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Chapter-2 RANDOM VARIABLES

Random Variables: A real variable X whose value is determined by the outcome of a random experiment is called a random variable.

* A random variable X can also be regarded as a real value function defined on the sample space S of a random experiment such that for each point x of the sample space, $f(x)$ is the probability of occurence of the event represented by x.

Eg: When 2 coins are tossed,

Let E be the event of getting a tail

The sample space $S = \{HH, HT, TH, TT\}$

The probability of getting no tail $= 0$

The probability of getting one tail $= 1$

The probability of getting two tails $= 2$

$\therefore$ The random variable $X = \{0, 1, 2\}$

* Random variables are of two types:

1) Discrete
2) Continuous

Discrete Random Variable: A random Variable X which can take only a finite number of discrete Values in an interval of a domain is called a Discrete Random Variable.

Eg: Tossing a coin, Throwing a die etc

Continuous Random Variable: A random Variable x which can take Values continuously i.e which takes all possible Values in a given interval is called Continuous Random Variable.

Eg: The height, weight and age of an individual; Temperature and Time etc

Probability Distribution Function: Let X be a random Variable. Then the Probability Distribution Function associated with X is defined as the probability that the outcome of an experiment will be one of the outcomes. The function F(x) is defined by

$$\boxed{F_X(x) = P(X \le x)}$$

Discrete Probability Distribution: Probability Distribution of a random Variable is the set of its Possible values together with their respective Probabilities, i.e

1) $P(x_i) \geq 0$ 2) $\sum_{i=1}^{n} P_i = 1$

* If the function P satisfies above 2 conditions then the function p is called Discrete Probability Distribution.

Probability Density Function: The Probability Density Function is defined as the derivative of the Probability Distribution Function of the random variable X i.e

$$f_x(x) = \frac{d}{dx}\left[F_x(i)\right]$$

Expectation (or) Mean (or) Expected Value of x: Suppose a random Variable X assumes the values $x_1, x_2 \ldots x_n$ w.r.to the probabilities $P_1, P_2 \ldots P_n$. Then the mean (or) expectation is denoted by E(X) or μ and is defined as The sum of products of different Values of x and corresponding probabilities i.e

$$\boxed{E(X) \text{ or } \mu = \sum_{i=1}^{n} x_i P_i}$$

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SAQ

Theorem: If X is a random Variable & K is a constant then $\boxed{E(X+K) = E(X) + K}$

Proof:
Given X is a random Variable

By definition of expectation, we have $E(x) = \sum_{i=1}^{n} x_i P_i$

Consider $E(X+K) = \sum_{i=1}^{n} (x_i + k) P_i$

$$= \sum_{i=1}^{n} x_i P_i + \sum_{i=1}^{n} K P_i$$

$$= \sum_{i=1}^{n} x_i P_i + K \sum_{i=1}^{n} P_i$$

$$= E(X) + K(1) \qquad \left(\because \sum_{i=1}^{n} P_i = 1\right)$$

$$E(X+K) = E(X) + K$$

Theorem: If X is a random variable then

$$\boxed{E(KX) = K.E(X)}$$

Proof:
Given X is a random Variable

By definition of expectation, we have

$$E(X) = \sum_{i=1}^{n} x_i P_i$$

Consider $E(KX) = \sum_{i=1}^{n} K x_i P_i$

$$= K \sum_{i=1}^{n} x_i P_i$$

$$E(KX) = K.E(X)$$

<u>Theorem</u>: If X is a random Variable and a, b are constants then $\boxed{E(aX+b) = a(E(X)) + b}$

<u>Proof</u>:

Given X is the random Variable and a, b are constants

By definition of expectation, $E(x) = \sum_{i=1}^{n} x_i P_i$

Consider $E(aX+b) = \sum_{i=1}^{n} (ax_i+b) P_i$

$$= a \sum_{i=1}^{n} x_i P_i + b \sum_{i=1}^{n} P_i$$

$$= a\, E(x) + b(1)$$

$$E(ax+b) = a\,E(X) + b$$

<u>Theorem</u>: If X and Y are two discrete random Variable then $\boxed{E(X+Y) = E(X) + E(Y)}$

Proof: Given X, Y are discrete random Variables

Consider

$$E(X+Y) = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i + y_j)P_{ij}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i P_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} y_j P_{ij}$$

$$= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} P_i \cdot P_j + \sum_{j=1}^{n} y_j \sum_{i=1}^{n} P_i P_j$$

$$= \sum_{i=1}^{n} x_i P_i \sum_{j=1}^{n} P_j + \sum_{j=1}^{n} y_j P_j \sum_{i=1}^{n} P_i$$

$$= E(X)(1) + E(Y)(1)$$

$$E(X+Y) = E(X) + E(Y)$$

Variance: The Variance of a discrete random Variable is denoted by "σ^2" and is defined as

$$\boxed{\sigma^2 = E[X - E(X)]^2}$$

Standard Deviation (S.D): The Standard Deviation is the square root of the variance.

$$\boxed{S.D = \sqrt{\sigma^2}}$$

Theorem: The variance (σ^2) of the discrete random variable is

$$\boxed{\sigma^2 = E(x^2) - [E(x)]^2}$$

Proof: By definition of variance,

$$\sigma^2 = E[x - E(x)]^2$$

$$= E[x - \mu]^2$$

$$= E(x^2 - 2x\mu + \mu^2)$$

$$= E(x^2) - 2\mu E(x) + \mu^2$$

$$= E(x^2) - 2\mu(\mu) + \mu^2$$

$$= E(x^2) - 2\mu^2 + \mu^2$$

$$= E(x^2) - \mu^2$$

$$= E(x^2) - [E(x)]^2$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

Theorem: If **x** is a discrete random variable then

$$\boxed{V(ax+b) = a^2 V(x)}$$ where $V(x)$ is Variance of **x** and a, b are constants

Proof: Given X is a random Variable

Let $Y = aX + b$ —①

$E(Y) = E(ax+b)$

$E(Y) = a(E(x)) + b$ —②

① − ② $\Rightarrow Y - E(Y) = a[X - E(X)]$

Squaring on Both Sides and taking expectation

$E[Y-E(Y)]^2 = a^2 E[X - E(X)]^2$

$Var(Y) = a^2 \, Var(X)$ (∵ By definition of Variance)

