CHAPTER - 3

MATHEMATICAL EXPECTATION

3.1 EXPECTATION, MEAN, VARIANCE AND STANDARD DEVIATION OF A PROBABILITY DISTRIBUTION

The behaviour of a random variable is completely characterized by the distribution function $F(x)$ or density function $f(x)$ or $P(x_i)$. Instead of a function, a more compact description can be made by a single numbers such as mean, median and mode known as measures of central tendency of the random variable X.

EXPECTATION

In this section, We discuss the application of the concepts of probability theory to real life situations when the decisions are based on expectations about the value of a variable like life of an item, say electric bulb, steel, cement, scooter battery, etc.

When a dice is thrown, we know that the variable representing the top number on the dice can be any value from 1 to 6 with probability 1/6. Now, suppose a person is not interested in listening to this statement giving the range of the variable, but all we wants to know that single number which is expected to come up when the dice is thrown. The answer to these types of situations when one wants the reply as a single value, is provided by the theory of expectation. The theory is discussed separately for discrete and continuous variables.

Expectation theory plays a very important role in decision - making because most of the time we take decisions based on what is expected to happen.

(1) Expectation of a Discrete Variable :

As defined earlier, a discrete variable takes only some finite values like number on a dice, number of children in a family, etc.

Suppose a random variable X assumes the values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$. Then the **Mathematical Expectation** or **Mean** or **Expected value** of X, denoted by $E(X)$, is defined as the sum of products of different values of x and the corresponding probabilities.

$$\therefore\ E(X) = x_1 p_1 + x_2 p_2 + ... + x_n p_n$$

$$i.e.,\ E(X) = \sum_{i=1}^{n} p_i\, x_i$$

$$\text{Similarly, } E(x^r) = \sum_{i=1}^{n} p_i \cdot x_i^r$$

In general, the expected value of any function $g(x)$ of a random variable X is defined as

$$E[g(x)] = \sum_{i=1}^{n} p_i\, g(x_i)$$

Note : Expected value of X is a population mean. If population mean is μ then $E(X) = \mu$.

Some Important Results on Expectation :

(i) If X is a random variable and K is a constant, then

$E(X + K) = E(X) + K$ **[JNTU 2005S, 2005, 2007S, (H) Nov. 2009 (Set No.1)]**

Proof : By definition, we have

$$E(X+K) = \sum_{i=1}^{n} (x_i + K)p_i = \sum_{i=1}^{n} x_i\, p_i + K\sum_{i=1}^{n} p_i$$

$$= E(X) + K(1) = E(X) + K \quad \left[\because \sum_{i=1}^{n} p_i = 1\right]$$

Note 1 : Expected value of constant term is constant, that is, if K is constant, then $E(K) = K$ $[\because E(K) = \Sigma p_i K = K\Sigma p_i = K(1) = K]$.

Note 2 : If K is constant, then $E(KX) = KE(X)$

$E(KX) = \sum_{i=1}^{n} (K\,x_i)p_i = K\sum_{i=1}^{n} x_i\, p_i = K\,E(X)$, where K is a constant.

(ii) If X is a random variable and a and b are constants, then

$E(aX \pm b) = a\,E(X) \pm b$

Proof : Proceed as in *(i)*

(iii) If X and Y are any two random variables, then

$E(X + Y) = E(X) + E(Y)$ provided $E(X)$ and $E(Y)$ exist.

[JNTU 2005 S, 2005, 2007S, (H) Nov. 2009 (Set No.1)]

Proof : Let X assume the values $x_1, x_2, \ldots, x_n$ and Y assume the values $y_1, y_2, \ldots, y_m$. Then by definition,

$$E(X) = \sum_{i=1}^{n} p_i\, x_i \text{ and } E(Y) = \sum_{j=1}^{m} p_j\, y_j$$

Let $p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j)$

[This is called the joint probability function of X and Y]

The sum $(X + Y)$ is also a random variable which can take $m \times n$ values $(x_i + y_j)$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

$\therefore$ By definition,

$$E(X+Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}(x_i + y_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}x_i + \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}y_j$$

$$= \sum_{i=1}^{n}\left[x_i \sum_{j=1}^{m} p_{ij}\right] + \sum_{j=1}^{m}\left[y_j \sum_{i=1}^{n} p_{ij}\right]$$

$$= \sum_{i=1}^{n} x_i p_i + \sum_{j=1}^{m} y_j p_j$$

$$= E(X) + E(Y)$$

Note: 1. $E(X + Y + Z) = E(X + (Y + Z))$

$= E(X) + E(Y + Z)$

$= E(X) + E(Y) + E(Z)$

Similarly proceeding, we have in general by the method of induction, if $X_1, X_2, \ldots, X_n$ are n random variables then

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

i.e., The mathemtical expectation of the sum of random variables is equal to the sum of their expectations, provided all the expectations exist.

2. $E(aX + bY) = a\,E(X) + b\,E(Y)$, where a and b are constants

3. $E(X - \overline{X}) = 0$

(iv) If X and Y are two independent random variables, then

$E(XY) = E(X)\,E(Y)$

Proof : Let X assume the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ and Y assume the values $y_1, y_2, \ldots, y_m$ with probabilities $p'_1, p'_2, \ldots, p'_m$.

Then by definition,

$$E(X) = \sum_{i=1}^{n} p_i\, x_i$$

and $$E(Y) = \sum_{j=1}^{m} p'_j\, y_j$$

Let $p_{ij} = P(X = x_i \cap Y = y_j)$

$= P(X = x_i) \cdot P(Y = y_j)$ (since X and Y are independent)

$= p_i p'_j$

Example 2 : Two dice are thrown. Let X assign to each point (a, b) in S the maximum of its numbers *i.e.*, $X(a, b) = max.(a, b)$. Find the probability distribution. X is a random variable with $X(s) = \{1, 2, 3, 4, 5, 6\}$. Also find the mean and variance of the distribution.

[JNTU 2004, 2007, 2008S, (A) Dec. 2009, Nov. 2010 (Set No. 4)]

(OR) A random variable X has the following distribution ?

x	1	2	3	4	5	6
P(x)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Find (a) the mean (b) variance (c) $P(1 < x < 6)$ **[JNTU (H) Apr. 2012 (Set No. 1)]**

Solution : The total number of cases are $6 \times 6 = 36$.

The maximum number could be 1,2, 3, 4, 5, 6 i.e., $X(s) = X(a, b) = \max(a, b)$.

The number 1 will appear only in one case (1, 1). So $p(1) = P(X = 1) = P(1, 1) = \frac{1}{36}$

For maximum 2, favourable cases are (2, 1), (2, 2), (1, 2)

So $p(2) = P(X = 2) = 3/36$

For maximum 3, favourable cases are (1, 3), (3, 1), (2, 3), (3, 2), (3, 3).

So $p(3) = P(X = 3) = 5/36$

For maximum 4, favourable cases are (1, 4),(4, 1),(2, 4),(4, 2),(3, 4),(4, 3),(4, 4)

So $p(4) = P(X = 4) = 7/36$

Similarly, $p(5) = P(X = 5)$

$= P((1, 5), (5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4),(5, 5))$

$= \frac{9}{36}$

and $p(6) = P(X = 6)$

$= P((1, 6), (6, 1), (2, 6), (6,2), (3, 6), (6, 3), (4, 6), (6, 4), (5, 6), (6, 5), (6, 6))$

$= \frac{11}{36}$

$\therefore$ The required discrete probability distribution is

$X = x_i$	1	2	3	4	5	6
$P(X = x_i) = p(x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(i) Mean, $\mu = \sum_{i=1}^{6} p_i x_i = 1.\frac{1}{36} + 2.\frac{3}{36} + 3.\frac{5}{36} + 3.\frac{5}{36} + 4.\frac{7}{36} + 5.\frac{9}{11} + 6.\frac{11}{36}$

$$= \frac{1}{36}(1 + 6 + 15 + 28 + 45 + 66) = \frac{161}{36} = 4.47$$

(ii) Variance, $\sigma^2 = \sum_{i=1}^{6} p_i x_i^2 - \mu^2$

$$= \frac{1}{36}(1)^2 + \frac{3}{36}(2)^2 + \frac{5}{36}(3)^2 + \frac{7}{36}(4)^2 + \frac{9}{36}(5)^2 + \frac{11}{36}(6)^2 - (4.47)^2$$

$$= \frac{1}{36}(1 + 12 + 45 + 112 + 225 + 396) - (4.47)^2$$

$$= \frac{791}{36} - 19.9808 = 21.97 - 19.981 = 1.9912$$

Example 3 : A random variable X has the following probability function :

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

(i) Determine K *(ii)* Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(0 < X < 5)$ and $P(0 \leq X \leq 4)$ *(iii)* if $P(X \leq K) > (1/2)$, find the minimum value of K and, *(iv)* Determine the distribution function of X *(v)* Mean *(vi)* Variance.

[JNTU 04S, 05S, 08S, (A) Nov. 11, (H) Dec. 11, (K) May 10S, Nov. 2012, Mar. 2014 (Set No. 2)]

Solution :

(i) Since $\sum_{x=0}^{7} p(x) = 1$, we have

$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

i.e., $10K^2 + 9K - 1 = 0$ *i.e.*, $(10K - 1)(K + 1) = 0$

$\therefore K = \frac{1}{10} = 0.1$ (since $p(x) \geq 0$, so $K \neq -1$)

(ii) $P(X < 6) = P(X = 0) + P(X = 1) + ... + P(X = 5)$

$= 0 + K + 2K + 2K + 3K + K^2 = 8K + K^2 = 0.8 + 0.01 = 0.81$ $[\because K = 0.1]$

$P(X \geq 6) = 1 - P(X < 6) = 1 - 0.81 = 0.19$

$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= K + 2K + 2K + 3K = 8K = \frac{8}{10} = 0.8$

$P(0 \leq X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= 0 + K + 2K + 2K + 3K = 8K = 8(0.1) = 0.8$

Note : $P(X \leq 5) = P(X = 0) + P(X = 1) + ... + P(X = 5)$

$= 0.81$ [Refer *(ii)*]

$P(X > 5) = 1 - P(X \leq 5) = 1 - 0.81 = 0.19$

$P(0 < X < 6) = P(X = 1) + P(X = 2) + ... + P(X = 5) = 0.81$

(iii) The required minimum value of K is obtained as below.

$P(X \leq 1) = P(X = 0) + P(X = 1) = 0 + K = \frac{1}{10} = 0.1$

$P(X \leq 2) = [P(X = 0) + P(X = 1)] + P(X = 2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10} = 0.3$

$P(X \le 3) = [P(X=0) + P(X=1) + P(X=2)] + P(X=3) = 0.3 + 0.2 = 0.5$

$P(X \le 4) = P(X \le 3) + P(X=4) = 0.5 + \frac{3}{10} = 0.8 > 0.5 = \frac{1}{2}$

$\therefore$ The minimum value of K for which $P(X \le K) > \frac{1}{2}$ is $K = 4$

(iv) The distribution function of X is given by the following table :

X	$F(x) = P(X \le x)$
0	0
1	$K = 1/10$
2	$3K = 3/10$
3	$5K = 5/10$
4	$8K = 8/10$
5	$8K + K^2 = 81/100$
6	$8K + 3K^2 = 83/100$
7	$9K + 10K^2 = 1$

(v) Mean, $\mu = \sum_{i=0}^{7} p_i x_i$

$= 0(0) + 1(K) + 2(2K) + 3(2K) + 4(3K) + 5(K^2) + 6(2K^2) + 7(7K^2 + K)$

$= 66K^2 + 30K = \frac{66}{100} + \frac{30}{10} = 0.66 + 3 = 3.66 \left(\because K = \frac{1}{10}\right)$

(vi) Variance $= \sum_{i=0}^{7} p_i x_i^2 - \mu^2$

$= K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K - (3.66)^2$

$= 440K^2 + 124K - (3.66)^2 = \frac{440}{100} + \frac{124}{10} - (3.66)^2$

$= 4.4 + 12.4 - 13.3956 = 3.4044$

Example 4 : A random variable X has the following probability distribution.

