UNIT - III : PART - B

*Ordered pair: An ordered pair is a pair of objects whose components occur in a special order. It is written by listing the components in the specified order, separating them by a comma and enclosing the pair in parenthesis. In the ordered pair (a, b), a is called first component and b, the second.

*Cartesian product of sets: Let A and B be sets. Cartesian product of A & B, denoted by $A \times B$ and is defined as $A \times B = \{(a, b) : a \in A \text{ \& } b \in B\}$ i.e. it is the set of all possible ordered pairs whose first component comes from A and whose 2nd component comes from B.

For example, if $A = \{a, b\}$ and $B = \{1\}$ then $A \times B = \{(a,1), (b,1)\}$; $B \times A = \{(1,a), (1,b)\}$; $A \times A = \{(a,a), (a,b) (b,a) (b,b)\}$

$B \times B = \{(1,1)\}$.

NOTE ①: $A \times B \neq B \times A$.

② If a set A has "m elements" and B has "n" elements then $|A \times B| = m \times n$; i.e., $|A \times B| = |A||B|$

The idea of cartesian product of sets can be extended to any ~~infinite~~ finite no. of sets. For any non-empty sets $A_1, A_2, \ldots, A_k$, the k-fold product $A_1 \times A_2 \times \ldots \times A_k$ is defined as the set of all ordered k-tuples $(a_1, a_2, \ldots, a_k)$ where $a_i \in A_i$, $i = 1, 2, \ldots, k$. That is

$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_k) \mid a_i \in A_i \; i = 1, 2, \ldots k\}$

and $|A_1 \times A_2 \times \ldots \times A_n| = |A_1||A_2||A_3| \ldots |A_k|$.

For example, if $A = \{0\}$ $B = \{2, -2\}$ $C = \{0, -1\}$ then

$A \times B \times C = \{(1,2,0), (1,-2,0) (1,2,-1) (1,-2,-1) (0,2,0) (0,2,-1) (0,-2,0) (0,-2,-1)\}$.

① Find x and y in each of the following cases

(i) $(2x-3, 3y+1) = (5, -1)$ $\quad (4, 2)$

(ii) $(x+2, 4) = (5, 2x+y)$ $\quad (3, -2)$

(iii) $(x, y) = (x^2, y^2)$ $\quad x = 0, 1$ & $y = 0, 1$

(iv) $(x, y) = (y^2, x^2)$

(v) $(2x, x+y) = (6, 1)$ $\quad (3, -2)$

(vi) $(y-2, 2x+1) = (x-1, y+2)$ $\quad (2, 3)$

(iv) ①

$x = x^4 \Rightarrow x^3 = \ldots$

$x(x - x^3) = 0$

② Let $A = \{1, 3, \ldots$

(i) $A \times B = \{(1, 2 \ldots$

(ii) $B \times A = \{(2, 1 \ldots$

(iii) $B \times C = \{(\ldots$

(iv) $A \times C = \{(\ldots$

(v) $(A \cup B) \times C = \{ \ldots$

$= \{ \ldots$

(vi) $A \cup (B \times C) = \{ \ldots$

$= \{ \ldots$

(vii) $(A \times B) \cup C = \{ \ldots$

(viii) $A \cap (B \times C) = \ldots$

(ix) $(A \times B) \cup (B \times \ldots$

(x) $(A \times B) \cap (B \times \ldots$

(xi) $(A \times B) \cap (B \times \ldots$

③ If $A = \{1, 2, 3\} \ldots$ Find (i) $A \times B \ldots$

④ Given $A = \{ \ldots$ $A \times A$, $B \times B \ldots$

⑤ Given $A = \{1, \ldots$ $A \times B \times C$ an…

⑥ Let $A = \{1, 2, \ldots$ down the fol…

*Properties of C…

1. $(A \cap B) \times (C \cap D) \ldots$

2. $(A - B) \times C = (A \ldots$

3. $(A \cup B) \times C = \ldots$

4. $A \times (B \cap C) = \ldots$

5. $(A \cap B) \times C = \ldots$

6. $A \times (B - C) = \ldots$

(iv) $\underset{①}{x = y^2}$ & $\underset{②}{y = x^2}$. sub ① in ② and ② in ①,

$x = x^4 \Rightarrow x^3 = 1 \Rightarrow x = 1,\ y = 1$

$x(x - x^3) = 0 \Rightarrow x = 0\ ;\ y = 0.$

② Let $A = \{1,3,5\}$ $B = \{2,3\}$ $C = \{4,6\}$. Write the foll.

(i) $A \times B = \{(1,2)(1,3)(3,2)(3,3)(5,2)(5,3)\}$

(ii) $B \times A = \{(2,1)(3,1)(2,3)(3,3)(2,5)(3,5)\}$

(iii) $B \times C = \{(2,4)(2,6)(3,4)(3,6)\}$.

(iv) $A \times C = \{(1,4)(1,6)(3,4)(3,6)(5,4)(5,6)\}$.

(v) $(A \cup B) \times C = \{\{1,2,3,4,5\} \times \{4,6\}$.

$= \{(1,4)(1,6)(2,4)(2,6)(3,4)(3,6)(5,4)(5,6)\}$.

(vi) $A \cup (B \times C) = \{1,3,5\} \cup \{(2,4)(2,6)(3,4)(3,6)\}$.

$= \{1,3,5\ (2,4)(2,6)(3,4)(3,6)\}$.

(vii) $(A \times B) \cup C = \{(1,2)(1,3)(3,2)(3,3)(5,2)(5,3);4,6\}$.

(viii) $A \cap (B \times C) = \phi$

(ix) $(A \times B) \cup (B \times C) = \{(1,2)(1,3)(3,2)(3,3)(5,2)(5,3),(2,4)(2,6)$
$(3,4)(3,6)\}$.

(x) $(A \times B) \cap (B \times A) = \{(3,3)\}$.

(xi) $(A \times B) \cap (B \times C) = \phi$

③ If $A = \{1,2,3\}$ $B = \{4,5\}$. $C = \{1,2,3,4,5\}$.
find (i) $A \times B$ (ii) $C \times B$ (iii) $B \times B$.

④ Given $A = \{a,b\}$ and $B = \{1,2,3\}$. Find $A \times B, B \times A$, $A \times A$, $B \times B$.

⑤ Given $A = \{1,2\}$ $B = \{a,b,c\}$ and $C = \{3,4\}$. Find $A \times B \times C$ and $B \times C \times A$.

⑥ Let $A = \{1,2,3,4\}$ $B = \{2,5\}$ $C = \{3,4,7\}$. Write down the foll. $A \times B$, $B \times A$, $A \cup (B \times C)$, $(A \cup B) \times C$, $(A \times C) \cup (B \times C)$.

*Properties of Cartesian Product : For 4 sets A, B, C and D

1. $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
2. $(A - B) \times C = (A \times C) - (B \times C)$
3. $(A \cup B) \times C = (A \times C) \cup (B \times C)$
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
5. $(A \cap B) \times C = (A \times C) \cap (B \times C)$
6. $A \times (B - C) = (A \times B) - (A \times C)$

① P.T $A\times(B\cap C) = (A\times B)\cap(A\times C)$

Sol: Let (x,y) be any element of $A\times(B\cap C)$, then $x\in A$ and $y\in(B\cap C)$.

$x\in A$ and $(y\in B$ and $y\in C) \Rightarrow (x\in A$ and $y\in B)$ and $(x\in A$ and $y\in C)$

$\Rightarrow (x,y)\in(A\times B)$ and $(x,y)\in(A\times C)$

$\Rightarrow (x,y)\in(A\times B)\cap(A\times C)$.

$\therefore A\times(B\cap C)\subseteq(A\times B)\cap(A\times C)$ — ①.

Ily $(A\times B)\cap(A\cap C)\subseteq A\times(B\cap C)$ — ②.

* from ① and ② : $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

② If A, B, C are three sets such that $A\subseteq B$.
S.T. $(A\times C)\subseteq(B\times C)$.

Sol: Let $(x,y)\in A\times C \Rightarrow x\in A$ and $y\in C$.

Since $A\subseteq B \Rightarrow x\in B$ and $y\in C \Rightarrow (x,y)\in(B\times C)$

$\therefore A\times C\subseteq(B\times C)$

*Relations: The word relation is used to indicate a relationship between two objects. There are many kinds of relationship in the world. We deal with relationship between student and teacher, an employee and his salary and so on. The often used relations in mathematics are less than (<), greater than (>), "subset of", and so on. A relation between 2 objects can be defined by listing the 2 objects as an ordered pair. A set of all such ordered pairs, in each of which the first member has some definite relationship to the 2nd, describes a particular relation. This method of specifying a relation does not require any special symbol or description and so is suitable for any relation between any 2 sets. In this topic, we discuss the mathematics of relations ~~dis~~ on sets, various ways of representing relations and explore various properties they may have.

*Definition: Let A and B be 2 sets. Then a subset of $A\times B$ is called a relation from A to B. Thus, if R is a relation from A to B, then R is a set

If $(a,b) \in R$, we say that "a is related to b by R", this is denoted by $a R b$.

If R is a relation from A to A, that is, if R is subset of $A \times A$, we say that R is a binary relation on A.

If $(a,b) \notin R$, then a is not related to b by R and is written as $a \not R b$.

* Domain and Range of a Relation: The set of first coordinates of every ordered pair (element) of R is called the domain of R and the set of 2nd coordinates of every ordered pair (element) of R is called Range of R. Symbolically we can write: If R is a relation from A to B, then domain of $R = d(R) = \{x : x \in A \text{ and } (x,y) \in R\}$ and range of $R = r(R) = \{y : y \in B, (x,y) \in R\}$.

* Inverse Relation: If R is relation from a set A to set B, then the inverse of R, denoted by R^{-1} is the relation from the set B to set A which contains all the ordered pairs (elements) of R in which first and second coordinates of each ordered pair are interchanged. It means that R^{-1} shall consists of all those ordered pairs which if reversed shall belong to R. that is $R^{-1} = \{(y,x) : (x,y) \in R\}$.

* Identity Relation: A relation R in a set A is said to identify relation generally denoted by I_A, if $I_A = \{(x,x) : x \in A\}$.

Ex: Let $A = \{1,2,3\}$ then $I_A = \{(1,1)(2,2)(3,3)\}$ is an identity relation in A.

* n-ary Relation: Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of sets. A subset R of $A_1 \times A_2 \times \cdots \times A_n$ is called n-ary relation on $A_1, A_2 \ldots A_n$.

* Representation of a Relations: There are 5 main methods to represent a relation R from a set A to set B.

① Roaster Method: In this method all the elements (ordered pairs) of the relation are enclosed within braces Ex: Let $A\{1,2,3\}$ & $B = \{x,y,z\}$ then $R = \{(1,y)(1,z)(3,y)\}$.

② Matrix method or matrix of a relation: Consider the sets $A=\{a_1, a_2, --- a_n\}$ and $B=\{b_1, b_2, --- b_n\}$ of orders m and n respectively, then $A \times B$ consists of all ordered pairs of the form (a_i, b_j) $1 \le i \le m$, $i \le j \le n$ which are m n in number.

Let R be a relation from A to B so that R is subset of $A \times B$. Now let us put $m_{ij} = (a_i, b_j)$ and assign the values 1 (or) 0 to m_{ij} as follows $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \\ 0 & \text{if } (a_i, b_j) \end{cases}$

The $m \times n$ matrix formed by these m_{ij}'s is called the
* adjacency matrix (or) the matrix of the relation R or the relation matrix for R, and is denoted by M_R (or) $M(R)$.

It is to be noted that the rows of M_R corresponding to the element of A and the columns to those of B when B=A, the matrix M_R is a square matrix with 'n elements' ~~when B=A, then matrix M_R is a square matrix with "n elements"~~.

It is to be noted that the rows of M_R corresponding to the elements of A and the columns to those of B.

Ex 1: The relation $R=\{(1,y)(1,z)(3,y)\}$ from $A=\{1,2,3\}$ to $B=\{x,y,z\}$ can be represented in matrix form as follows

$$M_R = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

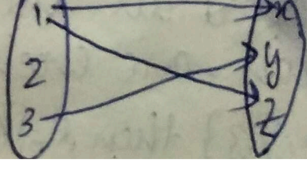
2: Define the relation R for the adjacency matrix given by

$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{3\times 4} \end{array}$$

(missed 2 points learn from pdf)

Sol: Let R be a relation from set A to set B. As M_R is 3×4, then let sets $A=\{a,b,c\}$ $B=\{1,2,3,4\}$ A and B are written in two disjoint plane figures (circles, rectangles, disc, etc). and then arrows are drawn from $x \in A$ to $y \in B$. if xRy.

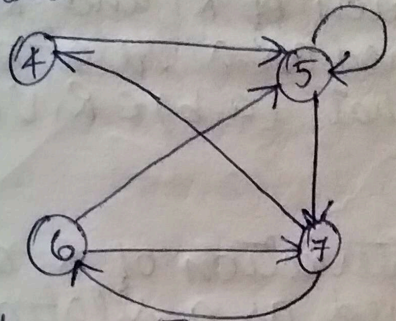
③ Arrow Diagram:



The diagram corresponding to relation $R=\{(1,x)(1,z)(3,y)\}$ is as shown.

Digraph of a relation on sets: when a relation is from a finite set A to itself can be represented by a digraph also First the elements of A are written down. Then arrows are drawn from each element ~~of~~ x to each element y $(x, y \in A)$ whenever x is related to y i.e., xRy.

Ex: Let the set $A = \{4,5,6,7\}$ and the relation R on $A = \{(4,5)(5,5)(5,7)(6,5)(6,7)(7,4)(7,6)\}$ this relation can be represented by a digraph as shown below:



* **Set Builder form:** In this method the rule that associates the first and 2nd coordinates of each ordered pair is given. Ex: Relation from A to B given by $R = \{(x,y): x \in A \text{ and } y \in B, x < y\}$. where $A = \{1,2,4\}$ $B = \{2,3,4\}$: $R = \{(1,2)(1,3)(1,5)(2,3)$
$(2,5)(4,5)\}$.

* **Composition of Relations:** If A, B and C are 3 non-empty sets and R and S are the relations from A to B and B to C respectively then $R \subseteq A \times B$ and $S \subseteq B \times C$. Then we define a relation from A to C denoted by RoS given by ~~RoS~~

$RoS = \{(x,z): \exists \text{ some } y \in B \ni (x,y) \in R \ \& \ (y,z) \in S\}$.

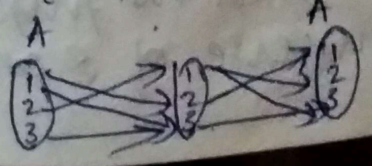
This relation is called a composition of R and S or a composite relation of R and S.

Ex: R and S are relations on $A = \{1,2,3\}$ $R = \{(1,1)$
$(1,2)(2,3)(3,1)(3,3)\}$, $S = \{(1,2)(1,3)(2,1)(3,3)\}$ then find RoS and $S^n = SoS$.

$(3,2)(3,3)\}\{1,3)\}$.

ⓐ To find $RoS = \{(1,2)(1,1)(2,3)(3,3)\}$.

$S^2 = SoS = \{(1,1)(1,3)(2,2)(2,3)(3,3)$



* Universal Relation: A relation R in a set A is called universal relation if $R = A \times A$.
For eg, if $A = \{1,2,3\}$ then $R = \{(1,1)(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)\}$.

* Operations on Relations: Since a relation is a subset of the cartesian product of 2 sets, the set-theoretic operations maybe used to construct new relations from given relations.

• Union: Given the relations R_1 and R_2 from a set A to set B, the union of R_1 and R_2, denoted by $R_1 \cup R_2$ is defined as a relation from A to B with the property that $(a,b) \in R_1 \cup R_2$ iff $(a,b) \in R_1$ or $(a,b) \in R_2$.

• Intersection: The intersection of R_1 and R_2, denoted by $R_1 \cap R_2$, is defined as a relation from A to B with the property that $(a,b) \in R_1 \cap R_2$ iff $(a,b) \in R_1$ and $(a,b) \in R_2$.
Evidently, $R_1 \cup R_2$ is the union of the sets R_1 and R_2 and $R_1 \cap R_2$ is the intersection of the sets R_1 and R_2 in the universal set $A \times B$.

* Complement of a relation: Given a relation R from a set A to set B the complement of R, denoted by $\bar{R}$ (or) R' is defined as a relation from A to B with the property that $(a,b) \in \bar{R}$ iff $(a,b) \notin R$. In other words $\bar{R}$ is the complement of the set R in the universal set $A \times B$.

* Difference of R and S: The difference of R and S is denoted by $R - S$ and is defined as a relation from A to B with the property that $(a,b) \in R-S$ iff $(a,b) \in R$ and $(a,b) \notin S$.

* Converse of a relation: Given a relation R from a set A to set B the converse of R is defined as a relation from B to A with the property that $(a,b) \in R$ iff $(b,a) \in$ converse of R and is denoted by R^c.

Note: If M_R is the matrix of R, then $(M_R)^T$ is the matrix of converse of R and $(R^c)^c = R$.

Consider the sets $A = \{a,b,c\}$ and $B = \{1,2,3\}$ and the relations $R = \{(a,1)(b,1)(c,2)(c,\underline{3})\}$ and $S = \{(a,1)(a,2)(b,1)(b,2)\}$ from A to B. Determine $\bar{R}$, $\bar{S}$, $R \cup S$, $R \cap S$, $R - S$, converse of R, S.

Sol: $A \times B = \{(a,1)(a,2)(a,3)(b,1)(b,2)(b,3)(c,1)(c,2)(c,3)\}$.

$\bar{R} = \{(a,2)(a,3)(b,2)(b,3)(c,1)\}$

$\bar{S} = \{(a,3)(b,3)(c,1)(c,2)(c,3)\}$.

$R \cup S = \{(a,1)(a,2)(b,1)(b,2)(c,2)(c,3)\}$

$R \cap S = \{(a,2)(b,2)\}$.

converse of $R = \{(1,a)(1,b)(2,c)(3,c)\}$

$S = \{(1,a)(2,a)(1,b)(2,b)\}$.

Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The ~~elements~~ relations R and S ~~from~~ from A to B are respectively represented by the matrices $M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Find the relations $\bar{R}$, $R \cup S$, $R \cap S$, and converse of S and their matrix representations.

Sol: $R = \{(1,1)(1,3)(2,4)(3,1)(3,2)(3,3)\}$.

(i) $\bar{R} = \{(1,2)(1,4)(2,1)(2,2)(2,3)(3,4)\}$.

$$M_{\bar{R}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$S = \{(1,1)(1,2)(1,3)(1,4)(2,4)(3,2)(3,4)\}$.

(ii) $R \cup S = \{(1,1)(1,2)(1,3)(1,4)(2,4)(3,2)(3,4)\}$.

(iii) $R \cup S = \{(1,1)(1,3)(2,4)(3,1)(3,2)(3,3)(1,2)(1,4)(3,4)\}$.

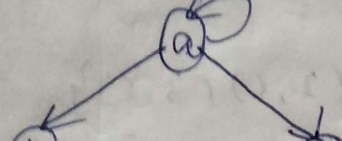
$$M_{R \cup S} \text{ from } A \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \text{ to } B \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad M_{R \cap S} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(iv) converse of $S = \{(1,1)(2,1)(3,1)(4,1)(4,2)(2,3)(4,3)\}$ which is a relation from B to A.

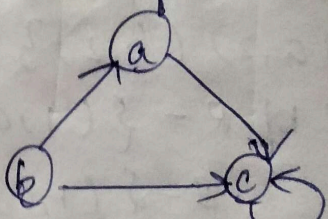
$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

(2) The digraphs of 2 relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\overline{R}$, $R \cup S$, $R \cap S$ and converse of R.

(i) R



(ii) S.



Sol: By examining the digraphs, WKT

$R = \{(a,a)(a,b)(b,c)(a,c)\}$.

$S = \{(b,a)(b,c)(a,c)(c,c)\}$.

$A \times A = \{(a,a)(a,b)(a,c)(b,a)(b,b)(b,c)(c,a)(c,b)(c,c)\}$.

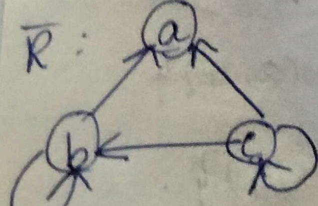
$\overline{R} = \{(c,a)(b,a)(b,b)(c,b)(c,c)\}$.

$R \cup S = \{(a,a)(a,b)(a,c)(b,c)(b,a)(c,c)\}$.

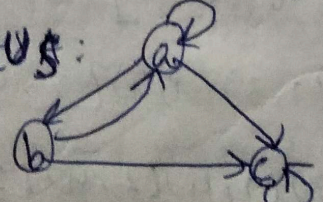
$R \cap S = \{(a,c)(b,c)\}$.

converse of $R = \{(a,a)(b,a)(c,b)(c,a)\}$.

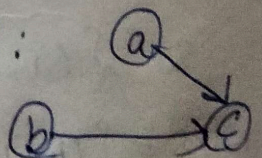
$\overline{R}$:



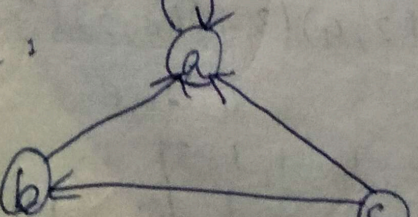
RUS:



$R \cap S$:



converse of R:



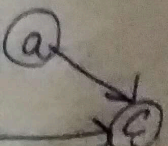
(4,3)}

on the set
digraphs

T

)(c,a)(c,b)
(c,c)}

,c)}



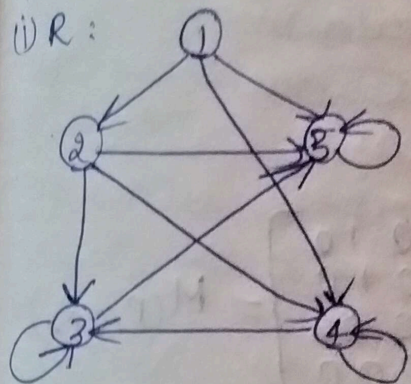
Let A = ... and R = {(1,1)(1,2)(2,3)(3,1)}
S = {(2,1)(3,1)(3,2)(3,3)} compute $\bar{R}$, $R\cap S$, $R\cup S$ and converse of S, $R-S$, $S-R$.

④ Let A = B = {a, b, c, d} R = {(a,a)(a,c)(b,c)(c,a)(d,b)(d,d)} and S = {(a,b)(b,c)(c,a)(c,b)(d,c)}. compute $M_{R\cap S}$, $M_{R\cup S}$, $M_{\text{converse of } R}$ and $M_{\bar{S}}$

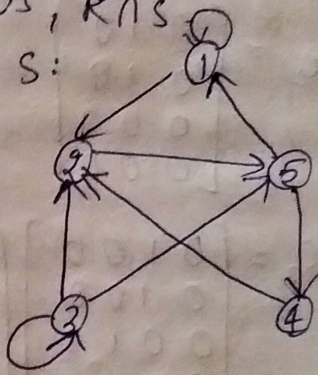
⑤ Let A = {1, 2, 3} and R and S be relations on A whose matrices are as given below. Find the matrices of $\bar{R}$, converse of R, $R\cap S$ and $R\cup S$ and $R-S$. $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

⑥ Let A = {1, 2, 3, 4, 5} and R and S be relations on A whose corresponding digraphs are as given below. Find $\bar{R}$, $R\cup S$, $R\cap S$

(i) R:



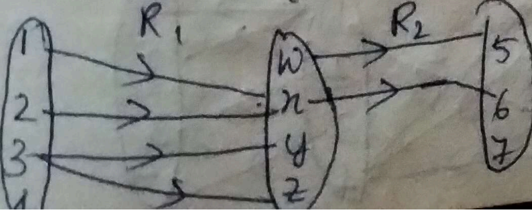
(ii) S:



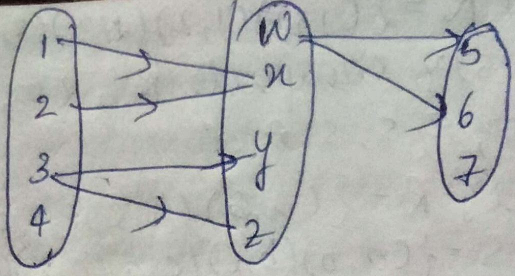
⑦ Let R and S be relations from a set A to set B. P.T. foll: (i) If $R \subseteq S$ then $R^c \subseteq S^c$ (ii) $(R\cap S)^c = R^c \cap S^c$
(iii) $(R\cup S)^c = R^c \cup S^c$

⑧ Let A = {1, 2, 3, 4} B = {w, x, y, z} C = {5, 6, 7}. Let R_1 be a relation from A to B defined by R_1 = {(1,x)(2,x)(3,y)(3,z)} and R_2, R_3 be the relations from B to C, defined by R_2 = {(w,5)(x,6)}. R_3 = {(w,5)(w,6)}. Find $R_1 \circ R_2$ & $R_1 \circ R_3$.

Sol: $R_1 \circ R_2$ = {(1,6)(2,6)} $R_1 \circ R_3$ = { } ϕ.



$R_1 \circ R_2$.



⑨ For the relations R_1 and R_2 in the above example find M_{R_1}, M_{R_2} and $M_{(R_1 o R_2)}$. Verify that $M_{(R_1 o R_2)} = M_{R_1} \cdot M_{R_2}$.

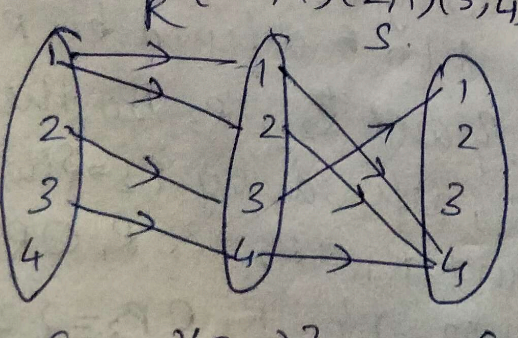
<u>Sol</u>: $M_{R_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $M_{R_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$M_{(R_1 o R_2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

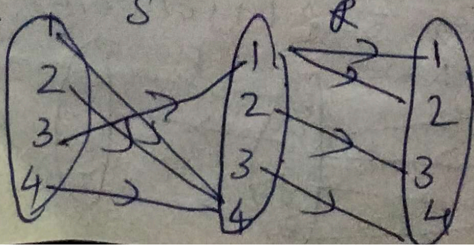
$M_{R_1} \cdot R_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M_{(R_1 o R_2)}$.

⑩ Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(1,2)(2,3)(3,4)\}$
$S = \{(3,1)(4,4)(2,4)(1,4)\}$ be relations on A. Find the relations ROS, SOR, R^2 and S^2.

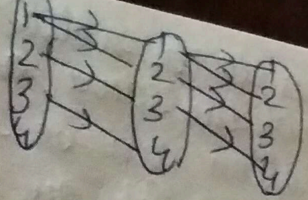
<u>Sol</u>: ROS $= \{(1,4)(2,1)(3,4)\}$.



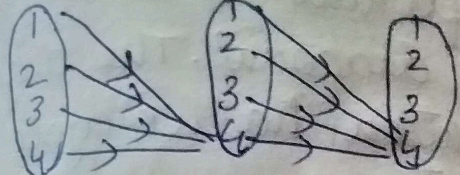
SOR: $\{(3,1)(3,2)\}$.



$R^2 = \{(1,1)(1,2)(1,3)(2,4)\}$. R^2:



$S^2 = \{(1,4)(2,4)(3,4)(4,4)\}$.



*Properties of Relations:

(1) Reflexive Relation: A relation R on a set A is said to have reflexive property (is said to be reflexive) if $(a,a) \in R \ \forall \ a \in A$. If follows that R is not reflexive if there is some $a \in A$ such that $(a,a) \notin R$.