$V(Y) = a^2 V(X)$

$\therefore V(aX+b) = a^2 V(X)$

NOTE: While solving the problems, we use the following formulae

1) Mean (μ) (or) $E(X) = \sum_{i=1}^{n} x_i P_i$

2) Variance $\sigma^2 = \sum_{i=1}^{n} x_i^2 P_i - \mu^2$

3) Standard Deviation $= \sqrt{\text{Variance}}$

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1) A random variable X has the following probability distribution

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Determine a) K

b) $P(x<6)$, $P(x\geq 6)$, $P(0<x<5)$

c) Mean

d) Variance

e) If $P(x\leq K) > \frac{1}{2}$ find minimum value of K

f) Determine the distribution function of X

Sol:

a) To find K

WKT, Sum of Probabilities = 1

$$\sum_{i=1}^{n} P_i = 1$$

$$\sum_{x=0}^{7} P_i = 1 \Rightarrow 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$\Rightarrow 10K^2 + 9K = 1$$

$$\Rightarrow 10K^2 + 9K - 1 = 0$$

$$K = \frac{-9 \pm \sqrt{81+40}}{2(10)}$$

$$= \frac{-9 \pm 11}{20}$$

$$= -1, \frac{1}{10}$$

$$K = \frac{1}{10}$$

b) $P(x<6) = P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4)+P(x=5)$

$$= 0 + K + 2K + 2K + 3K + K^2$$

$$= 8K + K^2$$

$$= 8\left(\frac{1}{10}\right) + \frac{1}{100}$$

$$= \frac{81}{100}$$

$$= 0.81$$

$P(x\geq 6) = P(x=6) + P(x=7)$

$$= 2K^2 + 7K^2 + K$$

$$= 9\left(\frac{1}{100}\right) + \frac{1\times10}{10\times10}$$

$$= \frac{19}{100}$$

$$= 0.19$$

$P(0<x<5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$$= K + 2K + 2K + 3K$$

$$= 8K$$

$$= 8\left(\frac{1}{10}\right)$$

$$= 0.8$$

c) Mean $= \sum_{i=1}^{n} x_i P_i$

$= 0 + K + 4K + 6K + 12K + 5K^2 + 12K^2 + 49K^2 + 7K$

$= 66K^2 + 30K$

$= 66\left(\frac{1}{100}\right) + 30\left(\frac{1}{10}\right)$

$= \frac{366}{100}$

$\mu = 3.66$

d) Variance $= \sum_{i=1}^{n} x_i^2 P_i - \mu^2$

$= \left[0 + K + 8K + 18K + 48K + 25K^2 + 72K^2 + 343K^2 + 49K\right] - (3.66)^2$

$= 440K^2 + 124K - (13.3956)$

$= 44\cancel{0}\left(\frac{1}{10\cancel{0}}\right) + 124\left(\frac{1}{10}\right) - 13.3956$

$= 16.8 - 13.3956$

$= 3.4044$

e) $P(X \leq 1) = P(X=0) + P(X=1) = 0 + K = \frac{1}{10} = 0.1 < \frac{1}{2}$

$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = K + 2K = \frac{3}{10} = 0.3 < \frac{1}{2}$

$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = K + 2K + 2K = \frac{5}{10} = 0.5 < \frac{1}{2}$

$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) = 8K = \frac{8}{10} = 0.8 < \frac{1}{2}$ (False)

$0.8 > \frac{1}{2}$

$\therefore$ Minimum Value of $K = 4$

f) The distribution function of X is given by the following table

X	$P(x) = P(X \le x)$
0	0
1	$K = \frac{1}{10}$
2	$K + 2K = \frac{3}{10}$
3	$5K = \frac{5}{10}$
4	$8K = \frac{8}{10}$
5	$K^2 + 8K = \frac{1}{100} + \frac{8}{10}$ $= \frac{81}{100}$
6	$3K^2 + 8K = \frac{3}{100} + \frac{8}{10}$ $= \frac{83}{100}$
7	$10K^2 + 9K = 1\cancel{0}\left(\frac{1}{10\cancel{0}}\right) + \frac{9}{10}$ $= \frac{10}{10}$ $= 1$

2) For the discrete probability distribution

x	0	1	2	3	4	5	6
P	0	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

a) Find K

b) Find Mean

c) Find Variance

Sol:

a) $\sum_{i=1}^{n} P_i = 1$

$\Rightarrow 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$10K^2 + 8K - 1 = 0$

$K = \dfrac{-8 \pm \sqrt{64 - 4(10)(-1)}}{2(10)}$

$= \dfrac{-8 \pm \sqrt{104}}{20}$

$= \dfrac{-8 \pm 10.19803903}{20}$

$K = 0.1099$

b) Mean $\mu = \sum_{i=1}^{n} x_i P_i$

$= 0 + 2K + 4K + 9K + 4K^2 + 10K^2 + 42K^2 + 6K$

$= 56K^2 + 21K$

$= 56(0.1099)^2 + 21(0.1099)$

$= 0.6763 + 2.3079$

$= 2.9842$

c) Variance $= \sum_{i=1}^{n} x_i^2 P_i - \mu^2$

$= 0 + 2K + 8K + 27K + 16K^2 + 50K^2 + 252K^2 + 36K - (2.9842)^2$

$= 318K^2 + 73K - 8.9054$

$= 318(0.1099)^2 + 73(0.1099) - 8.9054$

$= 3.8408 + 8.0227 - 8.9054$

$\sigma^2 = 2.9581$

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3) Let X denote the no. of heads in a single toss of 4 ~~points~~ coins. Determine

a) $P(X<2)$ b) $P(1<X\leq 3)$

Sol: Given 4 coins are tossed

Total no. of cases $= 2^4 = 16$

HHHH	HHHT	HHTT	HTTT	TTTT
	THHH	TTHH	TTTH	
	HTHH	HTHT	TTHT	
	HHTH	THTH	THTT	
		HTTH		
		THHT		

Let E be the event of getting heads

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

a) $P(X<2)$

$= P(X=0) + P(X=1)$

$= \frac{1}{16} + \frac{4}{16}$

$= \frac{5}{16}$

b) $P(1<X\leq 3)$

$= P(X=2)+P(X=3)$

$= \frac{6}{16} + \frac{4}{16}$

$= \frac{10}{16} = \frac{5}{8}$

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4) Two dice are thrown. Let X assign to each point (a,b) in S. The maximum of its numbers i.e $X(a,b) = \max(a,b)$. Find the probability distribution, the mean and variance of the distribution.

