Discrete Mathematics

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Sets: "Well defined collection of distinct objects"

Generally denoted by capital letters $A, B, C, \ldots$
$X, Y, Z, \ldots$

If x is an element of a set X, then we write $x \in X$ and $x \notin X$ means x does not belongs to X.

ex. Set of first 5 positive odd integers.

$A = \{1, 3, 5, 7, 9\}$.

Set-builder notation: $A = \{x \mid x \in S, P(x)\}$, means
A be a set of all elements x of S such that x satisfied $P(x)$ such as $A = \{x \mid x \in \mathbb{Z}, x > 0\}$.
$=$ Set of positive integers ...

ex. $B = \{x \mid x \in \mathbb{C}, x^4 = 1\} = \{1, -1, i, -i\}$.

Universal Sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Z}^*, \mathbb{E}, \mathbb{Q}, \mathbb{Q}^*, \mathbb{Q}^+, \mathbb{R}, \mathbb{R}^*, \mathbb{R}^+, \mathbb{C}, \mathbb{C}^*$.

($\mathbb{Z}^*$ ↓ non-zero integers)

Subset: $X \subseteq Y$ means every element of X is an element of Y. Y is said to be superset of X.

($\subseteq$ ↓ Subset of)

$X \nsubseteq Y$
↳ not subset of.

ex. $A = \{b, a, c\}$, $B = \{a, b, c\}$.

$A \subseteq B$. as well as $B \subseteq A$

ex. $A = \{0, 2, 4, 6\}$, $B = \{x \mid x \in \mathbb{Z}, 0 \leq x \leq 6\}$.

$A \subseteq B$ and $B \supseteq A$

Note: i) $X \subseteq X$

ii) $X \subset Y$ is called X is "proper subset" of Y i.e, $\exists$ atleast one element in Y which is not in X.

ex. $E = \{2n \mid n \in \mathbb{Z}\}$.

$E \subset \mathbb{Z}$

Equal Set: $X = Y$, if every elemnt of X is an elemnt of Y and vice versa.

ex. $A = \{1, 2, 3, 4\}$, $B = \{x \mid x \in \mathbb{Z}^+, x^2 < 18\}$.

$A = B$.

Empty or Null set: $\phi = \{\ \}$, no element.

Finite and infinite set:

X is said to be finite if $\exists$ a non-negative integer n such that X has n element otherwise X is called an infinite set.

ex. $A = \{a, b, c\} \to$ finite set,

$\mathbb{Z}^* \to$ infinite set.

$\phi \to$ empty set with zero elements is a finite set.

Cardinality: S is finite set, $|S| =$ number of elements in S or cardinality of S.

If $|S| = 1$, S is called singleton set.

Power set: For any set X, $P(X) = \{A \mid A \subseteq X\}$.

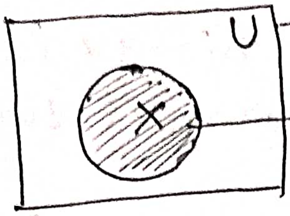
$=$ set of all subsets of X.

ex. $X = \{a, b, c\}$, $P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Note Let $|X| = n$ then $|P(X)| = 2^n$.

Universal Set (U): To avoid logical difficulties that arises in foundation of set theory, assume that all sets sets under consideration must be a subset of arbitrarily chosen ~~set~~ but fixed set called Universal set.

Venn diagrams (Pictorial representation of sets).

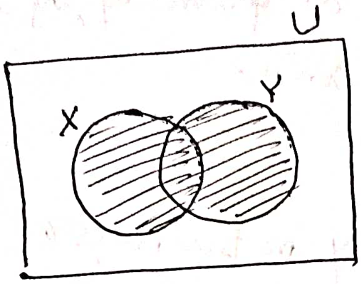


→ Universal set represented as rectangle.

→ subsets represented in circle with dark shade.

Operations on set

Union

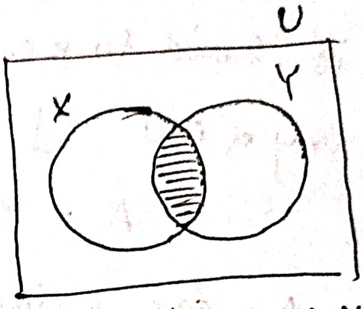


$X \cup Y$

$= \{x \mid x \in X \text{ or } x \in Y\}$

= x is a member of <u>atleast</u> one of the sets X and Y.