Ex 1: Let $A = \{1,2,3\}$, $R = \{(1,1)(1,2)(2,1)(3,1)(2,2)(3,3)\}$ is a reflexive relation.

Ex 2: The relation "$\leq$" or "$=$" on Z^{+ve} is a reflexive relation because $a = a$ for every $a \in Z^{+ve}$.

NOTE: The matrix of a reflexive relation must have 1's on its main diagonal. At every vertex of the digraph of a reflexive relation, there must be a cycle of length 1. On a set A, the relation $\Delta_A = \{(a,a) / a \in A\}$. is reflexive. Furthermore, Δ_A is subset of every reflexive relation on A. The matrix of Δ_A contains 1's on the main diagonal and 0's in all other positions.

(2) Irreflexive relation: A relation on A is said to be irreflexive if $(a,a) \notin R$ for any (a,a) ie, a relation R is irreflexive if not element of A is related to itself by R. For example, Let $A = \{1,2,3\}$ and relation $R = \{(1,2)(2,3)(3,1)(1,2)\}$ is an irreflexive relation. and $R = \{(1,1)(2,2)(1,3)(3,2)(2,3)\}$ is not reflexive. The relations "$<$", "$>$", are irreflexive on the set of real nos.

It is to be noted that an irreflexive relation is not the same as a non-reflexive relation. A relation can be neither reflexive nor irreflexive. The matrix of an irreflexive relation must have 0's on its main diagonal. The digraph of an irreflexive relation has no cycle of length 1 at any vertex.

(3) Symmetric Relation: A relation R on a set A is said to be symmetric (or said to have the symmetric property), if $(b, a) \in R$ whenever $(a, b) \in R$ $\forall \{a, b\} \in A$. Ex 1: $R = \{(1,1)(1,2)(2,1)(3,1)(1,3)(2,2)(3,3)\}$ is a symmetric relation.

→ Note that R is not symmetric if there exists $a, b \in A$ and $(a, b) \in R$ and $(b, a) \notin R$. A relation which is not symmetric is called an assymmetric relation. For example if $A = \{1,2,3\}$, $R_1 = \{(1,1)(1,2)(2,1)\}$, $R_2 = \{(1,2)(2,1)(1,3)\}$ are relations of A then R_1 is symmetric but R_2 is assymmetric because $(1,3) \in R_2$ but $(3,1) \notin R_2$.

NOTE: Matrix of a symmetric relation is a symmetric matrix.
In the digraph of symmetric relation, there is an edge from A to B then there is another edge from B to A. This means that if 2 vertices are connected by an edge, they must always be connected in both directions.

(4) Anti-Symmetric relation: A relation R on a set A is said to be anti-symmetric relation if whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$. For example, the relation "$\leq$" on the set of all real nos. is an antisymmetric relation because if $a \leq b$ and $b \leq a$, then $a = b$.
Ex 2: The relation "$\subseteq$" on the power set $P(A)$ is an anti-symmetric relation.

~~If there exists~~ Note that R is an antisymmetric relation if there exists $(a,b) \in A$ and $(a,b) \in R$, $(b,a) \in R$ but $a \neq b$.

It should be emphasised that assymmetric and antisymmetric relations are not same. A relation can be both symmetric and antisymmetric. A relation can be neither symmetric nor antisymmetric. For example $R_1 = \{(1,1)(2,2)\}$ is both symmetric & antisymmetric. $R_2 = \{(1,2)(2,1)(2,3)\}$ is neither symmetric nor antisymmetric.

* **Compatibility Relation**: A relation R on a set A which is both reflexive and symmetric is called a compatibility relation on A.

Ex: Let $A = \{1,2,3\}$; $R = \{(1,1)(1,2)(2,1)(3,1)(1,3)(2,2)(3,3)\}$ is a compatibility relation on A.

$R_1 = \{(1,1)(1,2)(1,3)\}$ is not a compatibility relation on A.

* **Transitive Relation**: A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall\, a,b,c \in A$.

Example: Let $A = \{1,2,3\}$. and $R = \{(1,2)(2,3)(1,3)(3,2)(3,3)(2,2)\}$ is a transitive relation.

It follows that R is not transitive if there exists $a,b,c \in A$ such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$. Example:

$R = \{(1,2)(2,3)(1,3)(3,2)\}$ is not transitive because $(3,2) \in R$, $(2,3) \in R$ but $(3,3) \notin R$.

Ex: The relations "$\leq$" and "$\geq$" are transitive relations on the set of real nos.

* **Equivalence Relations**: A relation R on a set A is said to be an equivalence relation on A if (i) R is reflexive (ii) R is symmetric (iii) R is transitive on A.

Ex 1: A trivial ex. of an equivalence relation is the relation "=" on the set of real nos. R.

A ex. of a relation which is not an equivalence relation is ~~not~~ the relation "$<$" on R.

NOTE: Every equivalence relation is a compatibility relation but vice versa need not be true.

*① Let $A=\{1,2,3,4\}$ and $R=\{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ be a relation on A. Verify that R is an equivalence relation.

*Sol: $(a,a) \in R \ \forall \ a \in A$ $\therefore R$ is reflexive.

$(b,a) \in R$ whenever $(a,b) \in R \ \forall \ a,b \in A$

$\therefore R$ is symmetric.

~~$\therefore R$~~

whenever $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$ $\forall \ a,b,c \in A$.

② Let $A=\{1,2,3,4\}$ and $R=\{(1,1)(1,2)(2,1)(2,2)(3,1)(3,3)(1,3)(4,1)(4,4)\}$ be a relation on A. Is R an equivalence relation.

Sol: ① NOTE that $(1,1)(2,2)(3,3)(4,4)$ belongs to R

$\Rightarrow (a,a) \in R \ \forall \ a \in A$.

$\Rightarrow R$ is reflexive.

② Note that $(4,1) \in R$ and $(1,4) \notin R$

$\Rightarrow R$ is not symmetric.

$\therefore R$ is not an equivalence Relation.

③ If $A = A_1 \cup A_2 \cup A_3$ where $A_1=\{1,2\}$; $A_2=\{2,3,4\}$; $A_3=\{5\}$. Define the relation R on A by xRy if and only if x & y are in the same set A_i, $i=1,2,3$ ~~is~~ is R an equivalence relation.

Sol (1) For any $a \in A$, aRa exists, because a,a both belongs in the same set $A_i = 1,2,3$.

$\Rightarrow R$ is reflexive.

(2) For any $a,b \in R$ $(a,b) \in R$ & $(b,a) \in R$

$\Rightarrow R$ is symmetric.

(3) Suppose $(1,2) \in R$ & $(2,3) \in R$ but $(1,3) \notin R$

because 1 & 3 does not belongs any one of A_1, A_2, A_3

$\Rightarrow R$ is not transitive.

(4) A relation R on a set $A = \{a, b, c\}$ is represented by the following matrix. Determine whether R is an equivalence relation.

$$M_R = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Sol: From M_R, the relation R is:

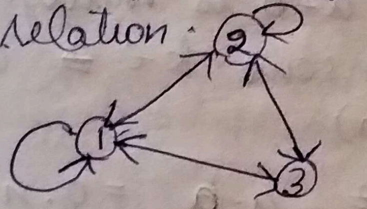
$$R = \{(a,a)(a,c)(b,b)(c,c)\}.$$

(1) $(a,a) \in R \ \forall\ a \in A \Rightarrow R$ is reflexive.

(2) $(a,c) \in R$ but $(c,a) \notin R \Rightarrow$ Not symmetric

$\therefore R$ is not an equivalence relation.

(5) The digraph of R on a set $A = \{1, 2, 3\}$ is given below. Determine whether R is an equivalence relation.



Sol: The relation R represented by digraph is:

$$R = \{(1,1)(1,2)(2,1)(2,2)(2,3)(3,2)(1,3)(3,1)\}.$$

(1) It is not reflexive since $\overset{(3,3)\notin R}{(a,a)} \notin R \ \forall\ a \in R$.

$\therefore$ It is Not an equivalence relation.

(6) Let S be the set of all non-zero integers and $A = S \times S$. On A define the relation R by $(a,b) R (c,d)$ iff $ad = bc$. S.T. R is an equivalence relation.

Sol: (1) For any $(a,a) \in A$; $(a,a) R (a,a)$ is $aa = aa$ exists. $\Rightarrow R$ is reflexive.

(2) For any (a,b) & $(c,d) \in A$, $(a,b) R (c,d)$ iff $ad = bc \Rightarrow da = cb \Rightarrow (d,c) R (a,b)$

$\Rightarrow bc = da \Rightarrow (b,a) R (d,c)$.

$\Rightarrow R$ is symmetric.

(3) $[(a,b)(c,d)] \in R$ & $[(c,d),(e,f)] \in R$.

$\Rightarrow ad = bc$ — ① & $cf = de$.

$\Rightarrow c = \frac{de}{f}$ — ②

from ① & ② : $a\cancel{d} = b\left(\frac{\cancel{d}e}{f}\right)$

$af = be$

$[(a,b)(e,f)] \in R$.

R is transitive.

$\therefore R$ is an equivalence relation.

⑦ For a fixed integer $n > 1$, P.T. the relation "congruent modulo n" is an equivalence relation.

Sol: For $(a,b) \in \mathbb{Z}$, we say that a is "congruent to b modulo n" if $a-b$ is multiple of n i.e., $a-b = kn$ for some $k \in \mathbb{Z}$ and is denoted by $a \equiv b \pmod{n}$

(1) For every $a \in \mathbb{Z}$, $a-a = 0 \cdot n \Rightarrow a \equiv a \pmod{n}$

$\therefore (a,a) \in R \Rightarrow R$ is reflexive.

(2) For any $a,b \in \mathbb{Z}$, and $(a,b) \in R$

$\Rightarrow a \equiv b \pmod{n}$

$\Rightarrow a-b = kn$

$\Rightarrow b-a = -kn$ for some $(-k) \in \mathbb{Z}$

$\Rightarrow b \equiv a \pmod{n}$

$\Rightarrow (b,a) \in R$.

$\Rightarrow R$ is symmetric.

(3) For any $a,b,c \in \mathbb{Z}$ and $(a,b) \in R$ $(b,c) \in R$

$\Rightarrow a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$

$\Rightarrow a-b = k_1 n$, $b-c = k_2 n$

consider $(a-b)+(b-c) = k_1 n + k_2 n$.

$(a-c) = (k_1+k_2)\, n$

$a \equiv c \pmod{n}$

$\Rightarrow (a,c) \in R \Rightarrow R$ is transitive

$\therefore R$ is equivalence relation.

Q) On the set of all integers, a relation R is defined by aRb (or) $(a,b) \in R$ iff $a^2 = b^2$. Verify that R is an equivalence relation.

Q) Let $A = \{1,2,3,4,5\}$. Define a relation R on $A \times A$ by $(x_1, y_1) R (x_2, y_2)$ iff $(x_1+y_1) = (x_2+y_2)$

i.e., $[(x_1, y_1), (x_2, y_2)] \in R$ iff $x_1 + y_1 = x_2 + y_2$.

P.T. R is an equivalence relation on $A \times A$.

* **Partial Order**: A relation R on a set A is said to be a partial ordering relation or a partial order on A if (i) R is reflexive.

(ii) R is anti-symmetric.

(iii) R is transitive on A.

* **Poset**: A set A with a partial order R defined on it is called a partial ordered set or ordered set (or) poset and is denoted by the pair (A, R)

Examples:

1) Ex: The most familiar poset is the set of all integers with the relation "$\leq$".

2) The set of all integers with the relation "$\geq$"

3) P.T. the divisibility relation a divides b on the set Z^{+ve} is a poset.

Sol: (1) Clearly a divides a $\Rightarrow (a,a) \in R \ \forall a \in Z^{+ve}$

$\Rightarrow R$ is reflexive.

(2) If $(a,b) \in R$ and $(b,a) \in R$ is not possible unless $a = b$. $\Rightarrow R$ is anti-symmetric.

(3) If $(a,b) \in R$ and $(b,c) \in R \Rightarrow$ a divides b and b divides c. $\Rightarrow$ a divides c. $\Rightarrow$

$\Rightarrow (a,c) \in R$. $\Rightarrow R$ is transitive.

$\therefore (Z^{+ve}, R)$ is a poset.

② S.T. the relation "$\subseteq$" defined on the powerset $P(A)$ of the set A is a partial order relation.

Sol: (1) For any element $A_1 \in P(A)$, $A_1 \subseteq A_1$

$\Rightarrow (A_1, A_1) \in R$.

$\Rightarrow R$ is reflexive.

(2) Let $A_1, A_2 \in P(A)$ be any 2 different subsets of A such that $A_1 \subseteq A_2$ but

* $A_2 \subseteq A_1$ is not possible unless $A_1 = A_2$. $\Rightarrow R$ is anti-symmetric.

(3) ~~For any~~ Let $A_1, A_2, A_3 \in P(A)$ such that $(A_1, A_2) \in R$; $(A_2, A_3) \in R$.

$\Rightarrow A_1 \subseteq A_2$ & $A_2 \subseteq A_3$.

$\Rightarrow A_1 \subseteq A_3$.

$\Rightarrow (A_1, A_3) \in R$.

$\Rightarrow R$ is transitive.

$\therefore R$ is a partial order relation & $(P(A), R)$ is a poset.

③ Verify that the relation $R = \{(1,1)(2,2)(3,3)\}$ defined on a set $A = \{1,2,3\}$ is an equivalence relation.

Sol: Clearly R is reflexive, symmetric and transitive.

$\therefore R$ is an equivalence relation.

* Equivalence class

Let R be an equivalence relation on a set A. Then the set of all those elements x of A which are related to a by R is called the equivalence class of A w.r.t. R. This equivalence class is denoted by $R(A)$ (or) $[a]$ (or) $\bar{a}$

Thus $[a] = \{x \mid (x, a) \in R\}$.

1) Consider the equivalence relation $R = \{(1,1)(1,3)(2,2)(3,1)(3,3)\}$ on $A = \{1,2,3\}$. Then find the equivalence class of $[1]$ $[2]$ $[3]$.

Sol: $[1] = \{x \mid (x,1) \in R\}$

$= \{1, 3\}$.

$[2] = \{2\}$ $[\because \{x \mid (x,2) \in R\}]$

$[3] = \{1,3\}$. $[\because \{x \mid (x,3) \in R\}]$.

2) Consider the relation R on Z defined by xRy (or) $(x,y) \in R$ iff $x-y$ is even.

(i) Verify that R is an equivalence relation.

(ii) Find equivalence class of a wrt R.

Sol: (i). For any $a \in R$, $a - a = 0 = 2k$ (even no.) $\underset{(0)}{\overset{\uparrow}{}}$

$\Rightarrow (a,a) \in R$

[Given that $R = \{(x,y) \mid x-y$ is even, that is $x-y=2k$ for some $k\}$.]

• For any $a,b \in Z$ $\ni$ $(a,b) \in R \Rightarrow a-b = 2k$

$\Rightarrow b-a = 2(-k)$

$\Rightarrow (b,a) \in R$

$\Rightarrow R$ is symmetric.

• For any $a,b,c \in Z \ni (a,b) \in R$ & $(b,c) \in R$

$\Rightarrow a-b = 2k_1$ & $b-c = 2k_2$

consider $(a-\cancel{b})+(\cancel{b}-c) = 2k_1 + 2k_2$

$a-c = 2(k_1+k_2)$

$(a,c) \in R$.

R is transitive.

$\therefore R$ is an equivalence relation.

(ii) The equivalence class of $a \in Z$ is:

$[a] = \{x \mid (x,a) \in R;\ x - a = \text{even}\}$

$= \{x \mid x - a = 2k\}$

$[a] = \{x \mid x = 2k + a \quad k = 0, \pm 1, \pm 2, \ldots\}$

$a=0 \quad [0]=\{0, 2(1)+0, 2(-1)+0, \dots\}$.

$[0]=\{0,-2,2,-4,4,\dots\}$

$[1]=\{1, 2(1)+1, 2(-1)+1, 2(2)+1, \dots\}$.

$=\{-3,-1,1,3,5\}$.

NOTE: (1) If R is an equivalence relation on a set A and $a \in A$ then $a \in [a]$

* (2) let R be an equivalence relation on a set A and let $(a,b) \in A$ (i) then aRb iff $[a]=[b]$.

(ii) If $[a] \cap [b] \neq \phi$ then $[a]=[b]$.

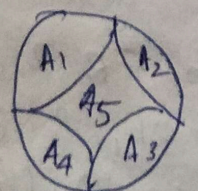
* Partitian of a set :

let A be a non-empty set. suppose there exists non-empty subsets $A_1, A_2, \dots, A_k$ of A. such that the foll. 2 conditions hold.

(i) $A = A_1 \cup A_2 \cup \dots \cup A_k$

(ii) $A_i \cap A_j = \phi \quad i \neq j$

then the set $P=\{A_1, A_2, \dots, A_k\}$ is called a partitian of A. Also $A_1, A_2, \dots, A_k$ are called the blocks (or) cells of the partition. A partition of set A with 5 blocks is as shown in the figure:



A.

NOTE: If A is a non-empty set then :

(i) any equivalence relation R induces a partition of A.

(ii) Any partition of A gives rise to an equivalence relation R on A.

1) For the equivalence relation $R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ defined on the set $A = \{1,2,3,4\}$. Determine the partition induced.

Sol: The equivalences classes corresponding to $\{1,2,3,4\}$ are $[1] = \{x/(x,1) \in R\} = \{1,2\}$

$[2] = \{x/(x,2) \in R\} = \{1,2\}$.

$[3] = \{x/(x,3) \in R\} = \{3,4\}$.

$[4] = \{x/(x,4) \in R\} = \{3,4\}$.

Note that ~~equivalen~~ $[1] = [2]$

and $[3] = [4]$.

$\therefore$ The partition of A is $P = \{[1],[3]\}$

$= \{\{1,2\},\{3,4\}\}$.

2) Consider the set $A = \{1,2,3,4,5\}$ and the equivalence relation $R = \{(1,1)(2,2)(2,3)(3,2)(3,3)(4,4)(4,5)(5,4)(5,5)\}$. Find the partition on A induced by R.

Sol:
$[1] = \{1\}$

$[2] = \{2,3\}$

$[3] = \{3,2\}$

$[4] = \{4,5\}$

$[5] = \{4,5\}$.

Note that $[4] = [5]$ and $[2] = [3]$.

$\therefore$ The partition of A is $P = \{[1],[2],[4]\}$.

$= \{\{1\},\{2,3\},\{4,5\}\}$.

3) Let $A = \{a,b,c,d,e\}$. Consider the partition $P = \{\{a,b\},\{c,d\},\{e\}\}$ of A. Find the equivalence relation inducing in partition.

Sol: (i) since a, b belongs to the block $\{a,b\}$

$\Rightarrow (a,a) \in R \quad (a,b) \in R \quad (b,a) \in R \quad (b,b) \in R$

$c, d \in \{c,d\} \Rightarrow (c,c)(c,d)(d,c)(d,d) \in R$.

since $e \in \{e\} \Rightarrow (e,e) \in R$.

∴ the required equivalence relation R is
$\{(a,a)(a,b)(b,c)(b,b)(c,c)(c,d)(d,c)(d,d)(e,e)\}$

② Let $A = \{1,2,3,4,5,6,7\}$ and R be the equivalence relation on A that induces the partition $P = \{\{1,2\}\{3\}\{4,5,7\}\{6\}\}$. Find R.

Sol: $R = \{(1,1)(1,2)(2,1)(2,2)(3,3)(4,5)(4,7)(4,4)$
$(5,5)(5,4)(5,7)(7,7)(7,4)(7,5)(6,6)\}$.

③ On the set Z of all integers a relation R is defined by aRb iff $a^2 = b^2$. Verify that R is an equivalence relation. Determine the partition induced by this relation.

Sol: $[0] = \{x/(x,0) \in R;\ x^2 = 0^2 \Rightarrow x = 0\}$.
$[0] = \{0\}$.
for any $a \in Z$ $[a] = \{x/(x,a) R;\ x^2 = a^2 \Rightarrow x = \pm a\}$.
$[a] = \{-a, a\}$.
The partition of $Z = \{[0],[a]\}$ where $a \in Z^+$

* ~~Tatal~~ Total Order (or) Linear Order: Let R be a partial order on a set A then R is called a total order or linear order on A if $\forall (x,y) \in A$ either $(x,y) \in R$ (or) $(y,x) \in R$. In this case, the poset (A, R) is called a totally ordered (or) linearly order set (or) a chain. For example, the relation $(I_+, \leq)$ is a partial as well as linear ordered on the set of positive integers.

NOTE: Every total order is a partial order but viceversa is not true. For example, the poset $(P(A), \subseteq)$ is a partial order but not a total order because $\{1\},\{2\} \in A$, $\{1\} \nsubseteq \{2\}$ (or) $\{2\} \nsubseteq \{1\}$.

* Hasse diagram: Let (A, R) be a poset and $(x,y) \in A$, we say that y covers x if $(x,y) \in R$

and $x \neq y$ and there are no elements $z \in A \ni$ $(x, z) \in R$ $(z, y) \in R$. Sometimes the term immediate predecessor is also used in place of cover.

A diagram that is used to describe partial order relation associated with a set is called Hasse diagram.

In such a digram, each element is represented by a dot (or) a small circle. The dot represented by x is drawn below the circle for y if $(x, y) \in R$, and line is drawn between x & y if y covers x. If y does not cover x directly then x & y are not connected directly (a single line). However they are connected through 1 or more elements of A. It is possible to obtain the set of ordered pairs in R from such a diagram.

A hasse diagram of a poset (A, R) (or) $(A, \leq)$ of finite elements can also be obtained from the digraph of the poset (A, R). To do this we use the foll. steps:

(1) In the ~~digraph~~ hasse diagram we place vertex A above vertex B if $(a, b) \in R$.

(2) We delete all loops from the digraph

(3) We delete all the directed edges that are implied by the transitive property.

for example: suppose $(a, b) \in R$ $(b, c) \in R$ then $(a, c) \in R$ we omit edge from A to C.

(4) We omit the arrow signs from the directed edges.

① Let $A = \{1,2,3\}$ and $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A\}$. Let R be the relation represent the set inclusion. P.T. $(P(A), R)$ is a poset. Also draw the hasse diagram.

Sol: Note that for any $A_1 \in P(A)$; $A_1 \subseteq A_1$.

$\Rightarrow (A_1, A_1) \in R$.

$\therefore R$ is reflexive.

For any $A_1, A_2 \in P(A) \ni A_1 \subseteq A_2$ & $A_2 \subseteq A_1$

$\Rightarrow A_1 = A_2$.

i.e., if $(A_1 A_2) \in R$ & $(A_2, A_1) \in R$

$\Rightarrow A_1 = A_2$.

$\therefore R$ is anti symmetric.

For any $A_1\ A_2\ A_3 \in P(A) \ni$

$A_1 \subseteq A_2$ & $A_2 \subseteq A_3 \Rightarrow A_1 \subseteq A_3$.

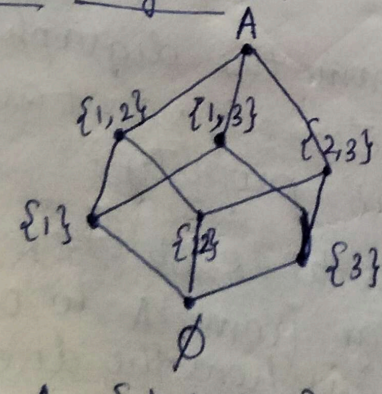
$\Rightarrow (A_1 A_2) \in R\ (A_2 A_3) \in R$

$\Rightarrow (A_1 A_3) \in R$.

$\therefore R$ is transitive.

$(P(A), R)$ is a poset.

Hasse Diagram:



② Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(1,2)(2,2)(2,4)(1,3)(3,3)(4,4)(1,4)\}$. Verify that R is partial order on A. Also draw the hasse diagram for R.

Sol: For every $a \in A$, $(a,a) \in R$

$\Rightarrow R$ is reflexive.

1}, {2}, {3}, {1,2}
e relation
T. (P(A), R) is
iagram.
$_1 \subseteq A_1$.

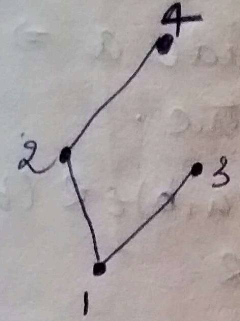
& $A_2 \subseteq A_1$

$\in R$

$_1 \subseteq A_3$.

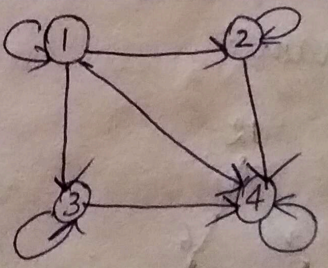
) (2,2) (2,4) (1,3)
order on A.

R does not contain any ordered pair $(a,b) \in R$, $(b,a) \in R$ & $(a \neq b)$.
$\Rightarrow$ R is anti symmetric.
finally for any $a,b,c \in A$ & $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) R$.
$\Rightarrow$ R is transitive
$\therefore$ R is partial order.
and A, R is a poset.

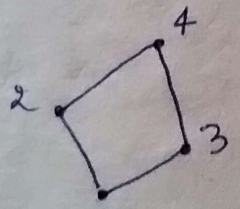


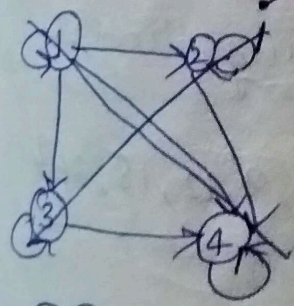
Q) A partial order R on a set A = {1, 2, 3, 4} represented by the foll. digraph. Draw the hasse diagram for all.

(i)

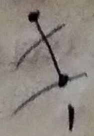


Sol:

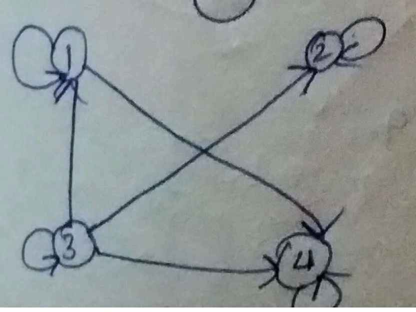




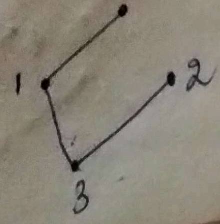
~~Hasse~~ diagram:



(ii)



Hasse diagram:



⑤ Let $A = \{1, 2, 3, 4, 6, 12\}$. On A define the relation R aRb iff a divides b. P.T. R is poset on A. Draw the hasse diagram for this relation.

Sol: $R = \{(1,1)(1,2)(1,3)(1,4)(1,6)(1,12)(2,2)(2,4)(3,3)(4,4)(2,6)(2,12)(3,6)(3,12)(4,12)(6,12)(6,6)(12,12)\}$

For any $a \in A$, a divides a $\Rightarrow (a,a) \in R$.

$\Rightarrow$ R is reflexive.

For any $(a, b) \in A \ni (a,b) \in R \;\&\; (b,a) \in R$.

a divides b & b divides a $\Rightarrow a = b$.

$\Rightarrow$ R is anti-symmetric.

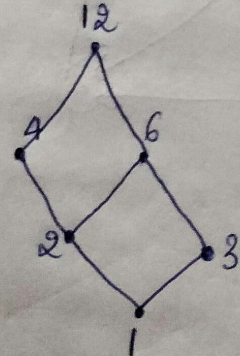
For any $a, b, c \in A \ni (a,b) \in R \; (b,c) \in R$

$\Rightarrow (a,c) \in R$.

$\Rightarrow$ R is transitive.