X :	1	2	3	4	5	6	7	8
P(X) :	K	2K	3K	4K	5K	6K	7K	8K

Find the value of

(i) K (ii) $P(X \le 2)$ (iii) $P(2 \le X \le 5)$ **[JNTU 2008 (Set No. 1)]**

Solution : (i) Since $\sum_{i=1}^{\infty} p(x_i) = 1$, we have

$K + 2K + 3K + 4K + 5K + 6K + 7K + 8K = 1 \Rightarrow 36K = 1.$

$\therefore K = \frac{1}{36}$

(ii) $P(X \le 2) = P(X=1) + P(X=2)$

$= K + 2K = 3K = \frac{3}{36} = \frac{1}{12} \left(\because K = \frac{1}{36}\right)$

(iii) $P(2 \le X \le 5) = P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 2K + 3K + 4K + 5K = 14K$

$= \frac{14}{36} = \frac{7}{18}$

Example 5 : A random variable X has the following probability distribution.

Values of x	0	1	2	3	4	5	6	7	8
P(x)	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the value of a.

(ii) Find $p(x<3)$, $p(x \ge 3)$ and $p(0<x<5)$

(iii) Find the distribution function F(x). **[JNTU(K) Dec. 2013 (Set No.1)]**

Solution : (i) Since $\sum_{i=0}^{8} p(x_i) = 1$, we have

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

i.e., $81a = 1$ or $a = \frac{1}{81}$

(ii) $p(x<3) = p(x=0) + p(x=1) + p(x=2)$

$= a + 3a + 5a = 9a = \frac{9}{81} = \frac{1}{9}$

$p(x \ge 3) = 1 - p(x<3) = 1 - \frac{1}{9} = \frac{8}{9}$

$p(0<x<5) = p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$= 3a + 5a + 7a + 9a = 24a = \frac{24}{81} = \frac{8}{27}$.

(iii) The distribution function F(x) of the discrete variate X is defined by

$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$ where x is any integer

X	$F(X) = P(X \le x)$
0	$a = \frac{1}{81}$
1	$4a = \frac{4}{81}$
2	$9a = \frac{9}{81}$
3	$16a = \frac{16}{81}$
4	$25a = \frac{25}{81}$
5	$36a = \frac{36}{81}$
6	$49a = \frac{49}{81}$
7	$64a = \frac{64}{81}$
8	$81a = \frac{81}{81} = 1$

Example 6 : The probability density function of a variate X is

X	0	1	2	3	4	5	6
P(X)	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(*i*) Find k (*ii*) Find $P(X<4)$, $P(X \ge 5)$, $P(3 < X \le 6)$ (*iii*) What will be the minimum value of k so that $P(X \le 2) > 0.3$? **[JNTU (K) Nov. 2009, Dec. 2013 (Set No. 3, 4)]**

Solution : (*i*) Since $\sum_{i=0}^{6} p(x_i) = 1$ so $k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

i.e., $49k = 1$ or $k = \frac{1}{49}$

(*ii*) $P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= k + 3k + 5k + 7k = 16k = \frac{16}{49} \left[\because k = \frac{1}{49}\right]$$

$P(X \ge 5) = P(X=5) + P(X=6) = 11k + 13k = 24k = \frac{24}{49}$

$P(3 < X \le 6) = P(X=4) + P(X=5) + P(X=6) = 9k + 11k + 13k = 33k = \frac{33}{49}$

(*iii*) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = k + 3k + 5k = 9k$

$P(X \le 2) > 0.3 \Rightarrow 9k > 0.3$ or $k > \frac{1}{30}$

$\therefore$ The minimum value of k is $\frac{1}{30}$.

Example 7 : For the discrete probability distribution

X	0	1	2	3	4	5	6
P(X)	0	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (*i*) k (*ii*) Mean (*iii*) Variance **[JNTU (H) 2009 (Set No. 1)]**

Solution : (*i*) Since the total probability is unity, we have $\sum_{x=0}^{6} p(x) = 1$

i.e., $0 + 2K + 2K + 3K + K^2 + 2K^2 + (7K^2 + K) = 1$

i.e., $10K^2 + 8K - 1 = 0$

$$\therefore K = \frac{-8 \pm \sqrt{64+40}}{20} = \frac{-8 \pm \sqrt{104}}{20} = \frac{-8 \pm 2\sqrt{26}}{20} = \frac{-4 \pm \sqrt{26}}{10}$$

Since, $p(x) \ge 0$,

$$\therefore K = \frac{-4 + \sqrt{26}}{10} = 0.1099$$

(*ii*) Mean, $\mu = \sum_{i=0}^{6} p_i x_i$

$$= (0)(0) + (1)(2K) + (2)(2K) + (3)(3K) + (4)(K^2) + (5)(2K^2) + (6)(7K^2 + K)$$

$$= 2K + 4K + 9K + 4K^2 + 10K^2 + 42K^2 + 6K$$

$$= 56K^2 + 21K = K(56K + 21)$$

$$= (0.1099)[56(0.1099) + 21] = 2.9842$$

(*iii*) Variance $= \sum_{i=0}^{6} p_i x_i^2 - \mu^2$

$$= 0 + 2K(1)^2 + 2K(2)^2 + 3K(3)^2 + K^2(4)^2 + 2K^2(5)^2 + (7K^2 + K)(6)^2 - (2.9842)^2$$

$$= 2K + 8K + 27K + 16K^2 + 50K^2 + 252K^2 + 36K - 8.9054$$

$$= 318K^2 + 73K - 8.9054 = K(318K + 73) - 8.9054$$

$$= (0.1099)[318(0.1099) + 73] - 8.9054 = 2.9581$$

Example 8 : A random variables X has the following probability function

X_i	−3	−2	−1	0	1	2	3
$P(X_i)$	K	0.1	K	0.2	$2k$	0.4	$2k$

Find (*i*) K (*ii*) Mean (*iii*) Variance **[JNTU (H) 2009 (Set No. 2)]**

Solution : (*i*) We know that, if X is a random variable, then $\sum p(x_i)=1$

i.e., $K+0.1+K+0.2+2K+0.4+2K=1$

i.e., $6K+0.7=1$

i.e., $6K=0.3$ or $K=\frac{0.3}{6}=\frac{0.1}{2}$

(*ii*) Mean, $\mu=\sum_{i=0}^{6} p_i x_i$

$$=(-3)(K)+(-2)(0.1)+(-1)(K)+0(0.2)+1(2K)+2(0.4)+3(2K)$$

$$=-3K-0.2-K+0+2K+0.8+6K$$

$$=4K+0.6=4\left(\frac{0.1}{2}\right)+0.6\ =0.8$$

(*iii*) Variance $=\sum_{i=0}^{6} p_i x_i^2-\mu^2$

$$=K(-3)^2+(0.1)(-2)^2+K(-1)^2+0.2(0)^2+2K(1)^2+(0.4)(2)^2+2K(3)^2-(0.8)^2$$

$$=9K+0.4+K+0+2K+1.6+18K-0.64$$

$$=30K+2-0.64=30\left(\frac{0.1}{2}\right)+1.36\ =1.5+1.36=2.86$$

Example 9 : From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Find the probability distribution of X when the sample is drawn without replacement.

[JNTU (K) May 2013 (Set No.4)]

Solution : Obviously X can take the values 0, 1, 2 or 3.

Given total number of items = 10

No. of good items = 7

No. of defective items = 3

$$P(X=0)=P(\text{no defective})=\frac{{}^7C_4}{{}^{10}C_4}=\frac{7!}{4!3!}\times\frac{4!6!}{10!}=\frac{1}{6}$$

$P(X=1)=P(\text{one defective and 3 good items})$

$$=\frac{{}^3C_1\times{}^7C_3}{{}^{10}C_4}=\frac{3\times 7!}{3!4!}\times\frac{4!6!}{10!}=\frac{1}{2}$$

$P(X=2)=P(\text{2 defective and 2 good items})$

$$=\frac{{}^3C_2\times{}^7C_2}{{}^{10}C_4}=\frac{3}{10}$$

$P(X=3)=P(\text{3 defective and 1 good item})$

$$=\frac{{}^3C_3\times{}^7C_1}{{}^{10}C_4}=\frac{7}{{}^{10}C_4}=\frac{4!}{8\times9\times10}=\frac{1}{30}$$

$\therefore$ The probability distribution of random variable X is as follows :

X	0	1	2	3
$P(X)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Example 10 : Let X denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the

(*i*) Discrete probability distribution (*ii*) Expectation

(*iii*) Variance **[JNTU 2006, (K) Nov. 2009, 2010S (Set No. 1)]**

Solution : When two dice are thrown, total number of outcomes is $6\times6=36$

In this case, sample space
$$S=\begin{Bmatrix}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{Bmatrix}$$

If the random variable X assigns the minimum of its number in S, then the sample space
$$S=\begin{Bmatrix}1&1&1&1&1&1\\1&2&2&2&2&2\\1&2&3&3&3&3\\1&2&3&4&4&4\\1&2&3&4&5&5\\1&2&3&4&5&6\end{Bmatrix}$$

The minimum number could be 1, 2, 3, 4, 5, 6.

For minimum 1, favourable cases are (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (1, 5), (5, 1), (1, 6), (6, 1).

So $P(X=1)=\frac{11}{36}$

For minimum 2, favourable cases are (2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (2, 5), (5, 2), (2, 6), (6, 2).

So $P(X=2)=\frac{9}{36}$

Similarly, $P(X = 3) = P((3, 3), (3, 4), (4, 3), (3, 5), (5, 3), (3, 6), (6, 3)) = \frac{7}{36}$

$$P(X = 4) = P((4, 4), (4, 5), (5, 4), (4, 6), (6, 4)) = \frac{5}{36}.$$

$$P(X = 5) = P((5, 5), (5, 6), (6, 5)) = \frac{3}{36}$$

$$P(X = 6) = P((6, 6)) = \frac{1}{36}$$

$\therefore$ The probability distribution is

X	1	2	3	4	5	6
$P(X)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(ii) Expectation = Mean = $\Sigma\, p_i x_i$

i.e., $E(X) = 1.\frac{11}{36} + 2.\frac{9}{36} + 3.\frac{7}{36} + 4.\frac{5}{36} + 5.\frac{3}{36} + 6.\frac{1}{36}$

or $\mu = \frac{1}{36}(11 + 18 + 21 + 20 + 15 + 6) = \frac{91}{36} = 2.5278$

(iii) Variance = $\Sigma\, p_i x_i^2 - \mu^2$

$$= \frac{11}{36}.1 + \frac{9}{36}.4 + \frac{7}{36}.9 + \frac{5}{36}.16 + \frac{3}{36}.25 + \frac{1}{36}.36 - \left(\frac{91}{36}\right)^2$$

$$= \frac{1}{36}(11 + 36 + 63 + 80 + 75 + 36) - \left(\frac{91}{36}\right)^2$$

i.e., $\sigma^2 = 8.\ 3611 - 6.3898 = 1.9713$

Note : Standard deviation, $\sigma = \sqrt{1.9713} = 1.404$

Example 11 : Calculate expectation and variance of X, if the probability distribution of the random variable X is given by

X	-1	0	1	2	3
f	0.3	0.1	0.1	0.3	0.2

[JNTU 2006 (Set No.3)]

Solution :

(i) Expectation = Mean = $\Sigma\, f_i x_i$

i.e., $E(X) = (-1)(0.3) + 0(0.1) + 1(0.1) + 2(0.3) + 3(0.2)$

or $\mu = -0.3 + 0.1 + 0.6 + 0.6 = 1$

(ii) Variance $= \Sigma\, f_i x_i^2 - \mu^2 = \Sigma\, x_i^2 f_i - [E(X)]^2$

$$= (-1)^2(0.3) + 0(0.1) + 1(0.1) + 2^2(0.3) + 3^2(0.2) - 1$$

$$= 0.3 + 0.1 + 1.2 + 1.8 - 1 = 2.4$$

Example 12 : Calculate the mean for the following distribution.

