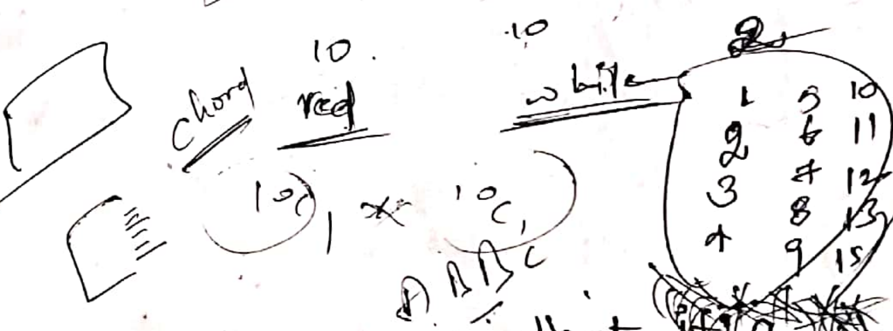
Elementary combinatorics:-

Basics of Counting, Combinations & permutations,
Enumeration of Combinations & permutations,
Enumeration of Combinations & Permutations with Repetition,
Enumerating Permutations with constrained Repetitions.
Binomial coefficients
Binomial & multinomial theorems,
The principles of Inclusion. Exclusion

Basics of Counting :-

The following are two basic counting principles.

(1) product rule.
(2) Sum rule.



product rule.

The product rule states that if a procedure can be broken down into a sequence of two tasks such that the first task can be done in n ways and the second task can be done in m ways after the first task has been done, then there are nm ways of carrying out the procedure.

$m \times n$ ways

Sum rule:-

According to sum rule, if a procedure can be broken down into a sequence of two tasks such that the first task can be done in n ways & the second task can be done in m ways & if these tasks cannot be done at the same time, then there are (n+m) ways of doing one of these tasks.

① There are 150 mathematics major student and 200 computer science students at a college.

(i) How many ways are there to select two representatives so that one is a mathematics major and the other is a computer science major.

150

Sol: ① $150 \times 200 = 30,000$.

② How many ways are there to pick one representative who is either a mathematics or a computer science?

Sol: $150 + 200 = 350$

② How many different bit strings are there of length 9?

1 or 0

Sol: Since each bit is either 1 or 0, each bit can be chosen in two ways.

The number of different bit strings of length 9 $= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$
$= 2^9 = 512$.

③ The chairs of an auditorium are to be labelled with a letter and a positive integer not exceeding 100. what is the largest number of chairs that can be labelled differently.

So: no of ways of labelling a chair with a letter = 26.

number not exceeding 100 = 100.

The largest number of chairs labeled differently = $26 \times 100 = 2600$.

④ How many different license plates are available if each plate contains a sequence of two letters followed by four digits.

$26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6,760,000$.

⑤ A student can choose a computer project from one of five lists. The five lists contain 15, 12, 9, 10 and 20 projects respectively. How many possible projects are there to choose from?

$15 + 12 + 9 + 10 + 20 = 66$.

5) How many bit strings of length 8 either start with bit 1 or end with the two bits 00?

1 or 00

Tasks 1: Starts with bit 1

first bit can be choosen in only one way. and each of the other seven bits can be chosen in two ways.

$1 \times 2^7 = 128$ way.

1×2

0 or 1 0 or 1

1 - 00

Task 2:

$2^6 \times (1 \times 1) = 64$ ways.

~~The~~ ~~is~~ Both the tasks $(T_1 \cap T_2)$ represen the construction of a bit string of length 8 beginning with bit 1 & ending with 00 is $2^5 = 32$ ways.

1 [ ] 00

Hence the number of bit string of length 8 that begin with 1 or end with 00.

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 128 + 64 - 32$

$= 160$

1 2^5 00

Permutations and Combinations:-

An ordered arrangement of objects from a set of distinct objects is called a permutation.

