

## UNIT-IV

(1)

### ELEMENTARY COMBINATORIAL

Basic rules of counting: (i) sum Rule (ii) The product Rule

The sum Rule: Suppose two tasks  $T_1$  and  $T_2$  are to be performed. If the task  $T_1$  can be performed in ' $m$ ' different ways and the task  $T_2$  can be performed in ' $n$ ' different ways and if these two tasks can't be performed simultaneously, then one of the two tasks i.e.  $T_1$  or  $T_2$  can be performed in  $m+n$  ways.

In generally, if  $T_1, T_2, \dots, T_k$  are  $k$  tasks such that no two of these tasks can be performed at the same time and if the task  $T_i$  can be performed in  $n_i$  different ways, then one of the  $k$ -tasks i.e.  $T_1$  or  $T_2$  or ... or  $T_k$  can be performed in  $n_1 + n_2 + \dots + n_k$  ways.

Ex.: Suppose there are 16 boys and 18 girls in a class and we wish to select one of these students (either a boy or a girl) as the class representative. The no. of ways of selecting a boy is 16 and the no. of ways of selecting a girl is 18.

$\therefore$  The no. of ways of selecting a student (boy or girl) is  $16+18=34$

- (P) How many ways can we get a sum of 4 or of 8 when two distinguishable dice (say one die is red and the other is white) are rolled?  
 - How many ways can we get an even sum?

Soln: We obtain the sum 4 from the outcomes  $(1,3), (2,2), (3,1)$

Thus there are 3 ways to obtain the sum 4

Likewise, obtain the sum 8 from the outcomes  $(2,6), (3,5), (4,4), (5,3), (6,2)$

Thus there are 5 ways to obtain the sum 8.

$\therefore 3+5=8$  outcomes whose sum is 4 or 8.

$\rightarrow$  The no. of ways to obtain an even sum is the same as the no. of ways to obtain either the sum 2, 4, 6, 8, 10 or 12

Sum 2  $\rightarrow (1,1)$ , Sum 4  $\rightarrow (2,2), (1,3), (3,1)$

Sum 6  $\rightarrow (1,5), (5,1), (2,4), (4,2), (3,3)$

Sum 8  $\rightarrow (2,6), (3,5), (4,4), (5,3), (6,2)$

Sum 10  $\rightarrow (4,6), (6,4), (5,5)$

Sum 12  $\rightarrow (6,6)$

$\therefore$  There are  $1+3+5+5+3+1 = 18$  ways to obtain an even sum

The product Rule: Suppose that two tasks  $T_1$  and  $T_2$  are to be performed one after the other. If  $T_1$  can be performed in  $n_1$  different ways and for each of these ways  $T_2$  can be performed in  $n_2$  different ways, then both of the tasks can be performed in  $n_1 n_2$  different ways.

In general, suppose that  $K$  tasks  $T_1, T_2, \dots, T_K$  are to be performed in a sequence. If  $T_1$  can be performed in  $n_1$  different ways and for each of these ways  $T_2$  can be performed in  $n_2$  different ways, and for each of  $n_1 n_2$  different ways of performing  $T_1$  and  $T_2$  in that order,  $T_3$  can be performed in  $n_3$  different ways, and so on, then the sequence of tasks  $T_1, T_2, \dots, T_K$  can be performed in  $n_1 n_2 n_3 \dots n_K$  different ways.

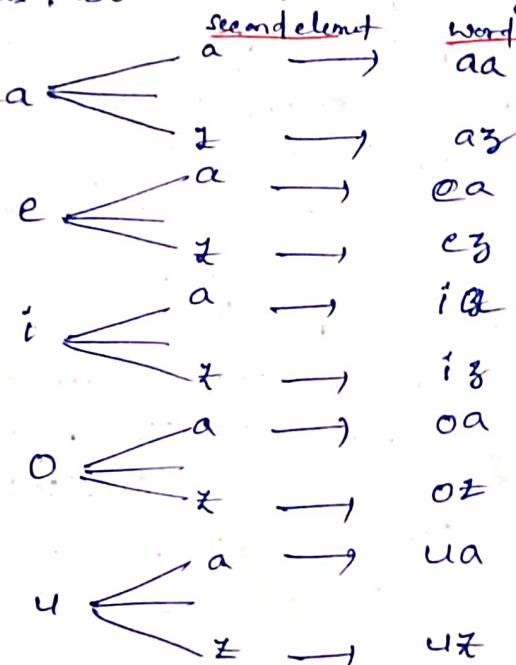
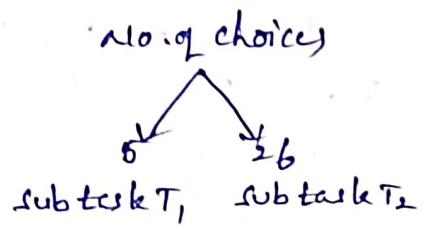
- ① find the number of two letter words that begin with a vowel a,e,i,o,u

Soln: The task of forming a two letter word consists of two subtasks,  $T_1$  and  $T_2$ .

$T_1$  consists of selecting the first letter and  $T_2$  consists of selecting the second letter.

Since each word must begin with a vowel  $T_1$  can be accomplished in 5 ways. There are no restrictions on the choice of the second letter so  $T_2$  can be done in 26 ways.

∴ By Multiplication principle, the task can be performed  $5 \times 26 = 130$  different ways. In otherwords, 130 two-letter words begin with a vowel.



(2) Show that a set 'S' with n elements has  $2^n$  subsets. (2)

Soln: Given that a set S with n elements

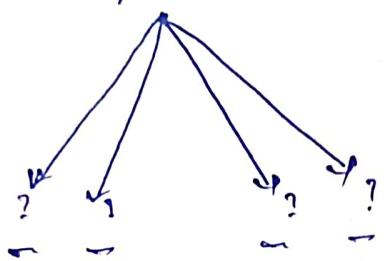
we prove that the set 'S' has  $2^n$  subsets

Every subset of 'S' can be uniquely identified by n-bit word  
The task of forming an n-bit word can be broken down to n sub-tasks, selecting a bit for each of the n positions

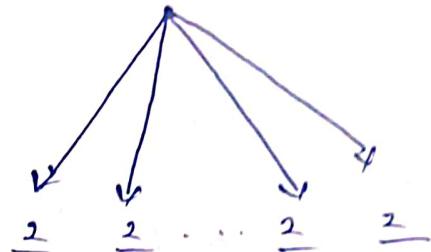
Each position in the word has two choices 0 or 1  
so by the multiplication principle, the total number of n-bit words  
that can be formed  $\underbrace{2 \cdot 2 \cdots 2}_{n\text{-times}} = 2^n$

In otherwords, S has  $2^n$  subsets

No. of choices



No. of choices



(3) If 2 distinguishable dice are rolled, in how many ways can they fall?

(4) Suppose that the license plate can be manufactured if repetitions of letters of a certain state require 3 English letters followed by 4 digits.

(a) How many different plates can be manufactured if repetitions of letters and digits are allowed

(b) How many plates are possible if only the letters can be repeated

(c) How many plates are possible if only the digits can be repeated.

(d) How many plates are possible if no repetitions are allowed at all

Soln: (a)  $26^3 \cdot 10^4$ , since there are 26 possibilities for each of 3 letters and 10 possibilities for each of 4 digits

(b)  $26^3 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

(c)  $26 \cdot 25 \cdot 24 \cdot 10^4$

(d)  $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

- (P) (a) How many 3-digit numbers can be formed using the digits 1, 3, 4, 5, 6, 8, 9  
 (b) How many can be formed if no digit can be repeated.