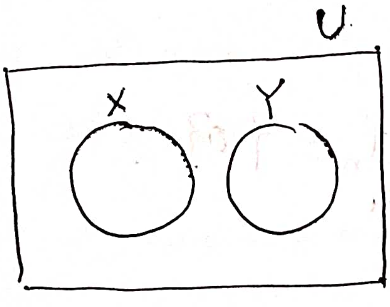
intersection



$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$

= x is an element of X as well as Y.

Disjoint



$X \cap Y = \phi$.

<u>Note</u>: i) $X \subseteq X \cup Y$; $Y \subseteq X \cup Y$

ii) $X \cap Y \subseteq X$, $X \cap Y \subseteq Y$.

iii) If $X \subseteq Y$, then
$X \cup Y = Y$, $X \cap Y = Y$.

iv) Laws of identity:
$X \cup \phi = X$, $X \cap \phi = X$.

v) idempotent law:
$X \cup X = X$, $X \cap X = X$

vi) Commutative law:
$X \cup Y = Y \cup X$, $X \cap Y = Y \cap X$

vii) Associative law:
$(X \cup Y) \cup Z = X \cup (Y \cup Z)$.
$(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

viii) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$
$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
Distributive law.

ix) Laws of absorptivity:
$X \cap (X \cup Y) = X$
$X \cup (X \cap Y) = X$

Remark: ✓ $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$
✓ $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.
* $x \in A - B \Rightarrow x \in A$ and $x \in B^c$
* $(x, y) \notin A \times B \Rightarrow x \in A$ and $y \in B$
✓ $x \in A \cup B \Rightarrow x \in A$ or $x \in B$
✓ $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

Index set: let us consider finite collection n sets,
$A_1, A_2, \cdots A_n$, then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x \mid x \in A_i, 1 \le i \le n\}$$
for some i such that

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$
$$= \{x \mid x \in A_i, \text{ for all } i \text{ such that } 1 \le i \le n\}.$$

A set I is said to be index set for a family A of sets, if for any $\alpha \in I$, $\exists$ a set $A_\alpha \in A$ and

$$A = \{A_\alpha \mid \alpha \in I\}.$$

The sets A_α are said to be mutually or pairwise disjoin if for $\alpha, \beta \in I$, $\alpha \neq \beta$ implies $A_\alpha \cap A_\beta = \phi$.

ex. let $n \in \mathbb{N}$, $I_n = \{x \in \mathbb{R} \mid -\frac{1}{n} < x < \frac{1}{n}\}$.

$I_1 = \{x \in \mathbb{R} \mid -1 < x < 1\}$

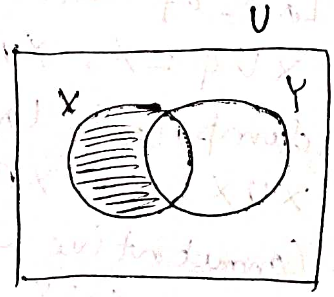
$I_2 = \{x \in \mathbb{R} \mid -\frac{1}{2} < x < \frac{1}{2}\}$

$I_3 = \{x \in \mathbb{R} \mid -\frac{1}{3} < x < \frac{1}{3}\}$

$\vdots$

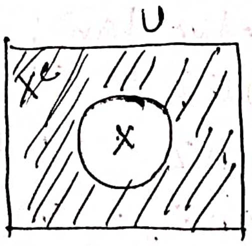
$\bigcup_{n \in \mathbb{N}} I_n = \{x \in \mathbb{R} \mid -1 < x < 1\}$; $\bigcap_{n \in \mathbb{N}} I_n = \{0\}$.

Difference:



$X - Y$ = difference of X and Y or relative complement of Y in X
$= \{x \mid x \in X \text{ but } x \notin Y\}$
$= X \cap Y^c$

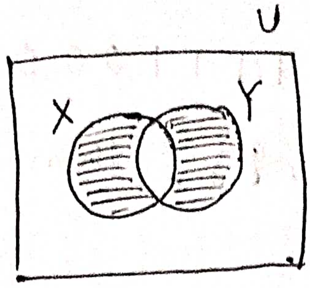
Complement



Complement of $X = X^c$ or X' or $\bar{X}$
$= \{x \mid x \in U, x \notin X\}$.
$= U - X$

Note: (De Morgan's Laws): $(X \cup Y)^c = X^c \cap Y^c$, $(X \cap Y)^c = X^c \cup Y^c$

Symmetric difference:



$X \Delta Y$

$= (X - Y) \cup (Y - X)$

= Set of elements either only in X or Y.

