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1. Mathematical Logic

Statements:- A declarative sentence which can be said to be either true or false, such sentences are called as statements or propositions.

A proposition is a declarative sentence in a given context which can be either true or false, but not both.

Ex:- 3 is a prime number.
7 is divisible by 3.

Consider an example $xy = yx$.. The
The above sentence is not a proposition because we cannot say whether it is true or not until and unless x and y values are given.

Logical connectives and truth tables:- New propositions are obtained by starting with given propositions with the help of words like "not, and, or, if, iff."

Such words or phrases are called "logical connectives".

→ The new propositions obtained by the use of connectives are called "compound propositions."

Negation:- A proposition obtained by inserting a word 'not' at an appropriate place in a given statement is called negation of that statement.

The negation of a proposition 'p' is denoted by "$\neg p$".

Ex:- 3 is a prime number is denoted by
Then, $\neg p$ is 3 is not a prime number.

Truth table for negation:-

p	$\neg p$
0	1
1	0

Conjunction:- A compound proposition obtained by combining two given propositions by inserting the word 'and' in between them is called the conjunction of given two statements.

It is denoted by the symbol

The following rule is adopted in deciding the truth value of a conjunction. i.e "the conjunction $(p \wedge q)$ is true only when p is true & q is true, in all the other cases, it is false."

Truth table of conjunction:-

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

Ex:- p: $\sqrt{2}$ is an irrational number

q: 9 is a prime number

$(p \wedge q)$: $\sqrt{2}$ is an irrational number and 9 is a prime number.

Disjunction:- A compound proposition obtained by combining a two statements by inserting the word 'or' in between them is called the disjunction of given statements.

It is denoted by the symbol '$\vee$'

The following rule is adopted in deciding the truth value of a disjunction i.e " the disjunction $(p \vee q)$ is false only when p and q both are false and in all the other cases it is true".

Truth table of disjunction:-

p	q	$p \vee q$
0	0	0
0	1	1

Ex:- R2 Y:-

Conditional :- A compound proposition obtained by combining two given propositions by using the words "if" and "then" at appropriate places is called a conditional.

The following rule is adopted in knowing the truth value of a condition $(p \to q)$: The condition $(p \to q)$ is false only when p is true and q is false, in all the other cases it is true.

Truth table of conditional:-

p	q	$p \to q$
0	0	1
1	0	0
0	1	1
1	1	1

Biconditional:- A compound proposition obtained by the "conjunction of the conditionals $(p \to q)$ and $(q \to p)$ is called biconditional of p and q.

$(p \leftrightarrow q)$ is read as "if p then q and if q then p."

Truth table for biconditional:-

p	q	$p \leftrightarrow q$
0	0	1
0	1	0

Well formed formulae:- while representing a statement using connectives in symbolic form, care has to be taken to ensure that statement conveys the intended meaning without any ambiguity. Appropriate parantheses are to be used at appropriate places.

Statements represented in symbolic forms which cannot be interpretted in more than one way are called well formed formulae.

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Ex:- Negation of conjunction of propositions p and q is symbolically represented as

$\Rightarrow \sim(p \wedge q)$

1. Consider the following propositions concerned with certain triangle ABC.

p: ABC is isosceles

q: ABC is equilateral

r: ABC is equiangular.

Write down the following compound propositions in words.

1) $p \wedge (\neg q)$

2) $((\neg p) \vee q)$ %

3) $p \rightarrow q$

4) $(\neg r) \rightarrow (\neg q)$

5) $p \leftrightarrow (\neg q)$

1) ABC is isosceles and ABC is not equilateral

2) ABC is not isosceles or ABC is equilateral

3) If ABC is isosceles then ABC is equilateral.

4) If ABC is not equiangular then ABC is not equilateral.

5) If ABC is isosceles then ABC is not equilateral and if ABC is not equilateral then ABC is isosceles.

2. Construct the truth tables for the following well formed formulae.

1) $p \vee (\neg q)$

2) $p \wedge (q \wedge p)$

3) $p \wedge (q \vee p)$

4) $(p \vee q) \wedge (\neg p)$

5) $p \rightarrow (q \rightarrow r)$

1) $p \vee (\neg q)$

p	q	$\neg q$	$p \vee (\neg q)$
0	0	1	1
1	0	1	1
0	1	0	0
1	1	0	1

2) $p \wedge (q \wedge p)$

p	q	$q \wedge p$	$p \wedge (q \wedge p)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

3) $p \wedge (q \vee p)$

p	q	$q \vee p$	$p \wedge (q \vee p)$
0	0	0	0
1	0	1	0
0	1	1	0
1	1	1	1

4) $(p \vee q) \wedge (\neg p)$

p	$\neg p$	q	$p \vee q$	$(p \vee q) \wedge (\neg p)$
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0

5) $p \to (q \to r)$

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1

Tautology & contradiction:-

A compound proposition which is true for all possible truth values of its components is called a tautology.

A compound proposition which is false for all possible truth values of its components is called a contradiction. (or) absurdity.

1. Prove that for any proposition p, the compound proposition $p \vee (\neg p)$ is a tautology and $p \wedge (\neg p)$ is a contradiction.

p	$\neg p$	$p \vee (\neg p)$	$p \wedge (\neg p)$
0	1	1	0
1	0	1	0

$\therefore$ $p \vee (\neg p)$ has only true values for its components. Therefore it is a tautology.

$\therefore$ $p \wedge (\neg p)$ has only false values. Therefore it is a contradiction.

2. Show that for any propositions p and q, $p \to (p \vee q)$ is a tautology & $p \wedge (\neg p \wedge q)$ is a contradiction.

p	q	$\neg p$	$p \vee q$	$\neg p \wedge q$	$p \to (p \vee q)$	$p \wedge (\neg p \wedge q)$
0	0	1	0	0	1	0
1	0	0	1	0	1	0
0	1	1	1	1	1	0
1	1	0	1	0	1	0

3. $(p \to q) \leftrightarrow (\neg p \vee q)$

p	q	$\neg p$	$p \to q$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
0	0	1	1	1	1
1	0	0	0	0	1
0	1	1	1	1	1
1	1	0	1	1	1

4. $[(p \to q) \wedge (q \to r)] \to (p \to r)$

p	q	r	$\overset{x}{p \to q}$	$\overset{y}{q \to r}$	$(x \wedge y)$	$(x \wedge y) \to x$
1	1	1	1	1	1	1
1	1	0	1	0	0	1
1	0	1	0	1	0	1
1	0	0	0	1	0	1
0	1	1	1	1	1	1
0	1	0	1	0	0	1
0	0	1	1	1	1	1
0	0	0	1	1	1	1

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Logical equivalence :- Two propositions u and v are said to be logically equivalent whenever u and v have the same truth values (or) whenever the biconditional $u \leftrightarrow v$ is a tautology.

Ex:- for any two propositions p and q,

Ex:- For any two propositions p and q, prove th
$p \to q$ is logically equivalent to $(\neg p) \vee q$.

p	q	$\neg p$	$p \to q$	$\neg p \vee q$	$(p \to q) \Leftrightarrow (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	0	1	1	1

2. P.T for any three propositions p, q, r
$[(p \vee q) \to r] \Leftrightarrow [(p \to r) \wedge (q \to r)]$

p	q	r	$(p \vee q)$	$(p \vee q) \to r$	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

$\therefore$ Logically equivalent.

Laws of logic:- The following results, known as laws of logic follow from the definition of logical equivalence. In these laws, T_0 denotes a tautology and F_0 denotes a contradiction.

* Law of double negation:- for any proposition p,

$(\neg\neg p) \Leftrightarrow p$

* Idempotent law:- For any proposition p,

a) $(p \vee p) \Leftrightarrow p$

b) $(p \wedge p) \Leftrightarrow p$

* Identity laws:- for any proposition p,

a) $(p \vee F_0) \Leftrightarrow p$

b) $(p \wedge T_0) \Leftrightarrow p$

* Inverse laws:- For any proposition p,

a) $(p \vee \neg p) \Leftrightarrow T_0$

b) $(p \wedge \neg p) \Leftrightarrow F_0$.

* Domination laws:- For any proposition p,

a) $(p \vee T_0) \Leftrightarrow T_0$

b) $(p \wedge F_0) \Leftrightarrow F_0$

* Commutative laws:- For any two propositions p & q,

a) $(p \vee q) \Leftrightarrow (q \vee p)$

b) $(p \wedge q) \Leftrightarrow (q \wedge p)$

* Absorption laws:- For any two propositions p & q,

a) $[p \vee (p \wedge q)] \Leftrightarrow p$

b) $[p \wedge (p \vee q)] \Leftrightarrow p$

* De morgan's laws:- For any two propositions p & q,

a) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

b) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

* **Associative laws:-** For any three propositions p, q & r

a) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

b) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

* **Distributive laws:-** For any three propositions p, q & r,

a) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

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Converse, inverse and contrapositive:- Consider a conditional $p \rightarrow q$, then

i) $q \rightarrow p$ is called converse of $p \rightarrow q$

ii) $\neg p \rightarrow \neg q$ is called inverse of $p \rightarrow q$

iii) $\neg q \rightarrow \neg p$ is called contrapositive of $p \rightarrow q$

Ex:- p: 2 is an integer

q: 9 is multiple of 3.

$p \rightarrow q$.

If 2 is an integer, then 9 is a multiple of 3.

Converse:- $q \rightarrow p$:

If 9 is a multiple of 3, then 2 is an integer.

Inverse:- $\neg p \rightarrow \neg q$.

If 2 is not an integer, then 9 is not a multiple of 3.

Contrapositive: $\neg q \rightarrow \neg p$

If 9 is not a multiple of 3, then 2 is not an integer.

Duality:- Suppose u is a compound proposition that contains connectives '$\wedge$' & '$\vee$'. Replacing each occurance of '$\wedge$' & '$\vee$' in u by '$\vee$' & '$\wedge$' respectively. If u has a tautology T_0 and F_0 as components, we replace each occurance of T_0 and F_0 by F_0 and T_0 respectively, then the result compound proposition is called duality of u and is denoted by u^d.

Ex:- $u: p\wedge(q\vee r)\vee(s\wedge T_0)$

$u^d: p\vee(q\wedge r)\wedge(s\vee F_0)$

Note:- The following two results are very important.

1) $(u^d)^d \Leftrightarrow u$

2) For any two propositions u & v, if $u \Leftrightarrow v$, then $u^d \Leftrightarrow v^d$ [This is called principle of duality]

Necessary and Sufficient Conditions:- Consider two propositions p and q whose truth values are interrelated.

Suppose that $p \Rightarrow q$, then in order that q may be true, it is sufficient that p is true. Also if p is true, then it is necessary that q is true.

Ex:- Let A denote a specified city. consider the following propositions

p: A is in Karnataka.

Then $p \Rightarrow q$ and $q \not\Rightarrow p$. Accordingly, p is Sufficient but not necessary condition for q. And q is necessary but not a sufficient condition of p.

Ex:- p: ABC is equiangular
q: ABC is equilateral.

Now, $p \Rightarrow q$ & $q \Rightarrow p$

1. Prove the following logical equivalences without using truth tables.

i) $p \vee (p \wedge (p \vee q)) \Leftrightarrow p$

$p((p \wedge p) \vee (p \wedge q)) \Leftrightarrow p$

$p \vee (p \vee (p \wedge q))$

$= (p \vee p) \vee (p \vee (p \wedge q))$

$= p \vee [(p \vee p) \wedge (p \wedge q)]$

LHS $=$ $p \vee (p \wedge (p \vee q))$

$\Rightarrow p \vee p$ [by using absorption law, $p \wedge (p \vee q) \Leftrightarrow p$]

$\Rightarrow p$

RHS:- p

$\therefore$ LHS $=$ RHS.

$p \vee (p \wedge (p \vee q)) \Leftrightarrow p$

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2. [C

Lt

Truth

Truth table

p	q
0	0
0	0
0	1
0	1
1	0
1	0
1	1
1	1

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2. $[(\neg p \vee \neg q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$.

LHS:- ~~E: u→v~~

$[\neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)]$

$[\because u \to v \Rightarrow \neg u \vee v]$

$= [(p \wedge q) \vee (p \wedge q \wedge r)]$ [De morgan's law & law of double negation]

$= [(p \wedge q) \vee [(p \wedge q) \wedge r]$

$= p \wedge q$ [Absorption laws].

~~Truth table~~ = RHS

$\therefore$ LHS = RHS

$\therefore$ They are logically equivalent.

Truth table:

p	q	¬p	¬q	(u) ¬p ∨ ¬q	r	(v) p ∧ q ∧ r	p ∧ q	u → v
0	0	1	1	1	0	0	0	0
0	0	1	1	1	1	0	0	0
0	1	1	0	1	0	0	0	0
0	1	1	0	1	1	0	0	0
1	0	0	1	1	0	0	0	0
1	0	0	1	1	1	0	0	0
1	1	0	0	0	0	0	1	1
1	1	0	0	0	1	1	1	1

$\therefore$ Truth values are equal.

They are logically equivalent

3. $(p\vee q)\wedge[\neg(\neg p\wedge q)]\Leftrightarrow p$.

LHS:- $(p\vee q)\wedge[p\vee\neg q]$ [Demorgan's law & law of double negation]

$=[p\vee(q\wedge\neg q)]$ [distributive law]

$= p\vee F_0$ [Inverse law]

$= p$

= RHS

LHS = RHS.

∴ They are logically equivalent

4. $(p\rightarrow(q\rightarrow r))\Leftrightarrow(p\wedge q)\rightarrow r$

LHS $\neg p\vee(q\rightarrow r)$ [$u\rightarrow r\Rightarrow\neg u\vee V$]

$\neg p\vee(\neg q\vee r)$

$(\neg p\vee\neg q)\vee r$ (Associative law)

$\neg(p\wedge q)\vee r$ (Demorgan's law)

$(p\wedge q)\rightarrow r$ ($\neg u\vee V\Rightarrow u\rightarrow r$)

= RHS

∴ LHS = RHS

They are logically equivalent

5. $[(p\vee q)\wedge(p\vee\neg q)]\vee q\Leftrightarrow p\vee q$

LHS:-

$[(p\vee q)\wedge(p\vee\neg q)]\vee q$

$[p\vee(q\wedge\neg q)]\vee q$ (distributive law & Inverse law)

$(p\vee F_0)\vee q$ (Identity law)

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Normal forms:- The problem of determining in a finite no. of steps whether a given statement formulae is a tautology or a contradiction is known as decision problem.

Disjunctive normal form:-

Note:- It will be convinient to use a word product in place of conjunction & sum in place of disjunction.

→ A product of variables and their negations in a formulae is called Elementary product.

→ A sum of variables and their negations is called Elementary sum.

Eg for elementary products :- $p \wedge \neg p$, $q \wedge \neg q$, $p \wedge q$.

Eg for elementary sum:- $p \vee \neg p$, $q \vee \neg q$, $p \vee q$.

→ A formula which is equivalent to a given formula and which consists of a sum of elementary products is called Disjunctive normal form.

Ex:- Obtain disjunctive normal form of $p \wedge (p \rightarrow q)$

$p \wedge (\neg p \vee q)$ $[u \rightarrow v \Rightarrow \neg u \vee v]$

$(p \wedge \neg p) \vee (p \wedge q)$ (distributive law)

1. $p \rightarrow \{(p \rightarrow q) \wedge \neg(\neg q \vee \neg p)\}$.

$p \rightarrow ((\neg p \vee q) \wedge (q \wedge p))$ (Demorgan's law)

$\neg p \vee ((\neg p \vee q) \wedge (q \wedge p))$

$\cancel{\neg q}\ \neg p \vee ((\neg p \vee q) \wedge (q \wedge p))$

$\cancel{= (\neg p \vee (\neg p \vee q)) \wedge (q \wedge p)}$

$= \neg p \vee ((\neg p \wedge (q \wedge p)) \vee (q \wedge (q \wedge p)))$

$= \neg p \vee ((F_0 \wedge q) \vee (q \wedge p))$

$= \neg p \vee (F_0 \vee (q \wedge p))$

$= \neg p \vee (q \wedge p)$

$= (\neg p \wedge T_0) \vee (q \wedge p)$ //

2) Conjuctive normal form:-
A formula which is equivalent to given formula & which consists of product of elementary sums is called conjunctive normal form.

1. Obtain a conjunctive normal form for $p \wedge (p \to q)$

$p \wedge (p \to q)$

$p \wedge (\neg p \vee q)$

$(p \vee F_0) \wedge (\neg p \vee q)$

2. $\neg(p \vee q) \leftrightarrow (p \wedge q)$ $\quad (R \leftrightarrow S = (R \to S) \wedge (S \to R))$

$(\neg(p \vee q) \to (p \wedge q)) \wedge ((p \wedge q) \to \neg(p \vee q))$

$((p \vee q) \vee (p \wedge q)) \wedge (\neg(p \wedge q) \vee \neg(p \vee q))$

$((p \vee q \vee p) \wedge (p \vee q \vee q)) \wedge ((\neg p \vee \neg q) \vee (\neg p \wedge \neg q))$

$= (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge ((\neg p \vee \neg q \vee \neg q) \wedge (\neg p \vee \neg q \vee \neg p))$

$= ((p \vee q) \wedge (p \vee q)) \wedge ((\neg p \vee \neg q) \wedge (\neg p \vee \neg q))$

Solve ·

2. $q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$

$\Rightarrow q \vee (p \wedge \neg q) \vee (\neg(p \vee q))$

$\Rightarrow ((q \vee p) \wedge (q \vee \neg q)) \vee (\neg(p \vee q))$

$\Rightarrow ((q \vee p) \wedge T_0) \vee (\neg(p \vee q))$

$\Rightarrow (q \wedge T_0) \vee (p \wedge T_0) \vee (\neg(p \vee q))$

$\Rightarrow (q \vee p) \vee (\neg(p \vee q)) = T_0$.

$\Rightarrow (q \vee (\neg p \wedge \neg q)) \vee \cancel{(p \wedge (\neg p \wedge \neg q))}$ $(q \vee p) \vee (\neg p \wedge \neg q)$

$(q \vee \neg p)$

$\cancel{(F_0 \vee \neg p) \vee (F_0 \wedge \neg q)}$

$\Rightarrow (F_0 \vee \neg p) \vee (F_0 \wedge \neg q)$

$\Rightarrow \neg p \vee F_0$

$= \neg p$

p	q	¬p	¬q	p∧¬q	¬p∧¬q	q∨(p∧¬q)∨(¬p∧¬q)
1	0	0	1	1	0	1
0	0	1	1	0	1	1
1	1	0	0	0	0	1
0	1	1	0	0	0	1
						1

2. conjuctive normal form = Product of sums

$q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$.

$= q \vee ((\neg q \wedge p) \vee (\neg q \wedge \neg p))$

$((p \wedge q) \vee (p \wedge r))$

$= q \vee [\neg q \wedge (p \vee \neg p)]$

$= (q \vee \neg q) \wedge (q \vee p \vee \neg p)$

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Principal normal forms:- Given two simple propositions p and q, a compound proposition $(p \wedge q)$, $(p \wedge \neg q)$, $(\neg p \wedge q)$, $(\neg p \wedge \neg q)$ are called min terms involving p and q.

The duals of min terms mainly $(p \vee q)$, $(p \vee \neg q)$, $(\neg p \vee q)$, $(\neg p \vee \neg q)$ are called ~~max~~ max terms.

