UNIT-I: Probability & Random Variables

Experiment: An experiment is a physical action or Process i.e observed and the result is noted.

Eg: Tossing a coin, Throwing a die

* Experiments are of two types:

Deterministic Experiment → An experiment is called deterministic experiment if the result can be predicted with certainity prior to the performance.

Eg: Throwing a stone will always touches the ground

Undeterministic Experiment → Its defined as if the result can't be predicted prior to the performance with certainity, but all the outcomes can be predicted before in-hand.

Eg: Tossing a coin The possible outcomes are Head or Tail but we can't predict the exact result.

Trial: Every single performance of an experiment is called a trial.

Outcome or Event: Result of an experiment is called an outcome or event.

* Events are of two types:

Elementary Events → The elementary event is an event which can't be broken further.

Eg: Getting a number 5 on die

Compound Events → The compound event is an event which can be broken further into smaller events.

Eg: Getting an odd number when throwing a die

Sample Space(S): The set of all possible outcomes of a random experiment is called a Sample Space. Each point in a Sample space is called a Sample Point

Eg: Tossing a coin
Sample Space $(S) = \{H, T\}$

Finite Sample Space → If the number of sample Points in a sample space are finite, then the sample Space is called finite Sample space

Infinite Sample Space → If the number of sample Points in a sample space are infinite, then the sample space is called infinite sample space.

Equally Likely Events: A set of events are said to be equally likely, if no one of them is expected to occur in preference to the other in any single trial of an experiment.

Eg: When tossing a coin getting head or tail are equally likely events.

Mutually Exclusive Events: Events which rejects the happening of other events.

Favorable Events: The events which are favorable to a particular event of an experiment are known as Favorable Events.

Eg: Getting 2,4,6 are favorable to the event of getting an even number.

Probability: If an experiment is performed, "n" is the no. of total cases and "m" is the no. of favorable cases of an event 'A'. Then, the probability of an event 'A' is defined by

$$P(A) = \frac{\text{No. of favorable cases}}{\text{Total no. of cases}}$$

$$\boxed{P(A) = \frac{m}{n}}$$

① Find the probability of getting a head in tossing a coin.

Sol: No. of ways $= 2^n$ $\quad n \rightarrow$ no. of coins tossed

Events $= 2$

Total events $(n) = 2 = \{H, T\}$

Probability of getting head $(m) = 1$

$P(A) = \frac{1}{2}$

② Find the probability of getting 1 head in tossing 2 coins.

Sol: No. of events $= 2^n$

$= 2^2$

$n = 4$

$m = \{HH, HT, TH, TT\}$

$m = 2$

$P(A) = \frac{m}{n} = \frac{2}{4}$

$P(A) = \frac{1}{2}$

③ If 3 coins are tossed. Find the probability of getting i) Three heads

ii) No head

iii) Two heads

Sol: $n = 2^3$

$n = 8$ events

which are $\{HHH, HTH, THH, HHT, TTT, TTH, THT, HTT\}$

i) $P(A) = \frac{1}{8}$

ii) $P(A) = \frac{1}{8}$

iii) $P(A) = \frac{3}{8}$

④ Find the probability of getting a sum 9 if 2 dice are thrown.

Sol: Events $= 6^n$

$= 6^2$

$n = 36$

$$\text{Sample Space } (S) = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

m = Sum of getting 9

$= \{(4,5)\ (5,4)\ (3,6)\ (6,3)\}$

$m = 4$

$P(A) = \frac{4}{36} = \frac{1}{9}$

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⑤ Find the probability of getting 1 Red King if we select a card from a pack of 52 cards.

Sol: 52 cards

Diamonds (13) | Hearts (13) | Spades (13) | Clubs (13)

Red (26) — (K, Q, J, A, 1, 2 ..... 10)

Black (26) — (K, Q, J, A, 1, 2 .... 10)

Total no. of cards (n) = 52

m = Probability i.e Favorable cases of getting 1 red king

= 2

$P(A) = \frac{m}{n} = \frac{2}{52}$

$P(A) = \frac{1}{26}$

⑥ Find the probability of getting 2 diamonds if we draw 2 cards at random from a pack of 52 cards.

Sol: Total no. of cards = 52

2 cards are picked from 52 cards in $n = {}^{52}C_2$ ways

2 diamonds are picked from 13 cards in $m = {}^{13}C_2$ ways

$$P(A) = \frac{^{13}C_2}{^{52}C_2} = \frac{\frac{13\times12}{2}}{\frac{52\times51}{2}}$$

$$= \frac{\cancel{13}^{1}\times\cancel{12}^{3}}{\cancel{52}_{4}\times\cancel{51}_{17}}$$

$$P(A) = \frac{1}{17}$$

⑦ 3 cards are drawn from a pack of 52 cards. Find the probability that

i) 3 are spades

ii) 2 spades one diamond

iii) 1 spade, 1 diamond, 1 heart

<u>Sol:</u> 3 cards are picked from 52 cards in $^{52}C_3$ ways

i) Getting 3 spades

$m = {}^{13}C_3$

$$P(A) = \frac{^{13}C_3}{^{52}C_3} = \frac{\frac{13\times12\times11}{\cancel{3\times2\times1}}}{\frac{52\times51\times50}{\cancel{3\times2\times1}}}$$

$$= \frac{\cancel{13}^{1}\times\cancel{12}^{3}\times11}{\cancel{52}_{4}\times\cancel{51}_{17}\times50}$$

$$P(A) = \frac{11}{850}$$

ii) 2 spades and 1 diamond

$m = {}^{13}C_2 \cdot {}^{13}C_1$

$$P(A) = \frac{{}^{13}C_2 \times {}^{13}C_1}{{}^{52}C_3} = \frac{\frac{13\times12}{2} \times 13}{\frac{52\times51\times50}{3\times2\times1}}$$

$$= \frac{\cancel{13}\times\cancel{12}\times 13}{\cancel{2}} \times \frac{3\times\cancel{2}}{\cancel{52}\times\cancel{51}\times50}$$

$$= \frac{39}{850}$$

iii) 1 Spade, 1 diamond, 1 heart

$m = {}^{13}C_1 \cdot {}^{13}C_1 \cdot {}^{13}C_1$

$$P(A) = \frac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_3} = \frac{\cancel{13}\times13\times13\times\cancel{3}\times\cancel{2}\times1}{\cancel{52}\times\cancel{51}\times50}$$

$$= \frac{169}{1700}$$

(8) What is the probability of drawing ace from a well shuffled pack of 52 playing cards.