Ordered pair and Cartesian cross product:

- ordered pair of the elements $x \in X$ and $y \in Y$ written as $(x, y) = \{\{x\}, \{x, y\}\}$.
  $(x, y) = (z, w)$ iff $x = z$ and $y = w$ for all $x, z \in X$ and $y, w \in Y$.

Cartesian product of X and Y denoted by

$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

Note: i) $X \times \phi = \phi \times Y = \phi$

ii) In general $X \times Y \neq Y \times X$.

iii) If $|X| = m$, $|Y| = n$, $|X \times Y| = mn$.

iv) $X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots x_n) \mid x_i \in X_i$ for all $i = 1, 2, \ldots n\}$.

is the set of all ordered n-tuples $(x_1, x_2, \ldots x_n)$.

Computer representation of Sets:

Sets are described as a sequence of bit string (sequence of 0's and 1's). Length is number of 0's and 1's in it.

Consider $A = \{a, b, c, d\}$, $U = \{a, b, c, d, e, f, g, h\}$.

Let us write $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$, $a_5 = e$, $a_6 = f$, $a_7 = g$, $a_8 = h$.

Hence a bit string of length 8 represents the subset of U.

Consider S_A denotes the bit string of A.

$$S_{A_i} = \begin{cases} 1, & \text{if } a_i \in A \\ 0, & \text{if } a_i \notin A \end{cases}$$

Hence bit string of A is 1 1 1 1 0 0 0 0.

Consider $B = \{a, b, d, f, g\}$, then bit string of B is

$S_B = 1\,1\,0\,1\,0\,1\,1\,0$.

<u>Note</u>

$$S_{(A\cup B)i} = \begin{cases} 1 & \text{if either } S_{Ai} = 1 \text{ or } S_{Bi} = 1 \\ 0 & \text{else} \end{cases}$$

$$S_{(A\cap B)i} = \begin{cases} 1 & \text{if } S_{Ai} = S_{Bi} = 1 \\ 0 & \text{else} \end{cases}$$

$$S_{A'i} = \begin{cases} 1 & \text{if } S_{Ai} = 0 \\ 0 & \text{if } A_i = 1 \end{cases}$$

<u>Ex.</u> $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{2, 4, 6, 8, 11, 12\}$, $B = \{1, 2, 5, 7, 8, 10\}$.

$S_A = 010101010011$, $S_B = 110010110100$

~~$S_B = 1\,1\,0\,0$~~

~~$S_{A\cup B} =$~~

$S_A = 0\;1\;0\;1\;0\;1\;0\;1\;0\;0\;1\;1$

$S_B = 1\;1\;0\;0\;1\;0\;1\;1\;0\;1\;0\;0$

$S_{A\cup B} = 1\,1\,0\,1\,1\,1\,1\,1\,0\,1\,1\,1$

$S_{A\cap B} = 0\,1\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0$

$S_{A'} = 1\,0\,1\,0\,1\,0\,1\,0\,1\,1\,0\,0$

<u>Principle of Inclusion - Exclusion</u> (<u>Subtraction principle</u>)

<u>Sum rule</u>: A_i, $i = 1, 2, \dots n$ are the finite numbers of sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|, \text{ when } \boxed{A_i \cap A_j = \phi}, i \neq j \text{ for all } i, j.$$

disjoint sets.

If $A_i \cap A_j \neq \phi \;\forall\, i, j$ then

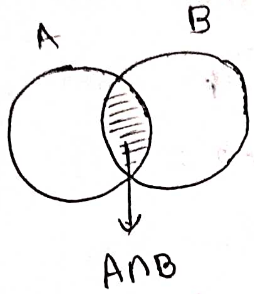
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |A_i \cap A_j| + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} |A_i \cap A_j \cap A_k| - \dots$$

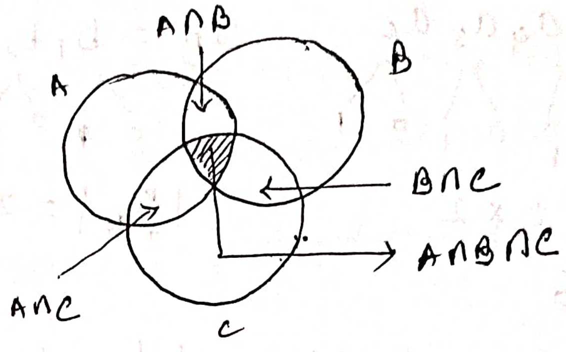
$+ (-1)^n \left| \bigcap_{i=1}^{n} A_i \right|$

For two sets, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

" Three ", $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2|$

$- |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$





<u>Ex. 1</u> A = Set of positive integers that are ≤ 30 and multiple of 4

B = Set of positive integers that are ≤ 30 and multiple of 6.