X = x :	0.3	0.2	0.1	0	1	2	3
P(X = x) :	0.05	0.10	0.30	0	0.30	0.15	0.1

[JNTU 2008 (Set No. 4)]

Solution : The mean value of the probability distribution of a variate X is commonly known as its expectation and is denoted by E(X). It is given by

$$E(X) = \sum_i x_i f(x_i)$$

$$= 0.3(0.05) + 0.2(0.10) + 0.1(0.30) + 0(0) + 1(0.30) + 2(0.15) + 3(0.1)$$

$$= 0.015 + 0.02 + 0.03 + 0 + 0.30 + 0.30 + 0.3 = 0.965$$

Example 13 : A random variable X is defined as the sum of the numbers on the faces when two dice are thrown. Find the mean of X. **[JNTU (A) 2009 (Set No. 4)]**

Solution : Let x be the sum of the numbers on the faces when two dice are thrown. x is a discrete random variable whose probability distribution is given by

x_i	2	3	4	5	6	7	....	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	....	2/36	1/36

$\therefore$ Mean of $X = E(X) = \sum_i p_i x_i$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right)$$

$$+ 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$

$$= \frac{252}{36} = 7$$

Example 14 : Find the mean and variance of the uniform probability distribution given by $f(x) = \frac{1}{n}$ for $x = 1, 2, 3, ..., n.$

[JNTU 2001, (H) Nov. 2009, Nov. 2010, Dec. 2011 (Set No. 3)]

Solution : The probability distribution is

x	1	2	3	...	n
$f(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

(i) Mean $= \sum_{i=1}^{n} x_i f(x_i) = 1.\frac{1}{n}+2.\frac{1}{n}+.....+n.\frac{1}{n}$

or $E(X) = \mu = \frac{1}{n}(1+2+3+\ldots+n) = \frac{1}{n}.\frac{n(n+1)}{2} = \frac{n+1}{2}$

(ii) Variance $= \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$

$$= 1^2.\frac{1}{n} + 2^2.\frac{1}{n} + 3^2.\frac{1}{n} + \ldots + n^2.\frac{1}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}(1^2+2^2+3^2+\ldots+n^2) - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{n}.\frac{n(n+1)(2n+1)}{6} - \frac{1}{4}(n+1)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)$$

$$= \frac{n+1}{12}(4n+2-3n-3) = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

Example 15 : A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected number E of defective items.

[JNTU (H) Nov. 2009 (Set No. 4), (A) Nov. 2010 (Set No. 1)]

Solution : Let X denote the number of defective items among 4 items drawn from 12 items.

Obviously X can take the values 0, 1, 2, 3, or 4.

No. of good items = 7

No. of defective items = 5

$$P(X=0) = P(\text{no defective}) = \frac{^7C_4}{^{12}C_4} = \frac{35}{495} = \frac{7}{99}$$

$P(X=1) = $ P(one defective and 3 good items)

$$= \frac{^7C_3 \times {^5C_1}}{^{12}C_4} = \frac{7\times6\times5\times5}{6} = \frac{175}{495} = \frac{35}{99}$$

$P(X=2) = P$(2 defective and 2 good items)

$$= \frac{^7C_2 \times {^5C_2}}{^{12}C_4} = \frac{210}{495} = \frac{42}{99}$$

$P(X=3) = P$(3 defective and 1 good item)

$$= \frac{^7C_1 \times {^5C_3}}{^{12}C_4} = \frac{70}{495} = \frac{14}{99}$$

$P(X=4) = P$(all are defective)

$$= \frac{^5C_4}{^{12}C_4} = \frac{5}{495} = \frac{1}{99}$$

Discrete probability distribution is

$X = x_i$	0	1	2	3	4
$P(X=x_i) = f(x_i)$	$\frac{7}{99}$	$\frac{35}{99}$	$\frac{42}{99}$	$\frac{14}{99}$	$\frac{1}{99}$

Expected number of defective items $= E(X) = \Sigma x_i f(x_i)$

$$= 0.\frac{7}{99} + 1.\frac{35}{99} + 2.\frac{42}{99} + 3.\frac{14}{99} + 4.\frac{1}{99} = \frac{165}{99}$$

Example 16 : Show that the variance of a random variable X is given by

$$\sigma^2 = E(X^2) - [E(X)]^2$$

Solution : We know that

$\sigma^2 = E[(X-\mu)^2] = E(X^2 - 2X\mu + \mu^2) = E(X^2) - 2\mu E(X) + E(\mu^2)$

$= E(X^2) - 2\mu.\mu + \mu^2 = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$

Example 17 : Given that $f(x) = K/2x$, is a probability distribution for a random variable X that can take on the values $x = 0,1,2,3$ and 4.

(i) Find K (ii) Mean and variance of X. **[JNTU (A) Sup., 2010]**

Solution : Given $f(x) = K/2x$, where $x = 0,1,2,3,4$. If $x = 0, f(x)$ is not defined.

Hence the problem is not correct.

We shall delete $x = 0$ and workout the problem.

(i) If $f(x)$ is a probability distribution function, then

1. $f(x) \geq 0 \Rightarrow f(x) = \frac{K}{2x} \geq 0 \Rightarrow K \geq 0$

2. $\sum_{\forall x} f(x) = 1 \Rightarrow \sum_{x=1}^{4} \frac{K}{2x} = 1 \Rightarrow \frac{K}{2}.\sum_{x=1}^{4}\frac{1}{x} = 1$

$$\Rightarrow \frac{K}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right]=1 \Rightarrow \frac{K}{2}\left(\frac{25}{12}\right)=1$$

$$\therefore K=\frac{24}{25}$$

The probability distribution is given in the following table :

x	1	2	3	4
$f(x)$	$\frac{K}{2}$	$\frac{K}{4}$	$\frac{K}{6}$	$\frac{K}{8}$

(ii) Mean $= E(X)=\sum_{x=1}^{4} x.f(x)$

$$=1\times\frac{K}{2}+2\times\frac{K}{4}+3\times\frac{K}{6}+4\times\frac{K}{8}$$

$$=\frac{K}{2}+\frac{K}{2}+\frac{K}{2}+\frac{K}{2}=4\left(\frac{K}{2}\right)=2K$$

$$=2\left(\frac{24}{25}\right)=\frac{48}{25}\ \left[\because K=\frac{24}{25}\right]$$

Now $E(X^2)=\sum_{x=1}^{4} x^2.f(x)$

$$=1\times\frac{K}{2}+4\times\frac{K}{4}+9\times\frac{K}{6}+16\times\frac{K}{8}$$

$$=\frac{K}{2}+K+\frac{3K}{2}+2K=5K$$

$\therefore$ Variance $= E(X^2)-[E(X)]^2$

$$=5K-(2K)^2=5K-4K^2=K(5-4K)$$

$$=\frac{24}{25}\left(5-\frac{96}{25}\right)\ \left[\because K=\frac{24}{25}\right]$$

$$=\frac{24}{25}\left(\frac{29}{25}\right)=\frac{696}{625}=1.1136$$

Example 18 : A variate X has the following probability distribution

x	-3	6	9
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find *(i)* $E(X)$ *(ii)* $E(X^2)$ *(iii)* $E[(2X+1)^2]$

Solution :

(i) $E(X)=\sum_{i=1}^{3} p_i x_i = (-3)\left(\frac{1}{6}\right)+6.\frac{1}{2}+9.\frac{1}{3}=-\frac{1}{2}+3+3=\frac{11}{2}$

(ii) $E(X^2)=\sum_{i=1}^{3} p_i x_i^2 = (-3)^2\left(\frac{1}{6}\right)+(6)^2.\frac{1}{2}+(9)^2.\frac{1}{3}$

$$=\frac{9}{6}+\frac{36}{2}+\frac{81}{3}=\frac{93}{2}$$

(iii) $E[(2X+1)^2]=E(4X^2+4X+1)=4E(X^2)+4E(X)+E(1)$

$$=4.\frac{93}{2}+4.\frac{11}{2}+1=\frac{418}{2}=209$$

Example 19 : A player wins if he gets 5 on a single throw of a die, he loses if he gets 2 or 4. If he wins, he gets Rs. 50, if he loses he gets Rs. 10, otherwise he has to pay Rs. 15. Find the value of the game to the player. Is it favourable ?

Solution : Range of $X=\{-15, 10, 50\}$

Since there are six numbers in a die, out of these 5 is only one number, the probability of getting Rs. 50 is

$$P(X=50)=\frac{1}{6}$$

Similarly, the probability of getting two numbers (2 or 4) to win Rs. 10 is

$$P(X=10)=\frac{2}{6}=\frac{1}{3}$$

The probability of getting three numbers (1 or 3 or 6) to loose Rs. 15 is

$$P(X=-15)=\frac{3}{6}=\frac{1}{2}$$

Discrete probability distribution is

$X=x$	-15	10	50
$P(X)=p(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Expected value of the game $=E(X)=\sum x_i p(x_i)=(-15).\frac{1}{2}+10.\frac{1}{3}+50.\frac{1}{6}=$ Rs. $\frac{25}{6}$

Game is favourable to the player since $E>0$.

Example 20 : A player tosses 3 fair coins. He wins Rs. 500 if 3 heads appear, Rs. 300 if 2 heads appear, Rs. 100 if 1 head occurs. On the other hand, he loses Rs. 1500 if 3 tails occur. Find the expected gain of the player.

Solution : Let X denote the gain. Then the range of X is $\{-1500, 100, 300, 500\}$

The sample space $S=n(s)=2^3=8$

$$=\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

The probability of all 3 heads (getting Rs. 500) = $P(X = 3) = \frac{1}{8}$

The probability of getting 2 heads (getting Rs. 300) = $P(X = 2) = \frac{3}{8}$

The probability of getting one head (getting Rs. 100) = $P(X = 1) = \frac{3}{8}$

The probability of getting 3 tails (lossing Rs. 1500) = $P(X = 0) = \frac{1}{8}$

Discrete probability distribution is

$X = x_i$	-1500	100	300	500
$P(X = x_i) = p(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\therefore$ Expected value of $X = E(X) = \Sigma x_i p(x_i)$

$$= (-1500)\cdot\frac{1}{8} + 100.\frac{3}{8} + 300.\frac{3}{8} + 500.\frac{1}{8}$$

$$= \frac{1}{8}(-1500 + 300 + 900 + 500) = \text{Rs. } 25.$$

Example 21 : A fair coin is tossed until a head or five tails occurs. Find the expecte number E of tosses of the coin. **[JNTU (H) Nov. 2009 (Set No.3)]**

Solution : If head occurs first time there will be only one toss. On the other hand, i first one is tail, second occurs. If head occurs there will be only two tosses. Suppose secon one is also tail third occurs. If head occurs there will be three tosses and so on.

$$\therefore \quad p(1) = p(\text{H}) = \frac{1}{2}, \quad p(2) = p(\text{TH}) = \frac{1}{4},$$

$$p(3) = p(\text{TTH}) = \frac{1}{8}, \quad p(4) = p(\text{TTTH}) = \frac{1}{16},$$

$$p(5) = p(\text{TTTTH}) + p(\text{TTTTT}) = \frac{1}{32} + \frac{1}{32} = \frac{1}{16}$$

The probability distribution function of X is

x	1	2	3	4	5
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

Hence $E(X) = \sum_i p_i x_i$

$$= 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 4\times\frac{1}{16} + 5\times\frac{1}{16} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{16}$$

$$= \frac{8+8+6+4+5}{16} = \frac{31}{16} = 1.9375$$

Example 22 : A player tosses two fair coins. He wins ₹ 100/- if head appears, ₹ 200/- if two heads appear. On the other hand he loses ₹ 500/- if no head appears. Determine the expected value E of the game and is the game favourable to the player?