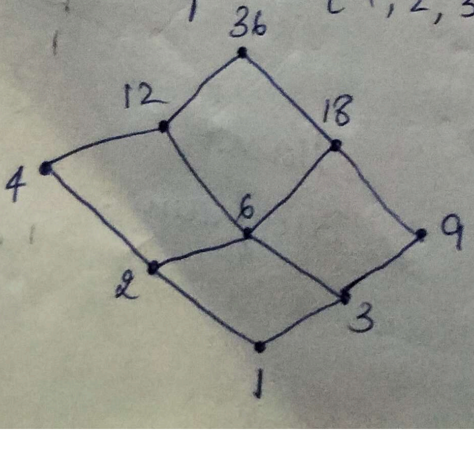
∴ R is a poset.

Hasse diagram:



⑥ Draw the hasse diagram representing the +ve divisors of 36.

Sol: +ve divisors of $36 = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



...ine the relation
...s poset on A.
...relation.

...12) (2,2) (2,4)
...) (6,12) (6,6) (12,12)}
...,a) $\in R$.

...(b,a) $\in R$.
a=b.

...,c) $\in R$

the +ve

...2,18,36}

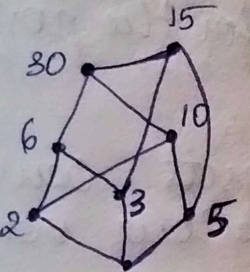
...S.T. the set of all +ve integers is not a totally ordered by the relation of divisibility.

...For any set A to be totally ordered by a partial order R, we should have either $(a,b) \in R$ or $(b,a) \in R$ $\forall a,b \in A$. In set of Z^{+ve}, we have 2 elements $a=2$, $b=3$ $\ni$ $(2,3) \notin R$ and $(3,2) \notin R$, $\therefore$ Z^{+ve} is not a totally ordered by the divisibility relation.

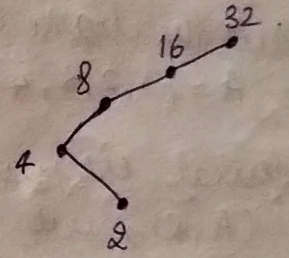
In the foll. cases, consider the partial order of divisibility on the set A. Draw the hasse diagram for the poset and determine whether the poset is totally ordered or not.

(i) $A=\{1,2,3,5,6,10,15,30\}$ (ii) $A=\{2,4,8,16,32\}$

Sol: (i)



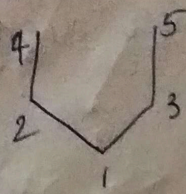
(ii)



In this diagram there is no relation between 2 & 3 and 3 & 5 so it is a partial order but not a total order.

It is a total order

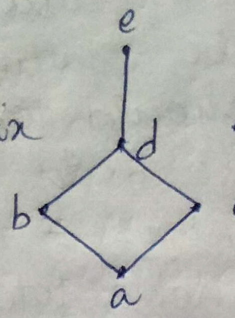
② Determine the matrix of partial order whose hasse diagram is given below:



Sol: $R=\{(1,1)(1,2)(2,4)(1,3)(3,5)(1,4)(1,5)(2,2)(3,3)(4,4)(5,5)\}$

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

⑩ For $A = \{a, b, c, d, e\}$, the hasse diagram for the poset (A, R) is . Find.



(i) Relation matrix for R.

(ii) Construct the digraph for R.

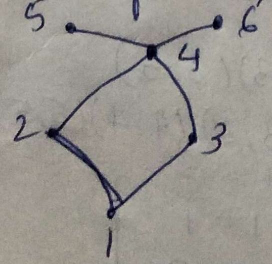
⑪ Draw the hasse diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$. whose matrix is given below. $M_R =$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

⑫ Draw the hasse diagram for the poset $(P(A), \subseteq)$ where $A = \{1, 2, 3, 4\}$.

* Extremal elements in posets. Consider a poset (A, R); we define below some special elements called extremal elements that may exist in A.

* Maximal element: A element $a \in A$ is called maximal element of A if there exists no element $x \neq a$ in $A \ni (a, x) \in R$. This means that a is a maximal element of A, in the hasse diagram of R, no edges starts at a



(i)

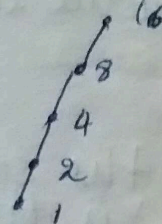
(i) maximal elements are 5, 6.

* Minimal element: An element $a \in A$ is called the minimal element of A if there exits no element $x \neq a$ in $A \ni (x, a) \in R$

This means that a is minimal element of A in the hasse diagram of R, no edge terminates at a.
In fig (i), minimal element is 1.

*Greatest element: An element $a \in A$ is called a greatest element of A if $(x, a) \in R \; \forall \; x \in A$.

Ex:



Greatest element = 16.

Least element = 1.

*Least element: A element $a \in A$ is called a least element of A if $(a, x) \in R \; \forall \; x \in A$.

*Upperbound: An element $a \in A$ is said to be an upper bound of subset B of A if $(x, a) \in R \; \forall \; x \in B$. In fig(i) for subset $B = \{2, 3, 4\}$, the upper bounds are $\{4, 5, 6\}$.

*Lower bound: An element $a \in A$ is called lower bound of a subset $B(A)$ if $(a, x) \in R$. $\forall \; x \in B$. $\{1\}$ is lower bound in fig (i).

*Lattice: Lattice is a partially ordered set (A, R) or (A, R) in which every pair of elements $(a, b) \in A$ has a greatest lowerbound (GLB) and least upperbound (LUB).
- If (A, R) is a lattice, the LUB of 2 elements subset $\{a, b\} \subseteq A$ and read as $a \vee b$ and a greatest LB (GLB) of the subset is denoted by $a \wedge b$ and read as a meed b.
- The GLB of a subset $\{a, b\} \subseteq A$ will be denoted $a * b$ & the LUB is denoted by $a \oplus b$.
- $a * b$ is known as the meed or product of a & b
- $a \oplus b$ is known as the join or sum of a & b.
- Other symbols such as 'dot' & '+' plus are also used to denote the meed & join of the 2 elements respectively.

① A totally ordered set is a trivially a lattice, but not all partially ordered sets are lattices.

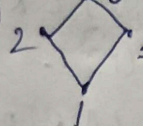
Ex 1: Let ~~I~~ denote the set of all Z^{+ve} and let D denote the relation of division such that $a D b$ iff (a divides b) then (I, D) is a lattice in which

$$a \vee b = LCM(a, b)$$
$$a \wedge b = GCD(a, b)$$

Ex 2: Let n be a Z^{+ve} and S_n be the set of all divisors of n. For example, S_6 is divisors of 6. S_8 is divisors of 8 = {1, 2, 4, 8}, S_{10} = {1, 2, 5, 10}, S_{20} = {1, 2, 4, 5, 10, 20}, S_{24} = {1, 2, 3, 4, 6, 8, 12, 24}.

Let D denote the relation of division such that for any $(a, b) \in S_n$ aDb iff a divides b are all lattices.

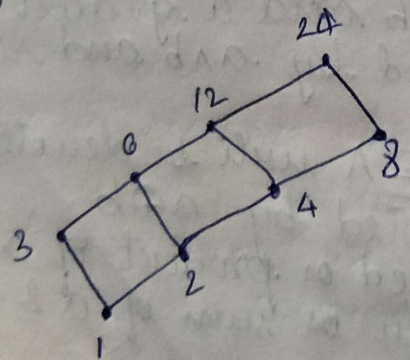
① Check which of the foll. hasse diagrams are lattices.

(i) 

a lattices.

$\langle S_6, D \rangle$

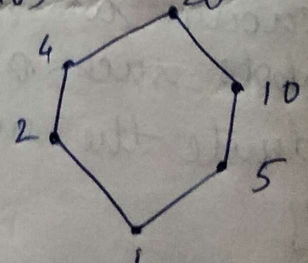
Here ~~GLB~~ & every pair elements has a GLB & LUB

(ii) 

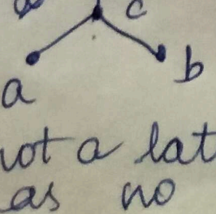
$\langle S_{24}, D \rangle$

(iii) 

$\langle S_8, D \rangle$

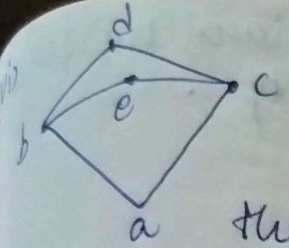
(iv) 

$\langle S_{20}, D \rangle$.

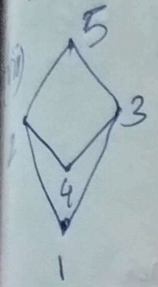
(v) 

(v) Is not a lattice because the pair of elements {e, f} has no upper bound and {a, b} has no LB.

vii)

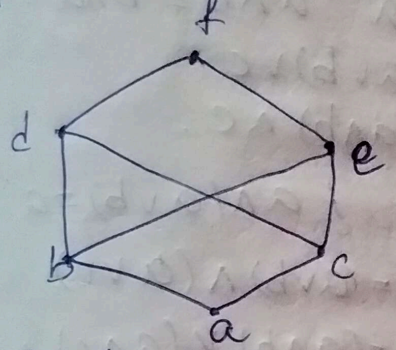


$\{b, c\}$ has no least upper bound even though it has upper bounds.
The UB of $\{b, c\}$ are e & d but there is no relation between e & d.
$\therefore$ Not a lattice.



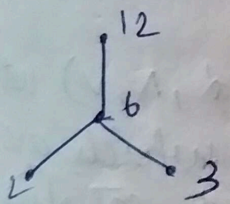
$\{2, 3\}$ has no GLB even though it has 2 lower bounds $\{2, 4\}$

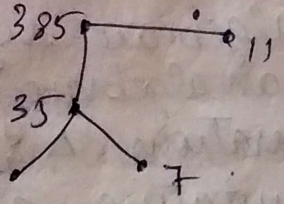
viii)



$\{b, e\}$ $\{d, c\}$ has no GLB & no LUB.
$\therefore$ Not a lattice.

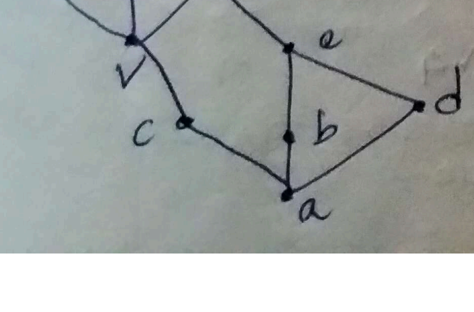
Q Let $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$ and a partial order R on A represented by the hasse diagram. Check whether it is a lattice or not.





Sol The given A, R is not a lattice because we have 2 elements 3, 35 in A which have no GLB and no LUB.

Q Let $A = \{a, b, c, d, e, v, w, x, y, z\}$ Consider the poset (A, R) whose hasse diagram is:



Sol : By examining the hasse diagram, we find that every 2 element subset of .

$\therefore (A, R)$ is a lattice.

* Some Properties of Lattice :

Let L, R be a lattice, then for any $a, b, c \in L$ the foll. are true,

(i) Idempotent : $a \wedge a = a$; $a \vee a = a$.

(ii) Commutative : $a \vee b = b \vee a$ $\quad a \wedge b = b \wedge a$.

(iii) Associative : $a \vee (b \vee c) = (a \vee b) \vee c$

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

(iv) Absorption : $a \vee (a \wedge b) = a$; $a \wedge (a \vee b) = a$.

(v) distributive : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

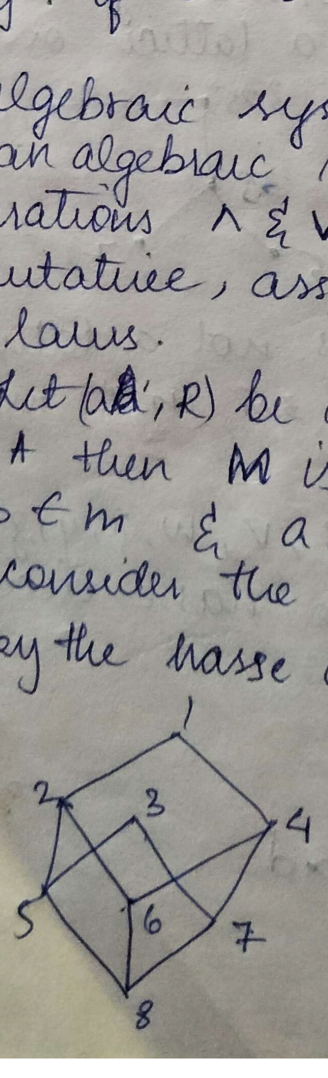
(vi) Isotonicity : if $b \leq c \Rightarrow \begin{cases} b \vee a \leq c \vee a \\ b \wedge a \leq c \wedge a \end{cases}$.

* Lattice as algebraic systems :

A lattice is an algebraic system $(L, \wedge, \vee)$ with 2 binary operations $\wedge$ & $\vee$ on L which are both commutative, associative & satisfies absorption laws.

* Sub lattice : Let (A, R) be a lattice and M be a subset of A then M is called a sublattice of A if $a \vee b \in M$ & $a \wedge b \in M$ $\forall a, b \in M$.

For example, consider the lattice L, R are represented by the hasse diagram .



ients Clearly $L = \{1,2,3,4,5,6,7,8\}$. Consider the foll. subsets:

$M_1 = \{1,2,4,6\}$.
$M_2 = \{3,5,7,8\}$
$M_3 = \{1,2,4,8\}$.

By examining the hasse diagram (M_1, R) & (M_2, R) are sub lattices of (A, R). but (M_3, R) is not a sub lattice. because $2 \wedge 4 = 6 \notin M_3$

*Bounded Lattice: A lattice L, R is said to be bounded if it has a greatest element and a least element. In a bounded lattice, a greatest element is denoted by 1, & least element is denoted by zero '0'.

Ex 1: In the lattice ~~for~~ $(P(S), \subseteq)$ a non-empty set S, the set S itself a greatest element and the null set ϕ is a least element. Therefore $(P(S), \subseteq)$ is a bounded lattice.

Ex 2: The lattice $(Z^+, /)$ is not a bounded lattice because it has least element as 1 and there is no greatest element. Therefore $(Z^+, /)$ is not a bounded lattice. Ily, $(Z^+, \leq)$ is not a bounded lattice because neither greatest nor least element exist.

*Complement lattice: Let L be a bounded lattice with greatest element 1 & least element 0. For a chosen element $a(L)$ if there exists an element $a' \in L$ such that $a \vee a' = 1$, $a \wedge a' = 0$, then a' is called a complement of a in L.

*A lattice L is said to be a complemented lattice if L is bounded & every element in L has a complemented in L.

Definition of Functions: Let A & B be two non-empty sets. Then a function or a mapping $f: A \to B$ is a relation from $A \to B$ $\ni$ for each a in A there is a unique b in B $\ni$ $(a,b) \in f$. Then we write ~~$B = f(A)$~~ $b = f(a)$, $a = f^{-1}(b)$. Here b is called the image of ~~A~~ a and a is called the pre image of b under f. The element a is also called an argument of the function f and b is called the value of function f for the argument a.

NOTE: Every function is a relation but a relation need not be a function.

$f: A \to B$.

$\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}a\\b\\c\end{pmatrix}$ $f = \{(1,a)(1,b)(2,c)(3,c)\}$.

① has no unique image in B.

NOTE: Every function is a relation. But every relation is not a function.

① Let $A = \{1,2,3\}$ and $B = \{-1,0\}$ and a relation $f: A \to B$ is defined by $R = \{(1,-1)(1,0)(2,-1)(3,0)\}$. Is R a function from A to B | $R: A \to B$.

$\begin{pmatrix}1\\2\\3\end{pmatrix}$ $\begin{pmatrix}-1\\0\end{pmatrix}$ For 1 we have 2 images it is not a function.

② Let $A = \{0, \pm 1, \pm 2, 3\}$ consider the function $f: A \to R$ defined by $f(x) = x^3 - 2x^2 + 3x + 1$ $\forall x \in A$. Find range of f.

Sol: Let $A = \{0, \pm 1, \pm 2, 3\}$.

$f: A \to R$, $f(x) = x^3 - 2x^2 + 3x + 1$

Range of f: $f(0) = 1$; $f(1) = 3$; $f(-1) = -5$.

$f(2) = 7$; $f(-2) = -2$; $f(3) = 19$.

$f = \{-2, 1, -5, 1, 3, 7, 19\}$.

③ Let A & B be 2 finite sets with order of $A = M$ and order of $B = N$. Then the possible no. of functions from A to B is n^m.

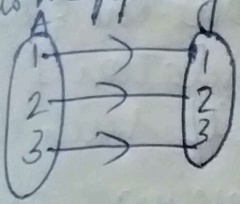
Ex: If there 2187 functions from A to B and order of $B = 3$. Then what is order of A.

Sol: Let order of A $O(A) = m$ then no. of possible functions from A to B $= 3^m = 2187 = 3^7 \Rightarrow \boxed{m = 7}$

* Types of Functions :

(1) Identity function : A function $f : A \to A$ such that $f(a) = a \ \forall \ a \in A$ is called identity function of A

1 is mapping to 1, 2 to 2 & 3 to 3.



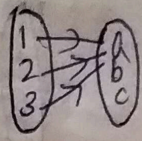
$f : A \to A$

Let $A = \{1, 2, 3\}$ then $I_A = \{(1,1)(2,2)(3,3)\}$.

(2) Constant function : A function $f : A \to B$ such that $\forall \ a \in A$ $f(a) = c$ where c is a constant of B is called a constant function.

Ex : Let $A = \{1, 2, 3\}$ $B = \{a, b, c\}$.

Then func. $f : A \to B$ is defined by

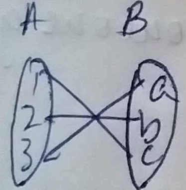


(3) One-One function / Injective function :

A function $f : A \to B$ is said to be one-one if distinct elements in A have distinct images in B i.e. In the form of if $f(a) = f(b) \Rightarrow a = b$.

Suppose of $a \neq b \Rightarrow f(a) \neq f(b)$.

Ex : Let $A = \{1, 2, 3\}$ & $B = \{a, b, c\}$ defined by.



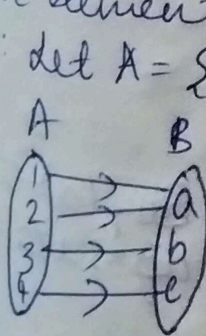
$\{(1, c)(2, b)(3, a)\}$.

$\therefore f$ is one-one function.

(4) Onto function / Surjective function :

A function $f : A \to B$ is said to be onto if every element of B has a preimage in A. i.e. $\forall \ b \in B \ \exists$ an element $a \in A \ni f(a) = b$

Ex : Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.



$f = \{(1, a)(2, a)(3, b)(4, c)\}$ which is an onto function.

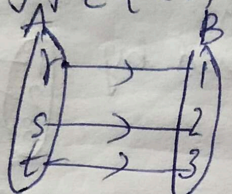
(5) Bijective Function : If the function is both one-one & onto function, f is said to be bijective function

NOTE : If $f : A \to B$ is a function and order of A $O(A) = M$ & $O(B) = n$

(1) f is one-one when $m \le n$
(2) f is onto when $n \le m$.
(3) f is bijective when $n = m$.

(6) Inverse Function: Let $f: A \to B$ is one-one & onto and $b \in B$ be an element then the relation $f: B \to A$ is called inverse function of f, if such element of $b \in B$ there exists a unique preimage $a \in A \ni f^{-1}(b)$ ...

* Ex: Suppose let $A = \{r, s, t\}$ $B = \{1, 2, 3\}$. definition $f: A \to B$ by $f c \{(r,1)(s,2)(t,3)\}$.



Clearly f is one-one & also onto.
$\therefore$ f is bijective and f^{-1} is defined as.
$f^{-1} = \{(1,r)(2,s)(3,t)\}$.
$f^{-1}(1) = r$, $f^{-1}(2) = s$; $f^{-1}(3) = t$.

① The function $f: R \to R$ and from $g: R \to R$ defined by (i) $f(x) = 3x + 7 \ \forall x \in R$
(ii) $g(x) = x(x^3 - 1) \ \forall x \in R$. Verify f is one one g is not one-one.

Sol: Consider $f(a) = f(b)$.
$\Rightarrow 3a + \cancel{7} = 3b + \cancel{7}$
$3a = 3b$
$a = b$
$\therefore f(a) = f(b) \Rightarrow a = b$. Hence, it is one-one.

NOTE: $g(0) = 0$
$g(1) = 0$
$0 \ne 1$ but $g(0) = g(1)$.
g is not a one-one function.

② The function ~~Z~~, $f: Z^+ \to Z^+$ defined by $f(a) = a^2$
Check if f is one-one and onto using suitable exp.

Sol: Consider $f(a) = f(b)$.
$a^2 = b^2 \Rightarrow a = b$ $(a, b \in Z^+)$
$\therefore f$ is one-one.
For the elements $\{2, 3, 5, 6\}$ --- in co-domain there exists no elements in the domain. $\therefore f$ is not onto.

1) If $f: R \to R$ defined by $f(x) = x+1$, check if f is one-one and onto and find f^{-1}.

One-One funct.

consider $f(a) = f(b)$.

$a + 1 = b + 1$

$a = b$.

$f(a) = f(b) \Rightarrow a = b$.

$\therefore f$ is one-one.

Onto funct

Let $y \in B \ni f(x) = y$.

$x + 1 = y \Rightarrow y - 1 = x$.

$y \in R \; \exists \; x = y - 1 \in R$.

f is onto.

conclusion: f is one-one & onto. f is bijective.

Inverse: Let $y \in R \ni f(y) = x$.

$f(x) = y$.

$x + 1 = y$.

$x = y - 1 \Rightarrow f^{-1}(y) = y - 1$

$f^{-1}(x) = x - 1$.

2) Find inverse of the foll. functions

(i) $f(x) = \dfrac{10}{\sqrt[5]{7-3x}}$.

Sol: Let $y \in R \ni f^{-1}(y) = x$.

$\Rightarrow f(x) = y$.

$\Rightarrow \dfrac{10}{\sqrt[5]{7-3x}} = y \Rightarrow \dfrac{10}{y} = \sqrt[5]{7-3x}$.

taking $(\;)^5$ on b.s.

$\left(\dfrac{10}{y}\right)^5 = \left(\sqrt[5]{7-3x}\right)^5$

$\left(\dfrac{10}{y}\right)^5 = 7 - 3x$.

$3x = 7 - \dfrac{10^5}{y^5}$

$x = \dfrac{1}{3}\left(\dfrac{7y^5 - 10^5}{y^5}\right)$

$f^{-1}(y) = x = \dfrac{1}{3y^5}(7y^5 - 10^5)$.

(ii) $f(x) = 4e^{6x+2}$

Sol: Let $y \in R \ni f^{-1}(y) = x$.

$f(x) = y \Rightarrow 4e^{6x+2} = y \Rightarrow e^{6x+2} = \dfrac{y}{4}$

$$\log_e e^{6x+2} = \log\left(\frac{y}{4}\right)$$

$$6x+2 = \log\left(\frac{y}{4}\right)$$

$$6x = \log\left(\frac{y}{4}\right) - 2$$

$$x = \frac{1}{6}\left(\log\left(\frac{y}{4}\right) - 2\right)$$

$$f^{-1}(y) = \frac{1}{6}\left[\log\left(\frac{y}{4}\right) - 2\right]$$

$$f^{-1}(x) = \frac{1}{6}\left[\log\left(\frac{x}{4}\right) - 2\right]$$

* (iii) $f(x) = \frac{x+1}{x}$

Let $y \in R \Rightarrow f^{-1}(y) = x$

$$f(x) = y$$

$$\frac{x+1}{x} = y \Rightarrow x+1 = yx \Rightarrow yx - x = 1$$

$$x(y-1) = 1$$

$$x = \frac{1}{y-1}$$

$$f^{-1}(y) = \frac{1}{y-1} \quad ; \quad f^{-1}(x) = \frac{1}{x-1}$$

(iv) $f(x) = x^3+2$

Let $y \in R \Rightarrow f^{-1}(y) = x$

$$f(x) = y$$

$$x^3 + 2 = y$$

$$x^3 = y-2$$

$$x = \sqrt[3]{y-2}$$

$$f^{-1}(y) = \sqrt[3]{y-2}$$

$$f^{-1}(x) = \sqrt[3]{x-2}$$

① Explain whether the foll functions are bijective or not. Find the inverse ~~of~~:

(i) $f(x) = 4x - 12$ where A = set of real nos.

Sol: One-One: Consider $f(a) = f(b)$

$$4a + \cancel{x} = 4b + \cancel{x}$$

$$a = b$$

$f(a) = f(b) \Rightarrow a = b$.

$\therefore f$ is one-one.

Onto:
Let $y \in A$ (codomain) & $x \in A$ (domain) be the preimage of y in A $\ni f(x) = y$.

$$4x+2 = y$$
$$4x = y-2$$
$$x = \frac{y-2}{4}$$

i.e. for every $y \in A$ in the codomain $\exists$ an element of the form $\frac{y-2}{4}$ in domain.

$\therefore f$ is onto.
$\rightarrow f$ is bijective

Inverse of f:
Let $y \in A$ (codomain) $\ni f^{-1}(y) = x$.

$$\Rightarrow f(x) = y.$$
$$4x+2 = y.$$
$$x = \frac{y-2}{4}.$$
$$f^{-1}(y) = x = \frac{y-2}{4}$$
$$f^{-1}(y) = \frac{y-2}{4}$$
$$f^{-1}(x) = \frac{x-2}{4}$$

(ii) $f(x) = 3 + \frac{1}{x}$ where A is set on non-zero real nos.

1: one-one: Consider $f(a) = f(b)$.

$$\cancel{3} + \frac{1}{a} = \cancel{3} + \frac{1}{b}$$
$$a = b.$$

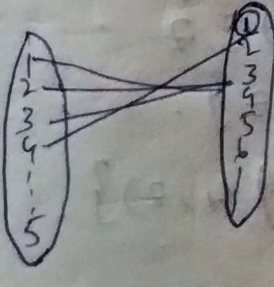
$f(a) = f(b) \Rightarrow a = b$ $\therefore f$ is one-one.

Onto: Let $y \in A$ & $x \in A$ be the preimage of y in A

$$f(x) = y$$
$$3 + \frac{1}{x} = y.$$
$$\frac{1}{x} = y-3$$
$$x = \frac{1}{y-3}.$$

$\frac{1}{y-3} \notin A$ for $y = 1, 2, 3,$ ($\because A$ is a set of real nos).

$\therefore f$ is not onto.



For elements $y = \{1, 2, 3\}$ in the codomain there exist no preimages in the domain.
$\therefore f$ is not onto function

① $f(x) = (2x+3) \pmod 7 \rightarrow$ contain 0 to 6.

Sol: $f(x) = (2x+3) \bmod 7 =$

$f(0) = (3) \bmod 7 =$

$f(1) = (2+3) \bmod 7 = 5$

$f(2) = (4+3) \bmod 7 = 0$

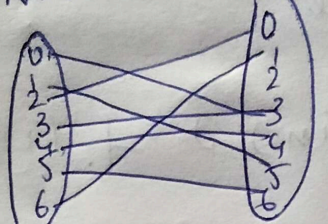
$f(3) = (9) \bmod 7 = 2$

$f(4) = (11) \bmod 7 = 4$

$f(5) = (13) \bmod 7 = 6$

$f(6) = (15) \bmod 7 = 1$

* N-1

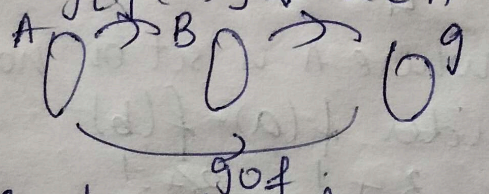


$f = \{(0,3)(1,5)(2,0)(3,2)$
$(4,4)(5,6)(6,1)\}$.

Clearly f is one-one & onto $\therefore f$ is bijective

$f^{-1} = \{(3,0)(5,1)(0,2)(2,3)(4,4)(6,5)(1,6)\}$.