Sol:

$n = 52$ cards

$m = 4$ (ace cards)

$$P(A) = \frac{m}{n} = \frac{\cancel{4}}{\cancel{52}}$$

$$P(A) = \frac{1}{13}$$

9) A bag contains 5 red balls, 8 blue balls, 11 white balls. 3 balls are drawn from a box. Find the probability that

i) 1 Red, 1 Blue and 1 white

ii) 2 whites and 1 Red

iii) 3 white

Sol: 3 balls are picked from 24 balls in $^{24}C_3$ ways

$n = {}^{24}C_3$

i) 1 Red, 1 blue & 1 white

$m = {}^5C_1 \times {}^8C_1 \times {}^{11}C_1$

$$P(A) = \frac{{}^5C_1 \times {}^8C_1 \times {}^{11}C_1}{{}^{24}C_3} = \frac{5 \times 8 \times 11}{\frac{24 \times 23 \times 22}{3 \times 2 \times 1}}$$

$$= \frac{5 \times \cancel{8} \times \cancel{11} \times \cancel{3} \times \cancel{2}}{\cancel{24} \times 23 \times \cancel{22}}$$

$$P(A) = \frac{5}{23}$$

ii) 2 whites & 1 Red

$m = {}^{11}C_2 \times {}^5C_1$

$$P(A) = \frac{{}^{11}C_2 \times {}^5C_1}{{}^{24}C_3} = \frac{\frac{11 \times \cancel{10}}{\cancel{2}} \times 5}{\frac{24 \times 23 \times 22}{3 \times 2 \times 1}} = \frac{25 \times \cancel{11} \times \cancel{6}}{\cancel{24} \times 23 \times \cancel{22}}$$

$$= \frac{25}{184}$$

$\boxed{iii}$ 3 white

$$P(A) = \frac{^{11}C_3}{^{24}C_3} = \frac{\frac{11\times10\times9}{3\times2\times1}}{\frac{24\times23\times22}{3\times2\times1}}$$

$$= \frac{\cancel{11}^{1}\times\cancel{10}^{5}\times\cancel{9}^{3}}{\cancel{24}_{8}\times23\times\cancel{22}_{2}}$$

$$= \frac{15}{184}$$

$\boxed{\text{SETS}}$

Set: A collection of well-defined objects is called a set. The objects containing the set, are called elements.

We use capital letters to represent sets and small letters to represent elements.

Sub-Set: ($\subseteq$) Suppose A is a set, B is a set such that $\forall$ every element of B belonging to set A then we say B is a sub-set of A and we write as $B \subseteq A$.

Union ($\cup$): Let A & B be two sets. Union of A & B is a set of all those elements which are belonging to either A or B or both.

We represent it as $A \cup B$

Intersection ($\cap$): Let A and B be two sets. The intersection of A and B is a set of all those elements which are common to A and B.

We represent it as $A \cap B$

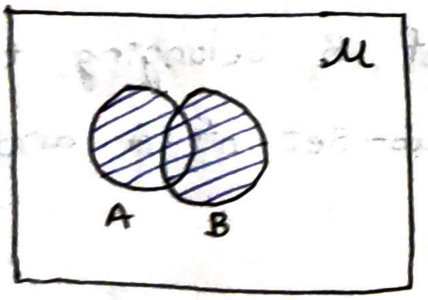
Destroying Sets:

Disjoint Sets: If $A \cap B = \phi$. That means A and B donot have any element in common. In this case we say that A and B are disjoint.

Complement of A (A^c): The set of elements which don't belong to A.

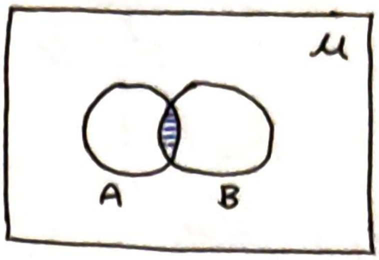
Venn-Diagrams: Rectangle represent universal set. Circles represent sets.

Eg: $A \cup B$



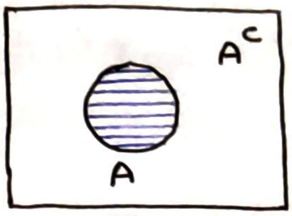
Venn Diagram - $A \cup B$

$A \cap B$



Venn Diagram – $A \cap B$

A^c



Venn Diagram – A^c

Formulae:

* $A \cup A = A$
* $A \cap A = A$
* $A \cup B = B \cup A$
* $(A \cup B) \cup C = A \cup (B \cup C)$
* $A \cap B = B \cap A$
* $A \cup \phi = A$
* $A \cap \phi = \phi$
* $A \cup U = U$
* $A \cap U = A$
* $(A^c)^c = A$
* $A \cap A^c = \phi$

De-Morgan Laws:

* $(A\cap B)^c = A^c \cup B^c$

* $(A\cup B)^c = A^c \cap B^c$

Distributive Laws:

* $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$
* $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Axioms of Probability:

The axioms of Probability are

1) $0 \le P(A) \le 1$

2) * $P(S) = 1$

3) * If A and B are any 2 mutually exclusive events then $\boxed{P(A\cup B) = P(A)+P(B)}$

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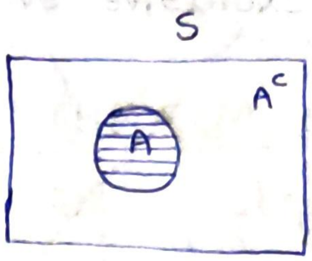
Some elementary theorems:

Theorem-1:

Statement → Probability of complementary event is

$P(A^c) = 1 - P(A)$

Proof →
* Let A and A^c be any two events



From the figure, we have

$$S = A \cup A^c$$

and also from the figure, A and A^c are mutually exclusive events.