* Given a compound proposition u involving two simple propositions p and q, an equivalent compound proposition v consisting of "Disjunction of min-terms involving ~~do~~ p and q is known as principle disjunctive Normal form (PDNF)". (or) Sum of products of canonical form

* Similarly, given a compound statement u involving two simple propositions. p and q, an equivalent proposition v consisting of "Conjunction of max terms involving p and q is called principal conjunctive Normal form (PCNF)" or ~~Sum~~ Product of sums of canonical form.

Ex:- Obtain the PDNF of $p \vee (p \wedge q)$

$(p \wedge T_0) \vee (p \wedge q)$

$(p \wedge (q \vee \neg q)) \vee (p \wedge q)$

$(p \wedge q) \vee (p \wedge \neg q) \vee (p \wedge q)$

PCNF

2) $(\neg p \to q) \wedge (q \leftrightarrow p)$

$(\neg p \to q) \wedge (q \to p) \wedge (p \to q)$

$(p \vee q) \wedge (\neg q \vee p) \wedge (\neg p \vee q)$

3) Obtain PDNF for $\neg p \vee q$

$(\neg p \wedge T_0) \vee (q \wedge T_0)$

$(\neg p \wedge (q \vee \neg q)) \vee (q \wedge (p \vee \neg p))$

$= (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg p)$

$= (\neg p \wedge q) \vee (\neg p \wedge \neg q) \vee (q \wedge p)$

4) Obtain PCNF for $(p \wedge q) \vee (\neg p \wedge q)$

$(p \wedge q) \vee (\neg p \wedge q)$

$[q \wedge (p \vee \neg p)]$

~~$(q \wedge T_0) \wedge (q \wedge$~~

$(q \vee F_0) \wedge (p \vee \neg p)$

$q \vee (p \wedge \neg p) \wedge (p \vee \neg p)$

$(q \vee p) \wedge (q \vee \neg p) \wedge (p \vee \neg p)$

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15/7/13 Universal quantifiers:- Quantifiers are derived into two types:-

1) Universal quantifier:- which indicates that something is true for all individuals

2) Existential quantifier which indicates that a statement is true for some individuals.

Definition:- Let, A represents an expression ~~a~~ and let x represents a variable. If we want to indicate that A is true for all possible values of x, then we write $\forall x A$.

Here, $\forall x$ is called a ~~qa~~ universal quantifier and A is called the scope of the quantifier.

The variable x is said to bound by the quantifier. The symbol '$\forall$' is pronounced as 'for all'.

Statements containing words like "every", "each", "every one" usually indicate universal quantification.

Ex:- Express "everyone gets a break once in a while". in ~~predicate~~ prediculate calculus.

Sol: We define B as gets a break once in a while.

Hence, $B(x)$ means that x gets a break once in a while.

The word everyone indicates that this is true for all x.

It can be expressed as '$\forall x B(x)$'.

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2. Predicates

Let us consider two statements

1) John is a bachelor.
2) Smith is a bachelor

If we express the statements by symbol, we require two different symbols to denote them.

If we introduce some symbol to denote "is a bachelor" and a method to join it with symbols denoting the names of the individuals, then we will have a symobolism to denote a statement.

The part "is a bachelor" is called a predicate.

We shall symbolize a predicate by a capital letter and the names of the individuals and objects by small letters.

In the above example, denote the Predicate "is a bachelor" symbolically by letter 'B' and John by small 'j' & Smith by small 's'. The statement can be written as '$B(j)$' & '$B(s)$'

Let 'R' denote a predicate "is red" and let 'p' denotes "this painting". Then the statement painting is red is symbolically denoted as $R(P)$.

Further, the connectives can be used to form the compound statements.

Note:- A predicate requiring "m names" is called m place predicate.

Ex:- In the given statements 1 and 2, B is a '1 place predicate'.

consider the example of statements involving two objects Jack is taller than Jill

⇒ 'two place predicate'

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Free & bound variables:- Given a formula containi a part of the form $x P(x)$ or $(\exists x) P(x)$, such a part is called an x bound part of the form

Any occurance of ~~is are~~ of any x an x bound part is called bound occurance Variable x.

While any occurance of x or of an variable that is not a bound occurance is called free occurance of a variable.

Futher the formula $P(x)$ either $(x) P(x)$ or $(\exists x) P(x)$ is described as the scope the quantifier.

Ex:- consider the following formulae

① $(x) P(x,y)$

In the above example, scope of the quantifier is $P(x, y)$' Both occurances of x are bound occurances while occurance of y is free occurance.

② $(x)(P(x) \rightarrow Q(x))$

In this example, $(P(x) \rightarrow Q(x))$ is scope of the quantifier and x is the bound occurance.

③ $(x)(P(x) \rightarrow (\exists y) R(x,y))$

In this example, $(P(x) \rightarrow (\exists y) R(x,y))$ is the scope of the quantifier and x & y. both are bound occuarances. ~~since occurar~~

And also the scope of $\exists y$ is $R(x,y)$.

Rules of inference:-

The process of derivation by which one demonstrates that a particular formula is a ~~validate~~ valid consequence of a given set of premises. Before this, we have two rules of inference which are called 'rule p' & 'rule T'

Rule P:- A premise may be introduced at any point in the derivation

Rule T:- A formula S may be introduced in a derivation if S is tautologically implied by anyone or more of the preceeding formulae in the derivation.

Following are some important implications and equivalences that are used in the process of derivation.

Implications:-

$$\left.\begin{aligned} I_1 &: P\land Q \Rightarrow P \\ I_2 &: P\land Q \Rightarrow Q \end{aligned}\right\} \text{(simplification)}$$

$$\left.\begin{aligned} I_3 &: P \Rightarrow P\lor Q \\ I_4 &: Q \Rightarrow P\lor Q \end{aligned}\right\} \text{(addition)}$$

$I_5: \neg P \Rightarrow P \to Q$

$I_6: Q \Rightarrow P \to Q$

$I_7: \neg(P\to Q) \Rightarrow P$

$I_8: \neg(P\to Q) \Rightarrow \neg Q$

$I_9: P, Q \Rightarrow P\land Q$

$I_{10}: \neg P, P\lor Q \Rightarrow Q$ (disjunctive syllogism)

$I_{11}: P, P\to Q \Rightarrow Q$ (modus ponens)

$I_{12}: \neg Q, P\to Q \Rightarrow \neg P$ (modus Tollens)

$I_{13}: P\to Q, Q\to R \Rightarrow P\to R$ (hypothetical syllogism)

$I_{14}: P\lor Q, P\to R, Q\to R \Rightarrow R$ (dilemma)

Equivalences:-

$E_1: \neg\neg P \Leftrightarrow P$ } $\to$ law of double negation.

$$\left.\begin{aligned} E_2 &: P\land Q \Leftrightarrow Q\land P \\ E_3 &: P\lor Q \Leftrightarrow Q\lor P \end{aligned}\right\} \to \text{commutative laws}$$

$$\left.\begin{aligned} E_4 &: (P\land Q)\land R \Leftrightarrow P\land(Q\land R) \\ E_5 &: (P\lor Q)\lor R \Leftrightarrow P\lor(Q\lor R) \end{aligned}\right\} \text{Associative laws}$$

$$\left.\begin{aligned} E_6 &: P\land(Q\lor R) \Leftrightarrow (P\land Q)\lor(P\land R) \\ E_7 &: P\lor(Q\land R) \Leftrightarrow (P\lor Q)\land(P\lor R) \end{aligned}\right\} \text{Distributive laws}$$

$$\left.\begin{aligned} E_8 &: \neg(P\land Q) \Leftrightarrow (\neg P\lor \neg Q) \\ E_9 &: \neg(P\lor Q) \Leftrightarrow \neg P\land \neg Q. \end{aligned}\right\} \text{DeMorgan's laws}$$

$E_{10}: P\lor P \Leftrightarrow P$

$E_{12}: R \vee (p \wedge \neg p) \Leftrightarrow R$

$E_{13}: R \wedge (p \vee \neg p) \Leftrightarrow R$

$E_{14}: R \vee (p \vee \neg p) \Leftrightarrow T$

$E_{15}: R \wedge (p \wedge \neg p) \Leftrightarrow F$

$E_{16}: P \rightarrow Q \Leftrightarrow \neg p \vee Q$

$E_{17}: \neg(p \rightarrow Q) \Leftrightarrow p \wedge \neg Q$

$E_{18}: P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

$E_{19}: P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

$E_{20}: \neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$

$E_{21}: P (\rightleftarrows Q) \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

$E_{22}: (P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

1. Demonstrate that R is a valid inference from the premises $P \rightarrow Q$, $Q \rightarrow R$ & P

Soln $P \rightarrow Q,\ Q \rightarrow R,\ P$

$\{1\}$	(1)	$P \rightarrow Q$	Rule P
$\{2\}$	(2)	P	Rule P
$\{1, 2\}$	(3)	$P, P \rightarrow Q = Q$	Rule T (modus Ponens, I_{11})
$\{4\}$	(4)	$Q \rightarrow R$	Rule P
$\{1, 2, 4\}$	(5)	R	Rule T (modus Ponens, I_{11})

2. Show that $R \vee S$ follows logically from the premises $C \vee D, ((C \vee D) \rightarrow \neg H), (\neg H \rightarrow (A \wedge \neg B)), (A \wedge (\neg B)) \rightarrow (R \vee S))$

Soln Given $C \vee D,\ ((C \vee D) \rightarrow \neg H),\ (\neg H \rightarrow (A \wedge \neg B)),$
$((A \wedge \neg B)) \rightarrow (R \vee S))$

$\{1\}$ (1) $C \vee D \rightarrow \neg H$ Rule P

$\{2\}$ (2) $\neg H \rightarrow (A \wedge \neg B)$ Rule P

$\{1,2\}$ (3) $C \vee D \rightarrow (A \wedge \neg B)$ Rule T (I_{13}: Hypothetical syllogism)

$\{4\}$ (4) $(A \wedge \neg B) \rightarrow R \vee S$ Rule P

$\{1,2,4\}$ (5) $C \vee D \rightarrow R \vee S$ Rule T (I_{13}: Hypothetical syllogism)

$\{6\}$ (6) $C \vee D$ Rule P

$\{1,2,4,6\}$ (7) $R \vee S$ Rule T (I_{11}: Modus ponens)

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3. Show that $I_{12}: \neg Q, P \rightarrow Q \Rightarrow \neg P$

$\{1\}$ (1) $P \rightarrow Q$ Rule P

$\{1\}$ (2) $\neg Q \rightarrow \neg P$ Rule T (E_{18}) ~~modus ponens~~

$\{3\}$ (3) $\neg Q$ Rule P

$\{1,3\}$ (4) $\neg P$ Rule T (I_{11}: modus ponens)

4. Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$

$\{1\}$ (1) $P \vee Q$ Rule P

$\{1\}$ (2) $\neg P \rightarrow Q$ Rule T (E_{16})

$\{3\}$ (3) $Q \rightarrow S$ Rule P

$\{1,3\}$ (4) $\neg P \rightarrow S$ Rule T (I_{12})

$\{1,3\}$ (5) $\neg S \rightarrow P$ Rule T

$\{1,3,6\}$ (7) $\neg B \to R$ Rule T (I_{13})

$\{1,3,6\}$ (8) $S \vee R$ Rule T (E_{16})

Consistency and proof of contradiction:-

A set of formulae $H_1, H_2, H_3 \ldots H_m$ is ~~set~~ said to be consistent if their conjunction has the truth value 'T' from some assignment of Truth values to atomic variables appearing in $H_1, H_2 \ldots H_m$.

Alternatively, a set of formulae $H_1, H_2 \ldots H_m$, is inconsistent if their conjunction implies a contradiction (i.e.) $H_1 \wedge H_2 \wedge H_3 \ldots \wedge H_m \Rightarrow F \Rightarrow R \wedge \neg R$

Where R is any formula. Note that $R \wedge \neg R$ is a contradiction, and is necessary and sufficient condition for implication that $H_1 \wedge H_2 \wedge \ldots \wedge H_m$ is a contradiction.

The notation of inconsistency is used in a procedure called proof by contradiction or indirect method of proof.

Ex:- Show that the following premises are inconsistent

1) If Jack misses many classes ~~the~~ due to illness, then he fails high school.

2) If Jack fails ~~to~~ high school, then he is uneducated

3) If Jack reads lot of books, then he is not uneducated.

4) Jack misses many classes due to illness and reads a lot of books

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Baye's theorem:-

Statement:- Let $E_1, E_2, E_3, \ldots E_n$ be n mutually exclusive & exhaustive with non zero probability of a random experiment. If A is any arbitrary event of the sample space of the above experiment, then

$$P(E_i/A) = \frac{P(E_i)\cdot P(A/E_i)}{\sum_{i=1} P(E_i)\cdot P(A/E_i)}$$

Proof:- Let S be the sample space of the random experiment. Then,

$$S = E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n$$

Now, $A = S \cap A$

$$A = (E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) \cap A$$

$$A = (E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A) \cdots \cup (E_n \cap A)$$

$$P(A) = P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2) + \cdots + P(E_n)\cdot P(A/E_n)$$

(By multiplication theorem)

$$P(A) = \sum_{i=1}^{n} P(E_i)\cdot P(A/E_i) \quad \text{--- (1)}$$

Then, for any event E_i,

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}$$

$$\boxed{P(E_i/A) = \frac{P(E_i)\cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\cdot P(A/E_i)}}$$

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Soln E: Jack misses many classes due to illness.

A: He fails high school

S: Jack is uneducated.

H: Jack reads a lot of books

1. $E \rightarrow A$
2. $A \rightarrow H$
3. $S \rightarrow \neg H$
4. $E \wedge S$.

$\{1\}$	(1)	$E \rightarrow A$	Rule P
$\{2\}$	(2)	$A \rightarrow H$	Rule P
$\{1,2\}$	(3)	$E \rightarrow H$	Rule T (I_{13})
$\{4\}$	(4)	$S \rightarrow \neg H$	Rule P
$\{4\}$	(5)	$H \rightarrow \neg S$	Rule T (E_{18})
$\{1,2,4\}$	(6)	$E \rightarrow \neg S$	Rule T (I_{13})
$\{1,2,4\}$	(7)	$\neg E \vee \neg S$	Rule T (E_{16})
$\{1,2,4\}$	(8)	$\neg(E \wedge S)$	Rule T (E_8)
$\{9\}$	(9)	$E \wedge S$	Rule P
$\{1,2,4,9\}$	(10)	F_0	Rule T

$\therefore$ Inconsistent.

2) 1. "If there was a ball game, then travelling was difficult.

2. "If they arrive on time, then travelling was not difficult.

3. "They arrive on time." Therefore there was no ball game.

Show that these statements constitute a Valid

Soln

3) Show

Soln P: There was a ball game.

Q: Travelling was difficult

R: They arrive on time.

{1,2} (1) $P \to Q$

(2) $R \to \neg Q$

(3) R.

$\{1\}$	(1)	$P \to Q$	Rule P
$\{2\}$	(2)	$R \to \neg Q$	Rule P
$\{2\}$	(3)	$Q \to \neg R$	Rule T (E_{18})
$\{1,2\}$	(4)	$P \to \neg R$	Rule T (I_{13})
$\{5\}$	(5)	R	Rule P
$\{5\}$	(6)	$\neg(\neg R)$	Rule T (E_1)
$\{1,2,5\}$	(7)	$\neg P$	Rule T (I_{12})

3) Show ~~that $S \vee R$ is tautologically implied by~~

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III. Relations

A set is a collection of objects.

Ex:- A collection of songs.

* An ordered pair consists of two objects in a given fixed order. An ordered pair is denoted by (x, y).
* A binary relation is a relation between two objects in which the first member has some definite relationship with the second.
* Any set of ordered pairs defines a binary relation.
* A binary relation is a relation which can be expressed as a particular ordered pair say $(x,y) \in R$, where R is the relation, by writing xRy it may be read as "x is in relation R to y"

Properties of binary relations:-

1. Reflexive
2. Symmetric
3. Transitive
4. Irreflexive
5. Antisymmetric (or) Asymmetric.

Reflexive relation:- A binary relation 'R' in a set 'X' is reflexive if for every $x \in X$, xRx i.e $(x,x) \in R$ (or)

R is reflexive in $X \iff (x)((x \in X) \rightarrow xRx)$

Ex:- If $A = \{a,b,c\}$, then $R = \{(a,a),(b,b),(c,c)\}$ is reflexive

Symmetric relation:- A relation R in a set X is symmetric $\forall$ x and y in X, whenever xRy, then yRx (or)

Ex:- If

Transitive

if $\forall$

xRz.

Ex:- I

Irreflexive

if $\forall$

Antisymm

antisym

yRx

Equivalenc

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$R = \{(1,1$

Ex:- If $A = \{a, b, c\}$, then $R = \{(a,a), (a,b), (a,c), (b,c), (c,b), (c,a), (b,a)\}$

Transitive relation:- A relation R in a set X is transitive if $\forall$ 'x', 'y', 'z' in X, whenever xRy & yRz then xRz. (or)

R is transitive in X iff;

$X \Leftrightarrow (x)(y)(z)(x \in X \wedge y \in X \wedge z \in X \wedge xRy \wedge yRz \rightarrow xRz)$

Ex:- If $A = \{a, b, c\}$, then $R = \{(a,b), (b,c), (a,c)\}$ is transitive.

Irreflexive relation:- The relation R in set X is irreflexive if $\forall x \in X$, $(x,x) \notin R$

Antisymmetric relation:- The relation R in set X is antisymmetric if $\forall$ x & y in X whenever xRy & yRx then $x = y$. (or)

R is assymetric in X iff;

$X \Leftrightarrow (x)(y)(x \in X \wedge y \in X \wedge xRy \wedge yRx \rightarrow x = y)$

Equivalence relation:- A relation R in a set X is called equivalence relation if it is reflexive, transitive & symmetric.

1. Write matrix of R and sketch its graph for the following example. Let $X = \{1, 2, 3, 4\}$ &

$R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$

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For a relation to be equivalent, it should be reflexive, ~~trans~~ symmetric & transitive.

1) Reflexive:-

$(x,x) \in R$

$\{(1,1),(2,2),(3,3),(4,4)\} \in R$

$\Rightarrow$ It is reflexive.

2) Symmetric:-

xRy ; yRx

$\{(1,4),(4,1),(2,3),(3,2)\} \in R$

$\Rightarrow$ It is symmetric.

3) Transitive:-

xRy ; $yRz \Rightarrow xRz$

$\{(1,1),(1,4),(2,2),(2,3)\}$

$\therefore$ It is transitive.

$\therefore$ ~~The~~ R is equivalence relation.

Matrix:-

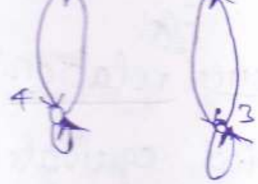
$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

2. Let $X = \{1,2,3,4\}$ & $R = \{(x,y) \mid x > y\}$. Draw the graph of R & also give its matrix.

Soln Matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



Transitive closure:- Transitive closure of a binary relation R denoted by R^+ is the smallest transitive relation that consists of R. It can be defined as set of unions of $R^1, R^2, R^3 \ldots R^n$

(i.e.;) $R^+ = R^1 \cup R^2 \cup R^3 \cup \ldots \cup R^n$.

Here, $R^2, R^3, R^4 \ldots R^n$ are the transitive extensions of $R^1, R^2, R^3 \ldots R^{n-1}$ respectively.