*Composition of two functions: Consider 3 non-empty sets A, B, C and the functions $f: A \rightarrow B$ & $g: B \rightarrow C$. The composition of these 2 functions is defined as $g o f: A \rightarrow C$ with $g(of)(a) = g[f(a)] \ \forall\, a \in A$



NOTE: for a function $f: A \rightarrow A$ fof denoted by f^2 & $fofof$ is denoted by f^3 and so on.

① Let $A = \{1,2,3,4\}$ & $B = \{a,b,c\}$ $C = \{w\, x\, y\, z\}$ & the functions f, g are defined as $f: A \rightarrow B$ & $g: B \rightarrow C$ are given by $f = \{(1,a)(2,a)(3,b)(4,c)\}$.

$g = \{(a,x)\ (b,y)(c,z)\}$.

Sol: $gof\,(1) = g[f(1)] = g(a) = x$.

$gof\,(2) = g[f(2)] = g(a) = x$

$gof\,(3) = g[f(3)] = g(b) = y$

$gof\,(4) = g[f(4)] = g(c) = z$.

$\therefore$ Range of $gof = \{x, y, z\}$.

$gof = \{(1,x)(2,x)(3,y\ 4,z)\}$.

Q. $f: R \to R$ is defined by $f(x) = \begin{cases} 3x-5 & \text{for } x>0 \\ -3x+1 & \text{for } x \le 0 \end{cases}$

find (i) $f(0)$, $f(-1)$, $f(5/3)$, $f(-5/3)$

(ii) $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$

(iii) $f^{-1}([-5,5])$, $f^{-1}[-6,5]$.

Sol (i) $f(0) = -3(0)+1 = 1$

$f(-1) = -3(-1)+1 = 4$.

$f(5/3) = 3(\frac{5}{3}) - 5 = 0$

$f(-5/3) = -3(-5/3)+1 = 6$.

(ii) Let $a \in R \ni f^{-1}(0) = a$

$\Rightarrow f(a) = 0$

(a) $3a-5=0$

$a = 5/3 > 0$ which exists.

(b) $-3a+1=0$

$a = \frac{1}{3} \not\le 0$ does not exist

$f^{-1}(0) = 5/3$.

$f^{-1}(1)$: Let $a \in R \ni f^{-1}(1) = a$.

$f(a) = 1$

• $3a-5=1$

$3a = 6$

$a = 2 > 0$ exists.

• $-3a+1=1$

$-3a = 0$

$a = 0 \le$ exists

$\therefore f^{-1}(1) = \{0, 2\}$

$f^{-1}(-1)$: $f(a) = -1$

• $3a-5=-1$

$3a = 4$

$a = \frac{4}{3} > 0$ exists

• $-3a+1=-1$

$3a = 2$

$a = \frac{2}{3} \not\le$ does not exist

$\therefore f^{-1}(-1) = 4/3$.

(iii) Let $x \in R \ni f(x) \in [-5,5] \Rightarrow -5 \le f(x) \le 5$

since $f(x) = 3x-5$ exists for $x>0$

if $3x-5 = -5$

$$3x = 0$$

$$\boxed{x=0}$$

$$\therefore 3x\cancel{-5} = 5$$

$$\boxed{x = \frac{10}{3}}$$

$$\therefore 0 \le x \le 5/3$$

* suppose $f(x) = -3x+1$.

$$-3x+1 = 5$$

$$-3x = 4$$

$$x = -4/3$$

$$-3x+1 = -5$$

$$-3x = -6$$

$$x = 2.$$

$$-4/3 \le x \le 2.$$

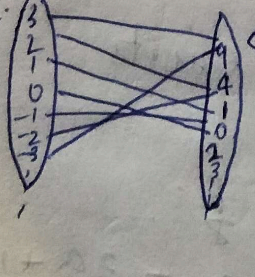
$f^{-1}([-5,5]) = \{x \,/\, 0 \le x \le 10/3$ (or) $-4/3 \le x \le 2\}$

$f^{-1}([-5,5]) = \{x \,/\, -\frac{4}{3} \le x \le \frac{10}{3}\}$.

$f^{-1}([-5,5]) = [-4/3, 10/3]$

② P.T. $f(x) = x^2$ where (f is defined from) $f: R \to R^+$ is onto but not one-one

Sol:



$f(3) = f(-3)$

but $3 \ne -3$.

$f(a) = f(-a)$

but $a \ne -a$.

$R^+ \ni 0, 2, 3, 4, \ldots \in R^+ \ni f^{-1}(2)$ does not exist in the domain.

Q) In each of the foll. cases a function $f: R \to R$ is given. Determine whether one-one or onto. If f is not onto, find its range.

(i) $f(x) = 2x - 3$ (iv) $f(x) = x^2 + x$

(ii) $f(x) = x^3$ (v) $f(x) = e^x$ oneone onto

(iii) $f(x) = x^2$ (vi) $f(x) = \sin x$ not one one but onto.

Sol: (i) one-one: consider $f(a) = f(b)$.

$2a - \cancel{3} = 2b - \cancel{3}$

$\cancel{2}a = \cancel{2}b$

$a = b$.

$\therefore f(a) = f(b)$

$\Rightarrow a = b \Rightarrow f$ is one-one.

onto: Let $y \in R$ (codomain) $\exists$ an element $x \in R$ (domain)

$\ni f^{-1}(y) = x$.

$f(x) = y$

$2x - 3 = y$

$x = \frac{y+3}{2}$.

$\therefore f^{-1}(y) = \frac{y+3}{2}$.

if $y \in R$ then $\frac{y+3}{2} \in R$.

$f^{-1}(y) \in R$ (domain).

$\therefore f$ is onto.

f is one-one and onto

$\Rightarrow f$ is bijective $\Rightarrow f^{-1}$ exists.

Inverse: Let $y \in R$ then $f^{-1}(y) = x$.

$f^{-1}(y) = \frac{y+3}{2}$

$\therefore f^{-1}(x) = \frac{x+3}{2}$.

(ii) one-one: $a^3 = b^3$.

$a = b$.

$f(a) = f(b) \Rightarrow a = b \Rightarrow f$ is one-one.

onto: Let $y \in R$ $\exists$ an element $x \in R$: $f^{-1}(y) = x$.

$f(x) = y$.

$y = x^3$

$x = y^{1/3}$

for every $y \in R \ \exists \ y^{1/3} \in R$ hence, f is onto.

(iii) one-one: $f(x) = x^2$

$a^2 = b^2$

$a = \pm b$.

[NOTE: $1 \neq -1$ but $f(1) = f(-1)$].

$\therefore f$ is not one-one.

* onto: Let $y \in R \ \exists \ f^{-1}(y) = x$

$f(x) = y$

$y = x^2$

$x = \sqrt{y}$.

whenever $y \in R$, $\sqrt{y} \notin R$

$[-1 \in R, \sqrt{-1} \notin R]$

$\therefore f$ is not onto.

④ Consider the function : $f(x) = x^3$, $g(x) = x^2+1$, $\forall x \in R$. Find $gof(x)$, fog, f^2, g^2.

Sol: $gof(x) = g(f(x)) = g(x^3) = (x^3)^2 + 1 = x^6 + 1$.

$fog(x) = f(g(x)) = f(x^2+1) = (x^2+1)^3$.

$f^2(x) = f(f(x)) = f(x^3) = (x^3)^3 = x^9$.

$g^2(x) = g(g(x)) = g(x^2+1) = (x^2+1)^2 + 1 = x^4 + 2 + 2x^2$.

⑤ Let f & g are any 2 functions from $R \to R$ defined as $f(x) = ax+b$, $g(x) = 1-x+x^2$. If $gof(x) = 9x^2 - 9x + 3$ then find a & b values.

Sol: $g(f(x)) = 9x^2 - 9x + 3$.

$g(ax+b) = 9x^2 - 9x + 3$.

$1 - (ax+b) + (ax+b)^2 = 9x^2 - 9x + 3$.

$a^2x^2 + (2ab - a)x + (1-b) = 9x^2 - 9x + 3$.

$\therefore 9 = a^2$ | $1 - b = 3$

$a = \pm 3$ | $b = -2$

f is onto.

, $g(x) = x^2+1$,

g^2.

$+1 = x^6+1$.

x^9.

$-1 = x^4+2+2x^2$.

from $R \to R$

x^2. If

b values.

3,

NOTE: If $gof = I_A$, $fog = I_B$ when $f: A \to B$ & $g: B \to A$ then g is called inverse of f and is written as $g = f^{-1}$.

Q. consider the functions $f: R \to R$ defined by $f(x) = 2x+5$ and $g: R \to R$ defined by $g(x) = \frac{1}{2}(x-5)$. P.T. g is the inverse function of f.

Sol: consider $gof(x) = g[f(x)] = g(2x+5)$

$$= \frac{1}{2}[2x+5-5]$$

$$= x.$$

$$fog(x) = f[g(x)] = f[\tfrac{1}{2}(x-5)].$$

$$= \frac{2}{2}(x-5)+5$$

$$= x.$$

$gof = fog = I$.

$\therefore$ g is inverse of f.

UNIT-4 :

*Sum rule : Suppose 2 tasks t_1 & t_2 are to be performed. If task t_1 can be performed in m different ways and task t_2 can be performed in n different ways, If these 2 tasks cannot be performed simultaneously then the 2 tasks can be performed in (m+n) ways. For example, suppose there are 16 boys & 18 girls in a class and we wish to select one of these students (either a boy (or) a girl) as the class representative. The no. of ways of selecting a boy is 16 & the no. of ways of selecting a girl is 18. Therefore no. of ways of selecting a student as a CR is $16+18=34$

*Product rule : Suppose that 2 tasks t_1 & t_2 are to be performed one after the other. If t_1 can be performed in n_1 different ways & for each of these ways, t_2 can be performed in n_2 different ways, then both the tasks can be performed in $m \times n$ different ways. For example, suppose a person has 3 shirts & 5 pants, the total no. of pairs he has $3 \times 15 = 15$ (i.e. 15 diff ways of choosing a shirt & pant).

*Permutations : The different arrangements which can be made out of a given set of things by taking some (or) all of them at a time is called permutations (arrangements). The no. of permutations of n diff. things taken ($r \leq n$) at a time is denoted by ^nP_r (or) $P(n,r)$ and is defined as $^nP_r = \frac{n!}{(n-r)!}$.

Suppose it is required to find the no. of permutations that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of type 2 and so on n_k are of type k such that $n_1+n_2+\cdots+n_k=n$ then the no. of permutations of the n objects is $\frac{n!}{n_1!n_2!n_3!\cdots n_k!}$

① How many different strings of length 4 can be formed using the letters of the word flower.

Sol: _ _ _ _ $n=6$; $r=4$.

$\therefore$ No. of strings of length 4 obtained from the word flower is ${}^nP_r = {}^6P_4$

$$\frac{6!}{(6-4)!} = \frac{6\cdot5\cdot4\cdot3\cdot\cancel{2}}{\cancel{2}} = 360$$

② Find the no. of permutations of the letters of the foll. words: (i) success (ii) difficult (iii) massasauga (iv) basic (v) pascal (vi) banana. (vii) pepper (viii) calculus (ix) discrete (x) structures (xi) engineering (xii) mathematics.

Sol: (i) ~~total~~ = 7.

s = 3

c = 2.

u = 1.

e = 1

total no. of diff. permutations $= \frac{7!}{3!2!1!1!}$

$$= \frac{7\cdot\cancel{6}\cdot5\cdot4\cdot3\cdot\cancel{2}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{2}} = 420$$

(ii) ~~total~~ = 9.

d = 1, i = 2, f = 2, c = 1, u = 1, l = 1, t = 1.

total no. of permutations $= \frac{9!}{2!2!1!1!1!1!1!} =$

(iii) $\frac{10!}{3!4!}$ (iv) $5!$ (v) $\frac{6!}{2!}$ (vi) $\frac{6!}{3!2!}$

③ Find the no. of permutations of the letters of the word MASSASAUGA such that (i) all four A's are together (ii) how many of them begin with S.

Sol: (i) $\underbrace{AAAA}_{1}$ $\underbrace{S\ S\ S}_{\text{1 type}}$ M U G

① ② ③ ④ ⑤ ⑥ ⑦

$$= \frac{7!}{\underset{S's}{3!}\,1!\,1!\,1!\,1!} = 840.$$

(ii) $\frac{9!}{4!\,2!\,1!\,1!\,1!}$ → (removing 1 S at beginning).

$= 7560$.

④ How many +ve integers of 6 digits can be formed using the digits 3, 4, 4, 5, 5, 6, 7 ~~if we~~ want n to exceed 50,00,000?

Sol: $\underset{5,6,7}{\_}$ _ _ _ _ _

to exceed 500000, 1st digit must be 5, 6, 7 ~~in a 6 digit no.~~

take 5 as 1st digit : $\frac{6!}{2!(1!)^4} = 360$ | 4 - 2 times, 3, 5, 6, 7 - 1 times

take 6 as 1st digit : $\frac{6!}{2!2!} = 180$ | (4 - 2 times, 5 - 2 times) 3, 6, 7 - 1 time.

take 7 as 1st digit $= \frac{6!}{2!2!} = 180$

Total = 360 + 180 + 180 = 720.

⑤ In how many ways can n persons be seated at a round table if arrangements are considered the same when one can be obtained from the others by rotation

Sol: Let one of them be seated anywhere, then the remaining $(n-1)$ persons can be seated in $(n-1)!$ ways. This is the total no. of ways of arranging the n persons in a circle.

6) It is required to seat 5 men & 4 women in a row so that the women occupy even places. How many such arrangements are possible?

Sol:

$$\_\ \overset{W}{\_}\ \_\ \overset{W}{\_}\ \_\ \overset{W}{\_}\ \_\ \overset{W}{\_}\ \_$$
M M M M M $= 4! \times 5!$

7) In how many ways can 6 men & 6 women be seated in a row
(a) If any person may sit next to any other. $= 12!$
(b) If men & women must sit alternatively.
$= 6! \times 6! + 6!6!$

start with men : MWMWMWMWMWMW
start with women : WMWMWMWMWMWM

8) 4 diff. mathematics books, 5 diff CSE books and 2 diff control theory books are to be arranged in a shelf.
(i) How many diff arrangements are possible if the books in each particular subject must all be together
(ii) Only mathematics books must be together.

Sol: (i) 4 math $= 4!$
5 CSE $= 5!$
2 CT $= 2!$
} $3!$ (all 3 units can be arranged in 3! ways)

total $= 3!4!5!2!$

(ii) $4!$ – Math – 1 unit
~~Remaining books in 7! ways~~ ~~} both units can be arranged in 2! ways~~
~~total = 4! 7! 2!~~
Remaining are 7 books (each as a seperate unit) then we have 8 books which can be

arranged in 8! ways. math books internally can be arranged in 4! ways keeping all of them together. so,

total $= 8!\,4!$

⑨ Find the value of n in each of the foll. cases. (i) $P(n,2) = 90$ (ii) $P(n,3) = 3P(n,2)$ (iii) $P(n,4) = 42P(n,2)$ (iv) $2P(n,2)+50 = P(2n,2)$

* Sol : ${}^{n}P_2 = 90$

$\frac{n!}{(n-2)!} = 90 \Rightarrow \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 90$

$\Rightarrow n(n-1) = 90$

$10 \times 9 = 90$

$\boxed{n = 10}$

$$\begin{array}{r|l} 2 & 90 \\ 5 & 45 \\ 3 & 9 \\ 3 & 3 \\ & 1 \end{array}$$

⑩ Find the total no. of +ve integers that can be formed from the digits 1, 2, 3, 4 if no digit is repeated in any one integers.

Sol: Since there are 4 digits, there are 4 integers containing exactly one digit. i.e.

$S_1 = 4$

There are 4×3 integers of 2 digits

$S_2 = \underset{4}{\_}\ \underset{3}{\_} = 12$

$S_3 = \underset{4}{\_}\ \underset{3}{\_}\ \underset{2}{\_} = 24$

$S_4 = \_\ \_\ \_\ \_ = 4! = 24$.

∴ Total no. of integers ~~by~~ which are formed by using 1, 2, 3, 4 $= S_1 + S_2 + S_3 + S_4 = 4 + 12 + 24 + 24$
$= 64$.

⑪ How many 8 digit telephone nos have 1 or more repeated digits.

Sol: The no of 8 digit nos. in which repitition is allowed is 10^8. Of these, ${}^{10}P_8$ nos. do not contain repititions. Therefore, the oks

4! ways

30,

he foll.

P(n,2)

0 = P(2n,2)

0.

2|90
5|45
3|9
3|3
1

that.

2,3,4 if

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required no. is $10^8 - {}^{10}P_8$.

(12) If k is a +ve integer an $n = 2k$. P.T. $\frac{n!}{2^k}$ is +ve integer.

(13) If k is a +ve integer and $n = 3k$, P.T. $\frac{n!}{6^k}$ is a +ve integer.

Sol (12) consider the symbols $x_1, x_2, \ldots x_k$ in which each of x_i is 2 in number. Therefore, the no. of permutations of these symbols is

$$\frac{n!}{2!\,2!\ldots 2!} = \frac{n!}{(2!)^k} = \frac{n!}{2^k}$$

since the no. of permutations is always +ve, therefore $\frac{n!}{2^k}$ is also positive.

Sol 13: Consider the symbols $x_1, x_2, \ldots x_k$ in which each of x_i is 3 in number. Therefore, the no. of permutations of ~~always~~ these symbols is

$$\frac{n!}{3!\,3!\ldots 3!} = \frac{n!}{(3!)^k} = \frac{n!}{6^k}$$

since, the no. of permutations is always +ve, therefore $\frac{n!}{6^k}$ is also +ve.

(14) P.T. for all integeres, $n^r \geq 0$ if $n+1 > r$ then $P(n+1, r)$ ~~equal~~ $= \frac{n+1}{(n+1-r)} \cdot P(n,r)$.

Sol: ${}^nP_r = \frac{n!}{(n-r)!}$

$${}^{(n+1)}P_r = \frac{(n+1)!}{(n+1-r)!}$$

Consider: $$\frac{P(n+1, r)}{P(n,r)} = \frac{\frac{(n+1)!}{(n-r+1)!}}{\frac{n!}{(n-r)!}} = \frac{\frac{(n+1)\,\cancel{n!}}{(n-r+1)\,\cancel{(n-r)!}}}{\frac{\cancel{n!}}{\cancel{(n-r)!}}}$$

$$= \frac{n+1}{n-r+1}$$

$$P(n+1, r) = \frac{n+1}{n-r+1} \cdot P(n,r)$$

(15) Similarly P. the foll results:

(i) $P(n+1, r) = (n+1) \cdot P(n, r-1)$

(ii) $P(n, r) = (n-r+1) P(n, r-1)$

(16) How many diff. arrangements of the letters in the word "bought" can be formed if the vowels must be kept next to each other.

* Sol: vowels = o, u = 2.

no. of arrangements of vowels = 2!

consider 'ou' as 1 unit and remaining letters as 1 unit each. = 5 units.

$\bar{5}\ \bar{4}\ \bar{3}\ \bar{2}\ \bar{1}$

arrangement $\cancel{s}$ = 5!

total arrangement = 5! 2!

= 240

(17) Find the no. of permutations of all letters of the word "baseball" if the words are to begin and end with a vowel.

Sol:

$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$

total alphabets = 8.

a = 2, b = 2, l = 2, e = 1, s = 1

Starting and ending can be filled in 3 ways (a, e) (e, a) (a, a).

Therefore the no. of permutations are

$= \dfrac{6!}{2!2!1!1!1!} \cdot 3 =$

(18) Find the no. of permutations of the letters of the word MISSISSIPPI. i) How many of these begin with 'I' ii) How many begin and end with 'S'.

Sol: _ _ _ _ _ _ _ _ _ _ _

$s = 4 \; ; \; p = 2 \; ; \; i = 4 \; ; \; m = 1$

total permutations $= \dfrac{11!}{4!\,4!\,2!}$

(i) $\underset{4}{\underset{\downarrow}{I}}$ _ _ _ _ _ _ _ _ _ _

$$\frac{10!}{4!\,2!\,3!}$$

First place can be filled in 4 ways.

total $= 4 \times \dfrac{10!}{4!\,2!\,3!}$

(ii)

$\underset{S}{\underset{\downarrow}{I}}$ _ _ _ _ _ _ _ _ _ $\underset{S.}{\underset{\downarrow}{I}}$

$$2 \times \frac{9!}{2!\,4!\,4!}$$

(19) How many diff signals each consisting of 6 flags hang in a vertical line can be formed from 4 identical red flags & 2 identical blue flags.

(20) In how many ways can the symbols a, b, c, d, e, e, e, e, e, be arranged so that no 'e' is adjacent to another 'e'.

Sol: $5!\,.\,4!$

*Combinations: Suppose we are interested in selecting a set of 'r' objects from a set of n objects n ≥ r without regard to order. The set of r objects being selected is traditionally called a combination of r objects. The total no. of combinations of r different objects that can be selected from 'n' different objects is $^{n}C_{r}$ or $C(n,r)$ and is defined as $\dfrac{n!}{r!\,(n-r)!}$.

NOTE: (i) ${}^nC_r = {}^nP_r / r!$

(ii) $C(n,n) = 1$ & $C(n,0) = 1$.

① How many committees of 5 with a given chairperson can be selected from 12 persons?

Sol: Chairperson can be chosen in 12 ways. The other four on the committee can be chosen in ${}^{11}C_4$ ways.

$\therefore$ total $= 12 \times {}^{11}C_4 = 3960$.

* ② Find the no. of committees of 5 that can be selected from 7 men & 5 women, if the committee is to consist of atleast 1 man & atleast 1 woman.

Sol: From the given 12 person, the no. of committees of 5 that can be formed ${}^{12}C_5$.
Among these possible committees there are ${}^7C_5$ consisting of 5 men & ${}^5C_5$ consisting of 5 women.
Accordingly, the no. of committees containing atleast 1 man & 1 woman is ${}^{12}C_5 - {}^7C_5 - {}^5C_5 = 770$

③ At a certain college, the housing office has decided to appoint for each floor, one male & 1 female resedential advisor. How many diff pairs of advisors can be selected for a 7 floor building from 12 male & 15 female candidates.

Sol: From 12 male candidates, 7 candidates can be selected in ${}^{12}C_7$ ways. From 15 female candidates 7 candidates can be selected in ${}^{15}C_7$ ways.
$\therefore$ Total no. of possible pairs of advisors of the required type is ${}^{12}C_7 \times {}^{15}C_7$.

④ A certain question paper contains 2 parts, A & B, each having 4 questions. How many diff ways a student can answer 5 questions by selecting atleast 2 questions from each part.

Sol: The diff ways a student can select his 5 questions are (i) 3 from A & 2 from B

This can be done in ${}^4C_3 \times {}^4C_2$ ways = 24 ways.

(ii) 2 Q from A & 3 Q from B.

This can be done in ${}^4C_2 \times {}^4C_3 = 24$ ways.

∴ Total no. of ways a student can answer 5 questions under the given restrictions = 24 + 24 = 48

⑤ A certain question paper contains 3 parts A, B, C with 4 Q in A, 5 Q in B and 6 Q in part C. It is required to answer 7 Q selecting atleast 2 questions from each part. In how many diff. ways can a student select his 7 Q's?

Sol: The diff ways in which a student can select:

(i) 2 Q from A, 2 Q from B & 3 Q from C.

This can be done in ${}^4C_2 \times {}^5C_2 \times {}^6C_3 = 1200$ ways.

(ii) 3 Q from A, 2 Q from B, 2 Q from C.

$\Rightarrow {}^4C_3 \cdot {}^5C_2 \cdot {}^6C_2 = 600$ ways.

(iii) 2 Q from A, 3 Q from B & 2 Q from C

$\Rightarrow {}^4C_2 \cdot {}^5C_3 \cdot {}^6C_2 = 900$ ways.

∴ Total no. of possible selections = 1200 + 600 + 900 = 2700 ways.

⑥ A woman has 11 close relatives & she wants to invite 5 of them to dinner. In how many ways can she invite them, in the foll. situations

(i) There is no restriction on the choice - ${}^{11}C_5$

(ii) Two particular persons will not attend separately

Since 2 particular persons will attend separately, they should both be invited (or) not invited.

If both of them are invited, then 3 more invitees are to be selected from the remaining 9 relatives in ${}^9C_3 = 84$ ways.

If both of them are not invited then 5 invitees are to be selected from 9 relatives in ${}^9C_5 = 126$ ways.

∴ Total no. of ways in which the invitees can be selected in this case = 84 + 126 = 210 ways.

(iii) 2 particular persons will not attend together

Sol: If the 2 particular persons are A & B, we have 3 possibilities;

(a) A attends, B not attends. (or)

B attends, A not attends

∴ No. of ways of choosing A = $^9C_4 = 126$

& B = $^9C_4 = 126$

(ii) Suppose both A & B do not attend.

No. of ways of choosing the 5 invitees = $^9C_5 = 126$.

Hence, total no. of ways in which the invitees can be selected in this case = 126 + 126 + 126 = 378 ways

(7) Find the no. of 5 digit +ve integers, such that in each of them, every digit is greater than the digit to the right.

Sol: A set of 5 diff digits can be selected from 10 digits in $^{10}C_5$ ways.

Once these digits are chosen, there is only one way of arranging them in a descending order from left to right.

So, the no. of 5 digit +ve integers of the required type is $1 \times {}^{10}C_5$.

En: select 0, 1, 2, 3, 4, --- $^{10}C_5$

arrangement - 43210 - 1 way only.

(8) Find the no. of arrangements of the letters "TALLAHASSEE" which have no adjacent A's.

Sol: Total no. of letters n = 11

A - 3, L = 2, E - 2, S - 2, T, H - 1

∴ First let us disregard A's, the remaining 8 letters can be arranged in $\frac{8!}{2!2!2!(1!)^2} = 5040$

In each of these arrangements, there are 9 possible locations for the 3 A's. These locations can be chosen in $^9C_3$ ways.

∴ Required no. of arrangements is $5040 \times {}^9C_3$
$= 4,23,360$

⑨ Find the no. of ways of seating 'r' out of n persons around a circular table & others around another circular table.

Sol: Choose a set of r persons for the first table. This can be done in ^nC_r ways. These r persons can be seated around the first table in $(r-1)!$ ways. The remaining $n-r$ persons can be seated around in $(n-r-1)!$ ways.

So, the required no. of ways is $^nC_r \cdot (r-1)!(n-r-1)!$

⑩ A party is attended by n persons. If each person in the party shakes hands with all the others in the party. Find the no. of hand shakes.

Sol: Each handshake is determined by exactly 2 persons. Therefore, if each person shakes hands with ~~othe~~ all the other person. total no. of handshakes = the no. of combinations of 2 persons that can be selected from n person.

This no. is $^nC_2 = \frac{n(n-1)}{2}$

⑪ There are 'n' married couples attending a party. Each person shakes hands with every person other than his or her spouse.

