DISCRETE MATHEMATICS

UNIT-1

Mathematical Logic

Propositions: A proposition is a declarative sentence, either it is true or false, but not both. It is also called a statement.

Propositions are usually represented by small letters such as p, q, r, s, ... The truth or the falsity of a proposition is called its truth value. If a proposition is true, we will indicate the truth value by the symbol '1' (or T) and if it is false by the symbol '0' (or F)

Ex:- ① p: $2+3=5$ then the truth value of p is 'T'

② q: Every rectangle is square, then the truth value of q is 'F'

③ x is an integer, it is not a proposition as we don't know the value of x.

④ Take a triangle ABC.
It is not at all a declaration, so, it is not a proposition

Logical connectives:- The words like NOT, OR, AND, If then, if and only if are called connectives

Two (or) more propositions (statement) can be combined by means of logical operators (or connectives) to form a single proposition called compound propositions

Logical Connectives

Symbol	Connective word	Name
$\sim$	not	Negation
$\wedge$	and	conjuction
$\vee$	OR	disjunction
$\Rightarrow$	implies (or) if ... then	implication (or) conditional
$\Leftrightarrow$	If and only if	Equivalence (or) Biconditional

If p and q are two propositions, then $\sim q$, $p \wedge q$, $p \vee q$, $p \Leftrightarrow q$ $\{\sim(p \Rightarrow q) \wedge (q \Rightarrow p)\}$ are all propositions

Truth tables:- A truth table is the convenient way of summarizing the truth values of logical statements. A truth table consits of columns and rows. The number of columns depends upon the no. of sample propositions and connectives used to form a compound proposition. The no. of rows in a truth table are found on the basis of simple propositions.

In general, for 'n' simple propositions, the total no. of rows will be 2^n. It is useful in
(i) finding out the validity of equivalence relation b/w functions
(ii) designing and testing the electronic circuits to perform a given operation based on certain relationship.

1) Negation:- A proposition obtained by inserting the word 'NOT' at an appropriate place in a given proposition is called the negation of the given proposition

The negation of proposition 'p' is denoted by $\sim p$ (Not p)

Ex:- p: 3 is a prime number
$\sim p$: 3 is not a prime number

Truth table:

p	$\sim p$
T	F
F	T

2) Conjunction:- A compound proposition obtained by using AND b/w two given propositions is called the conjunction

The conjuction of two propositions p and q is denoted by $p \wedge q$ (i.e p and q)

The conjunction $p \wedge q$ is true only when p is true and q is true, in all other cases it is false (F)

TRUTH table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Ex:- p: $\sqrt{2}$ is an irational number
q: All triangles are equilateral then
$p \wedge q$: $\sqrt{2}$ is an irrational number and all triangles are equilatral.

3) Disjunction :- A compound proposition obtained by combining (2)
two given propositions by inserting the word 'OR' in b/w them is called the disjunction

The disjunction of two propositions p and q is ~~obtained~~ denoted by $p \vee q$ (i.e p or q)

The disjunction $p \vee q$ is false only when p is false and q is false; in all other cases it is true

Truth table :-

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Ex :- p: 5 is a positive integer and q: $\sqrt{5}$ is a rational number

then $p \vee q$: 5 is a +ve integer (or) $\sqrt{5}$ is a rational number

4) Exclusive Disjunction :- The compound proposition $p \underline{\vee} q$ (Read as either p or q but not both) is called the exclusive of the propositions p and q

i.e we require that the compound proposition "p or q" to be true only when either p is true or q is true but not both

Truth table :-

p	q	$p \underline{\vee} q$
F	F	F
F	T	T
T	F	T
T	T	F

Ex :- Let p : $\sqrt{2}$ is irrational number, q : 2+3=5

then $p \underline{\vee} q$: Either $\sqrt{2}$ is irrational number (or) 2+3=5, but not both

5) Conditional :- A compound proposition obtained by combining two given propositions by using the words 'if' and 'then' at appropriate places is called a conditional.

If p and q are two propositions, we can form the conditionals "If p, then q" and "If q, then p". The conditional "If p, then q" is denoted by $p \Rightarrow q$ and the conditional "If q, then p" is denoted by $q \Rightarrow p$

* The conditional $q \Rightarrow p$ is not the same as the conditional $p \Rightarrow q$

*The conditional $p \Rightarrow q$ is false only when p is true and q is false, in all other cases it is true

Truth table:-

p	q	$p \to q$
T	F	F
F	T	T
T	T	T
F	F	T

Ex:- Let p: 4 is even number, q: 4 is divisible by '2'
then $p \to q$: If 4 is even number then 4 is divisible by '2'

6) Biconditional:- If p and q are two propositions then the conjuction of conditionals $p \Rightarrow q$ and $q \Rightarrow p$ is called the biconditional of p and q. It is denoted by $p \Leftrightarrow q$

i.e. $p \Leftrightarrow q$ is the same as $(p \to q) \wedge (q \Rightarrow p)$

Truth table:

p	q	$p \to q$	$q \to p$	$p \Leftrightarrow q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

* $p \Leftrightarrow q$ is true only when both p and q have the same truth values

(B) Well-formed formulas: Statements represented in symbolic forms which cannot be ~~interpted~~ interpreted in more than one way are called well-formed formulas. The following are regarded as well-formed formlas

(i) primitive statements. (ii) The negation of well-formed formula
(iii) The conjunction, disjunction, exclusive disjunction, conditional and the biconditional whose components themselves are well-defined formulas.

(P) p: A circle is a conic q: $\sqrt{5}$ is an irrational number
r: Exponential series is cgt. Express the following compound propositions in words

(i) $p \wedge (\sim q)$ (ii) $(\sim p) \vee q$ (iii) $p \veebar (\sim q)$ (iv) $p \Rightarrow (q \veebar r)$
(v) $\sim p \Leftrightarrow [q \wedge (\sim r)]$

Soln: (i) $p \wedge (\sim q)$: A circle is conic and $\sqrt{5}$ is not a ~~ration~~ irr. rational no.
(ii) $(\sim p) \vee q$: A circle is not a conic or $\sqrt{5}$ is an irrational number
(iii) $p \veebar (\sim q)$: Either A circle is a conic or $\sqrt{5}$ is not an irrational number
~~(iv) $p \Rightarrow$ (q~~

(iv) $q \Rightarrow (\sim p)$: If $\sqrt{5}$ is an irrational number then a circle is not a conic ③

(v) $p \Rightarrow (q \veebar r)$: If a Circle is a conic then either $\sqrt{5}$ is an irrational number or the exponential series is cgt.

(vi) $\sim p \Leftrightarrow \{q \wedge (\sim r)\}$; A circle is not a conic iff $\sqrt{5}$ is an irrational number and the exponential series is not cgt.

Ⓟ Construct the truth tables of the following compound propositions

(i) $\{p \vee (\sim q)\} \wedge q$:

p	q	$\sim q$	$p \vee (\sim q)$	$\{p \vee (\sim q)\} \wedge q$
T	T	F	T	T
F	T	F	F	F
F	F	T	T	F
T	F	T	T	F

(ii) $(p \wedge q) \Rightarrow (\sim r)$ (iii) $q \wedge \{(\sim r) \Rightarrow p\}$

Sol^n:

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \Rightarrow (\sim r)$	$(\sim r) \Rightarrow p$	$q \wedge \{(\sim r) \Rightarrow p\}$
F	F	F	T	F	T	F	F
F	F	T	F	F	T	T	F
F	T	F	T	F	T	F	F
F	T	T	F	F	T	T	T
T	F	F	T	F	T	T	F
T	F	T	F	F	T	T	F
T	T	F	T	T	T	T	T
T	T	T	F	T	F	T	T

(iv) $p \Rightarrow (q \Rightarrow r)$ (v) $\{(p \wedge q) \vee (\sim r)\} \Leftrightarrow p$ (vi) $q \Leftrightarrow (\sim p \vee \sim q)$

(vii) $\sim (p \vee \sim q)$ (viii) $\{p \vee [q \rightarrow (r \wedge \sim p)]\} \Leftrightarrow (q \vee \sim s)$

Tautology; Contradiction:

(Truth values)

A Compound proposition which is true for all possible situations of its components is called a tautology. It is denoted by T_0

A Compound propositions which is false for all possible truth values of its components is called a contradiction. It is denoted by F_0.

A Compound propositions that can be true or false (depending upon the truth values of the components) is called a contingency. In other words a contingency is a compound proposition which is neither a tautology nor a contradiction

Ex:- $p \vee \sim p$ is a tautology
$p \wedge \sim p$ is a contradiction

p	q	$p \vee \sim p$	$p \wedge \sim p$
F	T	T	0
T	F	T	0

(P) Show that (i) $(p \underline{\vee} q) \vee (p \Leftrightarrow q)$ is a tautology
(ii) $(p \underline{\vee} q) \wedge (p \Leftrightarrow q)$ is a contradiction
(iii) $(p \underline{\vee} q) \wedge (p \Rightarrow q)$ is a contingency

Sol:- (i)

p	q	$p \vee q$	$p \underline{\vee} q$	$p \Leftrightarrow q$	$(p \underline{\vee} q) \vee (p \Leftrightarrow q)$	$(p \underline{\vee} q) \wedge (p \Leftrightarrow q)$
F	F	F	F	T	T	F
F	T	T	T	F	T	F
T	F	T	T	F	T	F
T	T	T	F	T	T	F
					↳ Tautology	↳ contradiction

(iii)

p	q	$p \vee q$	$p \underline{\vee} q$	$p \Rightarrow q$	$(p \underline{\vee} q) \wedge (p \Rightarrow q)$
F	F	F	F	T	F
F	T	T	T	T	T
T	F	T	T	F	F
T	T	T	F	T	F

→ contingency

(P) Prove that, for any propositions p and q, the compound proposition
(i) $[(\sim q) \wedge (p \Rightarrow q)] \Rightarrow (\sim p)$ is a tautology
(ii) $[(p \to q) \wedge (q \to r)] \Rightarrow (p \to r)$ is a tautology
(iii) $\{p \to (q \to r)\} \Rightarrow \{(p \to q) \to (p \to r)\}$ is a tautology
(iv) $[(p \wedge \sim q) \to r] \Rightarrow [p \to (q \vee r)]$
(v) $\sim (p \vee q) \vee [(\sim p) \wedge q] \vee p$
(vi) $[(p \to r) \wedge (q \to r)] \to [(p \vee q) \to r]$

(P) Show that (i) $(p\wedge q)\wedge[\sim(p\vee q)]$ is a contradiction

Sol:

p	q	$p\wedge q$	$p\vee q$	$\sim(p\vee q)$	$(p\wedge q)\wedge[\sim(p\vee q)]$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

$\therefore$ It is a contradiction.

(ii) $[(p\wedge q)\wedge r]\to(s\vee t)$ (iii) $[p\wedge(q\wedge r)]\Rightarrow(s\vee t)$ contradiction

(P) Show that the following statements are contingency

(i) $p\to(p\Rightarrow q)$ (ii) $(p\wedge\sim q)\vee(\sim p\wedge q)$ (iii) $\sim(p\vee q)\wedge(\sim p\vee\sim q)$

(iv) $[p\Rightarrow(q\wedge r)]\Rightarrow\sim(p\to q)$

Sol: (i)

p	q	$p\Rightarrow q$	$p\Rightarrow(p\Rightarrow q)$
T	F	F	F
F	T	T	T
T	T	T	T
F	F	T	T

Since entries in the last column of the truth table depend on statements p, q and $p\Rightarrow q$

$\therefore$ given statement is contingency.

Logical Equivalence:- Two compound propositions u and v are said to be logical equivalent if u and v are having same truth values (OR) The biconditional $u\Leftrightarrow v$ is a tautology. Then, we write $u\Leftrightarrow v$ (or) $u\equiv v$

(P): P.T, for any propositions p and q, the compound propositions $p\underline{\vee}q$ and $(p\vee q)\wedge(\sim p\vee\sim q)$ are logically equivalent

Sol:

p	q	$p\vee q$	$p\underline{\vee}q$	$\sim p$	$\sim q$	$\sim p\vee\sim q$	$(p\vee q)\wedge(\sim p\vee\sim q)$
F	F	F	F	T	T	T	F
F	T	T	T	T	F	T	T
T	F	T	T	F	T	T	T
T	T	T	F	F	F	F	F

For columns 4 & 8 of the above truth table, we find that $p\underline{\vee}q$ and $(p\vee q)\wedge(\sim p\vee\sim q)$ have same truth values for all truth values of p and q

$\therefore (p\underline{\vee}q)\equiv(p\vee q)\wedge(\sim p\vee\sim q)$

(P) P.T for any three propositions p, q, r. $[p \to (q \wedge r)] \equiv [(p \to q) \wedge (p \to r)]$

Sol:

p	q	r	$q \wedge r$	$p \to (q \wedge r)$	$p \to q$	$p \to r$	$(p \to q) \wedge (p \to r)$
F	F	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	T	F	F	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	F	F	F	F	F
T	F	T	F	F	F	T	F
T	T	F	F	F	T	F	F
T	T	T	T	T	T	T	T

Columns 5 and 8 of the above table, show that $[p \to (q \wedge r)]$ and $[(p \to q) \wedge (p \to r)]$ have identical truth values in all possible situations.

$\therefore [p \to (q \wedge r)] \equiv [(p \to q) \wedge (p \to r)]$

(P) P.T, for any three propositions p, q, r

(i) $[(p \vee q) \to r] \equiv [(p \to r) \wedge (q \to r)]$

(ii) $[(p \to q) \wedge (\cancel{p \to r}\ r \to q)] \Leftrightarrow [(p \vee r) \to q]$

(iii) $[(p \to q) \wedge (p \to \sim q)] \Leftrightarrow \sim p$

(iv) $[p \wedge (\sim r \vee q \vee \sim q)] \vee [(r \vee t \vee \sim r) \wedge \sim q] \Leftrightarrow p \vee \sim q$

(v) $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \equiv [(p \wedge q) \wedge (q \to r) \wedge (r \to p)]$

(vi) $(p \underline{\vee} q) \Leftrightarrow [(p \wedge \sim q) \vee (\sim p \wedge q)] \Leftrightarrow \sim (p \leftrightarrow q)$

Logical Laws:

1) Law of Double Negation :- $(\sim \sim p) \Leftrightarrow p$
2) Idempotent law :- (a) $(p \vee p) \Leftrightarrow p$ (b) $(p \wedge p) \Leftrightarrow p$
3) Identity Laws :- (a) $(p \vee F_0) \Leftrightarrow p$ (b) $(p \wedge T_0) \Leftrightarrow p$
4) Inverse Laws : (a) $(p \vee \sim p) \Leftrightarrow T_0$ (b) $(p \wedge \sim p) \Leftrightarrow F_0$
5) Domination Laws : (a) $(p \vee T_0) \Leftrightarrow T_0$ (b) $(p \wedge F_0) \Leftrightarrow F_0$
6) Commutative laws :- (a) $(p \vee q) \Leftrightarrow (q \vee p)$ (b) $(p \wedge q) \Leftrightarrow (q \wedge p)$
7) Absorption laws :- (a) $[p \vee (p \wedge q)] \Leftrightarrow p$ (b) $[p \wedge (p \vee q)] \Leftrightarrow p$
8) Demorgans laws :- (a) $\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q$

(b) $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$

9) Associative laws:- (a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

(b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

10) Distributive laws:- (a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(P) Indicate the negation of the following Compound proposition

(a). 3 is a prime and 4 is even

Soln: p: 3 is a prime

q: 4 is even

Given, $p \wedge q$, negation for this is

$$\sim (p \wedge q) = \sim p \vee \sim q$$

i.e 3 is not a prime or 4 is not even

(b) If $\sqrt{2}$ is rational then $\sqrt{2}+1$ is rational

Soln: Let p: $\sqrt{2}$ is rational

q: $\sqrt{2}+1$ is rational

Given, $p \to q$, negation for this is $\sim (p \to q) = p \wedge \sim q$

i.e $\sqrt{2}$ is rational and $\sqrt{2}+1$ is not rational.

Law for the Negation of a Conditional :- $\sim (p \to q) \Leftrightarrow p \wedge (\sim q)$

Table for Negation

proposition	Negation
$\sim p$	p
$p \wedge q$	$\sim p \vee \sim q$
$p \vee q$	$\sim p \wedge \sim q$
$p \to q$	$p \wedge \sim q$

Transitive Rule: If u, v, w are propositions such that $u \Leftrightarrow v$ and $v \Leftrightarrow w$ then $u \Leftrightarrow w$

Substitution Rule:- (i) Suppose that a compound proposition u is a tautology and p is a component of u. If we replace each occurence of p in u by a proposition q, Then the resulting compound proposition v is also a tautology.

(ii) Suppose that ~~a~~ u is a compound proposition which contains a component p. Let q be a proposition such that $q \Leftrightarrow p$. Suppose we replace one or more occurences of p by q and obtain a compound proposition v. Then $v \Leftrightarrow u$

(P) Let x be a specified number. Write down the negation of the proposition
" If x is not a real number then it is not a rational number and not an irrational number"

Sol^n: Let p: x is a real number
q : x is a rational number
r : x is an irrational number

Given $\sim p \to (\sim q \wedge \sim r)$, negation for this is

$$\sim[\sim p \to (\sim q \wedge \sim r)] \equiv \sim p \wedge [\sim(\sim q \wedge \sim r)]$$
$$\equiv \sim p \wedge (\sim\sim q \vee \sim\sim r)$$
$$\equiv \sim p \wedge (q \vee r)$$

i.e x is not a real number and it is a rational number or an irrational number.

(P) Indicate the negation of the compound proposition
" If there is no cricket telecast this evening then either I visit a friend or I study"

Sol^n:- Let p: There is no cricket telecast this evening
q : I visit a friend
r : I study

Given $p \to (q \vee r)$, the negation for this is

$$\sim[p \to (q \vee r)] \Leftrightarrow p \wedge [\sim(q \vee r)]$$
$$\Leftrightarrow p \wedge (\sim q \wedge \sim r)$$

$\therefore$ There is no cricket telecast this evening and I do not visit a friend and I do not study.

(P) P.T the following logical equalences without using truth tables
(i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$ (ii) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow (p \vee q \vee r)$
(iii) $[(\sim p \vee \sim q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

Sol^n:- (i) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee p$ [$\because$ by Absorption Law]
$\Leftrightarrow p$ [$\because$ by an idempotent law

(ii) $\sim p \wedge \sim q \wedge r \Leftrightarrow (\sim p \wedge \sim q) \wedge r$ [by Associative law)
$\Leftrightarrow \sim(p \vee q) \wedge r$ [by DeMorgan law]

$\therefore p \vee q \vee (\sim p \wedge \sim q \wedge r) \Leftrightarrow (p \vee q) \vee [\sim(p \vee q) \wedge r]$
$\Leftrightarrow [(p \vee q) \vee \sim(p \vee q)] \wedge [(p \vee q) \vee r]$, by Distributive law

$\Leftrightarrow T_0 \wedge (p \vee q \vee r)$, [by Inverse law & Associative law]

$\Leftrightarrow p \vee q \vee r$ [by Commutative and Identity law]

(iii) $(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r) \Leftrightarrow \sim(\sim p \vee \sim q) \vee (p \wedge q \wedge r)$ [$\because u \rightarrow v \Leftrightarrow \sim u \vee v$]

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$ by Demorgan law & Associative law

$\Leftrightarrow p \wedge q$, by Absorption law

(iv) Prove the logical equivalences

(i) $[(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

(ii) $(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow \sim (q \vee p)$

Solⁿ:- (i) $(p \vee q) \wedge (p \wedge \sim q) \Leftrightarrow p \vee (q \wedge \sim q)$, by distributive law

$\Leftrightarrow p \vee F_0$ Inverse law

$\Leftrightarrow p$ by an Identity law

$\therefore [(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

(ii) $(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow (p \rightarrow q) \wedge [\sim q \wedge (\sim q \vee r)]$ by commutative

$\Leftrightarrow (p \rightarrow q) \wedge \sim q$, by absorption law

$\Leftrightarrow \sim[(p \rightarrow q) \rightarrow q]$. [$\because \sim(u \rightarrow v) \Leftrightarrow u \wedge \sim v$]

$\Leftrightarrow \sim[\sim(p \rightarrow q) \vee q]$ [$\because u \rightarrow v \Leftrightarrow (\sim u \vee v)$]

$\Leftrightarrow \sim[(p \wedge \sim q) \vee q]$

$\Leftrightarrow \sim[q \vee (p \wedge \sim q)]$, by Commutative law

$\Leftrightarrow \sim[(q \vee p) \wedge (q \vee \sim q)]$, by distributive law

$\Leftrightarrow \sim[(q \vee p) \wedge T_0]$

$\Leftrightarrow \sim (q \vee p)$, by an identity law

(B) P.T (i) $[\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

(ii) $\sim[\{(p \vee q) \wedge r\} \rightarrow \sim q] \Leftrightarrow \sim[\sim(p \vee q) \wedge r] \vee \sim q] \Leftrightarrow q \wedge r$

Solⁿ:- (i) $\sim p \wedge (\sim q \wedge r) \Leftrightarrow (\sim p \wedge \sim q) \wedge r$

$\Leftrightarrow [\sim(p \vee q)] \wedge r$

$\Leftrightarrow r \wedge [\sim(p \vee q)]$

and $(q \wedge r) \vee (p \wedge r) \Leftrightarrow (r \wedge q) \vee (r \wedge p)$

$\Leftrightarrow r \wedge (q \vee p) \Leftrightarrow r \wedge (p \vee q)$

$\therefore [\sim p \wedge (\sim q \wedge r)] \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow \{r \wedge [\sim(p \vee q)]\} \vee [r \wedge (p \vee q)]$

$\Leftrightarrow r \wedge \{[\sim(p \vee q)] \vee (p \vee q)\}$

$\Leftrightarrow r \wedge T_0 \Leftrightarrow r$.

Duality law:- Let u be a compound proposition having a connective $\vee, \wedge$ only then, a dual of the compound proposition u^d obtained by replacing $\vee$ by $\wedge$, $\wedge$ by $\vee$ (7)

NOTE:- ① If the compound proposition u is having T_0 and F_0 then u^d is obtained by replacing T_0 by F_0 and F_0 by T_0

② $(u^d)^d \Leftrightarrow u$

③ For any two propositions u and v, if $u \Leftrightarrow v$ then $u^d \Leftrightarrow v^d$ (principle of duality)

Ex: ① Write down the duals of the following propositions

(i) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim s)]$

(ii) $[(p \vee T_0) \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

Sol^n:- (i) $u^d = \sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim s)]$

(ii) $u^d = [(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$

② Find the dual of (i) $p \to q$ (ii) $p \leftrightarrow q$ (iii) $p \to (q \to r)$

Sol^n:- (i) $p \to q \Leftrightarrow \sim p \vee q$

$\therefore u^d = (\sim p \vee q)^d = \sim p \wedge q$

(ii) $u: p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

i.e. $u \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\therefore u^d = (\sim p \wedge q) \vee (\sim q \wedge p)$

(iii) $u \equiv p \to (q \to r) \Leftrightarrow \sim p \vee (q \to r) \Leftrightarrow \sim p \vee (\sim q \vee r)$

$\therefore u^d \equiv \sim p \wedge (\sim q \wedge r)$

③ Verify the principle of duality for the logical equivalence

$$[\sim(p \wedge q) \to \sim p \vee (\sim p \vee q)] \Leftrightarrow (\sim p \vee q)$$

Proof:- Given logical equivalence is $u \Leftrightarrow v$, where

$u = \sim(p \wedge q) \to [\sim p \vee (\sim p \vee q)]$ and $v = \sim p \vee q$

We note that $u \Leftrightarrow \sim\sim(p \wedge q) \vee [\sim p \vee (\sim p \vee q)]$

$\Leftrightarrow (p \wedge q) \vee [\sim p \vee (\sim p \vee q)]$

$\therefore u^d \Leftrightarrow (p \vee q) \wedge [\sim p \wedge (\sim p \wedge q)]$ Also $v^d = \sim p \wedge q$

$\Leftrightarrow (p \vee q) \wedge (\sim p \wedge q)$ $\therefore u^d = v^d$

$\Leftrightarrow [p \wedge (\sim p \wedge q)] \vee [q \wedge (\sim p \wedge q)]$ Hence verified

$\Leftrightarrow (F_0 \wedge q) \vee (q \wedge \sim p)$

$\Leftrightarrow F_0 \vee (q \wedge \sim p) \Leftrightarrow q \wedge \sim p$

(4) Verify the principle duality for the logical equivalence
$(P\vee q)\wedge[\sim P\wedge(\sim P\wedge q)]\Leftrightarrow\sim P\wedge q$

NAND and NOR Connectives :-

Let p, q be the propositions, the NAND connective is defined as "NOT (p and q)" i.e $\sim(P\wedge q)$ and is denoted by $P\uparrow q$

i.e $\boxed{P\uparrow q\Leftrightarrow\sim(P\wedge q)\Leftrightarrow\sim P\vee\sim q}$

The NOR connective is defined as "NOT (P or q)" i.e $\sim(P\vee q)$. It is denoted by $P\downarrow q$

i.e $\boxed{P\downarrow q\Leftrightarrow\sim(P\vee q)\Leftrightarrow\sim P\wedge\sim q}$

(P) P.T (i) $\sim(P\downarrow q)\Leftrightarrow(\sim p\uparrow\sim q)$ (ii) $\sim(P\uparrow q)\Leftrightarrow\sim p\downarrow\sim q$

Soln :- (i) $\sim(P\downarrow q)\Leftrightarrow\sim(\sim P\wedge\sim q)\Leftrightarrow(\sim p)\uparrow(\sim q)$

(ii) $\sim(P\uparrow q)\Leftrightarrow\sim[\sim(P\wedge q)]\Leftrightarrow\sim[\sim P\vee\sim q)=(\sim p)\downarrow(\sim q)$

(P) Express the following propositions interms of only NAND and only NOR connectives

(i) $\sim p$ (ii) $P\wedge q$ (iii) $P\rightarrow q$ (iv) $P\leftrightarrow q$

Soln :- (i) $\sim p\Leftrightarrow\sim(P\wedge P)\Leftrightarrow P\uparrow P$ Also $\sim p\Leftrightarrow\sim(P\vee P)\Leftrightarrow P\downarrow P$

(ii) $P\wedge q\Leftrightarrow\sim\sim(P\wedge q)\Leftrightarrow\sim(\sim p\vee\sim q)\Leftrightarrow(\sim p\vee\sim q)\uparrow(\sim p\vee\sim q)$
$\Leftrightarrow(P\uparrow q)\uparrow(P\uparrow q)$

UNIT – II

Predicate Calculus

Normal forms:-

(i) Disjunctive Normal form :- (DNF):- Let u be the compound proposition a proposition v is said to be disjunctive normal form s.t (i) $u \equiv v$ (ii) v is a disjunction of two or more compound propositions each of which is a conjuction involving the components of u or their negations.