The number of r- permutations of a set with n distinct elements is

$$P(n,r) = \frac{n!}{(n-r)!}$$ $${}^nP_r = \frac{n!}{(n-r)!}$$

eg: ① Find the number of 5- permutations of a set with nine elements.

$$P(9,5) = \frac{9!}{(9-5)!} = \frac{9!}{4!} = 9 \times 8 \times 7 \times 6 \times 5$$
$$= 15,120.$$

② How many permutations of {a,b,c,d,e,f,g} end with a?

Sol: The number of distinct objects in the given set without a = 6.

∴ The no. of permutations = 6! = 720.

③ Consider the six digits 1, 2, 3, 5, 6 and 7. Assuming that repetitions are not permitted, answer the following.

① How many four digit numbers can be formed from the six digits 1, 2, 3, 5, 6 and 7?

(2) How many of these numbers are less than 4000?

(3) How many of the numbers in ~~four~~ (i) are even?

(4) How many of the numbers in (i) are odd?

(5) How many of the numbers in (i) are a multiple of 5.

$^6P_4$

(i) $P(6,4)$ $\frac{6!}{(6-4)!}$

$= 6 \times 5 \times 4 \times 3$

$= 360.$

(ii) The first digit can be 1, 2, 3. $^3P_1$ +

$3 \times 5 \times 4 \times 3 = 60 \times 3 = 180$

$3 \times P(5,3)$

remaining there places is from 5 number

(iii) even means 4 digit number end with 2 or 6.

$5P_3 \times 2 = 2 \times 5 \times 4 \times 3$

$= 120.$

(iv) Odd

$= 4 \times P(5,3) = 240.$

(v) $1 \times P(5,3)$

$= 1 \times 5 \times 4 \times 3 = 60$

4000

3999

1) In how many ways can six men & four women sit in a row?

Sol: 10!

(2) In how many ways can they sit in a row if all the men sit together and all the women sit together?

women and men are 2 catagories.

2!

The men can be arranged among themselves in 6! way.

women in 4! way

The required number of ways

$= 2! \times 6! \times 4! = 34,560.$

(4) There are four bus lines between A & B and three bus lines between B and C.

In how many ways can a man travel.

(1) by bus from A to C via B?

(2) round trip by bus from A to C via B?

(3) round trip by bus from A to C via B if he does not want to use a bus line more than once.

Sol: ① $4 \times 3 = 12$ ways.

② 12 ways to go & 12 ways to return

$12 \times 12 = 144$.

③

$A \xrightarrow{4} B$

$B \xrightarrow{3} C$

$C \xrightarrow{2} B$

$B \xrightarrow{3} A$

$4 \times 3 \times 2 \times 3 = 72$.

$\frac{n!}{(n-r)!} = \frac{n!}{(n-2)!}$

$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$

④ Find n if

① $P(n,2) = 72$.

$P(n,2) = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1) = 72$

$n^2 - n = 72$

$n^2 - n - 72 = 0$

$n^2 - 9n + 8n - 72 = 0$

$n(n-9) + 8(n-9) = 0$

$n+8 = 0 \quad n-9 = 0$

$n = -8. \quad n = 9.$

Since n must be +ve.

$n = 9$.

(ii) $P(n,4) = 42\,P(n,2)$

$n(n-1)(n-2)(n-3) = 42\, n(n-1)$

$(n-2)\cdot(n-3) = 42$

$n^2 - 3n - 2n + 6 = 42$

$n^2 - 5n - 36 = 0$

$(n-9)(n+4) = 0$

$n = 9$

(3) $2P(n,2) + 50 = P(2n,2)$

$2(n(n-1)) + 50 = \frac{2n \cdot 2n-1 \cdot (2n-2)!}{(2n-2)!}$ $\quad {}^{2n}P_2$

$2(n^2 - n) + 50 = 2n(2n-1)$ $\quad \frac{2n!}{(2n-2)!}$

$2n^2 - 2n + 50 = 4n^2 - 2n$

$2n^2 - 2n + 2n - 50 = 0$

$2n^2 - 50 = 0$

$2n^2 = 50$

$n^2 = 25$

$n = 5$

$nC_r = \frac{n!}{(n-r)!\,r!}$

Combinations

Permutation deals with arrangement of objects of a particular set.