Sol: Two dice are thrown

Total no. of cases $= 6^2 = 36$

Let E be the event of getting $X(a,b) = \max(a,b)$

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

For maximum '1' favourable cases are $(1,1) = \frac{1}{36}$

For maximum '2' favourable cases are $(1,2)(2,1)(2,2) = \frac{3}{36}$

For maximum '3' favourable cases are $(1,3)\ (2,3)\ (3,3)$
$(3,1)\ (3,2) = \frac{5}{36}$

The probability distribution function

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

a) Mean $(\mu) = \sum_{i=1}^{6} x_i P_i$

$$= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} = \frac{161}{36}$$

$$= 4.472$$

b) Variance $(\sigma^2) = \sum_{i=1}^{6} x_i^2 P_i - \mu^2$

$$= \frac{1}{36} + \frac{12}{36} + \frac{45}{36} + \frac{112}{36} + \frac{225}{36} + \frac{396}{36} - (4.472)^2$$

$$= \frac{791}{36} - (4.472)^2$$

$$= 21.97 - 19.99$$

$$= 1.98$$

5) Let X denote to each point (a,b) in S. The min of its numbers i.e X (a,b) = min (a,b). Find the Probability distribution, mean & variance of the distribution

Sol: For minimum '1' favourable cases are

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) $= \frac{11}{36}$
(2,1) (3,1) (4,1) (5,1) (6,1)

For minimum '2' favourable cases are $= \frac{9}{36}$

x_i	1	2	3	4	5	6
$P(x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

i) Mean $\mu = \sum_{i=1}^{6} P_i x_i$

$$= \frac{11}{36} + \frac{18}{36} + \frac{21}{36} + \frac{20}{36} + \frac{15}{36} + \frac{6}{36}$$

$$= \frac{91}{36}$$

$$= 2.527$$

ii) Variance $\sigma^2 = \sum_{i=1}^{6} P_i x_i^2 - \mu^2$

$$= \frac{11}{36} + \frac{36}{36} + \frac{63}{36} + \frac{80}{36} + \frac{75}{36} + \frac{36}{36} - (2.527)^2$$

$$= 8.3611 - 6.3857$$

$$= 1.975$$

6) Given that $f(x) = \frac{K}{2x}$ is a probability distribution for a random Variable X that can take the Values $x = 0, 1, 2, 3, 4$

a) Find K

b) Mean & Variance of x

Sol: Given $f(x) = \frac{K}{2x}$ $x = 0,1,2,3,4$

x	1	2	3	4
$f(x)$	$\frac{K}{2}$	$\frac{K}{4}$	$\frac{K}{6}$	$\frac{K}{8}$

a) $\sum_{i=1}^{n} P_i = 1$

$\Rightarrow \frac{K}{2} + \frac{K}{4} + \frac{K}{6} + \frac{K}{8} = 1$

$\Rightarrow \frac{3K^{\times 6}}{4^{\times 6}} + \frac{\cancel{14}^{7}K}{\cancel{48}_{24}} = 1$

$\Rightarrow 25K = 24 \Rightarrow K = \frac{24}{25} = 0.96$

b) Mean $\mu = \sum_{i=1}^{4} x_i P_i$

$= \frac{K}{2} + \frac{2K}{4} + \frac{3K}{6} + \frac{4K}{8}$

$= \frac{K}{2} + \frac{K}{2} + \frac{K}{2} + \frac{K}{2}$

$= 2K$

$= 2\left(\frac{24}{25}\right)$

$\mu = 1.92$

Variance $\sigma^2 = \sum_{i=1}^{4} x_i^2 P_i - \mu^2$

$= \frac{K}{2} + \frac{\cancel{4}K}{\cancel{4}} + \frac{\cancel{9}^{3}K}{\cancel{6}_{2}} + \frac{16K}{8} - (1.92)^2$

$= 3K + 2K - (1.92)^2$

$= 5(0.96) - (1.92)^2$

$= 4.8 - 3.6864$

$\sigma^2 = 1.1136$

7) A random Variable X is defined as the sum of the numbers on the faces when two dice are thrown. Find the mean.

Sol: Total no. of cases $= 6^2 = 36$

x_i	2	3	4	5	6	7	8	9	10	11	12
$P(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\text{Mean}(\mu) = \sum_{i=2}^{12} P_i x_i$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= \frac{2+6+12+20+30+42+40+36+30+22+12}{36}$$

$$= \frac{\cancel{252}^{\cancel{42}\ 7}}{\cancel{36}\ \cancel{6}}$$

$\mu = 7$

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8) ~~Given that $f(x)$~~ Find the mean and Variance of the uniform probability distribution given by

$f(x) = \frac{1}{n}$ For $x = 1, 2, 3 \dots n$

sol: The probability distribution is

x	1	2	3	...	n
$f(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$		$\frac{1}{n}$

Mean $\mu = \sum_{i=1}^{n} x_i p_i$

$= \left(\frac{1}{n}\right)1 + 2\left(\frac{1}{n}\right) + 3\left(\frac{1}{n}\right) + \dots + n\left(\frac{1}{n}\right)$

$= \frac{1}{n}(1+2+3+\dots+n)$

$= \frac{1}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{2}\right)$

$\mu = \frac{n+1}{2}$

Variance $\sigma^2 = \sum_{i=1}^{n} x_i^2 p_i - \mu^2$

$= 1^2\left(\frac{1}{n}\right) + 2^2\left(\frac{1}{n}\right) + 3^2\left(\frac{1}{n}\right) + \dots + n^2\left(\frac{1}{n}\right) - \left(\frac{n+1}{2}\right)^2$

$= \frac{1}{n}(1^2+2^2+3^2+\dots+n^2) - \left(\frac{n+1}{2}\right)^2$

$= \frac{1}{\cancel{n}} \frac{(\cancel{n}(n+1)(2n+1))}{6} - \frac{(n+1)^2}{(2)^2}$

$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$

$= \frac{n+1}{2}\left[\frac{2(2n+1)-3n-3}{6}\right]$

$= \frac{n+1}{2}\left(\frac{4n+2-3n-3}{6}\right)$

$= \frac{n+1}{2}\left(\frac{n-1}{6}\right)$

$$\sigma^2 = \frac{n^2-1}{12}$$

9) A sample of 4 items are selected at random from a box containing 12 items of which 5 are defective. Find the expected no. of defective items.

