$

$= (0.65)^7\,(10.482) = 0.5139$

Example 16 : If the probability of a defective bolt is 0.2, find (*i*) mean (*ii*) standard deviation for the distribution of bolts in a total of 400.

Solution : Given $n = 400, p = 0.2. \therefore q = 1 - p = 1 - 0.2 = 0.8$

(*i*) Mean $= np = 400\ (0.2) = 80$ ... (1)

(*ii*) S.D. $= \sqrt{npq} = \sqrt{80(0.8)} = \sqrt{64} = 8$ [by (1)]

Example 17 : A die is tossed thrice. A success is getting 1 or 6 on a toss. Find the mean and variance of the number of successes.

Solution : The probability of success, $p = \frac{2}{6} = \frac{1}{3}$

$\therefore$ The probability of failure, $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

No. of trials, $n = 3$

Mean $= np = 3\left(\frac{1}{3}\right) = 1$

Variance $= npq = (np)\,q = (1)\,q = q = \frac{2}{3}$

Example 18 : If 10% of the rivets produced by a machine are defective, find the probability that out of 5 rivets chosen at random (*i*) none will be defective (*ii*) one will be defective, and (*iii*) at most two rivets will be defective.

Solution : Probability of defective rivets $= p = 10\% = 0.1$

Probability of non defective rivets $= q = 1 - p = 1 - 0.1 = 0.9$

Total number of rivets $= n = 5$

(*i*) Probability that none is defective = Probability of 0 defective bolt

$= P(0) = {}^5C_0\ (0.1)^0\ (0.9)^5 = (0.9)^5$

$= 0.5905$

(*ii*) Probability of 1 defective rivet $= P(1) = {}^5C_1\ (0.1)^1\ (0.9)^4 = 0.32805$

(*iii*) Probability of at most 2 defective $= P(0) + P(1) + P(2)$

$= 0.5905 + 0.32805 + {}^5C_2\ (0.1)^2\ (0.9)^3$

$= 0.5905 + 0.32805 + 0.0729$

$= 0.99145$

Example 19 : (*i*) Out of 800 families with 5 children each, how many would you expect to have (*a*) 3 boys (*b*) 5 girls (*c*) either 2 or 3 boys (*d*) atleast one boy ? Assume equal probabilities for boys and girls. **[JNTU 2008, (K) Dec. 2009, (A) Nov. 2010, (H) Nov. 2013]**

(*ii*) Out of 800 families with 4 children each, how many families would be expected to have (*a*) 2 boys and 2 girls (*b*) atleast one boy (*c*) no girl (*d*) at most two girls ? Assume equal probabilities for boys and girls ? **[JNTU (K) Nov. 2012 (Set No. 2)]**

Solution : **(*i*)** Let the number of boys in each family $= x$

p = The probability of each boy $= \frac{1}{2}$ (since equal probability for boys and girls)

Number of children, $n = 5$

The probability distribution is

$p\,(r) = {}^nC_r\ p^r\ q^{n-r}$

$= {}^5C_r\ \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r}$

$= \frac{1}{2^5}\ .\ {}^5C_r$ per family

(*a*) P (3 boys) $= P(r = 3) = p(3) = \frac{1}{2^5}\ .\ {}^5C_3 = \frac{10}{32} = \frac{5}{16}$ per family

Thus for 800 families the probability of number of families having 3 boys

$= \frac{5}{16}(800) = 250$ families

(*b*) P (5 girls) = P(no boys) $= P(r = 0) = p(0) = \frac{1}{2^5}\ .\ {}^5C_0 = \frac{1}{32}$ per family

Thus for 800 families, the probability of number of families having 5 girls

$= \frac{1}{32}(800) = 25$ families

(*c*) P (either 2 or 3 boys) $= P\,(r = 2) + P\,(r = 3) = p(2) + p(3)$

$= \frac{1}{2^5}\ .\ {}^5C_2 + \frac{1}{2^5}\ .\ {}^5C_3 = \frac{1}{2^5}(10 + 10)$

$= \frac{20}{32} = \frac{5}{8}$ per family

$\therefore$ Expected number of families with 2 or 3 boys $= \frac{5}{8}(800) = 500$ families

(*d*) P (at least one boy) $= P(r=1)+P(r=2)+P(r=3)+P(r=4)+P(r=5)$

$$= \frac{1}{2^5}({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5) = \frac{1}{2^5}(5+10+10+5+1) = \frac{31}{32}$$

$\therefore$ Expected number of families with atleast one boy $= \frac{31}{32}(800) = 775$

(*ii*) This is left as an exercise to the reader.

Example 20 : The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$. **[JNTU 2004, (A) Dec. 2009 (Set No. 3)]**

Solution : Mean of the binomial distribution = 4 *i.e.,* $np = 4$ ... (1)

and variance of the binomial distribution $= \frac{4}{3}$ *i.e.,* $npq = \frac{4}{3}$ ... (2)

(2) ÷ (1) gives $q = \frac{1}{3}$ $\therefore p = 1-q = 1-\frac{1}{3} = \frac{2}{3}$

From (1), $n = \frac{4}{p} = 4\left(\frac{3}{2}\right) = 6$

$\therefore P(X \geq 1) = 1 - P(X < 1)$

$$= 1 - P(X=0) = 1 - {}^6C_0\, p^0 q^{6-0}$$

$$= 1 - 1.\left(\frac{2}{3}\right)^0.\left(\frac{1}{3}\right)^6 = 1 - \left(\frac{1}{3}\right)^6 = 0.9986$$

Example 21 : A discrete random variable X has the mean 6 and variance 2. If it is assumed that the distribution is binomial find the probability that $5 \leq x \leq 7$. **[JNTU 2008 (Set No. 4)]**

Solution : We are given

The mean of the Binomial distribution = 6

i.e., $np = 6$ .... (1)

The variance of the Binomial distribution = 2

i.e., $npq = 2$ ..... (2)

$\Rightarrow q = \frac{2}{np} = \frac{2}{6}$, using (1) $\Rightarrow q = \frac{1}{3}$

$\therefore p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

From (1), $n = \frac{6}{p} = 6 \times \frac{3}{2} = 9$

$\therefore$ The Binomial distribution is $\left(\frac{2}{3}+\frac{1}{3}\right)^9$

Now $P(5 \leq X \leq 7) = P(5) + P(6) + P(7)$

$$= {}^9C_5 p^5 q^4 + {}^9C_6 p^6 q^3 + {}^9C_7 p^7 q^2$$

$$= \frac{9!}{5!\,4!}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^4 + \frac{9!}{6!\,3!}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^3 + \frac{9!}{2!\,7!}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^2$$

$$= 126\left(\frac{2^5}{3^9}\right) + 84\left(\frac{2^6}{3^9}\right) + 36\left(\frac{2^7}{3^9}\right)$$

$$= \frac{32}{3^9}(126 + 168 + 144) = \frac{14016}{3^9} = 0.7121$$

Example 22 : The mean and variance of a binomial distribution are 2 and 8/5. Find n. **[JNTU (H) III yr. Nov. 2015]**

Solution : We are given

Mean of the Binomial distribution = 2 *i.e.* $np = 2$ ... (1)

Variance of the Binomial distribution $= \frac{8}{5}$ *i.e.* $npq = \frac{8}{5}$ ... (2)

(2) ÷ (1) gives $q = \frac{8/5}{2} = \frac{4}{5}$

$\therefore p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$

From (1), $n = \frac{2}{p} = \frac{2}{1/5} = 10$

Example 23 : The mean and variance of a binomial variable X with parameters n and p are 16 and 8. Find $P(X \geq 1)$ and $P(X > 2)$. **[JNTU 2004 (Set No. 3)]**

Solution : The mean of the binomial distribution = 16 *i.e.,* $np = 16$ ... (1)

The variance of the binomial distribution = 8 *i.e.,* $npq = 8$ ... (2)

(2) ÷ (1) gives $\frac{npq}{np} = \frac{8}{16} \Rightarrow q = \frac{1}{2}$

$\therefore p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$

From (1), $n = \frac{16}{p} = \frac{16}{1/2} = 32$

Hence $P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0)$

$$= 1 - {}^{32}C_0\, p^0 q^{32} = 1 - {}^{32}C_0\left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^{32}$$

$$= 1 - 1 \,.\left(\frac{1}{2}\right)^{32} = 1 - \frac{1}{2^{32}}$$

and $P(X > 2) = 1 - P(X \le 2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$

$$= 1 - \left[{}^{32}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{32} + {}^{32}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{31} + {}^{32}C_2 \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^{30} \right]$$

$$= 1 - \left(\frac{1}{2}\right)^{32} \left[{}^{32}C_0 + {}^{32}C_1 + {}^{32}C_2 \right]$$

$$= 1 - \left(\frac{1}{2}\right)^{32} [1 + 32 + 496] = 0.9999$$

Example 24 : In eight throws of a die 5 or 6 is considered a success. Find the mean number of success and the standard deviation. **[JNTU 2004 (Set No. 3)]**

Solution : p = The probability of success $= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

q = The probability of failure $= 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

n = Number of throws = 8

$\therefore$ Mean $= np = 8\left(\frac{1}{3}\right) = \frac{8}{3}$

Variance $= npq = (np)\,q = \left(\frac{8}{3}\right)\left(\frac{2}{3}\right) = \frac{16}{9}$

Hence standard deviation $= \sqrt{\text{variance}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$

Example 25 : In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 success are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution. **[JNTU 2004 (Set No. 4)]**

Solution : Given n = Number of trials = 5

and $P(X = 1) = 0.4096$, $P(X = 2) = 0.2048$

$$\therefore \frac{P(X=1)}{P(X=2)} = \frac{0.4096}{0.2048} \quad i.e., \quad \frac{{}^5C_1\, pq^4}{{}^5C_2\, p^2 q^3} = \frac{0.4096}{0.2048}$$

$$i.e., \frac{5q}{10p} = \frac{0.4096}{0.2048} = 2 \Rightarrow \frac{q}{2p} = 2 \Rightarrow q = 4p \Rightarrow 1 - p = 4p \Rightarrow 5p = 1$$

$$\therefore p = \frac{1}{5} = 0.2$$

Note : $q = 1 - p = 1 - 0.2 = 0.8$

Example 26 : 20% of items produced from a factory are defective. Find the probability that in a sample of 5 chosen at random

(*i*) none is defective (*ii*) one is defective (*iii*) $p\,(1 < x < 4)$

[JNTU 2005, 2005S, 2007, (H) Nov. 2009, (A) Nov. 2010 (Set No. 4)]

Solution : Probability of defective items $= p = 20\% = 0.2$

Probability of non defective items $= q = 1 - p = 1 - 0.2 = 0.8$

Total number of items, $n = 5$

(*i*) Probability that none is defective = Probability of 0 defective item

$$= p(0) = {}^5C_0\ (0.2)^0\ (0.8)^5 = (0.8)^5 = 0.32768$$

(*ii*) Probability of 1 defective item $= p(1) = {}^5C_1\ (0.2)^1\ (0.8)^4$

$$= 5(0.2)\ (0.4096) = 0.4096$$

(*iii*) $p\,(1 < x < 4) = p(2) + p(3)$

$$= {}^5C_2\ (0.2)^2\ (0.8)^3 + {}^5C_3\ (0.2)^3\ (0.8)^2$$

$$= (0.2)^2\ (0.8)^2\ [10(0.8) + 10\ (0.2)]$$

$$= 0.0256\ (8 + 2) = 0.256$$

Example 27 : Find the maximum n such that the probability of getting no head in tossing a fair coin n times is greater than 0.1. **[JNTU 2004, 2007S (Set No. 4)]**

Solution : p = The probability of getting a head $= \frac{1}{2}$

q = The probability of not getting a head $= 1 - p = \frac{1}{2}$

Given that $P(X = 0) > 0.1$

$i.e.,\ {}^nC_0\ p^0\ q^n > 0.1 \quad i.e.,\ q^n > 0.1$

$i.e., \left(\frac{1}{2}\right)^n > 0.1 \Rightarrow 2^n < 10 \Rightarrow n < 4.$ So $n = 3$

[(or) For $n = 1$, $\frac{1}{2} = 0.5 > 0.1$

For $n = 2$, $\frac{1}{4} = 0.25 > 0.1$

For $n = 3$, $\frac{1}{8} = 0.125 > 0.1$

For $n = 4$, $\frac{1}{16} = 0.0625 < 0.1$]

Hence the required maximum $n = 3$.