The transitive extensions are binary relations on set A such that

1) R^{n+1} contains R^n (i.e); $R^n \subseteq R^{n+1}$
2) R^{n+1}, the transitive extension of R^n is transitive.
3) $R^{n+1} \subseteq R^{n+2}$ iff there exists R^{n+2} such that $R^n \subseteq R^{n+2}$ and R^{n+2} is transitive.

The transitive closure relation, R^+ can be represented in a matrix denoted as MR^+

Ex:- Consider a set $A = \{a, b, c, d\}$ and a relation, $R = \{(a,c), (b,c), (c,d), (d,a)\}$. Find the transitive closure of R and construct the matrix and draw the graph.

Sol: Transitive closure of R is denoted by $R^+ = R^1_{\otimes} \cup R^2_{\otimes} \cup \ldots \cup R^n$

$R = \{(a,c), (b,c), (c,d), (d,a)\}$

$R^2 = R \circ R = \{(a,c), (b,c), (c,d), (d,a)\} \circ \{(a,c), (b,c), (c,d), (d,a)\}$

$= \{(a,d), (b,d), (c,a), (d,c)\}$

$R^3 = R^2 \circ R = \{(a,d), (b,d), (c,a), (d,c)\} \circ \{(a,c), (b,c), (c,d), (d,a)\}$

$= (a,a), (b,a), (c,c), (d,d)\}$

$R^4 = R^3 \circ R = \{(a,a), (b,a), (c,c), (d,d)\} \circ \{(a,c), (b,c), (c,d), (d,a)\}$

$= \{(a,c), (b,c), (c,d), (d,a)\}$

$R^3 = R^4$

$R^+ = R_\phi^1 \cup R^2 \cup R^3$

$= \{(a,c), (b,c), (c,d), (d,a)\} \cup \{(a,d), (b,d), (c,a), (d,c)\}$

$\cup \{(a,a), (b,a), (c,c), (d,d)\}$

$R^+ = \{(a,c), (b,c), (c,d), (d,a), (a,d), (b,d), (c,a), (d,c),$
$(a,a), (b,a), (c,c), (d,d)\}$.

$$M_{R^+} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

2. Consider a set $A = \{1,2,3,4\}$ and a relation,

$R = \{(1,2), (2,3), (3,3)\}$

$R^2 = R \circ R = \{(1,2), (2,3), (3,3)\} \circ \{(1,2), (2,3), (3,3)\}$

$= \{(1,3), (2,3), (3,3)\}$

$R^3 = R^2 \circ R = \{(1,3), (2,3), (3,3)\} \circ \{(1,2), (2,3), (3,3)\}$.

$= \{(1,3), (2,3), (3,3)\}$

$R^2 = R^3$.

$\Rightarrow R^+ = R \cup R^2 = \{(1,2), (2,3), (3,3)\} \cup \{(1,3), (2,3), (3,3)\}$

$= \{(1,2), (2,3), (3,3), (1,3)\}$

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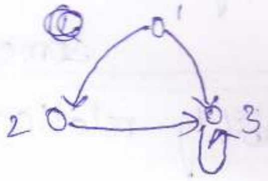
$$MR^{+} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & & \\ 3 & & & \end{array}\quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$



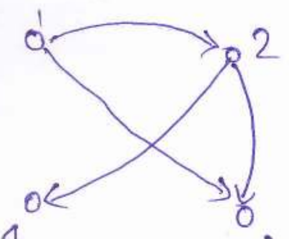
3. Let $R = \{(1,2),(2,3),(2,4)\}$ on the set $\{1,2,3,4\}$. Obtain the transitive closure of R.

$$R^2 = R \circ R = \{(1,2),(2,3),(2,4)\} \circ \{(1,2),(2,3),(2,4)\}$$
$$= \{(1,3)\} \cancel{(2,4)\}}.$$

$$R^3 = R^2 \circ R = \cancel{\{(1,3),(2,3)\}}\ \{(1,3)\} \circ \{(1,2),(2,3),(2,4)\}$$
$$= \emptyset$$

$$R^{+} = R \cup R^2$$
$$= \{(1,2),(2,3),(2,4)\} \cup \{(1,3)\}$$
$$= \{(1,2),(1,3),(2,3),(2,4)\}$$

$$MR^{+} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$



Compatibility and partial ordering relations:-

A relation R in X is said to be a compatibility relation if it is reflexive and symmetric.

All equivalence relations are compatibility relations

* Let $X = \{ball, bed, dog, let, egg\}$ and let the relation 2)

R be given by ~~$R=\{(x,y)/(x,y) \in R \wedge xRy\}$~~

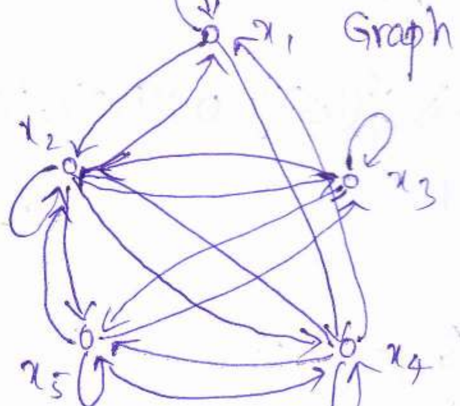
$R = \{(x,y) / x,y \in X \wedge xRy$ if x & y contains some common letters$\}$

→ Then R is a compatibility relation & x, y are called compatible if xRy.

→ compatibility relation is denoted by "≈".

<u>Note</u>:- ball ≈ bed ; bed ≈ egg

but, ball ≉ egg. Hence, it is not transitive.



$x_1(x_1, x_2, x_4)$ 3)

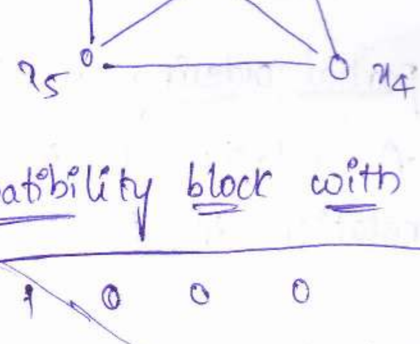
$x_2(x_2, x_1, x_3, x_4, x_5)$

$x_3(x_3, x_2, x_5)$

$x_4(x_4, x_1, x_2, x_5)$

$x_5(x_5, x_2, x_3, x_4)$

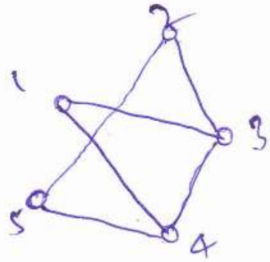
<u>Relation</u>



30/7/13 <u>Maximal</u> <u>compatibility</u> <u>block</u> <u>with</u> <u>relation</u> <u>matrix</u>:-

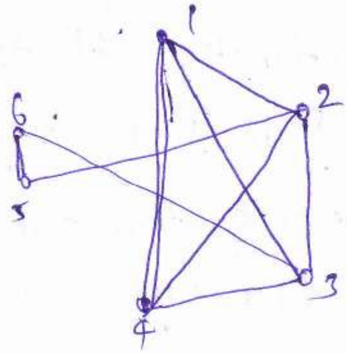
$$\begin{array}{c|cccc} x_2 & 1 & 0 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ x_4 & 1 & 1 & 0 & 0 \end{array}$$

2)



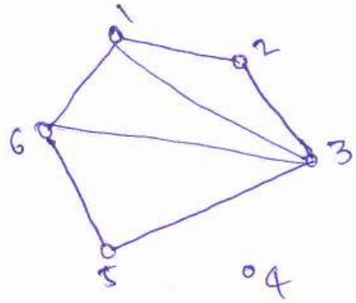
	1	2	3	4
2	0	0	0	0
3	1	1	0	0
4	1	0	1	0
5	0	1	0	1

3)



	1	2	3	4	5
2	1	0	0	0	0
3	1	1	0	0	0
4	1	1	1	0	0
5	0	1	0	0	0
6	0	0	1	0	1

4)



2	1	0	0	0	0
3	1	1	0	0	0
4	0	0	0	0	0
5	0	0	1	0	0
6	1	0	0	0	1
	1	2	3	4	5

Functions:- Let X & Y be any two sets. A relation 'f' from X to Y is called a function if $\forall x \in X$ there is a unique $y \in Y$ such that $(x,y) \in f$.

Terms such as mapping, transformation, correspondence & operation $f: x \to y$. are used as synonyms for functions.

The notations $f: X \to Y$ or $X \xrightarrow{f} Y$ are used as expressions for f as a function from X to Y.

Pictorially a function is shown as

For a function $f: X \to Y$, if $(x,y) \in f$, then x is called as argument & corresponding y is called image of x under f.

$(x,y) \in f$ can be expressed as $y = f(x)$,

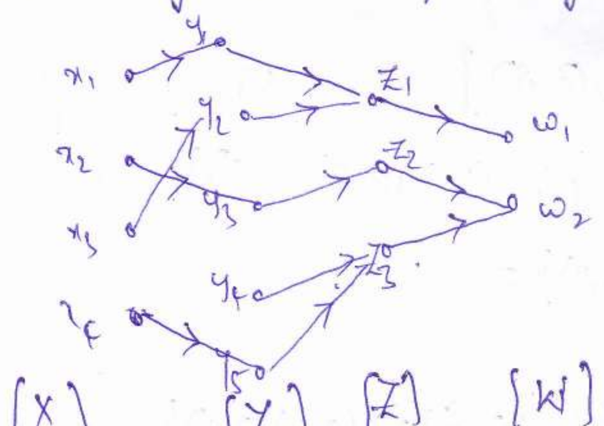
hence the range of f is defined as

Composition of functions:-

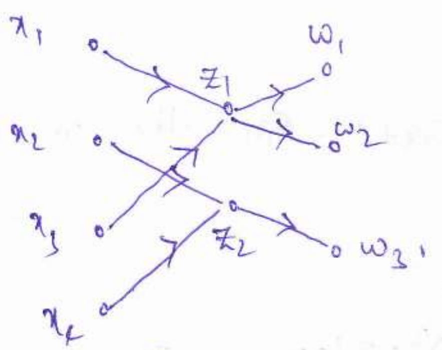
Consider 3 functions $f: X \to Y$, $g: Y \to Z$ and $h: Z \to W$. The composite functions $(g\circ f): x \to z$ & $(h\circ g): y \to w$ can be formed. Assuming, $y = f(x)$, $z = g(y)$, $w = h(z)$ & we have $(x,y) \in f$, $(y,z) \in g$, $(z,w) \in h$ & $(x,z) \in g\circ f$ & $(y,w) \in h\circ g$.

Continuing the same arguments $(x,w) = (h\circ g)\circ f$. Similarly ~~for any~~ $(x,w) \in h\circ(g\circ f)$

31/7/13 For any x corresponding to w, $(h\circ g)\circ f = h\circ(g\circ f)$



$x \xrightarrow{g\circ f} z \xrightarrow{h} w$.



Ex:- 1. Let $X = \{1,2,3\}$ & f, g, h & s be functions from X to X given by $f = \{(1,2),(2,3),(3,1)\}$

$g = \{(1,2),(2,1),(3,3)\}$

$h = \{(1,1),(2,2),(3,1)\}$

$s = \{(1,1),(2,2),(3,3)\}$.

i) $fog = \{(1,2),(2,1),(3,3)\} \circ \{(1,2),(2,2),(3,1)\}$
$= \{(1,3),(2,2),(3,1)\}$

ii) $gof = \{(1,2),(2,3),(3,1)\} \circ \{(1,2),(2,1),(3,3)\}$
$= \{(1,1),(2,3),(3,2)\}$
$\therefore fog \neq gof$

iii) $sog = \{(1,2),(2,1),(3,3)\} \circ \{(1,1),(2,2),(3,3)\}$
$= \{(1,2),(2,1),(3,3)\}$

(iv) $gos = \{(1,1),(2,2),(3,3)\} \circ \{(1,2),(2,1),(3,3)\}$
$= \{(1,2),(2,1),(3,3)\} \Rightarrow sog = gos = g$

(v) $sos = \{(1,1),(2,2),(3,3)\} \circ \{(1,1),(2,2),(3,3)\}$
$= \{(1,1),(2,2),(3,3)\}$
$sos = s$

(vi) $fos = \{(1,1),(2,2),(3,3)\} \circ \{(1,2),(2,3),(3,1)\}$
$= \{(1,2),(2,3),(3,1)\}$
$fos = f$

2. Let $f(x) = 2x-3$, $g(x) = x^2+3x+5$. Find the formula for gof & fog.

$gof = g(f(x)) = g(2x-3)$
$= (2x-3)^2 + 3(2x-3) + 5$
$= 4x^2 + 9 - 12x + 6x - 9 + 5 = 4x^2 - 6x + 5$

$fog = f(g(x)) = f(x^2+3x+5)$
$= 2(x^2+3x+5) - 3$
$= 2x^2 + 6x + 10 - 3$
$= 2x^2 + 6x + 7$

3. Let f
& (h

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4. Consid
P.T

3. Let $f(x) = 2x$, $g(x) = x^2$, $h(x) = 2x - 3$. Find $ho(gof)$ & $(hog)of$.

$$ho(gof) = ho(g(f(x))) = ho(g(2x))$$
$$= ho(4x^2)$$
$$= h(4x^2)$$
$$= 2(4x^2) - 3$$
$$= 8x^2 - 3.$$

$$(hog)of = (h(g(x))) of$$
$$= (h(x^2))of$$
$$= (2(x^2) - 3) of.$$
$$= 2(2x)^2 - 3$$
$$= 8x^2 - 3$$

$ho(gof) = (hog)of$

$$(hog) of = (hog) f(x)$$
$$= hog(2x)$$
$$= h(g(2x))$$
$$= h(2x)^2$$
$$= 2(2x)^2 - 3$$
$$= 8x^2 - 3$$

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$$(hog)of = ho(gof) \,//$$

4. Consider a function, $f: A \to B$, $g: B \to C$, $h: C \to D$.

P.T $(hog)of = ho(gof)$

$f(A) = B$ Let $x \in A$

$g(B) = C$ $y \in B$

$h(C) = D$ $z \in C$

$k \in D$.

$\Rightarrow f(x) = y$

$g(y) = z$

$h(z) = k$

$$(hog)of = hog(f(x)) = h(g(y)) = h(z) = k.$$
$$ho(gof) = ho(g(f(x))) = ho(g(y)) = h(z) = k.$$

$\therefore (hog)of = ho(gof)$.   LHS = RHS

Inverse functions:-

The converse of a relation R from $X \to Y$ was defined to be a Relation $\tilde{R}$ from $Y \to X$ such that $(y,x) \in \tilde{R} \Longleftrightarrow (x,y) \in R$ i.e the ordered pairs of $\tilde{R}$ are obtained from those of R by simply interchanging the numbers. The situation is not quiet same in case of functions.

Let $\tilde{F}$ denote the converse of F where F is considered as a relation from X to Y. Naturally, $\tilde{F}$ may not be a function because

1) The domain of $\tilde{F}$ may not be Y but only a subset of Y.

2) $\tilde{F}$ maynot be a function from D_f to X because it may not satisfy the uniqueness condition.

Note:- For a given function $f: X \to Y$, $\tilde{f}$ is a function only if f is one-one function.

Ex:- Let $X = \{1,2,3\}$ & $Y = \{p,q,r\}$ & $f: X \to Y$ be given by $\{(1,p),(2,q),(3,q)\}$

& $\tilde{f} = \{(p,1),(q,2),(q,3)\}$ & $\tilde{f}$ is not a function.

* $\tilde{f}$ is written as f^{-1} so that $f^{-1}: Y \to X$.

f^{-1} is called the inverse of the function if f^{-1} exists, f is called invertible.

1. Find the inverse of the following function.

$$f(x) = \frac{10}{\sqrt[5]{7-3x}}$$

Soln $y = f(x) \Rightarrow f^{-1}(y) = x$

2. Fin

3. Let

$$y^5(7-3x) = 10^5$$

$$7y^5 - 3xy^5 = 10^5$$

$$\frac{7y^5 - 10^5}{3y^5} = x$$

$$x = f^{-1}(y)$$

$$\Rightarrow f^{-1}(y) = \frac{7y^5 - 10^5}{3y^5}$$

$$f^{-1}(x) = \frac{7x^5 - 10^5}{3x^5}$$

2. Find the inverse of the function $4e^{(6x+2)} = f(x)$

$$y = f(x)$$

$$4e^{(6x+2)} = y$$

$$\cancel{4\log}(6x+2) = \log y$$

$$4e^{6x} \cdot e^2 = y$$

$$\log 4 + \log_e 6x + \log e^2 = \log y$$

$$0.602(6x \log e) + (2 \log e) = \log y$$

$$2 + (6x) + (0.602) = \log y$$

$$6x = \frac{\log y - 0.602 - 2}{6}$$

$$f^{-1}(y) = \frac{\log y - 2.602}{6}$$

$$f^{-1}(x) = \frac{\log x - 2.602}{6}$$

3. Let $f(x) = \frac{x+1}{x}$. Find f^{-1}.

$$y = \frac{x+1}{x}$$

$$xy = x + 1$$

$$xy - x = 1$$

$$x(y-1) = 1$$

Rule CP :- We have a third inference rule known as rule CP or rule of conditional proof.

rule cp: if we can derive S from R & a set of premises, then we can derive $R \to S$ from the set of premises alone.

Rule cp is not new because it follows from the equivalence E_{19} which states that $(P \wedge R) \to S \Leftrightarrow P \to (R \to S)$.

Let P denote the conjunction of set of premises and R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from P & R, then $R \to S$ can be derived from the premises P alone. The rule cp is also called as Deduction theorem & is generally used if the conclusion is of the form $R \to S$. In such cases, R is taken as an additional premise & S is derived from given premises & R.

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1 Show that $R \to S$ can be derived from premises $P \to (Q \to S)$, $\neg R \vee P$, Q

Soln

Instead of deriving $R \to S$, we should include R as an additional premise & show S first.

$\{1\}$	(1)	$\neg R \vee P$	Rule P
$\{2\}$	(2)	R	Rule P (Assumed premise)
$\{1,2\}$	(3)	P	Rule T (I_{10})
[illegible]	(4)	$P \to (Q \to S)$	Rule P

$\{1,2,4\}$	(5)	$Q \to S$	Rule T (I_{11})
$\{6\}$	(6)	Q	Rule P
$\{1,2,4,6\}$	(7)	S	Rule T (I_{11})
$\{1,4,6\}$	(8)	$R \to S$	Rule CP

Partial ordering relations:- A relation R on a set P is called a partial order relation or a partial ordering in P iff R is reflexive, antisymmetric & transitive.

* We denote partial ordering by the symbol "$\leq$".
* A set 'P' on which partial ordering "$\leq$" is defined is called a partially ordered set or Poset. It is denoted by $(P, \leq)$.

Ex:- Let $S = \{x, y, z\}$ & consider the power set $P(S)$ with relation R given by set inclusion. Is R a partial order relation?