Sol: Total no. of people attending the wedding $= 2n$

No. of handshakes (total) $= {}^{2n}C_2$

No. of handshakes in this case $= {}^{2n}C_2 - n$

No. of handshakes

(12) Prove the foll. identities :

(i) ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$ (or) $C(n+1, r) = C(n, r-1) + C(n, r)$

(ii) $C(m+n, 2) - C(m,2) - C(n,2) = mn$

(iii) $C(n,r) \cdot C(r,k) = C(n,k) \cdot C(n-k, r-k)$ for $n \geq r \geq k$

Sol: (i) consider $C(n, r-1) + C(n, r)$

$$= \frac{n!}{(r-1)!\,(n-r+1)!} + \frac{n!}{r!\,(n-r)!}$$

$$= n! \left[\frac{1}{(r-1)!\,(n-r+1)(n-r)!} + \frac{1}{r(r-1)!\,(n-r)!} \right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \left[\frac{1}{n-r+1} + \frac{1}{r} \right].$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \left[\frac{\cancel{r} + n - \cancel{r} + 1}{r(n-r+1)} \right].$$

$$= \frac{(n+1)\,n!}{[(n-r-1)(n-r)!]\,[r(r-1)!]}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!} = C(n+1, r)$$

(ii) consider : $C(m+n, 2) - C(m,2) - C(n,2)$

$$= \frac{(m+n)!}{2!\,(m+n-2)!} - \frac{m!}{2!\,(m-2)!} - \frac{n!}{2!\,(n-2)!}$$

$$= \frac{1}{2} \left\{ \frac{(m+n)(m+n-1)\cancel{(m+n-2)!}}{\cancel{(m+n-2)!}} - \frac{m(m-1)\cancel{(m-2)!}}{\cancel{(m-2)!}} - \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} \right\}$$

$$= \frac{1}{2} \left\{ (m+n)(m+n-1) - m(m-1) - n(n-1) \right\}.$$

$$= \frac{1}{2} \left\{ \cancel{m^2} + mn - \cancel{m} + mn + \cancel{n^2} - \cancel{n} - \cancel{m^2} + \cancel{m} - \cancel{n^2} + \cancel{n} \right\}$$

$= \frac{1}{2}\{2mn\} = mn$ // proved.

(iii) Consider $C(n,r) \cdot C(r,k)$

$$= \frac{n!}{r!(n-r)!} \cdot \frac{r!}{k!(r-k)!} = \frac{n!}{k!(r-k)!(n-r)!}$$

consider $C(n,k) \cdot C(n-k, r-k)$

$$= \frac{n!}{(n-k)!k!} \cdot \frac{(n-k)!}{(n-k)!(n-r)!} = \frac{n!}{k!(n-r)!(r-k)!}$$

LHS = RHS.

Hence, proved.

UNIT - 4 : Continuation :

* Binomial and Multinomial theorem :

* Binomial theorem :

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r y^{n-r}$$

One of the basic properties of ${}^nC_r$ is that it is the coefficient of $x^r \cdot y^{n-r}$ in the expansion of the expression $(x+y)^n$, where x & y are real nos. In other words $(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r y^{n-r}$. This result is known as the binomial theorem for a +ve integral index. The no. ${}^nC_r$, $r = 0, 1, 2, \ldots, n$ in the above result are known as binomial coefficients.

* Multinomial theorem :

For +ve integers n, & t, the coefficient of $x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}$ in the expression $(x_1 + x_2 + \ldots + x_t)^n$ is $\dfrac{n!}{n_1! n_2! \ldots n_t!}$ where each n_i is non-negative integer which is less than or equal to n where $n_1 + n_2 + \ldots + n_t = n$.

Proof : WKT in the expansion of $(x_1 + x_2 + \ldots + x_t)^n$ the coefficient of $x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} \cdot \ldots \cdot x_t^{n_t}$ is the no. of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the remaining $n - n_1$ factors, x_3 from n_3 of the remaining $n - n_1 - n_2$ factors and so on. Therefore, this coefficient is by product rule

$${}^nC_{n_1} \cdot {}^{n-n_1}C_{n_2} \cdot {}^{n-n_1-n_2}C_{n_3} \cdot \ldots \cdot {}^{n-n_1-n_2-n_3-\ldots-n_{t-1}}C_{n_t}$$

$$\frac{n!}{n_1!\cancel{(n-n_1)!}} \cdot \frac{\cancel{(n-n_1)!}}{n_2!\cancel{(n-n_1-n_2)!}} \cdot \ldots \cdot \frac{\cancel{n-n_1-n_2-n_3-n_{t-1}}}{n_t!(n-n_1-n_2-\ldots-n_t)!}$$

$$= \frac{n!}{n_1! n_2! \ldots n_t! \cdot 1} = \frac{n!}{n_1! n_2! \ldots n_t!}$$

$$\left[\begin{gathered} n = n_1 + n_2 + n_3 + \ldots + n_t \\ (n-n)! = 0! = 1 \end{gathered}\right]$$

Another way of stating the multinomial theorem is the general term in the expansion of $(x_1+x_2+x_3+\dots+x_t)^n$ is $\frac{n!}{n_1!n_2!\dots n_t!}(x_1^{n_1}x_2^{n_2}x_3^{n_3}\dots x_t^{n_t})$

where $n_1, n_2 \dots n_t$ are non-negative integers not exceeding n and $n_1+n_2+\dots+n_t = n$.
The expression $\frac{n!}{n_1!n_2!\dots n_t!}$ can also be written in the form $\binom{n}{n_1, n_2, \dots, n_t}$ and is called a multinomial coefficient.

① Prove the foll. identities for +ve integer 'n'. (i) $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

(or)

(i) $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$

(ii) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \dots + (-1)^n \binom{n}{n} = 0$.

Sol: WKT by binomial theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \quad — ①$$

(i) take $x=1$ & $y=1$ in ①:
$$(1+1)^n = \sum_{r=0}^{n} \binom{n}{r} 1^r 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

(ii) take $x=1$ & $y=-1$ in ①. (or) take $x=-1$ & $y=$

$$(1-1)^n = \sum_{r=0}^{n} \binom{n}{r} (1)^r (-1)^{n-r}$$
$$= \binom{n}{0}(-1)^{n-0} + \binom{n}{1}(-1)^{n-1} + \binom{n}{2}(-1)^{n-2} + \dots + \binom{n}{n}(-1)^{n}$$
$$= -1\left\{\binom{n}{0} + \binom{n}{1}\frac{1}{-1} + \binom{n}{2}\left(\frac{1}{(-1)^2}\right) + \dots + \frac{(-1)^n}{(-1)^n(-1)^n}\binom{n}{n}\right\}$$
$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \dots + \left(\frac{(-1)^n}{(-1)^{2n}}\binom{n}{n}\right) = \frac{0}{(-1)^n}$$
$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} \dots$$

② Find the coefficient of $x^9 \cdot y^3$ in the expansion of $(2x-3y)^{12}$

Sol: We have by binomial theorem, the general term in the expansion of $(x+y)^n$ is ${}^nC_r\, x^r y^{n-r}$

which is : $\left.\begin{array}{l} \dfrac{n!}{r!(n-r)!}\, x^r y^{n-r} \\ \text{given } x^9 y^3 . \end{array}\right\}$ compare.

$r=9, \; n-r=3 ; \; n=12 ;$

$\therefore$ The general term in $(2x-3y)^{12}$ is :

$$\frac{n!}{r!(n-r)!} (2x)^r (-3y)^{n-r}$$

$$= \frac{n!}{r!(n-r)!}\, 2^r x^r (-3)^{n-r}\, y^{n-r}.$$

given $r=9, \; n=12$.

The Coefficients of $x^9 y^3$ in $(2x-3y)^{12}$ is

$$\frac{12!}{9!\,3!}\, 2^9 (-3)^3 = 1946.$$

③ Find the coefficient of $x^5 y^2$ in the expansion $(2x-3y)^7$.

Sol : coeff. of $x^5 y^2$ in $(2x-3y)^7$ is $r=5, n=7$.

$$\frac{7!}{5!\,2!}\, 2^5 . (-3)^2 .$$

④ Find the coeff of $x^9 y^3$ in the expansion $(x+2y)^{12}$

Sol : $r=9, \; n=12$.

$$\frac{12!}{9!\,3!} (1)^9 (2)^3 =$$

⑤ Find the term which contains x^{11} & y^4 in the expansion of $(2x^3 - 3xy^2 + z^2)^6$.

Sol : By multinomial theorem, the general term of the expansion $(x_1 + x_2 + x_3 + \text{---} + x_t)^n$ is

$\frac{n!}{\quad}$ $x_1^{n_1} x_2^{n_2} \text{---} x_t^{n_t}$. Here all n, t are +ve and $n_1 + n_2 + \dots + n_t = n$ — ①.

The general term of $(2x^3 - 3xy^2 + z^2)^6$ is

$$\frac{6!}{n_1!\,n_2!\,n_3!}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$$

$$= \frac{6!}{n_1!\,n_2!\,n_3!}\,2^{n_1}x^{3n_1}(-3)^{n_2}x^{n_2}y^{2n_2}z^{2n_3}$$

$$= \frac{6!}{n_1!\,n_2!\,n_3!}\,2^{n_1}x^{3n_1+n_2}(-3)^{n_2}y^{2n_2}z^{2n_3}$$

given x^{11} & y^4 $\Rightarrow$ $3n_1 + n_2 = 11$

$2n_2 = 4$

$\therefore \boxed{n_2 = 2}$ & $\boxed{n_1 = 3}$

since $n_1 + n_2 + n_3 = 6$; $\boxed{n_3 = 1}$

$\therefore x^{11}$ & y^4 exists in $\frac{6!}{3!\,2!\,1!}\,2^3(-3)^2 x^{11}y^4z^2 = 4320x^{11}y^4z^2$

⑥ Find the coefficient of (i) xyz^2 in the expansion of $(2x - y - z)^4$. Ans: -24

(ii) $a^2b^3c^2d^5$ in the expansion $(a + 2b - 3c + 2d + 5)^{16}$

(iii) xyz^5 in the expansion $(x + y + z)^7$.

(iv) xyz^{-2} in the expansion $(x - 2y + 3z^{-1})^4$

(v) x^3z^4 in the expansion $(x + y + z)^7$

(vi) $x^3y^3z^2$ in the expansion $(2x - 3y + 5z)^8$

(vii) $w^3x^2yz^2$ in the expansion $(2w - x + 3y - 2z)^8$

(viii) $x_1^2x_3x_4^3x_5^4$ in the expansion $(x_1 + x_2 + x_3 + x_4 + x_5)^{10}$.

Sol: (ii) The general term of the expansion $(a + 2b - 3c + 2d + 5)^{16}$ is $\frac{n!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!}a^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$

and $n_1 + n_2 + n_3 + n_4 + n_5 = 16$.

Given: $a^2b^3c^2d^5$ $\Rightarrow$ $\boxed{n_1 = 2}$; $\boxed{n_2 = 3}$; $\boxed{n_3 = 2}$; $\boxed{n_4 = 5}$

$n_5 = n - (n_1 + n_2 + n_3 + n_4)$

$\boxed{n_5 = 4}$

The coeff of $a^2 b^3 c^2 d^5$ is $\cancel{\frac{16!}{2!\,3!\,2!\,5!\,4!}}$

$$\frac{16!}{2!\,3!\,2!\,5!\,4!} \cdot 2^3 (-3)^2\, 2^5\, 5^4 .$$

(7) Compute the foll.

(i) $\binom{12}{5\ 3\ 2\ 2}$ (ii) $\binom{7}{2\ 3\ 2}$ (iii) $\binom{8}{4\ 2\ 2\ 0}$ (iv) $\binom{10}{5\ 3\ 2\ 2}$

Sol :(iv) Meaningless since $n_1 + n_2 + n_3 + n_4 \neq n$.

$5 + 3 + 2 + 2 \neq 10$.

* Combinations with repetitions :

Suppose we wish to select, with repetition, a combination of r objects from a set of n distinct objects. The no. of such selections is given by $^{(n+r-1)}C_r = C(n+r-1, r)$

$$= \binom{n+r-1}{r}$$

This is also equal to $^{(r+n-1)}C_{(n-1)}$

NOTE 1 : $^{(n+r-1)}C_r$ represents the no. of ways in which 'r' identical objects can be distributed among 'n' distinct containers.

NOTE 2 : $^{(n+r-1)}C_r$ represents the no. of non-negative integer solutions of the equation $x_1 + x_2 + \dots + x_n = r$

(1) A bag contains coins of 7 different denominations with atleast one-dozen coins in each denomination. In how many ways can we select a dozen coins from the bag.

Sol : The selection consists in choosing with repetitions, $r = 12$ coins of $n = 7$ distinct denominations. The no. of ways of making this selection is $^{n+r-1}C_r$

$$^{7+12-1}C_{12} = {}^{18}C_{12} = \frac{18!}{12!\,6!} = 18564$$

(2) In how many ways can we distribute 10 identical marbles among 6 distinct containers.

Sol: $r = 10$ $n = 6$ $\Rightarrow {}^{n+r-1}C_r = {}^{10+6-1}C_{10} = {}^{15}C_{10}$

$= \dfrac{15!}{10!\,5!}$

(3) Find the no. of non-negative integer solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 8$

Sol: ${}^{n+r-1}C_r$ solutions exist (from NOTE (2))

$r = x_1 + x_2 + \cdots + x_n$.

$r = 8$; $n = 5$;

${}^{n+r-1}C_r = {}^{5+8-1}C_8 = {}^{12}C_8 = \dfrac{12!}{8!\,4!}$

(4) Find the no. of distinct terms in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{16}$.

Sol: By multinomial theorem; every term in the above expansion is of the form $\dfrac{16!}{n_1!\,n_2!\,n_3!\,n_4!\,n_5!} \times x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$. where each n_i is a non-negative integer and $n_1 + n_2 + n_3 + n_4 + n_5 = 16$.

Therefore, the no. of distinct terms in the above expansion is equal to the no. of non-negative integer solutions of the eqn: $n_1 + n_2 + n_3 + n_4 + n_5 = 16$ where $n = 5$, $r = 16$.

$\therefore$ no. of solutions $= {}^{5+16-1}C_{16} = {}^{20}C_{16}$

(5) Find the no. of non-negative integer solutions of the inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 < 10$.

Sol: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 9 - x_7$ where $9 - x_7 \le 9 \le 10$

$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 9$. (where x_7 is non-ve integer)

$r = 9$, $n = 7$.

${}^{n+r-1}C_r = {}^{7+9-1}C_9$

$= {}^{15}C_9$

$= \dfrac{15!}{9!\,6!}$

⑥ Find the no. of integer solutions of $x_1+x_2+x_3+x_4+x_5=30$ where $x_1 \geq 2$; $x_2 \geq 3$; $x_3 \geq 4$; $x_4 \geq 2$; $x_5 \geq 0$.

Sol: Take $x_1 - 2 = y_1 \geq 0$

$y_1 + 2 = x_1$ &

$x_2 - 3 = y_2 \geq 0$

$x_2 = y_2 + 3$.

$x_3 - 4 = y_3 \geq 0$

$x_3 = y_3 + 4$

$x_4 - 2 = y_4$

$x_4 = y_4 + 2$

$x_5 = y_5$.

eq ① becomes:

$$y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 30 - 11$$

$$= 19 \quad \text{where } y_i\text{'s} \geq 0.$$

∴ The no. of non-negative solutions is

$$^{(n+r-1)}C_r \Rightarrow \begin{matrix} n = 5 \\ r = 19 \end{matrix} \Rightarrow {}^{23}C_{19} = \frac{23!}{19!\,4!}$$

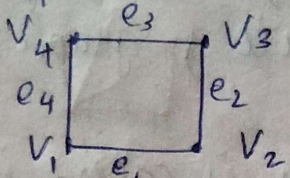
⑦ In how many ways can we distribute 12 identical pencils to 5 children so that every child gets atleast 1 pencil.

Sol: distribute 5 pencils to 5 childrean. Then there will remain 7 pencils to be distributed. the no. of ways of distributing these 7 pencils to 5 children is the required number.

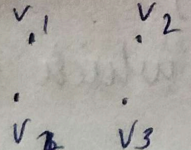
$n = 5$; $r = 7$; $^{n+r-1}C_r = {}^{5+7-1}C_7 = {}^{11}C_7$.

UNIT-5 : Graph Theory.

*Graph: A graph $G(V,E)$ consists of a set of objects $V=\{v_1, v_2, v_3 ---\}$ called vertices and another set $E=\{e_1, e_2, ---\}$ whose elements are called edges such that each edge is associated with an unordered pair of vertices. The vertex and edge sets, respectively are represented by $V(G)$ and $E(G)$. The below graph has 4 vertices and 4 edges.

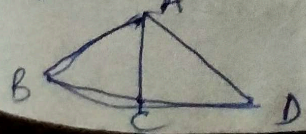


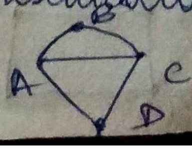
- The most common representation of a graph is by means of a diagram, in which the vertices are represented by points and each edge as a line segment joining its end vertices. Often this diagram itself is referred to as the graph.
- According to the definition of a graph, the vertex set in a graph has to be non-empty. Thus, a graph must contain atleast one vertex. But, the edge set can be empty. This means that a graph need not contain any edge.
- A graph containing no edges is called a null graph
- A null graph with only one vertex is called a trivial graph.



- a null graph with 4 vertices.

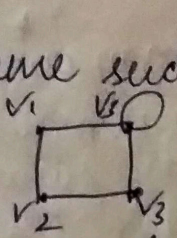
- The way one draws a diagram of a graph is basic-ally immaterial. There can be more than one diagram for the same graph. For example, the 2 diagrams in the foll. fig. look different, yet they represent the same graph since each conveys the same information. The 2 graphs have same vertices and edges but the representations are different.





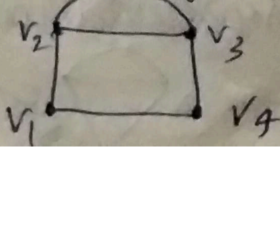
- The definition of a graph does not impose any upper limit for the no. of vertices and the no. of edges. Thus, a graph can have infinitely many vertices and edges.
- A graph with only a finite no. of vertices as well as only a finite no. of edges is called a finite graph. Otherwise it is called an infinite graph.
- The no. of vertices in a (finite) graph is called the order of the graph.
- The no. of edges in a graph is called its size.
- A graph of order 'n' and size 'm' is called a (n, m) graph.
- If v_i and v_j denote two vertices of a graph and if e_k denotes the edge joining v_i and v_j then v_i and v_j are called end vertices or adjacent vertices of e_k. This is symbolically written as $e_k = \{v_i, v_j\} = v_i v_j$.
- The vertices in a graph that are joined by an edge are known as adjacent vertices.
- Neighbours are 2 vertices in a graph that are adjacent. The neighbouring set of v is the collection of all neighbour vertices of a fixed vertex v in G. It is represented by N(V).
- Incident edges are the edges which have a common vertices.
- If end vertices of an edge are same such edge is known as self loop or loop.

A graph with self loop at v_3



- Two or more edges having common end points are known as parallel edges

A graph with parallel edges between v_2 & v_3

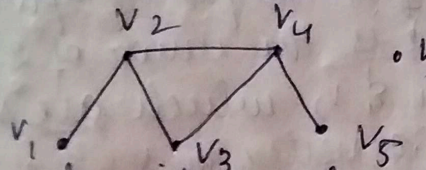


- The no. of edges incident with a vertex 'v' in a graph 'G', determines its degree, Taking 2 for a self loop, and it is denoted by $d(v)$ or $d_G(v)$ or $deg(v)$. The degrees of the vertices of a graph arranged in non-decreasing order is called the degree sequence of the graph. Also, the minimum of the degrees of vertices of a graph is called the degree of the graph

From the above diagram: $d(v_1) = 2$; $d(v_2) = 3$, $d(v_3) = 3$; $d(v_4) = 2$.

- If the degree of a vertex is 1 it is known as a pendent vertex.
- If the degree of a vertex is zero, it is known as isolated vertex.

Ex:

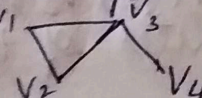


v_1, v_2 - pendent

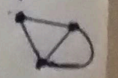
v_6 - Isolated vertex.

- A graph with only one vertex is known as a trivial graph.

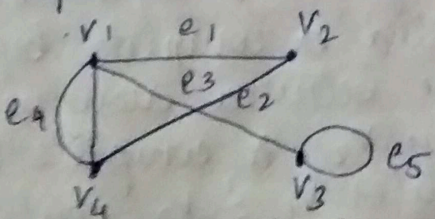
*Simple graph: A graph which does not contain loops and multiple edges is called a simple graph. Ex:



*Multi graph: A graph which contains multiple edges but no loops is called a multigraph.

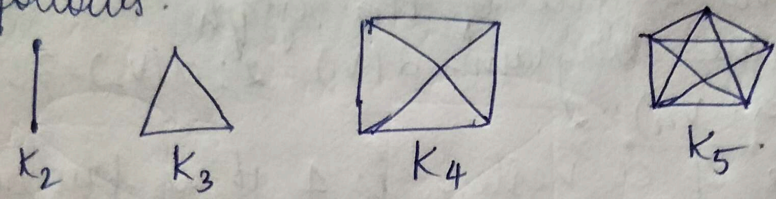


*General graph: A graph which contains multiple edges or loops or both is called a general graph



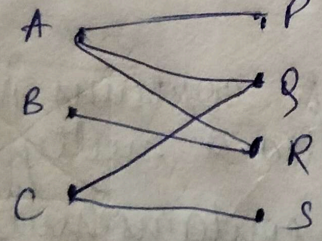
*Regular graph: A graph in which each vertex has same degree is known as a regular graph.

* Complete graph: A simple graph of order $n \geq 2$ in which there is an edge between every pair of vertices is called a complete graph. In other words, a complete graph is a simple graph in which every pair of distinct vertices are adjacent. A complete graph with $n \geq 2$ vertices is denoted by K_n. Complete graphs with 2, 3, 4 & 5 vertices are as follows.

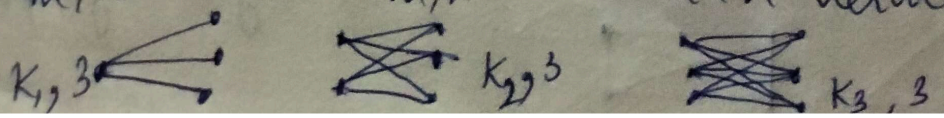


K_5 is also called the Kuratowski's first graph.

* Bipartite graph: Suppose a simple graph G is such that its vertex set V is the union of 2 mutually disjoint non-empty sets V_1 & V_2 which are such that every edge in G joins a vertex in V_1 & V_2 which are such that every edge in G joins a vertex in V_1 & a vertex in V_2 then G is called a bipartite graph. If E is the edge set of this graph, the graph is denoted by $G(V_1, V_2; E)$. The sets V_1 & V_2 are called bipartites of the vertex set V.



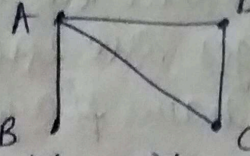
* Complete Bipartite graph: A bipartite graph $G(V_1, V_2; E)$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2. A complete bipartite graph in which the bipartites V_1 & V_2 contains m & n vertices respectively with $m \leq n$ is denoted by $K_{m,n}$. Thus $K_{m,n}$ has $m+n$ vertices and mn e



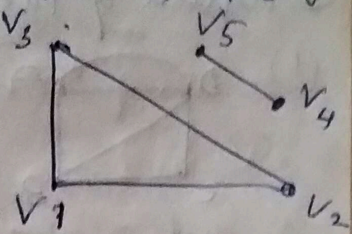
The graph $K_{3,3}$ is known as Kuratowski's 2nd graph

① Draw a diagram of the graph $G=(V,E)$ in each of the foll. cases:

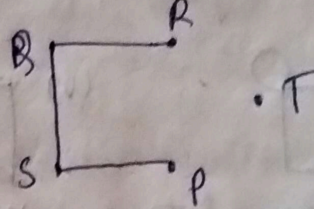
(i) $V=\{A,B,C,D\}$; $E=\{(A,B)(A,C)(A,D)(C,D)\}$.



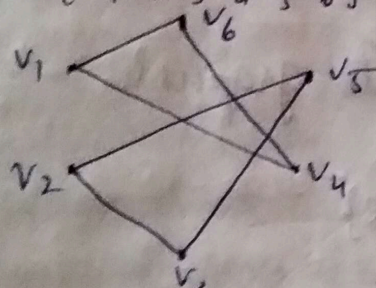
(ii) $V=\{V_1 V_2 V_3 V_4 V_5\}$; $E=\{(V_1 V_2)(V_1 V_3)(V_2 V_3)(V_4 V_5)\}$.



(iii) $V=\{P Q R S T\}$ $E=\{(P,S)(Q,R)(Q,S)\}$.

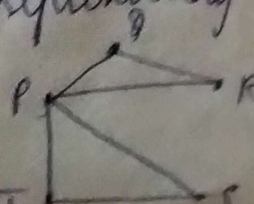


(iv) $V=\{V_1 V_2 V_3 V_4 V_5 V_6\}$ $E=\{(V_1 V_4)(V_1 V_6)(V_4 V_6)(V_3 V_2)(V_3 V_5)(V_2 V_5)\}$



② Let P Q R S T represent 5 cricket teams. Suppose that the teams P Q R have played one game with each other, and teams P S T have played one game with each other. Represent this situation in a graph. Hence determine (i) the teams that have not played with each other, and (ii) the no of games played by each team.

Sol: Let the teams be represented by vertices and an edge represent the playing. Then the graph representing the given situation is as shown:

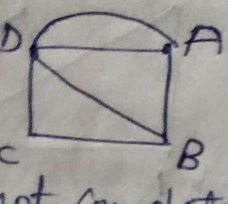


(i) We observe that there is no edge between (QS) (QT) (RS) (RT). ∴ The teams Q & S, Q & T, R & S, R & T have not played with each other.

(ii) From the graph, we note that 2 edges are incident on each of the vertices Q R S T and 4 edges are incident on P. Thus, the teams Q, R, S, T have played 2 each and team P has played 4.

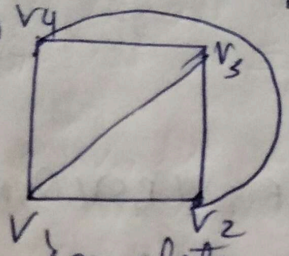
③ Which of the foll. are complete graphs?

(i)



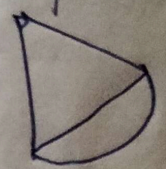
not complete

(ii)



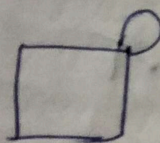
complete

(iii)



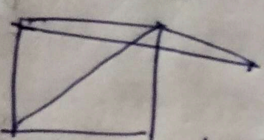
not complete

(iv)



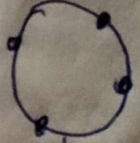
not complete

(v)



not complete

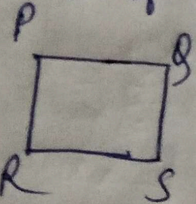
(vi)



not complete

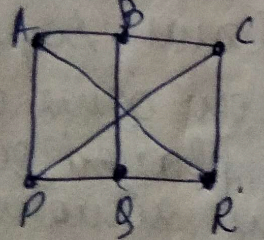
④ Which of the foll are bipartite graphs?

(i)



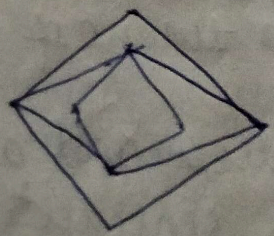
Bipartite $V_1 = \{P, S\}$
$V_2 = \{Q, R\}$.

(ii)



$V_1 = \{A, C, Q\}$
$V_2 = \{B, P, R\}$

(iii)



A procedure to check if a graph is a bipartite or not

(1) Arbitarily select a vertex from G and include it into set V_1

(2) Consider the edges directly connected to that vertex and put the other end vertices of these edges into the set V_2.

(3) Pick up one vertex from set V_2, and consider the edges directly connected to that vertex, and put the other end of these edges into set V_1.

(4) At each step, check if there is any edge among the vertices of set V_1 & set V_2. If so, the given graph is not bipartite graph and then return, else continue 2 & 3 alternately all the vertices are included in the union sets V_1 & V_2.

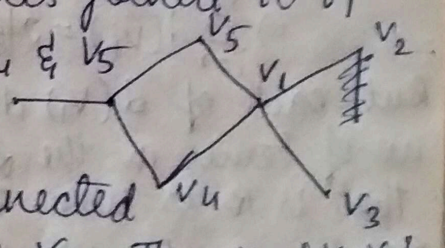
(5) If 2 computed sets are distinct, then the graph is bipartite.

① S.T. the foll. graph is bipartite.