Example 28 : Find the probability that at most 5 defective components will be found in a lot of 200. Experience shows that 2% of such components are defective. Also find the probability of more than five defective components. **[JNTU 2005 (Set No. 4)]**

Solution : p = The probability of defective component = 2% = 0.02

n = Number of components in a lot = 200

q = The probability of non defective component

$= 1 - p = 1 - 0.02 = 0.98$

(*i*) Probability of getting at most 5 defective components = $P(X \le 5)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$

$$= {}^{200}C_0\,(0.02)^0\,(0.98)^{200} + {}^{200}C_1\,(0.02)^1\,(0.98)^{199}$$
$$+ {}^{200}C_2\,(0.02)^2\,(0.98)^{198} + {}^{200}C_3\,(0.02)^3\,(0.98)^{197}$$
$$+ {}^{200}C_4\,(0.02)^4\,(0.98)^{196} + {}^{200}C_5\,(0.02)^5\,(0.98)^{195}$$

$$= (0.98)^{195}\,[(0.98)^5 + 200\,(0.02)\,(0.98)^4 + 19900\,(0.02)^2\,(0.98)^3$$
$$+ (6600)\,(199)\,(0.98)^2\,(0.02)^3 + 1650\,(199)\,(197)\,(0.98)\,(0.02)^4$$
$$+ 330\,(199)\,(197)\,(196)\,(0.02)^5]$$

$$= (0.98)^{195}\,[0.904 + 3.689 + 7.4918 + 10.0911 + 10.1426 + 8.1140]$$

$$= (0.98)^{195}\,(40.4325) = 0.7867$$

(*ii*) Probability of more than 5 defective components = $P(X > 5)$

$$= 1 - P(X \le 5) = 1 - 0.7867 = 0.2133$$

Example 29 : Assume that 50% of all engineering students are good in Mathematics. Determine the probabilities that among 18 engineering students

(*i*) exactly 10 (*ii*) at least 10 (*iii*) at most 8 (*iv*) at least 2 and at most 9 are good in mathematics. **[JNTU 2005 (Set No.2)]**

Solution : Let x be the number of engineering students who are good in Mathematics.

p = The probability of students good in mathematics = $50\% = \frac{1}{2}$

$q = 1 - p = \frac{1}{2}$

n = Number of students = 18

The probability distribution is

$$p(x) = {}^nC_x\, p^x\, q^{n-x} = {}^{18}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{18-x} = {}^{18}C_x \left(\frac{1}{2}\right)^{18}$$

(*i*) P (exactly 10) $= p(10) = \left(\frac{1}{2}\right)^{18}\,{}^{18}C_{10} = \left(\frac{1}{2}\right)^{18}(43758) = 0.1669235$

(*ii*) P (at least 10) $= P(X \ge 10) = \sum_{x=10}^{18} {}^{18}C_x \left(\frac{1}{2}\right)^{18} = \left(\frac{1}{2}\right)^{18}\left[{}^{18}C_{10} + {}^{18}C_{11} + \ldots + {}^{18}C_{18}\right]$

(*iii*) P (at most 8) $= P(X \le 8) = \sum_{x=0}^{8} {}^{18}C_x \left(\frac{1}{2}\right)^{18} = \left(\frac{1}{2}\right)^{18}\left[{}^{18}C_0 + {}^{18}C_1 + \ldots + {}^{18}C_8\right]$

(*iv*) P (at least 2 and at most 9) $= P(2 \le X \le 9) = \sum_{x=2}^{9} {}^{18}C_x \left(\frac{1}{2}\right)^{18}$

$$= \left(\frac{1}{2}\right)^{18}\left[{}^{18}C_2 + {}^{18}C_3 + \ldots + {}^{18}C_9\right]$$

Example 30 : The probability of a man hitting a target is $1/3$

(*i*) If he fires 5 times, what is the probability of his hitting the target at least twice ?

(*ii*) How many times must he fire so that the probability of his hitting the target at least once is more than 90% ? **[JNTU 2006S (Set No. 1)]**

Solution : p = The probability of hitting a target $= \frac{1}{3}$

q = The probability of not hitting a target $= 1 - \frac{1}{3} = \frac{2}{3}$

n = Number of trials = 5

(*i*) P(at least twice) $= P(X \ge 2) = 1 - P(X < 2) = 1 - [(P(X=0) + P(X=1)]$

$$= 1 - \left[{}^5C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^5 + {}^5C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^4\right]$$

$$= 1 - \left[\left(\frac{2}{3}\right)^5 + \frac{5}{3}\left(\frac{2}{3}\right)^4\right] = 1 - \left(\frac{2}{3}\right)^4\left[\frac{2}{3} + \frac{5}{3}\right]$$

$$= 1 - \frac{7}{3}\left(\frac{2}{3}\right)^4 = 0.5391$$

(*ii*) Given that P(at least once) > 90%

i.e., $P(X \ge 1) > 0.9$

i.e., $1 - P(X = 0) > 0.9$

i.e., $1 - {}^nC_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n > 0.9$

or $1 - \left(\frac{2}{3}\right)^n < 0.9$

This is satisfied for $n = 6$ because $1 - \left(\frac{2}{3}\right)^6 = 0.9122 > 0.9$

Example 31 : Determine the probability of getting a sum of 9 exactly twice in 3 throws with a pair of fair dice. **[JNTU 2006S, 2007 (Set No.1)]**

Solution : In a single throw of a pair of fair dice, a sum of 9 can occur in 4 ways : (3, 6), (4, 5), (5, 4), (6, 3) out of $6 \times 6 = 36$ ways. Thus

p = Probability of occurrence of a sum of 9 in one throw $= \frac{4}{36} = \frac{1}{9}$

$q = 1 - p = 1 - \frac{1}{9} = \frac{8}{9}$

n = Number of throws = 3

$\therefore$ Probability of getting a sum of 9 exactly twice in 3 throws

$$= b\left(2;3,\frac{1}{9}\right) = {}^3C_2\left(\frac{1}{9}\right)^2\left(\frac{8}{9}\right)^1$$

$$= 3 \times \frac{1}{81} \times \frac{8}{9} = \frac{8}{243} = 0.033$$

Example 32 : A coin is biased in a way that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, find the probability of getting 2 tail and 1 head

[JNTU (H) Dec. 2011 (Set No.2)]

Solution : Given $P(H) = 2P(T)$. We know that $P(H) + P(T) = 1$

Solving $3P(T) = 1 \Rightarrow P(T) = \frac{1}{3}$

$\therefore P(H) = 2P(T) = \frac{2}{3}$

Let getting a tail is a success and getting a head is a failure. Then $p = 1/3, q = 2/3$.

By Binomial distribution, $P(X = x) = {}^nC_x p^x q^{n-x}$

Here $n = 3, x = 2$

$\therefore$ Required probability $= {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{2}{9}$

Example 33 : Fit a binomial distribution to the following frequency distribution:

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

Solution : Here n = number of trials = 6 and N = total frequency = $\Sigma f_i = 200$

$$\therefore \text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{25+104+174+128+80+24}{200} = \frac{535}{200} = 2.675$$

Now mean of the binomial distribution = np

$i.e.,\ np = 6p = 2.675 \quad \therefore p = \frac{2.675}{6} = 0.446$

and $q = 1 - p = 1 - 0.446 = 0.554$

Hence the binomial distribution to be fitted is given by

$$N(q+p)^n = 200\,(0.554 + 0.446)^6$$

$$= 200\,[(0.554)^6 + {}^6C_1\,(0.554)^5\,(0.446) + {}^6C_2\,(0.554)^4\,(0.446)^2 + {}^6C_3\,(0.554)^3\,(0.446)^3 + {}^6C_4\,(0.554)^2\,(0.446)^4 + {}^6C_5\,(0.554)\,(0.446)^5 + {}^6C_6\,(0.446)^6]$$

$$= 200\,[0.02891 + 0.1396 + 0.2809 + 0.3016 + 0.1821 + 0.05864 + 0.007866]$$

$$= 5.782 + 27.92 + 56.18 + 60.32 + 36.42 + 11.728 + 1.5732$$

The expected frequencies can be rounded off to the nearest integer to get expected frequencies as whole numbers.

$\therefore$ The successive terms in the expansion give the expected or theoretical frequencies which are

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4
Expected or Theoretical frequency	6	28	56	60	36	12	2

(since frequencies are always integers).

It may be noted that the expected frequencies are close to observed frequencies. A measure of closeness called 'goodness of fit'.

Example 34 : Fit a binomial distribution to the following data

x	0	1	2	3	4	5
f	2	14	20	34	22	8

[JNTU 2004S, (A) Nov. 2011(Set No.1)]

Solution : Here $n = 5$ and $N = \Sigma f_i = 2 + 14 + 20 + 34 + 22 + 8 = 100$

$$\therefore \text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0+14+40+102+88+40}{100} = \frac{284}{100} = 2.84$$

Now the mean of the binomial distribution = np

$i.e.,\ np = 5p = 2.84 \quad \therefore p = \frac{2.84}{5} = 0.568$

and $q = 1 - p = 0.432$

Hence the Binomial Distribution to be fitted is given by

$$N(q+p)^n = 100\,(0.432 + 0.568)^5$$

$$= 100[{}^5C_0(0.432)^5 + {}^5C_1(0.432)^4(0.568) + {}^5C_2(0.432)^3(0.568)^2 + {}^5C_3(0.432)^2(0.568)^3 + {}^5C_4(0.432)(0.568)^4 + {}^5C_5(0.568)^5]$$

$$= 100\,[0.015 + 0.0989 + 0.260 + 0.341 + 0.224 + 0.059]$$

$$= 1.5 + 9.89 + 26 + 34.1 + 22.4 + 5.9$$

The respective terms of the binomial give the theoretical frequencies. Since frequencies are always integers, therefore, by converting them to the nearest integers, we get 2, 10, 26, 34, 22, 6.

x	0	1	2	3	4	5
f	2	14	20	34	22	8
Expected or Theoretical frequency	2	10	26	34	22	6

Example 35 : Four coins are tossed 160 times. The number of times x heads occur is given below.

x	0	1	2	3	4
No. of times	8	34	69	43	6

[JNTU 2004 S (Set No. 3)]

Fit a binomial distribution to this data on the hypothesis that coins are unbiased.

Solution : The coin is unbiased.

$\therefore \quad p = \frac{1}{2}, q = \frac{1}{2}$ and $n = 4$.

Also $N = \Sigma f_i = 8 + 34 + 69 + 43 + 6 = 160$

By Binomial distribution, $p(x) = {}^nC_x \, p^x q^{n-x}$

We have the recurrence relation

$$p(x+1) = \frac{(n-x)p}{(x+1)q} \cdot p(x) = \frac{4-x}{x+1} \cdot p(x) \left[\because n = 4, \frac{p}{q} = 1\right] \quad \dots (1)$$

$$\therefore p(0) = {}^4C_0 \, p^0 q^4 = {}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4$$

No. of heads x	Observed frequency f	Probability $p(x)$	Expected or Theoretical frequency $f(x) = N.p(x)$
0	8	$p(0) = \left(\frac{1}{2}\right)^4$	$f(0) = 160.p(0)$ $= 160 \times \frac{1}{16} = 10$
1	34	$p(1) = \frac{1}{4}$	$f(1) = 160.p(1)$ $= 160\left(\frac{1}{4}\right) = 40$
2	69	$p(2) = \frac{3}{8}$	$f(2) = 160.p(2)$ $= 160\left(\frac{3}{8}\right) = 60$
3	43	$p(3) = \frac{1}{4}$	$f(3) = 160.p(3)$ $= 160\left(\frac{1}{4}\right) = 40$
4	6	$p(4) = \frac{1}{16}$	$f(4) = 160\left(\frac{1}{16}\right) = 10$

Example 36 : The probability that John hits a target is $\frac{1}{2}$. He fires 6 times. Find the probability that he hits the target (*i*) exactly 2 times (*ii*) more than 4 times (*iii*) at least once.