Soln: (a) There are  $7^3$  such 3-digit numbers since each of the 3 digits can be filled with 7 possibilities

(b) There are  $7 \times 6 \times 5$  such 3-digit numbers

Since there are 7 possibilities for the hundreds place but digit 8 is used it is not available for the tens place (since no digit can be repeated in this place). Thus there only one '6' possibility for the tens place and then for the same reason there are only 5 possibilities for the units place

- (P) There are 20 married couple in a party. Find the number of ways of choosing one woman and one man from the party such that the two are not married to each other

Soln: From the party, a woman can be chosen in 20 ways

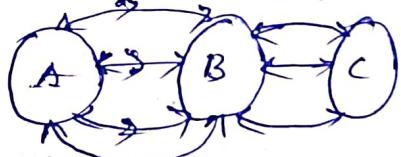
Among the 20 men in the party one is her husband out of the 19 other men, one can be chosen in 19 ways

$\therefore$  The required number is  $20 \times 19 = 380$

- (P) There are four bus routes b/w the places A and B and three bus routes b/w the places B and C. Find the no. of ways a person can make a round trip from A to A via B if he does not use a route more than once

Soln: The person can travel from A to B in four ways and from B to C in three ways, but only in two ways from C to B and only in three ways from B to A if he does not use a route more than once.

$\therefore$  The no. of ways he can make the round trip under the given condition is  $4 \times 3 \times 2 \times 3 = 72$



- (P) A license plate consists of two English letters followed by four digits. If repetitions are allowed, how many of the plates have only vowels A, E, I, O, U and even digits?

Soln: Each of the first two positions in a plate can be filled in 5 ways with vowels

And each of the remaining four places can be filled in 5 ways

i.e. 0, 2, 4, 6, 8.

$\therefore$  The no. of possible license plates of the given type is

$$(5 \times 5) \times (5 \times 5 \times 5 \times 5) = 5^6 = 15625 //$$

- (P) find the number of 3-digit even numbers with no repeated digits

Soln:  $(1 \times 9) + (4 \times 8) = 41$ .  $\therefore$  Desired number =  $41 \times 8 = 328$

factorial notation: The product of first 'n' natural numbers  $(1, 2, 3, \dots, n)$  is denoted by  $n!$  and it is read as factorial  $n$  (or)  $n$ -factorial.  
 i.e.  $n! = n(n-1)(n-2)\dots 3.2.1.$

NOTE:  $0! = 1$

Permutation: The ordered selection or arrangements of a given set of objects, taken some or all of them at a time is called permutation of the objects.

The total no. of permutations of  $n$  objects taken  $r$  at a time is denoted by  $P(n, r)$  or  ${}^nP_r$ .

Results: 1.  $P(n, r) = \frac{n!}{(n-r)!} = {}^nP_r$

2.  ${}^nP_n = n!$

3.  ${}^nP_{n-1} = {}^nP_{n-1}$

4. The number of permutations of elements of  $A_x$  taken  $r$  at a time, allowing repetition is  $n^r$ .

5. The number of permutations of  $n$  objects of which  $p$  objects are of one type,  $q$  are of second-type,  $r$  of third-type and remaining objects are different is  $\frac{n!}{p! \times q! \times r!}$

(P) find out how many 5-digit numbers greater than 30,000 can be formed from the digits 1, 2, 3, 4, 5?

Soln: In order to find the 5-digit numbers greater than 30,000 from the digits 1, 2, 3, 4, 5

The first digit can be taken as 3, 4 or 5. But, the first digit can be chosen in  ${}^3P_1 = 3$  ways.

The remaining 4 digits can be any of the 4 digits taken in  ${}^4P_4$  ways.  
 Hence, the total no. of 5-digit numbers greater than 30,000 will be

$$3P_1 \times 4P_4 = 3 \times 4!$$

(P) Find the no. of permutations of letters of the word ENGINEERING

Soln: The given word consists of 11 letters of which there are  $3E's$ ,  $3N's$ ,  $2G's$ ,  $2I's$  and rest all are different.

$\therefore$  The required no. of permutations is  $\frac{11!}{3! \times 3! \times 2! \times 2!} = 2,77,200$

(P) find the no. of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, 6 that are divisible by 3 when repetition is allowed.

Soln. consider 4 blank places                            

To prepare four digit numbers and repetition is allowed, we can fill up each place in 6 ways.

So, we get  $6^4$  numbers. Among these  $6^4$  numbers, some of them may not be divisible by '3'.

By fixing first 3 places with some fixed digits from 1, 2, 3, 4, 5, 6 we get the following five consecutive numbers

$$a_1 a_2 a_3 1$$

$$a_1 a_2 a_3 2$$

$$a_1 a_2 a_3 3$$

$$a_1 a_2 a_3 4$$

$$a_1 a_2 a_3 5$$

$$a_1 a_2 a_3 6$$

pick it out of any five consecutive integers exactly two are divisible by 3.

∴ One third of these  $6^4$  numbers will be divisible by 3.

∴ The no. of 4 digit numbers which are divisible by 3

$$= \frac{1}{3} 6^4 = 432/$$

(P) In how many ways can 7 women and 3 men be arranged in a row if the 3 men must be always stand next to each other

Soln. No. of men = 3, No. of women = 7

No. of ways of arranging the 3 men =  $3!$

" " " " " 7 women =  $7!$

since the '3' men always stand next to each other. We treat them as a single entity. which we denote by X

Thus if  $w_1, w_2, \dots, w_7$  represents the women, we next are interested in the no. of ways of arranging  $[X, w_1, w_2, \dots, w_7]$

There are  $8!$  ways.

Hence there are  $(3!) (8!)$  permutations altogether,

(P) How many 4 letter words can be formed using the letters of the word "ARTICLE" such that

a) Each word begin with vowel.

b) The word contains A but not E

c) Each word must contain atleast one vowel

Soln. Given that, word 'ARTICLE' Contains 7 letters  
From these 7 letters we have to select 4 and then these 4 letters has to be arranged. We get  ${}^7P_4$  words without any restriction.

a) vowels in the word 'ARTICLE' are A, I, E.

Since each 4-letter word must begin with a vowel in A or E or B (Q)  
This can be done in 3 ways

The remaining 3 places can be filled with any one of the remaining  
six letters in  $6P_3$  ways

By fundamental principle, required no. of 4 letter words

$$= 3 \times 6P_3$$

(b) Let us consider 4 blank places - - - -

Since the letter A 4 letter word must contain A, arrange A in  
any one of the 4 blank places. It can be done 4 ways

Since, the word must not contain E, the 3 remaining  
3 places can be filled from the remaining 5 letters. This can be  
done in  $5P_3$  ways.

By fundamental principle, the required no. of arrangements

$$= 4 \times 5P_3$$

(c) The no. of 4 letter words which contain atleast one vowel  
= (The no. of 4 letter words without any restriction) - (The number  
of 4 letter words which contain no vowel)

$$= 7P_4 - 4P_4 = 816$$

(P) Find the no. of ways of arranging the letters of the word  
'FATHER' so that

(a) The relative positions of vowels and consonants are not  
disturbed

(b) no vowel occupies even place

Sol? (a) This can be done  $4P_4 \cdot 2P_2 = 48$  ways

(b) Let us take 6 blanks  $\underline{1} \underline{2} \underline{3} \underline{4} \underline{5} \underline{6}$

Among these 6 places, 3 are even places. Since no vowel  
occupies even place, we have to arrange the two vowels in 3  
odd places. It can be done in  $3P_2$  ways

Now, we shall be left with 4 places (3 even and one odd)  
in which the consonants can be arranged. It can be done in  
 $4P_4$  ways.