Sol: Let X denote number of defective items,

Among 4 items drawn from 12 items

Let X can take the values 0, 1, 2, 3, 4

Given total no. of items = 12

No. of defective items = 5

No. of good items = 12 - 5 = 7.

$P(x=0) = P$ (no defective items)

$$= \frac{{}^7C_4}{{}^{12}C_4}$$

$$= \frac{7\times\cancel{6}\times\cancel{5}\times\cancel{4}}{\cancel{12}\times 11\times\cancel{10}\times 9}$$

$$= \frac{7}{99}$$

$P(x=1) = P$ (1 defective & 3 good items)

$$= \frac{{}^5C_1 \times {}^7C_3}{{}^{12}C_4}$$

$$= \frac{5\times 7\times\cancel{6}\times\cancel{5}\times\cancel{3\times2\times1}\times\cancel{4}}{\cancel{12}\times 11\times\cancel{10}\times 9\times\cancel{3\times2\times1}} = \frac{35}{99}$$

$P(x=2) = P(2 \text{ defective } \& \ 2 \text{ good items})$

$$= \frac{{}^5C_2 \times {}^7C_2}{{}^{12}C_4}$$

$$= \frac{\frac{5\times \cancel{4}^{2}}{\cancel{2}!} \times \frac{7 \times \cancel{6}^{3}}{\cancel{2}!} \times 4\times 3\times 2\times 1}{12\times 11\times 10\times 9}$$

$$= \frac{\cancel{10}\times \cancel{21}^{7}\times \cancel{12}\times 2}{\cancel{12}\times 11\times \cancel{10}\times \cancel{9}\,3}$$

$$= \frac{14}{33}$$

$P(x=3) = P(3 \text{ defective } \& \ 1 \text{ good item})$

$$= \frac{{}^5C_3 \times {}^7C_1}{{}^{12}C_4}$$

$$= \frac{\frac{\cancel{5}^{1}\times \cancel{4}^{2}}{\cancel{2}} \times 7 \times \cancel{4\times 3}\times\cancel{2}\times 1}{\cancel{12}\times 11\times \cancel{10}\times 9} \times 2!$$

$$= \frac{14}{99}$$

$P(x=4) = P(\text{all defective items})$

$$= \frac{{}^5C_4}{{}^{12}C_4}$$

$$= \frac{\cancel{5}\times \cancel{4}\times \cancel{3}^{1}\times \cancel{2}^{1}\times 1}{\cancel{12}_{4}\times 11\times \cancel{10}_{5}\times 9}$$

$$= \frac{1}{99}$$

The probability distribution table

x	0	1	2	3	4
$P(x_i)$	$\frac{7}{99}$	$\frac{35}{99}$	$\frac{14}{33}$	$\frac{14}{99}$	$\frac{1}{99}$

Mean $\mu = \sum_{i=1}^{4} x_i P_i$

$$= 0 + \frac{35}{99} + \frac{28}{33} + \frac{42}{99} + \frac{4}{99}$$

$$= \frac{81}{99} + \frac{28}{33}$$

$28 \times 3 = 84$, 81, 165

$$= \frac{81+84}{99}$$

$$= \frac{\cancel{165}\,\cancel{55}\,5}{\cancel{99}\,\cancel{33}\,3}$$

$$= \frac{5}{3} \text{ (or) } \frac{165}{99}$$

10) A player tosses 3 fair coins. He wins ₹ 500 if 3 heads appear, ₹ 300 if 2 heads appear, ₹ 100 if 1 head appear. On the other hand, he loses ₹ 1500 if 3 tails appear. Find the expected Gain of the Player.

Sol: Given 3 coins are tossed

Total no. of cases $= 2^3 = 8$

$S = \{HHH, HTT, HTH, HHT, TTT, THH, THT, TTH\}$

Let X denote the gain. The range of $X = \{-1500, 100, 300, 500\}$

The probability of getting 3 heads is

$P(\text{getting } 500 \text{ Rs}) = \frac{1}{8}$

The probability of getting 2 heads is

$P(\text{getting Rs } 300) = \frac{3}{8}$

The probability of getting 1 head is

$P(\text{getting Rs } 100) = \frac{3}{8}$

Probability of getting 3 tails is

$P(\text{losing Rs } 1500) = \frac{1}{8}$

The probability distribution table is

x_i	-1500	100	300	500
$P(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Gain = Mean $\mu = \sum_{i=1}^{n} x_i P_i$

$$= -1500\left(\frac{1}{8}\right) + 100\left(\frac{3}{8}\right) + 300\left(\frac{3}{8}\right) + 500\left(\frac{1}{8}\right)$$

$$= \frac{1}{8}\left[-1500 + 300 + 900 + 500\right]$$

$$= \frac{\cancel{200}\ \cancel{50}\ 25}{\cancel{8}\ \cancel{2}\ 1}$$

$$= 25$$

$\therefore$ The player gains Rs. 25

11) A fair die is tossed. Let the random Variable X denote twice sum of same number appearing on the die.

a) Write the probability distribution of X

b) Find Mean & Variance

Sol: Given a die is tossed

Total no. of cases $=6$

Let X denote twice the sum of same number appearing on the die

a)

The probability distribution of X

x	2	4	6	8	10	12
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b)

$$\text{Mean} = \sum_{i=1}^{n} x_i P_i$$

$$= \frac{1}{6}(2) + \frac{1}{6}(4) + \frac{1}{6}(6) + \frac{1}{6}(8) + \frac{1}{6}(10) + \frac{1}{6}(12)$$

$$= \frac{1}{6}(42)$$

$$\mu = 7$$

$$\text{Variance } \sigma^2 = \sum_{i=1}^{n} x_i^2 P_i - \mu^2$$

$$= 4\left(\frac{1}{6}\right) + 16\left(\frac{1}{6}\right) + 36\left(\frac{1}{6}\right) + 64\left(\frac{1}{6}\right) + 100\left(\frac{1}{6}\right) + 144\left(\frac{1}{6}\right) - (7)^2$$

$$= \frac{1}{6}(364) - 7^2$$

$= \frac{\cancel{364}^{182}}{\cancel{6}3} - 49$

$= \frac{182}{3} - 49$

$= 60.66 - 49$

$= 11.66$

12) A random sample with replacement of size 2 is taken from $S = \{1, 2, 3\}$. Let the random variable x denote the sum of two numbers taken. Find

a) Probability Distribution

b) Mean

c) Variance

Sol: Given, a sample of size 2 is taken from $S = [1, 2, 3]$ with replacement

i.e $S = \{(1,1)\ (1,2)\ (1,3)\ (2,1)\ (2,2)\ (2,3)\ (3,1)\ (3,2)\ (3,3)\}$

Total no. of cases $= 9$

a) Probability Distribution

x_i	2	3	4	5	6
$P(x_i)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

b) Mean $\mu = \sum_i x_i P_i$

$= 2\left(\frac{1}{9}\right) + 3\left(\frac{2}{9}\right) + 4\left(\frac{3}{9}\right) + 5\left(\frac{2}{9}\right) + 6\left(\frac{1}{9}\right)$