[JNTU 2007S, (H) May 2011 (Set No. 4)]

Solution : Probability of hitting a target $= p = \frac{1}{2}$

Probability of no hit (or failure) $= q = \frac{1}{2}$

Number of trials $= n = 6$

Number of hits (successes) $= X$

(*i*) P(exactly 2 times) $= P(X = 2)$

$$= {}^6C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4 = \frac{15}{2^6} = 0.2344$$

(*ii*) P(more than 4 times) $= P(X > 4) = P(X = 5) + P(X = 6)$

$$= {}^6C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) + {}^6C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^0$$

$$= \frac{6}{2^6} + \frac{1}{2^6} = \frac{7}{2^6} = 0.1094$$

(*iii*) P(at least once) $= P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - {}^6C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^6 = 1 - \frac{1}{2^6} = 0.9844$$

Example 37 : The mean of Binomial distribution is 3 and the variance is $\frac{9}{4}$. Find (*i*) the value of n (*ii*) $P(X \geq 7)$ (*iii*) $P(1 \leq X < 6)$.

[JNTU 2007, 2008S, (H) Dec. 2009 (Set No. 2)]

Solution : (*i*) We have

Mean of the Binomial distribution $= 3 \Rightarrow np = 3 \quad \dots (1)$

Variance of the Binomial distribution $= \frac{9}{4} \Rightarrow npq = \frac{9}{4} \quad \dots (2)$

(2) ÷ (1) gives

$$\frac{npq}{np} = \frac{9}{4 \times 3} \Rightarrow q = \frac{3}{4}$$

$$\therefore p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

From (1), $n = \frac{3}{p} = 12$

(*ii*) $P(X \geq 7) = 1 - P(X < 7)$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)]$$

$$= 1 - \left[{}^{12}C_0\left(\frac{1}{4}\right)^0 \cdot \left(\frac{3}{4}\right)^{12} + {}^{12}C_1\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{11} + {}^{12}C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{10}\right.$$

$$\left. + {}^{12}C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^9 + {}^{12}C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^8 + {}^{12}C_5\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^7 + {}^{12}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^6\right]$$

$$= 1 - \left[\left(\frac{3}{4}\right)^{12} + 12.\frac{1}{4}.\left(\frac{3}{4}\right)^{11} + 66.\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{10} + 220.\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^9\right.$$

$$\left. + 495.\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^8 + 792.\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^7 + 924.\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^6\right]$$

$$= 1 - \left[\frac{3^7}{4^{12}}(243+972+1782+1980+1485+792+308)\right] = 1 - \frac{3^7}{4^{12}}(7562)$$

$$= 1 - 0.9857 = 0.0142$$

(*iii*) $p(1 \leq x < 6) = p(x=1) + \ldots + p(x=5)$

$$= {}^{12}C_1 \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{11} + {}^{12}C_2\left(\frac{1}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^{10} + {}^{12}C_3\left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^9$$

$$+ {}^{12}C_4\left(\frac{1}{4}\right)^4 \cdot \left(\frac{3}{4}\right)^8 + {}^{12}C_5\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^7$$

$$= \frac{3^7}{4^{12}}(972 + 1782 + 1980 + 1485 + 72)$$

$$= \frac{3^7}{4^{12}}(6291) = 0.82$$

Example 38 : The probability that the life of a bulb is 100 days is 0.05. Find the probability that out of 6 bulbs (*i*) At least one (*ii*) greater than four (*iii*) none, will be having a life of 100 days.

[JNTU 2007S, (K) May 2013 (Set No.3)]

Solution : We have

p = Probability that a bulb is of 100 days life $= 0.05 = \dfrac{1}{20}$ and $n = 6$

$\therefore\ q = 1 - p = 1 - \dfrac{1}{20} = \dfrac{19}{20}$

By Binomial distribution,

$$P(X = x) = {}^nC_x\, p^x q^{n-x} = {}^6C_x\left(\frac{1}{20}\right)^x\left(\frac{19}{20}\right)^{6-x}$$

(*i*) $P(x \geq 1) = 1 - P(x=0) = 1 - {}^6C_0\left(\dfrac{1}{20}\right)^0\left(\dfrac{19}{20}\right)^6 = 1 - \left(\dfrac{19}{20}\right)^6 = 0.265$

(*ii*) $P(x > 4) = P(x=5) + P(x=6)$

$$= {}^6C_5\left(\frac{1}{20}\right)^5 \cdot \frac{19}{20} + {}^6C_6\left(\frac{1}{20}\right)^6$$

$$= \left(\frac{1}{20}\right)^6[114 + 1] = \frac{115}{(20)^6}$$

(*iii*) $P(x=0) = {}^6C_0\left(\dfrac{1}{20}\right)^0\left(\dfrac{19}{20}\right)^6 = \left(\dfrac{19}{20}\right)^6 = 0.7351$

Example 39 : If the chance that any of the 10 telephone lines is busy at an instant is 0.2. What is the most probable number of busy lines and what is the probability of this number.

[JNTU 2008 (Set No. 4)]

Solution : Given the probability that a line is busy is $p = 0.2$

$\therefore$ The probability that a line is not busy is $q = 1 - p = 1 - 0.2 = 0.8$

On average $np = 10(0.2) = 5$ lines are busy at an instant.

The probability that 'r' lines are busy is

$$P(r) = {}^{10}C_r\,(0.2)^r(0.8)^{10-r}$$

$\therefore$ P(5 lines are busy) = P(5) $= {}^{10}C_5(0.2)^5(0.8)^{10-5} = 252\,(0.2)^5(0.8)^5 = 0.0264$

Example 40 : Seven coins are tossed and the number of heads are noted. The experiment is repeated 128 times and the following distribution is obtained.

No. of heads	0	1	2	3	4	5	6	7	Total
Frequency	7	6	19	35	30	23	7	1	128

Fit a binomial distribution assuming. (*a*) the coin is unbiased.

(*b*) the nature of the coin is not known. **[JNTU 2009 (Set No. 1)]**

Solution : (*a*) The coin is unbiased.

$\therefore\ p = \dfrac{1}{2}, q = \dfrac{1}{2}$ and $n = 7$

$N = \Sigma f_i = 7 + 6 + 19 + 35 + 30 + 23 + 7 + 1 = 128$

By Binomial distribution, $p(x) = {}^nC_x\, p^x q^{n-x}$

We have the recurrence relation

$$p(x+1) = \frac{(n-x)p}{(x+1)q} \cdot p(x) = \frac{7-x}{x+1} \cdot p(x) \quad \left[\because n = 7, \frac{p}{q} = 1\right]$$

$$\therefore\ p(0) = {}^7C_0\, p^0 q^7 = {}^7C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^7 = \frac{1}{2^7}$$

No. of heads x	Observed frequency	Probability $p(x)$	Expected or Theoretical frequency $f(x) = N.p(x)$
0	7	$p(0)=\frac{1}{2^7}$	$f(0)=128.p(0)=128\times\frac{1}{2^7}=1$
1	6	$p(1)=7.p(0)=\frac{7}{2^7}$	$f(1)=128.p(1)=128\times\frac{7}{2^7}=7$
2	10	$p(2)=3.p(1)=\frac{21}{2^7}$	$f(2)=128.p(2)=128\times\frac{21}{2^7}=21$
3	35	$p(3)=\frac{35}{2^7}$	$f(3)=128.p(3)=35$
4	30	$p(4)=\frac{35}{2^7}$	$f(4)=128.p(4)=35$
5	23	$p(5)=\frac{21}{2^7}$	$f(5)=128.p(5)=21$
6	7	$p(6)=\frac{7}{2^7}$	$f(6)=128.p(6)=7$
7	1	$p(7)=\frac{1}{2^7}$	$f(7)=128.p(7)=1$

(*b*) The nature of the coin is not known.

We are given $n = 7$ and $N = \Sigma f_i = 128$

$$\therefore \quad \text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{6+38+105+120+115+42+7}{128} = \frac{433}{128} = 3.383$$

i.e., $np = 3.383$

$$\Rightarrow \quad 7p = 3.383 \quad \therefore p = \frac{3.383}{7} = 0.4833$$

and $q = 1 - p = 1 - 0.4833 = 0.5167$

Hence the Binomial Distribution to be fitted is given by the terms of

$$N(q+p)^n = 128\,(0.5167 + 0.4833)^7$$

This information can also be tabulated as in the earlier table.

Example 41 : 30% of items from a factory are defective. Find the probability that in a sample of 8 (*i*) one (*ii*) At least two (*iii*) $p(1 < x < 6)$ are defective.

[JNTU (H) 2010 (Set No. 1)]

Solution : Given probability of a defective item = 30%

$$= \frac{30}{100} = \frac{3}{10} = 0.3$$

$\therefore q = 1 - p = 1 - 0.3 = 0.7$

n = Sample size = 8.

For a Binomial Distribution, the probability function is

$$P(X=r) = f(x) = {}^nC_r\, p^r\, q^{n-r} = {}^8C_r (0.3)^r (0.7)^{8-r}$$

(*i*) $P(X=1) = {}^8C_1(0.3)(0.7)^{8-1} = 8(0.3)(0.7)^7 = 0.1976$

(*ii*) $P(X \geq 2) = 1 - P(X<2) = 1 - [P(X=0) + P(X=1)]$

$$= 1 - \left[{}^8C_0(0.3)(0.7)^8 + 0.1976\right] = 1 - \left[(0.3)(0.7)^8 + 0.1976\right]$$

$$= 1 - [0.0173 + 0.1976] = 1 - 0.3706 = 0.6294.$$

(*iii*) $P(1 < X < 6) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= {}^8C_2(0.3)^2(0.7)^{8-2} + {}^8C_3(0.3)^3(0.7)^{8-3} + {}^8C_4(0.3)^4(0.7)^{8-4} + {}^8C_5(0.3)^5(0.7)^{8-5}$$

$$= \frac{8!}{2!6!}(0.3)^2(0.7)^6 + \frac{8!}{3!5!}(0.3)^3(0.7)^5 + \frac{8!}{4!4!}(0.3)^4(0.7)^4 + \frac{8!}{3!5!}(0.3)^5(0.7)^3$$

$$= 0.2964 + 0.2541 + 0.1361 + 0.0467 = 0.493.$$

Example 42 : The probability that a man hitting a target is 1/3. If he fires 6 times, find the probability that he fires (*i*) At the most 5 times (*ii*) Exactly once (*iii*) At least two times

[JNTU (H) Nov. 2010 (Set No. 2)]

Solution : Given p = Probability of hitting the target $= \frac{1}{3}$ and $n = 6$.

$$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

For a Binomial Distribution, the probability function is

$$f(x) = P(X=x) = {}^nC_x \cdot p^x \cdot q^{n-x} = {}^6C_x \cdot p^x q^{6-x}$$

(*i*) $P(x \leq 5) = 1 - P(x=6) = 1 - {}^6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^{6-6} = 1 - \frac{1}{729} = \frac{728}{729} = 0.9986$

(*ii*) $P(x=1) = {}^6C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{6-1} = 2\left(\frac{2}{3}\right)^5 = 0.2634$

(*iii*) $P(x \geq 2) = 1 - [P(x=0) + P(x=1)] = 1 - \left[{}^6C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{6-0} + 0.2634\right]$ [From (*i*)]

$$= 1 - \left[\left(\frac{2}{3}\right)^6 + 0.2634\right] = 1 - [0.0877 + 0.2634] = 1 - 0.3511 = 0.6488.$$

Example 43 : A coin is biased in a way that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, find the probability of getting 2 tail and 1 head

[JNTU (H) Nov. 2010 (Set No. 2)]

Solution : When a coin is tossed, two mutually exclusive events H, T can occur.

We have P(H) + P(T) = 1

$P(H) = 2P(T) \Rightarrow 3P(T) = 1$

$\Rightarrow P(T) = \frac{1}{3} = p$ (say probability of success)

$\Rightarrow P(H) = \frac{2}{3} = q$ (say probability of failure)

In a Binomial distribution $n = 3$, Number of successes $= x = 2$.