Combinations deals with selection only.

$$c(n,r) = \frac{n!}{r!\,(n-r)!}$$

$nC_r = \frac{n!}{(n-r)!\,r!}$ 25

(1) A club has 25 members. How many ways are there to choose four members of the club to serve on an executive Committee?

Sol. $25C_4 = \frac{25!}{4!\,(21)!} = \frac{25 \times \cancel{24} \times 23 \times 22}{\cancel{4 \times 3 \times 2}} = 506 \times 25$

$= 12650$

① How many bit strings of length 8 contain

(i) exactly five 1s?

(ii) an equal number of 0s and 1s.

③ at least four 1s?

④ at least three 1s and at least three 0s?

A bit string of length 8 have eight positions

①

$$8C_5 = \frac{8!}{5!\,3!} = \frac{8!}{(8-5)!\,5!} = \frac{8!}{3!\times 5!}$$

$$= \frac{8\times 7\times \cancel{6}}{\cancel{3!}} = 56.$$

② $8C_4$

$$\frac{8!}{4!\,4!} = \frac{\cancel{8}\times\cancel{7}\times\cancel{6}\times 5}{4\times\cancel{3}\times\cancel{2}} = 70$$

$$\frac{8!}{(8-4)!\,4!}$$

③ These eight positions can be filled up with four 1s & four 0's.

five 1s & 3 0's

6 1s & 2 0's

7 1s & 1 0's.

8 1s.

$$8C_4 + 8C_5 + 8C_6 + 8C_7 + 8C_8.$$

$$= 163.$$

(4) 8 bit string can be

3 1's & 5 0's.
4 1's & 4 0's
5 1's & 3 0's.

$8C_3 + 8C_4 + 8C_5 = 182$.

(5) Suppose a department consists of 10 men & 15 women. How many ways are there to form a committe with six members if it must have three men & three women?

So!' $^{10}C_3 \times {}^{15}C_3 = 54,600$.

(6) Suppose a department consists of eight men & nine women. In how many ways can we select a committee of

(i) three men & four woment,

$8C_3 \times 9C_4 = 7056$

(ii) four persons that has atleast one women

$9C_1 * 8C_3 + 9C_2 \times 8C_2 + 9C_3 \times 8C_1 + 9C_4$
$= 2,310$

(3) four persons that has at most one man?

sol The committee must have at most one man.

Therefore possibilities are

4 w and 0 man
3 w and 1 man

$\cancel{8C_4} \; 9C_4 + 9C_3 \times 8C_1$

$= 798$

(4) four persons that has persons of both sexes?

$8C_3 \times 9C_1 + 8C_2 \times 9C_2 + 8C_1 \times 9C_3$

$= 2184$

(5) four persons so that two specific members are not included.

(1) 2 specific members can be selected in $15C_2$ way.

∴ The number of selections not including these two members =

$17C_4 - 15C_2$

$= \dfrac{17!}{4!\,13!} - \dfrac{15!}{2!\,13!}$

$= 2,275$

→ Binomial theorem:—

Let x and y be any two variables and let n be a non-negative integer then

$$\boxed{(x+y)^n = \sum_{r=1}^{n} {}^nC_r\, x^{r}\, y^{n-r}}$$

$$(x+y)^n = \sum_{r=1}^{n} {}^nC_r\, x^r\, y^{n-r}$$

① Find the expansion of $(x+y)^6$

$$(x+y)^6 = \sum_{i=0}^{6} {}^6C_i\, x^{i}\, y^{6-i}$$

$r=0 \quad n=6$

$\sum_{r=0}^{6} {}^nC_r\, x^r y^{n-r}$

$$= {}^6C_0 x^0 y^{6-0} + {}^6C_1 x^1 y^{6-1} + {}^6C_2 x^2 y^{6-2} + {}^6C_3 x^3 y^{6-3} + {}^6C_4 x^4 y^{6-4} + {}^6C_5 x^5 y^{6-5},\ {}^6C_6 x^6 y^{6-6}.$$

$$= {}^6C_0 y^6 + {}^6C_1 x y^5 + {}^6C_2 x^2 y^4 + {}^6C_3 x^3 y^3 + {}^6C_4 x^4 y^2 + {}^6C_5 x^5 y + {}^6C_6 x^6$$

$$= \boxed{y^6 + 6x y^5 + 15x^2y^4 + 20x^3y^3 + 15x^4y^2 + 6x^5y + x^6.}$$

② Find the co-efficient of $x^5 y^8$ in $(x+y)^{13}$.