$\therefore$  Required no. of arrangements  $= 3P_2 \cdot 4P_4 = 144$

(Q) If the letters of the word 'SIPIRON' are arranged in all possible ways and the words thus formed are arranged in dictionary order. find the rank of the word 'PRISON'

Soln. The letters of the word SIPIRON in dictionary order is

I, N, O, P, R, S

The no. of words which begin with I =  $5P_5 = 120$

" " " " N =  $5P_5 = 120$

" " " " O =  $5P_5 = 120$

The no. of words begin with PI =  $4P_4 = 24$

" " " " PN =  $4P_4 = 24$

" " " " PO =  $4P_4 = 24$

" " " " PRIN =  $2P_2 = 2$

" " " " PRIO =  $2P_2 = 2$

" " " " PRISONO =  $1P_1 = 1$

The next word is PRISON

$\therefore$  The rank of the word = 438

(P) If the letters of the word 'STREAM' are arranged in all possible ways and the words thus formed are arranged as in a dictionary and the word whose rank is 257.

Soln. The order of the letters of the word 'STREAM' is A, E, M, R, S, T

We know  $257 = 5! + 5! + 3! + 3! + 2! + 2! + 1!$

The first 5! words begin with A

Next 5! " " " B

Next 3! " " " MAE

Next 3! " " " MAR

Next 2! " " " MASB

Next 2! " " " MASR

The next word is MASTER

$\therefore$  The word with rank 257 is "MASTER"

## Permutation with repetition

(5)

The repetition is allowed then the number of permutations of  $n$  objects from a set of ' $n$ ' objects is  $n^n$ .

- ① Consider the 6 digits numbers 2, 3, 4, 5, 6 and 8 and repetitions of digits are allowed.

(a) How many 3 digit can be formed

(b) How many 3 digit numbers must contain the digit 5

- Soln: (a) For a 3 digit number we have to fill up three places. Since repetitions of the digit is allowed, each of the places can be filled up in 6 ways.

$$\text{Hence the required 3 digit numbers} = 6 \times 6 \times 6 = 6^3 = 216$$

- (b) Excluding the digit 5, the no. of 3 digit numbers that can be formed from the remaining 5 digits 2, 3, 4, 6 & 8

$$\text{is } 5 \times 5 \times 5 = 5^3 = 125$$

Hence the number must contain the digit 5

$$= \text{Total 3 digit numbers} - \text{the no. of 3 digit numbers that do not contain 5}$$

$$= 216 - 125 = 91$$

- ② There are 25 true or false questions on an examination. How many different ways can a student do the examination if he or she can also choose to leave the answer blank.

- Soln: Given that total number questions = 25

First question can answers 3 ways (T or F or Blank)

Second question can answers 3 ways (T or F or Blank)

.....

25<sup>th</sup> question can answers 3 ways (T or F or Blank)

∴ Different ways can a student do the examination =  $3^{25}$

- ③ How many integers b/w  $10^5$  and  $10^6$

(a) Have no digit other than 2, 5 or 8

(b) Have no digit other than 0, 2, 5 or 8

- Soln: (a) The integer b/w  $10^5$  and  $10^6$  will contain 6 digits

since only 3 digits 2, 5 or 8 are available and the repetition is allowed.

The six places can be filled up in  $3^6$  ways

Hence the required no. of integers is  $3^6$

- (b) Since only 4 digits 0, 2, 5 or 8 are available and the repetition is allowed.

The first place can be filled up in 3 ways and the remaining places can be filled up with  $4^5$  ways.

Hence the required no. of integers  $\underline{3} \ 4 \ 4 \ 4 \ 4 \ 4$   
is  $3 \times 4^5$

- (i) How many integers b/w 1 and  $10^4$  contain exactly one 8 and one 9

Soln:- (i) The number of 2-digit numbers contain exactly one 8 and one 9 i.e. 89 and 98

(ii) We arrange the digits 8, 9 in 3 places  $3P_2$  ways i.e. 6

(a) We arrange the digits 8, 9 in last two positions in 2 ways i.e. unit place and tenth position.

In this case we arrange the digits in  $100^{th}$  position in 7 ways i.e. we choose digits 1 to 7

$$\therefore \text{The total no. of ways } 7+7=14 \quad - \frac{8}{2} \frac{9}{2}$$

(b) We arrange the digits 8, 9 in 1<sup>st</sup> and last position in 2 ways i.e. units and  $10^{th}$  position

In this we arrange the digit in  $10^{th}$  position in 8 ways i.e. we choose digit 0 to 7  $\underline{8} - \underline{9}, \underline{2} - \underline{8}$

$$\therefore \text{The total number of ways is } 8+8=16$$

(c) We arrange the digits 8, 9 in 1<sup>st</sup> two positions in 2 ways i.e.  $10^{th}$  and  $100^{th}$  position  $\underline{\underline{8}} \underline{\underline{9}} - - , \underline{\underline{9}} \underline{\underline{8}} -$

In this case we arrange the digits in unit position in 8 ways i.e. (we choose digit 0 to 7)

$$\therefore \text{The total no. of ways is } 8+8=16$$

(iii) We arrange the digits 8, 9 in 4 places,  $4P_2$  ways i.e. 12  
The remaining 2 positions with the numbers 0-7 in  
 $7 \times 8$  ways and  $8 \times 8$  ways  
 $\therefore \text{The total no. of ways is } 6 \times 7 \times 8 + 6 \times 8 \times 8 = 720$

Q. The number of integers b/w 1 and  $10^4$  exactly one 8 and one 9 is  $2+14+16+16+720 = 768$  (Q)

(P) A ~~non~~-palindrome is word that reads the same forward or backward. How many 9-letter palindromes are possible using English alphabets?

Soln. Since 1<sup>st</sup> and 9<sup>th</sup>, 2<sup>nd</sup> and 8<sup>th</sup>, 3<sup>rd</sup> and 7<sup>th</sup>, 4<sup>th</sup> and 6<sup>th</sup> are equal.

We have to select five letters

There are 26 letters, so  $26 \times 26 \times 26 \times 26 \times 26 = 26^5 = 11,881,376$

(P) How many positive integers with non-repeated digits are there such that  $10 \leq n \leq 9999$

Soln. Initially let us consider the number 10

(i) First we find two digit numbers (non-repeated digits)

$\begin{array}{c} \overline{1} \quad \overline{2} \\ \text{We choose the } 1^{\text{st}} \text{ digit in 9 ways (from 1 to 9)} \text{ and the} \\ \text{second digit we choose in 9 ways (from 0 to 9) without repetition.} \\ \therefore \text{No. of two digit numbers are } 9 \times 9 = 81 \end{array}$

(ii) We find 3 digit numbers (non-repeated digits)  $\begin{array}{c} \overline{1} \quad \overline{2} \quad \overline{3} \end{array}$

We choose the 1<sup>st</sup> digit in 9 ways (from 1 to 9).

The 2<sup>nd</sup> digit in 9 ways (from 0 to 9),  
The 3<sup>rd</sup> digit in 8 ways (from 0 to 9) without repetition.

$\therefore$  The no. of three digit numbers are  $9 \times 9 \times 8 = 648$

(iii) We find 4 digit numbers (non-repeated digits)  $\begin{array}{c} \overline{1} \quad \overline{2} \quad \overline{3} \quad \overline{4} \end{array}$

We choose the 1<sup>st</sup> digit in 9 ways (from 1 to 9)

The 2<sup>nd</sup> digit in 9 ways (from 0 to 9 without repetition)

The 3<sup>rd</sup> digit in 8 ways

The 4<sup>th</sup> digit in 7 ways

$\therefore$  The no. of 4-digit numbers are  $9 \times 9 \times 8 \times 7 = 4536$

$\therefore$  The number of the integers b/w 10 and 9999 with non-repeated digits is  $81 + 648 + 4536 = 5265$

(P) How many 7-digit numbers are there with exactly one 5  
Soln. We have the no. of digits are 10 i.e. 0, 1, 2, ..., 9  
 we arrange the digit 5 in 7 places  ${}^7P_1$  ways i.e. 7 ways

(i) If we take the digit 5 in first position, then the remaining 6 positions can filled in  $9^6$  ways

$$\begin{array}{ccccccc} 5 & & & & & & \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

(ii) If we take the digit 5 in 2<sup>nd</sup> position, then the 1<sup>st</sup> position choose in 8 ways (using digits 1, 2, ..., 9) and the remaining position can filled in  $9^5$  way i.e.  $8 \cdot 9^5$  ways

$$\begin{array}{ccccccc} & 5 & & & & & \\ \hline & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

If we take the digit 5 in 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, 6<sup>th</sup> and 7<sup>th</sup> positions then the remaining position filled in  $8 \cdot 9^5$  way

$\therefore$  The 7-digit numbers are with exactly one 5 is

$$= 9^6 + 6(8 \cdot 9^5) = 9^5 \cdot 571$$

(P) find the number of 4 letter words that can be formed using the letters of the word 'ARTICLE' in which atleast one letter is repeated

Soln. We know that the number 4 letter words that can be formed using 7 letters =  $7^4$  (repetition is allowed)

These  $7^4$  words can be divided into two exclusive categories.