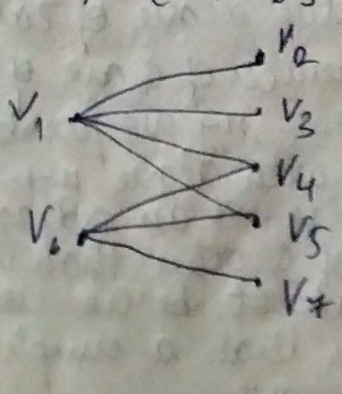
Sol: Select vertex v_1. The vertices joined to v_1 through direct edges are $v_2\, v_3\, v_4$ & v_5

take $V_1 = \{v_1\}$ $V_2 = \{v_2, v_3, v_4, v_5\}$

and vertices in V_2 are not connected to v_5 through direct edge is v_6. Then $V_1 = \{v_1, v_6\}$ and take v_7 in V_2.



$\therefore$ $V_1 = \{v_1\, v_6\}$ $V_2 = \{v_2\, v_3\, v_4\, v_5\, v_7\}$.



is the required bipartite graph.

* Handshaking property (or) First theorem of graph theory

① Statement: The sum of the degrees of all the vertices in a graph is an even number, and this number is equal to twice the no. of edges in the graph i.e., $\sum_{i=1}^{n} d(v_i) = 2|E|$

Proof: Let us consider a graph ~~G~~ with 'e' edges and n vertices $v_1, v_2, v_3, ---, v_n$.

Since each edge contributes 2 degrees, one at starting vertex and one at end of the edge. '' ——— ''

The sum of the degrees of all vertices in G is twice the no. of edges in G. That is $\sum_{i=1}^{n} d(v_i) = 2e$.

② Statement: In every graph, the no. of vertices of odd degree is even. (or) The no. of vertices of odd degree in a graph is always even.

Proof: Consider a graph with n vertices. Suppose k of these vertices are of odd degree, so that the remaining n-k vertices are of even degree. Denote the vertices with odd degree by $v_1, v_2, v_3, ---, v_k$, and the vertices with even degree by $v_{k+1}, v_{k+2}, ---, v_n$ ~~and~~. The sum of the degrees of the vertices is $\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{n} d(v_i) = 2|E|$

$$\Rightarrow \sum_{i=1}^{k} d(v_i) = 2|E| - \sum_{i=k+1}^{n} d(v_i) = \text{even} \quad \text{—①}$$

But each of $d(v_1)\, d(v_2)\, --- d(v_k)$ is odd. Therefore, the no. of terms in the LHS of eq ① must be even. That is k is even. This completes the proof of the theorem.

③ Statement: ~~A~~ simple graph with atleast 2 vertices has atleast 2 vertices of same degree.

Proof: Let G be a simple graph with $n \geq 2$ vertices. The graph G has no loop and ~~ll~~ edges. Hence the degree of each vertices is $\leq n-1$. Suppose all the vertices of G are of different degrees. Hence the foll. degrees $0, 1, 2, ---, n-1$ are possible for n vertices of G. Let v be the vertex with degree n-1. Then v has n-1 adjacent vertices. Since v is not an adjacent vertex of itself, therefore every vertex of G other than v is an adjacent vertex of G. Hence v cannot be an isolated vertex, this contradiction proves that a simple graph contains two vertices of same degree.

NOTE: The converse of above theorem is not true.

① statement: Show that the max. no. of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Proof: By handshaking theorem: $\sum_{i=1}^{n} d(V_i) = 2|E|$

where $|E|$ is the no. of edges with n vertices in the graph $G \Rightarrow d(V_1) + d(V_2) + \dots + d(V_n) = 2|E|$ — ①

Since the max. no. of edges & degree of each vertex in the graph G can be $(n-1)$.

∴ eq ① becomes $(n-1)+(n-1)+ \dots + n$ times $= 2|E|$

$$n(n-1) = 2|E| \Rightarrow |E| = \frac{n(n-1)}{2}$$

Hence, the max. no. of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

① Is there a simple graph corresponding to the foll. degree sequences?

(i) (1,1,2,3) (ii) (2,2,4,6) (iii) (1,1,1,1) (iv) (1,3,3,4,5,6,6).
(v) (1,1,3,3,3,4,6,7).

Sol: (i) Since the sum of degrees of vertices is odd, there exist no graph corresponding to this degree sequence.

(ii) No. of vertices in the graph sequence is four and the max. degree of a vertex is 6, which is not possible as in a simple graph the max. degree cannot exceed one less than the no. of vertices.

(iii) The sum of the degrees of all vertices is 4, even. The number of odd vertices is 4, even. Hence a simple disconnected graph is possible which has 4 vertices of degree 1 each. The no. of edges is $\frac{4}{2} = 2$.

(iv) Here the sum of the degrees is 28, even. The no. of vertices having odd degree is 4 even. The max. degree 6 does not exceed $7-1=6$. But 2 vertices have degree 6, each of these 2 vertices is adjacent with every other vertex. Hence, the degree of each vertex is atleast 2, so that the graph has no vertex of degree 1 which is a contradiction. Hence there does not exist a simple graph with the given degree sequence.

(v) Assume that there is such a graph, since the degree of vertices are 8 in no. the graph should have 8 vertices say P Q R S T U V W arranged in the order of degrees as given i.e. $d(P)=1$; $d(Q)=1$; $d(R)=3$;

$d(S)=3$; $d(U)=4$; $d(V)=6$; $d(W=7$.
Since $d(W)=7 \Rightarrow W$ is adjacent to P, Q, R, S, T, U, V. In particular W has an edge to both of the vertices P and Q which are of degree 1. Then P, Q are not joined to any other vertex in particular to the vertex V which is of degree 6 which is a contradiction ($\because$ G is simple). Hence there is no simple graph for which the degrees of vertices are as given.

② S.T. a simple graph of order $n=4$ and size $m=7$ does not exist.

Sol: WKT the no. of edges in a simple graph (size)
$$=\frac{n(n-1)}{2} = \frac{4(3)}{2} = 6.$$
but given size $m=7$.
$\therefore$ Not possible.

③ Can there be a graph consisting of the vertices A, B, C, D with $\deg(A)=2$; $\deg(B)=3$; $\deg(C)=2$; $\deg(D)=2$?

Sol: No. Since sum of degrees $=9$ is not even no.

④ Does there exist a graph with 12 vertices such that 2 of the vertices have degree 3 and the remaining vertices have degree 4 each?

Sol: Sum of the degrees of vertices $=(3\times 2)+(4\times 10)=46$
$\therefore$ By handshaking property $\sum_{i=1}^{n} d(v_i)=2|E|$
$$\Rightarrow 46 = 2|E| \Rightarrow |E|=23.$$
Hence, a graph of the desired type exists.

*Directed graph: A graph in which every edge is directed is called a digraph or a directed graph. In other words, if each edge of the graph G has a direction then the graph is called directed graph.

• In the diagram of a directed graph, each edge $e=(u,v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.
Suppose $e=(u,v)$ is a directed edge in a digraph, then:

(i) u is called the initial vertex of e &
v is called the terminal [illegible]

(ii) e is said to be initiating or originating in the node U and terminating or ending in the node V.

(iii) U is adjacent to V, and V is adjacent to U.

* In-degree and Out degree:

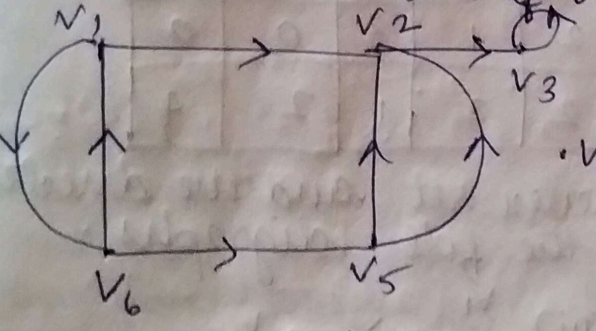
If v is a vertex of a digraph D, the no. of edges for which v is the initial vertex is called the out going degree or the out degree of v and the no. of no. of edges for which v is the terminal vertex is called the incoming degree or the in-degree of v.

The out degree of v is denoted by $d^+(v)$ or $od(v)$ and the indegree of v is denoted by $d^-(v)$ or $id(v)$.

It follows that (i) $d^+(v)=0$, if v is a sink

(ii) $d^-(v)=0$ if v is a source

(iii) $d^+(v)=d^-(v)=0$ if v is an isolated vertex



$d^+(v_1)=2$	$d^-(v_1)=1$
$d^+(v_2)=1$	$d^-(v_2)=3$
$d^+(v_3)=1$	$d^-(v_3)=2$
$d^+(v_4)=0$	$d^-(v_4)=0$
$d^+(v_5)=2$	$d^-(v_5)=1$
$d^+(v_6)=2$	$d^-(v_6)=1$

We note that, in the above digraph, there is a loop at the vertex v_3 and this loop contributes a count 1 to each of $d^+(v_3)$ & $d^-(v_3)$.

Theorem: In every digraph D, the sum of the out degrees of all vertices is equal to the sum of indegrees of all vertices, each sum being equal to the number of edges in D.

Proof: Suppose D has n vertices $v_1, v_2 \dots v_n$ and m edges. Let r_1 be the no. of edges going out of v_1, r_2 be the no. of edges going out of v_2 & so on. Then $d^+(v_1)=r_1$, $d^+(v_2)=r_2 \dots d^+(v_n)=r_n$

Since every edge terminates at some vertex and since there are m edges, we should have

$r_1 + r_2 + \dots + r_n = n$.

Accordingly, $d^+(v_1) + d^+(v_2) + \dots + d^+(v_n) = r_1 + r_2 + \dots = n$

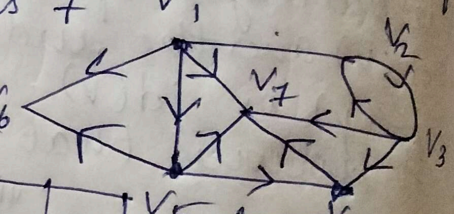
Similarly, if s_1 is the no. of edges coming in to v_1, s_2 is the no. of edges coming in v_2, and so on, we get

$d^-(v_1) + d^-(v_2) + \dots + d^-(v_n) = s_1 + s_2 + \dots + s_n = m$

Thus $\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = m$.

① Find the indegrees and out degrees of the vertices of the digraph shown in fig.

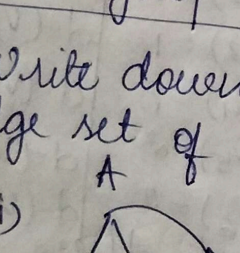
Sol: The given digraph has 7 vertices and 12 directed edges.



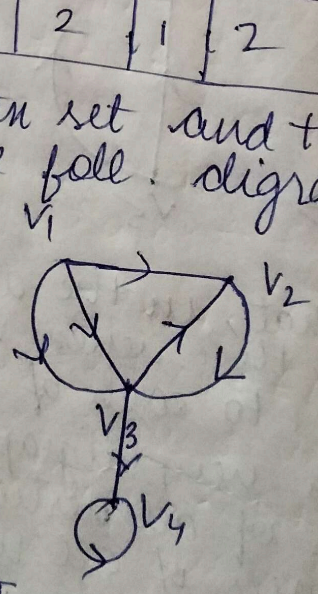
vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7
out degree	4	2	2	1	3	0	0
in-degree	0	1	2	2	1	2	4

② Write down the vertex set and the directed edge set of each of the foll. digraphs.

(i)



(ii)



Sol:

(i)

vertex	A	B	C
out degree	0	1	4
in degree	3	2	0

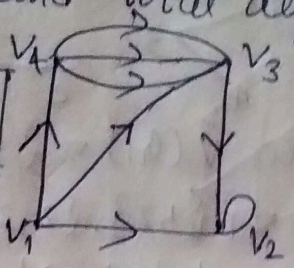
(ii)

vertex	v_1	v_2	v_3	v_4
out degree	3	1	2	1
in degree	0	2	3	2

* Total degree: In a directed graph, the sum of the outdegree and the indegree of v is called its total degree, i.e.,
* total degree of v = (indegree + outdegree) of v.

In case of an undirected graph, the total degree or the degree of a node v is equal to the number of edges incident with v. The total degree of an isolated vertex is 0.

① Find the indegree, outdegree and total degree of each vertex of the graph.



Sol:

Vertex	Indegree	Outdeg	total deg
v_1	0	3	3
v_2	3	1	4
v_3	3+1	1	5
v_4	1	3	4

② S.T. the degree of a vertex of a simple graph G on n vertices cannot exceed $n-1$.

Sol: Let v be a vertex of G, since G is simple, no multiple edges or loops are allowed in G. Thus, v can be adjacent to atmost all the remaining $n-1$ vertices of G. Hence, v may have max. degree $n-1$ in G. then $0 \le \deg(v) \le n-1$ $\forall\ v \in V(G)$.

* Isomorphic Graph (Isomorphism of two graphs).

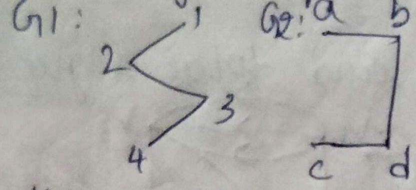
Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exists a function $f: V_1 \to V_2$ such that:

(i) f is one-to-one and onto i.e., f is bijective

(ii) $\{a, b\}$ is an edge in E_1, iff $\{f(a), f(b)\}$ is an edge in E_2 for any 2 elements $a, b \in V_1$. Here the function f is called an isomorphism btw G_1 and G_2 and we say that G_1 and G_2 are isomorphic graphs.

In other words, 2 graphs G_1 and G_2 are said to be isomorphic to each other) if there ~~exists~~ a one-to-one correspondence btw their vertices and btw their edges such that the adjacency of vertices is preserved (means that if (u, v) are adjacent vertices in G_1, then the corresponding vertices (u_1, v_1) are also adjacent in G_2.

① S.T. the given pair of graphs are isomorphic.



Sol: Here $V(G_1) = \{1, 2, 3, 4\}$; $V(G_2) = \{a, b, c, d\}$

$E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$; $E(G_2) = \{\{a,b\} \{b,d\} \{c,d\}\}$

$\therefore |V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$.

The vertices of degree 1 in G_1 are $\{1, 4\}$ and in G_2 are $\{a, c\}$.

The vertices of degree 2 in G_1 are $\{2, 3\}$ and in G_2 are $\{b, d\}$.

Define a function $f : V(G_1) \to V(G_2)$ as

$f(1) = a$ $f(2) = b$ $f(3) = d$ $f(4) = c$.

f is clearly one-to-one & onto.

Further, $\{1, 2\} \in E(G_1)$ and

$\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

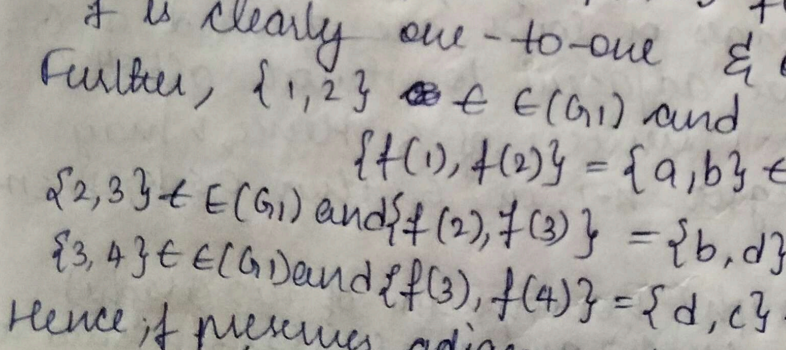
$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$.

$\{3, 4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$.

Hence, f preserves adjacency of the vertices.

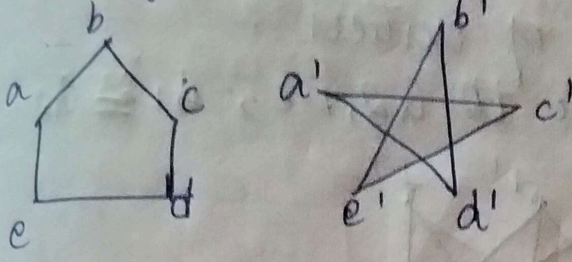
$\therefore G_1$ is isomorphic to G_2, i.e., $G_1 \cong G_2$.

② Check whether the given 2 graphs G_1 & G_2 are isomorphic, or not. Give reasons.



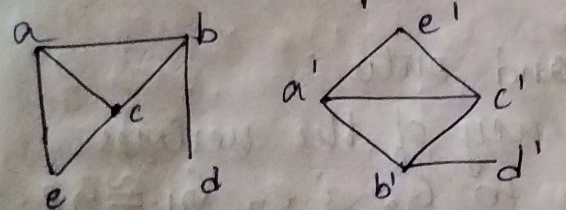
Sol: We observe that $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$
But G_1 has 2 vertices of degree 4 whereas G_2 has 3 vertices of degree 4.
The adjacency of vertices is not preserved.
The 2 graphs G_1 & G_2 are not isomorphic.

③



Sol: $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$
All vertices in G_1 are of degree 2 and also all vertices in G_2 are of degree 2.
Defining a function $f: V(G_1) \to V(G_2)$ as:
$f(a) = a'$ $f(b) = b'$ $f(c) = c'$ $f(d) = d'$ $f(e) = e'$
f is clearly one-to-one & onto.
Hence, f preserves adjacency of the vertices.
$\therefore$ G_1 is isomorphic to G_2, i.e. $G_1 \cong G_2$.

④

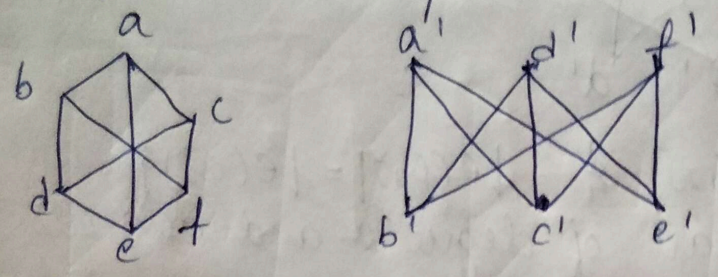


Sol: $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$
vertices a, b, c from G_1 have degree 3 and e, has degree 2 & d has degree 1.
Similarly in G_2, a', c', b' have degree 3, e' has degree 2 and d' has degree 1.
Defining a function: $f: V(G_1) \to V(G_2)$ as:
$f(a) = a'$ $f(b) = b'$ $f(c) = c'$ $f(d) = d'$ $f(e) = e'$.
f is clearly one to one & onto.
Further, $\{a, b\} \in E(G_1)$ and $\{f(a), f(b)\} = \{a', b'\} \in E(G_2)$
$\{b, d\} \in E(G_1)$ & $\{f(b), f(d)\} = \{b', d'\} \in E(G_2)$

$\{a, c\} \in E(G_1)$ & $\{f(a), f(c)\} = \{a', c'\} \in E(G_2)$.
$\{a, e\} \in E(G_1)$ & $\{f(a), f(e)\} = \{a', e'\} \in E(G_2)$
$\{e, c\} \in E(G_1)$ & $\{f(e), f(c)\} = \{e', c'\} \in E(G_2)$
$\{c, b\} \in E(G_1)$ & $\{f(c), f(b)\} = \{c', b'\} \in E(G_2)$.

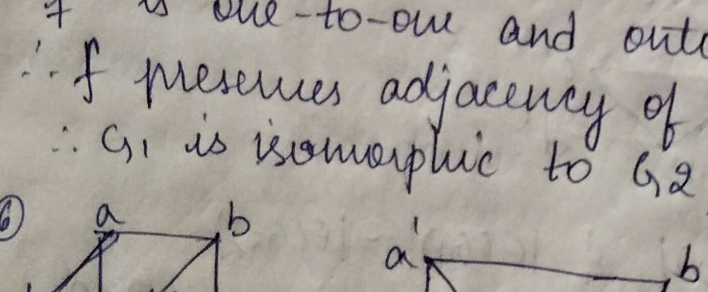
$\because$ f preserves adjacency of vertices.
$\therefore$ G_1 is isomorphic to G_2, i.e., $G_1 \cong G_2$.

⑤

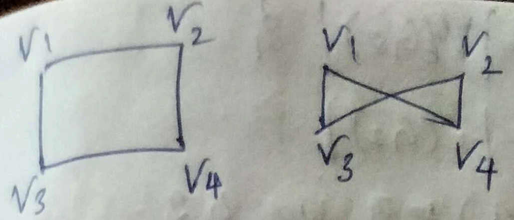


Sol: $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$
all vertices of G_1 have degree 3 and all vertices of G_2 also have degree 3.
$f(a) = a'$, $f(b) = b'$, $f(c) = c'$, $f(d) = d'$
$f(e) = e'$, $f(f) = f'$.
$f: V(G_1) \to V(G_2)$.
f is one-to-one and onto.
$\because$ f preserves adjacency of the vertices.
$\therefore$ G_1 is isomorphic to G_2, i.e., $G_1 \cong G_2$.

⑥



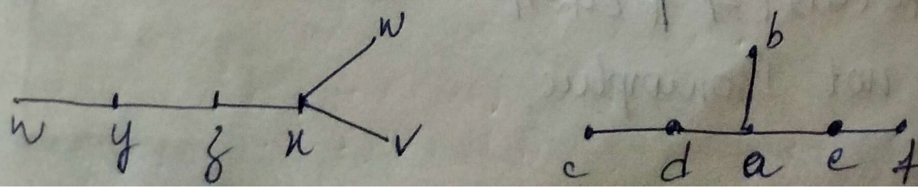
Sol: $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$.
all vertices of G_1 have degree 3 and same in G_2.
$f: V(G_1) \to V(G_2)$. $f(a) = a'$ $f(b) = b'$ $f(c) = c'$ $f(d) = d'$
$f(e) = e'$ $f(g) = g'$ $f(p) = p'$ $f(h) = h'$.
$\because$ f preserves adjacency of the vertices
$\therefore$ G_1 is isomorphic to G_2, i.e., $G_1 \cong G_2$.

7 

Sol: $|V(G_1)| = |V(G_2)|$ & $|E(G_1)| = |E(G_2)|$
All vertices of G_1 have degree 2 and all vertices of G_2 have degree 2 as well.
$f: V(G_1) \to V(G_2)$. $f(v_1) = v_1$ $f(v_2) = v_2$ $f(v_3) = v_3$ $f(v_4) = v_4$.
f is one to one and onto.
$\therefore$ f preserves adjacency of the vertices.
$\therefore G_1$ is isomorphic ~~to~~ G_2, i.e., $G_1 \cong G_2$.

* Determining when graphs are not isomorphic
We can prove that 2 graphs are not isomorphic by showing they do not share a property that isomorphic graphs ~~must~~ have, such a property is called an invariant with respect to the isomorphism of graphs. The invariants are :-
(i) The no. of vertices.
(ii) The no. of edges and
(iii) The degree ~~of~~ sequences of the 2 graphs.
If any ~~2~~ of these quantities in 2 graphs, those graphs cannot be isomorphic.
• When these invariants are the same, it doesn't mean that the 2 graphs are isomorphic. A part from these invariants we need a one-to-one and onto function which preserves adjacency of the vertices in simple graph and preserves the direction of edges in digraphs.

Q Determine whether the foll. graphs are isomorphic or not.

(a) 

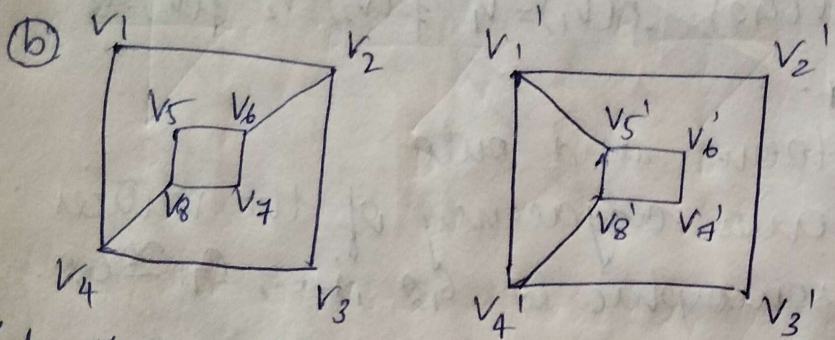
Sol: We observe that $|V(G_1)| = |V(G_2)|$ &

$$|E(G_1)| = |E(G_2)|$$

deg(x) in $G_1 = 3$ & deg(a) $= 3$ in G_2.

G_1 and G_2 are not isomorphic, because the vertex x is adjacent to 2 pendant vertices, whereas vertex a is adjacent to only one pendant vertex.

(b)

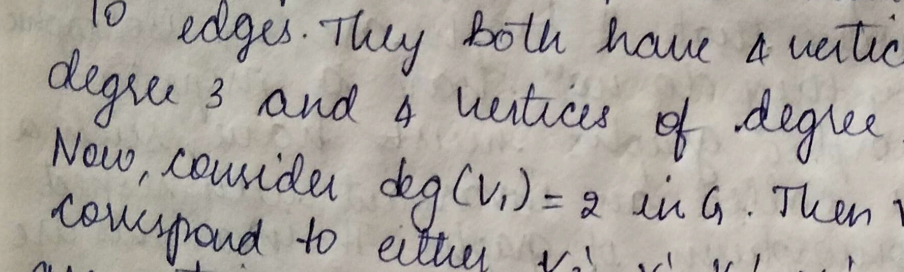


Sol: The graphs G and G' both have 8 vertices and 10 edges. They both have 4 vertices each of degree 3 and 4 vertices of degree 2.

Now, consider $\deg(v_1) = 2$ in G. Then v_1 must correspond to either v_2', v_3', v_6', v_7' since these are vertices of deg 2 in G'. However, each of these vertices in G' is adjacent to another vertex of deg 2 in G' but v_1 is adjacent to v_2 and v_4 in G which are of degree 3. Thus the preservation of adjacency of the vertices is not maintained.

∴ G & G' are not isomorphic.

(c)

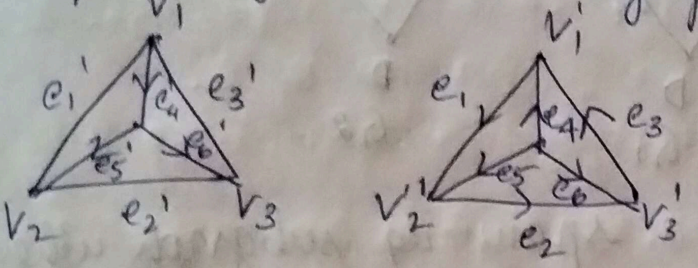


Sol: $|E(G_1)| \neq |E(G_2)|$.

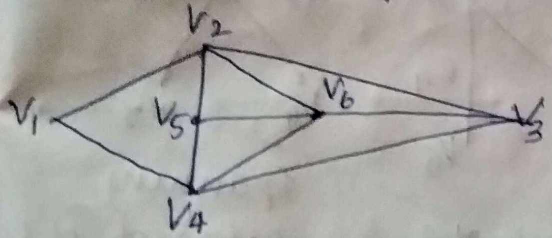
∴ not Isomorphic.

* Isomorphic Digraphs: Two digraphs are said to be isomorphic if (a) their corresponding undirected graphs are isomorphic.
(b) Directions of the corresponding edges are also agree.

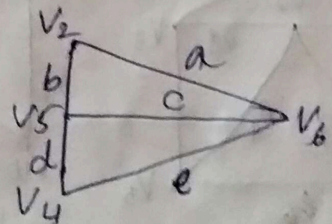
The foll. 2 digraphs are not isomorphic because the directions of the 2 corresponding edges e_4 & e_4' do not agree (although their corresponding undirected graphs are isomorphic)



* Sub graphs: Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$ then a graph $H(V(H), E(H))$ is said to be a subgraph of G if:
(i) All the vertices of H are in G i.e., $V(H) \subseteq V(G)$
(ii) All the edges of H are in G. i.e., $E(H) \subseteq E(G)$.
(iii) Each edge of H has the same end vertices in H as in G.