Then By Axiom (3)

$$P(S) = P(A \cup A^c)$$

$$P(S) = P(A) + P(A^c)$$

From axiom (2), $P(S) = 1$

$$1 = P(A) + P(A^c)$$

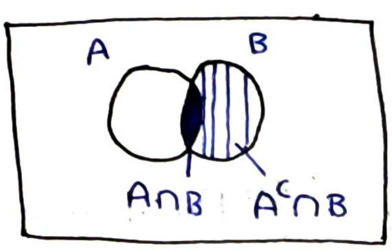
$$\boxed{P(A^c) = 1 - P(A)}$$

Theorem-2:

Statement → For any events A and B

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

Proof → Let A and B be any two events



From the figure, we have

$A \cap B$ & $A^c \cap B$ are mutually exclusive events

Also from figure we have

$$B = (A\cap B) \cup (A^c \cap B)$$

$$P(B) = P[(A\cap B) \cup (A^c\cap B)]$$

$$P(B) = P(A\cap B) + P(A^c \cap B) \quad \text{(From axiom-3)}$$

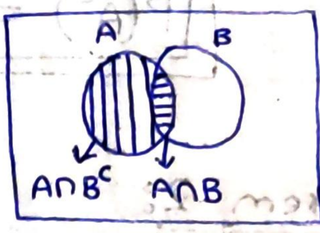
$$\boxed{P(A^c\cap B) = P(B) - P(A\cap B)}$$

Theorem-3:

Statement → If A and B are any two events then

$$P(A\cap B^c) = P(A) - P(A\cap B)$$

Proof → Let A and B be any two events



From the figure, we have

$A\cap B$ and $A\cap B^c$ are mutually exclusive events

Also from figure, we have

$$A = (A\cap B^c) \cup (A\cap B)$$

$$P(A) = P[(A\cap B^c)\cup(A\cap B)]$$

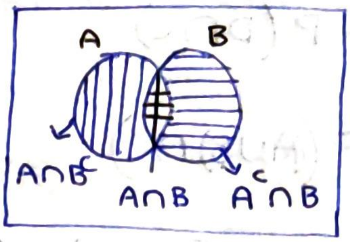
$$P(A) = P(A\cap B^c) + P(A\cap B) \quad (\because \text{From axiom-3})$$

$$\boxed{P(A\cap B^c) = P(A) - P(A\cap B)}$$

ADDITION THEOREM OF PROBABILITY: For any two events A and B

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

Proof → Let A and B be any two events



From the figure we have,

$A\cap B^c, A\cap B, A^c\cap B$ are mutually exclusive events and

$$A\cup B = (A\cap B^c)\cup(A\cap B)\cup(A^c\cap B)$$

$$P(A\cup B) = P\left[(A\cap B^c)\cup(A\cap B)\cup(A^c\cap B)\right]$$

$$P(A\cup B) = P(A\cap B^c) + P(A\cap B) + P(A^c\cap B) \quad (\because \text{By axiom } 3)$$

$$P(A\cup B) = P(A) - P(\cancel{A\cap B}) + P(\cancel{A\cap B}) + P(B) - P(A\cap B)$$

($\because$ From theorem ② & ③)

$$\boxed{P(A\cup B) = P(A) + P(B) - P(A\cap B)}$$

MULTIPLICATION THEOREM OF PROBABILITY: For any three events A, B and C

$$\boxed{P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)}$$

<u>Proof</u> → Let A, B and C are three events

Consider

$$P(\underbrace{A\cup B}_{D}\cup C) = P(D\cup C)$$

$$= P(D)+P(C)-P(D\cap C)$$

$$= P(A\cup B)+P(C)-P((A\cup B)\cap C)$$

$$= P(A)+P(B)+P(C)-P(A\cap B)-P[(A\cap C)\cup(B\cap C)]$$

$$\boxed{\because (A\cup B)\cap C = (A\cap C)\cup(B\cap C)}$$

$$= P(A)+P(B)+P(C)-P(A\cap B)-[P(A\cap C)+P(B\cap C)-P((A\cap C)\cap(B\cap C))]$$

($\because$ From Addition Theorem)

$$= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$$

$$\boxed{\therefore P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)}$$

Formulaes:

* $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$

* $P(A^c \cup B^c) = P(A \cap B)^c = 1 - P(A \cap B)$

* $P(B^c) = 1 - P(B)$

* $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

* $P(A/B^c) = \dfrac{P(A \cap B^c)}{P(B^c)}$

① If $P(A) = \frac{1}{5}$, $P(B) = \frac{2}{3}$ $P(A \cap B) = \frac{1}{15}$ then

find

i) $P(A \cup B)$

ii) $P(A^c \cap B)$

iii) $P(A \cap B^c)$

iv) $P(A^c \cap B^c)$

v) $P(A^c \cup B^c)$

Sol: Given $P(A) = \frac{1}{5}$

$P(B) = \frac{2}{3}$

$P(A \cap B) = \frac{1}{15}$

i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{5} + \frac{2}{3} - \frac{1}{15}$