(ii) Conjuctive Normal form :- (CNF) :- Let u be the compound proposition, a proposition v is said to be conjuctive normal form s.t (i) $u \equiv v$ (ii) v is a conjunction of two or more compound propositions each of which is a disjunction involving the component of u ~~and~~ or their negations.

(P). Find the DNF of the following

(i) $p \wedge (p \to q)$ (ii) $\sim[p \to (q \wedge r)]$ (iii) $\sim(p \vee q) \leftrightarrow (p \wedge q)$

(iv) $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$ (v) $p \vee [\sim p \to \{q \vee (q \to \sim r)\}]$

Sol^n:- (i) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$

$$\equiv (p \wedge \sim p) \vee (p \wedge q)$$

we observe that the R.H.S is the disjunction of $p \wedge \sim p$ and $p \wedge q$ each of which is a conjunction involving p or q or their negations.

(ii) $\sim[p \to (q \wedge r)] \equiv \sim[\sim p \vee (q \wedge r)]$

$$\equiv \sim[(\sim p \vee q) \wedge (\sim p \vee r)]$$

$$\equiv \sim[\sim p \vee q] \vee \sim(\sim q \vee r)$$

$$\equiv (p \wedge \sim q) \vee (q \wedge \sim r)$$

(iii) Take $r = \sim(p \vee q)$ and $s = p \wedge q$

$$\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv r \leftrightarrow s$$

$$\equiv (r \to s) \wedge (s \to r)$$

$$\equiv (\sim r \vee s) \wedge (\sim s \vee r)$$

$$\equiv [(\sim r \vee s) \wedge (\sim s)] \vee [(\sim r \vee s) \wedge r]$$

$$\equiv [(\sim r \wedge \sim s) \vee (s \wedge \sim s)] \vee [(\sim r \wedge r) \vee (s \wedge r)]$$

$$\equiv [(\sim r \wedge \sim s) \vee F_0] \vee [F_0 \vee (s \wedge r)]$$

$$\equiv (\sim r \wedge \sim s) \vee (s \wedge r) \equiv (r \wedge s) \vee (\sim r \wedge \sim s)$$

Now $r \wedge s \equiv (\sim p \wedge \sim q) \wedge (p \wedge q)$

$\equiv (\sim p \wedge p) \wedge (q \wedge \sim q)$

$\equiv F_0 \wedge F_0 \equiv F_0$

and $\sim r \wedge \sim s = \sim r \wedge (\sim p \vee \sim q)$

$\equiv (\sim r \wedge \sim p) \vee (\sim r \wedge \sim q)$

$\equiv [(p \vee q) \wedge \sim p] \vee [(p \vee q) \wedge \sim q]$

$\equiv [(p \wedge \sim p) \vee (q \wedge \sim p)] \vee [(p \wedge \sim q) \vee (q \wedge \sim q)]$

$\equiv [F_0 \vee (q \wedge \sim p)] \vee [(p \wedge \sim q) \vee F_0]$

$\equiv (q \wedge \sim p) \vee (p \wedge \sim q)$

$\therefore [\sim (p \vee q) \Leftrightarrow (p \wedge q)] \equiv F_0 \vee [(q \wedge \sim p) \vee (p \wedge \sim q)]$

$= (q \wedge \sim p) \vee (p \wedge \sim q)$

Hence R.H.S is a D.N.F of the given Compound proposition.

(iv) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)] \Leftrightarrow \sim p \vee [(p \to q) \wedge \sim (\sim q \vee \sim p)]$

$\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$

$\Leftrightarrow \sim p \vee [\{\sim p \wedge (q \wedge p)\} \vee \{q \wedge (q \wedge p)\}]$

$\Leftrightarrow \sim p \vee [\{(\sim p \wedge p) \wedge q\} \vee \{(q \wedge q) \wedge p\}]$

$\Leftrightarrow \sim p \vee [\{F_0 \wedge q\} \vee \{q \wedge p\}]$

$\Leftrightarrow \sim p \vee [F_0 \vee (p \wedge q)]$

$\Leftrightarrow \sim p \vee (p \wedge q) \Leftrightarrow (\sim p \wedge T_0) \vee (p \wedge q)$

$\therefore$ This is DNF of given Compound proposition.

(P) Obtain the PDNF of the following

(i) $p \vee (p \wedge q)$ (ii) $(\sim p) \vee q$ (iii) $p \to [(p \to q) \wedge \sim (\sim q \vee \sim p)]$

Soln:- (i) $p \vee (p \wedge q) \equiv (p \wedge T_0) \vee (p \wedge q)$

$\equiv [p \wedge (q \vee \sim q)] \vee (p \wedge q)$

$\equiv [(p \wedge q) \vee (p \wedge \sim q)] \vee (p \wedge q)$

$\equiv [(p \wedge q) \vee (p \wedge q) \vee (p \wedge \sim q)$

$\equiv (p \wedge q) \vee (p \wedge \sim q)$

(P) Find the conjuctive normal form of the compound propositions (2)

(Take the above problem)

Sol^n:- (i) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q) \equiv (p \vee p) \wedge (\sim p \vee q)$

(iii) Let $r = p \vee q$ and $s = \sim(p \wedge q)$, we find that

$$[\sim(p \vee q) \leftrightarrow (p \wedge q)] \equiv \sim r \leftrightarrow \sim s$$
$$\Leftrightarrow r \leftrightarrow s$$
$$\Leftrightarrow \equiv (r \to s) \wedge (s \to r)$$
$$\equiv (\sim r \vee s) \wedge (\sim s \vee r)$$
$$\equiv [(\sim p \wedge \sim q) \vee s] \wedge [(p \wedge q) \vee r]$$
$$\equiv [(\sim p \vee s) \wedge (\sim q \vee s)] \wedge [(p \vee r) \wedge (q \vee r)]$$
$$\equiv [\sim p \vee (\sim p \vee \sim q)] \wedge [\sim q \vee (\sim p \vee \sim q)]$$
$$\wedge [p \vee (p \vee q)] \wedge [q \vee (p \vee q)]$$
$$\equiv (\sim p \vee \sim q) \wedge (\sim q \vee \sim p) \wedge (p \vee q)$$
$$\equiv (\sim p \vee \sim q) \wedge (p \vee q)$$

This is C.N.F of given compound proposition.//

NOTE:- The DNF/CNF of compound proposition is not unique.

Principal Normal forms

Given p, q are two simple propositions, then the compound propositions $p \wedge q$, $p \wedge (\sim q)$, $(\sim p) \wedge q$ and $(\sim p) \wedge (\sim q)$ are called min-terms and the proposition $p \vee q$, $p \vee (\sim q)$, $(\sim p \vee q)$ and $(\sim p) \vee (\sim q)$ are called maxterms

principal disjunctive normal form (PDNF) :-

A Compound proposition u involving two simple propositions p and q, an equivalent compound proposition v consisting of disjunctions of the minterms involving p and q is known as its PDNF

principal Conjuctive Normal form (PCNF) :-

Given a compound proposition 'u' involving two simple propositions p and q, an equivalent compound proposition v consisting of conjunctions of the maxterms involving p and q only is known as it PCNF

(P) obtain the PDNF of the following

(i) $p \vee (p \wedge q)$ (ii) $(\sim p) \vee q$ (iii) $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$

Sol^n:- (i)

$$\begin{aligned} p \vee (p \wedge q) &\equiv (p \wedge T_0) \vee (p \wedge q) \\ &\equiv [p \wedge (q \vee \sim q)] \vee (p \wedge q) \\ &\equiv [(p \wedge q) \vee (p \wedge \sim q)] \vee (p \wedge q) \\ &\equiv (p \wedge q) \vee (p \wedge q) \vee (p \wedge \sim q) \\ &\equiv (p \wedge q) \vee (p \wedge \sim q) \end{aligned}$$

$\because$ R.H.S is a disjunction of minterms involving p and q

$\therefore$ It is PDNF of given compound proposition//

(ii)

$$\begin{aligned} (\sim p) \vee q &\equiv [(\sim p) \wedge T_0] \vee [q \wedge T_0] \\ &= [(\sim p) \wedge (q \vee \sim q)] \vee [q \wedge (p \vee \sim p)] \\ &\equiv [\{(\sim p) \wedge q\} \vee (\sim p \wedge \sim q)] \vee [(q \wedge p) \vee (q \wedge \sim p)] \\ &\equiv (\sim p \wedge q) \vee (q \wedge \sim p) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \\ &\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \end{aligned}$$

This is PDNF of given compound proposition//

(iii) $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)] \equiv \sim p \vee [(p \to q) \wedge \sim(\sim q \vee \sim p)]$

$\equiv \sim p \vee [\{(\sim p) \vee q) \wedge (q \wedge p)\}]$

$\equiv \sim p \vee \{(\sim p) \wedge (q \wedge p) \vee [q \wedge (q \wedge p)\}]$

$\equiv (\sim p \wedge T_0) \vee (F_0 \vee (q \wedge p)]$

$= [\sim p \wedge (q \vee \sim q)] \vee (q \wedge p)$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$

This PDNF of given Compound proposition//

(P) Obtain the PCNF of the following

(i) $(\sim p \to q) \wedge (q \Leftrightarrow p)$ (ii) $(p \wedge q) \vee (\sim p \wedge q)$

(iii) $p \wedge (p \vee q)$ (iv) $(\sim p) \wedge q$ (v) $\sim(p \vee q)$

Soln:- (i) $(\sim p \to q) \wedge (q \Leftrightarrow p) \equiv (p \vee q) \wedge [(q \to p) \wedge (p \to q)]$

$\equiv (p \vee q) \wedge [(\sim q \vee p) \wedge (\sim p \vee q)]$

$\equiv (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q)$

R.H.S is a conjunction of maxterms involvoving p and q

$\therefore$ It is PCNF of compound proposition//

(ii) $(p \wedge q) \vee (\sim p \wedge q) \equiv [(p \wedge q) \vee \sim p] \wedge [(p \wedge q) \vee q]$

$\equiv (p \vee \sim p) \wedge (q \vee \sim p) \wedge [(p \vee q) \wedge (q \vee q)]$

$\equiv T_0 \wedge (q \vee \sim p) \wedge [(p \vee q) \wedge T_0]$

$\equiv (q \vee \sim p) \wedge (p \vee q)$

Converse, Inverse and Contrapositive of a Conditional

Consider a conditional $p \to q$. Then

(1) $q \to p$ is called the converse of $p \to q$

(2) $\sim p \to \sim q$ is called the inverse of $p \to q$

(3) $\sim q \to \sim p$ is called the contrapositive of $p \to q$

Eg:- Let p: 2 is an integer q: 9 is a multiple of 3. Then

$p \to q$: If 2 is an integer then 9 is a multiple of 3

The converse of this conditional is

$q \to p$: If 9 is a multiple of 3 then 2 is an integer

The inverse is $\sim p \to \sim q$: If 2 is not an integer then 9 is not a multiple of 3

The contrapositive is $\sim q \to \sim p$: If 9 is not a multiple of 3 then 2 is not an integer

Truth-table

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
F	F	T	T	1	1	1	1
F	T	T	F	1	0	0	1
T	F	F	T	0	1	1	0
T	T	F	F	1	1	1	1

From the table shows that

$p \to q$ and $\sim q \to \sim p$ have identical truth values

$q \to p$ and $\sim p \to \sim q$ have identical truth values

i.e $p \to q \Leftrightarrow \sim q \to \sim p$

$q \to p \Leftrightarrow \sim p \to \sim q$

Rules of Inference

Argument:- Consider a set of propositions $p_1, p_2, \ldots p_n$ and a proposition 'q'. Then a compound proposition of the form $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is called an argument.

Here $p_1, p_2, \ldots p_n$ are called the premises of the argument and 'q' is called a conclusion of the argument.

It is a practice to write the above argument in following tabular form.

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

Consistent premises:- The premeses of the argument $p_1, p_2 \ldots p_n$ are said to be consistent if $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ is true in atleast one possible situation

Inconsistent premises:- The premesis of the argument $p_1, p_2 \ldots p_n$ are said to be inconsistent if $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ is false in all possible situations

Eg:- Show that the premises $p \to q$, $p \to r$, $q \to \sim r$, r are consistent

p	q	r	$\sim r$	$p \to q$	$p \to r$	$q \to \sim r$	$(p \to q) \wedge (p \to r) \wedge (q \to \sim r) \wedge r$
T	T	T	F	T	T	F	F
T	T	F	T	T	F	T	F
T	F	T	F	F	T	T	F
T	F	F	T	F	F	T	F
F	T	T	F	T	T	F	F
F	T	F	T	T	T	T	F
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	F

these premises are consistent if $(p \to q) \wedge (p \to r) \wedge (q \to \sim r) \wedge r$ is true atleast one situation //

Valid argument:- An argument with premises $p_1, p_2, \dots p_n$ and conclusion q is said to valid if whenever each of premises $p_1, p_2 \dots p_n$ is true, then the conclusion q is likewise true.

i.e the argument $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow q$ is valid when $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \Rightarrow q$. The premises are alway taken to be true where as the conclusion may be true or false.

The conclusion is true only in the case of a valid argument Otherwise it is said to be invalid ~~state~~ argument

To check the validy of the argument we use the following rules of inferensy.

(i) Rule of Conjuctive Simplication: for any two propositions p and q, if $p \wedge q$ is true then p is true i.e $\boxed{p \wedge q \Rightarrow p}$

(ii) Rule of disjunctive Simplication:- for any two propositions p and q, if p is true then $p \vee q$ is true i.e $\boxed{p \Rightarrow p \vee q}$

(iii) Rule of Syllogism:- for any three propositions p, q, r, if $p \rightarrow q$ is true and $q \rightarrow r$ is true then $p \rightarrow r$ is true

i.e $[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r)$ i.e
$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$$

(iv) Modus pones (Rule of detachment):- If p is true ~~then~~ and $p \rightarrow q$ is true then q is true i.e $\boxed{p \wedge (p \rightarrow q) \Rightarrow q}$

i.e
$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

(v) Modus Tollens:- If $p \rightarrow q$ is true and q is false then p is false i.e $\boxed{(p \rightarrow q) \wedge \sim q \Rightarrow \sim p}$ i.e
$$\begin{array}{l} p \rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

(vi) Rule of Disjunctive Syllogism:- If $p \vee q$ is true and p is false then q is true i.e $\boxed{[(p \vee q) \wedge \sim p] \Rightarrow q}$ i.e
$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore q \end{array}$$

(P) Test whether the following is a valid statement
"If sachin hits a century then he gets a free car
Sachin hit a century
$\therefore$ Sachin gets a free car

Soln:- Let p: Sachin hits a century q: Sachin gets a free car
Then, the given argument
$$\begin{array}{c} p \to q \\ \underline{p} \\ \therefore q \end{array}$$
This is the valid argument, by Modus pones Rule

(P) Test whether the following is a valid argument
If Sachin hits a century, then he gets a free car
Sachin gets a free car
$\therefore$ Sachin has hit a century

Soln:. Let p: Sachin hits a century q: Sachin gets a free car
Then, the given argument
$$\begin{array}{c} p \to q \\ \underline{q} \\ \therefore p \end{array}$$
We note that if $p \to q$ and q are true, there is no rule which asserts that p must be true. Indeed, p can be false when $p \to q$ and q are true
$\therefore$ The given argument is not a valid one. //

(P) Test the validy of the following argument
I will become famous or I will not become a musicn
I will become a musician
$\therefore$ I will become famous

Soln: Let p: I will become famous q: I will become a musian
Then, the given argument
$$\begin{array}{c} p \vee \sim q \\ \underline{q} \\ \therefore p \end{array}$$
is equivalent to
$$\begin{array}{c} q \to p \\ \underline{q} \\ \therefore p \end{array}$$
$[\because p \vee \sim q \Leftrightarrow \sim q \vee p \Leftrightarrow q \to p]$.

The view of the Modus pones Rule,
This argument is valid.

(P) Test whether the following is a valid argument
If I study then I do not fail in the examination
If I do not fail in the examination, my father gift a two-wheeler to me
$\therefore$ If I study then my father gifts a two-wheeler to me.

Soln:- Let p: I study q: I do not fail in the examination
r: My father gifts a two-wheeler to me
Then, the given argument
$$\begin{array}{c} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$
This is a valid argument, by Rule of syllogism.

P Test the validy of the following argument
If I study, I will not fail in the examination
If I do not watch TV in the evenings, I will study
I failed in the exam

∴ I must have watched TV in the evenings

Sol^n:- Let p: I study q: I fail in the examination
r: I watch TV in the evenings

Then, the given argument
$$\begin{array}{l} p \to \sim q \\ \sim r \to p \\ \hline q \\ \therefore r \end{array}$$
is equivalent to
$$\begin{array}{ll} q \to \sim p & (\because p \to \sim q \Leftrightarrow \sim\sim q \to \sim p) \\ \sim p \to r & (\because \sim r \to p \Leftrightarrow \sim p \to r) \\ \hline q & \\ \therefore r & \end{array}$$

This is equivalent to
$$\begin{array}{l} q \to r \\ \hline q \\ \therefore r \end{array}$$
(Using Rule of syllogism)

This argument is valid, by the Modus pony Rule.//

P Consider the following argument
I will get grade A in this course or I will not graduate
If I do not graduate, I will join the army
I got grade A

∴ I will not join the army. Is this a valid argument?

Sol^n:- Let p: I get grade A in the course q: I do not graduate
r: I joint the army

Then, the given argument,
$$\begin{array}{l} p \vee q \\ q \to r \\ \hline p \\ \therefore \sim r \end{array}$$

This argument is logically equivalent to
$$\begin{array}{ll} \sim q \to p & (\because p \vee q \Leftrightarrow q \vee p \Leftrightarrow \sim q \to p \\ \sim r \to \sim q & (\because \text{using contrapositive}) \\ \hline p & \\ \therefore \sim r & \end{array}$$

This is logically equivalent to
$$\begin{array}{l} \sim r \to p \\ \hline p \\ \therefore \sim r \end{array}$$
($\because$ Rule of syllogism)

This is not a valid argument//

(P) Test the validity of the following arguments

(i) $p \wedge q$
$p \to (q \to r)$
$\therefore r$

(ii) $p \to r$
$q \to r$
$\therefore (p \vee q) \to r$

(iii) $p \to q$
$r \to s$
$p \vee r$
$\therefore q \vee s$

(iv) $p \to q$
$r \to s$
$\sim q \vee \sim s$
$\therefore \sim (p \wedge r)$

Sol^n:- (i) Since $p \wedge q$ is true, both p and q are true
since p is true and $p \to (q \to r)$ is true, $q \to r$ has to be true
Since q is true and $q \to r$ is true, r has to be true.
Hence the given argument is valid.

(ii) $(p \to r) \wedge (q \to r) \Leftrightarrow (\sim p \vee r) \wedge (\sim q \vee r)$
$\Leftrightarrow (r \vee \sim p) \wedge (r \vee \sim q)$ (by Commutative law)
$\Leftrightarrow r \vee (\sim p \wedge \sim q)$ (by distributive law)
$\Leftrightarrow \sim (p \vee q) \vee r$ (by commutative and Demorgan
$\Leftrightarrow (p \vee q) \to r$
This logical equivalence shows that the given argument is valid

(iii) $(p \to q) \wedge (r \to s) \wedge (p \vee r)$
$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (\sim p \to r)$
$\Leftrightarrow (p \to q) \wedge (\sim p \to r) \wedge (r \to s)$, by Commutative law
$\Leftrightarrow (p \to q) \wedge (\sim p \to s)$, by rule of syllogism
$\Leftrightarrow (\sim q \to \sim p) \wedge (\sim p \to s)$, by Contrapositive
$\Leftrightarrow \sim q \to s \Leftrightarrow q \vee s$
This shows that the given argument is valid.

(iv) $(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s)$
$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (q \to \sim s)$
$\Leftrightarrow (p \to \sim s) \wedge (r \to s)$ (by Commutative law and Rule of syllogism)
$\Leftrightarrow (p \to \sim s) \wedge (\sim s \to \sim r)$ ($\because$ using Contrapositive)
$\Leftrightarrow (p \to \sim r)$ by the rule of syllogism
$\Leftrightarrow \sim p \vee \sim r \Leftrightarrow \sim (p \wedge r)$
This shows that the given argument is valid argument.

(P) P.T the validy of the following arguments

(i) $p \to r$
$\sim p \to q$
$q \to s$
$\therefore \sim r \to s$

(ii) $(\sim p \vee \sim q) \to (r \wedge s)$
$r \to t$
$\sim t$
$\therefore p$

(iii) $p \to (q \to r)$
$\sim q \to \sim p$
p
$\therefore r$

(iv) $\sim p \leftrightarrow q$
$q \to r$
$\sim r$
$\therefore p$

Sol^n:- (i) $(p \to r) \wedge (\sim p \to q) \wedge (q \to s)$

$\Leftrightarrow (p \to r) \wedge (\sim p \to s)$ (by rule of syllogism)

$\Leftrightarrow (\sim r \to \sim p) \wedge (\sim p \to s)$ (using contrapositive)

$\Leftrightarrow \sim r \to s$ [Rule of syllogism)

This proves the valid of the given argument

(ii) $[(\sim p \vee \sim q) \to (r \wedge s)] \wedge (r \to t) \wedge (\sim t)$

$\Leftrightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge (\sim r)$ [by Modus Tollens Rule]

$\Leftrightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge (\sim r \vee \sim s)$ [by the rule of disjunctive]

$\Leftrightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge \sim[(r \wedge s)]$ by D'Morgans law

$\Leftrightarrow \sim(\sim p \vee \sim q) \vee$, by the Modus Tollens Rule

$\Leftrightarrow p \wedge q$, by D'Morgan's Law

$\Leftrightarrow p$, by the Rule of Conjuctive simplication.

This proves that the validity of the given argument,,

(iii) $[p \to (q \to r)] \wedge [\sim q \to \sim p] \wedge p$

$\Leftrightarrow [\{p \to (q \to r)\} \wedge p] \wedge (\sim q \to \sim p)$

$\Leftrightarrow (q \to r) \wedge (\sim q \to \sim p)$, by Modus Pones Rule

$\Leftrightarrow (q \to r) \wedge (p \to q)$, by contrapositive

$\Leftrightarrow p \to r$, by Rule of syllogism

$\Leftrightarrow r$, because p is true (premise)

This proves that the given argument is valid.

(iv) $(\sim p \to q) \wedge (q \to r) \wedge (\sim r)$

$\Leftrightarrow (\sim p \leftrightarrow q) \wedge (\sim q)$ by Modus Tollens Rule

$\Leftrightarrow [(\sim p \to q) \wedge (q \to \sim p)] \wedge (\sim q)$

$\Leftrightarrow [(\sim p \to q) \wedge \sim q] \wedge (q \to \sim p)$

$\Leftrightarrow (\sim p \to q) \wedge \sim q$, by rule of conjuctive simplication

$\Leftrightarrow \sim(\sim p)$, by Modus Tollen's rule

$\Leftrightarrow p$

This proves that the given argument is valid //

(P) Prove the validity of the following arguments

(i)
$$\begin{array}{l} p \to q \\ q \to (r \wedge s) \\ \sim r \vee (\sim t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

(ii)
$$\begin{array}{l} u \to r \\ (r \to s) \to (p \vee t) \\ q \to (u \wedge s) \\ \sim t \\ \hline \therefore q \to p \end{array}$$

(iii)
$$\begin{array}{l} (\sim p \vee q) \to r \\ r \to (s \vee t) \\ \sim s \wedge \sim u \\ \sim u \to \sim t \\ \hline \therefore p \end{array}$$

Sol^n:-

(P) Consider the following argument

Aishwarya is playing tennis

If Aishwarya is playing tennis then she is not practicing her flute

If Aishwarya is not practising her flute, then her father will not buy her a car

---

∴ Aishwarya's father will not buy her a car

Is this valid argument?

Sol^n:- Let p: Aishwarya is playing tennis

q: Aishwarya is practicing her flute

r: Aishwarya's father will buy her a car

Then, the given argument may symbolically written as

$$\begin{array}{l} p \\ p \to \sim q \\ \sim q \to \sim r \\ \hline \therefore \sim r \end{array}$$

$$p \wedge (p \to \sim q) \wedge (\sim q \to \sim r)$$
$$\Rightarrow \sim q \wedge (\sim q \to \sim r)$$
$$\Rightarrow \sim r$$

(OR)

	Steps	Reason
{1}	1. p	premise
{2}	2. $p \to \sim q$	premise
{1,2}	3. $\sim q$	steps {1} and {2} and the rule of detachment
{4}	4. $\sim q \to \sim r$	premise
{1,2,4}	5. $\sim r$	steps {3} and {4} and the rule of detachment

This shows that the argument is valid //

(P) Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \sim H$, $\sim H \to (A \wedge \sim B)$ and $(A \wedge \sim B) \to (R \vee S)$

Sol^n.

	Steps	Reason
{1}	1. $(C \vee D) \to \sim H$	premise
{2}	2. $\sim H \to (A \wedge \sim B)$	premise
{1,2}	3. $(C \vee D) \to (A \wedge \sim B)$	steps {1} {2} and Rule of syllogism
{4}	4. $(A \wedge \sim B) \to (R \vee S)$	premise
{1,2,4}	5. $(C \vee D) \to (R \vee S)$	steps {3} and {4} and Rule of syllogism
{6}	6. $(C \vee D)$	premise
{1,2,4,6}	7. $R \vee S$	steps {5} and (6) and Modus ponens

∴ This shows that the given argument is valid //

Ⓟ Show that $[(P\to Q)\wedge(R\to S),\ (Q\to T)\wedge(S\to U),\ \sim(T\wedge U),\ P\to R] \Rightarrow \sim P$

Sol^n:-

	Steps	Reason
{1}	1. $(P\to Q)\wedge(R\to S)$	premise
{2}	2. $(Q\to T)\wedge(S\to U)$	premise
{1}	3. $P\to Q$	step (1) and rule of conjunction
{1}	4. $R\to S$	Step (1) and " "
{2}	5. $Q\to T$	step (2) and " "
{2}	6. $S\to U$	step (2) and " "
{1,2}	7. $P\to T$	steps (3) & (5) and rule of syllogism
{1,2}	8. $R\to U$	Steps (4), (6) and " "
{9}	9. $P\to R$	premise
{1,2,9}	10. $P\to U$	steps (8), (9) and " "
{1,2,9}	11. $P\to(T\wedge U)$	Steps (7), (10) and Logical equivalence $(p\to q)\wedge(p\to r)\Leftrightarrow p\to(q\wedge r)$
{12}	12. $\sim(T\wedge U)$	premise
{1,2,9,12}	13. $\sim P$	steps (11) (12) and Modus Tollens

∴ This shows that the argument is valid.