$A = \{4, 8, 12, 16, 20, 24, 28\}$, $B = \{6, 12, 18, 24, 30\}$.

$A \cap B = \{12, 24\}$, $A \cup B = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30\}$

$|A| = 7$, $|B| = 5$, $|A \cap B| = 2$, then,

$|A \cup B| = 7 + 5 - 2 = 10$.

<u>Another way</u>: $A = \{4n \mid n \in \mathbb{Z}, 1 \leq 4n \leq 30\}$.

$= \{4n \mid n \in \mathbb{Z}, 1 \leq n \leq \frac{30}{4}\}$.

$|A| = \lfloor \frac{30}{4} \rfloor = 7$.

$B = \{6n \mid n \in \mathbb{Z}, 1 \leq 6n \leq 30\}$.

$|B| = \lfloor \frac{30}{6} \rfloor = 5$

$A \cap B = \{12n \mid n \in \mathbb{Z}, 1 \leq 12n \leq 30\}$. min

$|A \cap B| = \lfloor A \cap B \rfloor = \frac{30}{12} = 2$.

$|A \cup B| = \underline{10}$.

Ex. 2 A = The set of bit string of length 6 that begin with 101

B = The set of bit string of length 6 that end with 00.

$S_A = 101\,a_4\,a_5\,a_6$, $S_B = b_1\,b_2\,b_3\,b_4\,00$

$a_4, a_5, a_6 \in \{0, 1\}$, $b_1, b_2, b_3, b_4 \in \{0, 1\}$

$|A| = 2 \times 2 \times 2$
$= 6$

$|B| = 2 \times 2 \times 2 \times 2 = 8$.

$A \cap B$ = set of bit string of length 6 that begin with 101 and end with 00.

$S_{A \cap B} = 101\,c_4\,00$

$c_4 \in \{0, 1\}$

$|A \cap B| = 2$

Hence $|A \cup B|$
$= 6 + 8 - 2 = 12$

Ex. 3 Determine all the +ve integers ~~that~~ less than 2102 and are divisible by at least one of the primes 2, 3, and 5.

$A = \{2n : | n \in \mathbb{Z}, 1 \leq 2n \leq 2102\}$, $|A| = \lfloor \frac{2102}{2} \rfloor = 1051$

$B = \{3n : | n \in \mathbb{Z}, 1 \leq 3n \leq 2102\}$, $|B| = \lfloor \frac{2102}{3} \rfloor = 700$

$C = \{5n : | n \in \mathbb{Z}, 1 \leq 5n \leq 2102\}$, $|C| = \lfloor \frac{2102}{5} \rfloor = 420$

$A \cap B = \{6n : | n \in \mathbb{Z}, 1 \leq 6n \leq 2102\}$, $|A \cap B| = 350$

$A \cap C = \{10n | n \in \mathbb{Z}, 1 \leq 10n \leq 2102\}$, $|A \cap C| = 210$

$B \cap C = \{15n | n \in \mathbb{Z}, 1 \leq 15n \leq 2102\}$, $|B \cap C| = 140$

$A \cap B \cap C = \{30n | n \in \mathbb{Z}, 1 \leq 30n \leq 2102\}$, $|A \cap B \cap C| = 70$

$|A \cup B \cup C| = 1051 + 700 + 420 - (350 + 210 + 140) + 70$
$= 1541$

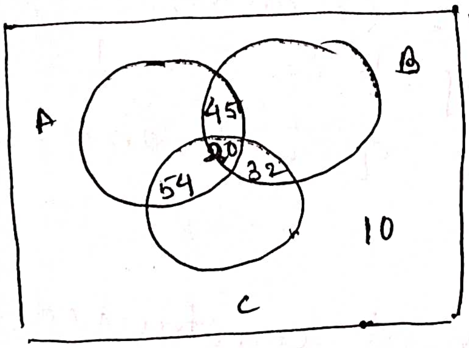
Ex.4 Consider the nested loops:
Assignment

for i := 1 to 10 do
    for j := 1 to 20 do
        print "Hello";

a. How many times the word "Hello" printed?
b. How many times does the inner loop execute?
c. How many times does the outer loop execute?