$= \frac{13}{15} - \frac{1}{15}$

$= \frac{\cancel{12}\,4}{\cancel{15}\,5}$

$= \frac{4}{5}$

ii) $P(A^c \cap B) = P(B) - P(A \cap B)$

$= \frac{2}{3} - \frac{1}{15}$

$= \frac{\cancel{27}\,\cancel{9}\,3}{\cancel{3}_1 \times \cancel{15}\,5}$

$= \frac{3}{5}$

iii) $P(A \cap B^c) = P(A) - P(A \cap B)$

$= \frac{1}{5} - \frac{1}{15}$

$= \frac{\cancel{10}\,2}{\cancel{5}_1 \times 15}$

$= \frac{2}{15}$

iv) $P(A^c \cap B^c) = P(A \cup B)^c$

$= 1 - P(A \cup B)$

$= 1 - \frac{4}{5}$

$= \frac{1}{5}$

v) $P(A^c \cup B^c) = P(A \cap B)^c$

$= 1 - P(A \cap B)$

$= 1 - \frac{1}{15}$

$= \frac{14}{15}$

② If $P(A\cup B)=\frac{4}{5}$, $P(B^c)=\frac{1}{3}$, $P(A\cap B)=\frac{1}{5}$ find

i) $P(B)$

ii) $P(A)$

iii) $P(A^c\cap B)$

Sol: Given $P(A\cup B)=\frac{4}{5}$

$P(B^c)=\frac{1}{3}$

$P(A\cap B)=\frac{1}{5}$

i) $P(B) = 1-P(B^c)$

$= 1-\frac{1}{3}$

$P(B)=\frac{2}{3}$

ii) We know $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\frac{4}{5} = P(A)+\frac{2}{3}-\frac{1}{5}$

$1 = P(A)+\frac{2}{3}$

$P(A) = 1-\frac{2}{3}$

$P(A)=\frac{1}{3}$

iii) $P(A^c\cap B) = P(B)-P(A\cap B)$

$= \frac{2}{3}-\frac{1}{5}$

$= \frac{7}{15}$

14/09/21

③ Two dice are thrown. Let A be the ~~sum of~~ event that sum of points on the faces is 9. Let B be the event that atleast one number is 6. Find the Probabilities of the following events.

i) $A \cap B$ ii) $A \cup B$ iii) $A^c \cap B$ iv) $A \cap B^c$ v) $A^c \cap B^c$

vi) $A^c \cup B^c$

Sol: Given two dice are thrown i.e $n = 36$

Sample Space (S) = $\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$

Let A be the event of getting sum as 9

$m = \{(3,6)\ (6,3)\ (4,5)\ (5,4)\}$

$m = 4$

$P(A) = \frac{m}{n} = \frac{4}{36} = \frac{1}{9}$

Let B be the event that atleast one number is 6

$m = \{(1,6)\ (2,6)\ (3,6)\ (4,6)\ (5,6)\ (6,6)\ (6,5)\ (6,4)\ (6,3)\ (6,2)$
$(6,1)\}$

$m = 11$

$P(B) = \frac{m}{n} = \frac{11}{36}$

$\boxed{i}$ $P(A \cap B) = \{(3,6)\ (6,3)\}$

$= 2$

$P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

$\boxed{ii}$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{4}{36} + \frac{11}{36} - \frac{2}{36}$

$= \frac{13}{36}$

$\boxed{iii}$ $P(A^c \cap B) = P(B) - P(A \cap B)$

$= \frac{11}{36} - \frac{1}{18}$

$= \frac{9}{36}$

$= \frac{1}{4}$

$\boxed{iv}$ $P(A \cap B^c) = P(A) - P(A \cap B)$

$= \frac{1}{9} - \frac{1}{18}$

$= \frac{1}{18}$

$\boxed{v}$ $P(A^c \cap B^c) = P(A \cup B)^c$

$= 1 - P(A \cup B)$

$= 1 - \frac{13}{36}$

$= \frac{23}{36}$

$\boxed{vi}$ $P(A^c \cup B^c) = P(A \cap B)^c$

$= 1 - P(A \cap B)$

$= 1 - \frac{1}{18} = \frac{17}{18}$

(4) If $P(A)=a$, $P(B)=b$, $P(A\cap B)=c$ Express the following probabilities in terms of a,b,c

i) $P(A^c \cup B^c)$ ii) $P(A\cap B^c)$ iii) $P(A^c \cap B)$ iv) $P(A^c \cap B^c)$

v) $P(A^c \cup B)$ vi) $P((A\cap B)^c)$ vii) $P((A\cup B)^c)$ viii) $P(A^c \cap (A\cup B$

ix) $P(A\cup(A^c\cap B))$

Sol: $P(A)=a$ $P(B)=b$ $P(A\cup B)=?$ $P(A\cap B)=c$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$
$$= a+b-c$$

i) $P(A^c \cup B^c) = P(A\cap B)^c$
$$= 1 - P(A\cap B)$$
$$= 1-c$$

ii) $P(A\cap B^c) = P(A) - P(A\cap B)$
$$= a-c$$

iii) $P(A^c\cap B) = P(B) - P(A\cap B)$
$$= b-c$$

iv) $P(A^c\cap B^c) = P(A\cup B)^c$
$$= 1 - P(A\cup B)$$
$$= 1 - [a+b-c]$$
$$= c-a-b+1$$

v) $P(A^c \cup B) = P(A^c) + P(B) - P(A^c \cap B)$

$= 1 - P(A) + P(B) - [b - c]$

$= 1 - a + \cancel{b} - \cancel{b} + c$

$= 1 - a + c$

vi) $P((A \cap B)^c) = 1 - P(A \cap B)$

$= 1 - c$

vii) $P((A \cup B)^c) = 1 - P(A \cup B)$

$= 1 - (a + b - c)$

$= c - a - b + 1$

viii) $P(A^c \cap (A \cup B)) = \cancel{P(A \cap B)}$

$P[(A^c \cap A) \cup (A^c \cap B)]$

$= P(A^c \cap A) + P(A^c \cap B) - P(A^c \cap A \cap B)$

$= 0 + P(A^c \cap B) - 0$

$= b - c$

ix) $P(A \cup (A^c \cap B)) = \cancel{P[(A \cup A^c) \cap (A \cup B)]}$

$= P(A) + P(A^c \cap B) - P[A \cap A^c \cap B]$

$= a + b - c$

⑤ Among 150 students 80 are studying Maths, 40 are studying Physics and 30 are studying Maths & Physics. If a student is chosen at random, find the probability that the student

a) Studying Maths or Physics

b) Studying Neither maths nor physics

Sol: Let A be the event of student studying Maths

Let B be the event of student studying Physics

$P(A) = \frac{80}{150}$ $\quad P(B) = \frac{40}{150}$ $\quad P(A \cap B) = \frac{30}{150}$

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{80}{150} + \frac{40}{150} - \frac{30}{150}$

$= \frac{\cancel{90}\,3}{\cancel{150}\,5}$

$= \frac{3}{5}$

b) $P(A^c \cap B^c) = P(A \cup B)^c$

$= 1 - P(A \cup B)$

$= 1 - \frac{3}{5}$

$= \frac{2}{5}$

⑥ Two bolts are drawn from a box containing 4 good and 6 bad bolts. Find the probability that the second bolt is good if the first one is found to be bad.