Ⓟ. Show that $S\vee R$ is tautologically implied by $(P\vee Q)\wedge(P\to R)\wedge(Q\to S)$

Sol^n:

	Steps	Reason
{1}	1. $P\to R$	premise
{2}	2. $P\vee Q$	premise
{2}	3. $\sim P\to Q$	step (2) and $p\to q\Leftrightarrow \sim p\vee q$
{4}	4. $Q\to S$	premise
{2,4}	5. $\sim P\to S$	steps (3), (4) and rule of syllogism
{1}	6. $\sim R\to\sim P$	step (1) and Contrapositive
{1,2,4}	7. $\sim R\to S$	Steps (5), (6) and Rule of syllogism

{1,2,4} 8. $R \vee S$ steps (7) and $(R \to S) \Leftrightarrow R \vee S$

{1,2,4} 9. $S \vee R$ step (8) and Commutative.

∴ The argument is valid //

(P). Show that 't' is a valid conclusion from the premises $p \to q$, $q \to r$, $r \to s$, $\sim s$ and $p \vee t$

Sol^n:

	Steps	Reason
{1}	1. $p \to q$	premise
{2}	2. $q \to r$	premise
{1,2}	3. $p \to r$	steps (1), (2) and rule of syllogism
{4}	4. $r \to s$	premise
{1,2,4}	5. $p \to s$	steps (3), (4) and rule of syllogism
{6}	6. $\sim s$	premise
{1,2,4,6}	7. $\sim p$	steps (5), (6) and Modus Tollens
{8}	8. $p \vee t$	premise
{1,2,4,6,8}	9. t	steps (7), (8) and rule of disjunctive

∴ The argument is valid //

(P) Determine the validity of the following argument

If two sides of a triangle are equal then two opposite angles are equal

Two sides of a triangle are not equal

∴ The opposite angles are not equal

Sol^n:- Let p : Two sides of a triangle are equal

q : The two opposite angles are equal

Then the symbolic form of the given premises are $p \to q$, $\sim p$ and the conclusion is $\sim q$ i.e.

$$\begin{array}{l} p \to q \\ \sim p \\ \hline \therefore \sim q \end{array}$$

There are two statement variables. so the truth table consist of 4 rows

Now we construct the truth table for $[(p \to q) \wedge (\sim p)] \to \sim q$

p	q	$\sim p$	$\sim q$	$p \to q$	$(p\to q)\wedge(\sim p)$	$[(p\to q)\wedge(\sim p)]\to\sim q$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

$\therefore$ This shows that $[(p\to q)\wedge(\sim p)]\to\sim q$ is not a tautology
Hence the conclusion $\sim q$ is not valid //

(b). Check the validity of the following argument
If today is Sunday then yesterday was saturday
Yesterday was saturday
$\therefore$ Today is sunday

Sol^n:- Let p: Today is Sunday q: yesterday was Saturday
The symbolical form of given argument is
$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

We construct the truth table for $[(p\to q)\wedge q]\to p$

p	q	$p\to q$	$(p\to q)\wedge q$	$[(p\to q)\wedge q]\to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

$\therefore$ This shows that $[(p\to q)\wedge q]\to p$ is not a tautology
Hence the given argument is not valid //

(P) Show that the following argument is valid

$$\begin{array}{l} p \to r \\ r \to s \\ t \vee \sim s \\ \sim t \vee u \\ \underline{\sim u} \\ \therefore \sim p \end{array}$$

Sol^n:-

	Steps	Reason
{1}	1. $p \to r$	premise
{2}	2. $r \to s$	premise
{1,2}	3. $p \to s$	steps (1), (2) and rule of syllogism
{4}	4. $t \vee \sim s$	premise
{4}	5. $\sim s \vee t$	step (4), commutative law
{4}	6. $s \to t$	step (5) & $\sim s \vee t \Rightarrow s \to t$
{1,2,4}	7. $p \to t$	step (3), (6), (7) & rule of syllogism
{8}	8. $\sim t \vee u$	premise
{8}	9. $t \to u$	step (8) and $\sim t \vee u \Leftrightarrow t \to u$
{1,2,4,8}	10. $p \to u$	step (7), (9) and rule of syllogism
{11}	11. $\sim u$	premise
{1,2,4,8,11}	12. $\sim p$	steps (10) (11) & Modus Tollens

∴ The argument is valid.//

∴ The argument is valid

(P) Show that the following argument is valid

If the band could not play rock music or the refreshments were not delivered on time, then the new year's party would have been cancelled and Aiswarya would have been angry.

If the party were cancelled, then refunds would have had to be made.

No refunds were made

∴ The band could play rock music

Sol^n: Symbolic form of the given argument

$$\begin{array}{l} (\sim p \vee \sim q) \to (r \wedge s) \\ r \to t \\ \underline{\sim t} \\ \therefore p \end{array}$$

(P) Establish the validity of the following arguments

(i) $[\{(\sim p \wedge q) \rightarrow r\} \wedge (r \rightarrow (s \vee t))] \wedge (\sim s \wedge \sim u) \wedge (\sim u \rightarrow \sim t)] \Rightarrow p$

(ii) $[p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \sim q)] \rightarrow (s \vee t)$

(P) Determine whether the conclusion 'c' is valid in the following premises : $P_1 : p \rightarrow (q \rightarrow r)$, $P_2 : p \wedge q$ $\quad c : r$

Sol:

	Steps	Reason
{1}	1. $p \rightarrow (q \rightarrow r)$	premise
{2}	2. $p \wedge q$	premise
{2}	3. p	step (2) and Rule of conjuctive
{1,2}	4. $q \rightarrow r$	steps (1) & (3) and Modus ponens
{2}	5. q	step (2) and conjuctive
{1,2}	6. r	step (4) (5) and Modus ponens

$\therefore$ This argument is valid //

(P) Show that 's' is a valid conclusion from the given premises
$p \rightarrow q, q \rightarrow r, r \rightarrow (s \wedge t), p$

(P) Establish the validity of the argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \sim r \vee (\sim t \vee u) \\ \underline{p \wedge t} \\ \therefore u \end{array}$$

(P) Show that the following argument is valid.

If Mrudula is a lawyer, then she is ambitious
If Mrudula is an early riser then she does not like Idlis
If Mrudula is ambitious then she is an early riser

$\therefore$ Mrudula is a lawyer then she does not likes idlis

Consistency of premises and Indirect Method of proof

A set of formulas $P_1, P_2, P_3 \cdots P_n$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $P_1, P_2, \cdots P_n$.

If for every assignment of the truth values to the atomic variables atleast one of the formulas $P_1, P_2, P_3, \cdots P_n$ is false so that their conjuction is identically false then the formulas $P_1, P_2 \cdots P_n$ are called inconsistent.

i.e a set of formulas $P_1, P_2, P_3 \cdots P_n$ is inconsistent if their conjunction implies a contradiction.

i.e $P_1 \wedge P_2 \wedge \cdots \wedge P_n \Rightarrow R \wedge \sim R$, where R is any formula. Note that $R \wedge \sim R$ is a contradiction and it is necessary and sufficient for the implication that $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ be a contradiction.

The notion of inconsistency is used in a procedure called proof of contradiction or reduction (or indirect method of proof. In order to show that a conclusion C follows logically from the premises $P_1, P_2, \cdots P_n$. We assume that C is false and consider $\sim C$ as an additional premise. If the new set of premises is inconsistent so that they imply a contradiction, then the assumption that $\sim C$ is true does not hold simultaneously with $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ being true

$\therefore$ C is true whenever $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is true

Thus C follows logically from the premises $P_1, P_2 \cdots P_n$.

(P) show that the following set of premises are inconsistent

$P \to Q$, $P \to R$, $Q \to \sim R$, P

Sol^n:

	Steps		Reason
{1}	1.	$P \to Q$	premise
{2}	2.	$P \to R$	premise
{1,2}	3.	$P \to (Q \wedge R)$	step (1), (2) and rule $(P \to Q) \wedge (P \to R) \Rightarrow P \to (Q \wedge R)$
{4}	4.	P	premise
{1,2,4}	5.	$Q \wedge R$	steps (3), (4) and rule of Modus ponens
{6}	6.	$Q \to \sim R$	premise
{1,2,4}	7.	Q	step (5) and conjuctive simplication
{1,2,4,6}	8.	$\sim R$	steps (6), (7) and Modulus ponens
{1,2,4}	9.	R	steps (5) and conjuctive simplication
{1,2,4,6}	10.	$R \wedge \sim R$	steps (8), (9) & Rule of conjunction.

Which is a contradiction

$\therefore$ The set of premises are inconsistent //

(P) Show that the following set of premises are inconsistent

$A \to (B \to C)$, $D \to (B \wedge \sim C)$, $A \wedge D$

Soln:

	Steps	Reason
{1}	1. $A \to (B \to C)$	premise
{2}	2. $A \wedge D$	premise
{2}	3. A	step (2) and Conjuctive Simplication
{1,2}	4. $B \to C$	steps (1),(3) & Modus ponens
{5}	5. $D \to (B \wedge \sim C)$	premise
{2}	6. D	step (2) and conjuctive Simplication
{2,5}	7. $B \wedge \sim C$	steps (5),(6) and Modus ponens
{1,2}	8. $\sim B \vee C$	step (4) and $B \to C \Leftrightarrow \sim B \vee C$
{1,2}	9. $\sim (B \wedge \sim C)$	step (8) and D'Morgan's law
{1,2,5}	10. $(B \wedge \sim C) \wedge \sim (B \wedge \sim C)$	steps (7),(9) ad Rule of Conjunction

which is contradiction

$\therefore$ The given set of premises are inconsistent //

(P) show that the following premises are inconsistent

If Rushika misses many classes through illness, then she fails high school

If Rushika fails high school, then she is uneducated

If Rushika reads a lot of books then she is not uneducated

If Rushika misses many classes through illness and reads a lot of books

Soln:- Let p: Rushika misses many classes

q: Rushika fails high school

r: Rushika is uneducated

s: Rushika reads a lot of books

The symbolic form of the given premises are $p \to q$, $q \to r$, $s \to \sim r$, $p \wedge s$

we p.T the premises are inconsistent

	Steps	Reason
{1}	1. $p \to q$	premise
{2}	2. $q \to r$	premise
{1,2}	3. $p \to r$	steps (1),(2) and Rule of Syllogism

{4}	4. $p \wedge s$	premise
{4}	5. p	step(4) and conjuctive simplication
{1,2,4}	6. r	steps (3),(5) and Modus ponens
{7}	7. $s \to \sim r$	premise
{4}	8. s	step (4) and conjuctive simplication
{4,7}	9. $\sim r$	steps (7), (8) and Modus ponens
{1,2,4,7}	10. $r \wedge \sim r$	steps (6), (9) ad Rule of conjunction

which is a contradiction

$\therefore$ The set of premises are inconsistent //

(P) Without construction a truth table, $A \vee C$ is not a valid consequence of $A \leftrightarrow (B \to C)$, $B \leftrightarrow (\sim A \wedge \sim C)$, $C \leftrightarrow (A \vee \sim B)$, B

Soln: Given that the premises are $A \leftrightarrow (B \to C)$, $B \to (\sim A \vee \sim C)$, $C \leftrightarrow (A \vee \sim B)$, B and the conclusion is $A \vee C$

	steps	Reason
{1}	1. $A \leftrightarrow (B \to C)$	premise
{2}	2. $C \leftrightarrow (A \vee \sim B)$	premise
{1}	3. $[A \to (B \to C)] \wedge [(B \to C) \to A]$	step(1) and $(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p)$
{1}	4. $A \to (B \to C)$	step(3) and conjuctive simplication
{1}	5. $(A \wedge B) \to C$	step(4) & $A \to (B \to C) \Leftrightarrow (A \wedge B) \to C$
{2}	6. $[C \to (A \vee \sim B)] \wedge [(A \vee \sim B) \to C]$	step(2) & $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
{2}	7. $C \to (A \vee \sim B)$	step (6) and conjuctive simplication
{1,2}	8. $(A \wedge B) \to (A \vee \sim B)$	step (5), (7) ad Rule of Syllogism
{1,2}	9. $\sim (A \wedge B) \vee (A \vee \sim B)$	step(8) and $p \to q \Leftrightarrow \sim p \vee q$
{1,2}	10. $(\sim A \vee \sim B) \vee (A \vee \sim B)$	step(9) & D'Morgan's law
{1,2}	11. $\sim B$	step (10) and $(\sim A \vee \sim B) \vee (A \vee \sim B) \Leftrightarrow \sim B$
{12}	12. B	~~step~~ premise
{1,2,12}	13. $B \wedge \sim B$	steps (11),(12) & Rule of conjuction

which is contradiction.

$\therefore A \vee C$ is not a valid consequence of $A \leftrightarrow (B \to C)$, $C \leftrightarrow (A \vee \sim B)$, $B \leftrightarrow (\sim A \vee \sim C)$, and B. //

METHOD OF INDIRECT PROOF

Indirect Method of proof:~ The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the indirect method of proof (or proof of contradiction (or reduction

Working procedure:-

step 1:- Introduce the negation of the desired conclusion as new premise i.e assume the conclusion 'c' is false and consider $\sim c$

~~step 2~~- as an additional premise

step 2:- From the additional or new premise, together with the given premises, derive a contradiction

i.e if the new set of premises is inconsistent, then they imply a contradiction. Therefore 'c' is true whenever $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is true

step 3:- Assert the desired conclusion as a logical inference from the premises. Thus c follows logically from the premises $P_1, P_2 \cdots P_n$

(P) prove by indirect method that, $(\sim Q), P \to Q, P \vee R \Rightarrow R$

Sol^n: Given that the premises are $\sim Q, P \to Q, P \vee R$

The desired result is R. Include its negation as a new premise

	Steps	Reason
{1}	1. $P \vee R$	premise
{2}	2. $\sim R$	additional premise
{1,2}	3. P	steps (1),(2) and disjunctive syllogism
{4}	4. $P \to Q$	premise
{1,2,4}	5. Q	steps (3),(4) and Modus ponens
{6}	6. $\sim Q$	premise
{1,2,4,6}	7. $Q \wedge \sim Q$	steps (5) (6) and contradiction

The additional premise, together with the given premises, leads to a contradiction

so R is derivable from $(\sim Q), P \to Q, P \vee R$

(P) Using indirect method of proof, derive $P \to \sim S$ from $P \to (Q \vee R)$, $Q \to \sim P$, $S \to \sim R$, P

Sol^n:- Given that the premises are $P \to (Q \vee R)$, $Q \to \sim P$, $S \to \sim R$, P

The desired result is $P \to \sim S$. Its negation is $\sim (P \to \sim S) \Leftrightarrow \sim(\sim P \vee \sim S)$

$\Leftrightarrow P \wedge S$

w

We include $P \wedge S$ is an additional premise

	Steps	Reason
{1}	1. $P \to (Q \vee R)$	premise
{2}	2. P	premise
{1,2}	3. $Q \vee R$	steps (1),(2) & Modus ponens
{4}	4. $S \to \sim R$	premise
{5}	5. $P \wedge S$	additional premise
{5}	6. S	step (5) & Conjuctive simplification
{4,5}	7. $\sim R$	step (4),(6) and Modus ponens
{1,2,4,5}	8. Q	steps (3),(7) and Disjunctive syllogism
{9}	9. $Q \to \sim P$	premise
{1,2,4,5,9}	10. $\sim P$	steps (8),(9) & Modus ponens
{1,2,4,5,9}	11. $P \wedge \sim P$	steps (2),(10) & contradiction

The additional premise $P \wedge S$ and the given premises together lead to a contradiction.

So, $\sim(P \wedge S)$ is derivable from $P \to (Q \vee R)$, $Q \to \sim P$, $S \to \sim R$, P

8. Establish the following argument by the method of contradiction

$$\begin{array}{l} \sim p \leftrightarrow q \\ q \to r \\ \underline{\sim r} \\ \therefore p \end{array}$$

Sol^n:- Given that the premises are $\sim p \leftrightarrow q$, $q \to r$ and $\sim r$, conclusion is p. We prove it by using method of contradiction. We assume that the negation of the conclusion is an additional premise i.e $\sim p$

	Steps	Reason
{1}	1. $\sim p \leftrightarrow q$	premise
{2}	2. $(\sim p \to q) \wedge (q \to \sim p)$	step(1) & $p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$
{2}	3. $\sim p \to q$	step (2) & Rule of conjunction
{4}	4. $q \to r$	premise
{2,4}	5. $\sim p \to r$	steps (3) (4) & rule of syllogism
{6}	6. $\sim p$	Additional premise
{2,4,6}	7. r	steps (5) (6) & modus ponens
{8}	8. $\sim r$	premise
{2,4,6,8}	9. $r \wedge \sim r$	steps (7),(8) & Rule of conjunction

Which is contradiction. we find that $[(\sim p \leftrightarrow q) \wedge (q \to r) \wedge \sim r \wedge \sim p] \Rightarrow F_0$

This requires the truth value of $[(\sim p \leftrightarrow q) \wedge (q \to r) \wedge (\sim r) \wedge (\sim p)]$ to be false, because $\sim p \leftrightarrow q$, $q \to r$ and $\sim r$ are the given premises each of these statements has the truth value true. Consequently for $[(\sim p \leftrightarrow q) \wedge (q \to r) \wedge \sim r \wedge \sim p]$ to have the truth value false, the statement $\sim p$ must have the truth value F. Therefore p has the truth value true and the conclusion p of the argument is true.//

(P) Establish the following argument by the method of proof by contradiction

$$\begin{array}{l} p \to (q \wedge r) \\ r \to s \\ \underline{\sim (q \wedge s)} \\ \therefore \sim p \end{array}$$

(P) show that the following using indirect method

$$\sim (P \to Q) \to \sim (R \vee S),\ [(Q \to P) \vee \sim R],\ R \Rightarrow P \leftrightarrow Q$$

Sol^n: Given that the premises are $\sim (P \to Q) \to \sim (R \vee S)$, $[(Q \to P) \vee \sim R]$, R and conclusion is $P \leftrightarrow Q$

We prove it by using indirect method. we introduce an additional premise $\sim (P \leftrightarrow Q)$ which is the negation of the conclusion of the argument.

$$\sim (P \leftrightarrow Q) \Leftrightarrow \sim [(P \to Q) \wedge (Q \to P)] \Leftrightarrow \sim (P \to Q) \vee \sim (Q \to P)$$

[illegible]. show that by indirect proof if $P \to (Q \land R)$, $(Q \lor S) \to T$ and $(P \lor S)$ then T (14)

[illegible]. show that, by indirect proof $E \to S, S \to H, A \to \sim H \Rightarrow \sim(E \land A)$

$$P \to Q,\ R \to S,\ P \lor R \Rightarrow Q \lor S$$

Quantifiers

Open statement:- The statements given for a variable 'x' are called open statements

(OR)

A declarative sentence is an open statement if

(i) it contains one or more variables

(ii) it is not a statement but

(iii) it becomes a statement when the variables in it are replaced by certain allowable choices.

Universe of Discourse:- If 'x' belongs to some set, which is called the universe of discourse

open statements containing a variable are denoted by $p(x)$, $q(x)$ etc

If U is the universe for the variable x in an open statement $p(x)$ and if $a \in U$, then the proposition got by replacing x by a in $p(x)$ and is denoted by $p(a)$.

Ex:- consider the open statement

$p(x)$: $x+2$ is an even integer

If the set of all integers is the universe for x in $p(x)$

$p(4)$: 6 is an even integer. We note that this proposition is true.

NOTE:- Like the compound propositions, compound open statements are formed by using logical connectives

If $p(x)$ and $q(x)$ are the open statements then

(i) $\sim p(x)$ is the negation of the open statement $p(x)$

(ii) $p(x) \wedge q(x)$ is the conjunction of ~~the~~ open statement $p(x)$, $q(x)$

(iii) $p(x) \vee q(x)$ " disjunction of " " " "

(iv) $p(x) \rightarrow q(x)$ " conditional " " " " "

(v) $p(x) \leftrightarrow q(x)$ " biconditional " " " " "

(1) Suppose the universe consists of all integers. Consider the following open statements: $p(x): x \leq 3$, $q(x): x+1$ is odd, $r(x): x > 0$

write down the truth values of the following.

(i) $p(2)$ (ii) $\sim q(4)$ (iii) $p(-1) \wedge q(1)$ (iv) $\sim p(3) \vee r(0)$

(v) $p(0) \rightarrow q(0)$ (vi) $p(1) \leftrightarrow \sim q(2)$ (vii) $p(4) \vee [q(1) \wedge r(2)]$

(vi) $p(2) \wedge [q(0) \vee \sim r(2)]$

Sol:- (i) $p(2)$ is the proposition "$2 \leq 3$", which is true

(ii) $q(4)$ is the proposition "$4+1$" is odd which is true. $\therefore \sim q(4)$ is false

(iii) $p(-1)$ is the proposition "$-1 \leq 3$" which is true and $q(1)$ is the proposition "$1+1$" is odd which is false $\therefore p(-1) \wedge q(1)$ is false.

(iv) $p(3)$ is true, so that $\sim p(3)$ is false and $r(0)$ is false
∴ $\sim p(3) \vee r(0)$ is false

(v) $p(0)$ is true and $q(0)$ is true . ∴ $p(0) \to q(0)$ is true

(vi) $p(1)$ is true and $q(2)$ is true . ∴ $p(1) \leftrightarrow \sim q(2)$ is false

(vii) $p(4)$ is true, ~~and~~ $q(1)$ is false and $r(2)$ is true
∴ $q(1) \wedge r(2)$ is false, so that $p(4) \vee [q(1) \wedge r(2)]$ is false

(viii) $p(2)$ is true, $q(0)$ is true and $r(2)$ is true.
∴ $q(0) \vee \sim r(2)$ is true, so that $p(2) \wedge [q(0) \vee \sim r(2)]$ is true.

Quantifier :- The phrase which indicates the quantity is called a quantifier

(OR)

The words 'all'; 'every'; "some"; there exists"; "None or one" are associated with the idea of quantity. Such words are called quantifiers.

Eg :- (i) Some men are tall
(ii) All birds have wings
(iii) There exists a real number whose square is equal to itself.

Types of Quantifiers :- There are two types of quantifiers
(i) Universal quantifier (ii) Existential quantifier

(i) Universal quantifier :- The quantifier 'all' is called universal quantifier and we shall denoted it by $\forall x$. We read it as "for all x", for any x" "for each x", (or "for every x", "for all x, y", For any x, y". "for every x, y" (or "for all x and y" is denoted by $\forall x \forall y$ (or $\forall x, y$

(ii) Existential quantifier :- The quantifier 'some' is the existential quantifier and we shall denote it by $\exists x$. We read it as "There exist an x" or 'for some x' or "for atleast one x"

Quantified statement :- A proposition involving quantifier is called the quantified statement

Ex :- (i) Each rectangle is a $\parallel^{gm}$
(ii) Some $\parallel^{gms}$ are squares

Free variable :- The variable, which are not bounded by the quantifiers are called the free variables

Ex :- $p(x), q(x)$, here x is free variable

Bounded variable :- The variable, which are bounded by the quantifiers are called the bounded variable

Ex :- $\exists x \in Z, p(x)$, $\forall x \in Z, q(x)$. Here x is a bounded variable

Predicate:— The part of the open statement $p(x)$ which makes $p(x)$ a proposition when x is replaced a chosen element of the universe is called a predicate. The logic involved in the analysis of predicates is referred to a predicate logic.

Ex:- Consider the open statement

$p(x)$: x is less than 10

Here the part " is less than 10 " is the predicate in $p(x)$ and 'x' is the subject to the statement.

A predicate requiring n $(n>0)$ names or objects is called an n-place predicate.

Ex: i) Aishwarya is a student

The predicate P: is a student is a 1-place predicate because it is ~~revealed~~ related to one object: Aishwarya

(ii) Sravya is taller than Mrudula

The predicate P: is less than is a 2-place predicate

The above statement can be represented as $T(n,a)$

Note that the order in which the names or object appear in the statements as well as in the predicate is important

In general, an n-place predicate requires 'n' names of objects to be inserted in fixed positions in order to obtain a statement. The position of these names is important.

If P is an n-place letter and $a_1, a_2, a_3 \ldots a_n$ are the names of objects then $P(a_1, a_2, \ldots a_n)$ is a statement.

Ex:- For the universe of all integers, let $p(x)$: $x>0$, $q(x)$: x is even $r(x)$: x is perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7.

Write down the following quantified statement in symbolic form

(i) Atleast one integer is even $\Rightarrow$ $\exists x, q(x)$

(ii) There exist a +ve integer that is even $\Rightarrow$ $\exists x\,[p(x) \wedge q(x)]$

(iii) Some even integers are divisible by 3 $\Rightarrow$ $\exists x, [q(x) \wedge s(x)]$

(iv) Every integer is either even or odd $\Rightarrow$ $\forall x, [q(x) \underline{\vee} \sim q(x)]$

(v) If x is even and a perfect square, Then x is not divisible by 3

$\Rightarrow$ $\forall x, [(q(x) \wedge r(x)) \rightarrow \sim s(x)]$

(vii) If x is odd or is not divisible by 7, then x is divisible by 3
$\Rightarrow \forall x, [\{\sim q(x) \vee \sim t(x)\} \to s(x)]$

Truth value of a Quantified statement :-

The following rules are emplyed for determining the truth value of a quantified statement.

Rule 1 :- The statement "$\forall x \in S, p(x)$" is true only when $p(x)$ is true for each $x \in S$

Rule 2 :- The statement "$\exists x \in S, p(x)$" is false only when $p(x)$ is false for every $x \in S$. [It is enough to exhibit one element a of S s.t $p(a)$ is false. This element a is called a counterexample]

Rule 3 :- If an open statement $p(x)$ is known to be true for all x in a universe 'S' and if $a \in S$, Then $p(a)$ is true. This is known as the Rule of Universal Simplication

Rule 4 :- If an open statement $p(x)$ is proved to be true for any x chosen from a set 'S' then the quantified statement $\forall x \in S, p(x)$ is true. This is known as the Rule of universal Generalization

Logical Equivalence :- Two quantified statements are said to be logical equivalent whenever they have the same truth values in all possible situations.