Ex.5 Find the number of palindromes over a set A of length 10.
Assignment

Ex.6 A = People go to resort for vacation
B = " took cruise " "
C = " go to national park "



Suppose, $|A| = 150$,
$|B| = 100$,
$|C| = 300$.

a) How many go to resort only?
b) How many go to either resort or cruise?
c) How many use one of the three?

Ex7. Find the number of integers between 1 and 10,000 inclusive, which are divisible by none of 5, 6 or 8.

8. Show that $A - (B \cap C) = (A - B) \cup (A - C)$

9. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

10. Out of a class of 153, 54 taken history, 63 geo, 62 eco and 43 geo & His, 45 His & eco, 46 geo & eco and 37 all three. How many have not taken any.

Mathematical Logic

Statement or proposition — a declarative statement that is either true or false, but not both.

Typically, we use small letters $a, b, c, \ldots p, q, r$ as the statements, such as

p : 4 is an integer

q : $\sqrt{5}$ is an integer

r : Washington, DC, is the capital of United States.

One of the values 'Truth' or 'False' that is assigned to a statement is called 'Truth value' or 'logical Value',

Truth (T or 1). ; False (F or 0)

p is true, hence truth value of p is T or 1.

q " false, " " " q " F or 0!

Connectives :

A molecular or compound statement constructed using connectives between atomic or primary Statement.

Negation ($\sim$ or $\neg$)

Truth table

P	$\sim P$
T	F
F	T

$\sim P$ → read as "not P"

P: 2 is positive

$\sim$P: It is not the case that 2 is positive
or
2 is not positive

Conjunction ($\wedge$)

p: 2 is even

q: 7 divides 14

consider r: 2 is even and 7 divides 14.

p	q	$p \wedge q = r$
T	T	T
T	F	F
F	T	F
F	F	F

→ read as "p and q"

$p \wedge q$ is true if both p and q are true.

Disjunction ($\vee$)

p: 2 is integer

q: 3 is greater than 5.

We can form the statement:

r: 2 is integer or 3 is greater than 5.

p	q	$r = p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

→ read as "either p or q"

$p \vee q$ is true if at least one of p or q is true.

Remark: $p \vee q$ means "either p or q or both p and q"

Implication ($\rightarrow$) or a condition:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

→ read as "if p then q" or "p implies q."

$p \rightarrow q$ is false when p is true and q is false.

Here, p is called hypotesis (antecedent) and q is called conclusion (consequent).

Definition: i) $q \rightarrow p$ is called converse of the implication $p \rightarrow q$

ii) $\sim p \rightarrow \sim q$ is called inverse of the implication $p \rightarrow q$

iii) $\sim q \rightarrow \sim p$ is called contrapositive of the implication $p \rightarrow q$.

Remark: In $p \rightarrow q$, p is sufficient for q

ex. p: Today is Sunday, q: I will go for a walk.

$p \to q$: If today is Sunday, then I will go for a walk.

$q \to p$: If I will go for a walk, then today is Sunday.

$\sim p \to \sim q$: If today is not sunday, then I will not go for walk.

$\sim q \to \sim p$: If I will not go for walk, then today is not Sunday.

Biimplication or biconditional ($\longleftrightarrow$)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ → read as "q iff p" or "p is ~~suf~~ necessary and sufficient for q" or "q is necessary or sufficient for p"

$p \leftrightarrow q$ is true only when both p and q has same truth value.

Statement Formula or Well formed formula (w.f.f)

$\sim, \wedge, \vee, \to, \leftrightarrow$ are called logical Connectives

$p, q, r, \ldots$ are called Statement Variable.

A wff is defined as:

i) A statement variable is a statement formula.

ii) If A & B are Statement formula, then $\sim A$, $A \wedge B$, $A \vee B$, $A \to B$, and $A \leftrightarrow B$ are Statement formulas. (PV is not wff)

iii) Those expressions are statement formula that are constructed only by using (i) and (ii).

Note $\sim$ is unary operator.

$\wedge, \vee, \to, \leftrightarrow$ are binary operators.