$\therefore$ Required probability is

$$P(X = x) = {}^nC_x p^x q^{n-x} = {}^3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 = 3 \times \frac{1}{9} \times \frac{2}{3} = \frac{2}{9}.$$

Example 44 : Traffic control engineer reports that 75% of the vehicles passing through a checkpost are from within state. What is the probability that fewer than 4 of the 9 are from out of the state? **[JNTU (H) Nov. 2010 (Set No. 4)]**

Solution : The probability that vehicle is from within state $= \frac{75}{100} = \frac{3}{4}$

Considering as Binomial distribution, $n = 9$

The probability more than three vehicles are from within state $= P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - P(X = 0) - P(X = 1) - P(X = 2)$$

$$= 1 - {}^9C_0 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^9 - {}^9C_1 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^8 - {}^9C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^7$$

Example 45 : Find the probability of getting an even number 3 or 4 or 5 times in throwing 10 dice, using Binomial Distribution. **[JNTU (H) Apr. 2012 (Set No. 2)]**

Solution : Let p be the probability of getting an even number in a throw of a die.

Then $p = \frac{3}{6} = \frac{1}{2}$ and $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$. Here $n = 10$

The probability of getting x even numbers in ten throws of a die is

$$P(X = x) = {}^{10}C_x\, p^x q^{10-x}, x = 0, 1, 2, \ldots, 10$$

$$= {}^{10}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = {}^{10}C_x \left(\frac{1}{2}\right)^{10} ; x = 0, 1, 2, \ldots, 10$$

$$\therefore P(X = 3) = {}^{10}C_3 \left(\frac{1}{2}\right)^{10} = \frac{120}{1024} = 0.117$$

$$P(X = 4) = {}^{10}C_4 \left(\frac{1}{2}\right)^{10} = \frac{210}{1024} = 0.2$$

$$P(X = 5) = {}^{10}C_5 \left(\frac{1}{2}\right)^{10} = \frac{252}{1024} = 0.246$$

Example 46 : Fit a binomial distribution to the following data

x	0	1	2	3	4
f	38	144	342	287	164

[JNTU (H) III yr. Nov. 2015]

Solution : Here n = no. of trials $= 4$ and

$$N = \text{total frequency} = \sum_{i=0}^{4} f_i = 38 + 144 + 342 + 287 + 164 = 975$$

Mean of the Binomial distribution is

$$\mu = np = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0(38) + 1(144) + 2(342) + 3(287) + 4(164)}{975}$$

$$i.e.,\ np = 4p = \frac{144 + 684 + 861 + 656}{975} = \frac{2345}{975} = 2.4051$$

$$\therefore\ p = \frac{2.4051}{4} = 0.6$$

So $q = 1 - p = 1 - 0.6 = 0.4$

Thus the binomial distribution that fits the given data is

$$b(x; n; p) = b(x; 4; 0.6) = {}^nC_x p^x q^{n-x} = {}^4C_x (0.6)^x (0.4)^{4-x}$$

$i.e.,\ p(x) = {}^4C_x (0.6)^x (0.4)^{4-x}$

$\therefore$ The theoretical frequencies are

x	0	1	2	3	4
$P(x)$	0.0256	0.1536	0.3456	0.3456	0.1296
Expected frequency $= N \times P(x) = 975P(x)$	24.96 ≈ 25	149.76 ≈ 150	336.96 ≈ 337	336.96 ≈ 337	126.36 ≈ 126

Since frequencies are always integers, therefore, by converting them to integers, we get the expected frequencies as 25, 150, 337, 337 and 126.

REVIEW QUESTIONS

1. Define Binomial Distribution. **[JNTU (H) Dec. 2014]**
2. Show that the mean of the Binomial distribution is the product of the parameter p and the number of times n.
3. What are the mean and variance of a Binomial distribution.
4. Find (or derive) the mean and variance of the Binomial distribution. **[JNTU (H) Dec. 2014]**
5. Prove that the variance of the Binomial distribution is npq.

ANSWERS

1. 0.6243 **2.** 0.2461 **3.** $\frac{3}{16}$ **4.** 0.9822 **6.** 6 **7.** 0.1937

8. (*i*) 0.3487 (*ii*) 0.3874 (*iii*) 0.9298

9. (*i*) 0.109375 (*ii*) 0.9648 **10.** (*i*) 0.890625 (*ii*) 0.65625

11. (*i*) 0.26 (*ii*) 0.0000244 (*iii*) 0.7674

12. (*i*) 0.2301 (*ii*) 0.3412 (*iii*) 0.2833

13. (*i*) 0.02642 (*ii*) 2, 0.0457 (*iii*) 1.023×10^{-7}

14. (*i*) $\frac{1}{5^{10}}$ (*ii*) 0.8926 (*iii*) 0.1074 (*iv*) 0.879128

15. (*i*) 0.1669 (*ii*) 0.9999 (*iii*) 0.999999

16. (*i*) 0.000004526 (*ii*) 0.00107374 (*iii*) $\frac{1}{5^{12}}$ (*iv*) 0.999999 (*v*) 0.422249

17. 0.65536 **18.** (*i*) 0.105 (*ii*) 0.00308

19. $\frac{1}{2^6}, \frac{6}{2^6}, \frac{15}{2^6}, \frac{20}{2^6}, \frac{15}{2^6}, \frac{6}{2^6}, \frac{1}{2^6}$

20.

x	0	1	2	3	4	5
$p(x)$	100	500	1000	1000	500	100

; $\mu = 2.5, \sigma = 1.118$

21. 33, 162, 316, 309, 151, 29 **22.** $80\,[0.8725 + 0.2175]^{10}$

23. (*i*) $128\,[0.517 + 0.483]^7$ (*ii*) 1, 7, 21, 35, 35, 21, 7, 1

4.7 GEOMETRIC DISTRIBUTION

Geometric Distribution is a discrete distribution. If p be the probability of success and x be the number of failures preceding the first success, then this distribution is

$P(x) = q^x p, x = 0, 1, 2, \ldots$ or $P(x) = q^{x-1} p, x = 1, 2, 3, \ldots$

where $p =$ probability of success of an outcome

$q = 1 - p =$ probability of failure of an outcome

$x =$ a random variable which denotes the no. of trials required to get a success

$$\text{Clearly sum of the probabilities} = \sum_{x=0}^{\infty} P(x)$$

$$= \sum_{x=0}^{\infty} q^x p = p \sum_{x=0}^{\infty} q^x = p(1 + q + q^2 + \ldots)$$

$$= p\left(\frac{1}{1-q}\right) \quad [\because \text{The series is in G. P.}]$$

$$= \frac{p}{1-q} = \frac{p}{p} = 1 \quad [\because p + q = 1]$$

Illustrations :

1. The number of job applicants interviewed by an employer until the first suitable candidate is found.
2. For a sequence of independent drilling for water, x could represent the number of drilling unitl the first successful hit.

4.7.1 MEAN OF THE GEOMETRIC DISTRIBUTION **[JNTU (H) Dec. 2019]**

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} kP(x) = \sum_{x=0}^{\infty} xq^x p = p\sum_{x=0}^{\infty} xq^x$$

$$= p(q + 2q^2 + 3q^3 + \ldots) = pq(1 + 2q + 3q^2 + \ldots)$$

$$= pq(1-q)^{-2}, \text{ using Binomial Theorem}$$

$$= \frac{pq}{(1-q)^2} = \frac{pq}{p^2} \quad [\because p + q = 1]$$

$$= \frac{q}{p}$$

Hence **Mean of the Geometric distribution is $\frac{q}{p}$.**

Similarly, **Variance of the Geometric distribution is $\frac{q}{p^2}$.**

REVIEW QUESTIONS

1. Define Geometric distribution
2. Derive mean of the Geometric distribution. **[JNTU (H) Dec. 2019]**
3. What are the mean and variance of the Geometric distribution.

POISSON DISTRIBUTION

4.8 POISSON DISTRIBUTION

Poisson Distribution due to French mathematician Simeon Denis Poisson (1837) is a discrete probability distribution.

S. D. Poisson introduced Poisson distribution as a rare distribution of rare events *i.e.*, the events whose probability of occurrence is very small but the number of trials which could lead to the occurrence of the event, are very large.

Suppose that, in a large number of parts manufactured by a machine, samples of 100 parts are taken from a product which is 2% defective. Then the probabilities of 0, 1, 2, ..., 99, 100 defectives in a single sample are given by the successive terms of the Binomial distribution $(0.98 + 0.02)^{100}$. Expanding the binomial in full will be a tedious process. This difficulty can be avoided by the Poisson distribution.

4.9 BINOMIAL APPROXIMATION TO POISSON DISTRIBUTION (DERIVATION OF THE POISSON DISTRIBUTION)

[JNTU 2004, 2006S, (H) May 2011, (H) Dec. 2011]

The Poisson distribution can be derived as a limiting case of the Binomial Distribution under the following conditions:

(i) p, the probability of the occurrence of the event is very small.

(ii) n is very very large, where n is number of trials i.e., $n \to \infty$.

(iii) np is a finite quantity, say $np = \lambda$, then λ is called the parameter of the Poisson distribution.

In the Binomial distribution, the probability $P(r)$ or $P(X = r)$ of r successes in a series of independent trials is given by

$$P(r) = {}^nC_r\, p^r q^{n-r} = {}^nC_r\, p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \cdot p^r \cdot \frac{(1-p)^n}{(1-p)^r} \quad \ldots (1)$$

Put $np = \lambda$. Then $n = \dfrac{\lambda}{p}$

Hence (1) becomes,

$$P(r) = \frac{\frac{\lambda}{p}\left(\frac{\lambda}{p}-1\right)\left(\frac{\lambda}{p}-2\right)\ldots\left(\frac{\lambda}{p}-r+1\right)}{r!} \cdot p^r \cdot \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{\lambda(\lambda-p)(\lambda-2p)\ldots[\lambda-(r-1)p]}{r!\,p^r} \cdot p^r \cdot \frac{(1-p)^n}{(1-p)^r}$$

$$= \frac{\lambda(\lambda-p)(\lambda-2p)\ldots[\lambda-(r-1)p]}{r!} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^n}{(1-p)^r} \quad \ldots (2)$$

As $n \to \infty, p \to 0$, so that $np = \lambda$, we have

$$P(r) = \frac{\lambda.\lambda\ldots r \text{ factors}}{r!} \cdot \underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^n \cdot \underset{p\to 0}{Lt} \frac{1}{(1-p)^r}$$

$$= \frac{\lambda^r}{r!} \cdot \underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^n \quad \left[\because \underset{p\to 0}{Lt}(1-p)^r = 1 \text{ for a given } r\right]$$

$$= \frac{\lambda^r}{r!} e^{-\lambda} \left[\because \underset{n\to\infty}{Lt}\left(1-\frac{\lambda}{n}\right)^n = \underset{n\to\infty}{Lt}\left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} = e^{-\lambda}\right]$$

$\therefore \quad P(r) = \text{Probability of } r \text{ successes} = \dfrac{e^{-\lambda}.\lambda^r}{r!}$

This is known as **Poisson distribution.**

Putting $r = 0, 1, 2, \ldots$ etc., the probabilities of 0, 1, 2, ..., r, ... successes are given by $e^{-\lambda}, \lambda e^{-\lambda}, \dfrac{\lambda^2}{2!}e^{-\lambda}, \ldots, \dfrac{\lambda^r}{r!}e^{-\lambda}, \ldots$ respectively, where $\lambda (> 0)$ is a parameter.

The sum of these probabilities is unity as it should be.

4.10 POISSON DISTRIBUTION

[JNTU 2006S, 2008S, (A), (H) Dec. 2009 (Set No. 4)]

Definition. A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability density function is given by

$$p(x, \lambda) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}.\lambda^x}{x!}; x = 0,1,2,3,\ldots \\ 0, \text{ otherwise} \end{cases} \quad \ldots (1)$$

Here $\lambda > 0$ is called the parameter of the distribution.