(i) $\forall x, [p(x) \wedge q(x)] \Leftrightarrow (\forall x, p(x)) \wedge (\forall x, q(x))$

(ii) $\exists x, [p(x) \vee q(x)] \Leftrightarrow [\exists x, p(x)] \vee [\exists x, q(x)]$

(iii) $\exists x, [p(x) \to q(x)] \Leftrightarrow \exists x, [\sim p(x) \vee q(x)]$

Rule for Negation of a Quantified statement :-

To construct the negation of a quantified statement, ~~forms an important part of a logical argument~~ change the quantifier from universal to existential and vice-versa and also replace the open statement by its negation.

i.e $\sim [\forall x, p(x)] \equiv \exists x, [\sim p(x)]$

and $\sim [\exists x, p(x)] \equiv \forall x, [\sim p(x)]$

statement	Negation
All true $\forall x, p(x)$	$\exists x, [\sim p(x)]$ atleast one false
atleast one false $\exists x, [\sim p(x)]$	$\forall x, p(x)$ all true
all false $\forall x, [\sim p(x)]$	$\exists x, p(x)$ atleast one true
atleast one true $\exists x, p(x)$	$\forall x, [\sim p(x)]$ all false

We observe that from the above table, to form the negation of a statement involving one quantifier we need only change the quantifier from universal to Existential (or from Existential to Universal and negate the statement which it quantifies.

Let us list these eight quantified statements and their abbreviated meaning

Statement	Abbriviated Meaning
$\forall x, p(x)$	All true
$\exists x, p(x)$	atleast one true
$\sim[\exists x, p(x)]$	None true
$\forall x, [\sim p(x)]$	all false
$\exists x, [\sim p(x)]$	atleast one false
$\sim[\exists x, (\sim p(x))]$	None false
$\sim[\forall x, p(x)]$	not all true
$\sim[\forall x, (\sim p(x))]$	not all false.

From the above table we conclude that

(i) 'all true' means the same as "non ~~true~~ false

(ii) "all-false" means the same as "~~atleast one false~~" "non-true"

(iii) "not all-true" means the same as "atleast one false"

(iv) "not all-false" means the same as "atleast one true"

(P) Consider the following statements with the set of all real numbers as the universe

$$p(x): x \geq 0,\ q(x): x^2 \geq 0,\ r(x): x^2-3x-4=0,\ s(x): x^2-3>0$$

Determine the trueness or falsity of the following statements

(i) $\exists x, p(x) \wedge q(x)$ (ii) $\forall x, p(x) \rightarrow q(x)$ (iii) $\forall x, q(x) \rightarrow s(x)$

(iv) $\forall x, r(x) \vee s(x)$ (v) $\exists x, p(x) \wedge r(x)$ (vi) $\forall x\ r(x) \rightarrow p(x)$

Soln:- (i) We note that, there exists a real number x for which both of $p(x)$ and $q(x)$ are true; for instance $x=1$

$\therefore \exists x, p(x) \wedge q(x)$ is a true statement

(ii) We note that, for every real number x, the statement $q(x)$ is true i.e. $q(x)$ cannot false for any real x. Hence $p(x) \rightarrow q(x)$ cannot be false ~~for~~ for any real x $\therefore \forall x, p(x) \rightarrow q(x)$ is true

(iii) We note that, $s(x)$ is false and $q(x)$ is true for $x=1$.

Thus $q(x) \to s(x)$ is false for $x=1$

i.e the statement $q(x) \to s(x)$ is not always true

Accordingly, $\forall x, q(x) \to s(x)$ is false.

(iv) We have $x^2-3x-4=(x-4)(x+1)$.

Hence $r(x)$ is true ~~for~~ only for $x=4$ or $x=-1$.

As such, $r(x)$ and $s(x)$ are false for $x=1$

Thus $r(x) \vee s(x)$ is not always true.

$\therefore \forall x, r(x) \vee s(x)$ is false

(v) We note that, for $x=4$, both of $p(x)$ and $r(x)$ are true.

$\therefore \exists x, p(x) \wedge r(x)$ is true.

(vi) We observe that $p(x)$ is false and $r(x)$ is true for $x=-1$.

Hence $r(x) \to p(x)$ is false for $x=-1$

Thus $r(x) \to p(x)$ is not always true

$\therefore \forall x, r(x) \to p(x)$ is false.

(P) Let $p(x): x^2-8x+15=0$, $q(x): x$ is odd, $r(x): x>0$

with the set of all integers as the universe. Determine the truth or falsity of each of the following statements

(i) $\forall x, [p(x) \to q(x)]$ (ii) $\forall x, [q(x) \to p(x)]$ (iii) $\exists x, [p(x) \to q(x)]$

(iv) $\exists x, (q(x) \to p(x)]$ (v) $\exists x, [r(x) \to p(x)]$ (vi) $\forall x, [\sim q(x) \to \sim r(x)]$

(vii) $\exists x, [p(x) \to \{q(x) \wedge p(x)\}]$ (viii) $\forall x, \{[p(x) \vee q(x)] \to r(x)\}$

Soln: (i) ~~T (ii) T (iii) F (iv) T (v) F (x=3b)~~

(i) T (ii) F $(x=7)$ (iii) T (iv) T (v) T (vi) F $(x=2)$ (vii) T

(viii) F $(x=-1)$

(P) Write down the converse, inverse and contrapositive of each of the following statements for which the set of all real numbers is the universe. Also, indicate their truth values

(i) $\forall x, [(x>3) \to (x^2>9)]$ (ii) $\forall x, [\{(x^2+4x-21)>0\} \to \{(x>3) \vee (x<-7)\}]$

Soln: (i) Converse : $\forall x, [(x^2>9) \to (x>3)]$, false

~~(i)~~ Inverse : $\forall x, [(x\le 3) \to (x^2 \le 9)]$, false

~~(i)~~ Contrapositive : $\forall x, [\{(x^2\le 9) \to (x\le 3)\}]$, True.

Ⓟ. Negate and Simplify each of the following

(i) $\exists x, [p(x) \vee q(x)]$ (ii) $\forall x, [p(x) \wedge \sim q(x)]$ (iii) $\forall x, [p(x) \to q(x)]$

(iv) $\exists x, [\{p(x) \vee q(x)\} \to r(x)]$ (v) $\exists x, [\sim p(x) \to \sim q(x)]$

(vi) $[\forall x, p(x)] \to [\exists (x) q(x)]$ (vii) $[\exists x, p(x)] \to [\exists x, \sim q(x)]$

Soln: By using the rule of negation for quantified statements and the Laws of logic

(i) $\sim [\exists x, \{p(x) \vee q(x)\}] \equiv \forall x, [\sim \{p(x) \vee q(x)\}]$
$\equiv \forall x, (\sim p(x) \wedge \sim q(x))$

(ii) $\sim [\forall x, \{p(x) \wedge \sim q(x)\}] = \exists x, [\sim \{p(x) \wedge \sim q(x)\}]$
$= \exists x, [\sim p(x) \vee q(x)]$

(iii) $\sim [\forall x, \{p(x) \to q(x)\}] \equiv \exists x, \sim [p(x) \to q(x)]$
$\equiv \exists x, [p(x) \wedge \sim q(x)]$

(iv) $\sim [\exists x, \{p(x) \vee q(x)\} \to r(x)] \equiv \forall x, [\sim \{p(x) \vee q(x)\} \to r(x)]$
$\equiv \forall x, [\{p(x) \vee q(x)\} \wedge \sim r(x)]$

(v) $\sim [\exists x, \{\sim p(x) \to \sim q(x)\}] \equiv \forall x, [\sim \{\sim p(x) \to \sim q(x)\}]$
$= \forall x, [\sim p(x) \wedge q(x)]$

(vi) $\sim [\{\forall x, p(x)\} \to \{\exists (x), q(x)\}] \equiv [\forall x, p(x)] \wedge \sim [\exists (x), q(x)]$
$= [\forall x, p(x)] \wedge [\forall (x), \sim q(x)]$

(vii) $\sim [\{\exists x, p(x)\} \to \{\exists x, \sim q(x)\}] \equiv [\exists x, p(x)] \wedge \sim [\exists x, \sim q(x)]$
$\equiv [\exists x, p(x)] \wedge [\forall x, q(x)]$

Ⓟ Write down the following proposition in symbolic form and find its negation

"All integers are rational numbers and some rational numbers are not integers"

Soln: Let $p(x)$: x is a rational number, $q(x)$: x is an integer

$\mathbb{Z}$: set of all integers $\mathbb{Q}$: set of all rational numbers

Then the symbolic form of given proposition is

$[\forall x \in \mathbb{Z}, p(x)] \wedge [\exists x \in \mathbb{Q}, \sim q(x)]$

Negation of this is $\sim [\{\forall x \in \mathbb{Z}, p(x)\} \wedge \{\exists x \in \mathbb{Q}, \sim q(x)\}]$
$\equiv [\exists x \in \mathbb{Z}, \sim p(x)] \vee [\forall x \in \mathbb{Q}, q(x)]$

(P) Write down the following proposition in symbolic form and find its negation

"If all triangles are right-angled then no triangle is equiangular"

Soln: Let T denote the set of all triangles. Also let

$p(x)$: x is right-angled ; $q(x)$: x is equiangular

Then the symbolic form is $[\forall x \in T, p(x)] \rightarrow [\forall x \in T, \sim q(x)]$

Negation of this is $[\forall x \in T, p(x)] \wedge [\exists x \in T, q(x)]$

In words, this reads as "All triangles are right-angled and some triangles are equiangular".

(P) Write down the following propositions in symbolic form and find its negation

(i) Every rational number is a real number and not every real number is a rational number [Ans: $\{\forall x, [p(x) \rightarrow q(x)]\} \rightarrow [\sim \{q(x) \rightarrow p(x)\}]$]

(ii) There is an integer which are not perfect square [$\therefore \forall x \in Z, \sim p(x)$]

(iii) Every element of group has an inverse ($\forall x \in G, p(x)$)

Logical Implication involving Quantifiers

A quantified statement P is said to be logically imply a quantified statement Q if Q is true whenever P is true. Then we write $P \Rightarrow Q$.

Given a set of quantified statements $P_1, P_2 \cdots P_n$ and Q, we say that Q is a valid conclusion from the premises $P_1, P_2 \cdots P_n$ (or) that $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \rightarrow Q$ is a valid argument if Q is true whenever each of $P_1, P_2 \cdots P_n$ is true or equivalently if $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \Rightarrow Q$

(P) prove the following (i) $\forall x, p(x) \Rightarrow \exists x, p(x)$

(ii) $\forall x, [p(x) \vee q(x)] \Rightarrow \forall x, p(x) \vee \exists x, q(x)$

Soln: Let S denote the universe

(i) $\forall x, p(x) \Rightarrow p(x)$ is true for every $x \in S$

$\Rightarrow p(a)$ is true for $x = a \in S$

$\Rightarrow p(x)$ is true for some $x \in S$

$\Rightarrow \exists x, p(x)$

(ii) $\forall x, [p(x) \vee q(x)] \Rightarrow p(x) \vee q(x)$ is true for every $x \in S$

$\Rightarrow \{p(x)$ is true for every $x \in S\} \vee \{q(x)$ is true for for every $x \in S\}$

$\Rightarrow \forall x, p(x) \vee q(a)$ is true for $a \in S$

$\Rightarrow \forall x, p(x) \vee \exists x, q(x)$

(P) P.T $[\forall x, p(x) \vee \forall x, q(x)] \Rightarrow \forall x, [p(x) \vee q(x)]$

Through a counter example, show that converse of this is not true?

Soln. Let 's' denote the universe. Take any $a \in S$.

~~Then any~~ Then $\forall x, [p(x) \vee q(x)]$ is true whenever $p(a) \vee q(a)$ is true.

i.e whenever $p(a)$ is true or $q(a)$ is true

i.e whenever $p(x)$ is true for any x or $q(x)$ is true for any x

i.e whenever $\forall x, p(x)$ is true or $\forall x, q(x)$ is true

i.e whenever $\forall x, p(x) \vee \forall x, q(x)$ is true.

This means that $[\forall x, p(x) \vee \forall x, q(x)] \Rightarrow \forall x, [p(x), \vee q(x)]$

Analyzing the converse

Let $p(x): x^2-4=0$, $q(x): x^2-1=0$, with $S=\{1,2\}$

We find that for $x=1$, $p(x)$ is false but $q(x)$ is true.

$\Rightarrow p(x) \vee q(x)$ is true

for $x=2$, $p(x)$ is true and $q(x)$ is false

$\Rightarrow p(x) \vee q(x)$ is true

Thus for every $x \in S$, $p(x) \vee q(x)$ is true

i.e $\forall x, [p(x) \vee q(x)]$ is true.

But $p(x)$ is not true for every $x \in S$ i.e $\forall x, p(x)$ is false

Likewise $\forall x, q(x)$ is false.

Consequently $[\forall x, p(x) \vee \forall x, q(x)]$ is false

Thus $\forall x, [p(x) \vee q(x)] \not\Rightarrow [\forall x, p(x) \vee \forall x, q(x)]$

The converse of the given implication is not true.

(P) prove that $\exists x, [p(x) \wedge q(x)] \Rightarrow \exists x, p(x) \wedge \exists x, q(x)$.

Is the converse true?

Soln. Let S denote the universe

$\exists x, [p(x) \wedge q(x)] \Rightarrow p(a) \wedge q(a)$ for some $a \in S$

$\Rightarrow p(a)$ for some $a \in S$ and $q(a)$ for some $a \in S$

$\Rightarrow \exists x, p(x) \wedge \exists x, q(x)$

Converse: We observe that $\exists x, p(x) \Rightarrow p(a)$ for some $a \in S$ and

$\exists x, q(x) \Rightarrow q(b)$ for some $b \in S$

(P) find whether the following is a valid argument for which the universe is the set of all students

No Engineering student is bad in studies
Anil is not bad in studies

$\therefore$ Anil is an Engineering student

Soln: Let $p(x)$: x is an engineering student

$q(x)$: x is bad in studies

Then the given argument in symbolic form is

$$\begin{array}{l} \forall x[p(x) \to \sim q(x)] \\ \sim q(a) \\ \hline \therefore p(a) \end{array}$$

We note that $\forall x, [p(x) \to \sim q(x)] \Rightarrow p(a) \to \sim q(a)$

by the rule of universal specification

$\therefore \forall x, [p(x) \to \sim q(x)] \wedge \sim q(a)$

$\Rightarrow [p(a) \to \sim q(a)] \wedge \sim q(a) \Rightarrow p(a)$

because $p(a)$ can be false when both of $p(a) \to \sim q(a)$ and $\sim q(a)$ are true

$\therefore$ The given argument is not valid//

(P) prove that the following argument is valid

(i)
$$\begin{array}{l} \forall x, [p(x) \to q(x)] \\ \forall x, [q(x) \to r(x)] \\ \hline \therefore \forall x, [p(x) \to r(x)] \end{array}$$

(ii)
$$\begin{array}{l} \forall x, [p(x) \to q(x)] \\ \forall x, [q(x) \to r(x)] \\ \sim r(c) \\ \hline \therefore \sim p(c) \end{array}$$

Soln: (i) Take any 'a' from the universe. Then

$[\forall x, \{p(x) \to q(x)\}] \wedge [\forall x, \{q(x) \to r(x)\}]$

$\Rightarrow [p(a) \to q(a)] \wedge [q(a) \to r(a)]$

$\Rightarrow p(a) \to r(a)$, by the rule of syllogism

$\Rightarrow \forall x, [p(x) \to r(x)]$, by the rule of universal Generalization

This prove that the argument is valid/

(ii) $\{\forall x, [p(x) \to q(x)]\} \wedge \{\forall x, [q(x) \to r(x)]\} \wedge \sim r(c)$

$\Rightarrow [\forall x, [p(x) \to r(x)]] \wedge \sim r(c)$

$\Rightarrow [p(c) \to r(c)] \wedge \sim r(c)$, by the rule of universal specification

$\Rightarrow \sim p(c)$ by the Modus Tollens Rule

$\therefore$ This proves that the given argument is valid//

(P) Verify the validity of the following argument

All women are mortal : $p(x)$

Deepthi is a women

$\therefore$ Deepthi is a mortal

Sol^n: Let $p(x)$: x is women, $Q(x)$: x is mortal, a : Deepthi

The symbolic form of the given argument is $\forall x, [P(x) \to Q(x)] \wedge P(a) \to Q(a)$

	Steps	Reason
{1}	1. $\forall x, [P(x) \to Q(x)]$	premise
{2}	2. $P(a)$	premise
{1}	3. $P(a) \to Q(a)$	step(1) and universal specification
{1,2}	4. $Q(a)$	step(2) (3) and Modus ponens

$\therefore$ The argument is valid //

(P) Given an argument which all establish the validity of the following inference.

All integers are rational numbers

Some integers are powers of 3

Therefore some rational numbers are powers of 3

Sol^n: Let $p(x)$: x is an integer, $Q(x)$: x is a rational number

$R(x)$: x is a power of 3

symbolic representation is
$$\begin{array}{l} \forall x, [P(x) \to Q(x)] \\ \exists x, [P(x) \wedge R(x)] \\ \hline \therefore \exists x, [Q(x) \wedge R(x)] \end{array}$$

	Steps	Reason
{1}	1. $\forall x, [P(x) \to Q(x)]$	premise
{1}	2. $p(a) \to Q(a)$	step(1) & universal specification
{3}	3. $\exists x, p(x) \wedge R(x)$	premise
{3}	4. $p(a) \wedge R(a)$	step(3) & universal specification
{3}	5. $p(a)$	step(4) and conjuctive
{1,5}	6. $Q(a)$	steps (2),(5) and Modus ponens
{3}	7. $R(a)$	step (4) and conjuctive
{1,3,5}	8. $Q(a) \wedge R(a)$	step(6),(7) & Rule of conjuctive
{1,3,5}	9. $\exists x, [Q(x) \wedge R(x)]$	step(8) & Rule Existential Generalization

$\therefore$ Argument is valid.

(P) Validity ~~the~~ of the following argument

All mathematics professors have studied calculus

Aishwarya is a mathematics professor

$\therefore$ Aishwarya has studied calculus

Soln: Let the universe is all people

$p(x)$: x is a mathematics professor, $q(x)$: x has studied calculus

Let 'a' represent the particular woman named Aishwarya

The symbolic form of given argument is $\forall x, [P(x) \to q(x)]$

$$\frac{p(a)}{\therefore q(a)}$$

Here the premises are $\forall x, (p(x) \to q(x))$, $p(a)$ and conclusion is $q(a)$

	Steps	Reason
{1}	1. $\forall x \{P(x) \to q(x)\}$	premise
{1}	2. $p(a) \to q(a)$	step ① & universal specification
{3}	3. $p(a)$	premise
{1, 3}	4. $q(a)$	step (2), (3) & Modus Ponens

$\therefore$ The argument is valid //

(P) Show that $\forall x, [P(x) \to Q(x)] \wedge \forall x, [Q(x) \to R(x)] \Rightarrow \forall x, [P(x) \to R(x)]$

(P) prove that $\forall x, [P(x) \vee Q(x)] \wedge [\forall x, \sim p(x)] \Rightarrow \exists x, q(x))$

Soln. Given that the premises are $\forall x, p(x) \to Q(x)$, $\forall x, \sim p(x)$ and the Conclusion is $\exists x, q(x)$

	Steps	Reason
{1}	1. $\forall x, p(x) \vee Q(x)$	premise
{1}	2. $P(a) \vee q(a)$	step ①, & universal specification
{3}	3. $\forall x, \sim p(x)$	premise
{3}	4. $\sim p(a)$	step ③ & universal specification
{1, 3}	5. $q(a)$	step (2)(4) & Disjunctive syllogism
{1, 3}	6. $\exists x, q(x)$	step (5) & Existential specification

Verified //

(P) Verify the validity of the following argument
Every living thing is a plant or an animal. Sranya's goldfish is alive and it is not a plant. All animals have heart.
Therefore, Sranya's goldfish has a heart.

Soln:- Let the Universe consist of all living things
Let $p(x)$: x is planet, $Q(x)$: x is an animal, $R(x)$: x has a heart
'a': Sranya's goldfish.

Symbolic form: $\forall x, [p(x) \vee Q(x)]$

$$\begin{array}{l} \forall x, [p(x) \vee Q(x)] \\ \sim p(a) \\ \forall x, [Q(x) \rightarrow R(x)] \\ \hline \therefore R(a) \end{array}$$

	Steps	Reason
{1}	1. $\forall x, (p(x) \vee Q(x))$	premise
{1}	2. $p(a) \vee Q(a)$	step① and universal specification
{3}	3. $\sim p(a)$	premise
{2,3}	4. $Q(a)$	steps (2)(3) & ~~Modus~~ Disjunctive syllogism
{5}	5. $\forall x, Q(x) \rightarrow R(x)$	premise
{5}	6. $Q(a) \rightarrow R(a)$	step⑤ & universal specification
{4,6}	7. $R(a)$	steps (4), (6) & Modus ponens

$\therefore$ The argument is valid //

(P) Let the universe is the all triangles in a plane.
$P(x)$: x has two sides of equal length
$q(x)$: x is an isosceles triangle
$r(x)$: x has two angles of equal measure.

Soln:- Let 'xyz' is a specific triangle with no two angles of equal measure and it will be denoted by 'a'
The symbolic form of given argument

$$\begin{array}{l} \sim r(a) \\ \forall x, [P(x) \rightarrow q(x)] \\ \forall x, [q(x) \rightarrow r(x)] \\ \hline \sim p(a) \end{array}$$

(P) Verify the validity of the following argument
"No junior or senior is enrolled in a physical education class
Rushika is enrolled in a physical education class
Therefor Rushika is not a senior.

(P) Verify the validity of the following arguments

(i) $\forall x, [p(x) \vee q(x)]$

$\exists x, \sim p(x)$

$\forall x, [\sim q(x) \vee r(x)]$

$\forall x, [s(x) \rightarrow \sim r(x)]$

$\therefore \exists x, \sim s(x)$

(ii) $\forall x, [p(x) \vee q(x)]$

$\forall x, [(\sim p(x) \wedge q(x)) \rightarrow r(x)]$

$\therefore \forall x, [\sim r(x) \rightarrow p(x)]$

(iii) $\forall x, [p(x) \rightarrow (q(x) \wedge r(x))]$

$\forall x, [p(x) \wedge s(x)]$

$\therefore \forall x, [r(x) \wedge s(x)]$

Principle of Mathematical Induction

Let $p(n)$ be a statement or proposition defined by the set of +ve integers $\mathbb{Z}^+$, such that it is either true or false for all $n \in \mathbb{Z}^+$. For the given statement $p(n)$, if we can p.T

(i) $p(n)$ is true for $n = n_0$ i.e $n_0 = 1, 2, 3 \ldots$

(ii) $p(n)$ is true for $n = k+1$, assuming that it is true for $n = k$

Then we can conclude that $p(n)$ is true for all natural numbers $n \geq n_0$.

(P) prove by the principle of mathematical induction

$$p(n) : 1\cdot 2 + 2\cdot 2^2 + \cdots + n\cdot 2^n = (n-1)2^{n+1} + 2$$

Soln: If $n=1$, The L.H.S of $P(n)$ is $1\cdot 2^1 = 2$ and

R.H.S of $P(n)$ is $(1-1)\cdot 2^{1+1} + 2 = 2$

Hence $P(n)$ is true for $n=1$

Let us assuming that $p(n)$ is true for $n=k$.

$$P(K) : 1\cdot 2 + 2\cdot 2^2 + \cdots + k\cdot 2^k = (k-1)\,2^{k+1} + 2 \rightarrow (1)$$

To P.T $p(n)$ is true for $n = k+1$

Adding the term $(k+1)2^{k+1}$ to both sides of (1), we get

$$1\cdot 2 + 2\cdot 2^2 + \cdots + k\cdot 2^k + (k+1)2^{k+1} = (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$

$$= 2^{k+1}[k-1+k+1] + 2$$

$$= 2^{k+1}\cdot 2k + 2$$

$$= k\cdot 2^{k+1+1} + 2$$

$$= [(k+1)-1]\,2^{k+1+1} + 2$$

This shows that if $p(n)$ is true for $n=k$, then it is also true for $n = k+1$

Hence by mathematical induction $p(n)$ is true for every integral value of n //

(P) (i) $P(n) : \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{1}{n+1}$

(ii) $p(n) : 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $P(n) : 1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \cdots + n(n+2) = \frac{n}{6}(n+1)(2n+7)$

(iv) $p(n) : 1^2 - 2^2 + 3^2 + \cdots + (-1)^{n+1} n^2 = \frac{(-1)^{n+1} n(n+1)}{2}$

(P) By the principle of mathematical induction, show that $3^{4n+2}+5^{2n+1}$ is a multiple of 14, for all positive integral value of n including zero.

Soln.: Let $p(n): 3^{4n+2}+5^{2n+1}$ be the multiple of 14

If $n=1$, then $p(1)=3^{4+2}+5^{2+1}=854=14\times 61$ which is multiple of 14

Assuming that the result is true for $n=k$. Then

$p(k)=3^{4k+2}+5^{2k+1}$ is multiple of 14 i.e. $14.t$

To P.T $p(n)$ is true for $n=k+1$

Replacing k by k+1 in $p(k)$, we get

$$3^{4(k+1)+2}+5^{2(k+1)+1}=3^{4k+6}+5^{2k+3}$$

$$=3^{4k+2}.3^4+5^{2k+1}.5^2$$

$$=3^{4k+2}(11+70)+5^{2k+1}(11+14)$$

$$=11(3^{4k+2}+5^{2k+1})+70\,3^{4k+2}+14.5^{2k+1}$$

$$=11.14t+14[5.3^{4k+2}+5^{2k+1}]$$

$$=14[11t+5.3^{4k+2}+5^{2k+1}]$$

which is multiple of 14. Hence the result is true for $n=k+1$

(P) (i) $a^{2n}-b^{2n}$ is divisible by '$a-b$'

(ii) $7^{2n}+16n$ is divisible by 64

(iii) $n(n^2-1)$ is divisible by 24

(iv) $10^{2n-1}+1$ is divisibly by 11 for each natural number

(v) $6^{n+2}+7^{2n+1}$ is divisible by 43 " " "

UNIT-3

Set theory, Relations and functions

Set:- A set is well-defined collection of distinct objects

Ex:- (i) Rivers of India

(ii) Students who speak either Hindi & English

The objects of set are called its elements or members. Usually sets are denoted by capital letters such as A, B, C --- and elements are denoted by small letters such as a, b, c, ---

If 'x' is an element of a set A, we write $x \in A$.