Sol: Total no. of bolts $= 4+6 = 10$

Probability that 2^{nd} bolt is good if the first bolt is bad is

$$= \frac{6_{C_1}}{10_{C_1}} \cdot \frac{4_{C_1}}{9_{C_1}}$$

$$= \frac{\cancel{6}^{2} \times 4^{1}}{\cancel{10}_{5} \times \cancel{9}_{3}}$$

$$= \frac{4}{15}$$

⑦ A class has 10 boys and 5 girls. 3 students are selected at random one after the other. Find the probability that

a) First two are boys and third one is girl

b) First and third are of same sex and 2^{nd} is of opposite sex

Sol: Total number of students = 15

a) Probability that first 2 are boys and third is girl $= \frac{10_{C_1}}{15_{C_1}} \cdot \frac{9_{C_1}}{14_{C_1}} \cdot \frac{5_{C_1}}{13_{C_1}}$

$$= \frac{\cancel{10}^{5} \times \cancel{9}^{3} \times \cancel{5}^{1}}{\cancel{15}_{3} \times \cancel{14}_{7} \times 13}$$

$\cancel{3}$ 1

$= \frac{15}{91}$

b) Probability that first & third are of same sex and second is of opposite sex

1st possibility

B G B

$$\text{Probability} = \frac{{}^{10}C_1}{{}^{15}C_1} \cdot \frac{{}^{5}C_1}{{}^{14}C_1} \cdot \frac{{}^{9}C_1}{{}^{13}C_1}$$

$$= \frac{10 \times 9 \times 5}{15 \times 14 \times 13}$$

$$= \frac{15}{91}$$

2nd possibility

G B G

$$\text{Probability} = \frac{{}^{5}C_1}{{}^{15}C_1} \cdot \frac{{}^{10}C_1}{{}^{14}C_1} \cdot \frac{{}^{4}C_1}{{}^{13}C_1}$$

$$= \frac{\overset{1}{\cancel{5}} \times 10 \times \overset{2}{\cancel{4}}}{\underset{3}{\cancel{15}} \times \underset{7}{\cancel{14}} \times 13}$$

$$= \frac{20}{13 \times 7 \times 3}$$

$$\text{Required Probability} = \frac{15}{13 \times 7} + \frac{20}{13 \times 7 \times 3}$$

$$= \frac{5}{13 \times 7}\left[\frac{3}{1} + \frac{4}{3}\right]$$

$$= \frac{5}{91}\left[\frac{13}{3}\right]$$

$$= \frac{\overset{5}{\cancel{65}}}{\underset{21}{\cancel{273}}}$$

$$= \frac{5}{21}$$

5/09/21

Conditional Probability:

Independent Events → Two events are said to be independent if the happening of an event is not affected by the happening of the other event.

In other words, happening of an event doesnot depend upon the happening of the other.

Eg: If we draw a card from a pack of 52 cards and replace it before we draw a second card. The second draw is independent of the first one.

Conditional Probability: If B is any arbitrary event in a Sample space and $P(B)>0$. The Probability of A is given and B is defined then

$$\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}}$$

Similarly,

$$\boxed{P(B/A) = \frac{P(A \cap B)}{P(A)}}$$

NOTE: 1) From conditional probability, we have

$$P(A \cap B) = P(B)\, P(A/B)$$

$$P(A \cap B) = P(A)\, P(B/A)$$

General Multiplication Rule: If A and B are any events in S then

$$P(A \cap B) = P(A)\, P(B/A)$$

NOTE: If A and B are independent events, then

$$\boxed{P(A \cap B) = P(A) . P(B)}$$

***Theorem - 1:** If A and B are independent events then A^c and B^c are also independent events.

Proof: Given A and B are independent events

$$P(A \cap B) = P(A) . P(B) \quad (1)$$

Now we have to prove

$$P(A^c \cap B^c) = P(A^c) . P(B^c)$$

Consider $P(A^c \cap B^c) = P(A \cup B)^c$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - P(A) - P(B) + P(A)P(B)$$

$$= 1[1 - P(A)] - P(B)(1 - P(A))$$

$$= (1 - P(A))(1 - P(B))$$

$\therefore P(A^c \cap B^c) = P(A^c).P(B^c)$

* **Theorem-2:** If A and B are independent events then A and B^c are independent.

Proof: Given A and B are independent events

We have to prove that

$$P(A \cap B^c) = P(A).P(B^c)$$

Consider $P(A \cap B^c) = P(A) - P(A \cap B)$

$$= P(A) - P(A)P(B)$$

$$= P(A)[1 - P(B)]$$

$$\therefore P(A \cap B^c) = P(A)P(B^c)$$

* **Theorem-3:** If A, B, C are mutually independent events then $A \cup B$ and C are also independent.