$${}^{13}C_8\, x^5 y^8$$

$$= {}^{13}C_8 = \frac{13!}{5!\,8!} = 1287.$$

$\sum {}^nC_r\, x^r y^{n-r}$

$$\boxed{{}^{13}C_5\, x^5 y^8}$$

$\frac{13!}{(13-5)!\,5!}$

${}^4C_r$

${}^nC_r = \frac{n!}{(n-r)!\,r!}$

${}^nP_r = \frac{n!}{(n-r)!}$

(3) what is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$, $n=200$

$r=101$

$$= 200C_{101}\,(2x)^{101}(-3y)^{99}$$

$nC_r\, x^r y^{n-r}$

$^{200}C_{101}\, x^{101}\, y^{200-101}$

$$= \frac{200!}{101!\,99!}\, 2^{101}(-3)^{99}\, x^{101}\, y^{99}$$

(4) If n is a non-negative integer, show that $\sum_{i=0}^{n} {}^nC_i = 2^n$

$(a+b)^n$

$$2^n = (1+1)^n$$

$$= nC_0 (1)^0 (1)^{n-0} + nC_1 (1)^1 (1)^{n-1} + nC_2 (1)^2 (1)^{n-2} \ldots$$

$$= nC_0 + nC_1 + nC_2 + nC_3 \ldots$$

$$= 1 + nC_i (1)^{n-1} + \ldots$$

$$= \sum_{i=0} {}^nC_i$$

(5) If n is a non-negative integer show that $\sum_{i=0}^{n} 2^i\, {}^nC_i = 3^n$.

$$3^n = (2+1)^n = \sum_{i=0} nC_i (2)^i (1)^{n-i}$$

$$= \sum_{i=0}^{n} nC_i\, 2^i$$

Permutation with Repetition

① The number of r permutations of a set of n objects with repetition allowed is n^r.

Diff color balls.

② If there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, and n_k indistinguishable objects of type k, then the number of different permutations of n objects is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$ where $n_1+n_2+n_3+\cdots n_k=n$

diff colors in one block.

③ Combinations with Repetition:→

→ If repetition of elements is allowed, then the number of r- combinations from a set with n elements is

$$^{n+r-1}C_r$$

① How many strings of six letters are there?

26

$n = 26$ letters.

$r = 6$ letters.

The strings of six letters $= n^r$

$= 26^6 = 26 \times 26 \times 26 \times 26 \times 26$.

② How many ways are there to assign three jobs to five employees if each employee can be given more than one job.

$n = 5$

$r = 3$

$n = 5$ ~~job~~

$r = 3$

The number of ways to assign three jobs to the five employees (when repetition is allowed)

$= 5^3 = 5 \times 5 \times 5 = 125$

(3) How many ways are there to select three unordered elements from a set of five elements when repetition is allowed?

Sol: we have to select three unordered elements from a set of five elements. repetition is allowed

$n = 5$

$r = 3$

$$\frac{2 \times \cancel{6} \times 5 \times \cancel{4}}{\cancel{4!} \times 3 \times \cancel{2} \times 1}$$

$${}^{n+r-1}C_r$$

$$= {}^{(5+3-1)}C_3$$

$${}^{n}C_r = \frac{n!}{(n-r)!\,r!} \qquad = {}^{7}C_3 = \frac{7 \times \cancel{6} \times 5 \times \cancel{4!}}{\cancel{3 \times 2}\ \cancel{4!}} = 35.$$

(4) How many different strings can be made from the letters of the word "SUCCESS" using all the letters?