$= \frac{2+6+12+10+6}{9}$

$$= \frac{36}{9}$$

$$\mu = 4$$

c) Variance $\sigma^2 = \sum x_i^2 p_i - \mu^2$

$$= 4\left(\frac{1}{9}\right) + 9\left(\frac{2}{9}\right) + 16\left(\frac{3}{9}\right) + 25\left(\frac{2}{9}\right) + 36\left(\frac{1}{9}\right) - (4)^2$$

$$= \frac{4+18+48+50+36}{9} - 16$$

$$= \frac{156}{9} - 16$$

$$= \frac{\cancel{12}\,4}{\cancel{9}\,3}$$

$$= 1.33$$

<u>H.W</u>

13) From a lot of 10 items containing 3 defective, a sample of 4 items are drawn at random. Let the random variable X denote no. of defective items. Find the probability distribution, mean & variance.

14) A player tosses two fair coins. He wins Rs 100 if one head appear, Rs 200 if 2 heads appear. On the other hand he loses ₹ 500 if no head appears. Determine the expected value of the game and is the game is favorable to the player.

3)

Sol: Let X denote the no. of defective items

Given, total number of items = 10

No. of defective items = 3

No. of good items = 10 − 3 = 7

Let X take the values 0, 1, 2, 3

$P(X=0) = P(\text{no defective item})$

$$= \frac{{}^{7}C_{4}}{{}^{10}C_{4}}$$

$$= \frac{\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}}{\cancel{10}\times\cancel{9}\times\cancel{8}\times\cancel{7}}$$

$$= \frac{1}{6}$$

$P(X=1) = P(1 \text{ defective } \& \text{ } 3 \text{ good items})$

$$= \frac{{}^{3}C_{1}\times{}^{7}C_{3}}{{}^{10}C_{4}}$$

$$= \frac{\cancel{3}\times\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times\cancel{1}}{\cancel{10}\times\cancel{9}\times\cancel{8}\times\cancel{7}\times\cancel{3}\times\cancel{2}\times\cancel{1}}$$

$$= \frac{1}{2}$$

$P(X=2) = P(2 \text{ defective } \& \text{ } 2 \text{ good items})$

$$= \frac{{}^{3}C_{2}\times{}^{7}C_{2}}{{}^{10}C_{4}}$$

$$= \frac{3 \times 7 \times 6 \times 4 \times 3 \times 2 \times 1}{10 \times 9 \times 8 \times 7 \times 2}$$

$$= \frac{3}{10}$$

$P(x=3) = P$ (3 defective items)

$$= \frac{{}^{3}C_{3} \times {}^{7}C_{1}}{{}^{10}C_{4}}$$

$$= \frac{1 \times 7 \times 4 \times 3 \times 2 \times 1}{10 \times 9 \times 8 \times 7}$$

$$= \frac{1}{30}$$

$\therefore$ Probability Distribution

X	0	1	2	3
P(X)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Expected numbers of defective items $= E(x) = \sum x_i P_i$

$$\text{Mean} = 0 + \frac{1 \times 5}{2 \times 5} + 2\left(\frac{3}{10}\right) + 3\left(\frac{1}{30}\right)$$

$$= \frac{5+6+1}{10}$$

$$= \frac{12}{10}$$

$$\mu = 1.2$$

$$\text{Variance } (\sigma^2) = \sum_i x_i^2 P_i - \mu^2$$

$$= 1\left(\frac{1}{2}\right) + 4\left(\frac{3}{10}\right) + \cancel{9}^{3}\left(\frac{1}{\cancel{30}_{10}}\right) - (1.2)^2$$

$$= \frac{1\times5}{2\times5} + \frac{12}{10} + \frac{3}{10} - (1.44)$$

$$= \frac{2\cancel{0}}{1\cancel{0}} - 1.44$$

$$\sigma^2 = 0.56$$

14)

Sol: Given 2 coins are tossed

Total no. of cases $= 2^2$

$= 4$

$S = \{HH, HT, TH, TT\}$

- Let X be the expected value of the game

The probability of getting 1 head is

$P(\text{getting Rs } 100) = \frac{2}{4}$

The probability of getting 2 heads is

$P(\text{getting Rs } 200) = \frac{1}{4}$

The probability of getting no heads is

$P(\text{losing Rs } 500) = \frac{1}{4}$

Range of $X = \{-500, 100, 200\}$

X	-500	100	200
P(X)	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$\text{Mean} = \sum x_i p_i$$

$$= -500\left(\frac{1}{4}\right) + 100\left(\frac{2}{4}\right) + 200\left(\frac{1}{4}\right)$$

$$= \frac{1}{4}(-500+400)$$

$$= \frac{-100}{4}$$

$$= -25 \text{ Rs}$$

$\therefore$ The game is not favourable to the player.

29/09/21

Continuous Random Variables: A random variable X which takes values continuously i.e which takes all possible values in a given interval is called Continuous Random Variable.

Probability Density Function: Consider the interval $\left[x - \frac{dx}{2}, x + \frac{dx}{2}\right]$ of length dx around a point x. Let $f(x)$ be any continuous function of x so that $f(x)dx$ represents the probability that the variable x falls in the interval $\left[x - \frac{dx}{2}, x + \frac{dx}{2}\right]$

Symbolically, it can be expressed as

$$P\left(x-\frac{dx}{2} \leq X \leq x+\frac{dx}{2}\right) = f(x)dx$$

Then $f(x)$ is called Probability Density Function.

