UNIT-3

Set Theory

Set:A set is collection of well defined objects.

In the above definition the words set and collection for all practical purposes are Synonymous. We have really used the word set to define itself.

Each of the objects in the set is called a member of an element of the set. The objects themselves can be almost anything. Books, cities, numbers, animals, flowers, etc. Elements of a set are usually denoted by lower-case letters. While sets are denoted by capital letters of English larguage.

The symbol ∈ indicates the membership in a set.

If –*a* is an element of the set *A*‖, then we write $a \in A$.

The symbol ∈ is read —is a member of ‖ or —is an element of ‖.

The symbol □ is used to indicate that an object is not in the given set. The symbol □ is read —is not a member of ‖ or —is not an element of ‖If *x* is not an element of the set *A* then we write *x* □ *A*.

Subset:

A set *A* is a subset of the set *B* if and only if every element of *A* is also an element of *B*. We also say that *A* is contained in *B*, and use the notation *A* □ *B*.

Proper Subset:

A set *A* is called proper subset of the set *B*. If (*i*) *A* is subset of *B* and (*ii*) *B* is not a subset *A* i.e., *A* is said to be a proper subset of *B* if every element of *A* belongs to the set *B*, but there is atleast one element of *B*, which is not in *A*. If *A* is a proper subset of *B*, then we denote it by *A* □ *B*.

Super set: If *A* is subset of *B*, then *B* is called a superset of *A*.

Null set: The set with no elements is called an empty set or null set. A Null set is designated by the symbol □ . The null set is a subset of every set, i.e., If *A* is any set then □ □ *A*.

Universal set:

In many discussions all the sets are considered to be subsets of one particular set. This set is called the universal set for that discussion. The Universal set is often designated by the script letter □ . Universal set in

not unique and it may change from one discussion to another.

Power set:

The set of all subsets of a set *A* is called the power set of *A*.

The power set of A is denoted by $P(A)$. If A has n elements in it, then $P(A)$ has 2^n elements:

Disjoint sets:

Two sets are said to be disjoint if they have no element in common.

Union of two sets:

The union of two sets A and B is the set whose elements are all of the elements in A or in B or in both. The union of sets A and B denoted by A □B is read as ―A union B‖.

For example:

A = {1,2,3,4,5}

B = {2,4,6,8,10}

The union of A and B (i.e. A∪B) is {1, 2, 3, 4, 5, 6, 8, 10}

Intersection of two sets:

The intersection of two sets A and B is the set whose elements are all of the elements common to both A and B. The intersection of the sets of ―A‖ and ―B‖ is denoted by A □B and is read as ―A intersection B‖

For example:

A = {1,2,3,4,5}

B = {2,4,6,8,10}

The intersection of A and B (i.e. A∩B) is simply {2, 4}

Difference of sets:

If A and B are subsets of the universal set U, then the relative complement of B in $A$is the set of all elements in

A which are not in A. It is denoted by $A – B$ thus: $A – B$ = {x | x ∈ A and x□B}

For example:

A = {1,2,3,4,5}

B = {2,3,5}

The difference of A and B (i.e. A-B) is {1,4}

NOTE: A-B ≠ B-A

Complement of a set:

If U is a universal set containing the set A, then $U - A$ is called the complement of A. It is denoted by A^1 . Thus

$A^1 = \{x : x \notin A\}$

For example:

U= {1,2,3,4,5}

A = {2,3,5}

The compliment of A is (i.e. U-A) is {1,4}

Cartesian Product of Sets

The Cartesian product of sets A and B, written A x B, is expressed as:

A x B = {(a,b) | a is every element in A, b is every element in B}

For example:

A = {1,2}

B = {4,5,6}

The Cartesian product of A and B (i.e. A x B) is {(1,4), (1,5), (1,6), (2,4), (2,5), (2,6)}

Now, lets us try doing some questions based on Set Theory.

Inclusion-Exclusion Principle:

The inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the unionof two finite sets; symbolically expressed as

$|A \cup B| = |A| + |B| - |A \cap B|$.