Proof: Given A, B, C are mutually independent events

We have to prove that,

$$P(A \cup B \cap C) = P(A \cup B).P(C)$$

Consider $P((A\cup B)\cap C) = P((A\cap C)\cup(B\cap C))$

$= P(A\cap C) + P(B\cap C) - P((A\cap C)\cap(B\cap C))$ (Addition Theorem)

$= P(A).P(C) + P(B).P(C) - P(A\cap B\cap C)$

$= P(A).P(C) + P(B).P(C) - P(A)P(B)P(C)$

$= P(C)\left[P(A) + P(B) - P(A)P(B)\right]$

$= P(C)\left[P(A) + P(B) - P(A\cap B)\right]$

$\therefore P((A\cup B)\cap C) = P(C)\,P(A\cup B)$

1) The probabilities that students A, B, C, D solve a Problem are $\frac{1}{3}, \frac{2}{5}, \frac{1}{5}$ and $\frac{1}{4}$ respectively. If all of them try to solve the problem. What is the probability that the problem is solved

<u>Sol:</u> Let $P(A)$ = Probability that A solved the problem

$P(A) = \frac{1}{3}$ $\quad P(A^c) = \frac{2}{3}$

Similarly,

$P(B) = \frac{2}{5}$ $\quad P(B^c) = \frac{3}{5}$

$P(C) = \frac{1}{5}$ $\quad P(C^c) = \frac{4}{5}$

$P(D) = \frac{1}{4}$ $\quad P(D^c) = \frac{3}{4}$

$$P(A\cup B\cup C\cup D) = 1-[P(A\cup B\cup C\cup D)]^c$$

$$= 1- P(A^c\cap B^c\cap C^c\cap D^c)$$

$$= 1 - P(A^c)\,P(B^c)\,P(C^c)\,P(D^c)$$

$$= 1-\left(\frac{2}{\cancel{3}}\right)\left(\frac{\cancel{3}}{5}\right)\left(\frac{\cancel{4}}{5}\right)\left(\frac{3}{\cancel{4}}\right)$$

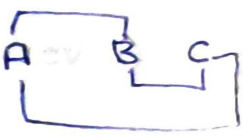
$$= 1-\frac{6}{25}$$

$$= \frac{19}{25}$$

2) A can hit a target once in 5 shots. B can hit 2 targets in 3 shots. C can hit one target in 4 shots. What is the probability that two shots hit the target

Sol: Let $P(A)$ be the probability that A hit the target is

$$P(A)=\frac{1}{5} \qquad P(A^c)=\frac{4}{5}$$



Similarly

$$P(B)=\frac{2}{3} \qquad P(C)=\frac{1}{4}$$

$$P(B^c)=\frac{1}{3} \qquad P(C^c)=\frac{3}{4}$$

Required Probability

$$= P[(A\cap B\cap C^c)\cup(A\cap B^c\cap C)\cup(A^c\cap B\cap C)]$$

$$= P(A\cap B\cap C^c)+P(A\cap B^c\cap C)+P(A^c\cap B\cap C)$$

$= P(A).P(B).P(C^c) + P(A)P(B^c)P(C) + P(A^c)P(B)P(C)$

$= \left(\frac{1}{5}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) + \frac{1}{5}\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)$

$= \frac{6+1+8}{5\times3\times4}$

$= \frac{\cancel{15}}{\cancel{15}\times4}$

$= \frac{1}{4}$

3) The probabilities of passing in subjects A, B, C, D are $\frac{3}{4}, \frac{2}{3}, \frac{4}{5}$ and $\frac{1}{2}$. To qualify in the examination a student should pass in A and two subjects among the three. What is the probability of qualifying in the exam.

Sol: Given $P(A) = \frac{3}{4}$ $P(A^c) = \frac{1}{4}$

$P(B) = \frac{2}{3}$ $P(B^c) = \frac{1}{3}$

$P(C) = \frac{4}{5}$ $P(C^c) = \frac{1}{5}$

$P(D) = \frac{1}{2}$ $P(D^c) = \frac{1}{2}$

There are four possibilities to qualify the exam

1) To pass in A, B, C and fail in D — 1) $A \; B \; C \; D^c$

2) To pass in A, C, D and fail in B — 2) $A \; C \; D \; B^c$

3) To pass in A, B, D and fail in C — 3) $A \; B \; D \; C^c$

4) To pass in A, B, C, D — 4) $A \; B \; C \; D$

Required Probability $= P\left[(A\cap B\cap C\cap D^c)\cup(A\cap C\cap D\cap B^c)\cup(A\cap B\cap D\cap C^c)\cup(A\cap B\cap C\cap D)\right]$

$= P(A\cap B\cap C\cap D^c) + P(A\cap C\cap D\cap B^c) + P(A\cap B\cap D\cap C^c) + P(A\cap B\cap C\cap D)$

$= P(A)P(B)P(C)P(D^c) + P(A)P(C)P(D)P(B^c) + P(A)P(B)P(D)P(C^c) + P(A)P(B)P(C)P(D)$

$= \frac{3}{4}\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{2}\right) + \frac{3}{4}\left(\frac{4}{5}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{5}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{2}\right)$

$= \frac{24+12+6+24}{120}$

$= \frac{\cancel{66}^{11}}{\cancel{120}_{20}}$

$= \frac{11}{20}$

4) Determine

a) $P(B/A)$ b) $P(A/B^c)$ if A and B are events with

$P(A) = \frac{1}{3}$ $P(B) = \frac{1}{4}$ $P(A\cup B) = \frac{1}{2}$

Sol: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{4} - P(A \cap B)$$

$$P(A \cap B) = \frac{7}{12} - \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{12}$$

a) $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

$$= \frac{\frac{1}{12}}{\frac{1}{3}}$$

$$= \frac{1}{12} \times \frac{3}{1}$$

$$= \frac{1}{4}$$

b) $P(A/B^c) = \dfrac{P(A \cap B^c)}{P(B^c)}$

$$= \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{\frac{1}{3} - \frac{1}{12}}{1 - \frac{1}{4}}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{3}$$

$$= \frac{1}{3}$$

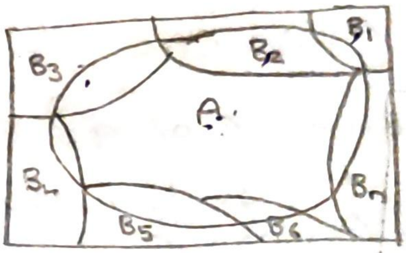
***

Baye's Theorem: Suppose $B_1, B_2, \ldots B_n$ are mutually exclusive events of a sample space S such that $P(B_i) > 0$ $i = 1, 2, 3 \ldots n$ and A is any arbitrary event of S such that $P(A) > 0$ and $A \subseteq \bigcup_{i=1}^{n} B_i$ $i = 1, 2 \ldots n$ then the conditional probability of B_i given $P(A)$ for $i = 1, 2 \ldots n$ is equal to

$$\boxed{P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}}$$

Proof: Suppose $B_1, B_2, B_3 \dots B_n$ are the set of exhaustive and mutually exclusive events.