[JNTU (K) Nov. 2009 (Set No. 3)]

Solution : Let X denote the number of heads occuring in tosses of two fair coins. The sample space S is

S = {H,T} × {H,T} = {HH, HT, TH, TT}

Probability of all 2 leads = $P(X = 2) = \frac{1}{4}$

Probability of all 2 tails = $P(X = 0) = \frac{1}{4}$

Probability of one head = $P(X = 1) = \frac{2}{4}$

$\therefore$ Discrete probability distribution is

$X = x_i$	0	1	2
$P(x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Expected value of the game = $100\times\frac{2}{4} + 200\times\frac{1}{4} - 500\times\frac{1}{4}$

$$= \frac{200+200-500}{4} = \frac{-100}{4} = -25 \text{ rupees}$$

$\therefore$ Game is not favourable to the player since $E < 0$

Example 23 : A discrete random variable X has the following distribution function :

$$F(x) = \begin{cases} 0, \text{ for } x < 1 \\ 1/3, \text{ for } 1 \le x < 4 \\ 1/2, \text{ for } 4 \le x < 6 \\ 5/6, \text{ for } 6 \le x < 10 \\ 1, \text{ for } x \ge 10 \end{cases}$$

Find (*i*) $P(2 < X \le 6)$ (*ii*) $P(X = 5)$ (*iii*) $P(X = 4)$(*iv*) $P(X \le 6)$ (*v*) $P(X = 6)$

[JNTU (K) Nov. 2009 (Set No. 3)]

Solution :

(*i*) $P(2 < X \le 6) = F(6) - F(2) = P(X \le 6) - P(X \le 2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(*ii*) $P(X = 5) = P(X \le 5) - P(X < 5) = F(5) - P(X < 5) = \frac{1}{2} - \frac{1}{2} = 0$

(*iii*) $P(X = 4) = P(X \le 4) - P(X < 4) = F(4) - P(X < 4) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(*iv*) $P(X \le 6) = F(6) = \frac{5}{6}$

(*v*) $P(X = 6) = F(6) - P(X < 6) = \frac{5}{6} - \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$

Example 24 : Find the distribution function which corresponds to the probability distribution defined by $f(x)=\frac{x}{15}$ for $x=1,2,3,4,5$. [JNTU (K) May 2013 (Set No.3)]

Solution :

Given $f(x)=\frac{x}{15}$. So $f(1)=\frac{1}{15}, f(2)=\frac{2}{15}, f(3)=\frac{3}{15}, f(4)=\frac{4}{15}$ and $f(5)=\frac{5}{15}$

Now $F(1)=f(1)=\frac{1}{15}$,

$$F(2)=F(1)+f(2)=\frac{1}{15}+\frac{2}{15}=\frac{1}{5}$$

$$F(3)=F(2)+f(3)=\frac{1}{5}+\frac{1}{5}=\frac{2}{5},$$

$$F(4)=F(3)+f(4)=\frac{2}{5}+\frac{4}{15}=\frac{2}{3}$$

and $F(5)=F(4)+f(5)=\frac{2}{3}+\frac{1}{3}=1$

Example 25 : A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items?

Solution : The probability of defective is $p=\frac{4}{10}=\frac{2}{5}$

Number of items chosen, $n=3$

The expected number of defective items is

$$E(X)=\mu=np=3\left(\frac{2}{5}\right)=\frac{6}{5}=1.2\approx 1$$

Alternative Method :

No. of Exhaustive cases $= {}^{10}C_3=\frac{10!}{3!7!}=120$

Probability that there are no defective items $=p(x=0)=\frac{{}^6C_3}{120}=\frac{1}{6}$

Probability that there is one defective item $=p(x=1)=\frac{{}^6C_2.{}^4C_1}{120}=\frac{1}{2}$

Probability that there are two defective items $=p(x=2)=\frac{{}^6C_1.{}^4C_2}{120}=\frac{3}{10}$

Probability that there are three defective items $=p(x=3)=\frac{{}^4C_3}{120}=\frac{1}{30}$

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

$\therefore$ Expected number of defective items $=E(x)=\Sigma\, p_i x_i$

$$=0.\frac{1}{6}+1.\frac{1}{2}+2.\frac{3}{10}+3.\frac{1}{30}$$

$$=0+\frac{1}{2}+\frac{3}{5}+\frac{1}{10}=\frac{12}{10}=1.2$$

Example 26 : Let X denote the sum of the two numbers that appear when a pair of fair dice is tossed. Determine the (*i*) Distribution function, (*ii*) mean and (*iii*) variance.

[JNTU (K) May 2010 (Set No.4)]

(or) Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Also find the mean of the distribution ? [JNTU (K) Nov. 2011 (Set No.4)]

Solution : (*i*) If two unbiased dice are thrown, then the sum X of the two numbers which turn up must be an integer between 2 and 12.

For $X=2$, there is only one favourable point (1, 1) and hence $P(X=2)=1/36$, since there are 36 sample points in all.

For $X=3$, there are two favourable sample points (1, 2) and (2, 1) and hence $P(X=3)=2/36$.

Similarly for $X=4$, there are three favourable sample points (1, 3,), (2, 2), (3, 1) and hence $P(X=4)=3/36$ and so on.

Now the probability distribution in this case is given by the following table :

$X=x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

(*ii*) Mean, $\mu=\Sigma x_i\, p(x_i)$

$$=2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+.....+12\times\frac{1}{36}$$

$$=\frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$$

i.e., $E(x)=\frac{252}{36}=7$

(*iii*) Variance, $\sigma^2=E(x^2)-[E(x)]^2$

$$=\Sigma x_i^2\, p(x_i)-(7)^2$$

$$=2^2\times\frac{1}{36}+3^2\times\frac{2}{36}+4^2\times\frac{3}{36}+5^2\times\frac{4}{36}+....+12^2\times\frac{1}{36}-49$$

$$= \frac{1}{36}[4+18+48+100+180+294+320+324+300+242+144] - 49$$

$$= \frac{1974}{36} - 49 = 54.83 - 49 = 5.83$$

Example 27 : A fair die is tossed. Let the random variable X denote the twice the number appearing on the die :

(*i*) Write the probability distribution of X

(*ii*) The mean (*iii*) The variance **[JNTU (H) Apr. 2012 (Set No. 2)]**

Solution : Let x denote twice the number appearing on the face when a die is thrown. Then x is a discrete random variable whose probability distribution is given by

(*i*)

$X = x_i$	2	4	6	8	10	12
$p(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(*ii*) Mean $= E(X) = p_i x_i$

$$= 2\times\frac{1}{6} + 4\times\frac{1}{6} + 6\times\frac{1}{6} + 8\times\frac{1}{6} + 10\times\frac{1}{6} + 12\times\frac{1}{6}$$

$$= \frac{1}{6}(2+4+6+8+10+12) = \frac{42}{6} = 7$$

Now $E(X^2) = \sum x_i^2 p(x_i)$

$$= 2^2\times\frac{1}{6} + 4^2\times\frac{1}{6} + 6^2\times\frac{1}{6}\times 8^2\times\frac{1}{6}\times 10^2\times\frac{1}{6}\times 12^2\times\frac{1}{6}$$

$$= \frac{1}{6}(2^2+4^2+6^2+8^2+10^2+12^2)$$

$$= \frac{1}{6}(4+16+36+64+100+144) = \frac{364}{6} = 60.67$$

(*iii*) Variance $= E(x^2) - [E(x)]^2$

$$= 60.67 - (7)^2 = 60.67 - 49 = 11.67$$

Example 28 : A random sample with replacement of size 2 is taken from $S = \{1, 2, 3\}$. Let the random variable X denote the sum of the two numbers taken:

(*i*) Write the probability distribution of X. (*ii*) Find the mean.

(*iii*) Find the variance. **[JNTU (H) Apr. 2012 (Set No. 3)]**

Solution : Sample space $S = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

$X(S) = (2, 3, 4, 5, 6)$

$p(2) = P(X=2) = p[(1,1)] = \frac{1}{9}$; $p(3) = P(X=3) = p[(1,2),(2,1)] = \frac{2}{9}$

$p(4) = P(X=4) = p[(1,3),(2,2),(3,1)] = \frac{3}{9}$; $p(5) = P(X=5) = p[(2,3),(3,2)] = \frac{2}{9}$

$p(6) = P(X=6) = p[(3,3)] = \frac{1}{9}$

(*i*) The probability distribution of X is

x	2	3	4	5	6
$p(x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(*ii*) Mean $= E(x) = \sum p_i x_i = 2\times\frac{1}{9} + 3\times\frac{2}{9} + 4\times\frac{3}{9} + 5\times\frac{2}{9} + 6\times\frac{1}{9}$

$$= \frac{1}{9}(2+6+12+10+6) = \frac{1}{9}(36) = 4$$

(*iii*) $E(x^2) = \sum p_i x_i^2$

$$= \frac{1}{9}\times 4 + \frac{2}{9}\times 9 + \frac{3}{9}\times 16 + \frac{2}{9}\times 25 + \frac{1}{9}\times 36$$

$$= \frac{1}{9}(4+18+48+50+36) = \frac{156}{9} = 17.33$$

Hence variance $= E(x^2) - [E(x)]^2 = 17.33 - (4)^2 = 17.33 - 16 = 1.33$

REVIEW QUESTIONS

1. What is the Mathematical Expectation ?
2. Write the properties of Mathematical Expectation.
3. Define expected value and variance of a random variable. **[JNTU (H) Dec. 2019]**

EXERCISE 3(A)

1.(*a*) The probability density function of a variate X is as follows :

$X = x$	0	1	2	3	4	5	6
$P(x)$	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

(*i*) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$

(*ii*) What will be the minimum value of K so that $P(X \leq 2) > 0.3$.

[JNTU (K) Nov. 2009, Mar. 2014 (Set No. 4)]

(*b*) A random variable X has the following probability function ;

X	0	1	3	4	5	6	7
$P(X)$	0	K	$2K$	$2K$	$3K$	K^2	$7K^2 + K$

(*i*) Find the value of K (*ii*) Evaluate $P(X < 6)$, $P(X \geq 6)$

(*iii*) Evaluate $P(0 < X < 5)$ **[JNTU (K) Mar. 2013, II Sem June 2015 (Set No. 3)]**

ANSWERS

1. (a) (i) $\frac{33}{49}$ (ii) $\frac{1}{30}$ 2. (i) 0.1 (ii) 0.8 (iii) 2.16

3. (i) 5.9 (ii) 1.49 (iii) 1.22 4. 0.8, $\frac{143}{50}$

5. (a) (i) $\frac{1}{36}$ (ii) 4.46 (iii) 2.08 6. 16, $2\sqrt{5}$

7. (i) 1 (ii) 4.622 (iii) 4.9971 (iv) 2.24

8. (i) 0.778 (ii) 0.2 (iii) 0.258 (iv) 3.698

9.

x	1	2	3	4	5
$f(x)$	1/2	1/4	1/8	1/16	1/16

; 1.9

10. $\frac{3}{4}$ 11. $\frac{7}{2}$ 12. 196

13.

x	0	1	2	3	4
$p(x)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

14.

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P_i = p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

; 7

15.

x	0	1	2
$p(x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$

16. 21.5,950.24 17. (i) 0 (ii) 1.6 (iii) –3 (iv) 6.4

3.3 MEASURES OF CENTRAL TENDENCY FOR CONTINUOUS PROBABILITY DISTRIBUTION

If a variable is continuous then it takes all possible values in its range like height of an individual, life of a car battery, etc., the expectation of the variable is defined as $E(x) = \int x\, f(x)\, dx$ where $f(x)$ is the probability function of the variable x.

On replacing p_i by $f(x)\,dx$, x_i by x and the summation over 'i' by integration over the specified range of the variate X in the formula of discrete probability distribution, we obtain the corresponding formulae for continuous probability distribution.

Let $f(x)$ be the probability density function of a continuous random variable X. Then

(i) **Mean :** Mean of a distribution is given by $\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$

If X is defined from a to b, then $\boxed{\mu = E(X) = \int_a^b x\, f(x) dx}$

In general, mean or expectation of any function $\phi(x)$ is given by

$$\boxed{E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x) dx}$$

(ii) **Median:** Median is the point which divides the entire distribution into two equal parts. In case of continuous distribution, median is the point which divides the total area into two equal parts. Thus if X is defined from a to b and M is the median, then

$$\int_a^M f(x) dx = \int_M^b f(x)\, dx = \frac{1}{2}$$

Solving for M, we get the median.

(iii) **Mode :** Mode is the value of x for which $f(x)$ is maximum. Mode is thus given by $f'(x) = 0$ and $f''(x) < 0$ for $a < x < b$.

(iv) **Variance :** Variance of a distribution is given by

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \quad \text{or} \quad \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

Suppose that the variate X is defined from a to b. Then

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)\, dx \quad \text{or} \quad \sigma^2 = \int_a^b x^2 f(x)\, dx - \mu^2$$

(v) **Mean deviation :** Mean deviation about the mean (μ) is given by

$$\int_{-\infty}^{\infty} |x-\mu|\, f(x)\, dx$$

SOLVED EXAMPLES

Example 1 : If a random variable has the probability density $f(x)$ as

$$f(x) = \begin{cases} 2e^{-2x}, \text{for } x > 0 \\ 0, \text{for } x \le 0 \end{cases}$$ find the probabilities that it will take on a value

(i) between 1 and 3 (ii) greater than 0.5.