(i) The 4 letter words without repetition

(ii) The 4 letter words with atleast one repetition of a letter.

The 4 letter words without repetition will be  ${}^7P_4$

$\therefore$  The 4 letter words with atleast one repetition of a letter =  $7^4 - {}^7P_4$

## Permutations of objects not all distinct

The number of permutations of  $n$  objects in which  $p$  objects are of one type,  $q$  objects are of second type,  $r$  objects are of 3rd type and rest are all distinct is

$$\frac{n!}{p! q! r!}$$

(P) How many different words can be formed with the letters of the word MISSISSIPPI?

Soln: There are 11 letters in the given word of which 4 are S's, 4 are I's, and 2 are P's, and rest are all distinct.

$$\therefore \text{Hence the total number of words} = \frac{11!}{4! 4! 2!} = 34650$$

(P) Find the no. of permutations of the letters of word "MASSASAUGA". In how many of these, all four A's are together? How many of them begin with S?

Soln: The given word has 10 letters of which 4 are A's, 3 are S's and rest are all distinct.

$$\text{The required no. of permutations is } \frac{10!}{4! 3!} = 25,200$$

(i) If, in a permutation all A's are to be together, we treat all of A's as one single letter, Then the letters to be permuted (AAAA), SS, M, U, G

$$\therefore \text{The no. of permutations} = \frac{4!}{1! 3! 1! 1! 1!} = 840$$

(ii) Permutations beginning with S, there occur nine position to fill. where two are S's, 4 are A's and one each are M, U, G

$$\text{The number of such permutations} = \frac{9!}{2! 4! 1! 1! 1!} = 7560$$

(P) Find the number of 5 letter words that can be formed (7)  
using the letters of the word 'DELHI' that begin and end with  
vowel when repetitions are allowed

Soln:- Given that the word 'DELHI'

No. of Letters in the word = 5 ; No. of vowel = 2

consider 5 blank places  $\underline{\quad}$   $\underline{\quad}$   $\underline{\quad}$   $\underline{\quad}$   $\underline{\quad}$

- (i) Suppose 1<sup>st</sup> and 5<sup>th</sup> positions filled with E and I and the remaining places can be filled in  $5 \times 5 \times 5 = 125$  ways
  - (ii) Suppose 1<sup>st</sup> and 5<sup>th</sup> positions filled with I and E and the remaining places can be filled in  $5^3 = 125$  ways
  - (iii) Suppose 1<sup>st</sup> and 5<sup>th</sup> positions filled with E and the remaining places can be filled in  $5^3 = 125$  ways
  - (iv) Suppose 1<sup>st</sup> and 5<sup>th</sup> positions filled with I and the remaining places can be filled in  $5^3 = 125$  ways

$$\therefore \text{Total no. of 5 letter words} = 125 + 125 + 125 + 125 = \underline{\underline{500}}$$

F find the number of 4-digit numbers that can be formed using the digits 0, 1, 2, 3, 4, 5 which are divisible by 6 when repetition of digits is allowed.

(P) How many positive integers 'n' can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000? (8)

Soln.: Let 'n' be must be of the form,  $n = x_1 x_2 x_3 x_4 x_5 x_6 x_7$  where  $x_1, x_2, x_3 \dots x_7$  are the given digits with  $x_1=5, 6 \text{ or } 7$

(i) Suppose we take  $x_1=5$  then  $x_2 x_3 x_4 x_5 x_6 x_7$  is an arrangement of the remaining 6 digits which contains ~~one each~~ two 4's and one each of 3, 5, 6, 7

$$\therefore \text{The no. of permutation} = \frac{6!}{2! 1! 1! 1! 1!} = 360$$

(ii) If we take  $x_1=6$  then  $x_2 x_3 x_4 x_5 x_6 x_7$  is an arrangement of the remaining 6 digits which contains two 4's, two 5's and one each of 3, 6, 7

$$\therefore \text{The no. of arrangement is } \frac{6!}{1! 2! 2! 1!} = 180$$

(iii) If we take  $x_1=7$ ,

$$\therefore \text{The no. of arrangements is } \frac{6!}{1! 2! 2! 1! 1!} = 180$$

By the sum rule, the number of 'n's of the desired type is  $360 + 180 + 180 = 720 //$

(P) A store has 25 flags to hang along the front of the store to celebrate a special occasion. If there are 10 red flags, 5 white flags, 4 yellow flags and 6 blue flags. Then how many distinguishable ways can the flags be displayed?

Soln.: Total no. of flags = 25

Red flags = 10, White flags = 5, Yellow flags = 4, Blue flags = 6  
 $\therefore$  No. of distinguishable ways can the flags be displayed is

$$\frac{25!}{10! 5! 4! 6!}$$

(P) find the no. of ways of arranging the letters of the word  $a^4 b^3 c^5$  in expanded form

Soln: Given word is  $a^4 b^3 c^5$

No. of Letters in the given word = 12

∴ The no. of arrangements by taking all at a time =  $\frac{12!}{3!4!5!}$

(P) find the number of ways of arranging the letters of the word 'BRINGING' so that they begin and end with 'I'

Soln: Given word 'BRINGING'

No. of letters in the given word = 8.

It contains 2 I's, 2 G's, 2 N's and B, R

Fix I at first and at last. Arranging the remaining 6 letters b/w these 2 I's

Total no. of arrangements =  $\frac{6!}{2!2!} = 180$

(P) find the number of ways of arranging the letters of the word 'SHIPPING' such that (i) 2 P's will come together

(ii) 2 I's do not come together

Soln: Given word 'SHIPPING'

No. of letters in the given word = 8.

It contains 2 P's, 2 I's and S, H, N, G one each

(i) Since P's must come together. Consider 2 P's as a unit  
Now it will have 7 letters of which there are 2 I's

∴ No. of ways of arrangements =  $\frac{7!}{2!}$

We need not arrange the 2 P's among themselves because they are alike.

∴ Required no. of arrangements =  $\frac{7!}{2!}$

(ii) Since the 2 I's do not come together, arrange the remaining letters at first.

It can be done in  $\frac{6!}{2!}$  ways. Because there are 2 P's among the remaining six

Now we get 7 gaps in which 2 I's can be arranged

It can be done =  $\frac{7!}{2!}$

∴ Total No. of arrangements =  $\frac{6!}{2!} \times \frac{7!}{2!} = 7560$

## Circular permutation

(7)

A circular permutation is an arrangement of objects arranged around a circle or other simple closed curve.

In circular permutation, clockwise and anticlockwise arrangements of objects are possible and both are distinguishable.

If objects are arranged in a circular order, then the circular permutations of 'n' different objects is  $(n-1)!$ .

If anti-clockwise and clockwise order of arrangements are not distinct.

If we have to arrange 'n' different objects around a table such that no two similar objects are neighbour, then the no. of permutations will be  $\frac{(n-1)!}{2}$ .

Ex:- Arrangement of beads in a necklace, arrangement of flowers in a garland, etc

- (P) In how many different ways can 5 gentlemen and 5 ladies can sit around a table if (i) there is no restriction  
(ii) no two ladies sit together.