If 'x' is not an element of a set A, we write $x \notin A$

The symbol $\in$ is read as "belongs to" and $\notin$ is reads as "does not belongs to"

For describing a set, two methods are commonly used

(i) The tabulation method (ii) The Rule method

In the tabulation method, all elements of a set are written down within flower brackets

In the rule method, we specify the set by stating a characteristic property which all the elements of the set posseses and which no other objects possesses.

ie If S is set of all +ve odd integers, then S can be described

$S = \{1, 3, 5, 7 --- \}$ → Tabulation Method

$S = \{x / x$ is a +ve odd integer$\}$ → Rule method.

The Singleton set consisting of the element x described by the tabulation method and is denoted by $\{x\}$,

Null set:- The set which contains no elements at all. This set is called the empty set or null set and it is denoted by $\{\ \}$, ϕ

Equal sets:- Two sets A and B are said to be equal if they have precisely the same elements, The we write $A = B$

Ex:- If $A = \{1, 2, 3, 4\}$ and $B = \{x / x$ is a +ve integer with $x^2 < 20\}$ then $A = B$

Finite and Infinite sets:- A set is finite if it contains finite no. of element

Ex:- $A = \{1, 2, 3, 4, 5\}$

A set which contains infinite no. of elements is known as infinite set

Ex:- $N = \{1, 2, 3, 4 --- \}$

Subset :- Let A and B be two non-empty sets. The set A is a subset of B if and only if every element of A is an element of B.

In other words, the set A is a subset of B if $x \in A \Rightarrow x \in B$.

Symbolically this can be written as

$$A \subseteq B \text{ if } x \in A \Rightarrow x \in B$$

If the set A is not subset of the set B i.e if atleast one element of A doesnot belongs to B and we write $A \nsubseteq B$

Properties :-

(i) If the set A is a subset of the set B, then the set B is called superset of the set A

(ii) If the set A is a subset of the set B and the set B is a subset of the set A, then the sets A and B are said to be equal

i.e $A \subseteq B,\ B \subseteq A \Rightarrow A = B$

(iii) $A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq C$

Proper Subset :- The set A is a proper subset of the set B or A is properly contained in B, if and only if

(i) every element of the set A is also an element of set B i.e $A \subseteq B$

(ii) there is atleast one element in set B that is not in set A

If A is a proper subst of B, then we write $A \subset B$

Power set :- If 'S' is any set, then the set of all subsets of 'S' is called the power set of S and is denoted by $P(S)$

i.e $P(S) = \{A : A \subset S\}$. obviously ϕ and S are both members of $P(S)$.

Ex:- If $A = \{a, b\}$ then $P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

NOTE :- If A is a finite set of ~~n~~ 'n' elements, then the no. of elements of $P(A)$ is 2^n.

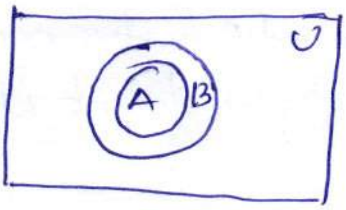
Universal set :- All the sets under discussion are assumed to be the subst of the fixed set. This set is called universal set. and it is denoted by 'U' (OR) E.

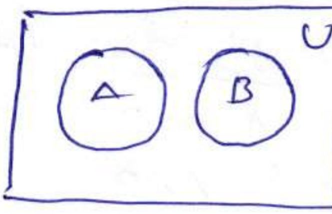
Ex :- A set of integers may be considered as a universal set for a set of odd or even integers.

Venn Diagrams:- A device is known as Venn-diagram is a pictorial notation of sets.

In venn diagram, a Universal set U is represented by the interior of the rectangle and each subset of U is represented by the circle inside the rectangle.

Ex:-



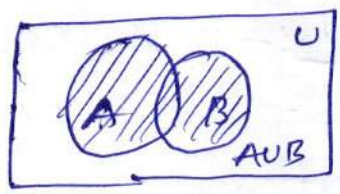


Operations on sets:-

Union of sets:- Let A and B be two non-empty sets. The union of A and B is the set of all elements which are either in A or in B or in both A and B. It is denoted by $A \cup B$

i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Venn diagram of $A \cup B$ is



property:-

1. The Union of sets is commutative i.e $A \cup B = B \cup A$

proof:- for proving $A \cup B = B \cup A$. we shall prove $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$

Let $x \in A \cup B$. then $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A \Rightarrow x \in B \cup A$

$\therefore A \cup B \subseteq B \cup A \rightarrow (i)$

Again let $y \in B \cup A$. then $y \in B \cup A \Rightarrow y \in B$ or $y \in A$

$\Rightarrow y \in A$ or $y \in B \Rightarrow y \in A \cup B$

$\therefore B \cup A \subseteq A \cup B \rightarrow (ii)$

from (i) & (ii) we have $A \cup B = B \cup A$

2. The union of sets associative i.e $A \cup (B \cup C) = (A \cup B) \cup C$

proof:- Let $x \in A \cup (B \cup C)$.

Then $x \in A \cup (B \cup C) \Rightarrow x \in A$ or $x \in B \cup C$

$\Rightarrow x \in A$ or $\{x \in B$ or $x \in C\}$

$\Rightarrow \{x \in A$ or $x \in B\}$ or $x \in C$

$\Rightarrow x \in A \cup B$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C \quad \therefore A \cup (B \cup C) = (A \cup B) \cup C$

3. The union of sets is idempotent. i.e $A \cup A = A$

4. If A and B are any sets then $A \subseteq A \cup B$ and $B \subseteq A \cup B$

5. If A is any set, then $A \cup \phi = A$, where ϕ is the null set

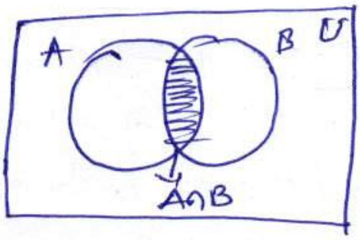
6. If A is any ~~set~~ subset of the universal set U, then $A \cup U = U$

7. If $A \subseteq B$, then $A \cup B = B$ and if $B \subseteq A$ then $A \cup B = A$

Intersection of sets:- Let A and B be two non-empty sets. The intersection of A and B is the set of all elements which are in both A and B. It is denoted by $A \cap B$

i.e $A \cap B = \{x / x \in A \text{ and } x \in B\}$

Venn diagram of $A \cap B$ is



Ex:- If $A = \{1, 2\}$ $B = \{2, 4\}$ then $A \cap B = \{2\}$

NOTE:- (i) $A \cap B \subseteq A \cup B$ (ii) $A \cap B \subseteq A$ (iii) $A \subseteq A \cup B$

properties:-

1. The intersection of sets is commutative i.e $A \cap B = B \cap A$

proof:- Let $x \in A \cap B$. Then $x \in A \cap B \Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in B$ and $x \in A \Rightarrow x \in B \cap A$

$\therefore A \cap B = B \cap A$

2. The intersection of sets is associative i.e $A \cap (B \cap C) = (A \cap B) \cap C$

proof:- Let $x \in A \cap (B \cap C)$. Then $x \in A \cap (B \cap C)$
$\Rightarrow x \in A$ and $x \in (B \cap C)$
$\Rightarrow x \in A$ and $[x \in B$ and $x \in C)$
$\Rightarrow \{x \in A$ and $x \in B\}$ and $x \in C$
$\Rightarrow \{x \in A \cap B\}$ and $x \in C$
$\Rightarrow x \in (A \cap B) \cap C$
$\therefore A \cap (B \cap C) = (A \cap B) \cap C$ //

3. The intersection of sets is idempotent i.e $A \cap A = A$

4. If A and B are any two sets, then $A \cap B \subseteq A$ and $A \cap B \subseteq B$

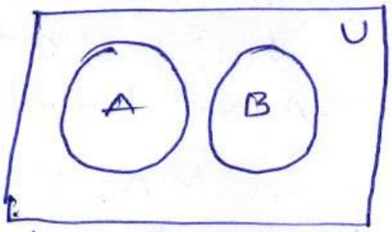
5. If A is any set then $A \cap \phi = \phi$, where ϕ is the null set

6. If A is any subst of the universal set, then $A \cap U = A$

Disjoint sets :- Let A and B be two non-empty sets. These two sets are said to be disjoint or mutually Exclusive if they have no common elements i.e their intersection is a null set

i.e $A \cap B = \phi$

Venn diagram of disjoint sets is

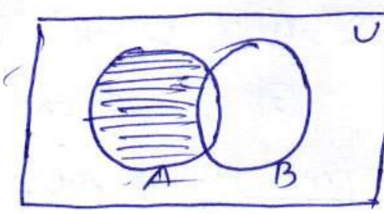


Ex :- If $A = \{1,2,3\}$, $B = \{a,b,c\}$ then $A \cap B = \{\phi\}$

Difference of Two sets :- If A and B are any two sets, then the difference of A and B i.e A-B is the set consisting elements which belongs to A but does not belongs to B. It is denoted by A-B

i.e $A-B = \{x \mid x \in A \text{ and } x \notin B\}$

Venn diagram of A-B is



Ex :- If $A = \{0,2,4,9\}$ and $B = \{0,3,6,8,9\}$ then $A-B = \{2,4\}$, $B-A = \{3,6,8\}$

$\therefore A-B \neq B-A$

properties :- 1) $A-A = \phi$ 2) $A-\phi = A$ 3) $A-B \subseteq A$

4) $A-B$, $A \cap B$ and $B-A$ are mutually disjoint

5) $(A-B) \cap B = \phi$ 6) $(A-B) \cup A = A$

proofs :- 4) $(A-B) \cap (A \cap B) = \phi$

Let $x \in (A-B) \cap (A \cap B) \Leftrightarrow x \in (A-B)$ and $x \in A \cap B$

$\Leftrightarrow (x \in A \text{ and } x \notin B)$ and $(x \in A \text{ and } x \notin B)$

$\Rightarrow x \in A$ and $x \in \phi$

$\Leftrightarrow x \in A \cap \phi \Leftrightarrow x \in \phi$

$||^{ly}$, we can P.T $(B-A) \cap (A \cap B) = \phi$

Now to P.T $(A-B) \cap (B-A) = \phi$

Let $x \in (A-B) \cap (B-A) \Leftrightarrow x \in (A-B)$ and $x \in (B-A)$

$\Leftrightarrow (x \in A \text{ and } x \notin B)$ and $(x \in B \text{ and } x \notin A)$

$\Rightarrow x \in \phi$

i.e there is no element satisfying both $x \in A$ and $x \notin A$

Hence $(A-B)$, $(B-A)$ and $(A \cap B)$ are disjoint sets //

5) To prove $(A-B) \cap B = \phi$

Let $x \in (A-B) \cap B \Leftrightarrow x \in (A-B)$ and $x \in B$

$\Leftrightarrow (x \in A \text{ and } x \notin B)$ and $x \in B$ $\quad \therefore (A-B) \cap B = \phi$

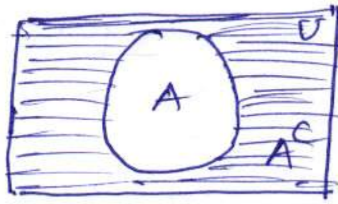
$\Leftrightarrow x \in A$ and $x \in \phi \Leftrightarrow x \in \phi$

Complement of a set:- Let A be any set. The complement of A is defined as the set of elements that are in the universal set but not in A.

Thus, if U is the universal set, the complement of A is the set $U-A$ and it denoted by A' or A^c

i.e. $A^c = U - A = \{x \mid x \in U \text{ and } x \notin A\} = \{x \mid x \notin A\}$

Venn diagram of Complement of A is



Eg:- If $N = \{1, 2, 3, 4 \dots\}$ is the universal set and $A = \{1, 3, 5, 7, \dots\}$ then

$A^c = N - A = \{2, 4, 6, 8, \dots\}$

NOTE ① If $U^c = R$, then $Q^c =$ The set of irrational numbers

② $U^c = \phi$ and $\phi^c = U$

properties:- 1) $A \cup A^c = U$ 2) $A \cap A^c = \phi$ 3) $U^c = \phi$ 4) $\phi^c = U$ 5) $(A^c)^c = A$ 6) $(A-B) = A \cap B^c$ 7) If $A \subseteq B$, then $A \cup (B-A) = B$.

Proof: 1) Since every set is a subset of the universal set, we have

$A \cup A^c \subseteq U \rightarrow$ (i)

Now let $x \in U \Rightarrow x \in A$ or $x \in A^c \Rightarrow x \in A \cup A^c$

$\therefore U \subseteq A \cup A^c \rightarrow$ (ii)

from (i) & (ii), we have $A \cup A^c = U$

2) Let $x \in A \cap A^c \Rightarrow x \in A$ and $x \in A^c$ [$\because$ no element x satisfying both $x \in A$ and $x \in A^c$]

$\Rightarrow x \in \phi$

$\therefore A \cap A^c = \phi$

6) Let $x \in (A-B) \Leftrightarrow x \in A$ and $x \notin B$

$\Leftrightarrow x \in A$ and $x \in B^c \Leftrightarrow x \in A \cap B^c$

Hence $A - B = A \cap B^c$

7) Given $A \subseteq B$, then to prove $A \cup (B-A) = B$

Let $x \in A \cup (B-A) \Leftrightarrow x \in A$ or $x \in (B-A)$

$\Leftrightarrow x \in A$ or $\{x \in B$ and $x \notin A\}$

$\Leftrightarrow \{x \in A$ or $x \in B\}$ and $\{x \in A$ ~~and~~ or $x \in A^c\}$

$\Leftrightarrow x \in A \cup B$ and $x \in U$ [$\because A \cup A^c = U$]

$\Leftrightarrow x \in A \cup B$

$\Leftrightarrow x \in B$

$\therefore A \cup (B-A) = B$

Distributive laws :-

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

proof :- Let $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in (B \cap C)$

$\Leftrightarrow x \in A$ or ($x \in B$ and $x \in C$)

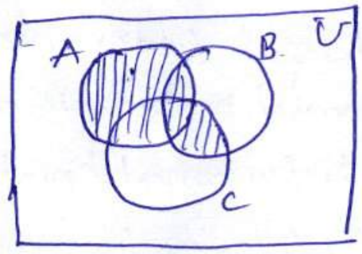
$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ and $x \in C$)

$\Leftrightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

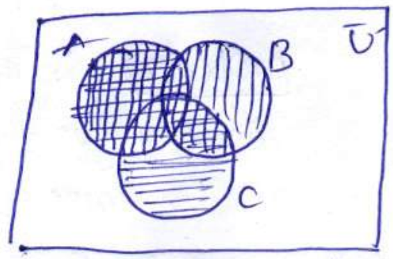
$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Venn diagram



$A \cup (B \cap C)$



$(A \cup B) \cap (A \cup C)$

2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

proof :- Let $x \in A \cap (B \cup C) \Leftrightarrow$ ~~$(A \cap B) \cup (A \cap C)$~~ $x \in A$ and $x \in (B \cup C)$

$\Leftrightarrow x \in A$ and $\{x \in B$ or $x \in C\}$

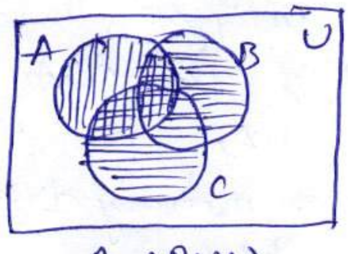
$\Leftrightarrow \{x \in A$ and $x \in B\}$ or $\{x \in A$ and $x \in C\}$

$\Leftrightarrow [x \in A \cap B]$ or $[x \in A \cap C]$

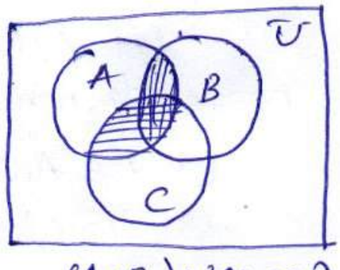
$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Venn Diagram:



$A \cap (B \cup C)$

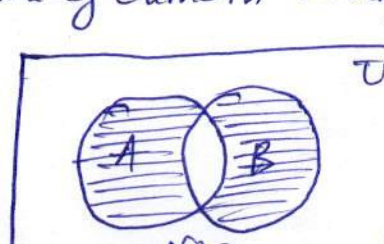


$(A \cap B) \cup (A \cap C)$

Symmetric Difference of sets :

Let A & B be two non-empty sets. Then the symmetric difference of A and B, denoted by $A \oplus B$, is the set containing elements which either belongs to A or B but not to both.

i.e. $A \oplus B = (A \cup B) - (B \cap A)$



$A \oplus B$

Ex: : If $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$

then $A \oplus B = (A \cup B) - (B \cap A)$

$= \{1, 2, 3, 4, 5\} - \{3\} = \{1, 2, 4, 5\}$

properties :- 1) $A \oplus B = B \oplus A$

2) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

3) $A \oplus A = \phi$ 4) $A \oplus (A \cap B) = \phi$

5) $A \oplus (A \cap B) = A \cup B$ 6) $(A \oplus B) \cup (A \cap B) = A \cup B$

7) $A \oplus B = (A - B) \cup (B - A)$ 8) $A \oplus B = \phi \Leftrightarrow A = B$

fundamental products: The fundamental product of 'n' distinct sets $A_1, A_2 \cdots A_n$ is a set defined as $A_1^* \cap A_2^* \cap \cdots \cap A_n^*$

where $A_i^* (i = 1, 2, \cdots n)$ is either A_i or A_i^C

properties: (i) The total such fundamental products are 2^n

(ii) Any two fundamental products are disjoint

(iii) The union of all the fundamental products is the Universal set U.

Eg:- Consider the following fundamental products of three distinct sets A, B, and C i.e $2^3 = 8$

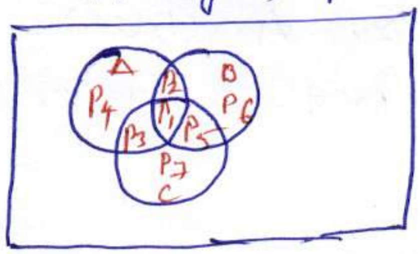
$P_1 = A \cap B \cap C$, $P_2 = A \cap B \cap C^c$

$P_3 = A \cap B^c \cap C$, $P_4 = A \cap B^c \cap C^c$

$P_5 = A^c \cap B \cap C$ $P_5 = A^c \cap B \cap C^c$

$P_7 = A^c \cap B^c \cap C^c$, $P_8 = A^c \cap B^c \cap C^c$

Venn diagram of fundamental products



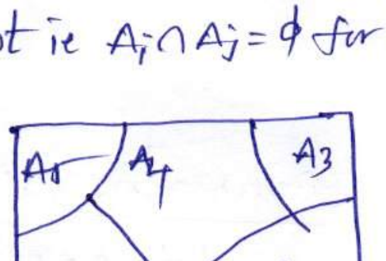
partition of sets:- A bigger set can be partitioned into smaller non-overlapping, non-empty subsets in order to make the task easy for analysing each part separately.

Let A be a non-empty set. The partition of A is any set of non-empty, non-overlapping subsets $A_1, A_2 \cdots A_n$ s.t

(i) $A = A_1 \cup A_2 \cup \cdots \cup A_n$

(ii) The subsets A_i are mutually disjoint i.e $A_i \cap A_j = \phi$ for $i \neq j$

venn diagram of partition of sets



Eg:- Let $A = \{a, b, c\}$ then $A_1 = \{a\}$

$A_2 = \{b, c\}$ are partitions of A.

Algebra of Sets and Duality

1. $(A\cup B)^c = A^c \cap B^c$

Proof:- Let $x \in (A\cup B)^c \Leftrightarrow x \in U$ and $x \notin (A\cup B)$

$\Leftrightarrow x \in U$ and $(x \notin A$ or $x \notin B)$

$\Leftrightarrow (x \in U$ but $x \notin A)$ and $(x \in U$ but $x \notin B)$

$\Leftrightarrow x \in A^c$ and $x \in B^c$

$\Leftrightarrow x \in A^c \cap B^c$

Hence $(A\cup B)^c = A^c \cap B^c$

2. $(A\cap B)^c = A^c \cup B^c$

Proof:- Let $x \in (A\cap B)^c \Leftrightarrow x \notin (A\cap B)$

$\Leftrightarrow x \notin A$ or $x \notin B$

$\Leftrightarrow x \in A^c$ or $x \in B^c$

$\Leftrightarrow x \in A^c \cup B^c$

Hence $(A\cap B)^c = A^c \cup B^c$

3. $A-(B\cup C) = (A-B)\cap(A-C)$

Proof:- Let $x \in A-(B\cup C) \Leftrightarrow x \in A$ and $x \notin B\cup C$

$\Leftrightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Leftrightarrow (x \in A$ but $x \notin B)$ and $(x \in A$ but $x \notin C)$

$\Leftrightarrow x \in (A-B)$ and $x \in (A-C)$

$\Leftrightarrow x \in (A-B)\cap(A-C)$

Hence $A-(B\cup C) = (A-B)\cap(A-C)$

4. $A-(B\cap C) = (A-B)\cup(A-C)$

Q. If A and B are any two sets, then (i) $A\cap(B-A)=\phi$ (ii) $(A-B)\cap B=\phi$

Sol^n:- (i) Let $x \in A\cap(B-A)$

$\Leftrightarrow x \in A$ and $x \in (B-A)$

$\Leftrightarrow x \in A$ and $\{x \in B$ and $x \notin A\}$

$\Leftrightarrow (x \in A$ and $x \notin A)$ and $x \in B$

$\Leftrightarrow x \in \phi$ and $x \in B$

$\Leftrightarrow x \in \phi$

Hence $A\cap(B-A)=\phi$

(ii) Let $x \in (A-B)\cap B$

$\Leftrightarrow x \in (A-B)$ and $x \in B$

$\Leftrightarrow (x \in A$ and $x \notin B)$ and $x \in B$

$\Leftrightarrow x \in A$ and $(x \notin B$ and $x \in B)$

$\Leftrightarrow x \in A$ and $x \in \phi$

$\Leftrightarrow x \in \phi$

Hence $(A-B)\cap B=\phi$ //

(P) If A and B are any two sets then (i) $A \cup B = (A-B) \cup B$
(ii) $A-B = A \cap B^C$ (iii) $A-B = B^C - A^C$ (iv) $A-(A \cap B) = A \cap B$

The inclusion and Exclusion principle

The no. of elements in finite sets such as $A \cup B$, $A \cap B$, etc are obtained by adding as well as excluding certain terms. This method of finding the no. of elements in a finite set (also called Cardinal number) is known as inclusion_exclusion principle. The Cardinal no. of a set A is denoted by $n(A)$

Eg:- If $A = \{a, b, c, d\}$ then $n(A) = 4$

Important Results:-

1) $n(A \cup B) \leq n(A) + n(B)$ 2) $n(A \cap B) \leq \min\{n(A), n(B)\}$

3) $n(A \oplus B) = n(A) + n(B) - 2n(A \cap B)$ 4) $n(A - B) \geq n(A) - n(B)$

5) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

6) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

If A, B, C are mutually disjoint sets, then $n(A \cup B \cup C) = n(A) + n(B) + n(C)$

7) $n(A^c) = n(U) - n(A)$

8) $n(A) = n[(A \cap B^c) \cup (A \cap B)] = n[(A - B) \cup (A \cap B)] = n(A - B) + n(A \cap B)$

9) $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

(P) In a group of 200 people, each of whom is atleast accountant or management consultant or sales manager, it was found that 80 are accountants, 110 are management consultants and 130 are sales managers, 25 are accountants as well as sales mangers, 70 are management consultants as well as sales managers, 10 are accountant, management consultants as well as sales managers. And the no. of those people who are accountant as well as management consultants but not sales mangers.

Sol. Suppose A, M and S denote the set of accountants, management consultants and sales managers resply.

Given that $n(A) = 80$, $n(M) = 110$, $n(S) = 130$, $n(A \cap S) = 25$

$n(M \cap S) = 70$ and $n(A \cap M \cap S) = 10$

We should find $n(A \cap M \cap S^c)$

W.K.T $n(A \cup M \cup S) = n(A) + n(M) + n(S) - n(A \cap M) - n(M \cap S) - n(A \cap S) + n(A \cap M \cap S)$

$\Rightarrow 200 = 80 + 110 + 130 - n(A \cap M) - 70 - 25 + 10$

$\Rightarrow n(A \cap M) = 35$

Now $n(A \cap M) = n(A \cap M \cap S^c) + n(A \cap M \cap S)$

$\Rightarrow 35 = n(A \cap M \cap S^c) + 10 \Rightarrow n(A \cap M \cap S^c) = 25$//

(P) A T.V survey shows that 60 percent people see programme, 50% see programme B, 50% see programm C, 30% see programme A and B, 20% see programme B and C, 30% see programme A and C and 10% donot see any programme. Find

(i) what percent see programme A, B and C ?

(ii) What persent see exactly two programmes?

Soln.- Suppose X, Y and Z denote the set of people who see programmes A, B and C resply.

Given that $n(X)=60$, $n(Y)=50$, $n(Z)=50$, $n(X\cap Y)=30$, $n(Y\cap Z)=20$
$n(X\cap Z)=30$, $n[(X\cup Y\cup Z)^c]=10$

(i) Let $n(X\cup Y\cup Z)+n[(X\cup Y\cup Z)^c]=100$. Then

$$n(X\cup Y\cup Z)=100-n[(X\cup Y\cup Z)^c]=100-10=90$$

But $n(X\cup Y\cup Z)=n(X)+n(Y)+n(Z)-n(X\cap Y)-n(Y\cap Z)-n(X\cap Z)+n(X\cap Y\cap Z)$

$$90=60+50+50-30-20-30+n(X\cap Y\cap Z)$$

$$\Rightarrow n(X\cap Y\cap Z)=90-80=10$$

Hence 10% people see programme A, B and C //

(ii) Since the set of people who see programme A and B but not C i.e $X\cap Y\cap Z^c$ and the set of people who see all the programmes A, B and C i.e $X\cap Y\cap Z$ are disjoint sets

$$\therefore n(X\cap Y\cap Z^c)+n(X\cap Y\cap Z)=n(X\cap Y)$$

$$\Rightarrow n(X\cap Y\cap Z^c)=n(X\cap Y)-n(X\cap Y\cap Z)=30-10=20$$

$||^y$ $n(X\cap Y^c\cap Z)+n(X\cap Y\cap Z)=n(X\cap Z)$

$$\Rightarrow n(X\cap Y^c\cap Z)=n(X\cap Z)-n(X\cap Y\cap Z)=30-10=20$$

and $n(X^c\cap Y\cap Z)+n(X\cap Y\cap Z)=n(Y\cap Z)$

$$\Rightarrow n(X^c\cap Y\cap Z)=n(Y\cap Z)-n(X\cap Y\cap Z)=20-10=10$$

Thus, the percentage of people who see exactly two programmes

$$=20+20+10=50$$

(iii) $n(X\cap Y^c\cap Z^c)=n(X)-n(X\cap Y)-n(X\cap Z)+n(X\cap Y\cap Z)$

$$=60-30-30+10=10$$

Thus, the percentage of people who see programme A only is 10.