Sol. SUCCESS = 7.

$s = 3$

$c =$

$$= \frac{7!}{2!\,3!\,1!\,1!} = 420.$$

(5) How many different strings can be made from the letters in ABRACADABRA using all the letters?

$$= \frac{11!}{5!\,2!\,2!\,1!\,1!}$$

$$= 83{,}160.$$

→ There are three boxes of identical red, blue & white balls, where each box contains at least 10 balls, How many ways are there to select 10 balls if

① there is no restriction.

$${}^{(3+10-1)}C_{10}$$

$$= {}^{12}C_{10} = 66.$$

② at least one white ball must be selected

$$1 \times {}^{(3+9-1)}C_9 = {}^{11}C_9 = 55$$

③ at least one red ball, at least two blue balls & at least three white balls must be selected

1 - red
2 - blue
3 - white

} 6 remaining
4

$${}^{3+4-1}C_4 = {}^{6}C_4 = 15$$

colors value

Circular permutations :—

If objects of a given set are arranged in a line, we obtain a linear permutation. if objects are arranged in a circle or any closed curve, we get a circular permutation.

The number of circular permutations will be different from number of linear permutations. 8

Note : The number of different circullar arrangements of n objects $= (n-1)!$

(1) Find the number of different circular arrangements of five element a, e, i, o and u elements.

So : we fix one of the elements.

Say a at the top of the circle. The other four elements e, i, o and u can be arranged in a circle is

$4 \times 3 \times 2 \times 1 = 4! = 20$ ways.

$(n-1)! = (5-1)! = 4! = 20$ ways.

Note : If the number of clockwise circular arrangements is equal to the number of counter-clockwise circular arrangements, then the number of different circular arrangements = $\frac{1}{2}(n-1)$ is

→ If eight people P, Q, R, S, T, U, V and W are seated around a round table.

(i) How many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation.

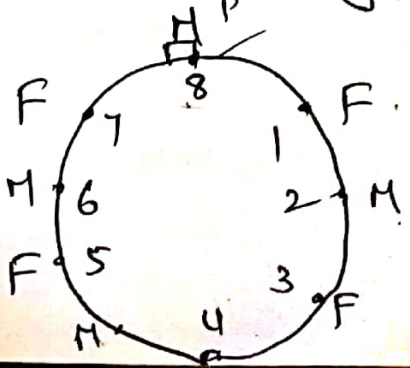
after rotation no change in adjacent elem

(2) If P, Q, R and S are males and T, U, V and W are females, in how many arrangements do the sexes alternate.

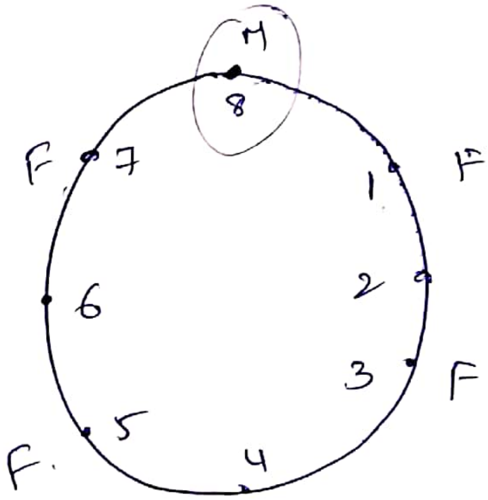
Sol. ① Since the rotation does not alter the circular arrangements,

The required number of different circular arrangements = $(8-1)! = 7! = 5040$.

②



Since the rotation does not alter the circular arrangements, we can assume that P occupies the top position.



The remaining position 1, 3, 5 and 7 must be occupied by four females.

The number of permutations = $4P_4 = 4! = 24$ way

The places 2, 4 and 6 must be occupied by the remaining three males.