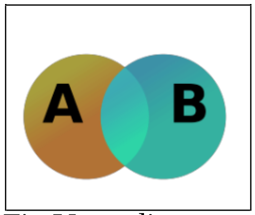


Fig.Venn diagram showing the

union of sets A and B

where *A* and *B* are two finite sets and |*S*| indicates the cardinality of a set *S* (which may be considered as the number of elements of the set, if the set is finite). The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

The principle is more clearly seen in the case of three sets, which for the sets *A*, *B* and *C* is given by

$|A \cup B \cup BC| = |A| + |B| + |C| - |A \cap B| - |C \cap B| - |A \cap C| + |A \cap B \cap C|$.

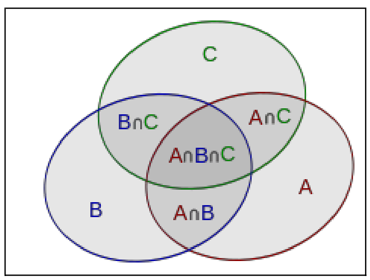


Fig.Inclusion–exclusion

illustrated by a Venn

diagram for three sets

This formula can be verified by counting how many times each region in the Venn diagram figure is included in the right-hand side of the formula. In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.

In general, Let A1, · · · , Ap be finite subsets of a set U. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_p| = \sum_{1 \le i \le p} |A_i| - \sum_{1 \le i_1 < i_2 \le p} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le p} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \ldots + (-1)^{p-1} |A_1 \cap A_2 \cap \cdots \cap A_p|,$$

Example: How many natural numbers $n \le 1000$ are not divisible by any of 2, 3? Ans: Let $A_2 = \{n \in N \mid n \le 1000, 2|n\}$ and $A_3 = \{n \in N \mid n \le 1000, 3|n\}$.

Then, $|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = 500 + 333 - 166 = 667$.

So, the required answer is $1000 - 667 = 333$.

Example: How many integers between 1 and 10000 are divisible by none of 2, 3, 5, 7? Ans: For $i \in \{2, 3, 5, 7\}$, let $A_i = \{n \in N \mid n \le 10000, i|n\}$.

Therefore, the required answer is $10000 - |A_2 \cup A_3 \cup A_5 \cup A_7| = 2285$.

1) If ∪ = {1, 3, 5, 7, 9, 11, 13}, then which of the following are subsets of U.

B = {2, 4}

A = {0}

C = {1, 9, 5, 13}

D = {5, 11, 1}

E = {13, 7, 9, 11, 5, 3, 1}

F = {2, 3, 4, 5}

Answer: Here, we can see that C, D and E have the terms which are there in ∪. Therefore, C, D and E are the subsets of ∪.

Question 2: Let A and B be two finite sets such that n(A) = 20, n(B) = 28 and n(A ∪ B) = 36, find n(A ∩ B).

Solution: Using the formula n(A ∪ B) = n(A) + n(B) - n(A ∩ B).

then n(A ∩B) = n(A) + n(B) - n(A ∪B)

= 20 + 28 - 36

= 48 - 36

= 12

Question 3: In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both coffee and tea?

Solution: Let A = Set of people who like cold drinks B = Set of people who like hot drinks Given,

$(A \cup B) = 60 \quad n(A) = 27 \quad n(B) = 42$ then;

$n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 27 + 42 - 60$

$= 69 - 60 = 9$

$= 9$

Therefore, 9 people like both tea and coffee.

Question 4: In a competition, a school awarded medals in different categories. 36 medals in dance, 12 medals in dramatics and 18 medals in music. If these medals went to a total of 45 persons and only 4 persons got medals in all the three categories, how many received medals in exactly two of these categories?

Solution: Let A = set of persons who got medals in dance.

B = set of persons who got medals in dramatics.

C = set of persons who got medals in music.