(P) Out of 450 students in a school, 193 students read science and 200 students read Commerce, 80 students read neither. Find out how many read both. (7)

Soln:- Suppose A and B denote the set of students who read science and Commerce resply.

Given that $n(A)=193$, $n(B)=200$, $n(U)=450$, $n(A^c \cap B^c)=80$

Now we should find $n(A\cap B)$

since $A^c \cap B^c = (A\cup B)^c$ $\therefore n(A\cup B)^c = 80$

$n(A\cup B)^c = n(U) - n(A\cup B)$

$\Rightarrow 80 = 450 - n(A\cup B) \Rightarrow n(A\cup B) = 450-80 = 370$

W.k.T $n(A\cup B) = n(A)+n(B)-n(A\cap B)$

$\Rightarrow 370 = 193+200 - n(A\cap B) \Rightarrow n(A\cap B) = 23$ //

(P) In a survey concerning the smoking habits of consumers, it was found that 55% smoke cigarette A, 50% smoke B, 42% smoke C 28% smoke A and B, 20% smoke A and C, 12% smoke B and C and 10% smoke all three cigarettes

(i) what percentage do not smoke? $n(A\cup B\cup C)^c = 100 - n(A\cup B\cup C)$

(ii) what percentage smoke exactly two brands of cigarettes?

(P) Determine the no. of integers b/w 1 and 250 that are divisible by any of the integers 2, 3, 5 and 7

Soln:- Suppose A_1, A_2, A_3 and A_4 denote the set of integers b/w 1 and 250 that are divisible by the integers 2, 3, 5 and 7 resply.

Given that $n(A_1) = 250/2 = 125$, $n(A_2) = \frac{250}{3} = 83$, $n(A_3) = \frac{250}{4} = 50$

$n(A_4) = \frac{250}{7} = 35$

We have $n(A_1\cap A_2) = \frac{250}{2}\times 3 = 41$; $n(A_1\cap A_3) = \frac{250}{2}\times 5 = 25$

$n(A_1\cap A_4) = \frac{250}{2}\times 7 = 17$, ; $n(A_2\cap A_3) = \frac{250}{3}\times 5 = 16$

$n(A_2\cap A_4) = \frac{250}{3}\times 7 = 11$; $n(A_3\cap A_4) = \frac{250}{5}\times 7 = 7$

and $n(A_1\cap A_2\cap A_3) = \frac{250}{2}\times 3\times 5 = 8$; $n(A_1\cap A_2\cap A_4) = \frac{250}{2}\times 3\times 7 = 5$

$n(A_1\cap A_3\cap A_4) = \frac{250}{2}\times 5\times 7 = 3$; $n(A_2\cap A_3\cap A_4) = \frac{250}{3}\times 5\times 7 = 2$

$n(A_1\cap A_2\cap A_3\cap A_4) = \frac{250}{2}\times 3\times 5\times\times 7 = 1$

Hence $n(A_1\cup A_2\cup A_3\cup A_4) = n(A_1)+n(A_2)+n(A_3)+n(A_4) - n(A_1\cap A_2)$
$- n(A_1\cap A_3) - n(A_1\cap A_4) - n(A_2\cap A_3) - n(A_2\cap A_4)$
$- n(A_3\cap A_4) + n(A_1\cap A_2\cap A_3) + n(A_1\cap A_2\cap A_4)$
$+ n(A_1\cap A_3\cap A_4) + n(A_2\cap A_3\cap A_4) - n(A_1\cap A_2\cap A_3\cap A_4) = 193$ //

(P) In a survey of 100 families, the numbers that read the most of various magazines were found to be as follows

Magazine	Number	Magazine	Number
A	28	AB	8
B	30	AC	10
C	42	BC	5
		ABC	3

(a) How many read none of three magazines
(b) How many read magazine C only
(c) How many read B if and only if they read C)

Relations

Cartesian product of sets:- Let A and B two sets. Then the set of all ordered pairs (a,b), where $a\in A$ and $b\in B$, is called the cartesian product of A and B and is denoted by $A\times B$

i.e $A\times B = \{(a,b) / a\in A, b\in B\}$

NOTE:- (i) $A\times A = A^2$

(ii) If A and B are finite sets, then $|A\times B| = |A|\,|B|$ and $|A\times A| = |A|^2$

(iii) for any set $A\subseteq U$, $A\times\phi = \phi\times A = \phi$

Ex:- (i) find x and y if a) $(2x, x+y) = (6,1)$

b) $(y-2, 2x+1) = (x-1, y+2)$

Soln: (a) $2x=6 \Rightarrow x=3$ and $x+y=1 \Rightarrow y=-2$

(b) $y-2 = x-1 \Rightarrow x-y=-1$

$2x+1 = y+2 \Rightarrow 2x-y=1$

$-x = -2 \Rightarrow x=2 \quad \therefore y=3$

(ii) Let $A=\{1,2,3,4\}$ $B=\{2,5\}$ and $C=\{3,4,7\}$ write down
(i) $A\times B$ (ii) $B\times A$ (iii) $A\cup(B\times C)$ (iv) $(A\cup B)\times C$ (v) $(A\times C)\cup(B\times C)$

Relation:- Let A and B be two sets. Then a subset of $A\times B$ is called a relation from A to B. Thus, if R is a relation from A to B, then R is a set of ordered pairs (a,b) where $a\in A$ and $b\in B$, and is denoted by 'aRb'

If R is a relation from A to A, i.e if R is a subset of $A\times A$ we say that R is a binary relation on A

Matrix of a Relation:- Consider the sets $A=\{a_1,a_2 \ldots a_m\}$ and $B=\{b_1,b_2 \ldots b_n\}$ of orders m and n resply. Then $A\times B$ consists of all ordered pairs of the form (a_i,b_j), $1\le i\le m$, $1\le j\le n$ which are mn in number.

i.e $m_{ij} = \begin{cases} 1, & (a_i,b_j)\in R \\ 0, & (a_i,b_j)\notin R \end{cases}$

The $m\times n$ matrix formed by these m_{ij}'s is called the matrix of the relation R and is denoted by M_R (or) $M(R)$

Ex:- Consider $A=\{1,2,3,4\}$ and a relation R defined on A by $R=\{(1,2),(1,3),(2,4),(3,2)\}$

Thus, here $A=\{a_1,a_2,a_3,a_4\}=B$ where $a_1=1, a_2=2, a_3=3, a_4=4$

Accordingly, $m_{ij}=(a_i,a_j)=(i,j)$, $i=1,2,3,4$; $j=1,2,3,4$

$m_{11}=(1,1)=0$ because $(1,1)\notin R$

$m_{12}=(1,2)=1$ " $(1,2)\in R$

$m_{13}=(1,3)=1$, $m_{14}=(1,4)=0$, $m_{21}=(2,1)=0$, $m_{22}=(2,2)=0$

$m_{23}=(2,3)=0$, $m_{24}=(2,4)=1$, $m_{31}=(3,1)=0$, $m_{32}=(3,2)=1$

$m_{33}=(3,3)=0$, $m_{34}=(3,4)=0$

$m_{41}=(4,1)=0$, $m_{42}=(4,2)=0$, $m_{43}=(4,3)=0$, $m_{44}=(4,4)=0$

Thus, the matrix R is $M_R=[m_{ij}]=\begin{bmatrix}0&1&1&0\\0&0&0&1\\0&1&0&0\\0&0&0&0\end{bmatrix}$

NOTE:- For any relation R from a finite set A to a finite set B

Then (i) M_R is the zero matrix if and only if $R=\phi$

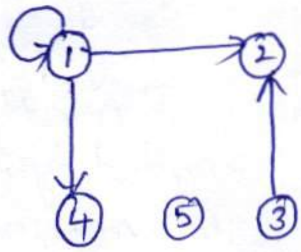
(ii) Every element of M_R is 1 if and only if $R=A\times B$

Digraph of a Relation:- Let R be a binary relation on a finite set A. ~~Draw~~ Then R can be represented pictorially.

Draw a small circle (or) a bullet for each element of A and label the circle with the corresponding element of A. These circles are called vertices or nodes. Draw an arrow, called an edge, from a vertex x to a vertex y if and only if $(x,y)\in R$. The resulting pictorial representation of R is called a directed graph (or) digraph of R.

If a relation is pictorially represented by a digraph, a vertex from which an edge leaves is called the origin or the source for that edge. and a vertex where an edge ends is called the terminus of that edge. A vertex which is neither a source nor a terminus of any edge is called a isolated vertex. An edge for which the source and terminus are one and the same vertex is called a loop. The number of edges (arrows) terminating at a vertex is called the in-degree of that vertex and the no. of edges leaving a vertex is called out-degree of that vertex

Ex:- Consider the set $A = \{1, 2, 3, 4, 5\}$ and the relation $R = \{(1,1), (1,2), (1,4), (3,2)\}$ defined on A. The diagraph of the relation is



In this graph has a Loop at the vertex '1'
The vertex '5' is an isolated vertex
In-degree of the vertices 1, 2, 3, 4 are 1, 2, 0, 1
out-degree of the vertices 1, 2, 3, 4 are 3, 0, 1, 0 Resply

(P) Let $A = \{1, 2, 3, 4, 6\}$ and R be a relation on A defined by aRb if and only if 'a' is a multiple of b. Represent the relation R as a matrix and draw its diagraph.

Soln:- from the def^n of given R

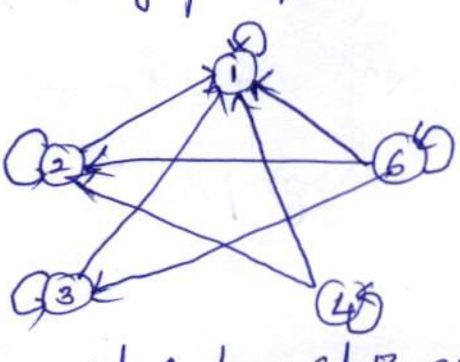
$R = \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), (4,4), (6,1), (6,2), (6,3), (6,6)\}$

By examining the elements of R,

Matrix of R

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Diagraph of R



(P) Determine the relation R from a set A to a set B as described by the following

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

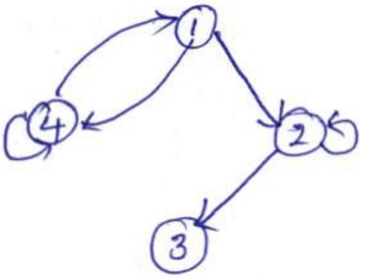
Soln. Given M_R is 4×3 matrix $\therefore |A| = 4$ and $|B| = 3$

If $|A| = \{a_1, a_2, a_3, a_4\}$ and $|B| = \{b_1, b_2, b_3\}$ Then by observing the element of R, we find that

$(a_1, b_1) \in R$, $(a_1, b_2) \notin R$, $(a_1, b_3) \in R$
$(a_2, b_1) \in R$, $(a_2, b_2) \in R$, $(a_3, b_3) \notin R$
$(a_3, b_1) \notin R$, $(a_3, b_2) \notin R$, $(a_3, b_3) \in R$
$(a_4, b_1) \in R$, $(a_4, b_2) \notin R$ $(a_4, b_3) \notin R$

$\therefore R = \{(a_1, b_1), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_1)\}$

(P) Find the relation represented by the graph given below. Also write down the matrix



Sol^n: By examining the given digraph which has four vertices we note that the relation R represented by it is defined on the set $A = \{1,2,3,4\}$ and is given by

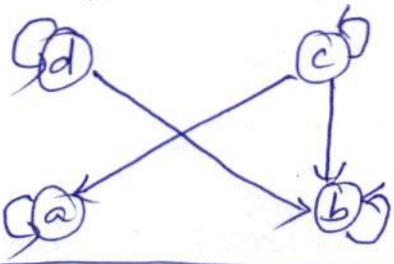
$$R = \{(1,2), (1,4), (2,2), (2,3), (4,1), (4,4)\}$$

Matrix of R is $M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

(P) Let $A = \{a,b,c,d\}$ and R be a relation on A that has the matrix $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ Construct the digraph of R and list the in-degrees and out-degrees of all vertices

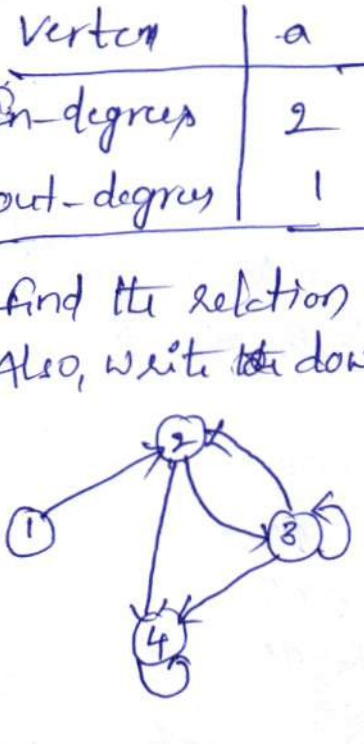
Sol^n. $R = \{(a,a), (b,b), (c,a), (c,b), (c,c), (d,b), (d,d)\}$

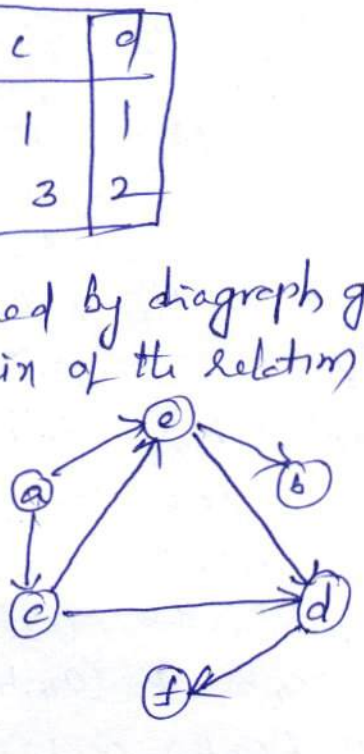
Diagraph



Vertex	a	b	c	d
In-degrees	2	3	1	1
out-degrees	1	1	3	2

(P) Find the relation R determined by diagraph given below Also, write down the matrix of the relation





Operations on Relations

Union and intersection of Relations

Given the relations R_1 and R_2 from a set A to a set B,

(i) then the union of R_1 and R_2, denoted by $R_1 \cup R_2$, is defined as relation from A to B with the property that $(a,b) \in R_1 \cup R_2$ if and only if $(a,b) \in R_1$ or $(a,b) \in R_2$

(ii) Then the intersection of R_1 and R_2, denoted by $R_1 \cap R_2$, is defined as relation from A to B with the property that $(a,b) \in R_1 \cap R_2$ if and only if $(a,b) \in R_1$ and $(a,b) \in R_2$

Complement of a Relation: Given a relation R from a set A to a set B, the complement of R, denoted by $\bar{R}$, is defined as a relation from A to B that $(a,b) \in \bar{R}$ if and only if $(a,b) \notin R$ $\boxed{\text{i.e. } \bar{R} = (A \times B) - R}$

Converse of a Relation: Given a relation R from a set A to a set B, the converse of is denoted by R^c, is defined as a relation from B to A that $(a,b) \in R^c$ if and only if $(b,a) \in R$

NOTE:- (i) If M_R is the matrix of R then $(M_R)^T$ is the matrix of R^c
(ii) $(R^c)^c = R$

(P) Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations R and S from A to B are represented by the following matrices. Determine the relation $\bar{R}$, $R \cup S$, $R \cap S$ and S^c and their matrix representations

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Soln: By examining M_R and M_S, we note that

$A \times B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4)\}$

$R = \{(1,1), (1,3), (2,4), (3,1), (3,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,2), (3,4)\}$

$\therefore \bar{R}$ = Complement of R in $(A \times B) - R$

$= \{(1,2), (1,4), (2,1), (2,2), (2,3), (3,4)\}$

$R \cup S = \{(1,1), (1,2), (1,3), (1,4), (2,4), (3,1), (3,2), (3,3), (3,4)\}$

$R \cap S = \{(1,1), (1,3), (2,4), (3,2)\}$

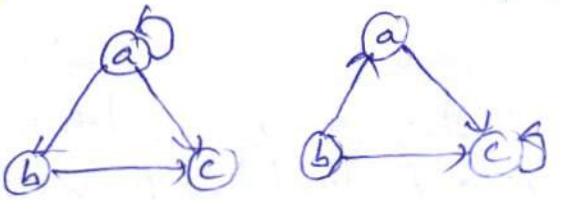
$S^c = \{(1,1), (2,1), (3,1), (4,1), (4,2), (2,3), (4,3)\}$

The matrix representations of the above relations are

$M(\bar{R}) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $M(R \cup S) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $M(R \cap S) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$M(S^c) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(P) The digraph of two relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\bar{R}$, $R \cup S$, $R \cap S$ and R^c



Sol: We first note that $A \times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

By examining the given graphs $R = \{(a,a), (a,b), (a,c), (b,c)\}$
$S = \{(a,c), (b,a), (b,c), (c,c)\}$

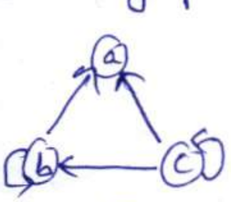
$\therefore \bar{R} = (A \times A) - R = \{(b,a), (b,b), (c,a), (c,b), (c,c)\}$

$R \cup S = \{(a,a), (a,b), (a,c), (b,c), (b,a), (c,c)\}$

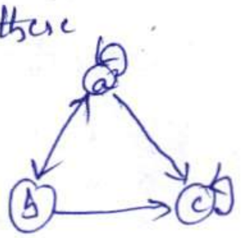
$R \cap S = \{(a,c), (b,c)\}$

$R^c = \{(a,a), (b,a), (c,a), (c,b)\}$

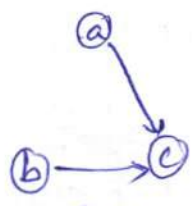
The digraph of these



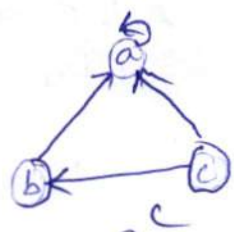
$\bar{R}$



$R \cup S$

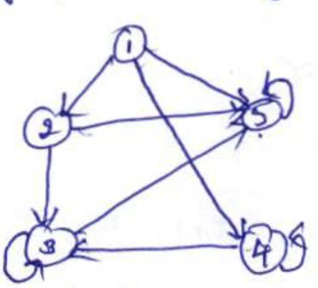


$R \cap S$

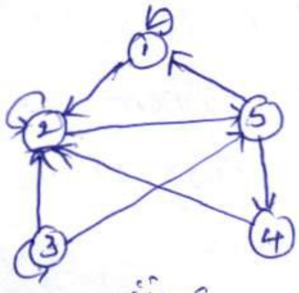


R^c

(P) Let $A = \{1, 2, 3, 4, 5\}$ and R and S be relations on A whose corresponding digraphs are as given below. Find $\bar{R}$, R^c and $R \cap S$



(i) R



(ii) S

Composition of Relations :-

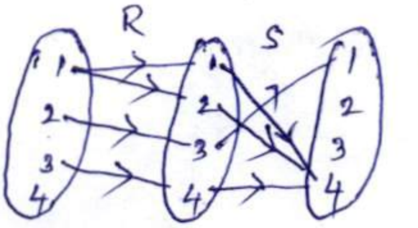
A relation R from a set A to a set B and a relation S from the set B to a set C, we can defined a new relation, called product or the composition of R and S from the set A to the set C. It is denoted by ROS

i.e. $ROS = \{(a,c) / a \in A, c \in C \text{ and } \exists\, b \in B \text{ with } (a,b) \in A \text{ and } (b,c) \in B\}$

(P) Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(1,2),(2,3),(3,4)\}$, $S = \{(3,1),(4,4),(2,4),(1,4)\}$ be relations on A. Determine the relations ROS, SOR, R^2 and S^2

Sol^n: Given $A = \{1,2,3,4\}$ and $R = \{(1,1),(1,2),(2,3),(3,4)\}$

$S = \{(3,1),(4,4),(2,4),(1,4)\}$.



from diagram $ROS = \{(1,4),(2,1),(3,4)\}$

$|||^{ly}$ $SOR = \{(3,1),(3,2)\}$

$ROR = R^2 = \{(1,1),(1,2),(1,3),(2,4)\}$

$SOS = S^2 = \{(1,4),(2,4),(3,4),(4,4)\}$

(P) If $A = \{1,2,3,4\}$ and R, S are relations on A defined by $R = \{(1,2),(1,3),(2,4),(4,4)\}$, $S = \{(1,1),(1,2)(1,3),(1,4),(2,3),(2,4)\}$ find ROS, SOR, R^2 and S^2. Write down their matrices

Sol^n: By examining the elements of R and S, we have

$ROS = \{(1,3),(1,4)\}$, $SOR = \{(1,2),(1,3),(1,4),(2,4)\}$

$R^2 = ROR = \{(1,4),(2,4)\}$, $S^2 = \{(1,1),(1,2),(1,3),(1,4)\}$

The matrices of these relations are

$$M(ROS) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(SOR) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(R^2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M(S^2) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(P) Let $A = \{a,b,c\}$ and R and S be relations on A whose matrices are as given below

$$M(R) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M(S) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

find the composite relations ROS, SOR, R^2, S^2 and their matrices

Sol^n: By examining the entries in M(R) and M(S), we find

$R = \{(a,a),(a,c),(b,a),(b,b),(b,c),(c,b)\}$

$S = \{(a,a),(b,b),(b,c),(c,a),(c,c)\}$

from these, we find that

$R\circ S = \{(a,a),(a,c),(b,a),(b,b),(b,c),(c,b),(c,c)\}$

$S\circ R = \{(a,a),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

$R\circ R = R^2 = \{(a,a),(a,c),(a,b),(b,a),(b,c),(b,b),(c,a),(c,b),(c,c)\}$

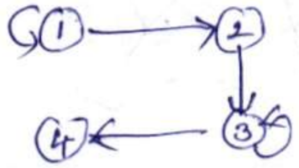
$S\circ S = S^2 = \{(a,a),(b,b),(b,c),(b,a),(c,a),(c,c)\}$

The matrices of the above Composite relations are

$M(R\circ S) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $M(S\circ R) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $M(R^2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $M(S^2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

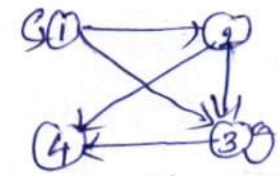
(P) Let $R = \{(1,1),(1,2),(2,3),(3,3),(3,4)\}$ be a relation on $A = \{1,2,3,4\}$. Draw the graph of R. Obtain R^2 and draw the graph of R^2

sol: The diagraph of given R is



By examining R, we note that $R^2 = \{(1,1),(1,2),(1,3),(2,3),(2,4),(3,3),(3,4)\}$

The diagraph of R^2 is



(P) Let $R = \{(1,2),(1,3),(2,4),(3,2)\}$ be a relation on $A = \{1,2,3,4\}$. Write down the relation matrix $M(R)$ of R. Compute $[M(R)]^2$ and hence obtain R^2

(P) If $A = \{1,2,3,4\}$ and R is a relation on A defined $R = \{(1,2),(1,3),(2,4),(3,2),(3,3),(3,4)\}$; find R^2 and R^3. Write down their diagraphs. And also find $M(R)$, $M(R^2)$, $M(R^3)$. verify that $M(R^2) = [M(R)]^2$

(P) Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(1,2),(2,3),(3,3),(3,4)\}$ be a relation on A. find R^2, R^3 and R^4 and draw their diagraph.

(P) Let $R = \{(1,2),(3,4),(2,2)\}$ and $S = \{(4,2),(2,5),(3,1),(1,3)\}$ be relations on the set $A = \{1,2,3,4,5\}$. find $R\circ(R\circ S)$, $R\circ(S\circ R)$, $S\circ(R\circ S)$, $S\circ(S\circ R)$

Properties of Relations

Reflexive Relation:- A relation R on a set A is said to be reflexive if $(a,a) \in R$, for all $a \in A$

Ex:- If $A = \{1,2,3\}$, then $R = \{(1,1),(2,2),(3,3)\}$ is Reflexive

Irreflexive Relation:- A relation R on a set A is said to be irreflexive if $(a,a) \notin R$, for any $a \in A$

Ex:- If $A = \{1,2,3\}$ then $R = \{(1,1),(3,3)\}$ is irreflexive because $(2,2) \notin R$

Symmetric Relation: A relation R on a set A is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$

Ex:- For, if $A = \{1,2,3\}$ and $R_1 = \{(1,1),(1,2),(2,1)\}$ and $R_2 = \{(1,2),(2,1),(1,3)\}$ are relation on A, then R_1 is symmetric and R_2 is asymmetric, because $(1,3) \in R_2$ but $(3,1) \notin R_2$

Compatibility Relation: A relation R on a set A which is both reflexive and symmetric is called compatibility relation on A

Ex:- $R_1 = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$ is compatibility relation on a set $A = \{1,2,3\}$

Antisymmetric Relation:- A relation R on a set A is said to be antisymmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$

Ex:- Let $A = \{1,2,3\}$ and $R_1 = \{(1,1),(2,2)\}$. Then R is both symmetric and antisymmetric

Transitive Relation:- A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall a,b,c \in A$

Ex:- If $A = \{1,2,3\}$ and $R = \{(1,1),(1,2),(2,3),(1,3),(3,1),(3,2)\}$ is relation on A the R is transitive relation.

(P) Let $R = \{(1,1),(2,2),(2,3),(3,2),(4,2),(4,4)\}$ and $S = \{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$ be relation on $A = \{1,2,3,4\}$

(i) Verify that R is not (a) reflexive (ii) symmetric (iii) transitive

(ii) Determine whether S is reflexive, irreflexive, symmetric, asymmetric antisymmetric or transitive.