[JNTU 2001, 2006S (Set No. 4), (H) III yr. Nov. 2015]

<u>**Solution :**</u>

(i) The probability that a variate takes a value between 1 and 3 is given by

$$P(1 \le X \le 3) = \int_1^3 f(x)\, dx = \int_1^3 2e^{-2x} dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_1^3 = -(e^{-6} - e^{-2}) = e^{-2} - e^{-6}$$

(ii) The probability that a variable takes a value greater than 0.5 is

$$P(X \geq 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty} = -\left(e^{-\infty} - e^{-1}\right) = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

Example 2 : If the probability density of a random variable is given by

$$f(x) = \begin{cases} k(1-x^2), \text{ for } 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$$

find the value of k and the probabilities that a random variable having this probability density will take on a value *(i)* between 0.1 and 0.2 *(ii)* greater than 0.5. **[JNTU 1999S, (H) May 2017]**

Solution : Given $f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$

i.e., $0 + \int_{0}^{1} k(1-x^2)\,dx + 0 = 1$

i.e., $k\left(x - \frac{x^3}{3}\right)_0^1 = 1$ or $k\left(1 - \frac{1}{3}\right) = 1$

$\therefore\ k = \frac{3}{2}$

(i) The probability that the variate will take on a value between 0.1 and 0.2 is

$$P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)\,dx = \int_{0.1}^{0.2} k(1-x^2)dx$$

$$= \frac{3}{2}\left(x - \frac{x^3}{3}\right)_{0.1}^{0.2} \quad \left(\because k = \frac{3}{2}\right)$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right]$$

$$= \frac{3}{2}\left[0.1 - \frac{0.007}{3}\right] = 0.2965$$

(ii) The probability that the variate will take on a value greater than 0.5 is

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= \frac{3}{2}\int_{0.5}^{1} (1-x^2)\,dx + 0 = \frac{3}{2}\left(x - \frac{x^3}{3}\right)_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= \frac{3}{2}\left(\frac{2}{3} - 0.4583\right) = 0.3125$$

Example 3 : The probability density $f(x)$ of a continuous random variable is given by $f(x) = c\,e^{-|x|}$, $-\infty < x < \infty$. Show that $c = 1/2$ and find the mean and variance of the distribution. Also find the probability that the variate lies between 0 and 4.

[JNTU Jan. 2007, (A) Nov. 2010 (Set No. 2), (K) May 2013 (Set No. 4)]

Solution : Given $f(x) = c\,e^{-|x|}$, $-\infty < x < \infty$

We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$ [since the total probability is unity]

i.e., $\int_{-\infty}^{\infty} c\,e^{-|x|}dx = 1$ *i.e.,* $c\int_{-\infty}^{\infty} e^{-|x|}dx = 1$

i.e., $2c\int_{0}^{\infty} e^{-|x|}dx = 1\left[\because e^{-|x|} \text{ is an even function}\right]$

i.e., $2c\int_{0}^{\infty} e^{-x}dx = 1$ [$\because$ in $0 \leq x \leq \infty$, $|x| = x$]

$\Rightarrow 2c\left(-e^{-x}\right)_0^{\infty} = 1 \Rightarrow -2c\,(0-1) \Rightarrow 2c = 1 \ \therefore c = \frac{1}{2}$

Hence $f(x) = c\,e^{-|x|} = \frac{1}{2}e^{-|x|}$

(i) Mean of the distribution, $E(x) = \mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}dx = 0, \text{ since integrand is odd.}$$

(ii) Variance of the distribution,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} (x-0)^2 . \frac{1}{2} e^{-|x|} dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx$$

$$= 2 . \frac{1}{2} \int_0^{\infty} x^2 e^{-|x|}\,dx \text{ , since integrand is even}$$

$$= \int_0^{\infty} x^2 e^{-x} dx = \left(x^2 \frac{e^{-x}}{-1} - 2x \frac{e^{-x}}{1} + 2\frac{e^{-x}}{-1}\right)_0^{\infty}$$

$$= [0 - (-2)] = 2$$

(iii) The probability between 0 and 4 = $P(0 \le X \le 4)$

$$= \frac{1}{2} \int_0^4 e^{-|x|} dx = \frac{1}{2} \int_0^4 e^{-x} dx \quad [\because \text{in } 0 < x < 4, |x| = x]$$

$$= -\frac{1}{2}\left(e^{-x}\right)_0^4 = -\frac{1}{2}\left(e^{-4} - 1\right)$$

$$= \frac{1}{2}(1 - e^{-4}) = 0.4908 \text{ (nearly)}$$

Example 4 : Probability density function of a random variable X is

$$f(x) = \begin{cases} \frac{1}{2} \sin x, \text{ for } 0 \le x \le \pi \\ 0, \text{elsewhere} \end{cases}.$$

Find the mean, mode and median of the distribution and also find the probability between 0 and $\pi/2$. **[JNTU 2004, 2008S, (A) Nov. 2010 (Set No. 4)]**

Solution : *(i)* Mean of the distribution $= \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-\infty}^{0} x(0)\,dx + \int_0^{\pi} x.\frac{1}{2} \sin x\; dx + \int_{\pi}^{\infty} x(0)\; dx$$

$$= \frac{1}{2}\int_0^{\pi} x \sin x\, dx = \frac{1}{2}(-x \cos x + \sin x)_0^{\pi} = \frac{\pi}{2}$$

(ii) Mode is the value of x for which $f(x)$ is maximum

Now $f'(x) = \frac{1}{2} \cos x$

For $f(x)$ to be maximum, $f'(x) = 0$

i.e., $\cos x = 0 \;\therefore\; x = \frac{\pi}{2}$

$f''(x) = -\frac{1}{2} \sin x$. At $x = \frac{\pi}{2}$, $f''(x) = -\frac{1}{2} < 0$

Hence $f(x)$ is maximum at $x = \frac{\pi}{2}$

$\therefore$ Mode of the distribution is given by $x = \frac{\pi}{2}$

(iii) If M is the median of the distribution, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

i.e., $\int_0^M \frac{1}{2} \sin x\, dx = \int_M^{\pi} \frac{1}{2} \sin x\, dx = \frac{1}{2}$

Solving $\int_0^M \frac{1}{2} \sin x\, dx = \frac{1}{2}$, we get

$$-\frac{1}{2}(\cos x)_0^M = \frac{1}{2} \Rightarrow -\frac{1}{2}(\cos M - 1) = \frac{1}{2}$$

$$\Rightarrow 1 - \cos M = 1 \Rightarrow \cos M = 0 \;\therefore\; M = \frac{\pi}{2}$$

$\therefore$ Median of the distribution $= \frac{\pi}{2}$

Thus Mean = Mode = Median $= \frac{\pi}{2}$

(iv) $P(0 < x < \frac{\pi}{2}) = \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{1}{2} \sin x\, dx$

$$= -\frac{1}{2}(\cos x)_0^{\pi/2}$$

$$= -\frac{1}{2}(0 - 1) = \frac{1}{2}$$

Example 5 : A continuous random vaiable has the probability density function

$$f(x) = \begin{cases} k\,x\,e^{-\lambda x}, \text{ for } x \ge 0, \lambda > 0 \\ 0, \text{ otherwise} \end{cases}$$

Determine (*i*) k (*ii*) Mean (*iii*) Variance

[JNTU (A) Dec. 2009, Nov. 2010, Dec. 2011, (H) May 2011, Nov. 2012, (K) May 2013 (Set No. 1)]

Solution : (*i*) Since the total probability is unity, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$i.e., \int_{-\infty}^{0} 0.\,dx + \int_{0}^{\infty} kx\,e^{-\lambda x} dx = 1 \quad i.e., \; k\int_{0}^{\infty} x\,e^{-\lambda x} dx = 1$$

$$i.e., \; k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1.\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = 1$$

$$i.e., \; k\left[(0-0)-\left(0-\frac{1}{\lambda^2}\right)\right] = 1 \text{ or } k = \lambda^2$$

Now $f(x)$ becomes

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, \text{ for } x \ge 0, \lambda > 0 \\ 0, \text{ otherwise} \end{cases}$$

(*ii*) Mean of the distribution, $\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$i.e., \mu = \int_{-\infty}^{0} 0.\,dx + \int_{0}^{\infty} x.\lambda^2 x\,e^{-\lambda x} dx = \lambda^2 \int_{0}^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda^2\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_0^{\infty}, \text{ using Bernoulli's Rule}$$

$$= \lambda^2\left[(0-0+0)-(0-0-\frac{2}{\lambda^3})\right] = \frac{2}{\lambda}$$

(*iii*) Variance of the distribution, $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$i.e., \sigma^2 = \int_{0}^{\infty} x^2 f(x)\,dx - \left(\frac{2}{\lambda}\right)^2 = \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x} dx - \frac{4}{\lambda^2} \text{ (Apply Bernoulli's Rule)}$$

$$= \lambda^2\left[x^3\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 6x\left(\frac{e^{-\lambda x}}{-\lambda^3}\right) - 6\left(\frac{e^{-\lambda x}}{\lambda^4}\right)\right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2\left[(0-0+0-0)-\left(0-0+0-\frac{6}{\lambda^4}\right)\right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

Example 6 : A continuous random variable X is defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, \text{ if } -3 \le x < -1 \\ \frac{1}{16}(6-2x^2), \text{ if } -1 \le x < 1 \\ \frac{1}{16}(3-x)^2, \text{ if } 1 \le x \le 3 \\ 0, \text{ elsewhere} \end{cases}$$

Verify that $f(x)$ is a density function and also find the mean of X.

(OR) Show that the area under the curve above x-axis is unity. Also find the mean of the distribution. **[JNTU 2003S, (K) May 2013, Dec. 2015 (Set No. 4)]**

Solution : $f(x)$ is clearly ≥ 0 for every x in $[-3, 3]$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-3} 0\,dx + \int_{-3}^{-1} \frac{1}{16}(3+x)^2 dx + \int_{-1}^{1} \frac{1}{16}(6-2x^2)dx + \int_{1}^{3} \frac{1}{16}(3-x)^2 dx + \int_{3}^{\infty} 0\,dx$$

$$= \frac{1}{16}\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \frac{1}{16}.2\int_{0}^{1}(6-2x^2)\,dx + \frac{1}{16}.\left[\frac{(3-x)^3}{-3}\right]_1^3$$

$$= \frac{1}{48}(8-0) + \frac{1}{8}\left(6x - \frac{2x^3}{3}\right)_0^1 - \frac{1}{48}(0-8)$$

$$= \frac{1}{6} + \frac{1}{8}\left(6-\frac{2}{3}\right) + \frac{1}{6} = \frac{1}{3} + \frac{1}{4}\left(3-\frac{1}{3}\right) = \frac{1}{3} + \frac{2}{3} = 1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

Mean of $f(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \frac{1}{16}\int_{-3}^{-1} x(3+x)^2\, dx + \frac{1}{16}\int_{-1}^{1} x(6-2x^2)\, dx + \frac{1}{16}\int_{1}^{3} x(3-x)^2\, dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(9+x^2+6x)\, dx + 0 + \frac{1}{16}\int_{1}^{3} x(9-6x+x^2)\, dx$$

[$\because$ The integrand of the second integral is odd function]

$$= \frac{1}{16}\int_{-3}^{-1} (9x+6x^2+x^3)\, dx + \frac{1}{16}\int_{1}^{3} (9x-6x^2+x^3)\, dx$$

$$= \frac{1}{16}\left(\frac{9x^2}{2}+\frac{6x^3}{3}+\frac{x^4}{4}\right)_{-3}^{-1} + \frac{1}{16}\left(\frac{9x^2}{2}-\frac{6x^3}{3}+\frac{x^4}{4}\right)_{1}^{3}$$

$$= \frac{1}{16}\left[\left(\frac{9}{2}-2+\frac{1}{4}\right)-\left(\frac{81}{2}-54+\frac{81}{4}\right)\right] + \frac{1}{16}\left[\left(\frac{81}{2}-54+\frac{81}{4}\right)-\left(\frac{9}{2}-2+\frac{1}{4}\right)\right]$$

$= 0$

Example 7 : Is the function defined by

$$f(x) = \begin{cases} 0, x<2 \\ \frac{1}{18}(2x+3), 2 \le x \le 4 \\ 0, x>4 \end{cases}$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $2 \le x \le 3$. **[JNTU (K) Nov. 2009, May 2013 (Set No. 2)]**

Solution : (*i*) For all points x in $-\infty \le x \le \infty, f(x) \ge 0$ and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{2} 0.dx + \int_{2}^{4} \frac{1}{18}(2x+3)\, dx + \int_{4}^{\infty} 0.dx$$

$$= \frac{1}{18}\int_{2}^{4} (2x+3)\, dx = \frac{1}{18}\left[\frac{(2x+3)^2}{4}\right]_2^4$$

$$= \frac{1}{72}(121-49) = 1$$

Hence $f(x)$ is a probability density function.