(Graph-G)

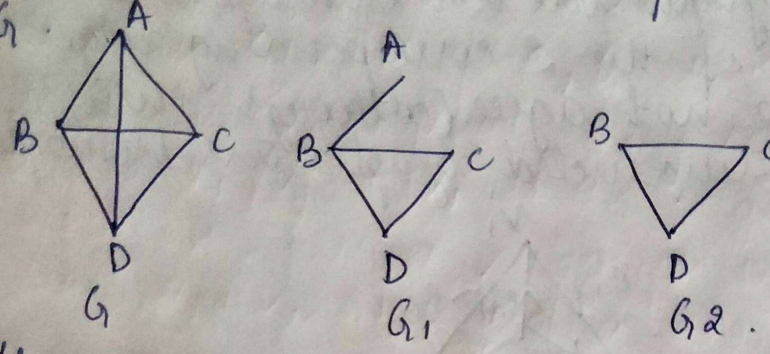


(Sub graph-H).

NOTE: ① Every graph is a subgraph of itself.
② Every simple graph of n vertices is a subgraph of the complete graph K_n.
③ If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G_L then G_1 is a subgraph of G_2.
④ A single vertex in a graph G is a subgraph of G.

⑤ A single edge in a graph G, together with its end vertices, is a subgraph of G.

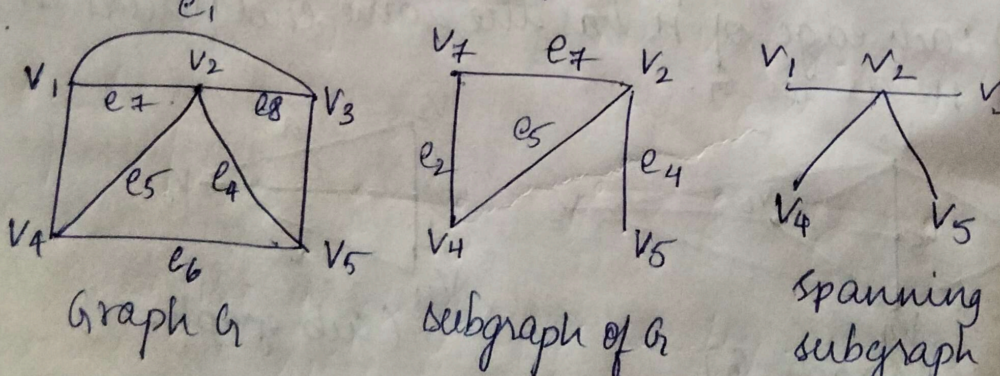
*Spanning subgraph: Given a graph $G=(V,E)$ if there is a subgraph $G_1=(V,E_1)$ of G such that $V_1=V$, then G_1 is called a spanning subgraph of G.

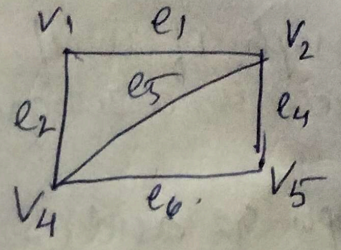


The graph G_1 is a spanning subgraph whereas the graph G_2 is a subgraph but not a spanning subgraph.

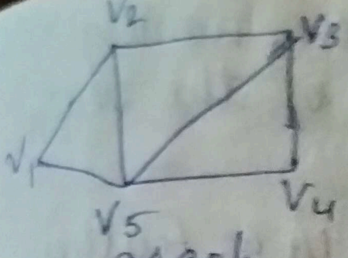
*Induced subgraph: If W is any subset of vertex set of G, then the subgraph generated or induced by W is the subgraph is of G obtained by taking $V(H)=W$ and $E(H)$ to be those edges of G that joins pair of vertices in W.

①.

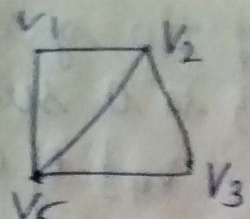




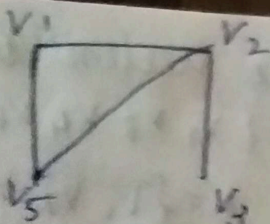
subgraph induced by $W=\{v_1,v_2,v_4,v_5\}$



graph



induced subgraph by $W = \{V_1, V_2, V_3, V_5\}$



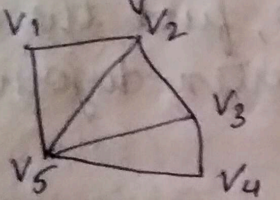
(not induced subgraph)

* Edge disjoint and vertex-disjoint subgraphs
Let G be a graph and G1 and G2 be 2 subgraphs of G. Then:

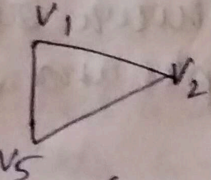
① G1 and G2 are said to be edge disjoint if they do not have any common edge.

② G1 and G2 are said to be vertex-disjoint if they do not have any common edge and any common vertex.

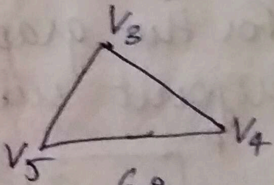
Ex:



G



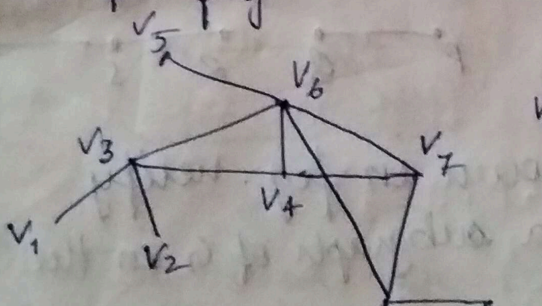
G1



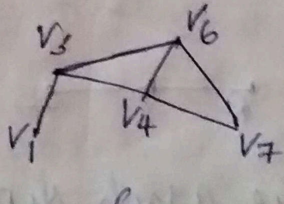
G2.

The graphs G1 and G2 are edge disjoint but not vertex-disjoint subgraphs.

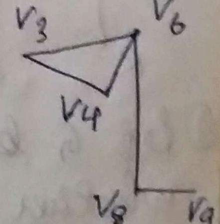
① Consider the graphs G and G1 as shown in the foll. fig.



G



G1



G2.

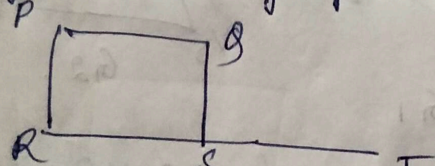
(a) Verify that the graph G1 is an induced subgraph of G. Is this a spanning subgraph of G.

(b) Draw the subgraph G2 of G induced by the set $V_2 = \{V_3, V_4, V_6, V_8, V_9\}$.

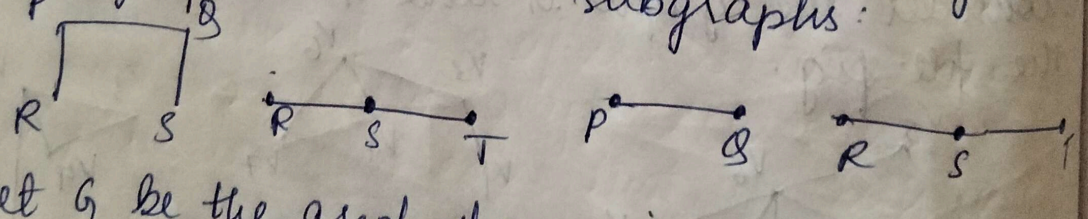
(a) The vertex set of the graph G_1, namely $V_1 = \{v_1\ v_3\ v_4\ v_6\ v_7\}$ is a subset of the vertex set $V = \{v_1, v_2, v_3, --- v_9\}$ of G.
Also, all the edges of G_1 are in G. Further each edge of G_1 has the same end vertices in G as in G_1. Therefore, G_1 is a subgraph of G.
We further check that every edge $\{v_i, v_j\}$ of G where $v_i, v_j \in V_1$ is an edge of G_1. Therefore G_1 is an induced subgraph of G.
Since $V_1 \neq V$, G_1 is not a spanning subgraph of G.

(b) $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$ is an induced subgraph of G.

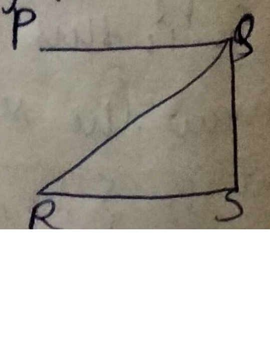
② For the graph shown below, find the 2 edge-disjoint subgraphs and 2 vertex-disjoint subgraphs.



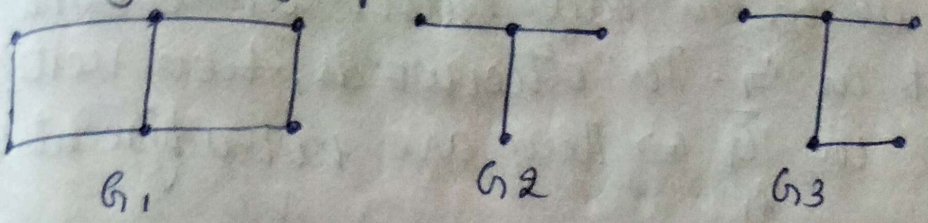
Sol: Two edge-disjoint subgraphs: Two vertex-disjoint subgraphs:



③ Let G be the graph shown in fig. verify whether $G_1 = (V_1, E_1)$ is a subgraph of G in the foll. cases:
(i) $V_1 = \{P, Q, S\}$ $E_1 = \{\{P,Q\}, \{P,S\}\}$ (ii) $V_1 = \{Q\}$; $E_1 = \phi$
(iii) $V_1 = \{P, Q, R\}$ $E_1 = \{(P,Q)\ (Q,R)\ (Q,S)\}$.
Sol: (i) No (ii) Yes (iii) No.



④ Three graphs G_1, G_2, G_3 are shown in fig. Are G_2 and G_3 induced subgraphs of G_1? Are they spanning subgraphs?



Sol: G_2 is an induced subgraph of G_1;
It is not a spanning subgraph.
G_3 is not an induced subgraph of G_1;
It is a spanning subgraph.

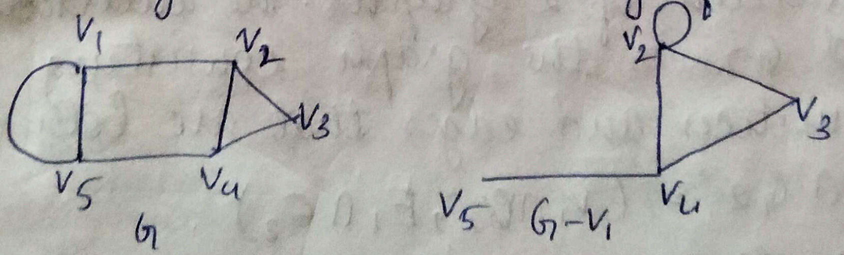
⑤ Can a finite graph be isomorphic to one of its subgraphs? (other than itself).

Sol : No.

* Operations on Graphs:

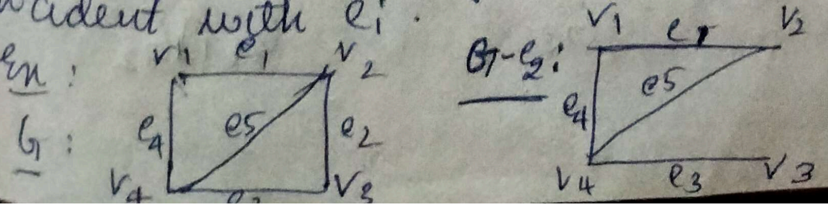
(1) Deleting a vertex: Let G be a graph with $V(G) = \{v_1, v_2, v_3 --- v_n\}$ then $G - v_i$ is the graph obtained deleting or removing the vertex v_i from G together with all edges incident on v_i.
More generally, we write $G - \{v_1, v_2 -- v_k\}$ for the graph obtained by deleting the vertices $v_1, v_2, -, v_k$. and all edges incident on any of them.

Ex:



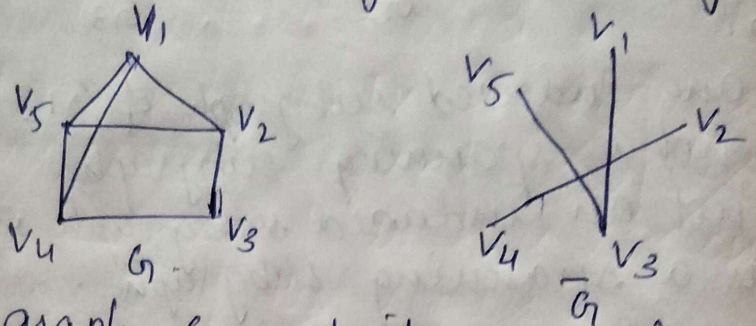
(2) Deleting and edge: Let G be a graph with $E(G) = \{e_1 e_2 --- e_n\}$ then $G - e_i$ is the graph obtained by removing or deleting the edge e_i with out deleting the vertices which are incident with e_i.

Ex:



(3) Complement of a graph: The complement of a graph G is the graph $\overline{G}$ with the same vertices as G. An edge exists in $\overline{G} \Leftrightarrow$ it does not exist in G. In otherwords, two vertices adjacent in $\overline{G} \Leftrightarrow$ they are not adjacent in G.

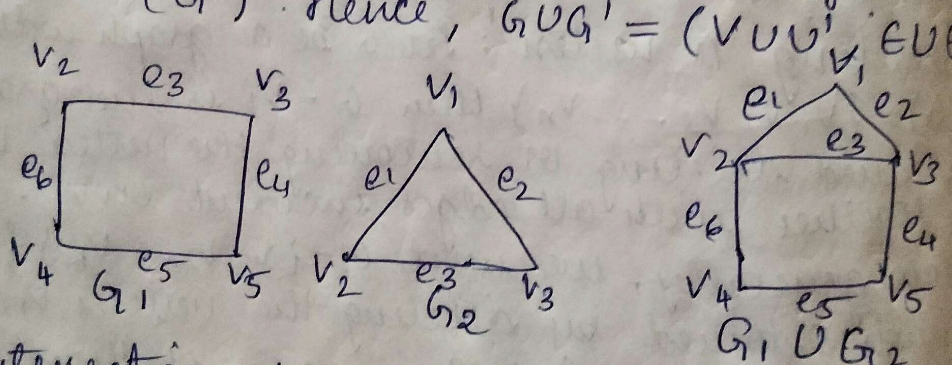
Ex:



A graph G and its complement $\overline{G}$.

(4) Union of 2 graphs: Let G and G' be 2 graphs the union of G and G' is the graph with vertices set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$. Hence, $G \cup G' = (V \cup V', E \cup E')$

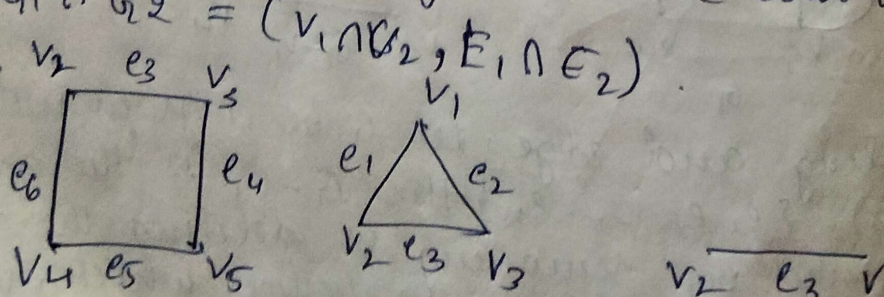
Ex:



(5) Intersection of 2 graphs: The intersection of G_1 and G_2 is the graph consisting only those vertices and edges that are both in G_1 & G_2

$\Rightarrow G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

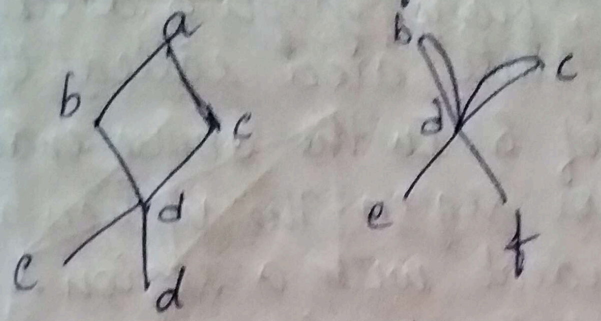
Ex:



G_1 G_2 $G_1 \cap G_2$

(6) Fusion: Fusion of 2 vertices a and b in a graph G is an operation G in which 2 vertices a and b are fused (merged) together without deleting any edge of G.

Ex:

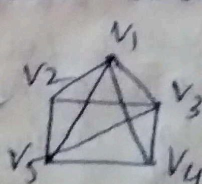


Fusion of the vertices a & b.

Note: Fusion of 2 adjacent vertices always produce a loop at the point of fusion and the no. of loops is equal to the no. of edges btw the vertices which are fused together

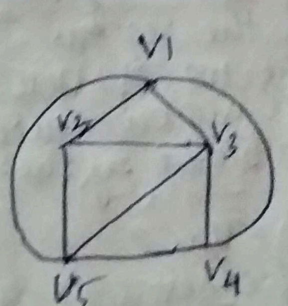
(7) Planarity: A graph G is said to be planar if it can be drawn in the plane without its edges crossing otherwise G is non-planar.

NOTE: A graph G may be planar even if its is usually drawn with edge crossings, since it may be possible to draw it in a different way without any edge crossings. We say that a planar graph is a plane graph if it is already drawn in the plane without edge crossings

Q Check whether the graph 

is planar or not.

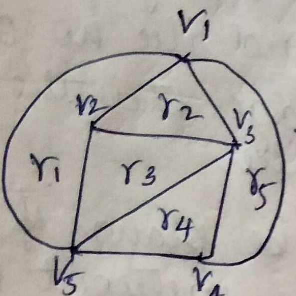
Sol:



There is no edge crossings, so the given graph is a planar graph.

NOTE: ① A plane graph divides the plane into regions. A region is characterized by the cycle that forms its boundary. These regions are connected portions of the plane.

② In each plane, plane graph G determines a region of infinite area called the exterior region of G. In the above example, r_6 is the exterior region. The vertices and edges of G incident with a region r make up boundary of the region r.



$v_1 - v_2 - v_5 - v_1$ is the boundary of the region r_1. The degree of the region r_1 is the length of its boundary. $\deg(r_1) = 3$.

$\deg(r_2) = 3$ $\deg(r_3) = 3$ $\deg(r_4) = 3$ $\deg(r_5) = 3$

$\deg(r_6) = 3$.

* Eulers Formula: If G is a connected planar graph, then ~~they~~ any drawing of G in the plane ~~of~~ as planar graph will always form $|R| = |E| - |V| + 2$ regions including the exterior region, where $|R|, |E|$ & $|V|$ denote respectively, the no. of regions edges and vertices of G.

Proof: We prove this result by induction on the no. of regions 'k' determined by G. It is obvious when $k=1$. Assume the result for $k>1$ and suppose that G is a connected plane graph having $(k+1)$ regions. Delete an edge common to both the regions. The resulting graph G' has the same no. of vertices, one fewer edge but also one fewer region since two previous regions have been combined by the removal of the edge.

$\therefore |E'| = |E| - 1$ & $|R'| = |R| - 1$; $|V'| = V$.

where $|E'|$ $|R'|$ and $|V'|$ are no. of edges, regions vertices of G'. Then $|V'| - |E'| + |R'| = |V| - |E| + 1 + |R| - 1$

$= |V| - |E| - |R| = 2$

(by induction hypothesis)

Hence, the theorem is proved by mathematical induction.

Theorem: A complete graph K_n is planar iff $n \le 4$

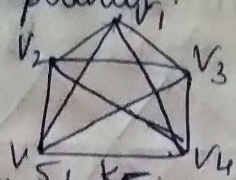
Proof: It is easy to see that K_n is planar for $n = 1,2,3,4$

Now, we have to S.T. when $n \le 4$, K_n is planar. For this, it is sufficient to show K_n is non-planar, when $n \ge 5$. In other words, we prove this by an indirect argument. Now, Assume that K_5 is planar,

then $|R| = |E| - |V| + 2$

$= 10 - 5 + 2$

$= 7$



Since K_n is simple and loop free, we have $3|R| \le 2|E|$

$3 \times 7 \le 2 \times 10$

$21 \le 20$ which is a contradiction.

$\therefore$ our assumption is wrong.

$\Rightarrow K_5$ is non-planar.

① A complete graph $K_{m,n}$ is planar iff $m \le 2$ or $n \le 2$.

Sol: It is clear that $K_{m,n}$ is planar if $m \le 2$ or $n \le 2$

Now let $m \ge 3$ & $n \ge 3$. To prove that $K_{m,n}$ is nonplanar it is sufficient to prove that $K_{3,3}$ is nonplanar. Since, $K_{3,3}$ has six vertices and 9 edges, if $K_{3,3}$ is planar, By Eulers formula $|R| = |E| - |V| + 2$

$= 9 - 6 + 2 = 5$

Since $K_{3,3}$ is simple and loop free, we have $3|R| \le 2|E|$,

$\Rightarrow 3 \times 5 \le 2 \times 9 \Rightarrow 15 \le 18$.

which is a contradiction, our assumption is wrong.

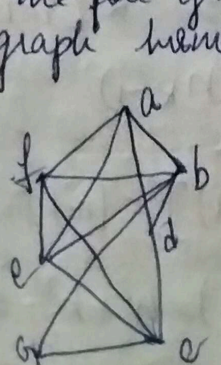
$\therefore$ when $m \ge 3$ and $n \ge 3$, $K_{m,n}$ is non-planar.

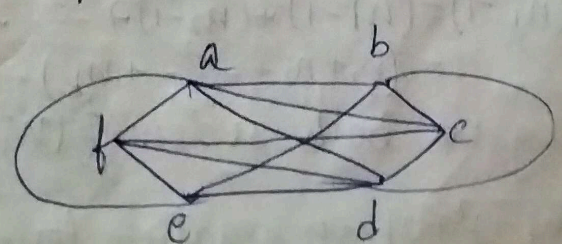
So, a complete bipartite graph $K_{m,n}$ is planar iff $m \le 2$ (or) $n \le 2$.

NOTE: A graph G is planar if and only if G does not contain a subgraph homomorphic to K_5 or $K_{3,3}$.

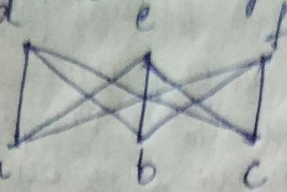
(or) A graph G is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5

② S.T. the foll. graphs are not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$.

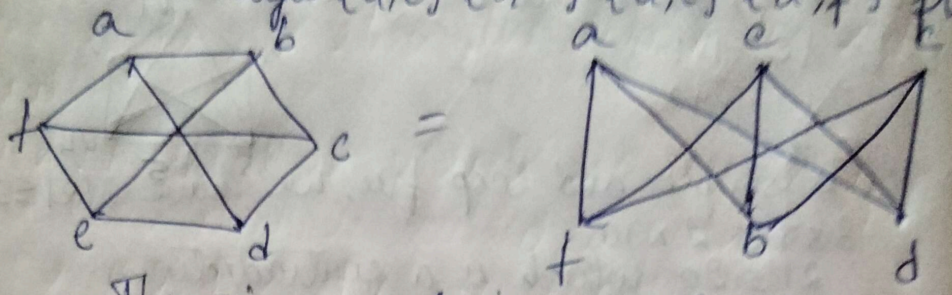




Sol: (i) Delete the edges {a, b} {f, e} {a, g} {g, h} {a, f} {c,



(ii) Delete the edges {a, e} {b, d} {a, c} {d, f} from G



The given graph is not planar.

(2) Theorem: A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let G be a simple graph of order n. Let $G_1, G_2, ----, G_k$ be the components of G. Let the no. of vertices in ith component be n_i i.e., $|V(G_i)| = |V_i| = n_i \quad 1 \le i \le k$.

Then $|V| = \sum_{i=1}^{k} n_i = n_1 + n_2 + --- + n_k$ and $n_i > 1$

Now, max. possible edges in ith components cannot exceed $= \frac{n_i(n_i-1)}{2}$.

$\Rightarrow \max |E(G_i)| = \max |E_i| \le \frac{n_i(n_i-1)}{2} \quad 1 \le i \le k$

Hence, $|E(G)| \le \sum_{i=1}^{k} \max |E_i| = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] = \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$$

Now, $\sum_{i=1}^{k}(n_i - 1) = (n_1 - 1) + (n_2 - 1) + ---- + (n_k - 1)$

$= (n_1 + n_2 + ---- + n_k) - k$.

squaring on both sides: $\left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n-k)^2 = n^2 + k^2 - 2nk$

$\Rightarrow \sum_{i=1}^{k}(n_i - 1)^2 + 2(\text{non negative terms}) = n^2 + k^2 - 2nk$ — (1)

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1)^2 = n^2 + k^2 - 2nk - 2 \text{ (non negative terms)}$$
$$\leq n^2 + k^2 - 2nk.$$
$$\Rightarrow \sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \leq n^2 + k^2 - 2nk.$$
$$\Rightarrow \sum_{i=1}^{k} n_i^2 + k - 2n \leq n^2 + k^2 - 2nk.$$
$$\Rightarrow \sum_{i=1}^{k} n_i^2 - n \leq n^2 - nk + n - nk + k^2 - k$$
$$= n(n-k+1) - k(n-k+1)$$
$$= (n-k)(n-k+1).$$
$$\therefore \sum_{i=1}^{k} n_i^2 - n \leq (n-k)(n-k+1) \quad - ②$$

From ① & ②: $|E(G)| \leq \frac{1}{2}(n-k)(n-k+1)$.

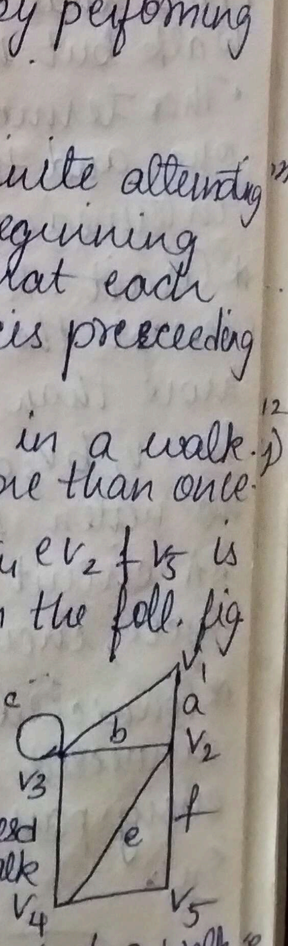
* Homeomorphic: Two graphs G_1 & G_2 are homeomorphic if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions.

* Walks, paths and circuits:

*Walk: A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceeding and foll. it.

- No edge appears more than once in a walk. A vertex, however may appear more than once.

Ex: For instance $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is a walk shown with heavy lines in the foll. fig.



- A walk is also referred to as an edge trail or a chain.
- Vertices with which a walk begins & end are called its terminal vertices. Vertices v_1 & v_5 are the terminal vertices of the walk shown in the above fig. It is possible for a walk to begin and end at the same vertex. Such a walk is called a closed walk.

* Closed walk: A walk that begins and ends at the same vertex is called a closed walk.

* Open walk: An open walk is a walk that begins and ends at ~~the~~ ~~two~~ different vertices.

* Trial: If in an open walk no edge appears more than one then the walk is called a trial.

* Path: A trial in which no vertex appears more than once is called a path.

En: $v_1 a v_2 b v_3 d v_4$ is a path whereas $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is not a path.

- A path does not intersect itself.

* Length of a path: The no. of edges in a path is called the length of a path.

- Note that a self-loop can be included in a walk but not in a path.
- The terminal vertices of a path are of degree one, and the rest of the vertices (called intermediate vertices) are of degree two.