Sol: $P(S) = \{\phi, \{x\}, \{y\}, \{z\}, \{x,x\}, \{x,y\}, \{x,z\}, \{y,y\}, \{y,z\}, \{z,z\}, \{x,y,z\}\}$

Define a relation R, by ARB iff $A \subseteq B$ $\forall$ $A,B \in P(S)$
~~$\{A,B\} \in P(S)$~~

1. We have $A \subseteq A$ for any $A \in P(S)$. Hence, 'R' is reflexive on $P(S)$
2. For any $A, B \in P(S)$, $A \nsubseteq B$, $B \nsubseteq A$, hence 'R' is not antisymmetric.
3. For any $A, B, C \in P(S)$, $A \subseteq B$, $B \subseteq C$ $\Rightarrow A \subseteq C$.
R is transitive on $P(S)$.

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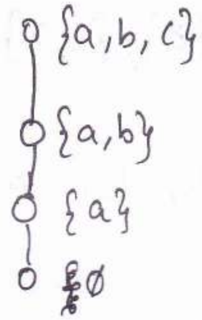
Hasse diagram or Poset diagram :-

A Partial ordering $\leq$ on a set 'P' can be represented by means of a diagram known as Hasse diagram (or) A partial ordered set diagram.

Following are the points to be remembered to work with Hasse diagram. Each element is represented by a small circle or a dot.

The circle for $x \in B$ is drawn below the circle for $y \in B$ if $x < y$, & a line is drawn b/w x & y if x covers y. If $x < y$, but y doesnot cover x, then x & y are not connected directly by a single line. They are connected through one or more elements of P.

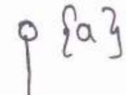
Ex:- 1. Consider $P = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$, $(P, \subseteq)$ is a Poset where '$\subseteq$' is a set inclusion, then Hasse diagram for $(P, \subseteq)$ is shown below.



2. Let A be the finite set & P(A) be the power set of A, '$\subseteq$' be the inclusion relation, draw the Hasse diagram of $(P(A), \subseteq)$ for i) $A = \{a\}$
ii) $A = \{a,b\}$.
iii) $A = \{a,b,c\}$

i) $A = \{a\}$

$P(A) = \{\phi, \{a\}\}$



ii) A=

iii) A=

3. Let

a R

A

Sol

P(A)

→ Here,

→ we

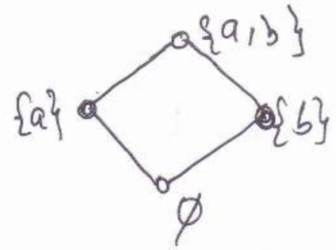
(a, b)

trans

→ Furt

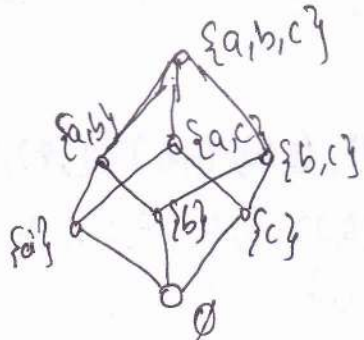
ii) $A = \{a, b\}$

$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$



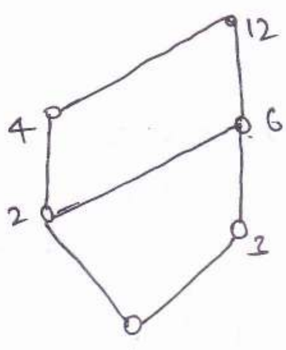
iii) $A = \{a, b, c\}$

$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$



3. Let $A = \{1, 2, 3, 4, 6, 12\}$, define the relation R, by aRb iff a divides b. Prove that R is a PO on A, draw the Hasse diagram for this relation.

Soln
$P(A) = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12), (3,6), (3,3), (3,12), (4,4), (4,12), (6,6), (6,12), (12,12)\}$



→ Here, $(a,a) \in R$ $\forall$ $a \in A$. Therefore R is reflexive.

→ We ~~check~~ check that elements of R such that $(a,b) \in R$ & $(b,c) \in R$, then $(a,c) \in R$. Therefore R is transitive.

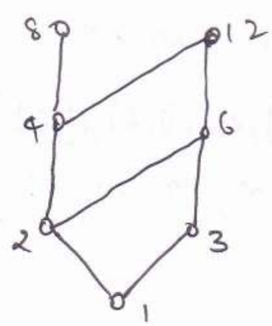
→ Furthur, $\forall$ $(a,b) \in A$, if a divides b & b divides a, Matrix:-

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 12 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 6 & 12 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

4. Let $A = \{1,2,3,4,6,8,12\}$., defined on the relation 'R' by aRb iff a divides b. Draw a hasse diagram & relation, matrix R.

$P(A) = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(12,12)\}$

Relation:



Matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \\ 12 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 6 & 8 & 12 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

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5. Draw
diviso

Soln

P(A) = {(1,
(2
(3,

6. Draw

A =

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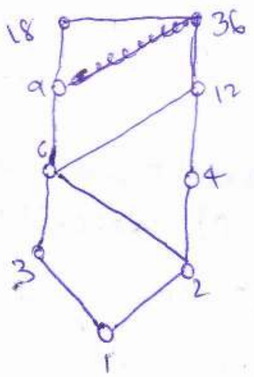
5. Draw the Hasse diagram representing the positive divisors of 36.

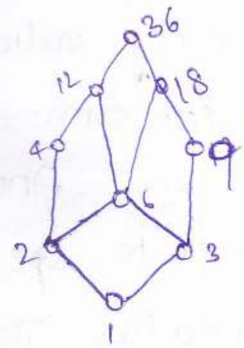
Soln

$A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$P(A) = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,9), (1,12), (1,18), (1,36), (2,2),$
$(2,4), (2,6), (2,12), (2,18), (2,36), (3,3), (3,6), (3,9), (3,12), (3,18),$
$(3,36), (4,4), (4,12), (4,36), (6,6), (6,12), (6,36), (9,9), (9,18),$
$(9,36), (12,12), (12,36), (18,18), (18,36), (36,36)\}.$



(or)

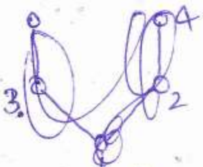


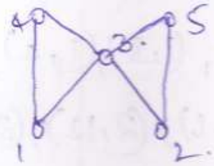
$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 6 & 9 & 12 & 18 & 36 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 9 \\ 12 \\ 18 \\ 36 \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

6. Draw the Hasse diagram of the relation R on A = $\{1, 2, 3, 4, 5\}$ whose relation matrix is given below

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} &
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

$P(A) = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4),$
$(3,5), (4,4), (5,5)\}$.





Recursive functions:- A function f of an argument 'x' is generally specified by indicating the value of $f(x)$ for every value of x in the domain of f in an explicit form.

Another way of describing a function $f(x)$ is to specify the value of $f(x)$ for some particular value of x and then to indicate how the remaining values of $f(x)$ can be got with these specified values. The latter method of describing 'f' is called "recursion".

Eg ⇒ Consider a factorial function $f(x) = n!$ which is defined for every natural number, 'n'. The explicit method of describing this function is

$f(n) = n f(n-1)$

$\Rightarrow f(1) = 1 \cdot f(1-1)$
$= 1 \cdot f(0) = 1 \cdot 1 = 1$

$f(2) = 2 f(2-1) = 2 \cdot f(1) = 2 \times 1 = 2$

$f(3) = 3 \cdot f(3-1) = 3 \times f(2) = 3 \times 2 = 6$.

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1. The fibonacci function $f(n) = f_n$ is defined recursively by $F_0 = 0$, $F_1 = 1$ & $F_n = f_{n-1} + f_{n-2}$ $(n \geq 2)$

Evaluate F_2 to F_{10}.

Lattices and properties of lattices:- A lattice is partially ordered set given by $(L, \leq)$ in which every pair of elements $(a,b) \in L$ has a greatest lower bound & least upper bound.

The greatest lower bound of a subset $\{a,b\} \subseteq L$ will be denoted by $a * b$ & the least upperbound by $a \oplus b$ (Ex-OR). We can call the $GLB\{a,b\} = a * b$ as the "meet" or "product" & $LUB\{a,b\} = a \oplus b$ as "join" or "sum" of a & b.

symbols such as '$\wedge$' & '$\vee$' (or) & + (or) are also used to denote meet or a join of two elements respectively.

Properties of lattices:-

- The properties of two binary operations of meet & join are denoted by $*$ & $\oplus$ on a lattice $(L, \leq)$ for any $a,b,c \in L$. Hence we have

	meet	Join	
L-1	$a * a = a$	(L-1)' $a \oplus a = a$	(Idempotent law)
L-2	$a * b = b * a$	(L-2)' $a \oplus b = b \oplus a$	(Commutative law)
L-3	$(a*b)*c = a*(b*c)$	(L-3)' $(a \oplus b) \oplus c = a \oplus (b \oplus c)$	(Associative)
L-4	$a * (a \oplus b) = a$	(L-4)' $a \oplus (a * b) = a$	(Absorption)

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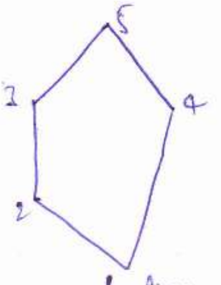
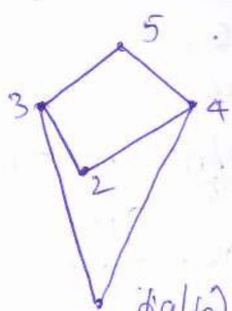
(*)



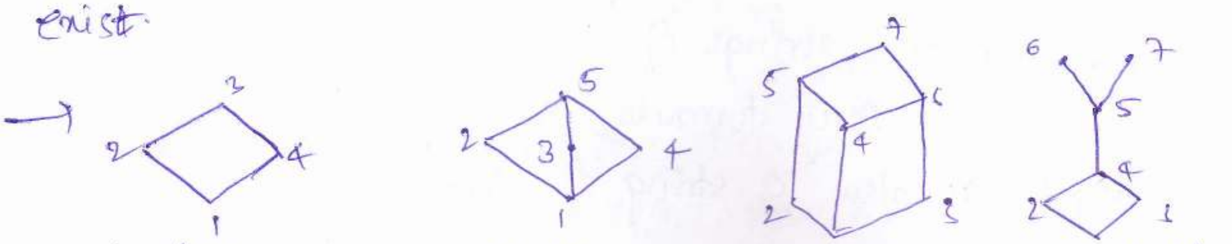
fig(a)



fig(b)

Consider the posets whose hasse diagrams are shown in the following figures. we observe that the Poset represented in fig a is a lattice because every pair

Greatest upper bound. The poset represented in fig b is not a lattice. Observe that, GLB of {3,4} doesn't exist.



Lattice Lattice Lattice Not lattice (2 GUB)

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Atomatic Theorem proving:-

In this theorem of proving, we reformulate the rules of inference theory for statement calculus and describe a procedure of derivation.

Rule 'p' permits the introduction of a premise at any point in the derivation, but doesnot suggest the step at which the premise is to be introduced.

Rule 'T' allows to introduce any formula which follows from the previous steps which is not particular.

Similarly Rule 'cp' doesnot tell anything about the stages at which an assumed premise is to be introduced.

The formulation described in this proving consists of 10 rules, axiom schema, sequent, variables, connectives.

Variables:- The capital letters A, B, C, P, Q, R are used as statement variables, statement formulae.

Connectives:- The connectives $\neg, \wedge, \vee, \rightarrow, \rightleftarrows$ appear in the formulae with the order of precedence.

String of formulae:- Any formula is string of formulae. If α & β are the strings of formulae, then α, β & β, α are also strings of formulae.

Any formula obtained by using step 1 & step 2 is also a string of formulae.

Sequent:- If α & β are the strings of formulae, then $\alpha \xrightarrow{S} \beta$ is called a sequent.

Axiom Schema:- If α & β are the strings of formulae such that every formula in both α & β is a variable, then the sequent $\alpha \xrightarrow{S} \beta$ is an axiom iff α & β have atleast one variable in common.

Ex:- $A, B, C \xrightarrow{S} P, B, R$

Rules:- The following rules are used to combine the formulae within strings by introducing connectives. It is derived into two categories.

1. Anticedent rules
2. Consequent rules

1. Anticedent rules:-

~~Rule ¬⇒ If~~

Rule $\neg\Rightarrow$: If $\alpha, \beta \xRightarrow{S} X, \gamma$, then $\alpha, \neg X, \beta \xRightarrow{S} \gamma$.

Rule $\wedge\Rightarrow$: If $X, Y, \alpha, \beta \xRightarrow{S} \gamma$, then $\alpha, X \wedge Y, \beta \xRightarrow{S} \gamma$

Rule $\vee\Rightarrow$: If $X, \alpha, \beta \xRightarrow{S} \gamma$ and $Y, \alpha, \beta \xRightarrow{S} \gamma$, then $\alpha, X \vee Y, \beta \xRightarrow{S} \gamma$

Rule $\rightarrow\Rightarrow$: If $Y, \alpha, \beta \xRightarrow{S} \gamma$ and $\alpha, \beta \xRightarrow{S} X, \gamma$, then $\alpha, X \rightarrow Y, \beta \xRightarrow{S} \gamma$

Rule $\leftrightarrows\Rightarrow$: If $X, Y, \alpha, \beta \xRightarrow{S} \gamma$ and $\alpha, \beta \xRightarrow{S} X, Y, \gamma$, then $\alpha, X \rightleftarrows Y, \beta \Rightarrow \gamma$

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2. Consequent rules:-

Rule $\Rightarrow\neg$: If $X, \alpha \overset{s}{\Rightarrow} \beta, \gamma$ then $\alpha \overset{s}{\Rightarrow} \beta, \neg X, \gamma$

Rule $\Rightarrow\wedge$: If $\alpha \overset{s}{\Rightarrow} X, \beta, \gamma$ and $\alpha \overset{s}{\Rightarrow} Y, \beta, \gamma$; then $\alpha \overset{s}{\Rightarrow} \beta, X\wedge Y, \gamma$

Rule $\Rightarrow\vee$: If $\alpha \overset{s}{\Rightarrow} X, Y, \beta, \gamma$; then $\alpha \overset{s}{\Rightarrow} \beta, X\vee Y, \gamma$.

Rule $\Rightarrow\rightarrow$: If $X, \alpha \overset{s}{\Rightarrow} Y, \beta, \gamma$; then $\alpha \overset{s}{\Rightarrow} \beta, X\rightarrow Y, \gamma$.

Rule $\Rightarrow\rightleftarrows$: If $X, \alpha \overset{s}{\Rightarrow} Y, \beta, \gamma$ and $Y, \alpha \overset{s}{\Rightarrow} X, \beta, \gamma$; then $\alpha \overset{s}{\Rightarrow} \beta, X \rightleftarrows Y, \gamma$

1. Show that $P\vee Q$ follows from P i.e $P \rightarrow P\vee Q$.

soln

(1) $\overset{s}{\Rightarrow} P \rightarrow (P\vee Q)$

(1) if (2) $P \overset{s}{\Rightarrow} P\vee Q$ $(\Rightarrow \rightarrow)$

(2) if (3) $P \overset{s}{\Rightarrow} P, Q$ $(\Rightarrow \vee)$

(a) $P \Rightarrow P, Q$ (Axiom).

(b) $P \overset{s}{\Rightarrow} P\vee Q$ $(\Rightarrow \vee)$

(c) $\overset{s}{\Rightarrow} P \rightarrow (P\vee Q)$ $(\Rightarrow \rightarrow)$

2. Show that $\overset{s}{\Rightarrow} (\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$

$\overset{s}{\Rightarrow} (\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$

(1) $\overset{s}{\Rightarrow} (\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$

(1) if (2) $\neg Q \wedge (P \rightarrow Q) \overset{s}{\Rightarrow} \neg P$ $(\Rightarrow \rightarrow)$

(2) if (3) $\neg Q, P \rightarrow Q \overset{s}{\Rightarrow} \neg P$ $(\wedge \Rightarrow)$

(3) if (4) $P \rightarrow Q \overset{s}{\Rightarrow} \neg P, Q$ $(\neg \Rightarrow)$

(4) if (5) $Q \overset{s}{\Rightarrow} \neg P, Q$ and (6) $\overset{s}{\Rightarrow} P, \neg P, Q$. $(\rightarrow \Rightarrow)$

(5) if (7) $P, Q \overset{s}{\Rightarrow} Q$ $(\Rightarrow \neg)$ (Axiom)

(6) if (8) $P \overset{s}{\Rightarrow} P, Q$ $(\Rightarrow \neg)$ (Axiom)

(a) $P \overset{s}{\Rightarrow} P, Q$ $(\Rightarrow \neg)$

(b) $P, Q \overset{s}{\Rightarrow} Q$ $(\Rightarrow \neg)$

(d) $Q \overset{s}{\Rightarrow} \neg P, Q$

(c) $\overset{s}{\Rightarrow} P, \neg P, Q$ $(\Rightarrow \Rightarrow)$

(e) $P \rightarrow Q \overset{s}{\Rightarrow} \neg P, Q$ $(\neg \Rightarrow)$

(f) [illegible] $(\wedge \Rightarrow)$

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IV Algebraic Structures

Algebraic system:- A system consisting of a set and one or more n-ary operations on the set will be called an algebraic system. (or) simply an algebra.

Algebraic system is denoted by $(S, f_1, f_2, f_3, \ldots)$ where 'S' is non-empty set & $f_1, f_2, f_3 \ldots$ are operations on 'S'.

Since the operations and relations on the set 'S' define a structure on the elements, hence algebraic system is called algebraic structure.

Ex:- Let I be the set of integers, consider the algebraic system $(I, +, \times)$ where $+, \times$ are the addition & multiplication operations on I.

The properties associated with the operations of addition & multiplication are labelled by the letters 'A' and 'M' respectively.

* (A-1) for any $a, b, c \in I$;

$$(a+b)+c = a+(b+c) \text{ (Associativity)}$$

* (A-2) for any $a, b \in I$;

$$a+b = b+a \text{ (Commutativity)}$$

* (A-3) there exist an element $0 \in I$ such that for any $a \in I$;

$$a+0 = 0+a = a \text{ [identity element]}$$

Here $0 \in I$ is the identity element w.r.t addition.

* (A-4) for each $a \in I$; there exist an element in I denoted by '$-a$' and called the negative of a such that $a+(-a) = 0$ [inverse element]

* (M-1)

* (M-2)

* (M-3)
for

* (M-4)

* (D)

* (C)

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Theorem:
Ex:- If
$(a*b$
algeb

Soln
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* (M-1) for any $a, b, c \in I$;

$$(a \times b) \times c = a \times (b \times c) \quad \text{(Associativity)}$$

* (M-2) for any $a, b \in I$;

$$a \times b = b \times a \quad \text{(commutativity)}$$

* (M-3). there exists an element $1 \in I$ such that for any $a \in I$;

$$a \times 1 = 1 \times a = a \quad \text{[identity element)}$$

~~* M-4~~ Here $1 \in I$ is the identity element w.r.t multiplication.

~~*(D-3)~~

* (D) for any $a, b, c \in I$;

$$a \times (b + c) = (a \times b) + (a \times c) \quad \text{[Distributivity]}$$

* (C) for any $a, b, c \in I$; $a \neq 0$

$$a \times b = a \times c \Rightarrow b = c \quad \text{[cancellation property)}$$

<u>Semi groups</u> & <u>monoids</u>:- Let 'S' be non empty set and 'O' be a binary operation on 'S'. The algebraic system (S, O) is called a semi group if the operation <u>O is associative</u>. In other words, (S, O) is a semi group if for any $x, y, z \in S$, then $(x \, o \, y) \, o \, z = x \, o \, (y \, o \, z)$.