Given,

$n(A) = 36$

$n(B) = 12$

$n(C) = 18$

$n(A \cup B \cup C) = 45$

$n(A \cap B \cap C) = 4$

We know that number of elements belonging to exactly two of the three sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3n(A \cap B \cap C)$

$= n(A \cap B) + n(B \cap C) + n(A \cap C) - 3 \times 4$ ……..(i)

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Therefore, $n(A \cap B) + n(B \cap C) + n(A \cap C) = n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C)$

From (i) required number

$= n(A) + n(B) + n(C) + n(A \cap B \cap C) - n(A \cup B \cup C) - 12$

$= 36 + 12 + 18 + 4 - 45 - 12$

$= 70 - 67$

$= 3$

Question 5: In a group of 100 persons, 72 people can speak English and 43 can speak French. How many can speak English only? How many can speak French only and how many can speak both English and French?

Solution: Let A be the set of people who speak English.

B be the set of people who speak French.

A - B be the set of people who speak English and not French.

B - A be the set of people who speak French and not English.

$A \cap B$ be the set of people who speak both French and English.

Given,

$n(A) = 72$

$n(B) = 43$

$n(A \cup B) = 100$

Now, $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 72 + 43 - 100$

$= 115 - 100$

$= 15$

Therefore, Number of persons who speak both French and English = 15

$n(A) = n(A - B) + n(A \cap B) \Rightarrow$

$n(A - B) = n(A) - n(A \cap B)$

$= 72 - 15$

$= 57$

and $n(B - A) = n(B) - n(A \cap B)$

$= 43 - 15$

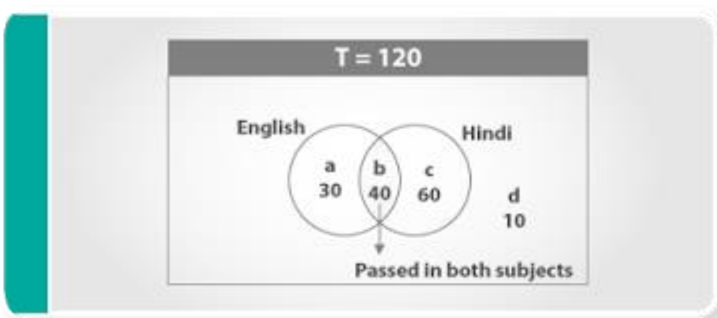
$= 28$

Therefore, Number of people speaking English only = 57

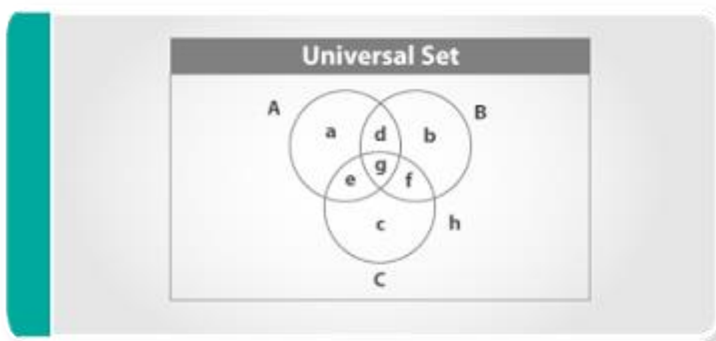
Number of people speaking French only = 28

Question: In a class of 120 students, 70 students passed in English, 80 students passed in Hindi and 40 students passes in both English and Hindi. How many students failed in both the students?

Solution:
Step 1:
To solve Set theory Questions by Using Set theory formulas, we need to first draw a Venn diagram.
First draw a rectangle which represents total number of students, then draw two circles which intersects each other. The region which represent number of students who passed in English label it as ‘a’, the region which represents number of students who passed in Hindi label it as ‘c’ and the intersection region which represents total number of students who passed in both English and Hindi label it as ‘b.’