Properties:

1) $f(x) \geq 0 \quad \forall x \in R$

2) $\int_{-\infty}^{\infty} f(x)dx = 1$

Probability Distribution Function: The distribution function of a continuous random variable is denoted by 'x' and is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{\infty} f(x)dx$$

Mean of a Continuous Random Variable: Mean of a continuous random variable is denoted as $E(x)$ or μ and is defined as

$$\boxed{\mu = \int_{-\infty}^{\infty} x f(x)\, dx}$$

Variance: Variance is denoted by σ^2 and is defined as

$$\boxed{\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2}$$

1) If a random variable has the probability density function as

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Find the probability

a) between 1 and 3

b) greater than 0.5

Sol: Given $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

a) $P(1 \leq X \leq 3) = \int_1^3 2e^{-2x}\,dx$

$$= 2\int_1^3 e^{-2x}\,dx$$

$$= \cancel{2}\left[\frac{e^{-2x}}{\cancel{-2}}\right]_1^3$$

$$= -(e^{-6} - e^{-2})$$

$$= e^{-2} - e^{-6}$$

b) $P(X > 0.5) = \int_{0.5}^{\infty} 2e^{-2x}\,dx$

$$= 2\int_{0.5}^{\infty} e^{-2x}\,dx$$

$$= \cancel{2}\left[\frac{e^{-2x}}{\cancel{-2}}\right]_0^{\infty}$$

$$= -\left(e^{-2x}\right)_{0.5}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-2(0.5)}\right]$$

$$= e^{-1} - \frac{1}{e^{\infty}}$$

$$= e^{-1} - \frac{1}{\infty}$$

$$= e^{-1}$$

$$= \frac{1}{e}$$

NOTE

1) $e^{\infty} = \infty$
2) $e^{-\infty} = 0$
3) $e^{0} = 1$
4) $e^{-0} = 1$

2) If the probability density of a random variable is given by

$$f(x) = \begin{cases} K(1-x^2) & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

Find a) the Value of K

b) Probability between 0.1 and 0.2

c) Greater than 0.5

d) Mean

e) Variance

Sol: a) To find K:

We Know That, $\int_{-\infty}^{\infty} F(x)dx = 1$

$$\int_0^1 F(x)dx = 1 \Rightarrow \int_0^1 K(1-x^2)dx = 1$$

$$\Rightarrow K\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow K\left[\left(1-\frac{1}{3}\right) - (0-0)\right] = 1$$

$$\Rightarrow \frac{2K}{3} = 1 \quad \Rightarrow K = \frac{3}{2}$$

b) $$P(0.1 \le X \le 0.2) = \int_{0.1}^{0.2} \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$$

$$= \frac{3}{2}\left[(0.2-0.1) - \frac{1}{3}\left((0.2)^3-(0.1)^3\right)\right]$$

$$= \frac{3}{2}\left[0.1 - \frac{1}{3}\left(\frac{7}{1000}\right)\right]$$

$$= \frac{3}{2}\left[0.1\right] - \frac{\cancel{3}}{2}\left(\frac{1}{\cancel{3}}\right)\frac{7}{1000} \qquad 00\,000.3\text{-}5, \times 10^{-6}$$

$$= 0.15 - 0.0035$$

$$= 0.1465$$

c) $$P(x \ge 0.5) = \int_{0.5}^{1} \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1-\frac{1}{2}\right) - \frac{1}{3}\left(1^3-\left(\frac{1}{2}\right)^3\right)\right]$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{3}\cdot\frac{7}{8}\right]$$

$$= \frac{3}{2}\left[0.2083\right]$$

$$= 0.3125$$

d) Mean $= \mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \int_0^1 x\, f(x)\,dx$$

$$= \int_0^1 x\, \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\left(\frac{1}{2}-0\right) - \left(\frac{1}{4}-0\right)\right]$$

$$= \frac{3}{2}\left(\frac{1}{4}\right)$$

$$= \frac{3}{8}$$

$$= 0.375$$

e) Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$$= \int_0^1 x^2\, \frac{3}{2}(1-x^2)\,dx - (0.375)^2$$

$$= \frac{3}{2}\int_0^1 (x^2 - x^4)\,dx - (0.375)^2$$

$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 - (0.375)^2$$

$$= \frac{3}{2}\left(\frac{1}{3} - \frac{1}{5}\right) - (0.375)^2$$

$$= \frac{3}{2}\left(\frac{2}{15}\right) - 0.140625$$

$$= 0.059375 = 0.06$$

3) If a random variable has the probability density function

$$f(x) = \begin{cases} K(x^2-1) & -1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Find the Value of K and $P\left(\frac{1}{2} < x < \frac{5}{2}\right)$

Sol: To find K

$$\int_{-1}^{3} f(x)\,dx = 1 \quad \Rightarrow \int_{-1}^{3} K(x^2-1)\,dx = 1$$

$$\Rightarrow K\left(\frac{x^3}{3} - x\right)_{-1}^{3} = 1$$

$$\Rightarrow K\left[\frac{1}{3}\left(3^3 - (-1)^3\right) - (3-(-1))\right] = 1$$

$$\Rightarrow K\left[\frac{28}{3} - 4\right] = 1 \Rightarrow K\left[\frac{16}{3}\right] = 1$$

$$\Rightarrow K = \frac{3}{16}$$

$$P\left(\frac{1}{2} < x < \frac{5}{2}\right) = \int_{\frac{1}{2}}^{\frac{5}{2}} K(x^2-1)\,dx$$

$$= \frac{3}{16}\left[\frac{x^3}{3} - x\right]_{\frac{1}{2}}^{\frac{5}{2}}$$

$$= \frac{\cancel{3}}{16} \times \frac{1}{\cancel{3}}\left[\left(\frac{5}{2}\right)^3 - \left(\frac{1}{2}\right)^3\right] - \frac{3}{16}\left[\frac{5}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{16}\left(\frac{\cancel{124}^{\,31}}{\cancel{8}_{\,2}}\right) - \frac{3}{16}(2)$$

$= \frac{31}{32} - \frac{6}{16}$

$= \frac{19}{32}$

$= 0.59$

Even Function: A function $y = f(x)$ is an even function if $\boxed{f(-x) = f(x)}$

Eg: Let $f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2$

$= f(x)$

Odd Function: A function $y = f(x)$ is an odd function if $\boxed{f(-x) = -f(x)}$

Eg: Let $f(x) = x$

$f(-x) = -x$

$= -f(x)$

NOTE:

1) If $f(x)$ is an even function in the interval $-\infty < x < \infty$, in this case convert the integral as $2\int_0^{\infty} f(x)\,dx$

2) If $f(x)$ is an odd function in the interval $-\infty < x < \infty$ then the integral value is always zero

****

(4) The probability density function $f(x)$ of a continuous random variable is given by

$$f(x) = c e^{-|x|} \quad -\infty < x < \infty$$

Show that $c = \frac{1}{2}$ and find mean & variance of the distribution also find the probability between 0 & 4.