A semi group (M, o) with an identity element w.r.t operation 'O' is called monoid.

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Theorem:

~~Ex~~:- If a, b are the elements of M and $a \times b = b \times a$, then $(a * b) * (a * b) = (a * a) * (b * b)$ where $M, *$ is an algebraic system.

Soln Given $M, *$ is an algebraic system, & $a, b \in M$ & $(a * b) = (b * a)$

Soln $(a*b)*(a*b) = a*(b$

$= (a*b)*(b*a)$

$= a*(b*b*a)$

$= a*(a*b*b)$

$= (a*a)*(b*b)$

Ex: verify which of the following are semi groups.

i) $(N,+)$. ii) $(Q,-)$ iii) $(R,*)$ iv) $(N,*)$

i) $N = 1,2,3,4 \ldots$

$A = 1,\ B = 2,\ C = 3$

$(A+B)+C = A+(B+C)$

$(1+2)+3 = 1+(2+3)$

$6 = 6$ //

ii) $(Q,-)$

$Q = \{\frac{1}{2}, \frac{1}{3}, \frac{4}{5} \ldots\}$

$A = 1/2,\ B = 1/3,\ C = 4/5$

$(A-B)-C = A-(B-C)$

$\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{4}{5} = \frac{1}{2}-\left(\frac{1}{3}-\frac{4}{5}\right)$

$\frac{1}{6}-\frac{4}{5} = \frac{1}{2}-\left(\frac{5-12}{15}\right)$

$\frac{5-24}{30} \neq \frac{15+2(7)}{30}$

iii) $(R,*)$

$R = \{\ldots -3,-2,-1,0,1,2,3 \ldots$

$A = -3,\ B = -2,\ C = -1$

$A*(B*C) = (A*B)*C$

$-3(-2\times-1) = (-3\times-2)\times-1$

$-3(2) = 6(-1)$

$-6 = -6$ /

(iv) $(N, *)$

$N = 1, 2, 3, \ldots$

$A = 1,\ B = 2,\ C = 3.$

$A * (B * C) = (A * B) * C$

$1 * (2 * 3) = (1 * 2) * 3$

$6 = 6$

$\therefore$ i, iii, iv are semi groups.

<u>Groups</u>:- A group $(G, *)$ is an algebraic system in which the binary operation $*$ on G satisfies the three conditions.

1) $\forall\ x, y, z \in G,\quad x * (y * z) = (x * y) * z$ (Associative)
2) $\exists$ an element, $e \in G \ /\ x * e = e * x = x$ (identity)
3) $\forall\ x \in G$, $\exists$ an element denoted by $x^{-1} \in G\ /$ $x^{-1} * x = x * x^{-1} = e$; where e is an identity element (inverse).

<u>Sub groups</u>:- Let $(G, *)$ be a group & $S \subseteq G$ such that it satisfies the following conditions:-

1. $e \in S$, where e is the identity element of $(G, *)$
2. $\forall\ a \in S,\ a^{-1} \in S$
3. For $a, b \in S$, $a * b \in S$, then $(S, *)$ is said to be a subgroup.

<u>Ex</u>:- Write the composite tables of i) $(Z_5, +_5)$, ii) $(Z_5, \times_5)$

i) $(Z_5, +_5)$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2

(sum ≥ 5 $\Rightarrow$ mod. div)

ii) $(Z_5, *_5)$

$*_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$\Rightarrow Z_5 - \{0\}$

2. $(Z_{11}, \times_{11})$

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	0	1	2	3	4	0
2	0	2	4	1	3	0	2	4	1	3	0
3	0	3	1	4	2	0	3	1	4	3	0
4	0	4	3	2	1	0	4	3			
5	0										
6	0										
7	0										
8	0										
9	0										
10	0										

3. Construct the group $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$ un[der] multiplication modulo 15.

a) Construct the multiplication table of G

b) Find the sub groups generated by 2, 7, 11

a)

$\times_{15}$	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4

b) 2:- $2^0 = 1 \bmod 15 = 1$

$2^1 = 2 \bmod 15 = 2$.

$2^2 = 4 \bmod 15 = 4$

$2^3 = 8 \bmod 15 = 8$

$2^4 = 16 \bmod 15 = 1$

$2^5 = 32 \bmod 15 = 2$.

$\Rightarrow \{1, 2, 4, 8\}$

11:- $11^0 = 1 \bmod 15 = 1$

$11^1 = 11 \bmod 15 = 11$

$11^2 = 121 \bmod 15 = 1$

$11^3 = 1331 \bmod 15 = 11$

$\Rightarrow \{1, 11\}$

7:- $7^0 = 1 \bmod 15 = 1$

$7^1 = 7 \bmod 15 = 7$.

$7^2 = 49 \bmod 15 = 4$

$7^3 = 343 \bmod 15 = 13$

$7^4 = 2401 \bmod 15 = 1$

$7^5 = 16807 \bmod 15 = 7$ $\Rightarrow \{1, 7, 4, 13\}$

Theorem:- In a group $(G, *)$, $\forall\ a, b, c \in G$ $b*a = c*a \Rightarrow b = c$; i.e right cancellation law holds

Proof:- By condition 2 of definition of group, $\forall a \in G$ $\exists$ an element, $d \in G$ such that $a*d = d*a = e$ Where d is an identity in G.

It is given that $b*a = c*a$

$(b*a)*d = (c*a)*d$.

$(b*(a*d)) = (c*(a*d))$

$b*e = c*e$

$\rightarrow \boxed{b = c}$

Isomorphism:- If $(G, *)$ & (G', o) are groups is an isomorphism if i) f is one-one

ii) f is onto

iii) f is homomorphism

Note :- If $f: X \to Y$ is onto, then homomorphism is defined as "epimorphism".

If $f: X \to Y$ is one-one, then homomorphism is defined as "monomorphism".

If it is both, then homomorphism is defined as "isomorphism".

Homomorphism: Let $(G, *)$ and (G', o) are groups, is called a homomorphism if for each $a, b \in G$, we have

$$(f(a*b) = f(a) \cdot f(b)) \text{ i.e } f(a \, o \, b) = f(a) \, o \, f(b)$$

Sub semi groups & Submonoids:- Let $(S, *)$ be a semi-group and $T \subseteq S$. If the set T is closed under the operation $*$, then $(T, *)$ is said to be sub-semi group of $(S, *)$

Let $(M, *, e)$ be a monoid & $T \subseteq M$.

If T is closed under the operation $*$, & $e \in T$, then $(T, *, e)$ is said to be a submonoid of $(M, *, e)$

Theorem:- Let $a, b \in G$ & $a o b$ is given by $a+b+ab$.

Prove that $\forall a, b, c \in G$ is associative. i.e ~~semi~~ group.

$$(a o b) o c = a o (b o c)$$

LHS:

$$(a o b) o c$$

$$(a+b+ab) o c$$

$$= a+b+ab+c+c(a+b+ab)$$

$$= a+b+ab+c+ac+bc+abc$$

$$= a+b+c+ab+bc+ac+abc$$

RHS:

$$a o (b o c) = a o (b+c+bc)$$

$$= a+b+c+b\cancel{c} + a(b+c+bc)$$

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1. Show that the four fourth roots of unity form a group w.r.t multiplication where $G = \{1, -1, i, -i\}$.

Sol: Closure property:- Since all entries in the composition table are the elements of G (set), G is satisfies closure property.

Associative property:- The elements of G consists of complex no's & multiplication of complex no's is associative.

Existance of identity:- Consider the top row of composition table, we have $1 \times 1 = 1$, $1 \times -1 = -1$, $1 \times i = i$, $1 \times -i = -i$ i.e $1 \times a = a \times 1 = a$, where '1' is the identity in G.

Existance of inverse:- Identity element is its own inverse. Therefore, inverse of "1" is '-1'.

Homomorphism:

Ex:- Consider the semigroups $(Z, +)$ and $(E, +)$. Define the function $f: Z \to E$ by $f(x) = 2x$ $\forall x \in Z$. Then we find that for any $a, b \in Z$,

$$\cancel{f(a+b)=}\ f(x) = 2x$$

$$f(a+b) = 2(a+b)$$

$$= 2a + 2b$$

$$f(a) = 2a, \ f(b) = 2b$$

$$\Rightarrow f(a+b) = f(a) + f(b)$$

Hence it is homomorphism.

If we take any 'y' $\in E$, then $y = 2x$ for some x in Z and $f(x) = 2x = y$.

Thus every y in E has a pre image in Z under f. Therefore f is onto.

For $a, b \in Z$, $f(a) = f(b)$

Theorem:- Let G be a group and let a, b, x be elements of G, then i) $x \cdot a = x \cdot b \Rightarrow a = b$ (Left cancellation property)

ii) $ax = bx \Rightarrow a = b$ (Right cancellation property)

Proof:- i) $x \cdot a = x \cdot b$

$$x^{-1}(x \cdot a) = x^{-1}(x \cdot b)$$
$$(x^{-1} \cdot x)a = (x^{-1} \cdot x)b$$
$$ea = eb$$
$$\Rightarrow a = b$$

ii) $a \cdot x = b \cdot x$

$$(a \cdot x)x^{-1} = (b \cdot x)x^{-1}$$
$$a(x \cdot x^{-1}) = b(x \cdot x^{-1})$$
$$ae = be.$$
$$\Rightarrow a = b.$$

3 Unit Contd)

Types of lattices:-

1. **Bounded lattice:-** A lattice (L, R) is said to be a bounded lattice if it has a greatest element & a least element.

In a bounded lattice, greatest element is denoted by I & least element is denoted by 0.

The following results are verified:-

$0Ra$, IRa, $a \vee 0 = a$, $a \wedge 0 = a$, $a \vee I = I$, $a \wedge I = I$.

2. **Distributive lattice:-** A lattice (L, R) is said to be distributive if for any $a, b, c \in L$, the following distributive law holds

1) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

2) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

A lattice which is not distributive is called a non distributive lattice.

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3. Complemented lattice:- Let L be a bounded lattice, with greatest element I and a least element 0. For a chosen element 'a' of L, if there exist an element $a' \in L$ such that $a \vee a' = I$, $a \wedge a' = 0$, then a' is called complement of a in L.

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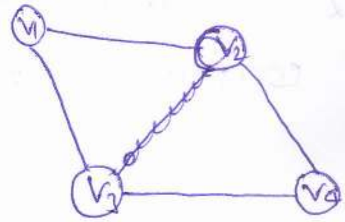
7. Graph Theory

A graph is defined as collection of nodes or vertices and edges. It is represented as $G=(V,E)$

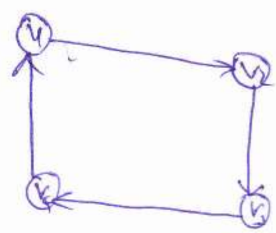
Ex:-



It contains 4 vertices V_1, V_2, V_3, V_4 and edges V_1V_2, V_1V_3, V_3V_4, V_2V_4

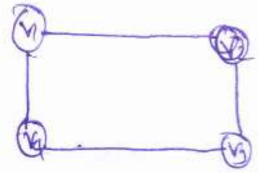
1) Directed edge graph:- A graph which consists of edges with a direction refers to directed edge graph. It is unidirectional.

Ex:-



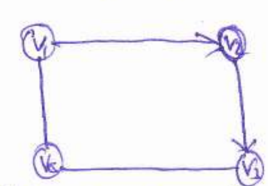
2) Undirected edge graph:- A graph with edges and no direction refers to undirected edge graph. It is always bidirectional.

Ex:-



3) Mixed graph:- A graph with directed and undirected edges refers to mixed graph.

Ex:-



4) Isolated node graph:- A graph with a node which doesn't have adjacent nodes refers Ex:-

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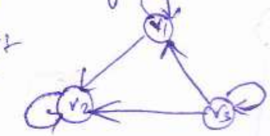
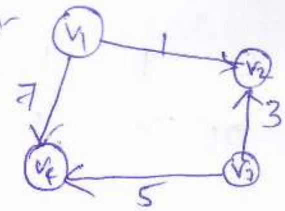
5. <u>loop edged graph</u>:- A node which itself gets pointed as an edge refers to a loop.

Ex:-



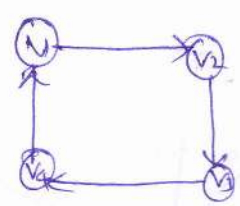
6. <u>weighted graph</u>:- A graph with edges ~~each~~ where each edge is associated ~~to~~ with some +ve real no. refers to weighted graph.

Ex:-



7. <u>cyclic graph</u>:- A circular relationship b/w different nodes of the graph

Ex:-



→ Graphs are represented in two procedures.

1) sequential representation / adjacency matrix representation.

2) Linked representation / adjacency list representation.

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1. <u>Adjacency matrix representation</u>:- In this it arranges the data in matrix format which contain n no. of rows and columns.

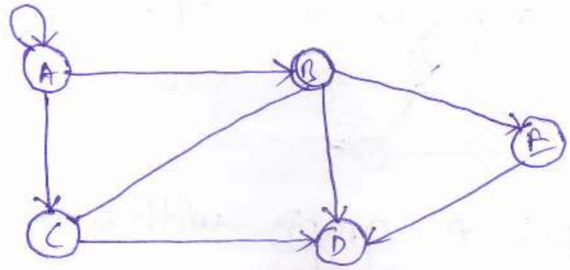
The matrix which it creates is called adjcency matrix. where it is enclosed by 0's and 1's.

If there is an edge, it is represented by 1, else represented by 0.

At weighted graph, value at adjacency matrix

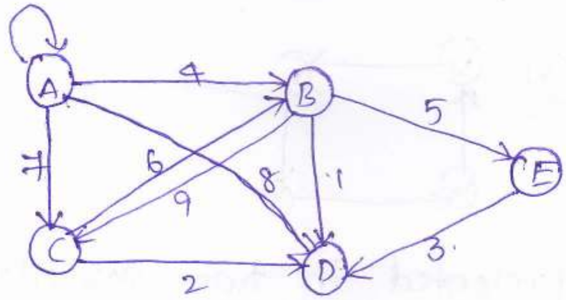
Adjacency matrix for the following graph:-

1)



	A	B	C	D	E
A	1	1	1	0	0
B	0	0	1	1	1
C	0	1	0	1	0
D	0	0	0	0	0
E	0	0	0	1	0

2)



	A	B	C	D	E
A	1	4	7	8	0
B	0	0	9	1	5
C	0	6	0	2	0
D	0	0	0	0	0
E	0	0	0	3	0

Adjacency list representation:- A graph can be represented in the form of linked list. There are types of list nodes.

1) Headers node

2) Arc node

* A Header node contains data and a pointer to next arc.

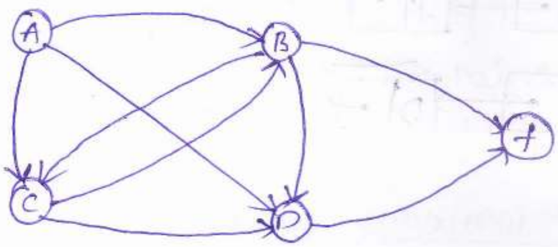
data	Arc pointer

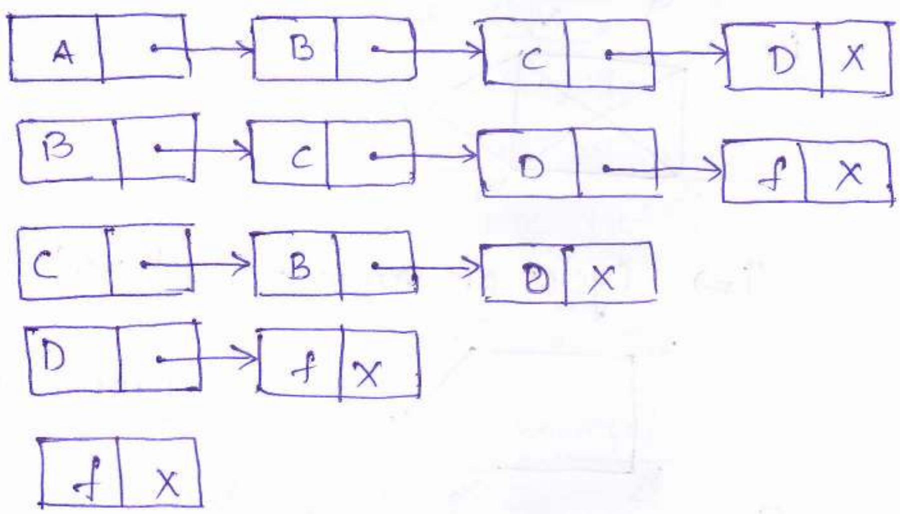
header node.

* An arc node contains a pointer to a node and a pointer to the next arc.

Node pointer	Next arc

Arc node.





Representation of a graph when the list of header nodes and arc nodes are circular:-

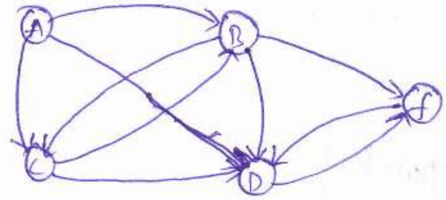
Next node	data	Arc pointer

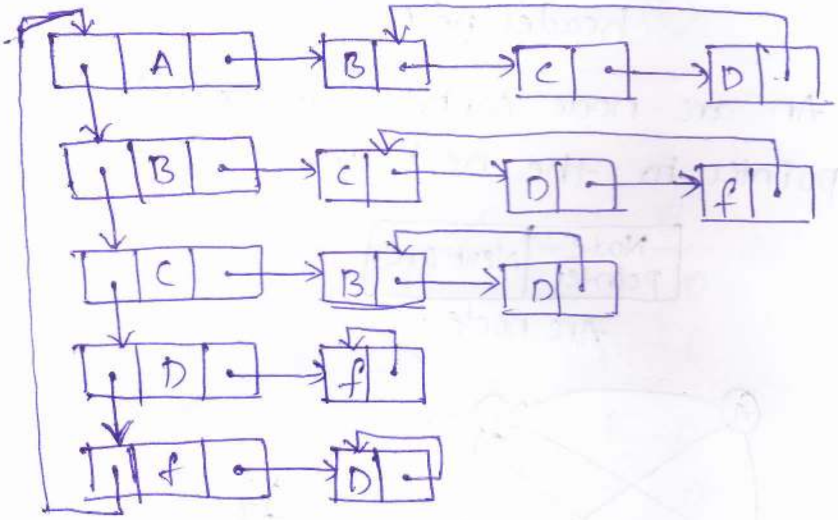
Header node.

Node pointer	next arc

Arc node

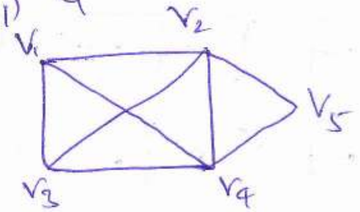
Ex1



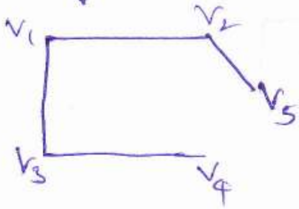


Spanning Tree:-

Ex:- 1) G

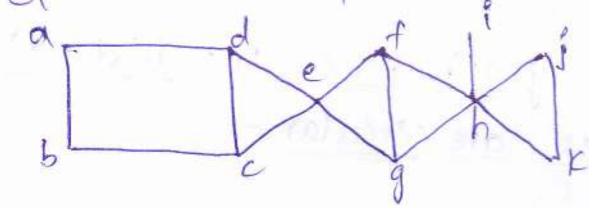


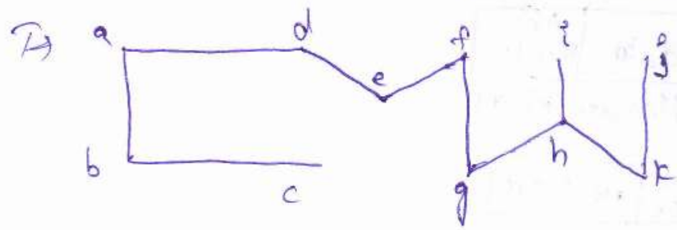
T ⇒ Cycles are removed & all vertices should be covered.