Step 2:
By substituting the value sin the set theory formulas,we get
Total = A + B – Both AB + none (AB)
120 = 70 + 80 – 40 + none
None = 120 – 110
None = 10 students
Therefore, there are 10 students who neither passed in English nor Hindi.
How to solve Set Theory Questions using Set Theory Formula when Three Sets are given.
In this module, we will discuss, how to draw Venn diagrams when three sets are given and solve this Set Theory Questions by using Set theory formulas.
In a Venn diagram, A rectangle represents the universal sets and the circles represent the subsets, the overlapping of the circles shows the intersection of two sets.

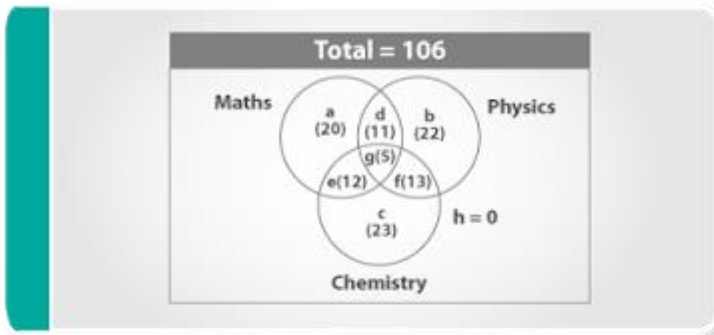


A = a + d+ e + g ; Only A = a
B = b + d + f + g ; Only B = b
C = c + e + f + g ; Only C = c
Both (AB) = d + g
Both (AC) = e + g
Both (BC) = f + g
All (ABC) = g
None (ABC) = h

Total = A + B + C - Both (AB) - Both (BC) - Both (CA) + All (ABC) + None (ABC)

An Example of Set Theory Question solved using Set Theory Formulas.

Question: In a class of 106 students, each student studies at least one of the three subjects Maths, Physics and Chemistry. 48 of them study Maths, 51 studies Physics and 53 Chemistry. 16 studies Maths and Physics, 17 study Maths and Chemistry and 18 study Physics and Chemistry.
Solution:
Step 1:
To solve this set theory question by set using set theory formula first draw a Venn diagram by using the details in the questions.



Step 2:
Before going ahead with the set theory questions, let's find out the value of all the elements.
=> g = all(ABC) = ?
Total= A + B + C + AB – Both (AB) – Both (BC) – Both (CA) + All (ABC) + None (ABC)
By substituting the values in the above formulas,
106 = 48 + 51 + 53 + -16 – 17 – 18 + All (ABC) – o
All ABC = 5 = g
=> e = No.of students who study chemistry and Maths – All (ABC)
e = 17 – 5 = 12
=> f = Number of students who study Physics and Chemistry – All (ABC)
f = 18 – 5 = 13
=> d = Number of students who study Maths and Physics – All (ABC)
d = 16 – 5 = 11
=> a = total number of students who study Maths – d – g – e
a = 48 – 11- 5 – 12 = 20
=> b = total number of students who study Physics – d – g – f
b = 51 – 11 – 5 – 13 = 22
=> c = total number of students who study Chemistry – e – g – f
c = 53 – 12 – 5 – 13 = 23

Question 1: The number of students who exactly study two subjects is?
Solution:
In the Venn diagram d, e and f represent number of students who study exactly two subjects.
Number of Students who study exactly two subjects = d + e + f
Number of Students who study exactly two subjects = 11 + 12 + 13 = 36
Therefore, 36 students study exactly two subjects.
Question 2: The number of Students who study more than two subjects?
Solution:
In the Venn diagram d, e, g and f represents the number of students who study exactly more than three subjects.
Number of Students who study more than one subject = d + e + g + f
Number of Students who study more than one subject = 11 + 12 + 13 + 5 = 41
Therefore, 41 students study more than one subject.
Question 3: The number of students who study all the three subjects?
Solution:

In the Venn diagram ‘g’ represents the number of students who study all the three subjects.
Therefore, 5 students study all the three subjects.
Question 4: The number of students who exactly study one subjects?
Solution:
In the Venn diagram a, b and c represents the number of students who study exactly one subject.
Number of Students who study exactly one subject = a + b + c
Number of Students who study exactly one subject = 20 + 22 + 23 = 65
Therefore, 65 students study exactly one subject.
Question 5: The number of students who study Physics and Maths but not Chemistry?
Solution:
The region that represents the students who study Physics and Maths is ‘d.’
Therefore, 11 students study only Physics and Maths.