Sol: Given $f(x) = ce^{-|x|}$

To find c:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} ce^{-|x|}\,dx = 1$$

$$\Rightarrow c\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1$$

Here $e^{-|x|}$ is an even function

$$\Rightarrow 2c\int_{0}^{\infty} e^{-|x|}\,dx = 1$$

$$\Rightarrow c\left[2\int_{0}^{\infty} e^{-x}\,dx\right] = 1$$

$$\Rightarrow 2c\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$$

$$\Rightarrow -2c\left(e^{-\infty} - e^{-0}\right) = 1$$

$$\Rightarrow -2c(0-1) = 1$$

$\Rightarrow 2c = 1$

$\Rightarrow c = \frac{1}{2}$

Mean $\mu = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{\infty} x c e^{-|x|}\,dx$

$= \frac{1}{2} \int_{-\infty}^{\infty} x e^{-|x|}\,dx$

$= \frac{1}{2} \cancel{\times 2} \int_{-\infty}^{\infty} x e^{-|x|}\,dx$

$xe^{-|x|}$ is an odd function

$\mu = 0$

Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \int_{-\infty}^{\infty} x^2 c e^{-|x|}\,dx - (0)^2$

$= \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx$

$x^2 e^{-|x|}$ is an even function

$= \frac{1}{\cancel{2}} \times \cancel{2} \int_{0}^{\infty} x^2 e^{-|x|}\,dx$

$= \int_{0}^{\infty} x^2 e^{-x}\,dx$

$$\boxed{\begin{aligned} \int_0^\infty e^{x} f(x)\,dx &= e^{x}\left[f(x) - f'(x) + f''(x) - \cdots\right] \\ \int_0^\infty e^{-x} f(x)\,dx &= -e^{-x}\left[f(x) + f'(x) + f''(x) + \cdots\right] \end{aligned}}$$

$$= -e^{-x}\left[x^2 + 2x + 2\right]_0^\infty$$

$$\cancel{= -\left(e^{-\infty} - e^{-0}\right)\left[(\infty - \infty + 2) - (0 - 0 - 2)\right]}$$

$$\cancel{= 1(}$$

$$= \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right)_0^\infty$$

$$= (\infty - \infty - \infty) - (0 - 0 - 2)$$

$$= 2$$

$$P(0 \le x \le 4) = \int_0^4 c\, e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_0^4 e^{-x}\,dx$$

$$= \frac{1}{2}\left(\frac{e^{-x}}{-1}\right)_0^4$$

$$= -\frac{1}{2}\left(e^{-4} - e^{-0}\right)$$

$$= -\frac{1}{2}\left[\frac{1}{e^4} - 1\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{e^4}\right] = \frac{1}{2}\left[1 - \frac{1}{54.59}\right] = 0.4908$$

5) A continuous random variable has the probability density function

$$f(x) = \begin{cases} Kxe^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Find K

b) Mean

c) Variance

Sol: To find K

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{0}^{\infty} Kxe^{-\lambda x}dx = 1$$

$$\Rightarrow K\int_{0}^{\infty} xe^{-\lambda x}dx = 1$$

$$\Rightarrow K\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = 1$$

$$\Rightarrow K\left[\frac{-xe^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right]_0^{\infty} = 1$$

$$\Rightarrow K\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$$

$$\Rightarrow K = \lambda^2$$

Mean $\mu = K\int_{0}^{\infty} x\,f(x)\,dx$

$$= \lambda^2\int_{0}^{\infty} x\,x\,e^{-\lambda x}dx$$

$$= \lambda^2\int_{0}^{\infty} x^2 e^{-\lambda x}dx$$

$$= \lambda^2 \left[x^2 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 2x \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 2 \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) \right]_0^\infty$$

$$= \lambda^2 \left[-\frac{x^2 e^{-\lambda x}}{\lambda} - \frac{2x e^{-\lambda x}}{\lambda^2} - \frac{2 e^{-\lambda x}}{\lambda^3} \right]_0^\infty$$

$$= \lambda^2 \left[(\infty - \cancel{\infty 2} \infty) - \left(0 - 0 - \frac{2}{\lambda^3} \right) \right]$$

$$= \frac{2\lambda^2}{\lambda^3}$$

$$= \frac{2}{\lambda}$$

Variance $\sigma^2 = \int_0^\infty x^2 f(x) dx - \mu^2$

$$= \int_0^\infty x^2 K x e^{-\lambda x} dx - \left(\frac{2}{\lambda} \right)^2$$

$$= K \int_0^\infty x^3 e^{-\lambda x} dx - \left(\frac{2}{\lambda} \right)^2$$

$$= \lambda^2 \left[x^3 \left(\frac{e^{-\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{\lambda^2} \right) + 6x \left(\frac{e^{-\lambda x}}{-\lambda^3} \right) - 6 \left(\frac{e^{-\lambda x}}{\lambda^4} \right) \right]_0^\infty - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[0 - \left(0 - 0 + 0 - \frac{6}{\lambda^4} \right) \right] - \frac{4}{\lambda^2}$$

$$= \frac{6\lambda^2}{\lambda^4} - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$\sigma^2 = \frac{2}{\lambda^2}$$

H.W

6) For the continuous random variable X whose probability density function is given by

$$f(x) = \begin{cases} c(2-x) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a) Find c b) mean c) Variance