2017. (i) There are 10 persons

The number of different arrangement is  $(10-1)! = 9!$

- (ii) The no. of ways 5 gentlemen can sit around a table is  $4! = 24$

Between any two men let a woman be seated. Then all the 5 ladies can be seated in 5 intermediate places in  $5!$  ways

∴ The required no. of ways is  $24 \times 5! = 2880$

- (P) If 10 persons were invited for a party, in how many ways can they and the host be seated at a circular-table? In how many ~~ways~~ of those ways will two particular persons be seated on either side of the host?

2017. (a) There are 11 persons, including the host to be seated around a circular-table i.e.  $(11-1)! = 10!$  ways

- (b) Let  $P_1, P_2$  be two particular persons and  $H$  be the host. There two particular persons can be seated on either side of the

host in the following two ways

$$(i) P_1 + P_2 \quad (ii) P_2 + P_1$$

Consider the two particular persons and the host as one person, we have 9 persons in all.

These 9 persons can be seated round a circular table in  $(9-1)! = 8$  ways. But two particular persons can be seated on either side of the host in 2 ways.

So the no. of ways of seating 11 persons at a ~~particular~~ circular table with particular persons on either side of the host =  $8! \times 2$ .

- (P) A family consists of father, mother, 2 daughters and 2 sons  
Q How many different ways can they sit at a round-table if  
2 daughters wish to sit on either side of the father.

(soln) First arrange father in any place.

Since the 2 daughters wish to sit on either side of the father they can be arranged in 2 places in  $2P_2 = 2$  ways

The remaining 3 persons can be arranged in 3! ways

∴ By product rule, the required no. of arrangements

$$= 2 \times 3! = 12$$

- (P) find the no. of ways to arrange 8 persons around circular table if (i) two specified persons wish to sit together  
(ii) never sit together

(soln) i) Since, in circular permutations, the person whom we are going to arrange at first and at which place we are arranging him is not important.

Let the special persons be A, B. So, arrange A in any place. Since A, B must sit together, B can be arranged in two ways i.e. in the two adjacent places on either side of A. It can be done in 2 ways

The remaining 6 persons can be arranged in 6! ways

∴ By product rule, the no. of arrangements =  $2 \times 6!$

(ii) first arrange A in any place. Since A, B do not come together, deleting the two places on either side of A, now B can be arranged in 5 ways. (10)

The remaining 6 persons can be arranged in  $6!$  ways

$\therefore$  The required number of arrangements =  $5 \times 6!$

(P) find the no. of ways of arranging 6 boys and 6 girls around a circle so that

(i) all the girls come together (ii) no two girls come together.

SOL: No. of girls = 6 ; No. of boys = 6

(i) since all the girls come together

consider '6' girls as a unit. Now we have to arrange 6 boys + 1 unit i.e. 7 persons along a round table in  $6!$  ways.

The girls can be arranged among themselves in  $6!$  ways

$\therefore$  The required no. of arrangements =  $6! \times 6!$

(ii) Since no two girls come together.

first arrange 6 boys along circle. It can be done in  $5!$  ways

we get exactly 6 gaps, in which the girls can be arranged. This can be done in  $6!$  ways

$\therefore$  The required no. of arrangements =  $5! \times 6!$

NOTE: (i) The number of circular permutations of  $n$  distinct things taken  $r$  at a time =  $\frac{r}{n} P_r$ ,  $1 \leq r \leq n$

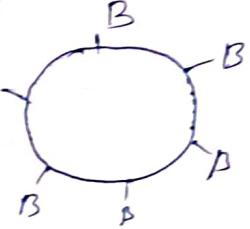
(ii) The number of permutations of  $n$  things taken  $r$  at a time in case of hanging type =  $\frac{1}{2} \cdot \frac{n}{r} P_r$  ( $\because$  we consider only one direction)

(iii) The number of permutations of  $n$  different objects taken  $r$  at a time in which  $p$  particular objects do not occur

in  $\frac{n-p}{r-p} P_r$

(iv) The no. of permutations of  $n$  different objects taken  $r$  at a time in which  $p$  particular objects are present

in  $\frac{n-p}{r-p} P_{r-p} \times \frac{r}{p} P_p$



- (Q). In how many ways 20 different coloured flowers can be arranged into a garland by taking 10 at a time, so that 2 specified colours must occur in the garland but not come together

Soln. Let the two specified colours be A, B  
Now fix A at any place of the 10 places along the circle  
Since A, B do not come together, we can arrange B only  
in 4 places. It can be done in 4 ways

Now the remaining 8 places can be filled with  
remaining 18 flowers by selecting 8 flowers from  
18 flowers.

This can be done in  ${}^{18}P_8$  ways

By product rule, The no. of hanging type of garlands

$$= 4 \times \frac{18P_8}{2}$$

- (Q). Find the number of ways of arranging 6 red roses and 3 yellow roses of different sizes into a garland. In how many of them all the yellow roses come together

Soln. The nine different roses can be arranged along a

circle in  $8!$  ways

Since it is hanging type circular arrangement, we consider  
only one direction.

$$\therefore \text{The no. of garlands} = \frac{8!}{2}$$

Since all the 3 different yellow roses come together,  
consider them as a unit.

The number of roses are 6+1 units. They can be  
arranged in  $6!$  ways. The yellow rose can be arranged  
among themselves in  $3!$  ways.

$$\therefore \text{The no. of garlands} = \frac{6! \times 3!}{2}$$

P.T (a)  $2^n P_n = 2^n [1, 3, 5, \dots, (2n-1)]$

(b)  $n P_n = n \times (n-1) P_{(n-1)}$

Soln: (a)  $2^n P_n = \frac{(2n)!}{n!} = \frac{(2n)(2n-1)(2n-2)\dots4 \cdot 3 \cdot 2 \cdot 1}{n!}$

$$= \frac{[2n(2n-2)\dots6 \cdot 4 \cdot 2]}{n!} [(2n-1)(2n-3)\dots3 \cdot 1]$$

$$= \frac{2 \cdot n! [(2n-1)(2n-3)\dots5 \cdot 3 \cdot 1]}{n!} = 2^n \frac{[(2n-1)(2n-3)\dots5 \cdot 3 \cdot 1]}{n!}$$

(b) R.H.S.  $= n \times (n-1) P_{(n-1)} = n \times \frac{(n-1)!}{((n-1)-(n-1))!}$   
 $= n \frac{(n-1)!}{(n-n)!} = \frac{n!}{(n-n)!} = n P_n = L.H.S.$

P If  $(2n+1)P_{n+1} : (2n-1)P_n = 3 : 5$ , find ' $n$ '

P find the value of ' $n$ ' if the number of permutations of ' $n$ ' objects taken 4 at a time is equal to 12 times the no. of permutation of  $n$  objects taken 2 at a time.

P find the value of ' $n$ ', if  $2^n P_3 = 2 \times n P_4$

## COMBINATIONS

(12)

Any unordered selection of  $n$  ( $\leq n$ ) objects from a set of  $n$ -objects is called an  $\alpha$ -combinations of  $n$  objects and is denoted by  $C(n, \alpha)$  or  $n_{C_\alpha}$

$$\text{i.e. } C(n, \alpha) = n_{C_\alpha} = \frac{n!}{\alpha! (n-\alpha)!}$$

### Important Results

$$(i) C(n, 0) = 1 \text{ i.e. } n_{C_0} = 1$$

$$(ii) n_{C_n} = 1, n_{C_\alpha} = n_{C_{n-\alpha}}$$

$$(iii) C(n+1, \alpha) = C(n, \alpha-1) + C(n, \alpha)$$

$$\text{i.e. } n+1_{C_\alpha} = n_{C_{\alpha-1}} + n_{C_\alpha}$$

$$(iv) n_1 = n, n_{C_2} = \frac{n(n-1)}{2!}, n_{C_3} = \frac{n(n+1)(n-2)}{3!} \text{ and so on}$$

Q How many committees of five with a given chairperson can be selected from 12 persons?