(*ii*) The probability that the density will fall in the interval $2 \le x \le 3$ is

$$P(2 \le x \le 3) = \int_2^3 f(x)\, dx = \frac{1}{18}\int_2^3 (2x+3)\, dx$$

$$= \frac{1}{18}\left(x^2+3x\right)_2^3 = \frac{1}{18}(18-10) = \frac{8}{18} = \frac{4}{9}$$

Example 8 : A random variable X gives measurements x between 0 and 1 with a probability function $f(x) = \begin{cases} 12x^3-21x^2+10x, \text{for } 0 \le x \le 1 \\ 0, \text{elsewhere} \end{cases}$

(*i*) Find $P\left(X \le \frac{1}{2}\right)$ and $P\left(X > \frac{1}{2}\right)$

(*ii*) Find a number k such that $P(X \le k) = \frac{1}{2}$ **[JNTU 2003]**

Solution : (*i*) $P\left(X \le \frac{1}{2}\right) = \int_0^{1/2} f(x)\, dx = \int_0^{1/2} (12x^3-21x^2+10x)\, dx$

$$= \left(12.\frac{x^4}{4}-21.\frac{x^3}{3}+10.\frac{x^2}{2}\right)_0^{1/2} = \left(3x^4-7x^3+5x^2\right)_0^{1/2}$$

$$= \left(\frac{3}{16}-\frac{7}{8}+\frac{5}{4}\right)-0 = \frac{1}{16}(3-14+20) = \frac{9}{16}$$

Now $P\left(X > \frac{1}{2}\right) = 1 - P\left(X \le \frac{1}{2}\right) = 1 - \frac{9}{16} = \frac{7}{16}$

(*ii*) Given $P(X \le k) = \frac{1}{2}$

i.e., $\int_0^k f(x)\, dx = \frac{1}{2}$ *i.e.,* $\int_0^k (12x^3-21x^2+10x)\, dx = \frac{1}{2}$ *i.e.,* $(3x^4-7x^3+5x^2)_0^k = \frac{1}{2}$

i.e., $3k^4 - 7k^3 + 5k^2 = \frac{1}{2}$ or $6k^4 - 14k^3 + 10k^2 - 1 = 0$

$\therefore$ $k = 0.452$

Example 9 : A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0, \text{ if } x \le 1 \\ k(x-1)^4, \text{ if } 1 < x \le 3 \\ 1, \text{if } x > 3 \end{cases}$$

Determine (*i*) $f(x)$ (*ii*) k (*iii*) Mean **[JNTU 2004S, 2007S, (A) Apr. 2012 (Set No. 2)]**

Solution : (*i*) We know that $f(x) = \frac{d}{dx}[F(x)]$

$$\therefore f(x) = \begin{cases} 0, \text{if } x \le 1 \\ 4k(x-1)^3, \text{if } 1 < x \le 3 \\ 0, \text{if } x > 3 \end{cases}$$

(ii) Since total probability is unity, we have

$$\int_1^3 f(x)\,dx = 1 \text{ i.e., } 4k \int_1^3 (x-1)^3\,dx = 1$$

$$\text{i.e., } 4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1 \text{ i.e., } k\,(16-0) = 1 \text{ or } k = \frac{1}{16}$$

$$\text{Hence } f(x) = \begin{cases} 0, \text{ if } x \le 1 \\ \frac{1}{4}(x-1)^3, \text{if } 1 < x \le 3 \\ 0, \text{ if } x > 3 \end{cases}$$

(iii) Mean of X, $\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \int_{-\infty}^{1} x.f(x)\,dx + \int_1^3 x.f(x)\,dx + \int_3^{\infty} x.f(x)\,dx$$

$$= 0 + \int_1^3 x.\frac{1}{4}(x-1)^3\,dx + 0 = \frac{1}{4}\int_1^3 x(x-1)^3\,dx$$

$$= \frac{1}{4}\int_0^2 (t+1)t^3\,dt \text{ (Putting } x - 1 = t)$$

$$= \frac{1}{4}\int_0^2 (t^4+t^3)\,dt = \frac{1}{4}\left(\frac{t^5}{5}+\frac{t^4}{4}\right)_0^2$$

$$= \frac{1}{4}\left(\frac{2^5}{5}+\frac{2^4}{4}\right) = \frac{2^4}{4}\left(\frac{2}{5}+\frac{1}{4}\right)$$

$$= 4\left(\frac{13}{20}\right) = \frac{13}{5} = 2.6$$

Example 10 : If X is a continuous random variable and $Y = aX + b$, prove that $E(Y) = a\,E(X) + b$ and $V(Y) = a^2.V(X)$, where V stands for variance and a, b are constants.

[JNTU 00, (A) Dec. 09 (Set No. 2), (H), (A) Nov. 2010, Nov. 2011, Apr. 2012, (H) Sept. 2017]

Solution : (*i*) By definition,

$$E(Y) = E\,(aX+b) = \int_{-\infty}^{\infty} (ax+b)\,f(x)\,dx \quad \left[\because E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)\,f(x)\,dx\right]$$

$$= a\int_{-\infty}^{\infty} x\,f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\,E\,(X) + b\,(1) \quad \text{[since total probability is unity]}$$

$$= a\,E\,(X) + b$$

(ii) From (*i*), we have $E(Y) = aE(X) + b$ ... (1)

where $Y = aX + b$ ... (2)

(2) – (1) gives $Y - E(Y) = a\,[X - E(X)]$

Squaring, $[Y - E(Y)]^2 = a^2[X - E(X)]^2$

Taking Expectation of both sides, we get

$E\{[Y - E(Y)]^2\} = a^2\,E\{[X - E(X)]^2\}$

$\therefore V(Y) = a^2V(X)$

Example 11 : If X is a continuous random variable and k is a constant, then prove that (*i*) $Var\,(X + k) = Var\,(X)$ (*ii*) $Var\,(kX) = k^2\,Var\,(X)$ **[JNTU 2006, 2007(Set No. 1)]**

Solution : By definition,

$$\text{Var }(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x\,f(x)\,dx\right]^2$$

(*i*) $$\text{Var }(X+K) = \int_{-\infty}^{\infty} (x+k)^2\,f(x)\,dx - \left[\int_{-\infty}^{\infty} (x+k)\,f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} (x^2+2kx+k^2)\,f(x)\,dx - \left[\int_{-\infty}^{\infty} x\,f(x)\,dx + k\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx + 2k\int_{-\infty}^{\infty} x\,f(x)\,dx + k^2 - \left[\int_{-\infty}^{\infty} x\,f(x)\,dx + k\right]^2$$

$$\left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$$

$$= E(X^2) + 2k\,E(X) + k^2 - [E(X) + k)]^2$$
$$= E(X^2) + 2k\,E(X) + k^2 - [E(X)]^2 - 2k\,E(X) - k^2$$
$$= E(X^2) - [E(X)]^2$$
$$= \text{Var}\,(X)$$

(ii) $\text{Var}\,(kX) = \int_{-\infty}^{\infty} k^2x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} kx\,f(x)\,dx\right]^2$

$$= k^2\int_{-\infty}^{\infty} x^2 f(x)\,dx - k^2\left[\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= k^2\left[E(X)^2 - \{E(X)\}^2\right] = k^2\,\text{Var}\,(X)$$

Example 12 : For the continuous random variable X whose probability density function is given by $f(x) = \begin{cases} cx\,(2-x), \text{ if } 0 \le x \le 2 \\ 0, \text{otherwise} \end{cases}$ where c is a constant.

Find c, mean and variance of X. **[JNTU 2003, 2004 (Set No.3)]**

(OR) The frequency function of a continuous random variable X is given by $f(x) = y_0 x\,(2-x), 0 \le x \le 2$. Find the value of y_0, mean and variance of X.

[JNTU (A) Dec. 2009 (Set No.3)]

Solution : *(i)* Since the total probability is unity, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1. \text{ So } \int_0^2 f(x)\,dx = 1$$

i.e., $\int_0^2 cx\,(2-x)\,dx = 1$ *i.e.,* $c\left(x^2 - \frac{x^3}{3}\right)_0^2 = 1$

i.e., $c\left(4 - \frac{8}{3}\right) = 1$ *i.e.,* $\frac{4c}{3} = 1$ or $c = \frac{3}{4}$

$$\therefore \quad f(x) = \begin{cases} \frac{3x}{4}(2-x), \text{ if } 0 \le x \le 2 \\ 0, \text{otherwise} \end{cases}$$

(ii) Mean of $X = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

i.e., $\mu = \int_0^2 x.\frac{3x}{4}(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$

$$= \frac{3}{4}\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^2 = \frac{3}{4}\left(\frac{2^4}{3} - \frac{2^4}{4}\right) = \frac{3}{4}(2^4)\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{12}{12} = 1$$

(iii) Variance of $X = V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 = \int_0^2 x^2.\frac{3x}{4}(2-x)\,dx - (1)^2$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx - 1 = \frac{3}{4}\left(2.\frac{x^4}{4} - \frac{x^5}{5}\right)_0^2 - 1$$

$$= \frac{3}{4}\left[\frac{32}{4} - \frac{32}{5}\right] - 1 = 24\left(\frac{1}{4} - \frac{1}{5}\right) - 1 = \frac{6}{5} - 1 = \frac{1}{5}$$

Example 13 : For the continuous probability function $f(x) = k\,x^2\,e^{-x}$ when $x \ge 0$, find *(i)* k *(ii)* Mean *(iii)* Variance **[JNTU 2005, 2005S, 2007, (K) Dec. 2013 (Set No. 4)]**

(OR) Is $f(x) = \frac{1}{2}x^2 e^{-x}$ when $x \ge 0$ can be regarded as a probability function for a continuous random variable? If, so find Mean and Variance of the random variable. **[JNTU (H) Nov. 2015]**

[Hint : In order that $f(x)$ should be a probability function $\int_{-\infty}^{\infty} f(x)\,dx = 1$**]**

Solution : *(i)* We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\therefore \int_0^{\infty} kx^2 e^{-x}\,dx = 1 \quad (\because x \ge 0)$$

i.e., $k\left[x^2(-e^{-x}) - 2x\,(e^{-x}) + 2(-e^{-x})\right]_0^{\infty} = 1$, **Using Bernoulli's Rule.**

i.e., $k\left\{-e^{-x}(x^2 + 2x + 2)\right\}_0^{\infty} = 1$

i.e., $k\,(0 + 2) = 1$ or $k = \frac{1}{2}$

(ii) Mean $= E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^{\infty} kx^3 e^{-x}\,dx$

$$= k\left[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6e^{-x}\right]_0^{\infty}, \text{ **Using Bernoulli's Rule**}$$

$$= k\left[-e^{-x}(x^3 + 3x^2 + 6x + 6)\right]_0^{\infty} = k\,[0 + 6] = 6k$$

$$\therefore \quad \mu = 6\left(\frac{1}{2}\right) = 3 \quad \left(\because k = \frac{1}{2}\right)$$

(iii) Variance $= E(x^2) - [E(x)]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$= \int_0^{\infty} x^2 . kx^2 e^{-x} dx - (3)^2 = k \int_0^{\infty} x^4 e^{-x} dx - 9$$

$$= k\left[x^4(-e^{-x}) - 4x^3(e^{-x}) + 12x^2(-e^{-x}) - 24x(-e^{-x}) + 24e^{-x}\right]_0^{\infty} - 9$$

$$= \frac{1}{2}\left[-e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24\right]_0^{\infty} - 9 \quad \left[\because k = \frac{1}{2}\right]$$

$$= \frac{1}{2}[0 + 24] - 9 = 12 - 9 = 3$$

Example 14 : The trouble shooting capability of an IC chip in a circuit is a random variable X whose distribution function is given by

$$F(X) = \begin{cases} 0, \text{ for } x \le 3 \\ 1 - \dfrac{9}{x^2}, \text{ for } x > 3 \end{cases}$$

where x denote the number of years. Find the probability that the IC chip will work properly

(i) Less than 8 years (ii) Beyond 8 years

(iii) Between 5 to 7 years (iv) Anywhere from 2 to 5 years

[JNTU (K) Nov. 2009 (Set No.4)]

Solution : We have $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$

$$\therefore \quad F(X) = \int_0^x f(t)\,dt \; (\because x > 0) = \begin{cases} 0, \text{if } x \le 3 \\ 1 - \dfrac{9}{x^2}, \text{if } x > 3 \end{cases}$$

(i) $P(X \le 8) = \int_0^8 f(t)\,dt = 1 - \dfrac{9}{8^2} = 0.8594$

(ii) $P(X > 8) = 1 - P(X \le 8) = 1 - 0.8594 = 0.1406$

(iii) $P(5 \le x \le 7) = F(7) - F(5) = \left(1 - \dfrac{9}{7^2}\right) - \left(1 - \dfrac{9}{5^2}\right)$

$$= 9\left(\frac{1}{25} - \frac{1}{49}\right) = \frac{24 \times 9}{25 \times 49} = 0.1763$$

(iv) $P(2 \le x \le 5) = F(5) - F(2) = \left(1 - \dfrac{9}{5^2}\right) - 0 = \dfrac{16}{25} = 0.64$

Example 15 : If the probability density function of a random variable X is given by

$$f(x) = \begin{cases} 2k\,x\,e^{-x^2}, \text{for } x > 0 \\ 0, \text{ for } x \le 0 \end{cases}$$

Determine (i) k (ii) the distribution function for X.