It is a distribution suitable for rare events for which the probability of occurrence p is very small and the number of trials n is very large where np is finite. For example, consider the number of persons born blind per year in a large city. Individually, being born blind is a rare event and its probability is very small. But in a large city the total number of births every year is high. As a result, we may get some blind births.

Note 1. It should be noted that

$$\sum_{x=0}^{\infty} P(X = x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}.\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}.\,e^{\lambda} = 1$$

Hence equation (1) is a probability function.

2. The Poisson Distribution function is

$$F_X(x) = P(X \le x) = \sum_{r=0}^{x} p(r) = e^{-\lambda}\sum_{r=0}^{x}\frac{\lambda^r}{r!}, x = 0, 1, 2, \ldots$$

Examples of Poisson Distribution :

(i) The number of defective electric bulbs manufactured by a reputed company.

(ii) The number of telephone calls per minute at a switch board.

(iii) The number of cars passing a certain point in one minute.

(iv) The number of printing mistakes per page in a large text.

(v) The number of particles emitted by a radio-active substance.

(vi) The number of persons born blind per year in a large city.

Conditions of Poisson Distribution :

The Poisson Distribution is used under the following conditions :

1. The variable (number of occurrences) is a discrete variable.
2. The occurrences are rare.
3. The number of trials (n) is large.
4. The probability of success (p) is very small (very close to zero)
5. $np = \lambda$ is finite.

4.11 CONSTANTS OF THE POISSON DISTRIBUTION

1. **The Mean of the Poisson Distribution**

[JNTU 2008S, (H) Nov. 2009 (Set No. 4), (K) May 2012, Dec. 2013, Mar. 2014 (Set No. 1)]

$$\text{Mean} = E(X) = \sum_{x=0}^{\infty} x.p(x) = \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}.\lambda^x}{x!} = e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!} \quad [\because x! = x(x-1)!]$$

$$= e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^{y+1}}{y!} \quad (\text{Putting } y = x-1)$$

$$= \lambda e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^y}{y!} = e^{-\lambda}.\lambda.e^{\lambda}\left[\because \sum_{y=0}^{\infty}\frac{\lambda^y}{y!} = e^{\lambda}\right]$$

$$= \lambda \; (= np)$$

Thus the parameter λ is the Arithmetic Mean of the Poisson distribution.

2. **Variance of Poisson Distribution**

[JNTU 2006S, 2008S, (H) Nov. 2009 (Set No. 4), (K) May 2012, Dec. 2013, Mar. 2014 (Set No. 1)]

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \sum_{x=0}^{\infty} x^2 p(x) - \lambda^2 \quad [\because \lambda = \text{mean of P.D.}]$$

$$= \sum_{x=0}^{\infty} x^2.\frac{e^{-\lambda}.\lambda^x}{x!} - \lambda^2 = e^{-\lambda}\sum_{x=1}^{\infty}\frac{x.\lambda^x}{(x-1)!} - \lambda^2$$

$$= e^{-\lambda}\sum_{x=1}^{\infty}[(x-1)+1].\frac{\lambda^x}{(x-1)!} - \lambda^2$$

$$= e^{-\lambda}\left[\sum_{x=1}^{\infty}(x-1).\frac{\lambda^x}{(x-1)(x-2)!} + \sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!}\right] - \lambda^2$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty}\frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty}\frac{\lambda^x}{(x-1)!}\right] - \lambda^2$$

$$= e^{-\lambda}\left[\sum_{y=0}^{\infty}\frac{\lambda^{y+2}}{y!} + \sum_{z=0}^{\infty}\frac{\lambda^{z+1}}{z!}\right] - \lambda^2 \quad (\text{Putting } y = x-2,\ z = x-1)$$

$$= e^{-\lambda}\left[\lambda^2\sum_{y=0}^{\infty}\frac{\lambda^y}{y!} + \lambda\sum_{z=0}^{\infty}\frac{\lambda^z}{z!}\right] - \lambda^2 = e^{-\lambda}\left[\lambda^2.e^{\lambda} + \lambda e^{\lambda}\right] - \lambda^2 = (\lambda^2+\lambda) - \lambda^2 = \lambda$$

Thus Variance $= \lambda$.

Hence the Variance of the distribution = Mean of the distribution = λ

Further, Standard deviation of the Poisson distribution, $\sigma = \sqrt{\lambda}$

3. **Mode of the Poisson Distribution**

Mode is the value of x for which the probability $p(x)$ is maximum.

$\therefore\ p(x) \geq p(x+1)$ and $p(x) \geq p(x-1)$

$$\text{Now } p(x) \geq p(x+1) \Rightarrow \frac{e^{-\lambda}.\lambda^x}{x!} \geq \frac{e^{-\lambda}.\lambda^{x+1}}{(x+1)!}$$

$$\Rightarrow 1 \geq \frac{\lambda}{x+1} \text{ or } \frac{\lambda}{x+1} \leq 1 \Rightarrow \lambda \leq x+1 \text{ or } x+1 \geq \lambda$$

$$\Rightarrow x \geq \lambda - 1 \qquad \ldots (1)$$

Similarly, $p(x) \geq p(x-1) \Rightarrow x \leq \lambda \qquad \ldots (2)$

Combining (1) and (2), we have

$$\lambda - 1 \leq x \leq \lambda$$

Hence mode of the Poisson distribution lies between $\lambda - 1$ and λ.

Case 1 : If λ is an integer then $\lambda - 1$ is also an integer. So we have two maximum values and the distribution is bimodel and the two modes are $(\lambda - 1)$ and λ.

Case 2 : If λ is not an integer, the mode of Poisson distribution is integral part of λ.

4. **Recurrence Relation for the Poisson Distribution**

We have $p(x) = \dfrac{e^{-\lambda}.\lambda^x}{x!}$

$$\therefore \quad p(x+1) = \frac{e^{-\lambda}.\lambda^{x+1}}{(x+1)!} = \frac{\lambda}{x+1}.\frac{e^{-\lambda}.\lambda^x}{x!} = \frac{\lambda}{x+1}p(x)$$

Thus $\boldsymbol{p(x+1) = \left(\dfrac{\lambda}{x+1}\right)p(x)}$

or $\boldsymbol{p(x) = \dfrac{\lambda}{x}.p(x-1)}$

which is the required relation. With this formula we can find $p(1), p(2), p(3), \ldots$ if $p(0)$ is given.

Note : 1. On successive application of the formula, we get

$$p(x+1) = \frac{\lambda^{x+1}}{(x+1)!}.p(0)$$

2. $P(X \leq t) = \left[1 + \dfrac{\lambda}{1!} + \dfrac{\lambda^2}{2!} + \ldots + \dfrac{\lambda^t}{t!}\right].p(0)$

3. $P(a \leq X \leq b) = \left[\dfrac{\lambda^a}{a!} + \dfrac{\lambda^{a+1}}{(a+1)!} + \ldots + \dfrac{\lambda^b}{b!}\right].p(0)$, for all $0 \leq a, b, a, b \in Z^+$

Properties of Poisson Distribution

1. Range of the variable is from 0 to ∞.
2. Mean and Variance are equal.
3. Distribution gets more and more symmetrical about the mean as λ increases and tends to normal distribution, described in the next section.

SOLVED EXAMPLES

Example 1 : If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals (*i*) exactly (*ii*) more than 2 individuals (*iii*) none (*iv*) more than one individual suffer a bad reaction.

Solution : Given $p = 0.001$ and $n = 2000$

$\therefore$ Mean, $\lambda = np = 2000\ (0.001) = 2$

Thus $P(r) = \dfrac{e^{-\lambda}.\lambda^r}{r!} = \dfrac{e^{-2}.2^r}{r!} = \dfrac{2^r}{e^2 r!}$

(*i*) $P(3) = \dfrac{1}{e^2} \cdot \dfrac{2^3}{3!} = \dfrac{8}{6(2.718)^2} = 0.1804$

(*ii*) $P(\text{more than } 2) = P(r \geq 2) = P(3) + P(4) + \ldots + P(2000)$

$$= 1 - [P(0) + P(1) + P(2)] = 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{4}{2e^2}\right]$$

$$= 1 - \frac{1}{e^2}(1 + 2 + 2) = 1 - \frac{5}{e^2} = 1 - 0.67667 = 0.3233$$

(*iii*) $P(\text{none}) = P(0) = \dfrac{2^0}{e^2 0!} = \dfrac{1}{e^2} = 0.1353$

(*iv*) $P(\text{more than one}) = P(r > 1) = P(2) + P(3) + \ldots + P(2000) = 1 - [P(0) + P(1)]$

$$= 1 - \left(\frac{1}{e^2} + \frac{2}{e^2}\right) = 1 - \frac{3}{e^2} = 1 - 0.406 = 0.594$$

Example 2 : A car-hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (*i*) on which there is no demand (*ii*) on which demand is refused.

[JNTU 2003]

Solution : Given mean, $\lambda = 1.5$

We have $P(r) = \dfrac{e^{-\lambda}.\lambda^r}{r!}$

(*i*) $P(\text{no demand}) = P(0) = \dfrac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5} = 0.2231$

Note : Number of days in a year there is no demand of car = 365 (0.2231) = 81 days

(*ii*) Some demand is refused if the number of demands is more than two i.e., $r > 2$

$P(\text{demand refused}) = P(r > 2) = 1 - [P(0) + P(1) + P(2)]$

$$= 1 - \left[e^{-1.5} + \frac{e^{-1.5}(1.5)}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$$= 1 - e^{-1.5}[1 + 1.5 + 1.125]$$

$$= 1 - 3.625\ (e^{-1.5}) = = 1 - 0.8088 = 0.1913$$

Note : Number of days in a year when some demand is refused

$= 365 \times 0.1913 = 69.82 = 70$ days

Example 3 : A hospital switch board receives an aveage of 4 emergency calls in a 10 minute interval. What is the probability that (*i*) there are at most 2 emergency calls in a 10 minute interval (*ii*) there are exactly 3 emergency calls in a 10 minute interval.

[JNTU 2000]

Solution : Mean, λ = (4 calls / 10 minutes) = 4 calls

$\therefore\ P(X = x) = p(x) = \dfrac{e^{-\lambda}.\lambda^x}{x!} = \dfrac{e^{-4}.4^x}{x!} = \dfrac{1}{e^4} \cdot \dfrac{4^x}{x!}$

(*i*) $P\ (\text{at most 2 calls}) = P(X \leq 2)$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{1}{e^4} + \frac{1}{e^4} \cdot 4 + \frac{1}{e^4} \cdot \frac{4^2}{2!}$$

$$= \frac{1}{e^4}(1 + 4 + 8) = 13\ e^{-4} = 0.2381$$

(*ii*) $P\ (\text{exactly 3 calls}) = P(X = 3) = \dfrac{1}{e^4} \cdot \dfrac{4^3}{3!} = \dfrac{32}{3}\ e^{-4} = 0.1954$

Example 4 : A manufacturer knows that the condensers he makes contain on average 1% defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers ?

Solution : Here p = Probability of defective condensers = 1% = 0.01

n = Total number of condensers = 100

$\therefore\ \lambda = \text{mean} = np = 100\ (0.01) = 1$

From Poisson Distribution,

$$P(X = x) = p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-1}.1^x}{x!} = \frac{e^{-1}}{x!}$$

$$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - [e^{-1} + e^{-1} + \frac{e^{-1}}{2}] = 1 - e^{-1} \cdot \frac{5}{2}$$

$$= 1 - 0.9197 = 0.0803$$

Example 5 : If a bank received on the average 6 bad cheques per day, find the probability that it will receive 4 bad cheques on any given day.

Solution : We have $P(X = x) = \dfrac{e^{-\lambda}.\lambda^x}{x!}$

Here $\lambda = 6$

$$\therefore\ P(X=4) = \frac{e^{-6}6^4}{4!} = \frac{54}{e^6} = 0.1339$$

Example 6: A manufacturer of cotter pins knows that 5% of his product is defective. Pins are sold in boxes of 100. He guarantees that not more than 10 pins will be defective. What is the approximate probability that a box will fail to meet the guaranteed quality ?