→ spanning tree

2) G





→ Spanning tree

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Spanning tree:- A tree is a simple graph 'G' which is connected and has no cycles. Let 'G' be a connected graph. A subgraph 'T' of 'G' is called a spanning tree of 'G', if it satisfies the following conditions.

i) 'T' is a tree

ii) 'T' contains all the vertices of 'G'

* Because a spanning tree of 'T' of a graph 'G' is a subgraph of 'G' which contains all vertices of 'G', 'T' is maximal subgraph of 'G'. Therefore, a spanning tree is also called as a maximal tree.

* The edges are called branches.

* If a graph has n vertices, then a spanning tree should have n vertices and (n-1) edges.

* The edges of 'G' which are not in 'T' are called 'chords'

* A set of all chords of 'G' is the complement of 'T' in 'G'. This is called a chordset (or) ~~ed~~ cotree. and is denoted by $\bar{T}$.

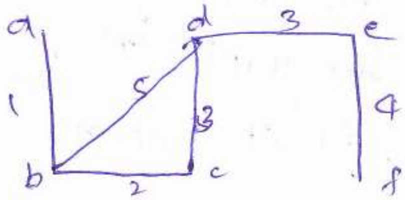
Minimal spanning tree:- If 'G' is a graph and there is a positive real number associated with each edge of 'G', then 'G' is called 'weighted graph'.

Let us consider a weighted graph which is connected. 'T' is a spanning tree of this Graph. Every branch of 'T' has a weight.

"The sum of weights of all the branches of T is called the weight of 'T'." Let us suppose

The spanning tree that has the least weight is called a minimal spanning tree.

Ex:-



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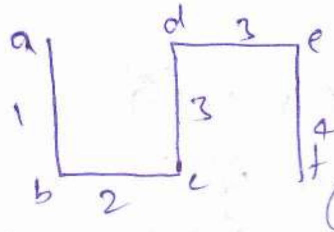

(13)

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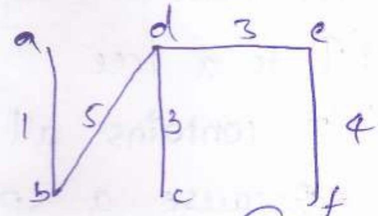

(16)

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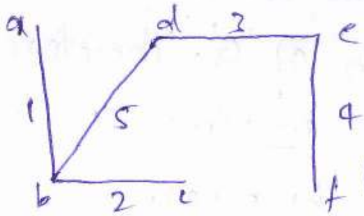

(15)

~~1st graph is~~ 1st spanning tree is the minimal spanning tree.

Methods to generate spanning tree :-

There are two algorithms or methods to find a spanning tree of a connected graph. They are 1) Depth first search (DFS)

2) Breadth first Search (BFS)

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Breadth first search :- The idea of BFS is to visit all vertices sequentially on a given level before going on to the next level.

BFS algorithm:- The input to this algorithm is a connected graph 'G' with vertex labelled $V_1, V_2, \ldots V_n$ and output will be a spanning tree 'T' for 'G'.

Step 1:- Let V_1 be the root of ~~of~~ 'T' from the set $V = \{V_1\}$

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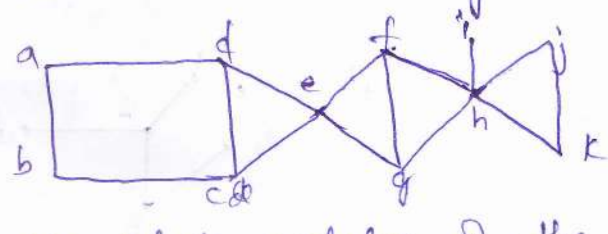
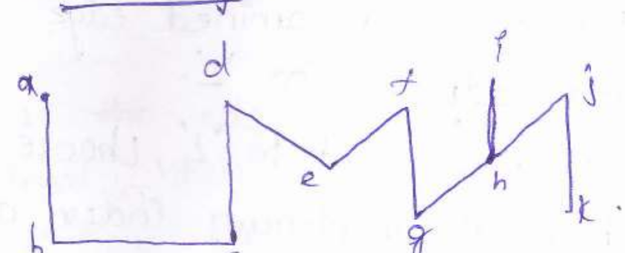
consistent with the original labelling. For each vertex, add the edge to 'T' such that adding the edge to 'T' wont produce a cycle, stop if no edge can be added.

<u>Step 3</u>:- After ~~the~~ all the vertices of V have been considered in order, the output is a spanning tree 'T' for 'G'.

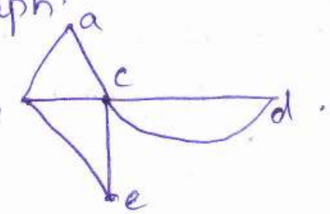
<u>Ex</u>:- <u>Illustrate BFS on a given graph.</u>

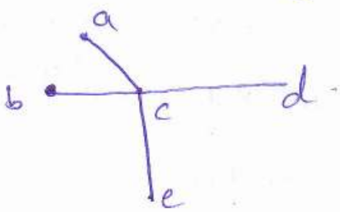


1. First we select ordering of the vertices abcdefghijk.
2. We select 'a' as the root vertex of 'T'. At this point, 'T' contains single vertex 'A'.
3. Now, add all edges to 'T' $\{a,x\}$, as x runs in order from a to k which doesnot produce a cycle.
4. We must repeat this process for all vertices on level 1 from the root by examining each vertex in an order. We include $\{b,c\}$ and we reject $\{c,d\}$
5. Now we consider the vertices at level 2. We include $\{e,f\}$ and $\{f,g\}$ but reject $\{c,e\}$
6. The edges which are rejected in BFS are known as <u>cross edges</u>.



Ex:- Derive all the possible spanning trees for the given graph.



1)


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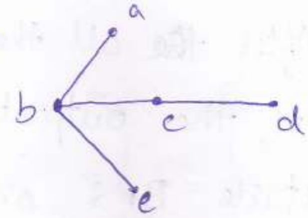
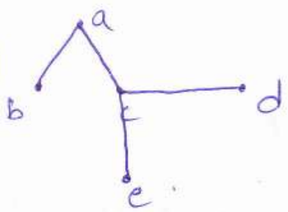


3)


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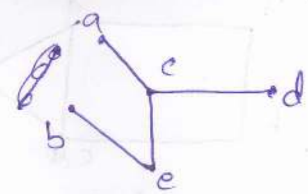

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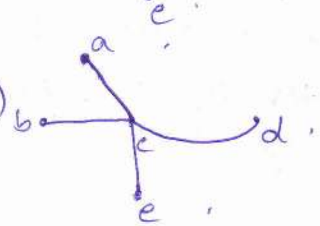
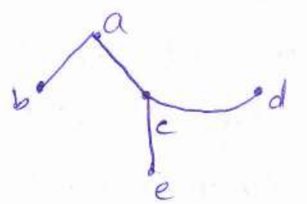


6)


7)


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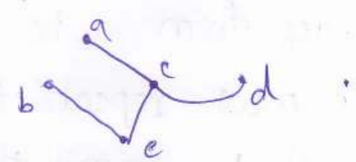

Depth for search:- The input to this algorithm is a connected graph 'G' with vertices $V_1, V_2, V_3 --- V_n$ and output will be a spanning tree for 'G'.

step 1:- Visiting a vertex. Let V_1 be the root of 'T' and set $L = V_1$ (L means vertex visited last)

step 2:- should find an unexamined edge and an unvisited vertex adjacent to 'L'.

For all vertices adjacent to 'L', choose the edge $[L, V_k]$ where k is the minimum index and also confirm that adding $[L, V_k]$ to 'T' doesnot produce a cycle.

suppose that type of edge exists, then add edge $[L, V_K]$ to 'T' and set $L = V_K$. Repeat this step 2 at the new value for L.

step 3:- Back tracking or terminating:- If x is a parent of 'L' in 'T', set $L = x^r$, apply step 2 at the new value of L. If has no parent in 'T', so that $L = V_1$, the search terminates and 'T' is the spanning tree for 'G'.

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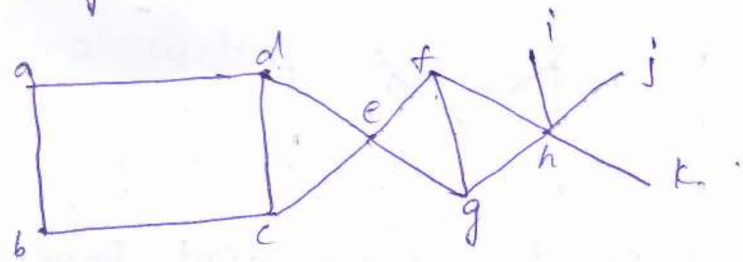
Ex:-



Initially, the vertices are traversed in the order abcdefghijk.

1. Now, we select a as root vertex of T. At this point the vertex A is visited.

$\cdot a$

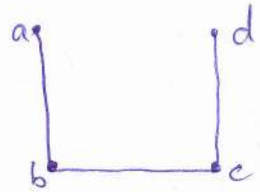
2. Now, we should select the edge $\{a, x\}$ where x is the first label in the designated order and which doesnot form a cycle

3. Since edge $\{a,b\}$ doesnot form a cycle, we select edge $\{a,b\}$.

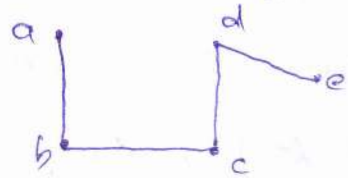
a
$|$
b

4. If all the edges incident on x are already examined, then we come back to the parent of x and continue the search from parent of x. This process is called back tracking

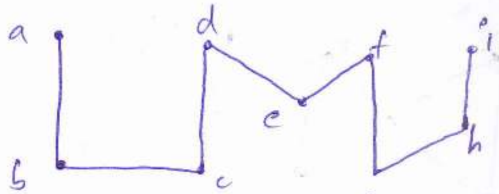
5. Now, we select $\{b,c\}$ and continue the search at C and select $\{c,d\}$. The search continues at d, donot select the edge $\{d,a\}$ since it forms a cycle.



6. Therefore, we select {d, e}, now we reject {e, c}

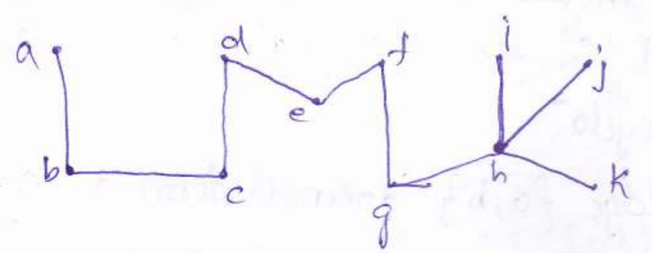


7. Therefore, we select the edge, {e, f}. Similarly continuing the search, we select {f, g}, {g, h}, {h, i} & reject {g, e}, {h, f}

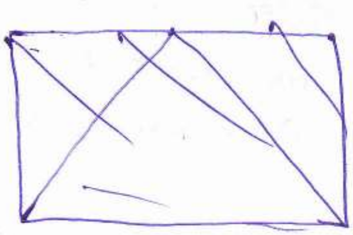


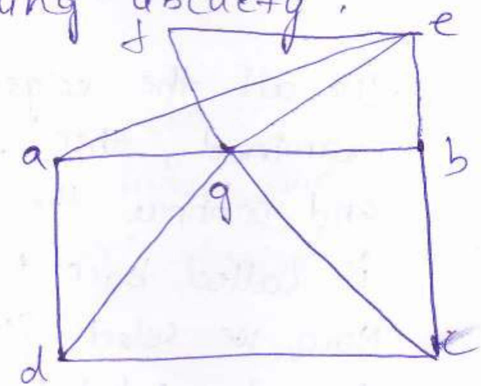
8.* Now, we select i and we dont have unexamined edges at i. So, we backtrack to h and continue from h once again.

9. Finally we select (h, j) & (h, k). So, we backtrack
Now there are no more edges at k. So, we backtrack to h and so on until we come back to a. Now the search terminates.



Ex: Use BFS to find spanning tree shown in the following fig with vertex ordering abcdefg.



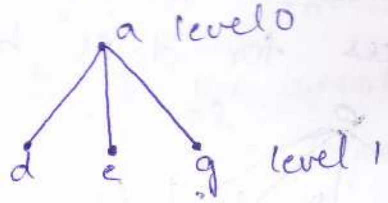


1. Let the vertex ordering abcdefg

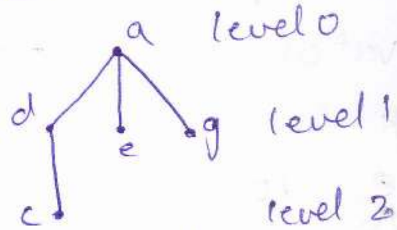
2. Start with vertex a as root vertex

a

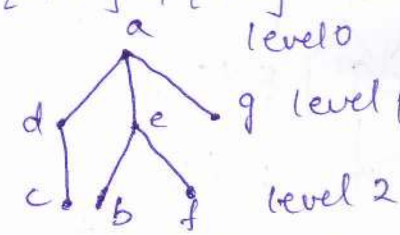
3. Add all the edges incident on a. They are $\{a,d\}$, $\{a,e\}$, $\{a,g\}$. The vertices d, e, g are at level 1 in the tree



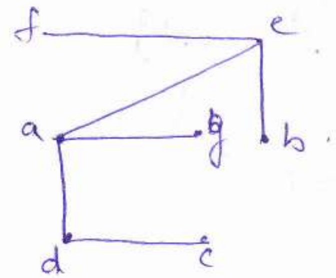
4. Add d to c i.e. $\{d,c\}$ but reject $\{d,g\}$ as it forms a cycle.



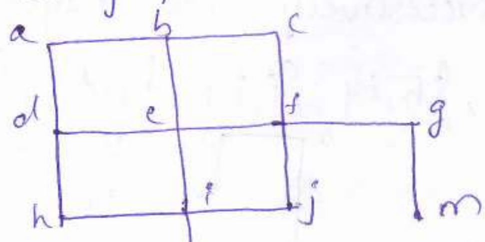
5. At e, add $\{e,b\}$ & $\{e,f\}$ but not $\{e,g\}$



6. It contains all the vertices, hence it is a spanning tree.

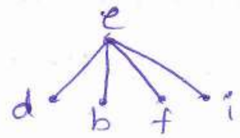


Ex:- Using BFS & DFS method, determine the spanning tree 'T' for graph 'G' with 'E' as the root of 'T'.



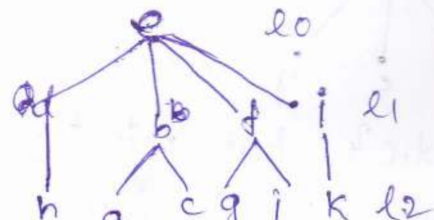
Using BFS:- 1. Let the vertex ordering be eabcdfghijklm

2. we take e as the root vertex of T, we find b, d, f, i at level 1

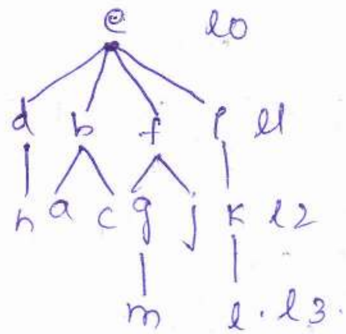


3. The remaining vertices are acghjklm

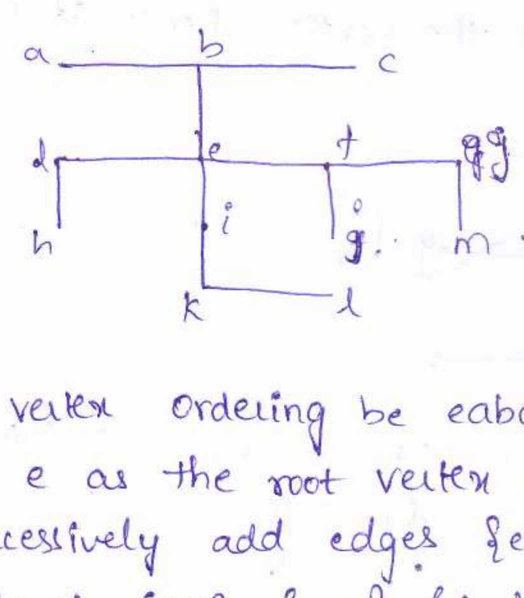
4 The level 2 vertices for d is h



5. The remaining vertices are l, m. At level 3, we add m to g & l to k.



Finally,

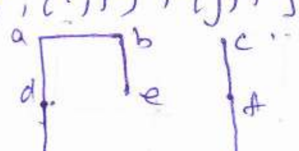


16/9/13

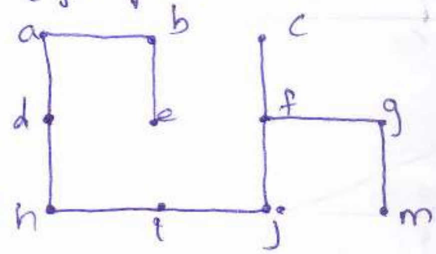
Using DFS:-

1. Let the vertex ordering be eabcdfghijklm.

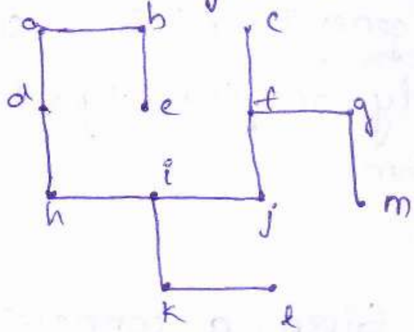
2. We take e as the root vertex of T, we start at e and successively add edges {e,b}, {b,a}, {a,d}, {d,h}, {h,i}, {i,j}, {j,f}, {f,c}



2. We backtrack to the vertex f since no more edges can be added at c. We find the edges {f, g} and {g, m}.



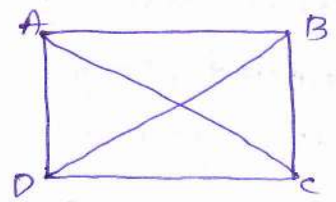
3. Now, backtrack to g, we cannot add any edges at g, so backtrack to f, no more edges can be added at f. Again backtrack to j, no edges are found. Now backtrack to i, add successively {i, k} and {k, l}



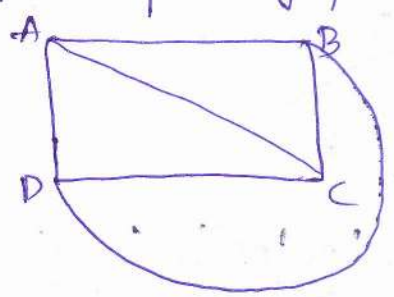
This is the required spanning tree.