Soln.: The chairperson can be chosen in 12 ways  
The other four on the committee can be chosen in  $^{11}C_4$  ways  
∴ The possible no. of such committee is

$$12 \times {}^{11}C_4 = 3960$$

Q Out of 5 men and 2 women a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?

Soln.: There are two possible ways

(i) 2 men and 1 woman (ii) 1 men and 2 women  
(i) The no. of ways of selecting the 2 men and 1 woman is

$$= {}^5C_2 \cdot {}^2C_1 = 20$$

(ii) The no. of ways of selecting the 1 men and 2 women is

$$= {}^5C_1 \cdot {}^2C_2 = 5$$

∴ The no. of ways of forming the committee =  $20+5=25$

(P) A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting atleast 2 questions from each part?

Soln: There are two possible ways

$$(i) \quad 4C_3 \times 4C_2 = 24 \quad (ii) \quad 4C_2 \times 4C_3 = 24$$

$$\therefore \text{Total no. of ways} = 24 + 24 = 48$$

(P) In how many ways can four students be selected out of twelve students, if (a) two particular students are not included at all (b) two particular students are included

Soln: The number ways in which 4 students can be selected out of 12 students are  $12C_4 = 495$

(a) When two particular students are not included, 4 are selected out of 10 in  $10C_4 = 210$

(b) When two particular students are included, they can be selected only one way. The number of ways of selecting 2 out of 10 students are  $10C_2 = 45$

$$\text{Hence total no. of ways are } = 1 \times 10C_2 = 45 \text{ if}$$

(P) How many ways can committees of 5 or more can be chosen from 9 people?

Ans:  $9C_5 + 9C_6 + 9C_7 + 9C_8 + 9C_9$

(P) How many 5 card hands consist of cards from a single suit?

Soln: For each of the 4 suits, spades, hearts, diamonds or clubs there are  $13C_5$  5-card hands

Hence, there are a total of  $4 \times 13C_5$  such hands.

(P) How many 5-card hands have 2 cards of one suit and 3 cards of a different suit?

Soln: For a fixed choice of 2 suits there are  $2 \times 13C_2 \times 13C_3$  ways to choose 2 from one of the suits and 3 from the other. We can choose the 2 suits in  $4C_2$  ways.

Thus there are  $2 \times 13C_2 \times 13C_3 \times 4C_2$  such 5-card hands

- Q. Find the number of committees of 5 that can be selected from 4 men and 5 women if the committee is to consist of at least 1 man and at least 1 woman. (13)

Soln: Given that there are 12 persons (5 women and 7 men) from the given 12 persons the no. of committees of 5 that can be formed is  ${}^{12}C_5$

Among these possible committees, there are  ${}^7C_5$  committees consisting of 5 men and  ${}^5C_5$  committee consisting of 5 women.

∴ The no. of committees containing at least one man and one woman is

$${}^{12}C_5 - {}^7C_5 - {}^5C_5 = 770 \text{ ways}$$

- Q. A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations

(a) There is no restriction on the choice

(b) Two particular persons will not attend separately

(c) Two particular persons will not attend together

Soln (a) Since there is no restriction on the choice of invitees, five out of 11 can be invited in  ${}^{11}C_5 = 462$  ways

(b) Since two particular persons will not attend separately they should both be invited or not invited.

If both of them are invited, then three more invitees are to be selected from the remaining 9 relatives.

This can be done in  ${}^9C_3 = 84$  ways

If both of them are not invited, then five invitees are to be selected from 9 relatives. This can be done in  ${}^9C_5 = 126$  ways.

∴ The total no. of ways in which the invitees can be selected in this case is  $84 + 126 = 210$

- (c) Since two particular persons will not attend together (say A and B), only one of them can be invited or none of them can be invited.

The number of ways of choosing the invitees with A invited is

$${}^9C_4 = 126$$

Similarly, the number ways of choosing the invitees with B invited is 126

If both A and B are not invited, the number of ways of choosing the invitees is

$${}^9C_5 = 126$$

Thus, the total no. of ways in which the invites can be selected in this case is  $126 + 126 + 126 = 378$

- (P) find the number of arrangements of the letters in TALLAHASSEE which have no adjacent A's

Soln.: No. of letters is 11 of which 3 are A's, 2 each are L's, 1 S, 1 E's and 1 each are T and H  
First, let us disregard the A's. The remaining 8 letters can be arranged in  $\frac{8!}{2!2!2!1!1!1!} = 5040$  ways

In each of these arrangements, there are 9 possible locations for the three A's.  
 $-T-L-L-H-I-S-E-E-$

These locations can be chosen in  $C(9,3)$  i.e.  ${}^9C_3$  ways.

$\therefore$  By the product rule, the required number of arrangements is

$$5040 \times {}^9C_3 = 423,360$$

- (P) A party is attended by n persons. If each person in the party shake hands with all the others in the party, find the number of handshakes.

Soln.: Each handshake is determined by exactly two persons. If each person shakehands with all other persons, the total number of handshakes is equal to the total number of combination of two persons that can be selected from the n persons.

$$\text{This number is } {}^nC_2 = \frac{n!}{(n-2)!2!} = \frac{1}{2}n(n-1)$$

## Combinations with Repetitions

(15)

The number of combinations of  $n$  objects among  $n'$  objects if the repetitions are allowed and  $a$  is not important  
 $\Rightarrow C(n+a-1, n-1) \text{ i.e. } \frac{(n+a-1)!}{(n-1)!} = \frac{(n+a-1)!}{a! (n-1)!}$

In other words  $C(n+a-1, n) = C(n+a-1, n-1)$  represents the number of combinations of  $n$  distinct objects taken  $n'$  at a time with repetitions allowed.

NOTE: (i)  $C(n+a-1, n) = C(n+a-1, n-1)$  represents the no. of ways  $'a'$  identical objects can be distributed among  $n$  distinct containers.

(ii)  $C(n+a-1, n) = C(n+a-1, n-1)$  represents the number of non-negative integer solutions of the eqn  $\gamma_1 + \gamma_2 + \dots + \gamma_n = n$ .

(iii) The number of ways of distributing  $'a'$  chocolates to  $n$  children so that each child gets atleast one is  $(a-1)C(a-n)$

(iv) The no. of the integer solutions of  $\gamma_1 + \gamma_2 + \dots + \gamma_n \leq a$  is  $C(a-1, a-n)$

(P) The bag contains coins of 7 different denominations with atleast one dozen coins in each denominations. In how many ways can we select a dozen coin from the bag.

Soln: The selection consists in choosing with repetitions  $n=12$  coins of  $n=7$  distinct denominations.