Solution : (i) We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = 1$ i.e., $0 + \int_0^{\infty} 2k\,x\,e^{-x^2} dx = 1$

i.e., $k \int_0^{\infty} 2x.e^{-x^2} dx = 1$ i.e., $k \int_0^{\infty} e^{-t} dt = 1$ (Putting $x^2 = t$)

or $k\left(-e^{-t}\right)_0^{\infty} = 1$ or $-k(0 - 1) = 1$

$\therefore \; k = 1$

(ii) The distribution function is

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = 0 \text{ if } x \le 0$$

and $F(X) = \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt$, if $x > 0$

$$= 0 + \int_0^x 2t\,e^{-t^2} dt = \int_0^{x^2} e^{-u}\,du \quad (\text{Putting } t^2 = u)$$

$$= -\left(e^{-u}\right)_0^{x^2} = -(e^{-x^2} - 1) = 1 - e^{-x^2}$$

$$\therefore \quad F(x) = \begin{cases} 1 - e^{-x^2}, \text{if } x > 0 \\ 0, \text{otherwise} \end{cases}$$

Example 16 : Suppose a continuous random variable X has the probability density $f(x) = K(1 - x^2)$ for $0 < x < 1$, and $f(x) = 0$ otherwise. Find (i) K (ii) Mean (iii) Variance

[JNTU 2007S(Set No.1), (H) Sept. 2017]

Solution : (i) Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e., $\int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^{\infty} f(x)\,dx = 1$

i.e., $0 + \int_0^1 K(1-x^2)\,dx + 0 = 1$

i.e., $K\left(x - \frac{x^3}{3}\right)_0^1 = 1$ or $K\left(1 - \frac{1}{3}\right) = 1$

$\therefore \quad K = \frac{3}{2}$

(ii) Mean of $X = E(X) = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^1 x.f(x)\,dx$

$$= \int_0^1 x.K(1-x^2)dx = K\int_0^1 (x-x^3)dx$$

i.e., $\mu = K\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 = K\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{K}{4} = \left(\frac{3}{2}\right)\left(\frac{1}{4}\right) = \frac{3}{8} \quad \left[\because K = \frac{3}{2}\right]$

(iii) Variance of $X = E(X^2) - [E(X)]^2 = \int_0^1 x^2 f(x)dx - \mu^2$

$$= \int_0^1 x^2.K(1-x^2)\,dx - \left(\frac{3}{8}\right)^2$$

$$= K\int_0^1 (x^2 - x^4)dx - \frac{9}{64} = K\left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1 - \frac{9}{64}$$

i.e., $\sigma^2 = K\left(\frac{1}{3} - \frac{1}{5}\right) - \frac{9}{64} = \frac{2K}{15} - \frac{9}{64} = \frac{2}{15}\left(\frac{3}{2}\right) - \frac{9}{64}$

$$= \frac{1}{5} - \frac{9}{64} = \frac{19}{320} = 0.06$$

Example 17 : The daily consumption of electric power (in millions of kW-hours) is a random variable having the probability density function (p.d.f)

$$f(x) = \begin{cases} \frac{1}{9}x\,e^{-x/3}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

If the total production is 12 million KW-hours, determine the probability that there is power cut (shortage) on any given day. **[JNTU Jan. 2008, (K) May 2010S]**

Solution : Probability that the power consumed is between 0 to 12 is

$$P(0 \le x \le 12 \text{ million KW-hours}) = \int_0^{12} f(x)\,dx = \frac{1}{9}\int_0^{12} x\,e^{-x/3}\,dx$$

$$= \frac{1}{9}\left[x\frac{e^{-x/3}}{(-1/3)} - 1.\frac{e^{-x/3}}{1/9}\right]_0^{12}$$

$$= \frac{1}{9}[-36\,e^{-4} - 9\,e^{-4} + 9]$$

$$= \frac{1}{9}(9 - 45e^{-4})$$

$$= 1 - 5e^{-4}$$

Power supply is inadequate if daily consumption exceeds 12 million kW, *i.e.,*

$P(x > 12) = 1 - P(0 \le x \le 12) = 1 - (1 - 5\,e^{-4})$

$= 5e^{-4} = 0.0915781$

Example 18 : If probability density function is

$$f(x) = \begin{cases} Kx^3 \text{ in } 0 \le x \le 3 \\ 0 \quad \text{elsewhere} \end{cases}$$

Find the value of K and find the probability between $x = 1/2$ and $x = 3/2$.

[JNTU Nov. 2008 (Set No. 1), (H) Sept. 2017]

Solution : Given $f(x) = \begin{cases} Kx^3 \text{ in } 0 \le x \le 3 \\ 0 \quad \text{elsewhere} \end{cases}$

We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^3 Kx^3 dx = 1 \Rightarrow K\left(\frac{x^4}{4}\right)_0^3 = 1 \Rightarrow \frac{81K}{4} = 1 \qquad \therefore K = \frac{4}{81}$$

Now $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx = \int_{1/2}^{3/2} \frac{4}{81}x^3 dx = \frac{4}{81}\left(\frac{x^4}{4}\right)_{1/2}^{3/2}$

$$= \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right] = \frac{1}{81 \times 16}[81 - 1]$$

$$= \frac{80}{81 \times 16} = \frac{5}{81} = 0.0617$$

Example 19 : Find the constant K such that

$$f(x) = \begin{cases} Kx^2, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

is a probability function.

(i) Find the distribution function F(x) **[JNTU Nov. 2008 (Set No. 2)]**

(ii) $P(1 < X \le 2)$

Solution : (i) Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow K\int_0^3 x^2\,dx = 1 \Rightarrow K\left(\frac{x^3}{3}\right)_0^3 = 1 \Rightarrow K = \frac{1}{9}$$

The distribution function is

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = 0 \text{ if } x \le 0$$

and $$F(X) = \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt \text{ if } x > 0$$

$$= 0 + \int_0^x \frac{1}{9}t^2\,dt = \frac{1}{9}\left(\frac{t^3}{3}\right)_0^3 = \frac{1}{27}x^3$$

$$\therefore F(x) = \begin{cases} \frac{1}{27}x^3, \text{ if } x > 0 \\ 0, \text{ otherwise} \end{cases}$$

(ii) $$P(1 < X \le 2) = \int_1^2 f(x)\,dx = \int_1^2 Kx^2\,dx$$

$$= \frac{1}{9}\int_1^2 x^2\,dx \quad \left[\because k = \frac{1}{9}\right]$$

$$= \frac{1}{9}\left(\frac{x^3}{3}\right)_1^2 = \frac{1}{27}(8-1) = \frac{7}{27}$$

Example 20 : If the probability density function of X is given by

$$f(x) = \begin{cases} x/2, \text{for } 0 < x \le 1 \\ 1/2, \text{for } 1 < x \le 2 \\ (3-x)/2, \text{for } 2 < x < 3 \\ 0, \text{ elsewhere} \end{cases}$$

Find the expected value of $f(x) = x^2 - 5x + 3$ **[JNTU Nov. 2008 (Set No. 2)]**

Solution : The expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)\,f(x)\,dx$$

$$E(x^2 - 5x + 3) = \int_{-\infty}^{\infty} (x^2 - 5x + 3)\,f(x)\,dx$$

$$= \int_0^1 (x^2-5x+3).\frac{x}{2}dx + \int_1^2 (x^2-5x+3).\frac{1}{2}dx + \int_2^3 (x^2-5x+3).\left(\frac{3-x}{2}\right)dx$$

$$= \frac{1}{2}\int_0^1 (x^3-5x^2+3x)\,dx + \frac{1}{2}\int_1^2 (x^2-5x+3)\,dx + \frac{1}{2}\int_2^3 (-x^3+8x^2-18x+9)\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{5x^3}{3} + \frac{3x^2}{2}\right)_0^1 + \frac{1}{2}\left(\frac{x^3}{3} - \frac{5x^2}{2} + 3x\right)_1^2$$

$$+ \frac{1}{2}\left(-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{18x^2}{2} + 9x\right)_2^3$$

$$= \frac{1}{24} - \frac{13}{12} - \frac{19}{24} = \frac{-44}{24} = \frac{-11}{6}$$

Example 21 : Let the continuous random variable X have the probability density function,

$$f(x) = \begin{cases} 2/x^3, \text{ if } 1 < x < \infty \\ 0, \text{ otherwise} \end{cases}$$

Find F(x). **[JNTU Apr. 2009 (Set No. 3)]**

Solution : We are given

$$f(x) = \begin{cases} 2/x^3, \text{ if } 1 < x < \infty \\ 0, \text{ otherwise} \end{cases} \quad i.e., \quad f(x) = \begin{cases} 2/x^3, \text{if } x > 1 \\ 0, \text{ if } x \le 1 \end{cases}$$

We have $$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

If $x > 1$, $$F(x) = \int_1^x f(x)\,dx$$

$$= \int_1^x \frac{2}{x^3}\,dx = 2\left(\frac{x^{-2}}{-2}\right)_1^x = -\left(\frac{1}{x^2}\right)_1^x$$

$$= -\left(\frac{1}{x^2} - 1\right) = 1 - \frac{1}{x^2}$$

If $x < 1$, $F(x) = 0$

$$\therefore F(x) = \begin{cases} 0, \text{ if } x \le 1 \\ 1 - \frac{1}{x^2}, \text{ if } 1 < x < \infty \end{cases}$$

Example 22 : The probability density function is

$$y=\begin{cases}k(3x^2-1), -1\le x\le 2\\ 0,\ \text{else where}\end{cases}$$

Find the value of k and find $P(-1\le x\le 0)$. **[JNTU Apr. 2009 (Set No. 4)]**

Solution : Since $\int_{-\infty}^{\infty} f(x)dx=1$, we have $\int_{-\infty}^{-1} f(x)dx+\int_{-1}^{2} f(x)dx+\int_{2}^{\infty} f(x)dx=1$

i.e., $\int_{-\infty}^{-1} 0\cdot dx+\int_{-1}^{2} K(3x^2-1)dx+\int_{2}^{\infty} 0\cdot dx=1$

i.e., $K\left(3\cdot\frac{x^3}{3}-x\right)_{-1}^{2}=1$ *i.e.,* $K(x^3-x)_{-1}^{2}=1$

i.e., $K[(8-2)-(-1+1)]=1$ or $6K=1$. $\therefore K=\frac{1}{6}$.