Let A is any arbitrary event of S



We Know That, $P(A \cap B_i) = P(B_i)\,P\left(\frac{A}{B_i}\right) = P(A)\,P(B_i/A)$ —— (1) —— (2)

By definition of Conditional Probability, we have

$$P(B_i/A) = \frac{P(A \cap B_i)}{P(A)} \quad \text{—— ②}$$

Sub ① in ②

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{P(A)} \quad \text{—— ③}$$

Given $A \subseteq \bigcup_{i=1}^{n} B_i$

From the figure we have

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)$$

$$P(A) = P[(A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)]$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)$$

$$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + \cdots + P(B_n \cap A)$$

$$P(A) = P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + \cdots + P(B_n)\,P(A/B_n)$$

(By Definition of Conditional Probability)

$$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A/B_i) \quad \text{--- (4)}$$

Sub $P(A)$ in (3)

$$\boxed{P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}}$$

~~Sub (1)~~

20/09/21

1) In a bolt factory machines A, B, C manufacture 20%, 30%, 50% of the total of their output. 6%, 3% and 2% are defective. A bolt is drawn at random and found to be defective. What is the probability that is manufactured by machines A, B and C.

Denominator

Sol: Let A be the probability that the bolt was manufactured by machine A

$$P(A) = 20\% = \frac{20}{100} = 0.2$$

$$P(B) = 30\% = \frac{30}{100} = 0.3$$

$$P(C) = 50\% = \frac{50}{100} = 0.5$$

Let D be the probability of the defective bolt manufactured by machine A, B & C

$P(D/A) = 6\%$

$P(D/B) = 3\%$

$P(D/C) = 2\%$

By Baye's Theorem,

$$P(A/D) = \frac{P(A)\,P(D/A)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{0.2\,(0.06)}{0.2(0.06) + (0.3)(0.03) + (0.5)(0.02)}$$

$$= \frac{0.012}{0.012 + 0.009 + 0.01}$$

$$= \frac{0,012}{0,031}$$

$$= \frac{12}{31}$$

$$P(B/D) = \frac{P(B)\,P(D/B)}{P(A)P(D/A) + P(B)P(D/B) + P(C)P(D/C)}$$

$$= \frac{(0.3)\,(0.03)}{(0.2)(0.06) + (0.3)(0.03) + (0.5)(0.02)}$$

$$= \frac{0,009}{0,031}$$

$$= \frac{9}{31}$$

$$P(C/D) = \frac{P(C)\,P(D/C)}{P(A)\,P(D/A) + P(B)\,P(D/B) + P(C)\,P(D/C)}$$

$$= \frac{0.5\,(0.02)}{(0.2)(0.06) + (0.3)(0.03) + (0.5)(0.02)}$$

$$= \frac{0.01 \times 10^{3}}{0.031 \times 10^{3}}$$

$$= \frac{10}{31}$$

2) There are three boxes (I, II, III). Box I contains 4 red, 5 blue & 6 white balls. Box II contains 3 red, 4 blue & 5 white balls. Box III contains 5 red, 10 blue & 5 white balls. One box is chosen and one ball is drawn from it. What is the probability that

a) The red ball is drawn from Box I

b) The blue ball is drawn from Box II

c) The white ball is drawn from Box III

Sol: Let B_1, B_2, B_3 are the three boxes i.e

$P(B_1) = \frac{1}{3}$

$P(B_2) = \frac{1}{3}$

$P(B_3) = \frac{1}{3}$

$P(R/B_1) = \frac{4}{15}$ $\quad$ $P(B/B_1) = \frac{5}{15}$ $\quad$ $P(W/B_1) = \frac{6}{15}$

$P(R/B_2) = \frac{3}{12}$ $\quad$ $P(B/B_2) = \frac{4}{12}$ $\quad$ $P(W/B_2) = \frac{5}{12}$

$P(R/B_3) = \frac{5}{20}$ $\quad$ $P(B/B_3) = \frac{10}{20}$ $\quad$ $P(W/B_3) = \frac{5}{20}$

By Baye's Theorem

a) Probability of getting a red ball from Box-I

$$P(B_1/R) = \frac{P(B_1)\,P(R/B_1)}{P(B_1)\,P(R/B_1) + P(B_2)P(R/B_2) + P(B_3)\,P(R/B_3)}$$

$$= \frac{\cancel{\frac{1}{3}}\left(\frac{4}{15}\right)}{\cancel{\frac{1}{3}}\left[\frac{4}{15} + \frac{\cancel{3}^{1}}{\cancel{12}_{4}} + \frac{\cancel{5}^{1}}{\cancel{20}_{4}}\right]}$$

$$= \frac{\frac{4}{15}}{\frac{4}{15} + \frac{1}{2}}$$

$$= \frac{\frac{4}{15}}{\frac{23}{30}}$$

$$= \frac{4}{\cancel{15}_{1}} \times \frac{\cancel{30}^{2}}{23}$$

$$= \frac{8}{23}$$

b) Probability of getting a blue ball from Box-II

$$P(B_2/B) = \frac{P(B_2)\,P(B/B_2)}{P(B_1)P(B/B_1) + P(B_2)P(B/B_2) + P(B_3)P(B/B_3)}$$

$$= \frac{\cancel{\frac{1}{3}}\left(\frac{1}{3}\right)}{\cancel{\frac{1}{3}}\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{2}\right]}$$