[JNTU 2000S]

Solution : The probability of cotter pins to be defective $= p = 5\% = 0.05$

Total number of cotter pins, $n = 100$

$\therefore$ Mean, $\lambda = np = 100\ (0.05) = 5$

We have $P(X=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}$

$\therefore\ P(X=x) = \dfrac{e^{-5}5^x}{x!}$

P(a box will fail to meet the guarantee)

$$= P(X>10) = 1 - P(X \le 10)$$

$$= 1 - [P(X=0) + P(X=1) + \ldots + P(X=10)]$$

$$= 1 - \left[\frac{e^{-5}(5)^0}{0!} + \frac{e^{-5}(5)^1}{1!} + \frac{e^{-5}(5)^2}{2!} + \frac{e^{-5}(5)^3}{3!} + \ldots + \frac{e^{-5}(5)^{10}}{10!}\right]$$

$$= 1 - 0.9863 = 0.0137$$

Example 7: If the probability is 0.05 that a certain wide-flange column will fill under a given axial load. What are the probability that among 16 such columns

(*i*) at most two will fail.

(*ii*) at least four will fail.

[JNTU 2008 (Set No. 4)]

Solution : Probability that a certain column will fail under a given axial load = $.05 = p$

Number of columns = $16 = n$

Now $\lambda = np = 16\ (.05) = 0.80$

$\therefore$ P(r columns will fail) $= \dfrac{e^{-\lambda}\lambda^r}{r!}$

(*i*) P(at most two will fail) = P(0) + P(1) + P(2)

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!}\right) \text{ where } \lambda = 0.8$$

(*ii*) P(at least four will fail) = P(4) + P(5) ..... + P(16)

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 1 - e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!}\right]$$

(It is assumed that the probability distribution is Poisson; as p is very small).

Example 8 : It has been found that 2% of the tools produced by a certain machine are defective. What is the probability that in a shipment of 400 such tools

(*a*) 3% or more

(*b*) 2% or less will prove defective.

[JNTU (A) May 2010S]

Solution : We are given $n = 400$.

Let p = Probability of a defective tool $= 2\% = 0.02$.

$\therefore\ \lambda$ = Mean number of defective tools in a shipment of 400

$$= np = 400 \times 0.02 = 8$$

Since 'p' is small, we may use Poisson distribution.

Probability of x defective tools in a shipment of 400 is

$$P(X=x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-8}8^x}{x!}, x = 0,1,2,......$$

(*a*) $P(X \ge 3\%) = P(X \ge 12) = 1 - P(X < 12)$

$$= 1 - [P(X=0) + P(X=1) + ... + P(X=11)]$$

$$= 1 - \left[e^{-8} + \frac{e^{-8}.8}{1!} + \frac{e^{-8}.8^2}{2!} + ... + \frac{e^{-8}.8^{11}}{11!}\right]$$

$$= 1 - e^{-8}\left[1 + 8 + 32 + \frac{8^3}{3!} + \frac{8^4}{4!} + \frac{8^5}{5!} + \frac{8^6}{6!} + \frac{8^7}{7!} + \frac{8^8}{8!} + \frac{8^9}{9!} + \frac{8^{10}}{10!} + \frac{8^{11}}{11!}\right]$$

$$= 1 - (e^{-8})[41 + 85.33 + 170.66 + 273.06 + 364.09 + 416.10 + 416.10 + 369.87 + 295.89 + 215.19]$$

$$= 1 - (e^{-8})(2647.29) = 0.1119$$

Example 9 : If a random variable has a poisson distribution such that P(1) = P(2), find (*i*) mean of the distribution (*ii*) $P(4)$ (*iii*) $P(x \ge 1)$ (*iv*) $P(1 < x < 4)$.

[JNTU 2007S (Set No. 2, 3)]

Solution : Given $P(1) = P(2) \Rightarrow \dfrac{P(2)}{P(1)} = 1$ ... (1)

By recurrence relation, we have

$$p(x+1) = \frac{\lambda}{x+1}\, p(x)$$

Put $x = 1$. Then $p(2) = \dfrac{\lambda}{2}\, p(1)$

$$\Rightarrow \frac{p(2)}{p(1)} = \frac{\lambda}{2} \Rightarrow \frac{\lambda}{2} = 1, \text{ using (1)}$$

$\therefore\ \lambda = 2$

Hence mean of the distribution = 2

Alternate Method :

$P(1) = P(2)$

Hence $\frac{e^{-\lambda}\lambda^1}{1!} = e^{-\lambda}\frac{\lambda^2}{2!}$

$\Rightarrow \lambda^2 = 2\lambda \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda-2) = 0$

$\therefore \lambda = 0$ or 2

But $\lambda \neq 0 \quad \therefore \lambda = 2$

(*ii*) $p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-2}.2^x}{x!}$

$$P(X=4) = p(4) = \frac{e^{-2}.2^4}{4!} = \frac{2}{3e^2} = 0.09022$$

(*iii*) $P(x \geq 1) = 1 - P(x < 1) = 1 - P(x = 0)$

$$= 1 - \frac{e^{-2}.2^0}{0!} = 1 - \frac{1}{e^2} = 0.8647$$

(*iv*) $P(1 < x < 4) = P(x = 2) + P(x = 3)$

$$= \frac{e^{-2}.2^2}{2!} + \frac{e^{-2}.2^3}{3!} = 2e^{-2} + \frac{4}{3}e^{-2} = e^{-2}\left(\frac{10}{3}\right) = 0.4511$$

Example 10 : Using Poisson's distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials.

Solution : Here $p = \frac{1}{52}$ and $n = 104$

$\therefore$ Mean of the distribution, $\lambda = np = \frac{104}{52} = 2$

$P(\text{at least once}) = P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - \frac{e^{-2}(2)^0}{0!} = 1 - 0.1353 = 0.8647$$

Example 11 : The average number of phone calls / minute coming into a switch board between 2 p.m and 4 p.m is 2.5. Determine the probability that during one particular minute there will be (*i*) 4 or fewer (*ii*) more than 6 calls. **[JNTU 2006 S, 2007 (Set No. 1)]**

Solution : Let x be the number of phone calls / minute coming into a switch board.

Given mean, $\lambda = 2.5$

Now the Poisson distribution is

$$p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-2.5}.(2.5)^x}{x!}$$

(*i*) $p(x \leq 4) = p(x=0) + p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$$= e^{-2.5}\left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!}\right]$$

$$= e^{-2.5}[1 + 2.5 + 3.125 + 2.6042 + 1.6276]$$
$$= (0.080209)(10.8568) = 0.8912$$

(*ii*) $p(x > 6) = 1 - p(x \leq 6) = 1 - e^{-2.5}\left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1!} + ... + \frac{(2.5)^6}{6!}\right]$

$$= 1 - e^{-2.5}[1 + 2.5 + 3.125 + 2.6042 + 1.6276 + 0.8138 + 0.3391]$$
$$= 1 - (0.080209)(12.01) = 0.01416$$

Example 12 : 2% of the items of a factory are defective. The items are packed in boxes. What is the probability that there will be

(*i*) 2 defective items (*ii*) at least three defective items

in a box of 100 items ? **[JNTU 2006S, (K) Nov. 2011 (Set No. 4)]**

Solution : Given $n = 100$

and p = The probability of defective items = 2% = 0.02

λ = Mean number of defective items in a box of 100 = $np = (0.02)(100) = 2$

Since p is small, we may use Poisson distribution. Probability of x defective items in a box of 100 is

$$P(X = x) = p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-2}.2^x}{x!};$$

(*i*) $p(\text{2 defective items}) = p(2) = \frac{e^{-2}.2^2}{2!} = \frac{2}{e^2} = 0.2706$

(*ii*) $p(\text{at least 3}) = p(x \geq 3) = 1 - [p(x=0) + p(x=1) + p(x=2)]$

$$= 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right]$$
$$= 1 - e^{-2}[1 + 2 + 2]$$
$$= 1 - 5e^{-2} = 0.3233$$

Example 13 : Average number of accidents on any day on a national highway is 1.8. Determine the probability that the number of accidents are (*i*) at least one (*ii*) at most one **[JNTU 2004S, 2005, 2007(Set No. 3)]**

Solution : Mean, $\lambda = 1.8$

We have $P(X = x) = p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-1.8}.(1.8)^x}{x!}$

(*i*) $P(\text{at least one}) = P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1.8} = 1 - 0.1653 = 0.8347$

(*ii*) $P(\text{at most one}) = P(X \leq 1) = P(X = 0) + P(X = 1)$

$$= e^{-1.8} + e^{-1.8}(1.8) = e^{-1.8}(2.8) = 0.4628$$

Example 14 : Average number of accidents on any day on a national highway is 1.6. Determine the probability that the number of accidents are **[JNTU 2008 (Set No. 2)]**

(*i*) at least one (*ii*) at most one

Solution : Given Mean, $\lambda = 1.6$. We have

$$p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-1.6}(1.6)^x}{x!}$$

(*i*) P(at least one) $= P(X \geq 1) = 1 - P(X = 0)$

$$= 1 - e^{-1.6} = 1 - 0.2019 = 0.7981$$

(*ii*) P(at most one) $= P(X \leq 1) = P(X = 0) + P(X = 1)$

$$= e^{-1.6} + e^{-1.6}(1.6) = e^{-1.6}(2.6)$$

$$= (0.2019)(2.6) = 0.5249$$

Example 15 : Number of monthly breakdowns of a computer is a random variable having Poisson distribution with mean equal to 1.8. Find the probability that the computer will function for a month (*i*) without a breakdown (*ii*) with only one breakdown and (*iii*) with at least one. **[JNTU (K) 2009 (Set No. 3)]**

Solution : Given mean, $\lambda = 1.8$

From Poisson distribution, $P(X = x) = p(x) = \frac{e^{-\lambda}\cdot\lambda^x}{x!} = \frac{(e^{-1.8})(1.8)^x}{x!}$

(*i*) $p(X = 0) = \frac{(e^{-1.8})(1.8)^0}{0!} = e^{-1.8} = 0.1653$

(*ii*) $p(X = 1) = (e^{-1.8})(1.8) = 0.2975$

(*iii*) $p(X \geq 1) = 1 - p(X < 1) = 1 - p(X = 0) = 1 - 0.1653 = 0.8347$

Example 16 : If X is a Poisson variate such that $p(x = 0) = p(x = 1)$, find $p(x = 0)$ and using recurrence formula find the probabilities at $x = 1, 2, 3, 4$, and 5.