Thus $y=\begin{cases}\frac{1}{6}(3x^2-1), -1\le x\le 2\\ 0,\ \text{else where}\end{cases}$

Hence $P(-1\le x\le 0)=\int_{-1}^{0} f(x)dx=\frac{1}{6}\int_{-1}^{0}(3x^2-1)dx=\frac{1}{6}\left[3\cdot\frac{x^3}{3}-x\right]_{-1}^{0}$

$$=\frac{1}{6}(x^3-x)_{-1}^{0}=\frac{1}{6}[(0-0)-(-1+1)]=0$$

Example 23 : The density function of a random variable X is $f(x)=\begin{cases}e^{-x}, x\ge 0\\ 0,\ \text{otherwise}\end{cases}$

Find $E(X), E(X^2)$, Var (X). **[JNTU (K) Nov. 2009 (Set No. 1)]**

Solution : Given $f(x)=\begin{cases}e^{-x}, x\ge 0\\ 0,\ \text{otherwise}\end{cases}$

(*i*) $E(X)=\int_{-\infty}^{\infty} x\cdot f(x)dx=\int_{-\infty}^{0} x(0)dx+\int_{0}^{\infty} xe^{-x}dx=\int_{0}^{\infty} xe^{-x}dx$. Apply Integration by parts

$$=\left[-(x+1)e^{-x}\right]_0^{\infty}=(-1)\lim_{x\to\infty}\frac{(x+1)}{e^x}+(0+1)=0+1=1.$$

(*ii*) $E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_{0}^{\infty} x^2e^{-x}dx$. Apply Integration by parts

$$=\left[x^2(-e^{-x})-2x(e^{-x})+2(-e^{-x})\right]_0^{\infty}=\left[-e^{-x}(x^2+2x+2)\right]_0^{\infty}$$

$$=(-1)\lim_{x\to\infty}\frac{x^2+2x+2}{e^x}+2=2$$

(*iii*) $Var(X)=E(X^2)-\{E(X)\}^2=2-(1)^2=2-1=1$

Example 24 : Is the function defined as follows a density function

$$f(x)=\begin{cases}e^{-x}, x\ge 0\\ 0, x<0\end{cases}$$

If so determine the probability that the variate having this density will fall in the interval (1,2)? Find the cumulative probability F(2)? **[JNTU (K) Nov. 2009 (Set No. 2)]**

Solution : Given $f(x)=\begin{cases}e^{-x}, x\ge 0\\ 0, x<0\end{cases}$

(*i*) Clearly $f(x)\ge 0, \forall x$ in (1,2) and

$$\int_{-\infty}^{\infty} f(x)dx=\int_{-\infty}^{0} 0\cdot dx+\int_{0}^{\infty} e^{-x}dx=\int_{0}^{\infty} e^{-x}dx=-(e^{-x})_0^{\infty}=-(0-1)=1$$

Hence the function $f(x)$ is a density function.

(*ii*) Required probability $=P(1\le x\le 2)=\int_{1}^{2} f(x)dx$

$$=\int_{1}^{2} e^{-x}dx=-(e^{-x})_1^2=-(e^{-2}-e^{-1})=e^{-1}-e^{-2}$$

$$=0.368-0.135=0.233.$$

(*iii*) Cumultative probability function,

$$F(2)=\int_{-\infty}^{2} f(x)dx=\int_{-\infty}^{0} 0\cdot dx+\int_{0}^{2} e^{-x}dx$$

$$=-(e^{-x})_0^2=-(e^{-2}-1)=1-e^{-2}$$

$$=1-0.135=0.865.$$

Example 25 : The cumulative distribution function for a continuous random variable X is $F(x)=\begin{cases}1-e^{-2x}, x\ge 0\\ 0, \quad x<0\end{cases}$

Find (*i*) the density function $f(x)$, (*ii*) mean and (*iii*) variance of the density function. **[JNTU (K) May 2010 (Set No. 2)]**

Solution : (*i*) The density function $f(x)$ is given by $f(x)=\frac{d}{dx}[F(x)]$

$$\therefore f(x)=\begin{cases}\frac{1}{2}e^{-2x}, x\ge 0\\ 0, x<0\end{cases}$$

(ii) Mean $= E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{0} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx$

i.e. $\mu = 0 + \int_0^{\infty} x.\frac{1}{2}e^{-2x}dx = \frac{1}{2}\int_0^{\infty} x\, e^{-2x}dx$

$$= \frac{1}{2}\left[x\left(\frac{e^{-2x}}{-2}\right) - 1.\left(\frac{e^{-2x}}{4}\right)\right]_0^{\infty} = -\frac{1}{8}\left[e^{-2x}(2x+1)\right]_0^{\infty}$$

$$= -\frac{1}{8}[0-(0+1)] = \frac{1}{8}$$

(iii) Variance $= \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \int_{-\infty}^{0} x^2 f(x)\, dx + \int_0^{\infty} x^2 f(x)\, dx - \mu^2$

$$= 0 + \int_0^{\infty} x^2.\frac{1}{2}e^{-2x}dx - \left(\frac{1}{8}\right)^2 = \frac{1}{2}\int_0^{\infty} x^2 e^{-2x}dx - \frac{1}{64}$$

$$= \frac{1}{2}\left[x^2\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right)\right]_0^{\infty} - \frac{1}{64}$$

$$= -\frac{1}{2}\left[e^{-2x}\left(\frac{1}{2}x^2+\frac{1}{2}x+\frac{1}{4}\right)\right]_0^{\infty} - \frac{1}{64} = -\frac{1}{8}\left[e^{-2x}(2x^2+2x+1)\right]_0^{\infty} - \frac{1}{64}$$

$$= -\frac{1}{8}[0-(0+0+1)] - \frac{1}{64} = \frac{1}{8} - \frac{1}{64} = \frac{-9}{64}$$

Example 26 : If X is a continuous random variable with p .d .f $f(x)=\begin{cases}x^2, 0\le x\le 1,\\ 0, \text{elsewhere}\end{cases}$

If $p(a\le x\le 1)=\frac{19}{81}$, find the value of '$a$'. **[JNTU (K) May 2010 (Set No. 3)]**

Solution : Given $p(a\le x\le 1)=\frac{19}{81} \Rightarrow \int_a^1 f(x)\, dx = \frac{19}{81} \Rightarrow \int_a^1 x^2 dx = \frac{19}{81}$

$$\Rightarrow \left(\frac{x^3}{3}\right)_a^1 = \frac{19}{81} \Rightarrow \frac{1-a^3}{3} = \frac{19}{81} \Rightarrow 1-a^3 = \frac{19}{27}.$$

$$\Rightarrow a^3 = 1+\frac{19}{27} = \frac{46}{27}. \quad \therefore a\left(\frac{46}{27}\right)^{1/3} = 1.19$$

Example 27 : If a random variable has the probability density function

$$f(x)=\begin{cases}k(x^2-1), -1\le x\le 3\\ 0, \text{elsewhere}\end{cases}$$

Find the value of 'k' and $p\left(\frac{1}{2}\le x\le\frac{5}{2}\right)$. **[JNTU (K) May 2010 (Set No. 4)]**

Solution : In order that $f(x)$ should be a probability density function

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

i.e., $\int_{-\infty}^{-1} f(x)\, dx + \int_{-1}^{3} f(x)\, dx + \int_{3}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-1}^{3} k(x^2-1)\, dx = 1 \Rightarrow k\left(\frac{x^3}{3}-x\right)_{-1}^{3} = 1$$

$$\Rightarrow k\left[(9+1)-\left(-\frac{1}{3}+1\right)\right] = 1 \Rightarrow k\left(9+\frac{1}{3}\right) = 1$$

$$\Rightarrow \frac{28}{3}k = 1 \quad \therefore k = \frac{3}{28}$$

Now $p\left(\frac{1}{2}\le x\le\frac{5}{2}\right) = \int_{1/2}^{5/2} \frac{3}{28}(x^2-1)\, dx$

$$= \frac{3}{28}\left(\frac{x^3}{3}-x\right)_{1/2}^{5/2} = \frac{3}{28}\left[\left(\frac{(5/2)^3}{3}-\frac{5}{2}\right)-\left(\frac{(1/2)^3}{3}-\frac{1}{2}\right)\right]$$

$$= \frac{3}{28}\left[\frac{65}{24}+\frac{11}{24}\right] = \frac{3}{28}\times\frac{76}{24} = \frac{19}{56}$$

Example 28 : If X is the continuous random variable whose density function is

$$f(x)=\begin{cases}x & \text{if } 0<x<1\\ 2-x & \text{if } 1\le x<2\\ 0, & \text{elsewhere}\end{cases}$$

Find $E(25X^2+30X-5)$. **[JNTU (H) Nov. 2010 (Set No. 2)]**

Solution : Given $f(x)=\begin{cases}x & \text{if } 0<x<1\\ 2-x & \text{if } 1\le x<2\\ 0, & \text{elsewhere}\end{cases}$

$$\therefore E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^1 x.x\, dx + \int_1^2 x(2-x)\, dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2)\, dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(2.\frac{x^2}{2} - \frac{x^3}{3}\right)_1^2 = \frac{1}{3} + \left(x^2 - \frac{x^3}{3}\right)_1^2$$

$$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 4 + \frac{1}{3} - \frac{8}{3} - \frac{2}{3} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^1 x^2.x\, dx + \int_1^2 x^2(x-2)\, dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (x^3 - 2x^2)dx = \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{x^4}{4} - 2.\frac{x^3}{3}\right)_1^2$$

$$= \frac{1}{4} + \left(4 - \frac{16}{3}\right) - \left(\frac{1}{4} - \frac{2}{3}\right) = -\frac{2}{3}$$

$$\therefore E(25X^2 + 30X - 5) = 25.E(X^2) + 30.E(X) - E(5)$$

$$= 25\left(-\frac{2}{3}\right) + 30(1) - 5 \quad [\because E(k) = k]$$

$$= -\frac{50}{3} + 25 = \frac{-50+75}{3} = \frac{25}{3}$$

Example 29 : Find the value of K and the distribution function $F(x)$ given the probabi density function of a random variable X as :

$$f(x) = \frac{K}{x^2+1}, -\infty < x < \infty$$

[JNTU(K) March 2014 (Set No.

Solution : Since total probability is unity, we have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{K}{x^2+1} dx = 1 \Rightarrow K \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 1 \Rightarrow K(\tan^{-1} x)_{-\infty}^{\infty} = 1$$

$$\Rightarrow K(\tan^{-1}\infty - \tan^{-1}(-\infty)) = 1 \Rightarrow K\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1 \Rightarrow K\pi = 1.$$

$$\therefore K = \frac{1}{\pi}.$$

By definition, the distribution function is given by

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{K}{1+x^2} dx = \frac{1}{\pi} \int_{-\infty}^{x} \frac{1}{1+x^2} dx \quad \left[\because K = \frac{1}{\pi}\right]$$

$$= \frac{1}{\pi}(\tan^{-1} x)_{-\infty}^{x} = \frac{1}{\pi}[\tan^{-1} x - \tan^{-1}(-\infty)] = \frac{1}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right).$$

Example 30 : The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the function

$$f(x) = \begin{cases} A\, e^{-x/5}, \text{ for } x \ge 0 \\ 0, \text{ otherwise} \end{cases}$$

(*i*) Find the value of A that makes $f(x)$ a probability density function.

(*ii*) What is the probability that the number of minutes that she will take over the phone is more than 10 minutes? **[JNTU(K) Dec. 2013 (Set No. 2)]**

Solution : (*i*) In order that $f(x)$ should be a probability density function,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \quad i.e., \int_0^{\infty} f(x)\, dx = 1 \quad \text{or} \quad \int_0^{\infty} A\, e^{-x/5}\, dx = 1 \Rightarrow A\left(\frac{e^{-x/5}}{-1/5}\right)_0^{\infty} = 1$$

$$\Rightarrow -5A(e^{-\infty} - e^{-0}) = 1 \Rightarrow -5A(0-1) = 1 \Rightarrow 5A = 1 \text{ or } A = \frac{1}{5}.$$

(*ii*) $$P(X > 10) = \int_{10}^{\infty} f(x)\, dx = \int_{10}^{\infty} A\, e^{-x/5}\, dx = A\left(\frac{e^{-x/5}}{-1/5}\right)_{10}^{\infty}$$

$$= -5A(e^{-\infty} - e^{-2}) = -5\left(\frac{1}{5}\right)(0 - e^{-2}) = \frac{1}{e^2}.$$

REVIEW QUESTIONS

1. Define expectation for discrete and continuous random variables.

EXERCISE 3(B)

1. The probability density function $f(x)$ of a continuous random variable is given by

$$f(x) = \begin{cases} k\, x^3, \text{ for } 0 < x < 1 \\ 0, \text{ elsewhere} \end{cases}$$

Find the value of k and the probability that the random variable takes on a value

(*i*) between $\frac{1}{4}$ and $\frac{3}{4}$ (*ii*) greater than $\frac{2}{3}$.