Equivalence Relations:- A relation R on a set A is said to be an equivalence relation on A if (i) R is reflexive (ii) R is symmetric and (iii) R is transitive on A (13)

Ex:- Let $A = \{1,2,3,4\}$ and $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ be a relation on A. Verify that R is an equivalence relation

Sol^n:- Given $A = \{1,2,3,4\}$ and $R = \{(1,1)(1,2),(2,1)(2,2),(3,4),(4,3),(3,3),(4,4)\}$

(i) We note that $(1,1),(2,2),(3,3),(4,4) \in R$ i.e $(a,a) \in R \; \forall \; a \in A$

$\therefore$ R is reflexive

(ii) We note that $(1,2),(2,1),(1,1) \in R$, $(2,1),(1,2),(2,2) \in R$, $(4,3),(3,4),(4,4) \in R$

i.e $(a,b) \in R$ then $(b,a) \in R$ for $a,b \in A$ $\therefore$ R is symmetric

(iii) If whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for $a,b,c \in A$

$\therefore$ R is a transitive relation

Hence R is an equivalence relation

Equivalence classes:- Let 'R' be an equivalence relation on a set A and $a \in A$. Then the set of all those elements x of A which are related to a by R is called the equivalence class of a w.r.to R. This equivalence class is denoted by $R(a)$ or $[a]$ or $\bar{a}$

i.e $\bar{a} = [a] = R(a) = \{x \in A / (x,a) \in R\}$

Ex:- Consider the equivalence relation $R = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$ defined on the set $A = \{1,2,3\}$

We find the element x of A for which $(x,1) \in R$ are $x=1, x=3$

$\therefore$ $\{1,3\}$ is the equivalence class of 1 i.e $[1] = \{1,3\}$

$||^{ly}$ $[2] = \{2\}$, $[3] = \{1,3\}$

partition of a set:- Let A be a non-empty set. Suppose there exist non-empty subsets $A_1, A_2, \ldots A_k$ of A such that the following conditions hold

(i) $A = A_1 \cup A_2 \cup \ldots \cup A_k$

(2) Any two of the subsets $A_1, A_2 \ldots A_k$ are disjoint i.e $A_i \cap A_j = \phi$ for $i \neq j$

Then the set $P = \{A_1, A_2 \ldots A_k\}$ is called partition of A. Also $A_1, A_2 \ldots A_k$ are called the blocks or cells of the partition

(P) For the equivalence relation $R=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ defined on the set $A=\{1,2,3,4\}$, determine the partition induced

Sol: Given $R=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ is an equivalence relation on the set $A=\{1,2,3,4\}$

By examining the given relation R, we find that equivalence classes by the elements of A w.r.to R are

$[1]=\{1,2\}$, $[2]=\{1,2\}$. $[3]=\{3,4\}$, $[4]=\{3,4\}$

Of these, only $[1]$ and $[3]$ are distinct. These two distinct equivalence classes constitute the partition

$$P=\{[1],[3]\}=\{\{1,2\},\{3,4\}\}$$

This is the partion of the given A induced by the given R

We observe that $A=[1]\cup[3]=\{1,2\}\cup\{3,4\}$

(P). Verify that $R=\{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1)\}$ is an equivalence relation on the set $A=\{1,2,3,4\}$. Find the corresponding partition of A

(P) Let $A=\{1,2,3,4,5,6,7\}$ and R be the equivalence relation on A that induces the partition $A=\{1,2\}\cup\{3\}\cup\{4,5,7\}\cup\{6\}$. Find R.

Sol: Given partition of A are four blocks i.e $\{1,2\},\{3\},\{4,5,7\},\{6\}$

Let R be the equivalence relation inducing this partition

Since the elements 1, 2 are in the same block, we have $1R1$, $2R1$, $2R2$

Since 3 belongs to the block $\{3\}$ which contains only 3, we have $3R3$

Since 4, 5, 7 belongs to the same block, we have

$4R4$, $4R5$, $5R4$, $5R5$, $5R7$, $7R4$, $7R5$, $7R7$

Since 6 belongs to $\{6\}$ which contains only 6, we have $6R6$

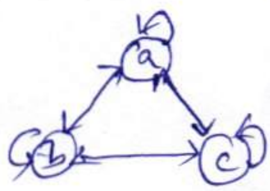
$\therefore$ The required relation $R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,5),(5,4),$
$(5,5),(5,7),(7,4),(7,5),(7,7),(6,6)\}$

(P) For the set $A=\{1,2,3,4,5,6\}$. Consider the partition $P=\{A_1,A_2\}$ where $A_1=\{1,3,5\}$, $A_2=\{2,4,6\}$. Determine the corresponding equivalence relation R

(P) If $A=\{1,2,3,4,5\}$ and R is the equivalence relation on A that induces the partition $A=\{1,2\}\cup\{3,4\}\cup\{5\}$, find R.

(P) The matrix of a relation R on the set $A=\{1,2,3\}$ is given by $M(R)=\begin{bmatrix}1&0&0\\0&1&1\\0&1&1\end{bmatrix}$ show that R is an equivalence relation

(P) A relation R on the set $\{a,b,c\}$ represented by the digraph given below. show that R is an equivalence relation.



partial orders :- A relation R on a set A is said to be a partial ordering or a partial order on A if R is reflexive, antisymmetric and transitive on A. i.e (i) $aRa \; \forall a \in A$ (ii) $aRb, bRa \Rightarrow a=b \; \forall a,b \in A$,

A set A with a partial order R defined on it is called a partially ordered set or poset and is denoted by the pair (A, R)

Total Order :- Let 'R' be a partial order on a set A. Then R is called a total order on A if for all $x, y \in A$, either xRy or yRx.

In this case, the poset (A, R) is called a totally ordered set (or) a chain

NOTE: Every total order is a partial order but not every partial order is a total order.

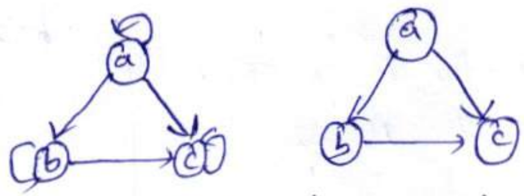
Hasse Diagram :- A diagram that is used to describe partial order relation associated with a set is called Hasse diagram

(OR)

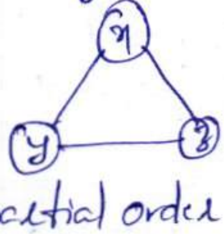
A partial order $\leq$ on a set x can be represented by means of a diagram known as Hasse diagram of $(X, \leq)$

The following tips are neccessary to read Hasse diagram for a relation R defined on a set A

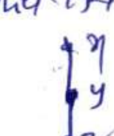
(i) since the partial order is reflexive, therefore every vertex is related to itself. so the arrows from a vertex to itself in a digraph are removed

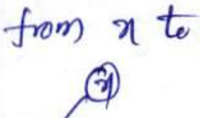


(ii) since the partial order is antisymmetric, therefore all arrows connecting two vertex are removed. thus if $x \leq y$ and $y \leq x$ is not neccessarly unless $x = y$. In this case direction of arrows b/w x and y is not neccessary



(iii) Since the partial order is transitive, all edges that are implied this property are removed i.e if $x \leq y$ and $y \leq z$ then it implies that $x \leq z$ So the edge from x to z be removed





(iv) circles in the graph representing vertices are replaced by dots. i.e the Hasse diagram is final form of the digraph

(P) Let $A = \{2,4,8,16,32\}$ on A, define the relation R by aRb if and only if a divides b. Draw the Hasse diagram for the poset and determine whether the poset is total order or not. Write the relation matrix for R.

Soln: Given that $A = \{2,4,8,16,32\}$

The Relation R is given by aRb iff $a|b$

$\therefore R = \{(2,2),(2,4),(2,8),(2,16),(2,32),(4,8),(4,16),(4,32),(8,16)$
$(8,32),(4,4),(16,16),(16,32),(32,32)\}$

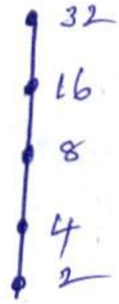
(i) Reflexive: for every $a \in A$, aRa $\therefore$ R is reflexive

(ii) Antisymmetric: for every $a,b \in A$, aRb and bRa then $a=b$
$\therefore$ R is antisymmetric

(iii) Transitive: for every $a,b,c \in A$, aRb, bRc then aRc
$\therefore$ R is transitive
$\therefore$ R is partial order relation

We observe that, for any $a,b \in A$, we have $a|b$ or $b|a$
$\therefore (A,|)$ is total ordered set

The Hasse diagram for the poset is



The relation matrix for R is given by

$$M(R) = \begin{array}{c} \\ 2 \\ 4 \\ 8 \\ 16 \\ 32 \end{array}\begin{array}{c} \begin{array}{ccccc} 2 & 4 & 8 & 16 & 32 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(P) Let $S = \{1,2,3\}$ and P(S) be the power set of S on P(S), define the relation R by xRy iff $x \subseteq y$. Show that this relation is a partial order on P(S). Draw its Hasse diagram

Soln: Given that $S = \{1,2,3\}$ $\therefore P(S) = \{\phi, \{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}$
$\{1,2,3\}\}$

The relation R defined on P(S) is given by '$\subseteq$'

We P.T is a partial order on P(S)

It is enough to P.T $\subseteq$ is reflexive, antisymmetric and transitive.

(i) Reflexive :- for every $A \subseteq P(S)$, $A \subseteq A$

$\therefore \subseteq$ is reflexive

(ii) Antisymmetric :- for every $A, B \subseteq P(S)$, $A \subseteq B$ and $B \subseteq A$ imply $A = B$

$\therefore \subseteq$ is antisymmetric

(iii) Transitive: for every $A, B, C \subseteq P(S)$, $A \subseteq B$, $B \subseteq C$ imply $A \subseteq C$

$\therefore \subseteq$ is transitive

~~It follows that~~ '$\subseteq$' is a partial ordering on $P(S)$

$\therefore (P(S), \subseteq)$ is a poset

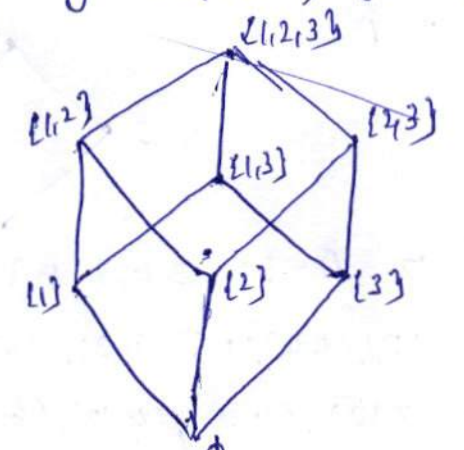
We observe that ϕ is subset of remaining all subsets of 'S'

$\{1\} \subseteq \{1,2,3\}$, $\{2\} \subseteq \{1,2,3\}$, $\{3\} \subseteq \{1,2,3\}$, $\{1,2\} \subseteq \{1,2,3\}$, $\{2,3\} \subseteq \{1,2,3\}$

$\{1,3\} \subseteq \{1,2,3\}$, $\{1,2,3\} \subseteq \{1,2,3\}$

$\{1\} \subseteq \{1,2\}$, $\{1\} \subseteq \{1,3\}$, $\{2\} \subseteq \{1,2\}$, $\{2\} \subseteq \{2,3\}$, $\{3\} \subseteq \{1,3\}$, $\{3\} \subseteq \{2,3\}$

The Hasse diagram of R is given by



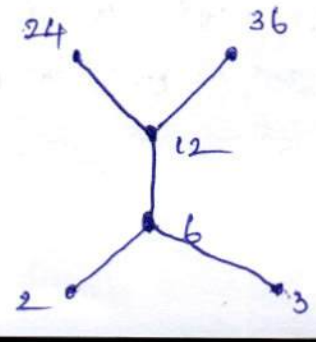
(P) Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ if x divides y. Draw the Hasse diagram of $(X, \leq)$

Sol:- Given that $X = \{2, 3, 6, 12, 24, 36\}$

The relation $\leq$ is $x \leq y$ if x divides y

$R = \{(2,2), (2,6), (2,12), (2,24), (2,36), (3,3), (3,12), (3,24), (3,36)$
$(6,6), (6,12), (6,24), (6,36), (12,12), (12,24), (12,36), (24,24), (36,36)\}$

The Hasse diagram of the given relation



(P) Let A be the set of factor of a perticular positive ~~order~~ integer m and let $\leq$ be the relation divides i.e $\leq = \{(x,y) / x \in A \wedge y \in A \wedge (x \text{ divides } y)\}$ Draw Hasse diagram for (a) $m=2$ (b) $m=6$ (c) $m=12$ (d) $m=30$ (e) $m=210$ (f) $m=45$

Sol^n: Given A be the set of factor of a perticular +ve integer Given that $\leq$ be the relation divides

i.e $\leq = \{(x,y) / x \in A \wedge y \in A \wedge (x \text{ divides } y)\}$

(i) $m=2$: Factors of 2 is $A = \{1,2\}$

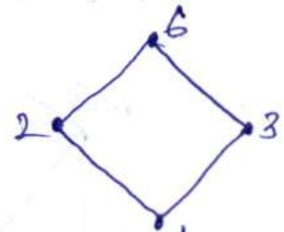
$\therefore R = \{(1,1), (1,2), (2,2)\}$

The Hasse diagram of R is

(ii) $m=6$: Factors of 6 is $A = \{1,2,3,6\}$

$\therefore R = \{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}$

$\therefore$ Hasse diagram of R is

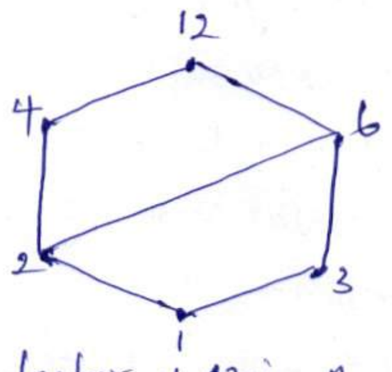


(iii) $m=12$.

factors of 12 $= A = \{1,2,3,4,6,12\}$

$\therefore R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,12),(2,2),(2,4),(2,6),(2,12)$
$(3,3),(3,6),(3,12),(4,4),(4,12),(6,6),(6,12),(12,12)\}$

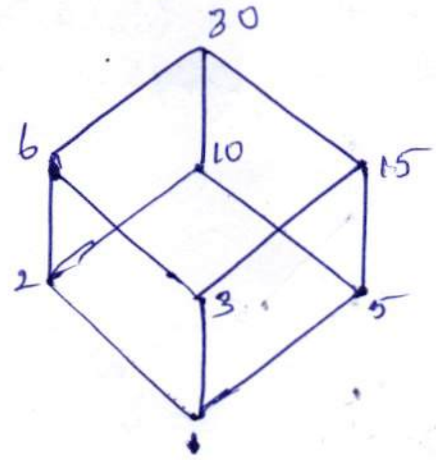
$\therefore$ Hasse diagram of the given relation R is



(iv) $m=30$. factor of 30 is $A = \{1,2,3,5,6,10,15,30\}$

$\therefore R = \{(1,2),(1,3),(1,5),(1,6),(1,10),(1,15),(1,30),(2,2),(2,6),(2,10),(2,30)$
$(3,3),(3,6)(3,15),(3,30),(5,5),(5,10),(5,15),(5,30),(6,6),(6,30)$
$(10,10),(10,30),(15,15),(15,30),(30,30)\}$

The Hasse diagram of given relation R is

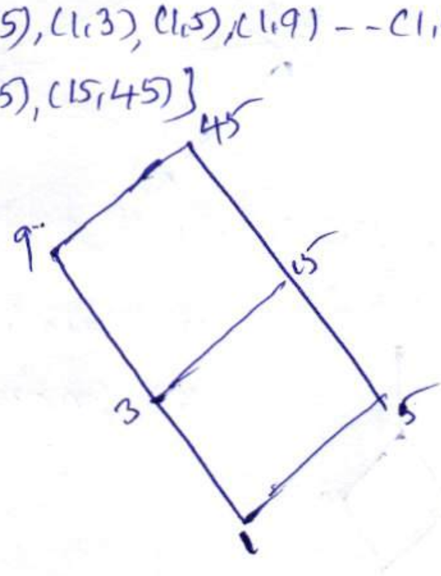


(v) $\underline{m = 210}$: factors of 210 are $A = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$

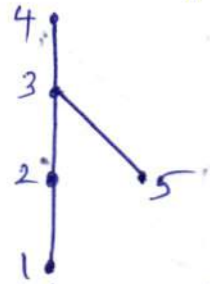
$\therefore R = \{(1,1), (1,2) \ldots (1,210), (2,6), (2,10) \ldots (2,210), (3,6), (3,15) \ldots (3,210)$
$(5,10), (5,15) \ldots (5,210), (6,30), \ldots (6,210), (7,14), (7,21) \ldots (7,210)$
$(10,30) \ldots (10,210), (14,42) \ldots (14,210), (15,30) \ldots (15,210), (21,42) \ldots$
$(21,210), (30,210), (35,70), (35,210), (42,210), (70,210), (105,210)\}$

(vi) $\underline{m = 45}$, factors of 45 are $A = \{1, 3, 5, 9, 15, 45\}$

$\therefore R = \{(1,1), (3,3) \ldots (45,45), (1,3), (1,5), (1,9) \ldots (1,45), (3,9), (3,15) \ldots (3,45)$
$(5,15), (5,45), (9,45), (15,45)\}$



(P) Describe the order pairs in the relation determined by the Hasse diagram of a poset $(A, \leq)$ on the set $A = \{1,2,3,4\}$ and determine the matrix of the partial order and draw the digraph of a relation



Soln: Since the relation on A is a partial order.

All reflexive pairs $(1,1), (2,2), (3,3), (4,4)$ $(5,5)$ must $\leq$

All edges when converted to upwards their vertices give the ordered pairs

$(1,2), (2,3), (3,4), (5,3)$

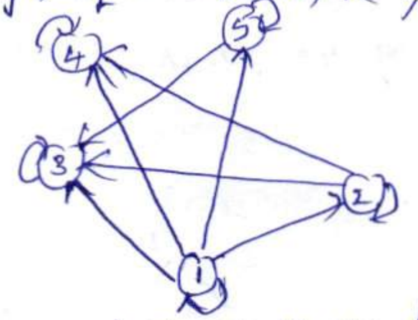
The transitivity implied arcs give the ordered pairs $(1,3), (2,4), (5,4), (1,4)$

$\therefore$ The ordered pairs in the relation represented by Hasse diagram is

$\{(1,1), (2,2), (3,3), (4,4)\ (5,5), (1,2), (2,3), (3,4)\ (5,3), (1,3)\ (2,4)\ (5,4)\ (1,4)\}$

$\therefore$ Matrix of the relation is
$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

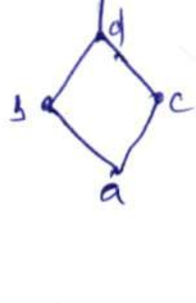
The digraph of a relation R is given by



(P) For $A = \{a,b,c,d,e\}$ the Hasse diagram for the poset (A, R) is as shown below

a) Determine the relation matrix for R

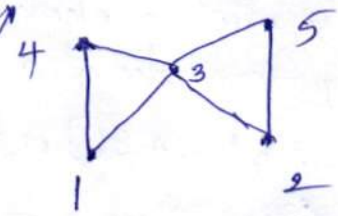
b) Construct the digraph for R



(P) Draw the Hasse diagram of the relation R on $A = \{1,2,3,4,5\}$ (18)
whose matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Soln: By examining the given M_R, we note that
$R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3) (3,4) (3,5)$
$(4,4)), (5,5)\}$
By looking at the elements of R, we can drawn the Hasse diagram
of R, if it is



(P) Draw the Hasse diagram for the partial order R on the set
$A = \{1,2,3,4,5\}$ whose matrix is

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

External Elements in Posets

Maximal element: An element $a \in A$ is called a maximal element of A if there exist no element $x \neq a$ in A s.t aRx. In otherwords, $a \in A$ is a maximal element of A if whenever there is $x \in A$ s.t aRx then $x = a$

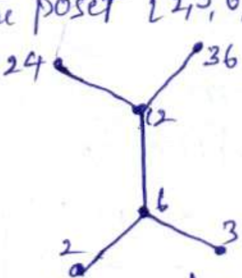
This means that 'a' is maximal element of A if and only if in the Hasse diagram of R no edge start at 'a'

Minimal Element:- An element $a \in A$ is called a minimal of A if there exist no element $x \neq a$ in A s.t xRa.

In otherwords, $a \in A$ is minimal element of A if whenever there is $x \in A$ s.t xRa then $x = a$

This means that, 'a' is minimal element of A if and only if in the Hasse diagram of R no edge terminates at 'a'

Ex:- In the poset $\{2, 3, 6, 12, 24, 36\}$ with divisibility relation



Here 2 and 3 are minimal elements and 24, 36 are maximal elements [$\because$ TOP & bottom elements in the diagram]

NOTE: ① A poset may have more than one maximal element and more than one minimal element

② A poset need not have any maximal or minimal elements

③ A poset may have a maximal element but no minimal elements

Greatest element:- An element $a \in P$ is called a greatest element of P if xRa for $x \in P$, $(P, \leq)$ be a poset

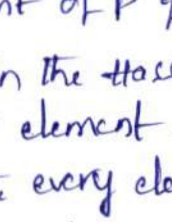
Least element: Let $(P, \leq)$ be a poset. An element $b \in P$ is called a least element of P if bRx for all $x \in P$.

NOTE:- ① In the Hasse diagram, a greatest element is connected to every other element by a path leading down and a least element is connected to every element by a path leading up.

② Maximal element may not be the greatest element

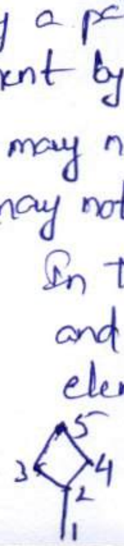
③ Minimal element may not be the least element.

Ex: (i)



In this Hasse diagram, 5 & 6 are maximal element and 1 is the minimal element. 1 is the least element. There is no greatest element

(ii)



Upper and Lower bounds :- Let $(A, \leq)$ be a poset and B be a subset of A

→ An element $a \in A$ is called an upper bound of a subset B of A if xRa for all $x \in B$

An element $a \in A$ is called a least upper bound (LUB) of a subset B of A if (i) 'a' is an upper bound of B

(ii) If 'a'' is an upper bound of B then aRa'

A least upper bound is also called a supremum

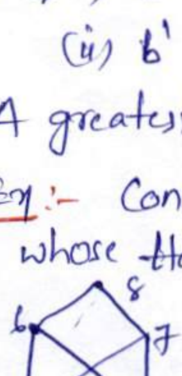
→ An element $b \in A$ is called lower bound of a subset ~~of~~ B of A if bRx for all $x \in B$

An element $b \in A$ is called a greatest lower bound (GLB) of a subset B of A if (i) b is a Lower bound of B

(ii) b' is a lower bound of B then $b'Rb$

A greatest lower bound is also called a ~~minimum~~ infimum.

Ex :- Consider the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and a partial order on A whose Hasse diagram is



consider the subset $B_1 = \{1, 2\}$, $B_2 = \{3, 4, 5\}$ of A

We observe that from the Hasse diagram

(i) $1R3$, $2R3$. Therefore 3 is an upperbound of B_1

$||^{ly}$ 4, 5, 6, 7, 8 are also upper bounds of B_1

The upper bound 3 of B_1 is s.t $3Rx$ for all upper bound of x of B_1

∴ 3 is a least upper bound of B_1 i.e $LUB(B_1) = 3$

In A, there is an element x s.t $xR1$ and $xR2$

∴ B_1 has no lower bounds

∴ B_1 has no greatest lower bound

(ii) for each $x \in B_2$, we have $xR6$

∴ 6 is an upper bound of B_2. similarly 7 and 8 are also upper bounds of B_2

6 is not related to the upper bound 7

∴ B_2 has no least upper bound

→ for each $x \in B_2$, we have $1Rx$, Therefore, 1 is a lower bound for B_2

$||^{ly}$ 2 and 3 are also lower bounds of B_2

The lower bound 3 of B_2 is such that $xR3$ for all lower bounds x of B_2 i.e $1R3$, $2R3$ ∴ 3 is the GLB of B_2 //

Lattice:- A lattice is a partially ordered set (A,R) i.e poset. This poset is called a lattice if every two-element subset of A has LUB and a GLB in A.

If (A,R) is a Lattice, the LUB of the two element subset $\{a,b\} \subseteq A$ is denoted by $a \vee b$

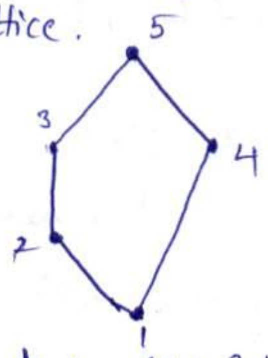
i.e $LUB = \{a,b\} = a \vee b$, read as join or sum of a and b

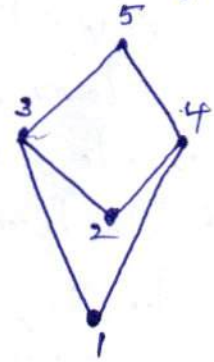
and the GLB of the two element subset $\{a,b\} \subseteq A$ is denoted by $a \wedge b$

i.e $GLB = \{a,b\} = a \wedge b$, read as meet or product of a and b

NOTE :. $x \vee y = LUB(x,y) = y$, $x \wedge y = GLB(x,y) = x$

(P) Determine whether the posets represented by each of the Hasse diagram are Lattice.