Planar Graphs:- A graph G is said to be a planar graph if it can be drawn in a plane so that its edges donot cross over.

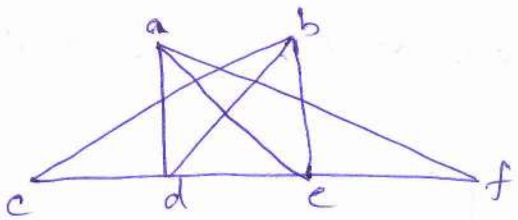
Ex:- 1)

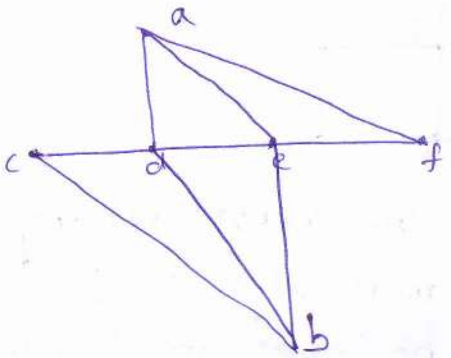


The required planar graph is



2)





Methods to generate minimal spanning trees:- There are various methods to generate minimal spanning trees. The two famous & widely accepted algorithms are

1) Kruskal's algorithm
2) Prim's algorithm.

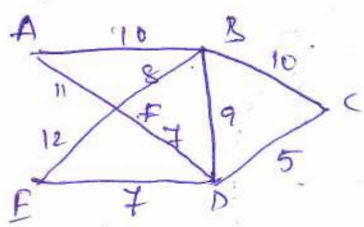
1. **Kruskal's algorithm:-** Given a connected weighted graph 'G' with n vertices. List the edges of 'G' in order of non decreasing weights.

→ start with the smallest weighted edge and continue sequentially by selecting one edge at a time so that it doesnot form a cycle.

→ When (n-1) edges are selected, stop step2 process. these (n-1) edges constitute a minimal spanning tree.

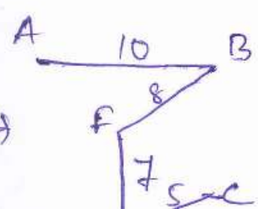
→ The step2 process is called as a greedy process.

Ex:- 1)



CD	DE	DF	FB	BD	BC	AB	AF	EF
5	7	7	8	9	10	10	11	12
Y	Y	Y	Y	N	N	Y	N	N

→

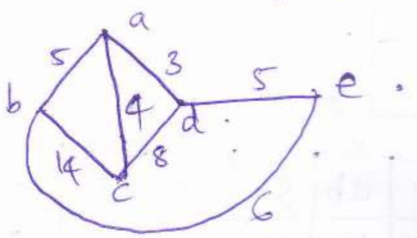


17/9/13
2. Generate
tree for

3.

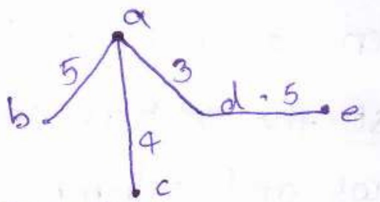
17/9/13

2. Generate a method to construct a minimal spanning tree for the following graph.



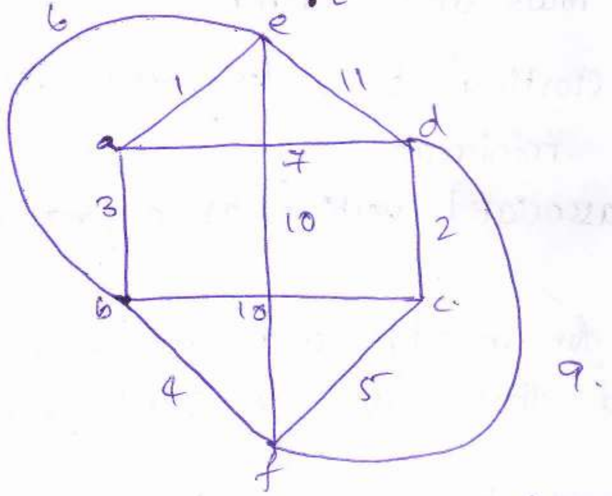
n = 5
E = 7

ad	ac	ab	de	eb	cd	cb
3	4	5	5	6	8	14
Y	Y	Y	Y	N	N	N

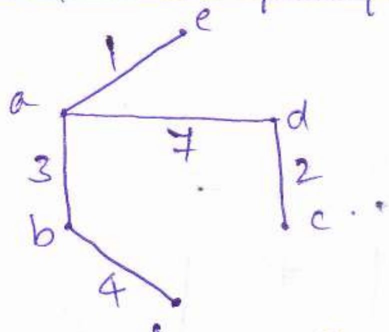


⇒ 17

3.



ae	dc	ab	bf	cf	be	da	df	ef	bc	ed
1	2	3	4	5	6	7	9	10	10	11
Y	Y	Y	Y	N	N	Y	N	N	N	N



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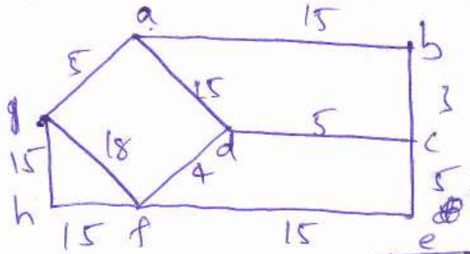
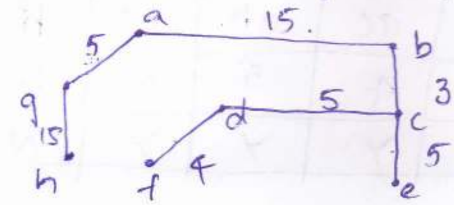
4'



bc	df	cd	ag	ce	ab	gh	hf	ef	ad	gf
3	4	5	5	5	15	15	15	15	15	18



$\Rightarrow$ 49

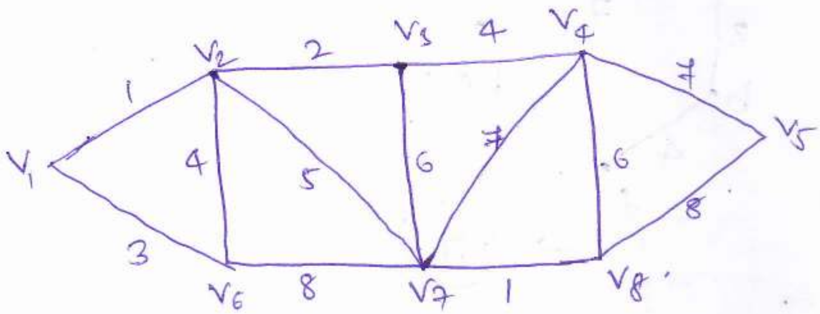
2. Prim's algorithm:- 1. Given a connected weighted graph 'G', with n vertices. List the vertices and weights of graph in the form of rows and columns.

2. The algorithm starting at a designated vertex chooses an edge with minimum weight and considers this edge and its associated vertices as a part of desired tree.

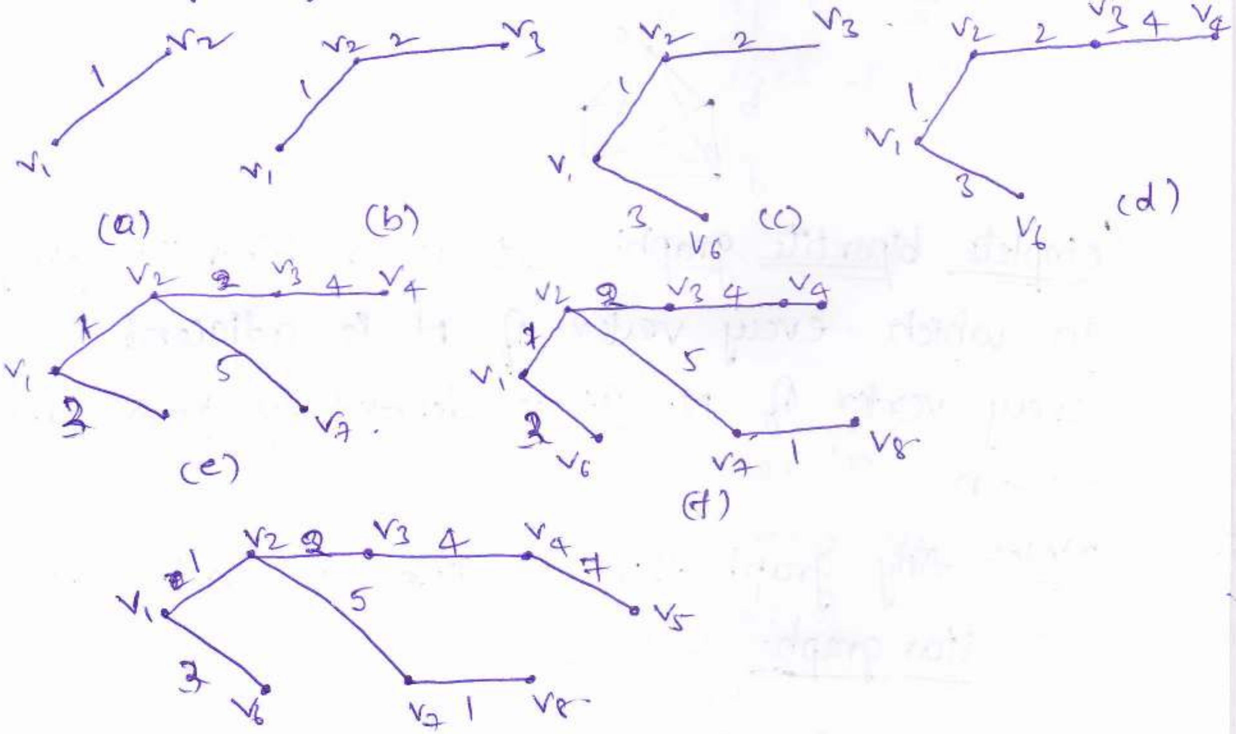
3. Iterate looking for an edge with minimum weight not yet selected that has one of its nodes in the tree.

4. The process terminates when (n-1) edges have been selected from a graph of 'n' nodes to form a minimal spanning tree. This process is called as Prim's method.

Ex:-



1. Assume that the algorithm begins at vertex V_1. The edge with the lowest weight incident to this vertex is $\{V_1, V_2\}$ which has a weight of 1.
2. Enter the loop in step 2. During the first iteration, there are 4 edge choices. $\{V_1, V_6\}$, $\{V_2, V_3\}$, $\{V_2, V_6\}$, $\{V_2, V_7\}$
3. The edge $\{V_2, V_3\}$ is selected because its weight is the least of 4 possibilities.
4. In the second iteration, edge $\{V_1, V_6\}$ is chosen. The process continues until 7 edges forming the tree are selected. The minimum spanning tree has a weight of 23.



	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	–	1	–	–	–	3	–	–
V_2	1	–	2	–	–	–	5	–
V_3	–	2	–	4	–	–	–	–
V_4	–	–	4	–	7	–	–	–
V_5	–	–	–	7	–	–	–	–
V_6	3	–	–	–	–	–	–	–
V_7	–	5	–	–	–	–	–	1
V_8	–	–	–	–	–	–	1	–

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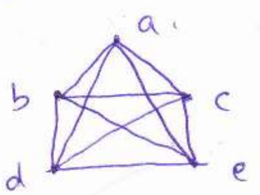
VIII. Graph theory and applications

Non planar graphs:- A graph which cannot be represented in a plane in which it contains cross overs of edges is called non planar graph.

Bipartite graph:- It is a non directed graph whose set of vertices can be partitioned into two sets M and N such a way that each edge joins a vertex in M to a vertex in N. It is represented by K_n.

Ex:- Take $n = 5 \Rightarrow K_5$

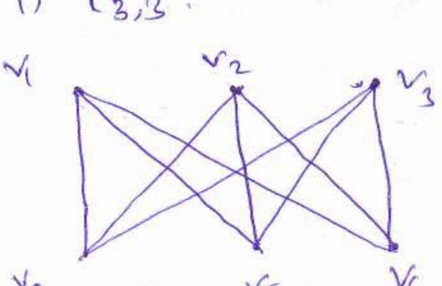


Complete bipartite graph:- It is a bipartite graph in which every vertex of M is adjacent to every vertex of N. It is denoted by $K_{m,n}$ where $m \leq n$.

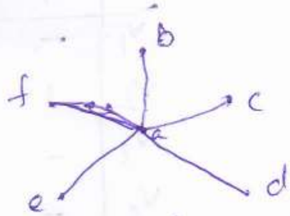
Note:- Any graph that is $K_{1,n}$ is called as a star graph.

Ex:- $m=3, n=3$

1) $K_{3,3}$



3) $K_{1,5}$

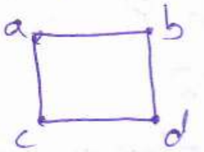


star graph.

2) $K_{2,4}$

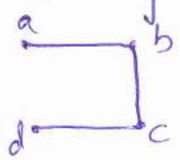
Cycle graph:- A cycle graph of order 'n' is a connected graph whose edges form a cycle of length 'n'. Cycle graphs are denoted by C_n

Ex:- C_4



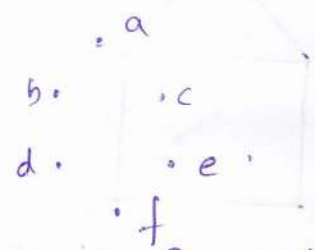
Path graph:- A path graph of order 'n' is obtained by removing an edge from C_n.

Ex:-



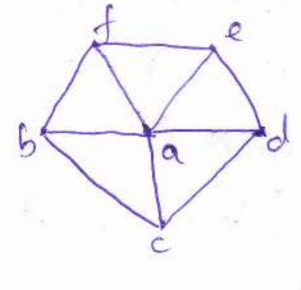
Null graph:- A null graph of 'n' is a graph with 'n' vertices and no edges. Null graphs of order 'n' are denoted by N_n.

Ex:-



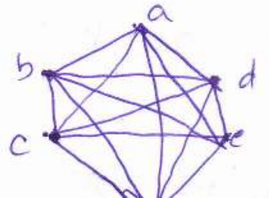
wheel graph:- A wheel of order 'n' is a graph obtained by joining a single new vertex to each vertex of a cycle graph of order $(n-1)$. Wheels of order 'n' are denoted by W_n.

Ex:



Ex:-

1) K_6.



2) P

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Criteria to detect planarity of a connected graph:-

1. A graph is planar iff it has a drawing without crossings.

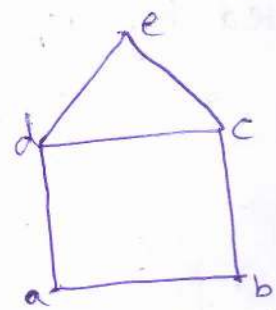
2. According to Euler's formula, if G is a connected planar graph, $|V|-|E|+|R|=2$.

3. A complete graph K_n is planar iff $n \leq 4$, where n is no. of vertices.

4. A complete bipartite graph $K_{m,n}$ is planar iff $m \leq 2$ or $n \leq 2$.

5. A connected simple graph is planar iff a) $|E| > 1$
   b) $|E| \leq 3|V|-6$
   c) There is a vertex v of G such that degree of $v \leq 5$

Ex:- 1)



1) $v=5,\ E=6,\ R=3$

$$|V|-|E|+|R| = 5-6+3 = 2$$

$\therefore |V|-|E|+|R| = 2$.

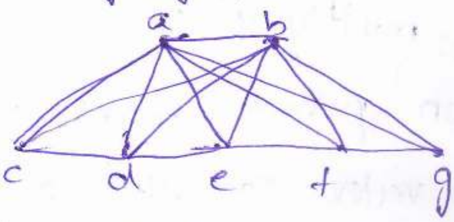
According to Euler's formula, a graph is planar if it satisfies the equation $|V|-|E|+|R|=2$ where V = set of vertices, E = set of ~~eg~~ edges R = set of regions.

$\therefore$ As it satisfied Euler's condition, the above graph is planar.

Note:- The no. of vertices in a graph is called order of the graph.

... is called size

2) P.T following graph is planar.



$V = 7$
$E = 15$

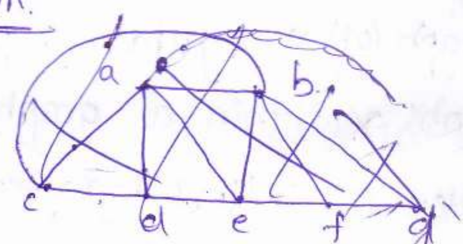
As the graph is nonplanar, the condition to be checked is $|E| \le 3|V| - 6$.

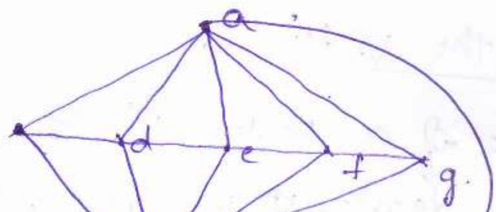
$15 \le 3(7) - 6$
$15 \le 15$

$\therefore$ True

Hence planar graph can be drawn from the given graph.

Planar graph:





Degree of vertex:-

If $G = (V, E)$ is a graph with v as a vertex of G. Then the no. of edges of G which are incident on v with the loops counted twice is called the degree of the vertex and is denoted by $d(v)$ or $deg(v)$

Multigraph:- A graph that contains multiple edges but no loops is called multigraph.

Adjacent edges:- Two non parallel edges which are incident on a common vertex is called adjacent edges.

Adjacent vertices:- Two vertices which have an edge joining them are called adjacent vertices.

Isolated vertex:- An isolated vertex is the one whose degree is zero.

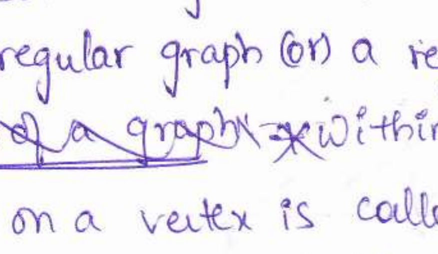
Pendant vertex:- A vertex of degree 1 is called a pendant vertex.

Regular graph:- In a given graph, if all the vertices are of same degree 'K', then it is called "K-regular graph (or) a regular graph of degree K".

Indegree of a graph:- Within a graph, no. of edges incident on a vertex is called indegree of a vertex & the no. of edges incident from it is called Outdegree of the vertex.

* The Indegree of a vertex 'V' in a graph 'G' is denoted by $\text{degree}_G^{+}(V)$ and outdegree is denoted by $\text{degree}_G^{-}(V)$.

Ex:-



Vertex	Indegree	Outdegree
V_1	0	4
V_2	1	2
V_3	0	3
V_4	3	0
V_5	1	2
V_6	2	0
V_7	3	1

Isomorphism:- The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 is correct iff $f(a)$ & $f(b)$ are adjacent in G_2 $\forall$ a and b in V_1. Such a function is known as isomorphism.

When two graphs i.e G and G' are isomorphic, it is represented as $G \cong G'$.
Inorder to show that the edge AB of G and A'B' of G' correspond to each other, we write $\{A, B\} \longleftrightarrow \{A', B'\}$

* If two graphs G and G' are isomorphic, then they should have 1) same no. of vertices
2) equal no. of vertices with given degree.