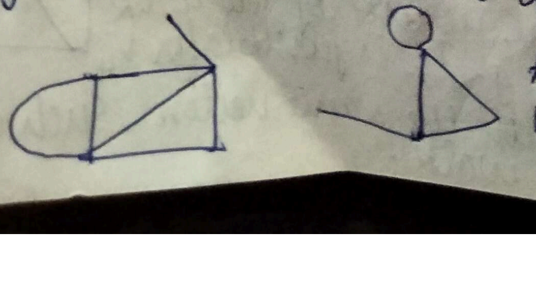
* Circuit: A closed walk in which no ~~vertex~~ edge (except the initial and the final vertex) appears more than once is called a circuit.

En: $v_2 b v_3 d v_4 e v_2$ is a closed walk in which no vertex repeated implies it is a circuit.

- A circuit is also called a cycle, elementary cycle, circular path, and polygon.

* Connected graph: A graph G is said to be connected if there is atleast one path between every pair of vertices in G. Otherwise G is disconnected.

En:



A disconnected graph with 2 components.

It is easy to see that a disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

The foll. facts are to be emphasized:

(1) A walk can be open or closed. In a walk, a vertex &/or an edge can appear more than once.

(2) A trial is a open walk in which a vertex can appear more than once but an edge cannot appear more than once.

(3) A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.

(4) A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trial; but a trial need not be a path.

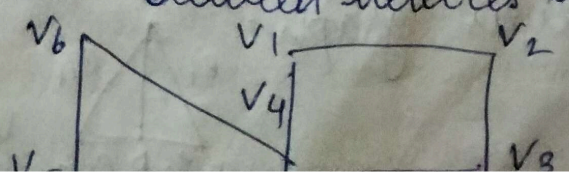
(5) A cycle is a closed walk in which neither a vertex nor an edge can appear more than once. Every cycle is a circuit; but a circuit need not be a cycle.

(6) If a cycle contains only one edge, it has to be a loop.

(7) Two parallel edges (when they occur) form a cycle.

(8) In a simple graph, a cycle must have atleast three edges. (A cycle formed by 3 edges is called a triangle).

<u>Ex ①</u>: For the graph, indicate the nature of the foll walks (the edges in between vertices being understood).



~~Sol~~:

(i) $V_1V_2V_3V_2$ (ii) $V_4V_1V_2V_3V_4V_5$ (iii) $V_1V_2V_3V_4V_5$ (iv) $V_1V_2V_3V_4$

(v) $V_6V_5V_4V_3V_2V_1V_4V_6$

Sol: (i) Open walk which is not a trial.

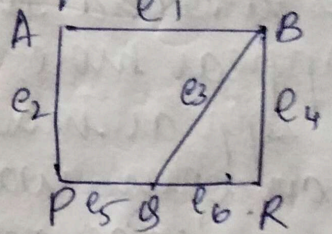
(ii) Trial which is not a path

(iii) Trial which is a path

(iv) Closed walk which is a cycle.

(v) Closed walk which is a circuit but not a cy

② Find all paths from A to R. Also indicate their lengths.



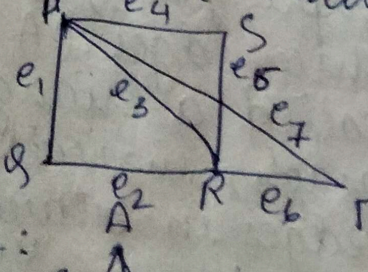
Sol: 4 paths: Ae_1Be_4R ; $Ae_1Be_3Qe_6R$;
$Ae_2Pe_5Qe_6R$; $Ae_2Pe_5Qe_3Be_4R$;

2, 3, 3, & 4 edges respectively are their lengths

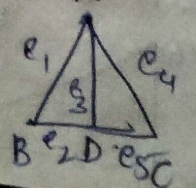
*Euler circuits: consider a connected graph G. If there is a circuit in G that contains all the edges of G, then that circuit is called an euler circuit (or Eulerian line (or) Euler tour) in G.

*Euler trial: If there is a trial in G, that contains all the edges of G, then that trial is called an euler trial (unicursal line) or

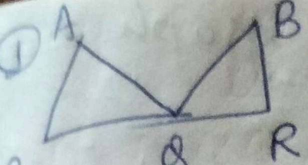
Ex: Euler graph

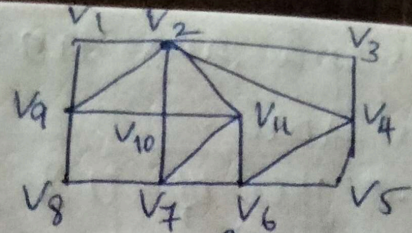


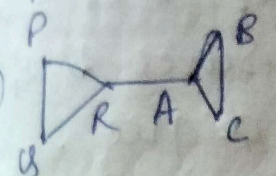
Not a euler graph:

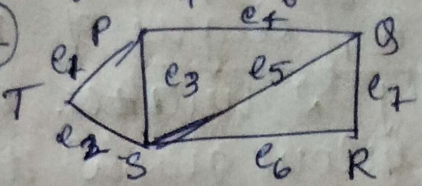


Starting at any vertex, does not end at same vertex without repeating

① 

② 

③ 

④ 

check if the aboves graphs are euler circuits.

Sol : (1) Yes. PAQBRQP

(2) Yes. $V_1V_2V_9V_{10}V_2V_{11}V_7V_{10}V_{11}V_6V_4V_2V_3V_4V_5V_6V_7V_8V_9V_1$

(3) No. cannot start and end at same vertex without repeating an edge.

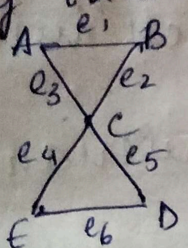
(4) ~~Yes. PTSPQSRQ~~ No.

① Find euler trial in ② after removing (V_9, V_{10}) and ~~before removing~~ in ④.

Sol : ~~After~~ : ~~P~~ ② : $V_{10}V_{11}V_6V_4V_2V_3V_4V_5V_6V_7V_8V_9V_1V_2V_9$

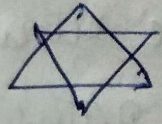
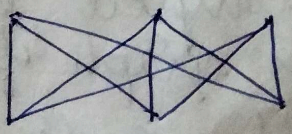
~~Before~~ : ④ : $Pe_1Te_2Se_3Pe_4Qe_5Se_6Re_7Q$.

② Verify that the graph G has an euler circuit.



$\underline{A}e_1Be_2Ce_5De_6Ee_4Ce_3\underline{A}$.

② S.T. the graph in the foll. figure contain no euler circuit.

(i)  (ii)  (iii) 

Sol : (i) Graph 1 is not a connected graph.
∴ That graph is not a euler graph.

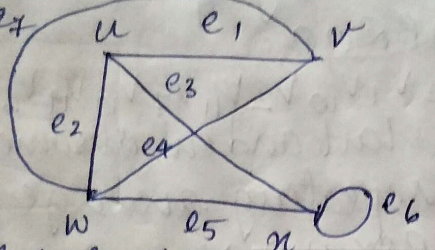
(ii) It is a line that can't form a circuit.
∴ It is not a euler circuit.

(iii) The degree of each vertex in a given graph e
... it does not contain euler ckt.

③ S.T. graph has no euler graph but has eulerian trial.

Sol: Here degree of U & degree of V is 3 & degree of W is 4, degree of $x = 4$. Since U & V have only 2 vertices of odd degree.

$\therefore$ The below graph doesn't contain ckt but it has eulerian trial.



$v e_1 u e_2 w e_7 v e_4 w e_5 x e_6 x e_3 v$ is a eulerian trial.

④ For what values of the complete graph K_n is eulerian.

Sol: WKT complete graph K_n on vertices is connected graph in which degree of each vertex is $n-1$.

Since a graph is euler graph iff it is connected and degree of each vertex is even.

We conclude that K_n is eulerian graph iff n is odd.

⑤ For what values of m, n, The complete biparted graph $K(m,n)$ is eulerian?

Sol: The complete biparted graph $K(m,n)$ is eulerian graph if both m, n are even.

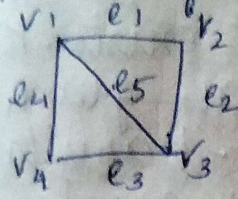
* Hamilton cycles and Hamilton paths:

Let G be a connected graph, if there is cycle in G that contains all the vertices of G then that cycle is called a hamilton cycle in G.

- A hamilton cycle in graph of n vertices consists of exactly n edges.
- A hamilton cycle in graph G must include all vertices in G. This does not mean that it should include all edges of G.
- A graph that contains hamilton cycle is called hamilton graph (or) hamiltonian cycle.

① Which of the foll. graphs are hamiltonian graphs

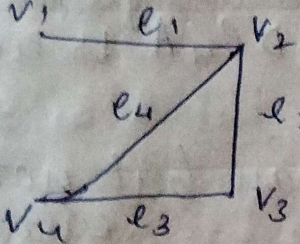
(i)



$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$

No edge repetition, no vertex repetition except end points, so it is hamiltonian graph.

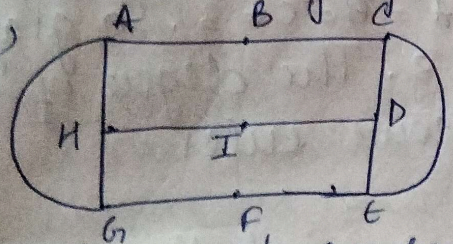
(ii)



The graph is not hamilton graph because there is no hamilton cycle in G but there is hamilton path.

② Check whether the foll. graphs are hamilton graphs.

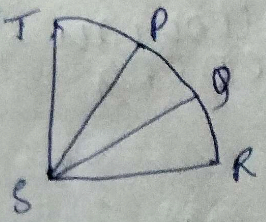
(i)



A B C D E F G H I.

The given graph isn't hamilton graph because it does not contain hamilton cycle but it contains hamilton path.

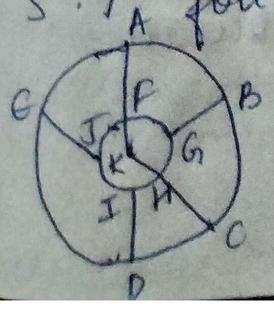
(ii)



- T – P – Q – R – S – T
- P – Q – R – S – T – P.

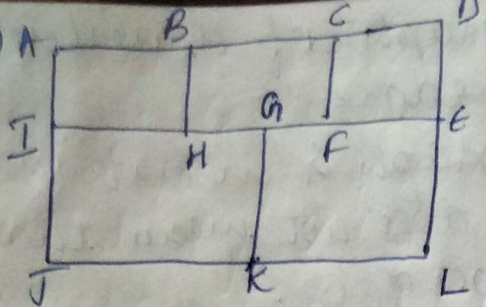
③. S.T foll. graphs are hamiltonian.

(i)



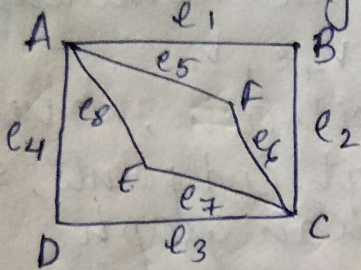
C D I J E A F K H G B C.

Is hamiltonian.

(ii)


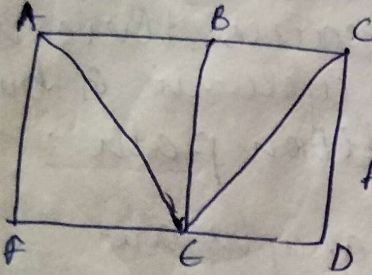
K – J – I – A – B – H – G – F – C – D – E – L – K.
Is hamiltonian.

④ Which of the foll. are euler's graph? Hamiltonian graphs :

(i)


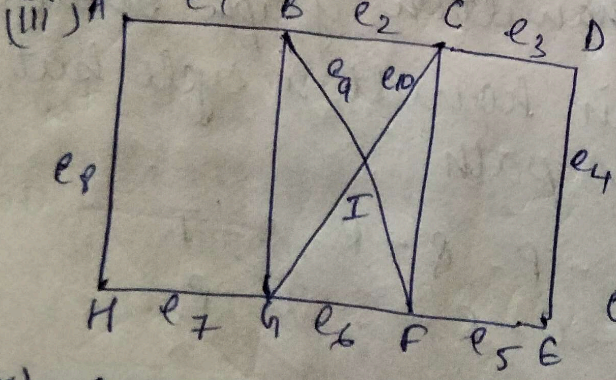
euler's circuit exists. Hence it is eulerian graph.
A – B – C – D – A – hamiltan cycle
∴ Hamiltonian graph.

A e_1 B e_2 C e_6 F e_5 A e_8 E e_7 C e_3 D e_4 A is euler c

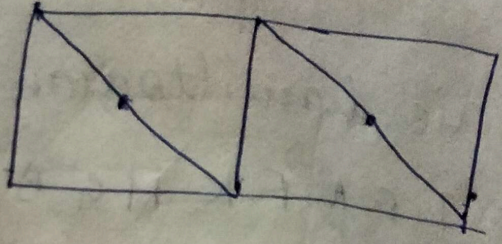
(ii)


Hamiltonian but not euler.
ABCDEFA – hamilton cycle.

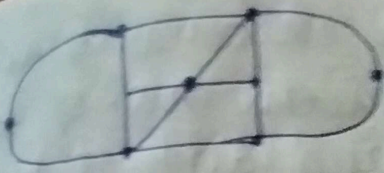
In this graph degree of A = B = C = 3. So graph has 3 vertices of degree 3. The degree of each vertex is even to become euler.

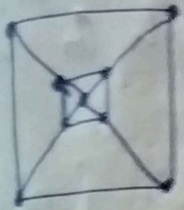
(iii)


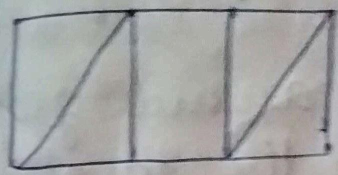
It is both eulerian and hamiltonian.
GHABICDEFG.

(iv)


Neither eulerian nor hamiltonian.

(v)  Hamiltonian.

(vi)  Hamiltonian not eulerian.

(vii)  Hamiltonian not eulerian.

* Graph Coloring: It is an assignment of colour to an elements of a graph subject to certain constraints.

· The starting point of graph colouring is the vertex colouring. The other colouring problems like edge colouring and region colouring can be transformed into vertex colouring.

· An edge colouring of a graph is just a vertex colouring of its line graph.

· The region colouring of a planar graph is the problem of colouring of its planar or dual graph.

* Vertex colouring or proper colouring:

The assignment of colours to the vertices of G, one colour to each vertex so that the adjacent vertices are assigned different colours is called the proper colouring of G (or) simply vertex colouring of G.

A graph in which every vertex has been assigned a colour according to a proper colouring is called a properly coloured graph.

· The n colouring of G is a proper colouring of G using n colours. If G has n-colouring then G is said to be n-colourable.

* Chromatic number: The chromatic no. of a graph G is the minimum no. of colours needed for proper colouring and it is denoted by $\chi(G)$. Thus, a graph G is K-chromatic if $\chi(G) = K$.

NOTE 1: A graph consisting of only isolated vertices is 1-chromatic.

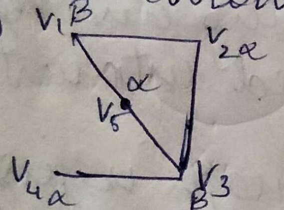
2: The chromatic no. of a null graph is 1

3: $\chi(G) \leq n$ where n is the no. of vertices of G.

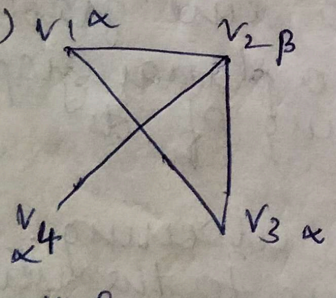
4: A graph with 1 or more edges is atleast 2-chromatic.

5: If G_1 is subgraph of G the $\chi(G_1) \leq \chi(G)$

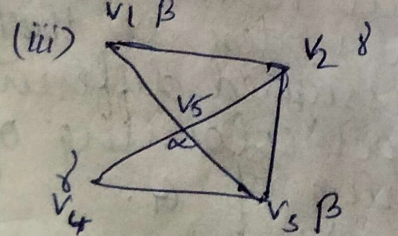
① Find chromatic no. for the foll. graphs.

(i)


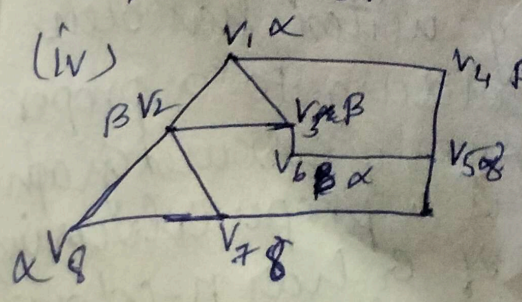
$\chi(G) = 2$

(ii)


$\chi(G) = 2$

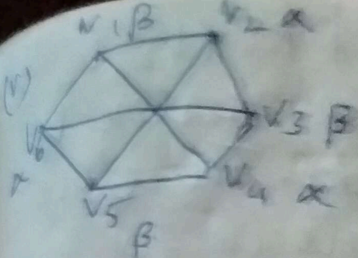
(iii)


$\chi(G) = 3$

(iv)


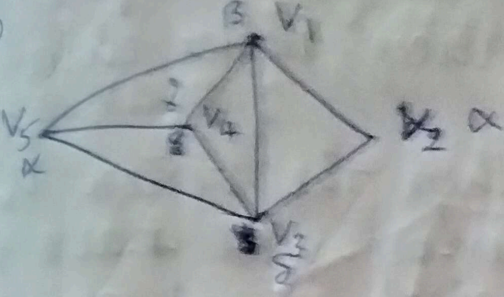
$\chi(G) = 3$

(v)



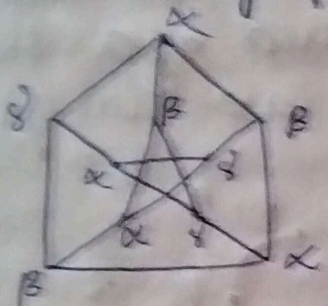
$\chi(G) = 2$

(vi)



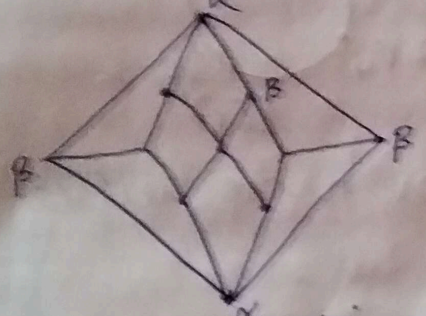
$\chi(G) = 4$

(vii) Peterson graph:



$\chi(G) = 3$.

(viii)



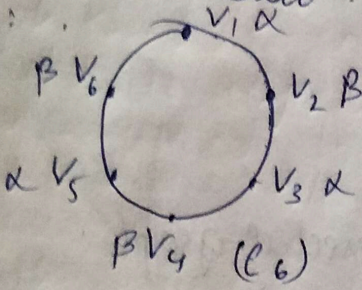
Herschel graph.

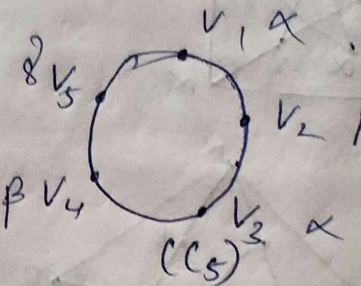
Q. P.T. the chromatic no. of cycle C_n with n vertices is.
(i) 2 if n is even.
(ii) 3 if n is odd

Sol. Let the cycle C_n has the vertices $v_1, v_2, \ldots, v_n$ appearing in order of the cycle. If we assign colour 'α' to v_1, v_2 must be coloured with different colour say colour 'β' because v_2 is adjacent to v_1. Assign 2 colours alternatively to the vertices, starting with v_1 assign 'α' to odd vertices ~~and~~ i.e. $(v_1, v_3, v_5, \ldots)$

and 'β' to even vertices i.e. $(v_2, v_4, v_6, \ldots)$
Suppose n is even then the vertex v_n is even and vertex v_1 is odd. Therefore we have 2 different colours for every pair of adjacent vertices. Hence chromatic no. of G is 2.
Suppose n is odd, then the vertex v_n is odd vertex and therefore will have the colour α, and the graph is not properly coloured. To make it properly coloured, it is enough if v_n is assigned by a 3rd colour 'γ'. Thus in this case the graph is 3-chromatic.

Ex:

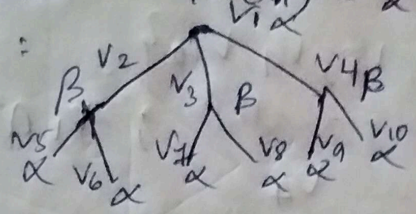




• Every tree with 2 or more vertices is 2-chromatic.

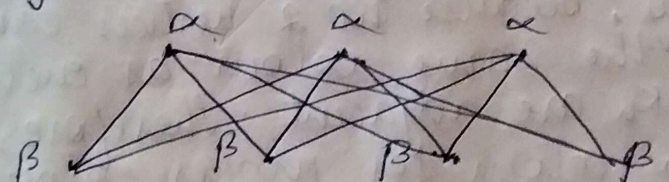
① Consider any vertex v in the given tree. Consider T as a rooted tree at vertex v, and paint v with colour α. Paint all vertices adjacent to v using colour 'β'. Next paint the vertices adjacent to these vertices using colour α. Continue this colouring process till every vertex in T has been painted. Thus all vertices at odd distances from v have colour 'β' while v & vertices at even distances from v have colour 'α'. Now along any path in T, the vertices are of alternating colours. WKT there is 1 and only 1 path btw. any two vertices in a tree. So no 2 adjacent vertices of T have the same colour. Thus T can be properly coloured using 2 colours $\therefore \chi(T) = 2$.

Ex :

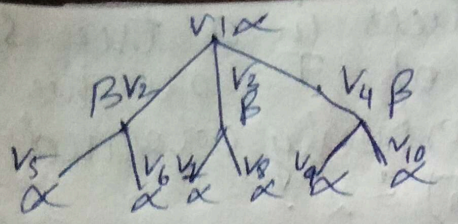


② The chromatic no. of a nonnull graph is 2 iff the graph is biparted.

Sol : (i) Let G be a biparted graph. We have to prove $\chi(G) = 2$.
Since G is biparted, its vertex set V can be partioned into 2 sets V_1 & V_2 such that every edge of G joins some vertex in V_1 to some vertex in V_2. Since not 2 of the vertices in V_1 & V_2 are adjacent. We can assign a colour 'α' to vertices in V_1 & colour β to vertices in V_2. Therefore the graph is coloured with only 2 colours. Hence $\chi(G) = 2$.

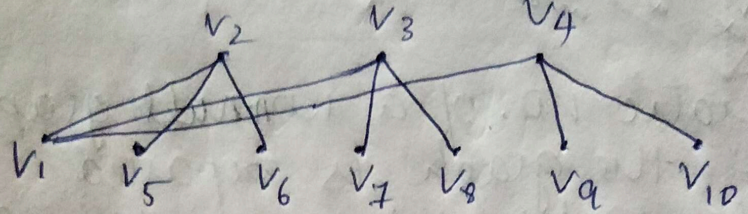


(ii) Let G be a 2-chromatic graph i.e. $\chi(G) = 2$. Let V_1 denote the set of all vertices for which 1st colour is assigned and V_2 be the set of all vertices for which the 2nd colour is assigned. Thus V_1 & V_2 are disjoint and $V_1 \cup V_2 = V$. Hence the 2 colour classes form a partition of G otherwise atleast 2 vertices in V_1 have the same colour. Therefore G is a biparted graph.



$V_1 = \{v_1, v_5, v_6, v_7, v_8, v_9, v_{10}\}$

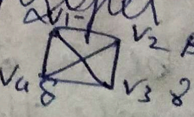
$V_2 = \{v_2, v_3, v_4\}$; $V_1 \cap V_2 = \phi$



③ The chromatic no. of a complete graph with n vertices K_n is 'n'.

Sol: Let $v_1, v_2, \ldots, v_n$ are n vertices of a complete graph K_n. Let a colour 'α' be assigned to v_1 and since v_2 is adjacent to v_1 a different colour 'β' is required to assign to v_2. Next, since v_3 is adjacent to both v_1 & v_2, so another colour 'γ' is required to be assigned to v_3. In this way the different n colours are required to be assigned to the n vertices respectively, because no 2 vertices can be assigned the same colour as every 2 vertices of K_n are adjacent. Therefore we require 'n' colour for proper colouring.

Thus $\chi(K_n) = n$. Ex:



$\chi(K_4) = 4$.

④ If $\Delta(G)$ is the maximum of the degrees of the vertices of a graph G, then prove that $\chi(G) \leq 1 + \Delta(G)$ using mathematical induction

Sol: Suppose G contains $n=2$ vertices. Then the degree of each vertex is 1 so that $\Delta(G) = 1$. Also for proper colouring we require minimum 2 colour $\therefore \chi(G) = 2$. Hence

$\chi(G) = 1 + \Delta(G)$

Hence the result is true for $n=2$.
Assume that eq ① satisfies the inequality for all graphs with k vertices.
Let G' be a graph with $k+1$ vertices. If we remove any vertex v from G' then the result in graph H will have k vertices.
$$\Delta(H) \leq \Delta(G') \quad - ②$$
Since H has k vertices, $\chi(H) \leq 1+\Delta(H)$ - ③
from ② & ③: $\chi(H) \leq 1+\Delta(G')$ - ④
Now, a proper colouring of G' can be acheived by removing the colours assigned to the vertices in H and by assigning a colour to v that is diffelent from the colours assigned to the vertices adjacent to it. The colour to be assigned to v can be one of the colours already assigned to a vertex in H that is not adjacent to v.
Thus a proper colouring of G' can be done without using a new colour.
Therefore $\chi(G') = \chi(H)$ - ⑤
from ④ & ⑤: $\chi(G') = \chi(H) \leq 1+\Delta(G')$
$$\Rightarrow \chi(G') \leq 1+\Delta(G')$$
Hence the result is true for $n=k+1$.
Therefore by the method of mathematical induction it follows that the inequality ① holds for all graphs.

* Adjacent ~~lines~~ regions: An assignment of colours to the regions of a map such that adjacent regions have different colours is known as regional colouring.

① Four colour Problem / 4-Colour Theorem:
Every simple, connected planar graph is 4-colourable. (or)

If the regions of a planar graph are colour[ed] so that adjacent regions have different colours then no more than 4-colours are required, that is $\chi(G) \leq 4$.

<u>Proof</u>: • Take any map and divide it into a set of connected regions $R_1, R_2, \ldots, R_n$ wit[h] continuous boundaries.

(i) Represent each region by a point contain[ed] in the region. (R_i & v_i)

(ii) If 2 regions share a boundary, draw [a] simple curve between the points.

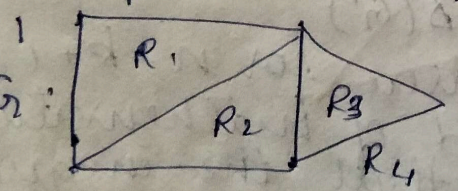
(iii) Points representing regions are vertice[s] and curves joining them are edges.

(iv) Edges are drawn so that their interio[rs] are disjoint.

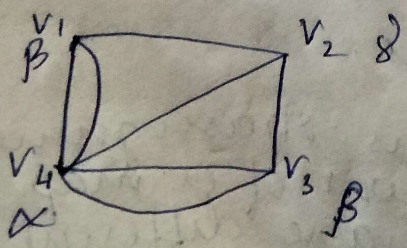
(v) This gives a planar graph which is known as the duel of the original map. From this duel graph, we can observ[e] that $\chi(G) \leq 4$.

Therefore, every simple connected planar graph is 4-colourable.

<u>Example</u>: Consider the planar graph G:

G : 

. Its duel is G'.

G' : 

$\chi(G') = 3$.