$\therefore$  The number of ways of making this selection is

$$C(7+12-1, 12) = C(18, 12) = {}^{18}C_{12} = 18,564$$

(P) In how many ways can we distribute 10 identical marbles among 6 distinct containers?

$$\text{Here } n=6, a=10 \quad \therefore C(n+a-1, n) = {}^{15}C_{10} = 3003$$

Soln: Here  $n=6, a=10$   $\therefore C(n+a-1, n) = {}^{15}C_{10} = 3003$

(P) Find the number of non-negative integer solutions of the equation  $\gamma_1 + \gamma_2 + \gamma_3 = 17$ , where  $\gamma_1, \gamma_2, \gamma_3$  are non-negative integers

Soln: Given that the eqn  $\gamma_1 + \gamma_2 + \gamma_3 = 17$

Each solution of the given eqn. is equivalent to distribution of 17 identical balls in 3 numbered boxes

with repetitions. where  $a_i$  represents the number of balls in the  $i^{th}$  box.

$$\text{Here } n=3, \lambda=17$$

$\therefore$  The required no. of soln's  $= C(n+\lambda-1, \lambda) = 19C_{17} = 1711$

- (P) find the number of distinct terms in the expansion of  $(x_1+x_2+x_3+x_4+x_5)^{16}$

Soln. Each term in the expansion is of the form  $\binom{16}{n_1 n_2 n_3 n_4 n_5}$  where each  $n_i$  is a non-ve integer and these  $n_i$ 's sum to 16

$\therefore$  The no. of distinct terms in the expansion is precisely equal to the number non-ve integer solutions of the eqn  $n_1+n_2+n_3+n_4+n_5=16$

$$\text{Here } n=5, \lambda=16$$

$\therefore$  The number is  $C(n+\lambda-1, \lambda) = 20C_{16} = 4845$

- (P) find the no. of non-ve integer solutions of the inequality  $x_1+x_2+\dots+x_6 < 10$

Soln. Given that the inequality  $x_1+x_2+\dots+x_6 < 10$

we have to find the no. of non-negative integer solutions of the eqn  $x_1+x_2+\dots+x_6 = 9-x_7$

where  $9-x_7 \leq 9$ , so that  $x_7$  is non-ve integer.

Thus the required number is the no. of non-ve solutions of the eqn  $x_1+x_2+\dots+x_7 = 9$

$$\text{Here } n=7, \lambda=9$$

$\therefore C(n+\lambda-1, \lambda) = 15C_9 = 5005$

- (P) find the number of distinct triply  $(x_1, x_2, x_3)$  of non-ve integers satisfying  $x_1+x_2+x_3 < 15$

Soln. Given that  $x_1+x_2+x_3 < 15$

Since the values are only integers  $x_1+x_2+x_3 = 0$  to 14

$$\text{Here } \lambda=14, n=3$$

$\therefore$  No. of soln.  $= C(n+\lambda-1, \lambda) = 16C_{14} = 120$

$$\begin{array}{ll}
 x=13, C(15, 13)=105 & x=12, n=3 C(14, 12)=91 \quad (15) \\
 x=11, n=3, C(13, 11)=78 & x=10, n=3, C(12, 10)=66 \\
 x=9, n=3, C(11, 9)=55 & x=8, n=3, C(10, 8)=45 \\
 x=7, n=3, C(9, 7)=36 & x=6, n=3, C(8, 6)=28 \\
 x=5, n=3, C(7, 5)=21 & x=4, n=3, C(6, 4)=15 \\
 x=3, n=3, C(5, 3)=10 & x=2, n=3, C(4, 2)=6 \\
 x=1, n=3, C(3, 1)=3 & x=0, n=3, C(2, 0)=1
 \end{array}$$

$\therefore$  The total no. of soln = 680.

(P) Find the no. of integer solutions of  $x+y+z=9$

Soln: Given that  $x+y+z=9$

This is equal to arranging 'n' stones in '3' boxes such that each box should have atleast one stone.

The no. of ways of arranging 'n' stones in '3' boxes such that there will be atleast one stone in each box is

$$C(x-1, x-n) = (x-1)C_{(x-n)} \quad \text{Here } x=9, n=3 \\ = 8C_2 = 28$$

The possible 28 solutions are  $(7, 1, 1), (1, 7, 1), (1, 1, 7), (6, 1, 2)$   
 $(6, 2, 1), (1, 6, 2), (2, 6, 1), (1, 2, 6), (2, 1, 6), (5, 2, 2), (2, 5, 2), (2, 2, 5)$   
 $(5, 1, 3), (5, 3, 1), (1, 5, 3), (3, 5, 1), (1, 3, 5), (3, 1, 5), (4, 4, 1), (4, 1, 4)$   
 $(1, 4, 4), (4, 1, 3), (4, 3, 2), (2, 4, 3), (3, 4, 2), (2, 3, 4), (3, 2, 4), (3, 3, 3)$

(P) How many no. of integer solutions will be these for  $x+y+z=100$

Soln: Given  $x+y+z=100$

This is equal to assigning 'n' stones in '3' boxes such that each box contains atleast one stone  $\therefore (x-1)C_{(x-n)} = 4851$

Here  $x=100, n=3 \rightarrow$  i.e

(P) And the no. of integer solutions of  $x_1+x_2+x_3+x_4+x_5+x_6 < 10$

Soln: Given that  $x_1+x_2+x_3+x_4+x_5+x_6 < 10$

Each  $x_i$  value should be  $\geq 1$ . So minimum value is 1

$$\therefore x_1+x_2+x_3+x_4+x_5+x_6 = 6 \text{ or } 7 \text{ or } 8 \text{ or } 9$$

$x=6, n=6$  The possible solution is  $(1,1,1,1,1,1) = 1$

$x=7, n=6$  " " are  $(1-1)_{C_{(x-n)}} = 6C_1 = 6$

$x=8, n=6$  " " " "  $= 7C_2 = 21$

$x=9, n=6$  " " " " " "  $= 8C_3 = 56$

$$\therefore \text{Total no. of the integer solutions} = 1+6+21+56 = 84.$$

(P) Find the number of the integer solutions of the equation

$$x_1+x_2+x_3 = 17$$

Soln: Given that  $x_1+x_2+x_3 = 17$

Here we require  $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$ . Thus  $x=17, n=3$

∴ The required no. of the integer solutions are

$$(x-1)_{C_{(x-n)}} = 16C_{14} = 120 //$$

(B) find how many solutions are there to the given equation  
that satisfies the given condition  $x_1+x_2+x_3 = 20$   
where each  $x_i$  is a non integer

Soln: Given that  $x_1+x_2+x_3 = 20 \rightarrow ①$

We require  $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

Let us set  $y_1 = x_1 - 1, y_2 = x_2 - 1, y_3 = x_3 - 1$

Then  $y_1, y_2, y_3$  are all non-negative integers

Sub.  $x_1 = y_1 + 1, x_2 = y_2 + 1, x_3 = y_3 + 1$  in ①

$$\text{we get } y_1 + y_2 + y_3 = 17 \rightarrow ②$$

The no. of non-negative integer solutions of this eqn  
is the required number.

Here  $n=3, x=17$

∴ No. of non-negative integer solutions of ③ is

$$(x-1)_{C_{(x-n)}} = 16C_{17} //$$

(P) In how many ways can five different messages be delivered by three messenger boys if no messenger boy left unemployed. (16) (20)

Soln: Let first messenger delivered a message

Second	"	"	Y	"
Third	"	"	Z	"

$$\therefore x+y+z=5 \text{ for } x \geq 1, y \geq 1, z \geq 1 \rightarrow (1)$$

$$\text{Let us set } y_1 = x-1, y_2 = y-1, y_3 = z-1$$

Then  $y_1, y_2, y_3$  are all non-negative integers

Sub.  $x=1+y_1, y=1+y_2, z=1+y_3$  in (1), we get

$$\cancel{x+y+z} = y_1 + y_2 + y_3 = 2 \rightarrow (2)$$

Here  $n=3, \lambda=2$

$\therefore$  No. of non-negative integer solutions of (2) is

$$C(n+\lambda-1, \lambda) = C(2, 2) = 6$$

There are  $(1, 2, 2), (2, 1, 2), (2, 2, 1), (1, 1, 3), (1, 3, 1), (3, 1)$  ways delivered by three messengers boys,

(P) How many integers b/w 1 and 1000 have a sum of digits of integer numbers equal to 10.

Soln: Let  $x_1, x_2, x_3$  be the position of the digits b/w integers 1 and 1000

$$\text{Let } x_1+x_2+x_3=10 \text{ for } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

Here  $\lambda=10, n=3$

The number of non-negative integer solutions gives the number of integers b/w 1 and 1000 and sum of digits of integer equal to 10.

$\therefore$  The no. of non-negative integer solutions is  
 $C(n+\lambda-1, \lambda) = C(12, 10) = 66$

$\therefore$  The required no. of integers is 66 //

(P) Determine the no. of ways possible to wear 5 rings on 4 fingers.