Set Identities:
Sets: A, B, C
Universal set: I
Complement: $\overline{A}$
Proper subset: $A \subset B$

Empty set: $\emptyset$
Union of sets: $A \cup B$
Intersection of sets: $A \cap B$
Difference of sets: $A \setminus B$

1. $A \subset I$
2. $A \subset A$
3. $A=B$, if $A \subset B$ and $B \subset A$
4. Empty set $\emptyset \subset A$
5. Union of sets $C=A \cup B= \{x|x \in A or x \in B\}$
6. Commutativity of union $A \cup B=B \cup A$
7. Associativity of union $A \cup (B \cup C)= (A \cup B) \cup C$
8. Intersection of sets $C=A \cap B= \{x|x \in A and x \in B\}$
9. Commutativity of intersection $A \cap B=B \cap A$
10. Associativity of intersection $A \cap (B \cap C)=(A \cap B) \cap C$

11. Distributivity

    $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$

    $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

12. Idempotency

    $A\cap A=A$

    $A\cup A=A$

13. Domination (Intersection of any set with the empty set) $A\cap\emptyset=\emptyset$

14. Union of any set with the universal set $A\cup I=I$

15. Union of any set with the empty set $A\cup\emptyset=A$

16. Complement $\overline{A}=\{x\in I|x\notin A\}$

17. Properties of the Complement

    $A\cup\overline{A}=I$

    $A\cap\overline{A}=\emptyset$

18. De Morgan's laws

    $\overline{(A\cup B)}=\overline{A}\cap\overline{B}$

    $\overline{(A\cap B)}=\overline{A}\cup\overline{B}$

19. Difference of sets $C=B\backslash A=\{x|x\in B \text{and} x\notin A\}$

20. $B\backslash A=B\backslash(A\cap B)$

21. $B\backslash A=B\cap\overline{A}$

22. Difference of a set from itself $A\backslash A=\emptyset$

23. $A\backslash B=A$, if$A\cap B=\emptyset$

24. $(A\backslash B)\cap C=(A\cap C)\backslash(B\cap C)$

25. $\overline{A}=I\backslash A$

26. Cartesian product $C=A\times B=\{(x,y)|x\in A \text{and} y\in B\}$

Identity 1. Let A and B be sets. Show that $A\cup(B-A)=A\cup B$

Proof.

$A\cup(B-A)=A\cup(B\cap A^c)$ set difference

$= A \cup (A^c \cap B)$ commutative

$= (A \cup A^c) \cap (A \cup B)$ distributive

$= U \cap (A \cup B)$ complement

$= A \cup B$ identity

Identity 2. Let A and B be sets. Show that $(A \cap B^c)^c \cup B = A^c \cup B$

Proof.

$(A \cap B^c)^c \cup B = (A^c \cup (B^c)^c) \cup B$ de Morgan's

$= (A^c \cup B) \cup B$ double complement

$= A^c \cup (B \cup B)$ associative

$= A^c \cup B$ idempotent

Identity 3. Let A, B and C be sets. Show that $(A - B) - C = A - (B \cup C)$

Proof.

$(A - B) - C = (A \cap B^c) - C$ set difference

$= (A \cap B^c) \cap C^c$ set difference

$= A \cap (B^c \cap C^c)$ associative

$= A \cap (B \cup C)^c$ de Morgan's

$= A - (B \cup C)$ set difference

Identity 4. Let A, B and C be sets. Show that $(B - A) \cup (C - A) = (B \cup C) - A$

Proof.

$(B - A) \cup (C - A) = (B \cap A^c) \cup (C \cap A^c)$ set difference

$= (B \cup C) \cap A^c$ distributive

$= (B \cup C) - A$ set difference