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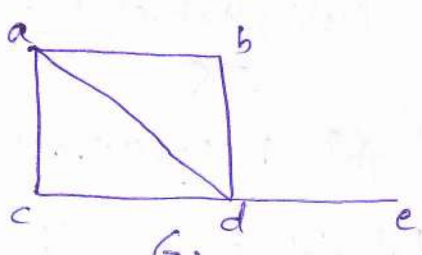
Note:- The two graphs G_1 and G_2 are said to be isomorphic if they satisfy the following conditions:-

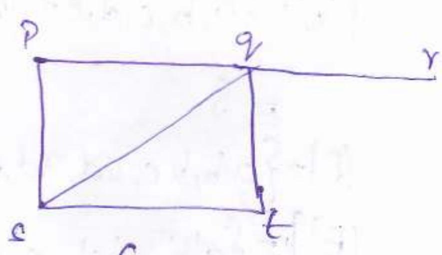
1. same no. of vertices
2. same no. of edges
3. same no. of cycles of ~~different~~ same (or) different length.
4. same degree of the sequence.

Ex:-

1.



G_1



G_2

$|V_1| = \{a, b, c, d, e\} = 5$ $\quad$ $|V_2| = \{p, q, r, s, t\} = 5$

$|C_1| = 3$ $\qquad$ $|C_2| = 3$

Degrees:-

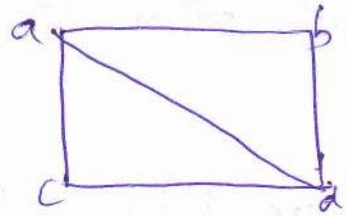
V_1	a	b	c	d	e
degree	3	2	2	4	1

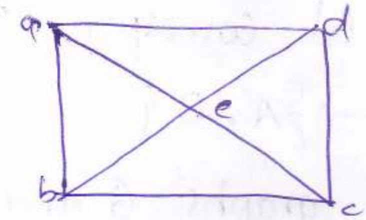
V_2	p	q	r	s	t
degree	2	4	1	3	2

$a \leftrightarrow s$
$b \leftrightarrow p$
$c \leftrightarrow t$ $\longrightarrow$ same sequence of degrees.
$d \leftrightarrow q$
$e \leftrightarrow r$

Hence as G_1, G_2 satisfied all the four conditions
They are isomorphic.

2.

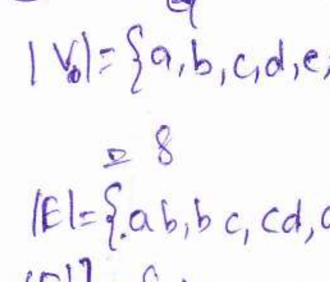




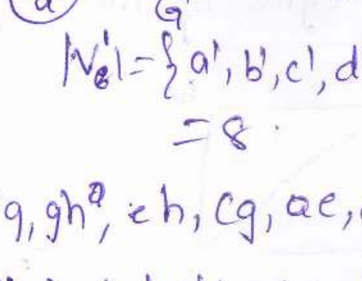
$|V_1| = 4$ $\qquad$ $|V_2| = 5$

∴ The first condition is not satisfied
→ Not isomorphic.

3. P.T the following two graphs are not isomorphic.



G



G'

$|V| = \{a,b,c,d,e,f,g,h\}$ $\qquad$ $|V'| = \{a',b',c',d',e',f',g',h'\}$
$= 8$ $\qquad$ $= 8$

$|E| = \{ab, bc, cd, ad, ef, fg, gh, eh, cg, ae, d\} = 11$

$|E'| = \{a'b', b'c', c'd', d'a', e'f', f'g, h'g', e'h', a'e', b'f', d'\} = 11$

$|E| = |E'|$

Degrees:-

V	a'
Deg	3

V'	a
Deg	

11 edge
vertex

adjace
of d
the
verti
reas
there

Subgraph
Said
a) all
in
b) each
as i
Note:- A
and
Ex

Degrees:-

v	a	b	c	d	e	f	g	h
Deg	3	2	3	4	3	2	3	2

V'	a'	b'	c'	d'	e'	f'	g'	h'
Deg	3	3	2	4	3	3	2	2

$a \leftrightarrow a'$
$b \leftrightarrow c'$
$c \leftrightarrow b'$
$d \leftrightarrow d'$
$e \leftrightarrow e'$
$f \leftrightarrow g'$
$g \leftrightarrow f'$
$h \leftrightarrow h'$

Above two graphs have 8 vertices & 11 edges and both have 3 vertices of degree 2, 1 vertex of degree 4 and 4 vertices of degree 3.

In G pair of vertices of degree 2 are not adjacent, in G', g and h' are vertices are adjacent of degree 2 and also there is a difference in the structure of the subgraphs introduced by the vertices of degree 3 in each graph. The other reason to say that they are not isomorphic is, there is a variation in the no. of cycles.

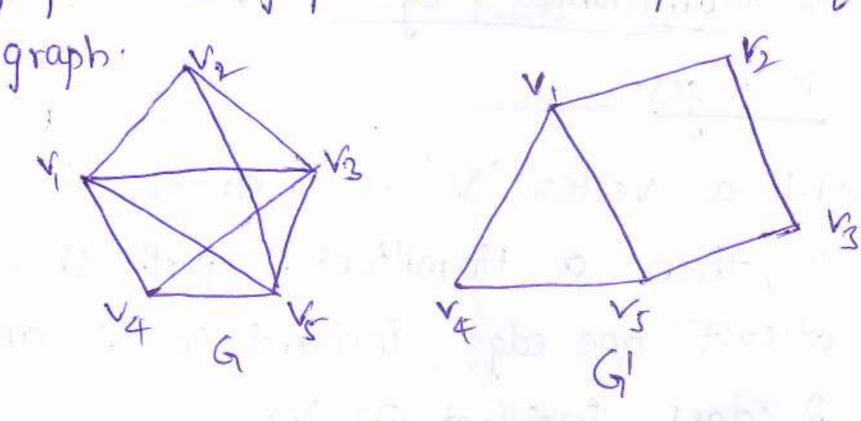
Subgraph:- Consider two graphs G and G_1, G_1 is said to be subgraph of G if

a) all the vertices ~~of G~~ and edges of G_1 should be in G

b) each edge of G_1 has the same end vertices in G as in G_1.

Note:- A subgraph is a graph which is a part of another graph.

Ex:-



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All the vertices and edges of G_1 are in graph G and also every edge in G_1 has the same end vertices, in G. Hence, G_1 is a subgraph of G.

Hamiltonian ~~Graph~~ cycle:- A graph G is said to be Hamiltonian if there exists a cycle containing every vertex of G. This cycle is called Hamiltonian cycle.

Hamiltonian Graph:- A Hamiltonian graph is a graph which has Hamiltonian cycle.

→ A path in a connected graph which includes every vertex of a graph is called a Hamiltonian path.

→ Hamiltonian cycle traverses each vertex exactly once.

Rules for construction of Hamiltonian paths and cycles:-

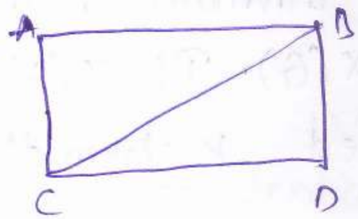
1. If a graph G has 'n' vertices, then Hamiltonian path should contain exactly (n-1) edges. and a Hamiltonian cycle should contain exactly n edges.

2. If a vertex 'V' in a graph 'G' has degree 'K', then a Hamiltonian path should contain atleast one edge incident on V and atmost 2 edges incident on V.

3. In Hamiltonian cycle, there should not be 3 or ~~edges~~ more edges incident with 1 vertex.

4. Once a Hamiltonian cycle has been constructed by traversing all the vertices, then all the unused edges incident on v can be deleted.

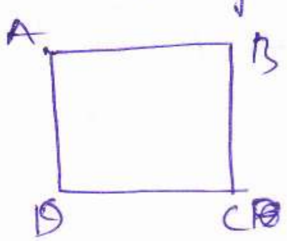
Ex:- 1.



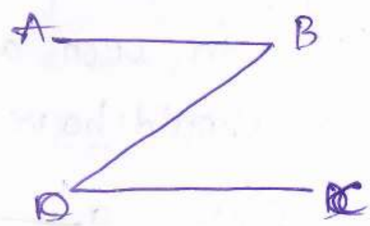
$\Rightarrow n=4$

Hamiltonian cycle

4 edges

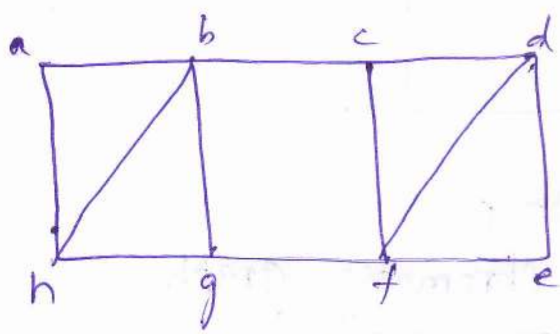


Path



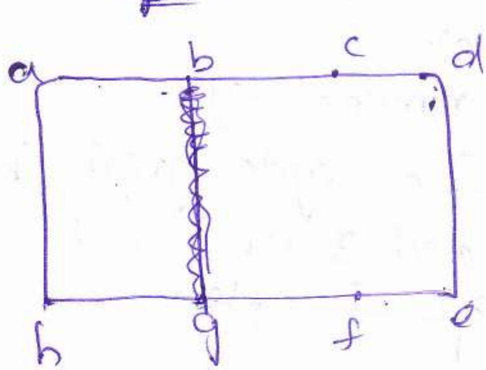
A - B - C - D

2.

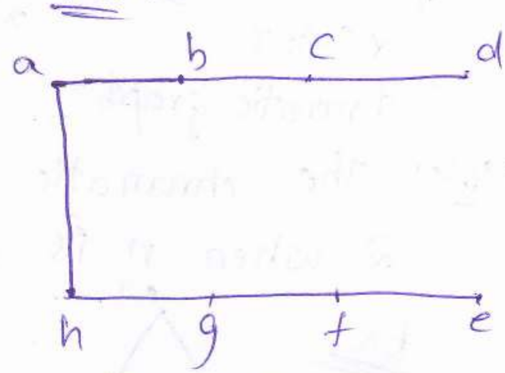


$n = 8$
$E = 12$

Cycle



Path



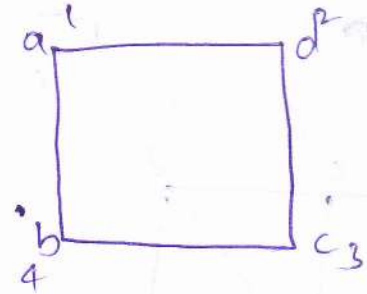
Chromatic numbers:-

The chromatic number of a graph G is defined as the minimum no. of different colours required for colouring each vertex of a graph provided no adjacent vertices should have same colour.

The chromatic number of a graph G is denoted by $\chi(G)$. If $\chi(G) = K$, then the graph G is called K chromatic graph.

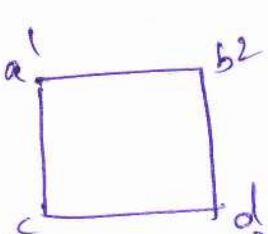
Graph colouring:- A graph G is said to be properly coloured if we assign colours to its vertices in such a way that no two adjacent vertices should have the same colour.

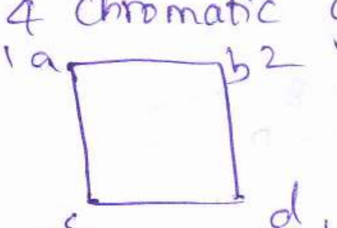
Ex:-



$\chi(G) = 4$

$\Rightarrow$ 4 Chromatic graph.



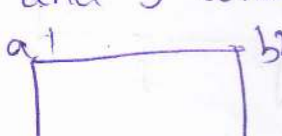


$\chi(G) = 3$ — 3 chromatic graph

$\chi(G) = 2$ — 2 Chromatic graph.

Note:- The chromatic no. of a cycle graph C_n is 2 when n is even and 3 when n is odd.

Ex:-



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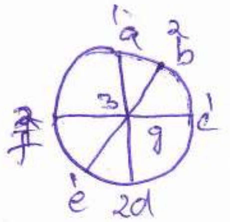


⇒ 3 chromatic graph

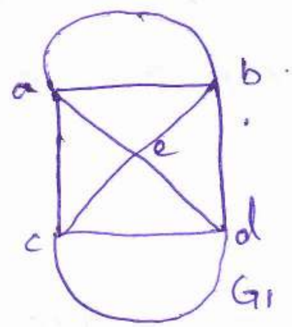
Vertex / Degree	g	a	b	c	d	e	f
Degree vertex	6	3	3	3	3	3	3
chromatic no.	3	1	2	1	2	1	2

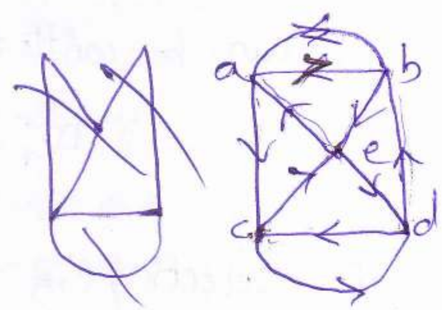
<u>Euler's Circuits:-</u>

An Euler's Circuit is an Euler path whose end points are identical.

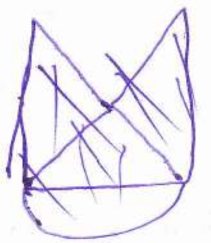
Euler's path:- A Euler path in a multigraph is a path which includes each edge of the multigraph exactly once and intersects each vertex atleast once.

Ex:- Give an example of an Eulerian graph.





~~c-e-d-c-d-b-e-a~~
~~b~~



c-d-c-e-d-b-e-a-b-a-c

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5. Elementary Combinatories

Permutations:- A permutation is a set of distinct objects in an ordered, arrangement of these objec

A permutation of 'n' objects taken 'r' at a time in an ordered selection of 'r' objects which is denoted by $p(n,r)$ or ^nP_r

$$^nP_r \text{ is given by, } {}^nP_r = \frac{n!}{(n-r)!}$$

Ex:- Arranging 5 persons from the set of 8 persons in a given order is

$$P(n,r) = P(8,5) = \cancel{\frac{8!}{8}}\ {}^8C_5 = \frac{8!}{(8-5)!}$$

$$= \frac{8!}{3!} = \cancel{8\times\times6}\ 6720.$$

Combination:- A combination of n objects taken 'r' at a time in an unordered selection of 'r' objects.

It can be written as $c(n,r)$ or ^nC_r

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

Ex:- selecting 5 cards from a deck of 52 cards

$C(52,5)$

$$^{52}C_5 = \frac{52!}{(52-5)!\,5!} = 2598960$$

Basics of Counting:- There are two fundamental principl of all Counting problems.

Note

1) Sum rule (or) disjunctive rule.

2) Product rule (or) Sequential rule

1. Sum rule :- Consider two events, if one event occur in m ways and other event occur in n ways and if these two events cannot occur simultaneously, then one of the two events can occur in (m+n) ways

Ex:- If there are 5 boys and 4 girls in a class, what is the way of selecting either a boy or a girl as class representative.

Soln
$5c_1$ ways to select a boy.
$4c_1$ ways to select a girl
$\Rightarrow 5+4=9$.

2. Product rule:- If an event can occur in m ways and second event can occur in n ways and if the no of ways the second event occurs does not depend upon how the first event occurs. Then the two events can occur simultaneously in (m×n) ways.

Ex:- 1. If two distinguishable dice are rolled, then the first die can fall (event E_1) in 6 ways and second event can fall (event E_2) in 6 ways. Hence, when two dice are rolled, the possible outcomes are

Soln
$n \times n = 6 \times 6$
$= 36$ ways

2. A group of 8 scientists is composed of 5 psychologists and 3 socialogists In how many ways i) a comittee of 5 can be formed. ii) In how many ways a committee of 5 can be formed that has 3 psychologists and 2 socialogists.

Soln (i) $S = 8$ A committe of 5 can be formed
$p = 5$, $r = 3$. from a group of 8 scientists in $^8C_5$
$^8C_5 = 56$. ways.

(ii) $S = 8$.

$\Rightarrow 5 \to p = 3 \Rightarrow$ 3 psychologists & 2 Sociologists
$r = 2$. in $^5C_3 \times {}^3C_2$ ways.

$^5C_3 \times {}^3C_2 = 30$.

3. How many ways are there to roll two distinguishable dice to yield a sum that is divisible by 3.

Soln $3, 6, 9, 12 \Rightarrow$ divisible by 3 =

$S = 6 \times 6 = 36$. $\begin{bmatrix} (1,1), (1,2) \cdots \\ \vdots \\ \cdots (6,6) \end{bmatrix}$

$\{(1,2), (2,1), (1,5), (5,1), (2,4), (4,2), (6,3), (3,6), (5,4), (4,5), (6,6), (3,3)\} \Rightarrow 12$

$\Rightarrow 36/12$ are the possible ways.

4. Find the no. of different ways in which 4 boys and 6 girls may be arranged in a row so that no two boys shall be together.

Soln 4 boys, 6 girls
$4!$, $6!$

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Enumerating permutations with constrained Repetitions:-

Theorem:- Enumerating permutations with constrained Repeatitions is given by $x = \dfrac{n!}{q_1!\,q_2! \cdots q_r!}$

Proof:- Let $x = P(n; q_1, q_2 \cdots q_r)$. If q_1 is different event, then the possible permutation is $q_1!x$. If q_2 and q_1 are replaced by different

We know that there are $n!$ permutations of n distinct objects. Hence it is given by also

$$n! = q_1!\, q_2! \cdots q_m!\, x.$$

From this, $x = \dfrac{n!}{q_1!\, q_2! \cdots q_r!}$

Ex:- 1. How many different arrangements are made to arrange the letters of the word ~~MISSISSIPPI~~ MISSISSIPPI

Soln They can be arranged in $11!$ ways.

$M-1,\ S-4,\ I-4,\ P-2$.

$$\therefore x = \frac{11!}{4!\,4!\,2!\,1!} = 34650.$$

2. The no. of arrangements in the word ENGINEERING is given by

Soln $11!$ ways.

$E-3,\ N-3,\ G-2,\ I-2,\ R-1$

$$x = \frac{11!}{3!\,3!\,2!\,2!\,1!} = 277200.$$

3. Find the arrangement of letters in the word TALLAHASSEE

Soln $11!$ ways

$T-1,\ A-3,\ L-2,\ H-1,\ S-2,\ E-2$.

$$x = \frac{11!}{3!\,2!\,2!\,2!} = 831600.$$

Note:- The no. of arrangements of these letters that begin with T and end with E is given by

⇒ Using the digits 1, 3, 4, 5, 6, 8, 9, how many 3 digit no's can be formed.

$$^{7}P_{3} = 210.$$

⇒ How many 3 digit no's can be formed if no digit is repeated.

$$^{10}P_{3} = 720$$

→ How many 3 digit no's can be formed if 3 & 4 are adjacent to each other.

Soln Total no. of digits are 10. Let us consider 3 & 4 as one digit, then total no. of digits in this case are 9 and 3, 4 can be arranged among themselves in 2! ways. Hence the no. of 3 digit no's can be formed if 3 & 4 are adjacent to each other is $2! \times {}^{9}P_{3}$