[JNTU 2004S (Set No. 4)]

Solution : Probability distribution is $p(x) = \frac{e^{-\lambda}.\lambda^x}{x!}$

Given $p(x = 0) = p(x = 1)$

$$\Rightarrow \frac{e^{-\lambda}.\lambda^0}{0!} = \frac{e^{-\lambda}.\lambda}{1!} \quad i.e., e^{-\lambda} = \lambda.e^{-\lambda} \quad \therefore \lambda = 1$$

(*i*) Now $p(x = 0) = \frac{e^{-\lambda}.\lambda^0}{0!} = e^{-1} = 0.3678$

(*ii*) By Recurrence formula, we have

$$p(x+1) = \frac{\lambda}{x+1}p(x)$$

$$\therefore \quad p(x+1) = \frac{1}{x+1}p(x) \quad (\because \lambda = 1) \qquad \ldots (1)$$

Now $p(1) = p(0) = 0.3678$

Put $x = 1$ in (1). Then

$$p(2) = \frac{1}{2}p(1) = \frac{1}{2}(0.3678) = 0.1839$$

Put $x = 2$ in (1). Then

$$p(3) = \frac{1}{3}p(2) = \frac{1}{3}(0.1839) = 0.0613$$

Put $x = 3$ in (1). Then

$$p(4) = \frac{1}{4}p(3) = \frac{1}{4}(0.0613) = 0.015325$$

Put $x = 4$ in (1). Then

$$p(5) = \frac{1}{5}p(4) = \frac{1}{5}(0.015325) = 0.003065$$

Example 17 : Using recurrence formula find the probabilities when $x = 0, 1, 2, 3, 4$ and 5, if the mean of Poisson distribution is 3. **[JNTU 2008, (A) Nov. 2010 (Set No. 3)]**

Solution : Given mean of the Poisson distribution is $3 \Rightarrow \lambda = 3$. Now the Poisson distribution is

$$p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-3}.3^x}{x!}$$

$$\therefore p(x = 0) = \frac{e^{-3}.3^0}{0!} = e^{-3} = 0.0498$$

By recurrence formula, we have

$$p(x+1) = \frac{\lambda}{x+1}p(x)$$

$$\Rightarrow p(x+1) = \frac{3}{x+1}p(x)[\because \lambda = 3] \qquad \ldots\ldots (1)$$

Put $x = 0$ in (1). Then

$p(1) = 3p(0) = 3(0.0498) = 0.1494$

Put $x = 1$ in (1). Then

$$p(2) = \frac{3}{2}p(1) = \frac{3}{2}(0.1494) = 0.2241$$

Put $x = 2$ in (1). Then

$$p(3) = \frac{3}{3}p(2) = p(2) = 0.2241$$

Put $x = 3$ in (1). Then

$$p(4) = \frac{3}{4}p(3) = \frac{3}{4}(0.2241) = 0.1681$$

Put $x = 4$ in (1). Then

$$p(5) = \frac{3}{5}p(4) = \frac{3}{5}(0.1681) = 0.1008$$

Example 18 : A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a particular packet will violate the guarantee.
[JNTU 2005, JNTU (A) Dec. 2009 (Set No. 4)]

Solution : p = The probability of a seed not germinating = 5% = 0.05

λ = mean number of seeds in a sample of 200

$= np = 200 \times 0.05 = 10$

Let x be the number of seeds that do not germinate. Then

$$p(x) = \frac{e^{-10}.10^x}{x!}$$

A packet will violate guarantee if it contains more than 20 germination seeds.

Probability that the guarantee is violated $= P(X > 20) = 1 - P(X \le 20)$

$$= 1 - \sum_{x=0}^{20} \frac{e^{-10}10^x}{x!} = 1 - 0.9984 = 0.0016$$

Example 19 : Suppose 2% of the people on the average are left handed. Find (*i*) the probability of finding 3 or more left handed (*ii*) the probability of finding none or one left handed.
[JNTU 2005 (Set No.3)]

Solution : Let x be the number of left handed.

Given mean, $\lambda = 2\% = 0.02$

We have $p(x) = \dfrac{e^{-\lambda}.\lambda^x}{x!} = \dfrac{e^{-0.02}(0.02)^x}{x!}$

(*i*) $p(x \ge 3) = 1 - p(x < 3) = 1 - [p(0) + p(1) + p(2)]$

$$= 1 - e^{-0.002}\left[1 + 0.02 + \frac{(0.02)^2}{2}\right]$$

$$= 1 - e^{-0.02}(1.0202) = 1 - 0.9802\ (1.0202) = 1.3077 \times 10^{-6}$$

(*ii*) $p(x \le 1) = p(0) + p(1)$

$$= e^{-0.02} + e^{-0.002}(0.02) = e^{-0.02}(1.02) = 0.9998$$

Example 20 : If a Poisson distribution is such that $P(X = 1)\,.\,\dfrac{3}{2} = P(X = 3)$, find

(*i*) $P(X \ge 1)$ (*ii*) $P(X \le 3)$ (*iii*) $P(2 \le X \le 5)$ [JNTU 2005 S (Set Nos. 1, 2,3)]

Solution : Given $\dfrac{3}{2}.\,P(X = 1) = P(X = 3)$

i.e., $\dfrac{3}{2}\cdot\dfrac{e^{-\lambda}.\lambda^1}{1!} = \dfrac{e^{-\lambda}.\lambda^3}{3!}$ *i.e.*, $\dfrac{3\lambda}{2} = \dfrac{\lambda^3}{6}$

i.e., $\lambda^3 - 9\lambda = 0$ or $\lambda(\lambda^2 - 9) = 0$ or $\lambda(\lambda - 3)(\lambda + 3) = 0$

$\therefore \lambda = 0, 3, -3$

$\Rightarrow \lambda = 3\ (\because \lambda > 0)$

Hence $P(X = x) = p(x) = \dfrac{e^{-3}3^x}{x!}$

(*i*) $P(X \ge 1) = 1 - P(x < 1) = 1 - P(X = 0) = 1 - \dfrac{e^{-3}.3^0}{0!}$

$$= 1 - e^{-3} = 0.950213$$

(*ii*) $P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= e^{-3}\left[\frac{3^0}{0!} + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right] = e^{-3}\left(1 + 3 + \frac{9}{2} + \frac{9}{2}\right) = 13\,e^{-3} = 0.6472318$$

(*iii*) $P(2 \le X \le 5) = P(X = 2) + P(X = 3) + (P(X = 4) + P(X = 5)$

$$= e^{-3}\left[\frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!}\right]$$

$$= 9e^{-3}\left(\frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{9}{40}\right) = 9e^{-3}(1.6) = 0.7169337$$

Example 21 : If the variance of a Poisson variate is 3, then find the probability that (*i*) $x = 0$ (*ii*) $0 < x \le 3$ (*iii*) $1 \le x < 4$.
[JNTU 2007, 2007S, (Set No. 4)]

Solution : Given variance of the Poisson distribution = 3

For a Poisson distribution, we have mean = variance

Hence mean of the Poisson distribution = 3 *i.e.*, $\lambda = 3$

So $P(X = x) = p(x) = \dfrac{e^{-\lambda}.\lambda^x}{x!} = \dfrac{e^{-3}.3^x}{x!}$

(*i*) $p(x = 0) = \dfrac{e^{-3}.3^0}{0!} = e^{-3} = 0.04979$

(*ii*) $p(0 < x \le 3) = p(x = 1) + p(x = 2) + p(x = 3)$

$$= e^{-3}.3 + e^{-3}.\frac{3^2}{2!} + e^{-3}.\frac{3^3}{3!}$$

$$= e^{-3}\left(3 + \frac{9}{2} + \frac{9}{2}\right) = e^{-3}(12) = 0.5974$$

(*iii*) $p(1 \le x < 4) = p(x = 1) + p(x = 2) + p(x = 3)$

$= 0.5974$, as in (*ii*)

Example 22 : If two cards are drawn from a pack of 52 cards which are diamonds, using Poisson distribution find the probability of getting two diamonds at least 3 times in 51 consecutive trials of two cards drawing each time.
[JNTU (A) Dec. 2009 (Set No. 3)]

Solution : The probability of getting two diamonds from a pack of 52 cards =

$$p = \frac{{}^{13}C_2}{{}^{52}C_2} = \frac{3}{51}$$

Number of trials, $n = 51$

$\therefore \lambda = \text{Mean} = np = \frac{3}{51}(51) = 3$

Thus the Poisson distribution is

$$p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-3}.3^x}{x!}$$

$$P(\text{at least 3 times}) = P(X \geq 3) = 1 - P(X < 3)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - [e^{-3} + 3e^{-3} + e^{-3} . \frac{3^2}{2!}]$$

$$= 1 - e^{-3}\left(1 + 3 + \frac{9}{2}\right) = 1 - e^{-3}\left(\frac{17}{2}\right)$$

$$= 1 - 0.42319 = 0.57681$$

Example 23 : Prove that the three successive values of a Poisson variate can not have equal probability of success.

Solution : Let $x - 1$, x and $x + 1$ be three successive Poisson variates.

Suppose that $p(x-1) = p(x) = p(x+1)$ .... (1)

Then by recurrence formula,

$$p(x+1) = \frac{\lambda}{x+1} p(x)$$

(1) gives $\frac{\lambda}{x} p(x) = p(x) = \frac{\lambda}{x+1} p(x)$

This implies $\frac{\lambda}{x} = 1 = \frac{\lambda}{x+1} \Rightarrow \lambda = x$ and $\lambda = x + 1$.

But as x and $x + 1$ are distinct and mean λ is unique for a poisson distribution, this is not admissible. Hence three successive values of a poisson variate can not be have equal probability of success.

Example 24 : If X is a Poisson Variate such that $3p(x = 4) = \frac{1}{2} p(x = 2) + p(x = 0)$, find (*i*) the mean of x (*ii*) $p(x \leq 2)$ **[JNTU 2008S, (A) Nov. 2010, (H) May 2011 (Set No. 1)]**

Solution : (*i*) If X is a Poisson Variate with parameter λ, then

$$P(X = x) = \frac{e^{-\lambda}.\lambda^x}{x!}, x = 0, 1, 2, ..., \lambda > 0$$

Here we are given $3p(x = 4) = \frac{1}{2} p(x = 2) + p(x = 0)$

$$\Rightarrow 3.\frac{e^{-\lambda}\lambda^4}{4!} = \frac{1}{2} . \frac{e^{-\lambda}.\lambda^2}{2!} + e^{-\lambda}$$

$$\Rightarrow \lambda^4 - 2\lambda^2 - 8 = 0 \Rightarrow (\lambda^2 - 4)(\lambda^2 + 2) = 0$$

$$\Rightarrow \lambda = \pm 2 \Rightarrow \lambda = 2 \quad (\because \lambda > 0)$$

$\therefore$ Mean of the Poisson variate *i.e.*, $\lambda = 2$

(*ii*) Now we have $P(X = x) = \frac{e^{-2}.2^x}{x!}$

$$p(x \leq 2) = p(x = 0) + p(x = 1) + p(x = 2)$$

$$= e^{-2} + \frac{1}{4} e^{-1} + \frac{e^{-2}.2^2}{2!}$$

$$= 3e^{-2} + \frac{1}{4}e^{-1} = 3(0.135) + \frac{1}{4}(0.3679)$$

$$= 0.498$$

Example 25 : If 2% of light bulbs are defective, find the probability if

(*i*) At least one is defective. (*ii*) Exactly 7 are defective.

(*iii*) $P(1 < x < 8)$ in a sample of 100 **[JNTU (H) Dec. 2011 (Set No. 1)]**

Solution : We are given $n = 100$

Let p = The probabilityof a defective bulb = 2% = 0.02

$\therefore \lambda$ = Mean number of defective bulbs in a sample of 100

$$= np = 100 \times 0.02 = 2$$

Since 'p' is small, we may use Poisson distribution.

Probability of x defective bulbs in a sample of 100 is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}; x = 0, 1, 2, ...$$

(*i*) P(At least one is defective) $= p(x \geq 1)$

$$= 1 - p(x = 0)$$

$$= 1 - \frac{e^{-2}.2^0}{0!} = 1 - e^{-2}$$

$$= 1 - 0.1353 = 0.8647$$

(*ii*) P(Exactly 7 are defective) $= P(x = 7)$

$$= \frac{e^{-2}2^7}{7!} = \frac{0.1353 \times 128}{5040} = 0.0034$$

(*iii*) $P(1 < x < 8) = P(x = 2) + P(x = 3) + P(x = 4) + P(x = 5) + P(x = 6) + P(x = 7)$

$$= \frac{e^{-2}.2^2}{2!} + \frac{e^{-2}.2^3}{3!} + \frac{e^{-2}.2^4}{4!} + \frac{e^{-2}.2^5}{5!} + \frac{e^{-2}.2^6}{6!} + \frac{e^{-2}.2^7}{7!}$$

$$= \frac{e^{-2}}{2}\left(4 + \frac{8}{3} + \frac{16}{12} + \frac{32}{60} + \frac{64}{360} + \frac{128}{2520}\right)$$

$$= \frac{0.1353}{2}(4 + 2.6666 + 1.3333 + 0.5333 + 0.1777 + 0.0508)$$

$$= \frac{0.1353}{2}(8.7617) = 0.593$$

Example 26 : Given that $p(x=2) = 9p(x=4) + 90p(x=46)$ for a Poisson variate X. Find :

(i) $P(x<2)$ (ii) $P(x>4)$ (iii) $P(x \geq 1)$ **[JNTU (H) Dec. 2011 (Set No. 2)]**

Solution : If X is a Poisson variate with parameter λ, then

$$P(X=x) = \frac{e^{-\lambda}.\lambda^x}{x!}, x = 0,1,2,.....;\lambda > 0$$

Since $P(X=2) = 9P(X=4) + 90P(X=6)$, we have

$$\frac{e^{-\lambda}.\lambda^2}{2!} = 9.\frac{e^{-\lambda}.\lambda^4}{4!} + 90.\frac{e^{-\lambda}.\lambda^6}{6!}$$

$$= \frac{e^{-\lambda}.\lambda^2}{8}[3\lambda^2 + \lambda^4]$$

or $4 = 3\lambda^2 + \lambda^4$ or $\lambda^4 + 3\lambda^2 - 4 = 0$ or $(\lambda^2+4)(\lambda^2-1) = 0$

$\therefore \lambda^2 + 4 = 0$ or $\lambda^2 - 1 = 0$. But λ cannot be imaginary.