Soln: (a) Let $A = \{1,2,3,4,5\}$

(i) Let $B_1 = \{1,2\}$ be a subset of a set A

a) $1 \in A$ be a lower bound of a subset B_1 of A, since $1Rx$, $\forall x \in B_1$

i.e $1R1$, $1R2$

1 is also GLB of subset B_1 of A $\therefore GLB\{B_1\} = 1$

(ii) b) $2 \in A$ be an upper bound of a subset B_1 of A, since $xR2$, $\forall x \in B_1$

i.e $2R2$, $1R2$

3 and 5 are also upper bounds of a subset B_1 of A, since $xR3$ $\forall x \in B_1$

i.e $2R3$ and $2R5$

$\therefore$ 2 is the LUB of B_1 $\therefore LUB\{B_1\} = 2$

(iii) Let $B_2 = \{3,4\}$ be a subset of a set A

a) $1 \in A$ be a lower bound of a subset B_2 of A, since $1Rx$ $\forall x \in B_2$

i.e $1R3$. and $1R4$

1 is also a GLB of B_1 of A $\therefore GLB = \{3,4\} = 1$

b) $5 \in A$ is an upper bound of a subset B_2 of A, since $xR5$ $\forall x \in B_2$

i.e $3R5$ and $4R5$

5 is also a LUB of a set B_2 $\therefore LUB\{B_2\} = 5$

$\therefore$ Every two-element subset of A has GLB and LUB

$\therefore (A, \leq)$ is Lattice //

(2) Let $A = \{1, 2, 3, 4, 5\}$

Let $B_1 = \{3,4\}$ be a subset of a set A

$2 \in A$ be a lower bound of subset B_1 of A, since $2Rx \ \forall x \in B_1$

i.e $2R3$, and $2R4$

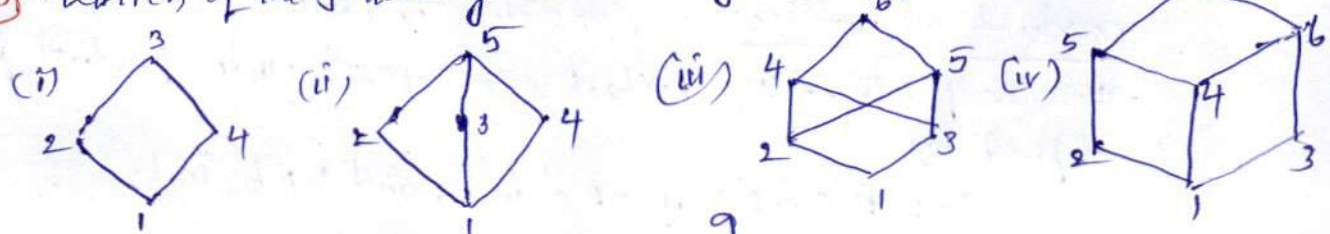
1 is also a lower bound of subset B_1 of A, since $1Rx \ \forall x \in B_1$

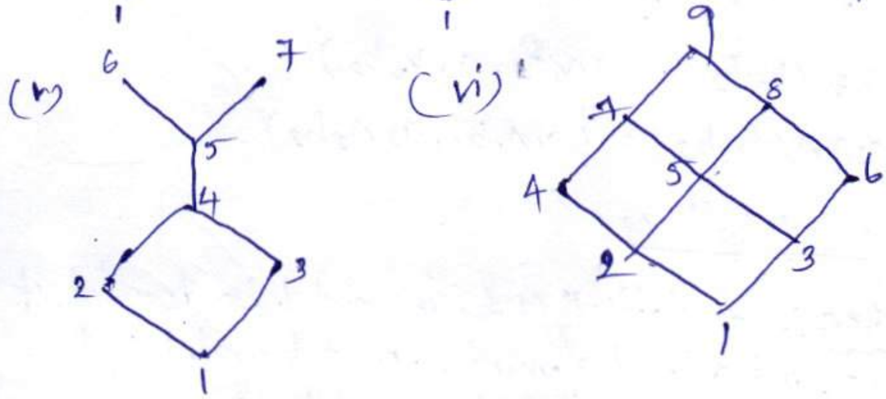
i.e $1R3$ and $1R4$

The GLB of B_1 does not exist because $1 \not R 2$ or $2 \not R 1$

$\therefore (A, \leq)$ is not a lattice.

(P) Which of the following Hasse diagrams represent Lattices?





properties of Lattices: (1) Let (L, R) be a Lattice. Then, for any $a, b, c \in L$ the follow are true

(i) Idempotent properties:- (i) $a \vee a = a$ (ii) $a \wedge a = a$

(ii) Commutative properties : (i) $a \vee b = b \vee a$ (ii) $a \wedge b = b \wedge a$

(iii) Associative properties : (i) $a \vee (b \vee c) = (a \vee b) \vee c$ (ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(iv) Absorption properties (i) $a \vee (a \wedge b) = a$ (ii) $a \wedge (a \vee b) = a$

(2) Let (L, R) be a lattice. Then for all a and b in L, the following is true

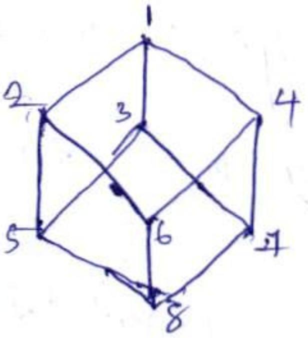
(i) $a \vee b = b \Leftrightarrow aRb$

(ii) $a \wedge b = a \Leftrightarrow aRb$

(iii) $a \wedge b = a \Leftrightarrow a \vee b = b$

Sublattice: Let (L,R) be a lattice and M be a subset of L. Then M is called a sublattice of L if $a \vee b \in M$ and $a \wedge b \in M$ whenever $a \in M$ & $b \in M$

Ex: Consider the lattice (L,R) represented by the Hasse diagram



Evidently $L = \{1,2,3,4,5,6,7,8\}$.

Consider $M_1 = \{1,2,4,6\}$, $M_2 = \{3,5,7,8\}$, $M_3 = \{1,2,4,8\}$

By examining the Hasse diagram, we check that (M_1,R) and (M_2,R) are sublattices of (L,R). But (M_3,R) is not a sublattice of (L,R), because $2 \wedge 4 = 6 \notin M_3$

Product of Lattices: Consider the lattices (L_1,R) and (L_2,R). Then these are posets. Also $(L_1 \times L_2, R)$ is a poset under product partial order defined by

$(a,b) R (a',b')$ if $a R a'$ in L_1 and $b R b'$ in L_2

i.e $(a_1,a_2) \vee (b_1,b_2) = (a_1 \vee_1 b_1, a_2 \vee_2 b_2)$

$(a_1,a_2) \wedge (b_1,b_2) = (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2)$

Special Types of Lattices:

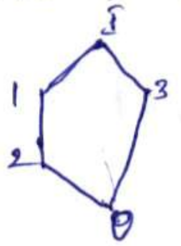
Bounded Lattice: A lattice (L,R) is said to be bounded if it has a greatest element and a least element. In a bounded lattice, a greatest element is denoted by I and a least element by 'O'

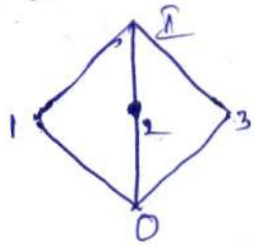
i.e $O R a$, $a R I$, $a \vee O = a$, $a \wedge O = O$, $a \vee I = I$, $a \wedge I = a$, for $a \in L$

Distributive Lattice: Let lattice (L,R) is said to be distributive, if, for any $a,b,c \in L$, the following distributive laws hold

(1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Ex:-





Both of these Hasse diagrams, lattices are bounded but not distributive

Because, from fig (1) $1 \wedge (2 \vee 3) = 1 \wedge I = 1$

$(1 \wedge 2) \vee (1 \wedge 3) = 2 \vee O = 2$

and from fig (2), $1 \wedge (2 \vee 3) = 1 \wedge I = 1$

$(1 \wedge 2) \vee (1 \wedge 3) = O \vee O = O$

Complemented Lattice:

Let 'L' be a bounded lattice with greatest element I and least element 'O'. For a chosen element 'a' of L, if $\exists$ an element $a' \in L$ such that $a \vee a' = I$ and $a \wedge a' = O$ Then a' is called a complement of a in L.

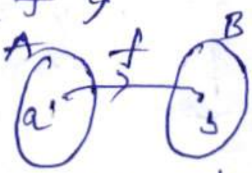
i.e A lattice is said to be a complemented lattice if L is bounded and every element in L has a complement in 'L'

Functions

Function:- Let A and B be two non-empty sets. Then a function f from A to B is a relation from A to B such that for each a in A there is a unique in b such that $(a,b) \in f$. Then we write $b = f(a)$

Here 'b' is called the image of a and 'a' is called pre image of b under f. The element 'a' is called an argument of f and $b = f(a)$ is then called the value of the function f for argument 'a'

A function f from A to B is denoted by $f: A \to B$. The pictorial representation of f is



for this function $f: A \to B$, A is called domain of f and B is called the co-domain of f.

The subset of B consisting of the images of all elements of A under f is called the range of f and is denoted by $f(A)$

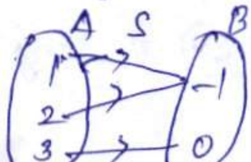
i.e. $f(A) = \{f(x) \mid x \in A\}$

Note:- 1) for $f: A \to B$ if $b \in B$ and $f^{-1}(b)$ is defined by

$$f^{-1}(b) = \{x \in A \mid f(x) = b\} \text{ then } f^{-1}(b) \subseteq A$$

(P) Let $A = \{1,2,3\}$ and $B = \{-1,0\}$ and S is a relation from A to B defined by $S = \{(1,-1),(2,-1),(3,0)\}$. Is S is a function?

Soln.



we observe that, under S, each element of A is related to a unique element of B

$\therefore$ S is a function from A to B

(P) Let $A = \{1,2,3,4\}$. Determine whether or not the following relations on A are functions.

(i) $f = \{(2,3),(1,4),(2,1),(3,2),(4,4)\}$ (ii) $g = \{(3,1),(4,2),(1,1)\}$

(iii) $h = \{(2,1),(3,4),(1,4),(2,1),(4,4)\}$

(P) Let $A = \{1,2,3,4,5,6\}$ and $B = \{6,7,8,9,10\}$. If a function $f: A \to B$ is defined by $f = \{(1,7),(2,7),(3,8),(4,6),(5,9),(6,9)\}$ determine $f^{-1}(6)$, $f^{-1}(9)$. If $B_1 = \{7,8\}$, $B_2 = \{8,9,10\}$ find $f^{-1}(B_1)$ and $f^{-1}(B_2)$

Soln:- Given $A = \{1,2,3,4,5,6\}$ and $B = \{6,7,8,9,10\}$. If a function $f: A \to B$ is defined by $f = \{(1,7),(2,7),(3,8),(4,6),(5,9),(6,9)\}$

then $f^{-1}(6) = \{x \in A \mid f(x) = 6\} = \{4\}$

$f^{-1}(9) = \{x \in A \mid f(x) = 9\} = \{5,6\}$

for $B_1 = \{7,8\}$, $f(x) \in B_1$ when $f(x) = 7$ & $f(x) = 8$. from the def^n of f, we note that $f(x) = 7$ when $x=1$ & $x=2$ and $f(x) = 8$ when $x=3$ $\therefore f^{-1}(B_1) = \{1,2,3\}$

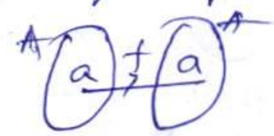
(P) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x+7, & x \le 0 \\ -2x+5, & 0 < x < 3 \\ x-1, & x \ge 3 \end{cases}$

Find $f^{-1}(-10)$, $f^{-1}(0)$, $f^{-1}(4)$, $f^{-1}(6)$, $f^{-1}(7)$ and $f^{-1}(8)$. Also determine $f^{-1}([-5,1])$, $f^{-1}([-5,0])$, $f^{-1}([-2,4])$, $f^{-1}([5,10])$ and $f^{-1}([11,17])$

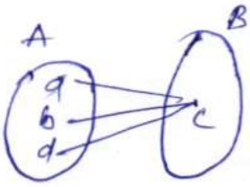
Types of functions:

Identity function: A function $f: A \to A$ s.t $f(a) = a$ for every $a \in A$ is called the identify function on A

i.e A function f on a set A is an identity function if the image of every element of A is itself i.e $f(A) = A$. It is denoted by $I_A : A \to A$

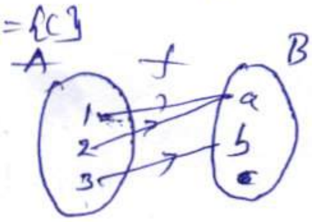


Constant function: A function $f: A \to B$ s.t $f(a) = c$ for every $a \in A$, where c is a fixed element of B



i.e A function f from A to B is a constant function if all elements of A have the same image in B. i.e $f(A) = \{c\}$

Onto function: A function $f: A \to B$ is said to be an onto function if every element of B has pre-image in A, under f.



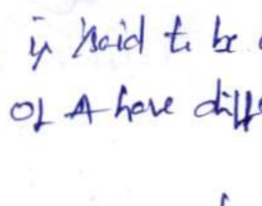
i.e f is an onto function from A to B if the range of f is equal to B

i.e $f(A) = B$

One-to-one function: A function $f: A \to B$ is said to be one-to-one function i.e one-one function. If different elements of A have different images in B under f.

i.e if $a_1, a_2 \in A$ with $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$ (or if whenever $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$, then $a_1 = a_2$

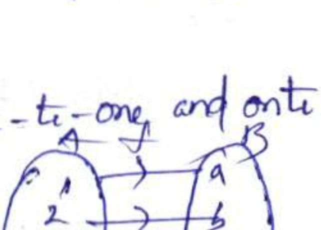
i.e If $f: A \to B$ is one-to-one function, then every element of A has a unique image in B and every element of $f(A)$ has a unique preimage in A



It is also called an injective function.

Bijection: A function $f: A \to B$ is both one-to-one and onto is called a bijective function

A bijective function is also called one-to-one correspondence



(P) The function $f: R \to R$ and $g: R \to R$ are defined by $f(x) = 3x+7 \ \forall x \in R$, $g(x) = x(x^3-1) \ \forall x \in R$. Verify that f is one-to-one but g is not.

Sol^n: Given that $f(x) = 3x+7 \ \forall x \in R$

For any $a_1, a_2 \in R$, $f(a_1) = 3a_1+7$, $f(a_2) = 3a_2+7$

$\therefore f(a_1) = f(a_2) \Rightarrow 3a_1+7 = 3a_2+7 \Rightarrow 3a_1 = 3a_2$

$\Rightarrow a_1 = a_2$

$\therefore f$ is one-to-one function

Given that $g(x) = x(x^3-1)$

At $x=0$, $g(0)=0$ We have $g(x_1) = g(x_2)$ but $x_1 \neq x_2$

At $x=1$, $g(1)=0$ $\therefore g$ is not a one-to-one function //

(P) Let $f: Z \to Z$ be defined by $f(a) = a+1$ for $a \in Z$. Find whether f is one-to-one or onto or both or neither.

Sol^n: Given that $f: Z \to Z$ be defined by $f(a) = a+1$

For any $x_1, x_2 \in Z$, $f(x_1) = x_1+1$, $f(x_2) = x_2+1$

$\therefore x_1 \neq x_2 \Rightarrow x_1+1 \neq x_2+1 \Rightarrow f(x_1) \neq f(x_2)$

Different elements of Z have different images under f

$\therefore f$ is one-to-one

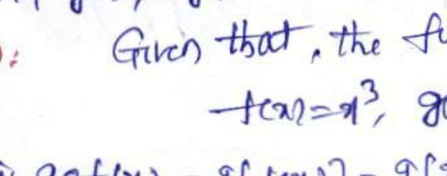
Take any $b \in Z$, we observed that b has $b-1$ as its pre-image under f because $f(b-1) = (b-1)+1 = b$

$\therefore$ Every element of Z has a pre-image

$\therefore f$ is onto

$\therefore f$ is both one-to-one and onto i.e. bijection //

Composite functions:- Let A, B, C be three non-empty sets. The functions $f: A \to B$ and $g: B \to C$. Then the composition of these two functions is defined as the function $gof: A \to C$ with $(gof)(a) = g[f(a)] \ \forall a \in A$



For function $f: A \to A$, fof is denoted by f^2, $fofof$ is denoted by f^3 and so on.

(P) Consider the functions f and g defined by $f(x) = x^3$ and $g(x) = x^2+1 \ \forall x \in R$. Find gof, fog, f^2 and g^2

Sol^n: Given that, the functions f and g are defined by

$f(x) = x^3$, $g(x) = x^2+1 \ \forall x \in R$

(i) $gof(x) = g[f(x)] = g[x^3] = [(x^3)^2+1] = (x^6+1)$

(ii) $fog(x) = f[g(x)] = f(x^2+1) = (x^2+1)^3$

(iii) $f^2(x) = f[f(x)] = f[x^3] = (x^3)^3 = x^9$

(iv) $g^2(x) = g[g(x)] = g(x^2+1) = (x^2+1)^2+1$

(P) Let f and g be functions from R to R defined by $f(x) = ax+b$ and $g(x) = 1-x+x^2$. If $(gof)(x) = 9x^2-9x+3$, determine a, b

Soln: Given that $f(x) = ax+b$, $g(x) = 1-x+x^2$

$$\therefore gof(x) = 9x^2-9x+3 \Rightarrow g[f(x)]$$
$$\Rightarrow g[ax+b]$$
$$\Rightarrow 1-(ax+b)+(ax+b)^2$$
$$= a^2x^2+(2ab-a)x+(1-b+b^2)$$

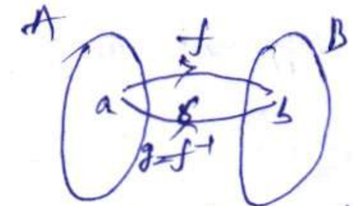
Comparing corresponding coeff, we get

$9 = a^2$, $9 = a-2ab$, $3 = 1-b+b^2$

$\therefore a = 3, b = -1$ and $a = -3, b = 2$

Invertible functions: A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $gof = I_A$ and $fog = I_B$ where I_A is the identity function on A and I_B is the identity function on B

Then, g is called an inverse of f and we write $g = f^{-1}$



(P) Let $A = \{1,2,3,4\}$ and f and g be functions from A to A given by $f = \{(1,4),(2,1),(3,2),(4,3)\}$ and $g = \{(1,2),(2,3),(3,4),(4,1)\}$. Prove that f and g inverse of each other

Soln: Given $A = \{1,2,3,4\}$ and $f = \{(1,4),(2,1),(3,2),(4,3)\}$
$g = \{(1,2),(2,3),(3,4),(4,1)\}$

$\therefore gof(1) = g[f(1)] = g(4) = 1 = I_A(1)$	$fog(1) = f[g(1)] = f(2) = 1 = I_A(1)$
$gof(2) = g[f(2)] = g(1) = 2 = I_A(2)$	$fog(2) = f[g(2)] = f(3) = 2 = I_A(2)$
$gof(3) = g[f(3)] = g(2) = 3 = I_A(3)$	$fog(3) = f[g(3)] = f(4) = 3 = I_A(3)$
$gof(4) = g[f(4)] = g(3) = 4 = I_A(4)$	$fog(4) = f[g(4)] = f(1) = 4 = I_A(4)$

Thus, $\forall x \in A$, we have $gof(x) = I_A(x)$ and $fog(x) = I_A(x)$.

$\therefore g$ is an inverse of f and f is an inverse of g //

(P) Let $A=B=R$, the set of all real numbers and the function $f: A\to B$ and $g: B\to A$ be defined by $f(x)=2x^3-1\ \forall x\in A$, $g(x)=\left[\frac{1}{2}(x+1)\right]^{1/3}$, $x\in A$ show that each of f and g is the inverse of the other.

Soln: Given that $f(x)=2x^3-1$, $g(x)=\left[\frac{1}{2}(x+1)\right]^{1/3}$

(i) $g\circ f(x)=g[f(x)]=g(2x^3-1)]=\left[\frac{1}{2}(2x^3-1+1)\right]^{1/3}=x=I_A(x)$

$\therefore g\circ f=I_A$

(ii) $f\circ g(x)=f[g(x)]=f\left[\left[\frac{1}{2}(x+1)\right]^{1/3}\right]=2\left[\frac{1}{2}(x+1)^{1/3}\right]^3-1=x=I_B(x)$

$\therefore f\circ g=I_A$

$\therefore \forall x$, $f\circ g(x)=I_B$ and $g\circ f(x)=I_A$

$\therefore g$ is invertible of f and f is invertible of g.

Recursive function:

Recursive function is a function that repeats its own previous term to calculate subsequent terms and thus forms a sequence of terms.

i.e. A function which calls itself from its previous value to generate subsequent value.

(P) Obtain a recursive definition for the function $f(n)=a_n$ in each of the following cases (i) $a_n=5n$ (ii) $a_n=6^n$ (iii) $a_n=3n+7$ (iv) $a_n=2-(-1)^n$ (v) $a_n=n(n+2)$ (vi) n^2

Soln: (i) Here $a_0=0$, $a_1=5$, $a_2=10$, $a_3=15\ldots$

We can rewrite these are $a_0=0$. and $a_n=a_{n-1}+5$ for $n\geq 1$

This is a recursive definition of the given fn

(ii) Here $a_0=1$, $a_1=6$, $a_2=6^2$, $a_3=6^3\ldots$

We can rewrite these are $a_0=1$, $a_{n+1}=6\times a_n$ for $n\geq 0$

This is a recursive defn of the given fn

(iii) Here $a_0=7$, $a_1=10$, $a_2=13$, $a_3=16$, $a_4=19\ldots$

These can be written as $a_0=7$ and $a_n=a_{n-1}+3$ for $n\geq 1$

This is recursive defn of the given fn

(iv) Here $a_0=1$, $a_1=3$, $a_2=1$, $a_3=3,\ldots a_n=2-(-1)^n$ and $a_{n+1}=2-(-1)^{n+1}$.

These give $a_{n+1}-a_n=-(-1)^{n+1}+(-1)^n=2(-1)^n$

i.e. $a_{n+1}=a_n+2(-1)^n$ for $n\geq 0$

Thus, $a_0=1$ and $a_{n+1}=a_n+2(-1)^n$ for $n\geq 0$

is a recursive defn of the given sequence//

(v) Here $a_0 = 0,\ a_1 = 3,\ a_2 = 8,\ a_3 = 15 \ldots$

We observe that $a_1 - a_0 = 3,\ a_2 - a_1 = 5 = 2\times1+3,\ a_3 - a_2 = 7 = 2\times2+3, \ldots$

We take $a_{n+1} - a_n = 2n+3$ ie $a_{n+1} = a_n + 2n + 3,\ n \geq 0$

Thus $a_0 = 0,\ a_{n+1} = a_n + 2n + 3$ for $n \geq 0$

is recursive defn of given f(n)

(vi) Here $a_0 = 0,\ a_1 = 1,\ a_2 = 4,\ a_3 = 9 \ldots$ We observe that

$a_1 - a_0 = 1,\ a_2 - a_1 = 3 = 2\times1+1,\ a_3 - a_2 = 5 = 2\times2+1$ and so on

We take $a_{n+1} - a_n = 2n+1$ ie $a_{n+1} = a_n + 2n + 1$ for $n \geq 0$

Thus $a_0 = 0$ and $a_{n+1} = a_n + 2n + 1$ for $n \geq 0$ is a recursive defn of the given f(n) //

(11) A function $f(n) = a_n$ is defined recursively by $a_0 = 4$. and $a_n = a_{n-1} + n$ for $n \geq 1$. find f(n) in explicit form

Soln: Using the given recursive formula repeatedly, we find that

$$a_n = a_{n-1} + n = [a_{n-2} + (n-1)] + n$$

$$= [a_{n-3} + (n-2)] + (n-1) + n$$

$$[a_{n-4} + (n-3)] + (n-2) + (n-1) + n$$

$$\vdots$$

$$= a_1 + 2 + 3 + 4 + \cdots + n$$

$$= (a_0 + 1) + 2 + 3 + 4 + \cdots + n$$

Using $a_0 = 4$, and standard result $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

~~= 1+2+3~~

It becomes $a_n = 4 + \frac{1}{2} n(n+1)$

This is the explicit formula for $f(n) = a_n$ for $n \geq 0$ //

(12) find an explicit defn of the function $f(n) = a_n$ defined recursively by $a_0 = 3,\ a_n = 2a_{n-1} + 1$ for $n \geq 1$

(13) For the function $f(n) = a_n$ defined recursively by $a_0 = 8,\ a_1 = 22,\ a_n = 4(a_{n-1} - a_{n-2})$ for $n \geq 2$ prove that $a_n = (8+3n)2^n$ for $n \geq 0$

Mod function: for each +ve integer number, m the mod-m function defined as

$f_m(a) = b$ or $a \equiv b \pmod{m}$: $0 \le b < m$, a is non-negative integer

The term mod is used for mathematical congruence relation and is defined as

$a \equiv b \pmod{m}$ if and only m divides b-a

- Here 'm' is called the modulus and $a \equiv b \pmod{m}$ read as "a is congruent to b modulo m."

Ex:- (i) $f_6(39) = 3$ because $39 = 6\times 6+3$ and $39 \equiv 3 \pmod{6}$

(ii) $f_6(3) = 3$ because $3 = 0\times 6+3$ and $3 \equiv 3 \pmod{6}$

Logarithmic function: Let 'a' be a +ve number. The exponential function $y = a^x$ can be expressed as the logarithmic function as $x = \log_a y$. The subscript 'a' is called the base of the logarithm.

Floor and Ceiling function:-

Let 'x' be any real number. The floor function f(x) defined for x is largest integer less than or equal to x. The notation $\lfloor x \rfloor$ is used for f(x)

Ex:- $f(4.25) = \lfloor 4.25 \rfloor = 4$ (iii) $f(-3.75) = \lfloor -3.75 \rfloor = -4$

$f(\sqrt{7}) = \lfloor \sqrt{7} \rfloor = 2$ (iv) $f(-3) = \lfloor -3 \rfloor = -3$

A ceiling function, c(x) defined for x is the smallest integer greater than or equal to x. The notation $\lceil x \rceil$ is used for c(x)

Ex: (i) $C(4.25) = \lceil 4.25 \rceil = 5$ (ii) $C(-3.75) = \lceil -3.75 \rceil = -3$

Fibonacci sequence: The sequence in which each succeeding term is the sum of the two ~~prece~~ preceding terms is called Fibonacci sequence and its ~~usually~~ terms are usually denoted by $f_1, f_2, f_3, \ldots$

The formal defn of a sequence is

(i) If $n = 0$ or 1, then $f_n = n$

(ii) If $n > 1$ then $f_n = f_{n-2} + f_{n-1}$

This defn uses recursive formula to describe the sequence, where

(i) The base (initial) values are 0 and 1

(ii) The value of f_n is defined in terms of smaller values of n which are closer to the initial values

Ex:- 0, 1, 1, 2, 3, 5, 8 — is the fibonacci sequence

(P) The fibonacci function $F(n) = F_n$ is defined recursively by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ Evaluate F_2 to F_{10}

soln: Using the given defn $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

$F_2 = F_1 + F_0 = 1 + 0 = 1$, $F_3 = F_2 + F_1 = 1 + 1 = 2$, $F_4 = F_3 + F_2 = 2 + 1 = 3$

$F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$.

(P) For the Fibonacci sequence $F_0, F_1, F_2 \ldots$ prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

soln. For $n = 0$, and $n = 1$, then $F_0 = 0$, $F_1 = 1$ which are true.

Thus, the required result is true for $n = 0$ and $n = 1$.

We assume that the result is true for $n = 0, 1, 2 \ldots k$ where $k \geq 1$

Then $F_{k+1} = F_k + F_{k-1}$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1+\sqrt{5}}{2} + 1\right\} + \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1-\sqrt{5}}{2} + 1\right\}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

This shows that the required result is true for $n = k+1$

Hence, by mathematical induction, the result is true for all non-negative integers n

Lucas function. $L(n) = L_n$ is defined recursively by

$L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$