Soln. Let  $x_1, x_2, x_3, x_4$  are fingers and no. of rings are 5  
i.e.  $x_1 + x_2 + x_3 + x_4 = 5 \rightarrow (1)$

The no. of non-negative integer solutions of above eqn. gives the required result.

Here  $n=5, r=4$

$\therefore$  The no. of non-negative integer solutions of (1) is  
 $C(n+r-1, r) = C(8, 5) = 56$

(P) If  $x > 2, y > 0, z > 0$  then find the no. of solutions of  $x+y+z+w=21$

Soln. Given that the eqn  $x+y+z+w=21 \rightarrow (1)$  and the given constraints  $x > 2, y > 0, z > 0$  i.e.  $x \geq 3, y \geq 1, z \geq 1$

Let  $x_1 = x-3, y_1 = y-1, z_1 = z-1$   
so that  $x_1, y_1, z_1$  are non-negative integers

Sub.  $x = x_1 + 3, y = y_1 + 1, z = z_1 + 1$  in (1), we get

$$x_1 + y_1 + z_1 + w = 16 \rightarrow (2)$$

Here  $n=4, r=16$

$\therefore$  The no. of non-negative integer solutions of the eqn (2)  
 $\therefore C(n+r-1, r) = C(16, 16) = 969 //$

(P) Find the number of integer solutions of  $x_1 + x_2 + x_3 + x_4 + x_5 = 30$  where  $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$  [Ans: 8855]

(P) In how many ways can we distribute 12 identical pencils to 5 children so that every child get at least one pencil [Ans: 330]

(P) Find the number of distinct triples  $(x_1, x_2, x_3)$  of non-negative integers satisfying  $x_1 + x_2 + x_3 \leq 15$  [Ans: 680]  $n=4, r=14$

(P) How many different outcomes are possible by tossing 10 similar coins (17)

Soln. This is same as the placing 10 similar balls into two boxes labeled 'heads' and 'tails'

$$n+r-1 = 10 \cdot \text{ Here } n=10, r=2$$

$$\therefore \text{Total no. of outcomes } C(n+r-1, r) = C(11, 10) = 11$$

(P) How many different outcomes are possible from tossing 10 similar dice. Ans: 3003

(P) Out of Large Supply of pennies, nickels, dimes and quarters, in how many ways can 10 coins be selected.

$$\text{Ans: } C(13, 10)$$

(P) Enumerate the number of ways of placing 20 indistinguishable balls into 5 boxes when each box is non-empty.

$$\text{Ans. } 19C4$$

(P) How many integral solutions are there to  $x_1 + x_2 + x_3 + x_4 + x_5 = 20$  when each  $x_i \geq 2$  Ans:  $C(14, 10)$

(P) Five balls are to be placed in three boxes. Each box can hold all the five balls. In how many ways can we place the balls in that no box is empty, if

- (i) balls and boxes are different
- (ii) balls are identical and boxes are different
- (iii) balls are different and boxes are identical
- (iv) balls and boxes are identical

Sol: (i) If balls and boxes are different, then the five balls can be distributed in the following ways

Distribution of five balls in three boxes

	<u>Box 1</u>	<u>Box 2</u>	<u>Box 3</u>
No. of Balls	3	1	1
	1	3	1
	1	1	3
	2	2	1
	2	1	2
	1	2	

If boxes I, II and III contain 3, 1 and 1-balls respectively. Then the no. of ways of distributing the balls

$$\frac{5!}{3!1!1!} = 20$$

ii) If boxes I, II, III contain 1, 3 and 1 balls respectively. Then the no. of ways of distributing the balls =  $\frac{5!}{3!1!1!} = 20$

If boxes I, II and III contain 1, 1 and 3 balls respectively. Then the no. of ways =  $\frac{5!}{3!1!1!} = 20$

Also the boxes I, II and III contain 2, 2 and 1 balls respectively.

Then the no. of ways =  $\frac{5!}{2!2!1!} = 30$

$$i.e. 2, 1, 2 \rightarrow \frac{5!}{2!1!2!} = 30$$

$$i.e. 1, 2, 2, \rightarrow \frac{5!}{2!2!1!} = 30$$

$$\therefore \text{Hence the total no. of required ways} = 20 + 20 + 20 + 30 + 30 = 150$$

(i) Balls are identical and boxes are different

(18)

Since the balls are identical, repetitions are allowed

We select three balls and put one ball each in each of the three boxes. We must distribute two more identical balls

Hence the no. of ways distributing five identical balls in three different boxes =  $(3+2-1)c_2 = 4c_2 = 6$

(ii) Balls are different and boxes are identical

Hence, the distributions of balls i.e. 1,1,3; 1,3,1 and 3,1,1 are identical.

∴ The no. of ways of distributing one ball in each of any two boxes and three balls in the third box

$$= \frac{5!}{3!1!1!} = 20$$

11) the number of ways of distributing 2 balls in each of any two boxes and one ball in the third box

$$= \frac{5!}{2!2!1!1!} = 30$$

Thus the required no. of ways =  $20 + 30 = 50$

(iv) Balls are identical and boxes are identical

Hence the required no. of ways = 2 (i.e. 1,1,3 and 1,2,2)

P How many words of 4 letters can be formed from the letters of the word INFINITE

Soln: Given word INFINITE. There are 8 letters of which 3 are I's, 2 are N's and F,T,E are distinct.

We have to form 4 letter words

Case (i) All distinct selection of 4 distinct letters from 5 letter is  $5c_4$  ways

No. of permutations of them are  $4!$

Hence the total no. of words =  $4! \cdot 5c_4 = 120$

Case (ii) 3 alike, 1 distinct

No. of words in this case are  $1c_1 \cdot 4c_1 \cdot \frac{4!}{3!} = 16$

Case (iii) 2 alike, 2 alike

No. of words  $2c_2 \cdot \frac{4!}{2!2!} = 6$

(P) How many 4 letter words can be formed using the letters of the word 'PROPOSITION' Ans:  $20+18+360+360=758$

(P) find the number of 4 letters that can be formed using the letters of the word 'KAMALA' Ans:  $24+12+36=72$

(P)

## The pigeonhole principle

(19) (20)

If  $n$  pigeons are accommodated in  $m$  pigeonholes and  $n > m$  then one pigeonhole must atleast contain  $\lceil \frac{n}{m} \rceil$  pigeons i.e atleast one pigeonhole contains two or more pigeons.

(P) P.T in any set of 29 persons atleast 5 persons must have born on same day of week.

Soln: No. of persons (Pigeons) =  $29 = n$   
A week contain 7 days (pigeonholes) if  $m = 7$   
i.e  $\text{ceil}(\frac{29}{7}) = \text{ceil}(4.1) = 5$

(P) Suppose there are 26 students and 4 cars to transport them, show that atleast one car must have 4 or more passengers

Soln: No. of students (Pigeons) =  $26 = n$   
No. of cars (pigeonholes) =  $4 = m$   
 $\text{ceil}(\frac{26}{4}) = \text{ceil}(3.7) = \lceil 3.7 \rceil = 4$

(P) If there are 6 possible grades A, B, C, D, E & F. What is the minimum no. of students required in a class to be sure that atleast 7 will receive the same grade

Soln: No. of students (Pigeons)  $n = ?$

$$\lceil \frac{n}{6} \rceil = 7 \Rightarrow \frac{n}{6} = 7 \Rightarrow n = 42$$

(P) Show that if four numbers from 1 to 7 are chosen, then two of them will add upto 8

Soln: Make three sets, each containing two numbers whose sum is 8 if  $A_1 = \{1, 7\}$ ,  $A_2 = \{2, 6\}$ ,  $A_3 = \{3, 5\}$

The four numbers chosen must belong to one of these sets  
As there are only three sets, two of the chosen numbers belong to the same set whose sum is 8.