$\therefore$ $3P_3 = 6$ way. 3! ways

The total number of required circular arrangments = $24 \times 6 = 144$

$4! \times 3! =$

Coefficient

Multinomial ~~Coefficient~~ Theorem:

Given non negative integers $k_1, k_2, \ldots k_m$ and $n = k_1 + k_2 + \cdots k_m$, the multinomial Coeffient $\begin{bmatrix} n \\ k_1, k_2, \ldots, k_m \end{bmatrix}$ is defined by

$$\begin{pmatrix} n \\ k_1, k_2, \ldots, k_m \end{pmatrix} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

$n_\phi = n_1 + n_2 + n_3$

eg: Compute $\begin{bmatrix} 10 \\ 3, 2, 5 \end{bmatrix}$ (n_1, n_2, n_3)

$n_1 + n_2 + n_3 = 7$

$$= \frac{10!}{3!\, 2!\, 5!} = 2520.$$

Note: Given nonnegative integers $k_1, k_2, k_3 \ldots k_m$ and $n = k_1 + k_2 + k_3 \cdots + k_m$. The multinomial coefficient $\begin{bmatrix} n \\ k_1, k_2, \ldots k_m \end{bmatrix}$ Counts the number of ways to split n distinct items into m distinct categories. of sizes $k_1, k_2, \ldots k_m$

Multinomial theorem:-

Let $x_1, x_2, \dots x_m \in R$ and $n \in Z$ with $n \geq 1$

then.

$$(x_1 + x_2 + \dots x_m)^n = \frac{n!}{n_1! \, n_2! \dots n_m!} \left[x_1^{n_1} \, x_2^{n_2} \dots x_m^{n_m} \right.$$

when $n_1 + n_2 + n_3 + \dots + n_m = n$.

① Find the term which contains x^{11} and y^4 in the expansion of $(2x^3 - 3xy^2 + z^2)^6$

so!

$$\binom{6}{n_1, n_2, n_3} (2x^3)^{n_1} (-3xy^2)^{n_2} (z^2)^{n_3}$$

$$= \binom{6}{n_1, n_2, n_3} 2^{n_1} (-3)^{n_2} (x)^{3n_1 + n_2} y^{2n_2} z^{2n_3}.$$

$n_1 + n_2 + n_3 = 6$

$\therefore$ For the term containing x^{11} and y^4 we should have $3n_1 + n_2 = 11$

$2n_2 = 4$

$2n_2 = 4$
$n_2 = 2$

and $3n_1 + 2 = 11$
$3n_1 = 9$
$n_1 = 3$

$n_1 + n_2 + n_3 = 6$

$3 + 2 + n_3 = 6$
$n_3 = 1$

$\therefore$ the term Containing x^{11} and y^4 is

$$\begin{bmatrix} 6 \\ 3,2,1 \end{bmatrix} 2^3 (-3)^2 x^{11} y^4 z^2 = 4320 x^{11} y^4 z^2.$$

$\rightarrow$ Determine the coefficient of $a^2 b^3 c^2 d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

$$\begin{bmatrix} 16 \\ n_1, n_2, n_3, n_4, n_5 \end{bmatrix} (a)^{n_1} (2b)^{n_2} (-3c)^{n_3} (2d)^{n_4} (5)^{n_5}$$

$$= \begin{bmatrix} 16 \\ n_1, n_2, n_3, n_4, n_5 \end{bmatrix} a^{n_1} (2)^{n_2} (b)^{n_2} (-3)^{n_3} (c)^{n_3} (2)^{n_4} (d)^{n_4} (5)^{n_5}$$

$n_1 = 2$ $\quad$ $n_3 = 2$ $\quad$ $n_5 = 16 - (2+3+2+5)$

$n_2 = 3$ $\quad$ $n_4 = 5$ $\quad$ $= 4$

$$= \begin{bmatrix} 16 \\ 2,3,2,5,4 \end{bmatrix} a^2 (2)^3 b^3 (-3)^2 (c)^3 (2)^5 d^5 5^4$$

The required Cofficient is

$$\frac{16!}{2! \times 3! \times 2! \times 5! \times 4!} (2)^3 (-3)^2 (2)^5 (5)^4$$

→ Determine the coefficient of xyz^2 in the expansion of $(2x-y-z)^4$.