H.W

7) For the continuous probability function

$f(x) = kx^2 e^{-x}$ when $x \ge 0$. Find a) k b) Mean

c) Variance

**

8) If X is a continuous random variable and $Y = aX + b$

Prove that $E(Y) = aE(X) + b$

Sol:- Given X is a continuous random variable

By definition of expectation,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$E(Y) = E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x) dx$$

$$= \int_{-\infty}^{\infty} ax f(x) dx + \int_{-\infty}^{\infty} b f(x) dx$$

$$= a \int_{-\infty}^{\infty} x f(x) dx + b \int_{-\infty}^{\infty} f(x) dx$$

$$= a E(X) + b(1)$$

$$E(aX+b) = a E(X) + b$$

*****

9) If X is a continuous random Variable and K is any constant then prove that

a) $Var(X+K) = Var(X)$

b) $Var(Kx) = K^2 \, Var(X)$

Sol: PROOF - a) By definition of Variance, we have

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\int_{-\infty}^{\infty} x f(x)dx\right]^2$$

$$Var(X+K) = \int_{-\infty}^{\infty} (x+K)^2 f(x)dx - \left[\int_{-\infty}^{\infty} (x+K) f(x)dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx + \int_{-\infty}^{\infty} 2Kx f(x)dx + \int_{-\infty}^{\infty} K^2 f(x)dx - \left[\int_{-\infty}^{\infty} x f(x)dx + \int_{-\infty}^{\infty} K f(x)dx\right]$$

$$= E(x^2) + 2K E(x) + K^2(1) - \left[E(x) + K\right]^2$$

$$= E(x^2) + \cancel{2K E(x)} + \cancel{K^2} - \left[(E(x))^2 + \cancel{K^2} + \cancel{2K E(x)}\right]$$

$$= E(x^2) - \left[E(x)\right]^2$$

$\therefore \; Var(X+K) = Var(X)$

b) $Var(Kx) = \int_{-\infty}^{\infty} (Kx)^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} Kx\, f(x)\,dx\right]^2$

$= K^2 \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[K\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$

$= K^2 E(x^2) - \left[K E(x)\right]^2$

$= K^2 E(x^2) - K^2 [E(x)]^2$

$= K^2 \left[E(x^2) - [E(x)]^2\right]$

$= K^2 \, Var(x)$

⑥

Sol: a) To find $c \rightarrow$

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} c(2-x)\,dx = 1$

$\Rightarrow \int_{0}^{2} c(2-x)\,dx = 1$

$\Rightarrow c\left[2x - \frac{x^2}{2}\right]_0^2 = 1$

$\Rightarrow c\left[2(2-0) - \frac{1}{2}(2^2-0^2)\right] = 1$

$\Rightarrow c\left[4-2\right] = 1 \quad \Rightarrow 2c = 1$

$\Rightarrow c = \frac{1}{2}$

b) Mean $= \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{0}^{2} x \left(\tfrac{1}{2}\right)(2-x)\, dx$$

$$= \frac{1}{2} \int_{0}^{2} (2x - x^2)\, dx$$

$$= \frac{1}{2} \left[\cancel{2}\left(\frac{x^2}{\cancel{2}}\right) - \frac{x^3}{3} \right]_0^2$$

$$= \frac{1}{2} \left[(2^2) - \frac{1}{3}(2^3) \right]$$

$$= \frac{1}{2} \left[4 - \frac{8}{3} \right]$$

$$= \frac{1}{\cancel{2}_{1}} \times \frac{\cancel{4}^{2}}{3}$$

$$= \frac{2}{3}$$

c) Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$$= \int_{0}^{2} \left(\tfrac{1}{2}\right) x^2 (2-x)\, dx - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} \int_{0}^{2} (2x^2 - x^3)\, dx - \frac{4}{9}$$

$$= \frac{1}{2} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 - \frac{4}{9}$$

$$= \frac{1}{2} \left[\frac{2}{3}(2^3 - 0^3) - \frac{1}{4}(2^4) \right] - \frac{4}{9}$$

$$= \frac{1}{2} \left[\frac{16}{3} - 4 \right] - \frac{4}{9}$$

$$= \frac{1}{2}\left[\frac{4}{3}\right] - \frac{4}{9}$$

$$= \frac{4}{6} - \frac{4}{9}$$

$$= \frac{\cancel{12}\,2}{\cancel{6}\times 9}$$

$$= \frac{2}{9}$$

(7)

Sol: a) To find K →

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} Kx^2 e^{-x}\,dx = 1$$

$$\Rightarrow K \int_0^{\infty} x^2 e^{-x}\,dx = 1$$

$$\Rightarrow K\left[(-e^{-x})(x^2+2x+2)\right]_0^{\infty} = 1$$

$$\Rightarrow K\left[(\infty)(0) - (-1)(0+0+2)\right] = 1$$

$$\Rightarrow K(2) = 1 \quad \Rightarrow K = \frac{1}{2}$$

b) Mean $= \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_0^{\infty} x\, Kx^2 e^{-x}\,dx$$

$$= K\int_0^{\infty} x^3 e^{-x}\,dx$$

$$= \frac{1}{2}\left[(-e^{-x})(x^3+3x^2+6x+6)\right]_0^{\infty}$$

$$= \frac{1}{2}\left[0(\infty) - (-1)(0+6)\right]$$

$= \frac{1}{2}(6)$

$= 3$

c) Variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

$= \int_{0}^{\infty} x^2 K x^2 e^{-x}\,dx - (3)^2$

$= \frac{1}{2} \int_{0}^{\infty} x^4 e^{-x}\,dx - 9$

$= \frac{1}{2}\left[-e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24)\right]_0^{\infty} - 9$

$= \frac{1}{2}\left[(0) - (-1)(0+0+0+0+24)\right] - 9$

$= 12 - 9$

$= 3$

Exhaustive Events: A set of events are called exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

If A starts the game

Probability of A's winning $= \frac{p}{1-q^2}$

Probability of B's winning $= \frac{pq}{1-q^2}$

04/10/21

UNIT - II

Probability Distributions

* We have already learnt frequency distributions which are based on the actual observations. In this chapter, we shall discuss theoretical distributions in which the variates are distributed as per some law which can be expressed mathematically.

* There are 2 types of probability distributions:

a) Discrete Probability Distribution

i) Binomial Distribution

ii) Poisson Distribution

b) Continuous Probability Distribution

Normal Distribution

Binomial Distribution: It was discovered by James Bernoulli in the year 1700 and it is a discrete Probability distribution.

* The probability of the no. of successes so obtained is called Binomial Probability Distribution.

Definition: A random variable X has a binomial distribution if it assumes only non-negative values & its probability density function is given by