[(or) This is a quadratic equation in λ^2

$$\therefore \lambda^2 = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm 5}{2} = 1, -4]$$

Since $\lambda > 0$, we get

$\lambda^2 = 1 \Rightarrow \lambda = 1$, taking positive sign

Hence $P(X=x) = \frac{e^{-1}.1^x}{x!}, x = 0,1,2,.....$

(i) $P(x<2) = P(x=0) + P(x=1)$

$$= e^{-1} + e^{-1} = \frac{2}{e} = 0.7358$$

(ii) $P(x>4) = 1 - P(x \leq 4)$

$$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)]$$

$$= 1 - \left[e^{-1} + e^{-1} + \frac{e^{-1}}{2} + \frac{e^{-1}}{3!} + \frac{e^{-1}}{4!}\right]$$

$$= 1 - e^{-1}\left[2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\right] = 1 - e^{-1}\left[\frac{65}{24}\right]$$

$$= 1 - 0.9963 = 0.0037$$

(iii) $P(x \geq 1) = 1 - p(x=0) = 1 - e^{-1} = 1 - 0.3679 = 0.6321$

Example 27 : If the Mean of a Poisson variable is 1.8, then find

(i) $p(x>1)$ (ii) $p(x=5)$ (iii) $p(0<x<5)$ **[JNTU (H) Dec. 2011 (Set No.3)]**

Solution : Given mean of a Poisson variate, $\lambda = 1.8$

For Poisson distribution, the probability of x successes is

$$P(X=x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-1.8}(1.8)^x}{x!}; x = 0,1,2,.....$$

(i) $p(x>1) = 1 - p(x \leq 1)$

$$= 1 - [p(x=0) + p(x=1)]$$

$$= 1 - [e^{-1.8} + e^{-1.8}(1.8)] = 1 - e^{-1.8}(1+1.8) = 1 - 0.4628 = 0.5372$$

(ii) $p(x=5) = \frac{e^{-1.8}(1.8)^5}{5!} = \frac{3.1234}{120} = 0.026$

(iii) $p(0<x<5) = p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$$= e^{-1.8} + \frac{e^{-1.8}(1.8)^2}{2!} + \frac{e^{-1.8}(1.8)^3}{3!} + \frac{e^{-1.8}(1.8)^4}{4!}$$

$$= e^{-1.8}\left[1 + \frac{(1.8)^2}{2} + \frac{(1.8)^3}{6} + \frac{(1.8)^4}{24}\right] = e^{-1.8}[1 + 1.62 + 0.972 + 0.4374]$$

$$= e^{-1.8}(4.0294) = 0.666$$

Example 28 : If 2% of light bulbs are defective. Find

(i) At least one is defective. (ii) Exactly 7 are defective

(iii) $p(1<x<8)$ in a sample of 100 **[JNTU (H) Apr. 2012 (Set No. 3)]**

Solution : Given $n = 100$ and

p = The probability of defective light bulbs = 0.02

$\therefore$ Mean $= \lambda =$ Mean number of defective light bulbs in a sample of 100.

$$= np = 100(0.02) = 2$$

Since p is small, we use Poisson distribution.

Probability of x defective light bulbs in a sample of 100 is

$$P(X = x) = p(x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-2}.2^x}{x!}; x = 0,1,2,......$$

(i) p (at least one is defective) $= p(x \geq 1) = 1 - p(x = 0)$

$$= 1 - \frac{e^{-2}.2^0}{0!} = 1 - e^{-2} = 0.8646$$

(ii) p (exactly 7 are defective) $= p(x = 7)$

$$= \frac{e^{-2}.2^7}{7!} = \frac{128\,e^{-2}}{5040} = 0.0034$$

(iii) $p(1 < x < 8) = p(x = 2) + p(x = 3) + p(x = 4) + p(x = 5) + p(x = 6) + p(x = 7)$

$$= e^{-2}\left[\frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + \frac{2^7}{7!}\right]$$

$$= e^{-2}\left[\frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \frac{64}{720} + \frac{128}{5040}\right]$$

$$= e^{-2}(2 + 1.33 + 0.66 + 0.27 + 0.09 + 0.02)$$

$$= e^{-2}(4.37) = 0.59$$

Example 29 : Poisson variable has a double mode at $x = 2$ and $x = 3$, find the maximum probability and also find $p(x \geq 2)$. **[JNTU (H) Apr. 2012 (Set No. 4)]**

Solution : We know that the Poisson distribution is bimodal and two modes are at the points $x = \lambda - 1$ and $x = \lambda$.

Since we are given that the two modes are at the points $x = 1$ and $x = 2$, we find that $\lambda = 2$.

$$\therefore P(X = x) = \frac{e^{-\lambda}.\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}; x = 0,1,2,.....$$

Now $P(X = 1) = 2e^{-2}$ and $P(X = 2) = \dfrac{e^{-2}.2^2}{2!} = 2e^{-2}$

Hence the required probability $= P(X = 1) + P(X = 2) = 2e^{-2} + 2e^{-2} = 4e^{-2} = 0.5413$

Now $p(x \geq 2) = 1 - p(x < 2)$

$$= 1 - [p(x = 0) + p(x = 1) + p(x = 2)]$$

$$= 1 - [e^{-2} + 2e^{-2} + 2e^{-2}] = 1 - 5e^{-2} = 1 - 0.676$$

$$= 0.3233$$

Example 30 : A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items? **[JNTU (K) May 2010 (Set No. 2)]**

Solution : The probability of defective is $p = \dfrac{4}{10} = \dfrac{2}{5}$

Number of items chosen, $n = 3$

The expected number of defectives is $E(X) = \mu = np = 3\left(\dfrac{2}{5}\right) = \dfrac{6}{5} = 1.2 \approx 1$

Alternative Method :

No. of Exhaustive cases $= {}^{10}C_3 = \dfrac{10!}{3!\,7!} = 120$

Probability that there are no defective items $= p(x = 0) = \dfrac{{}^6C_3}{120} = \dfrac{1}{6}$

Probability that there is one defective item $= p(x = 1) = \dfrac{{}^6C_2.{}^4C_1}{120} = \dfrac{1}{2}$

Probability that there are two defective items $= p(x = 2) = \dfrac{{}^6C_1.{}^4C_2}{120} = \dfrac{3}{10}$

Probability that there are three defective items $= p(x = 3) = \dfrac{{}^4C_3}{120} = \dfrac{1}{30}$

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

$\therefore$ Expected number of defective items $= E(x) = \Sigma\, p_i x_i$

$$= 0.\frac{1}{6} + 1.\frac{1}{2} + 2.\frac{3}{10} + 3.\frac{1}{30}$$

$$= 0 + \frac{1}{2} + \frac{3}{5} + \frac{1}{10} = \frac{12}{10} = 1.2$$

Example 31 : Wireless sets are manufactured with 25 soldered joints each. On the average 1 joint in 500 is defective. How many sets can be expected to be free from defective joints in a consignment of 10,000 sets. **[JNTU Nov. 2008 (Set No. 3)]**

Solution : On average, one joint in 500 joints is defective

Number of soldered joints $= 25 = n$

Probability that a joint is defective $= \dfrac{1}{500}$

$\therefore$ Mean $= np = 25 \times \dfrac{1}{500} = \dfrac{1}{20} = .05$

Thus $\lambda = .05$

P(There are r defective joints) $= \dfrac{e^{-\lambda}.\lambda^r}{r!}$

$\therefore \quad P(r = 0) = e^{-\lambda} = e^{-.05}$

Hence expected number of sets free of defective joints among 10000 sets

$$= 10000 \,.\, e^{-.05} = 10000\,(0.951229) = 9512.29 \approx 9512$$

Example 32 : Fit a Poisson distribution for the following data and calculate the expected frequencies

x	0	1	2	3	4
$f(x)$	109	65	22	3	1

[JNTU 2004S]

Solution : Here N = total frequency $= \Sigma f_i = 109 + 65 + 22 + 3 + 1 = 200$

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0+65+44+9+4}{200} = \frac{122}{200} = 0.61$$

$\therefore$ Mean of Poisson distribution, $\lambda = 0.61$

Hence the theoretical frequencies are given by N.P(x), where $x = 0, 1, 2, 3, 4$

i.e., $\quad 200 \,.\, \dfrac{e^{-0.61}(0.61)^x}{x!}$, where $x = 0, 1, 2, 3, 4$

i.e., $\quad 200\, e^{-0.61},\ 200\, e^{-0.61}(0.61),\ 200\, e^{-0.61}.\dfrac{(0.61)^2}{2!},$

$\qquad 200\, e^{-0.61}.\dfrac{(0.61)^3}{3!},\ 200\, e^{-0.61}.\dfrac{(0.61)^4}{4!}$

i.e., 108.67, 66.29, 20.22, 4.11, 0.63

Since frequencies are always integers, therefore by converting them to nearest integers, we get

Observed frequency	109	65	22	3	1
Expected frequency	109	66	20	4	1

Example 33 : Fit a poisson distribution to the following data

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400

[JNTU (K) Nov. 2009, May 2013 (Set No. 1)]

Solution : Mean $= \dfrac{\Sigma f_i x_i}{\Sigma f_i} = \dfrac{0+156+138+81+20+5}{400} = \dfrac{400}{400} = 1$

$\therefore$ Mean of Poisson distribution *i.e,* $\lambda = 1$

Hence the theoretical frequency for x successes is given by

N.P(x) where $x = 0,1,2,3,4,5$

i.e., $400 \,.\, \dfrac{e^{-1}(1)^x}{x!}$, where $x = 0,1,2,3,4,5$

i.e., $400\,(e^{-1}),\ 400\,(e^{-1}),\ 200\,(e^{-1}),\ 66.67\,(e^{-1}),\ 16.67\,(e^{-1}),\ 3.33(e^{-1})$

i.e., 147.15, 147.15, 73.58, 24.53, 6.13, 1.23

$\therefore$ The theoretical frequencies are

x	0	1	2	3	4	5
f	147	147	74	25	6	1

Example 34 : The distribution of typing mistakes committed by a typist is given below. Assuming the distribution to be Poisson, find the expected frequencies.

x	0	1	2	3	4	5
$f(x)$	42	33	14	6	4	1

Solution : $N = \Sigma f_i = 100$

Mean of the distribution, $\lambda = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

$$= \frac{0+33+28+18+16+5}{100}$$

$$= \frac{100}{100} = 1$$

By Poisson distribution, $p(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-1}}{x!}\ [\because \lambda = 1]$

$\therefore f(0) = N.p(0) = \dfrac{100}{e} = 36.79,\ f(1) = N.p(1) = \dfrac{100}{e} = 36.79$

$f(2) = N.p(2) = \dfrac{100}{2e} = 18.39,\ f(3) = N.p(3) = \dfrac{100}{6e} = 6.13$

$f(4) = \dfrac{100}{24e} = 1.53,\ f(5) = \dfrac{100}{120e} = 0.31$

Since frequencies are always integers, by converting them to nearest integers, we get

x	0	1	2	3	4	5
Observed frequency	42	33	14	6	4	1
Expected frequency	37	37	18	6	2	0

Note : In the above example we observe that the frequencies of first two values are same. Is it true for all the data ? Observe. Moreover, is it possible to get more than two same frequncies.