$$= \frac{\frac{1}{3}}{\frac{2}{3}+\frac{1}{2}}$$

$$= \frac{\frac{1}{3}}{\frac{7}{6}}$$

$$= \frac{1}{\cancel{3}_{1}} \times \frac{\cancel{6}^{2}}{7}$$

$$= \frac{2}{7}$$

c) Probability of getting white ball from Box – III

$$P(B_3/W) = \frac{P(B_3)P(W/B_3)}{P(B_1)P(W/B_1)+P(B_2)P(W/B_2)+P(B_3)P(W/B_3)}$$

$$= \frac{\cancel{\frac{1}{3}}\left(\frac{1}{4}\right)}{\cancel{\frac{1}{3}}\left[\frac{2}{5}+\frac{5}{12}+\frac{1}{4}\right]}$$

$$= \frac{\frac{1}{4}}{\frac{2}{5}+\frac{\cancel{8}^{2}}{\cancel{12}_{3}}}$$

$$= \frac{\frac{1}{4}}{\frac{16}{15}}$$

$$= \frac{1}{4} \times \frac{15}{16}$$

$$= \frac{15}{64}$$

3) In a class, 2% of boys and 3% of girls are having blue eyes. There are 30% girls in the class. If a student is selected and having blue eyes, What is the probability that the student is a girl.

Sol: Let G be the probability of girls i.e

$$P(G) = 30\% = \frac{30}{100} = 0.3$$

Let B be the probability of boys

$$P(B) = 70\% = \frac{70}{100} = 0.7$$

$$P(B.E/G) = 3\% = \frac{3}{100} = 0.03$$

$$P(B.E/B) = 2\% = \frac{2}{100} = 0.02$$

By Baye's Theorem

$$P(G/B.E) = \frac{P(G)\,P(B.E/G)}{P(G)\,P(B.E/G) + P(B)\,P(B.E/B)}$$

$$= \frac{0.3(0.03)}{0.3(0.03) + (0.7)(0.02)}$$

$$= \frac{\frac{9}{1000}}{\frac{9+14}{1000}}$$

$$= \frac{9}{23}$$

4) In a certain college, 25% of boys & 10% of girls are studying Mathematics. The girls constitute 60% of the students. If a student is selected & is found to be studying Mathematics. Find the probability that the student

a) is a girl b) is a boy

Sol: $P(B) = 40\% = \frac{40}{100} = 0.4$

$P(G) = 60\% = \frac{60}{100} = 0.6$

$P(M/B) = \frac{25}{100} = 0.25$

$P(M/G) = \frac{10}{100} = 0.1$

By Baye's Theorem

b) $P(B/M) = \frac{P(B)P(M/B)}{P(B)P(M/B) + P(G)P(M/G)}$

$= \frac{0.4(0.25)}{0.4(0.25) + (0.6)(0.1)}$

$= \frac{0.1}{0.1+0.06} = \frac{0.1 \times 100}{0.16 \times 10} = \frac{10}{16}$

$= \frac{5}{8}$

a) $P(G/M) = \dfrac{P(G)\,P(M/G)}{P(B)P(M/B) + P(G)P(M/G)}$

$= \dfrac{0.06}{0.16}$

$= \dfrac{6}{16}$

$= \dfrac{3}{8}$

21/09/21

5) A businessman goes to hotels X, Y, Z 20%, 50% and 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels have faulty plumbing. What is the probability that the businessman room having faulty plumbing is assigned to hotel Z.

Sol: X, Y, Z are the 3 given hotels

$P(X) = 20\% = \frac{20}{100} = 0.2$

$P(Y) = 50\% = \frac{50}{100} = 0.5$

$P(Z) = 30\% = \frac{30}{100} = 0.3$

$P(F.P/X) = 5\% = 0.05$

$P(F.P/Y) = 4\% = 0.04$

$P(F.P/Z) = 8\% = 0.08$

By Baye's Theorem,

$$P(Z/F.P) = \frac{P(Z)\,P(F.P/Z)}{P(X)P(F.P/X) + P(Y)P(F.P/Y) + P(Z)P(F.P/Z)}$$

$$= \frac{(0.3)(0.08)}{(0.2)(0.05)+(0.5)(0.04)+(0.3)(0.08)}$$

$$= \frac{\frac{24}{\cancel{1000}}}{\frac{54}{\cancel{1000}}}$$

$$= \frac{\cancel{24}\,12\,4}{\cancel{54}\,27\,9}$$

$$= \frac{4}{9}$$

6) A Box 1 contains 11 cards numbered 1 to 11, Box 2 contains 7 cards numbered 1 to 7. A box is selected at random and a card is drawn. If the no. is even, find the probability that the card is from Box 1.

Sol: Let B_1, B_2 are two boxes

$P(B_1) = \frac{1}{2}$

$P(B_2) = \frac{1}{2}$

Let E_1 be the event of getting even number from box 1 i.e $P(E_1/B_1) = \frac{5}{11}$

Let E_2 be the event of getting even number from box 2 i.e $P(E_2/B_2) = \frac{3}{7}$

By Baye's Theorem,

$$P\left(B_1/E_1\right) = \frac{P(B_1)P(E_1/B_1)}{P(B_1)P(E_1/B_1) + P(B_2)P(E_2/B_2)}$$

$$= \frac{\cancel{\frac{1}{2}}\left[\frac{5}{11}\right]}{\cancel{\frac{1}{2}}\left(\frac{5}{11} + \frac{3}{7}\right)}$$

$$= \frac{\frac{5}{11}}{\frac{68}{77}}$$

$$= \frac{5}{\cancel{11}_{1}} \times \frac{\cancel{77}^{7}}{68}$$

$$= \frac{35}{68}$$