Precedence of logical connectives

$\sim$ Highest
$\wedge$ second highest
$\vee$ Third "
$\rightarrow$ Fourth "
$\leftrightarrow$ Fifth "

Example ① $A : \sim(P \vee Q) \rightarrow (Q \wedge P)$

P	Q	$P \vee Q$	$\sim(P \vee Q)$	$Q \wedge P$	A
T	T	T	F	T	T
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	T	F	F

② $B : (\sim P \wedge Q) \rightarrow r$

Assignment

Tautology ; Contradiction :

A wff A is said to be a tautology if truth value of A is T for any assignment of the truth values T & F to the statement variables occuring in A. If the truth value of A is F, then it is called Contradiction.

$\models A$ means, the statement formula A is a tautology.

Ex. $A : (\sim P \wedge Q) \rightarrow (\sim(Q \rightarrow P))$

P	Q	$\sim P$	$\sim P \wedge Q$	$Q \rightarrow P$	$\sim(Q \rightarrow P)$	A
T	T	F	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	F	T	F	T

Remark

P	$\sim P$	$\sim P \vee P$	$\sim P \wedge P$
T	F	T	F
F	T	T	F

$\boxed{\sim P \vee P}$ is Tautology
$\boxed{\sim P \wedge P}$ " Contradiction

Logical implication and logical equivalence

A and B are statement formulas.

$A \to B$ is said to logically imply B if $A \to B$ is a tautology.

A is said to logically equivalent to B if $A \leftrightarrow B$ is a tautology. Symbolically we write

$$A \equiv B \ (\text{or } A \Leftrightarrow B).$$

ex. $A : P \wedge (P \to q)$, $B : q$

P	q	$P \to q$	A	$A \to B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Hence A logically implies B

ex. $A : P \to q$, $B : \sim P \vee q$

P	q	$\sim P$	$P \to q$ (A)	$\sim P \vee q$ (B)	$A \leftrightarrow B$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence, $A \equiv B$ i.e,

$$\boxed{P \to q \equiv \sim P \vee q}$$

Laws:

1. Commutative law: $P \wedge q \equiv q \wedge P$, $P \vee q \equiv q \vee P$
2. Associative law: $(P \wedge q) \wedge r \equiv P \wedge (q \wedge r)$
   $(P \vee q) \vee r \equiv P \vee (q \vee r)$
3. Distributive law: $P \vee (q \wedge r) \equiv (P \vee q) \wedge (P \vee r)$
   $P \wedge (q \vee r) \equiv (P \wedge q) \vee (P \wedge r)$
4. Absorption law: $P \wedge (P \vee q) \equiv P$, $P \vee (P \wedge q) \equiv P$
5. Idempotent law: $P \wedge P \equiv P$, $P \vee P \equiv P$
6. Double negation law: $\sim(\sim P) \equiv P$
7. DeMorgan's law: $\sim (P \wedge q) \equiv \sim P \vee \sim q$
   $\sim (P \vee q) \equiv \sim P \wedge \sim q$.

(8) A is tautology and B is any statement,
$A \vee B \equiv A$ i.e, $T \vee B \equiv T$
$A \wedge B \equiv B$ i.e, $T \wedge B \equiv B$
If A is contradiction
$A \vee B \equiv B$ i.e, $F \vee B \equiv B$
$A \wedge B \equiv A$ i.e, $F \wedge B \equiv F$

Duality Law: Two formulas A & A^* are duals to each others if either one can be obtained from the other by replacing '$\vee$' by '$\wedge$' and '$\wedge$' by '$\vee$'.

ex. 1. $A: (p \vee q) \wedge r$, $A^*: (p \wedge q) \vee r$

2. $A: \sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim s))$

$A^*: \sim(p \wedge q) \vee (p \wedge \sim(q \vee \sim s))$.

3. Show that

a) $\sim(p \wedge q) \rightarrow (\sim p \vee (\sim p \vee q)) \Leftrightarrow (\sim p \vee q)$

b) $(p \vee q) \wedge (\sim p \wedge (\sim p \wedge q)) \Leftrightarrow \sim p \wedge q$.

$\sim(p \wedge q) \rightarrow (\sim p \vee (\sim p \vee q))$

$\Leftrightarrow \sim(\sim(p \wedge q)) \vee (\sim p \vee (\sim p \vee q))$, as $A \rightarrow B \equiv \sim A \vee B$.

$\Leftrightarrow (p \wedge q) \vee (\sim p \vee (\sim p \vee q))$ —— (*)

$\Leftrightarrow (p \wedge q) \vee (\sim p \vee q)$

$\Leftrightarrow (p \vee \sim p) \wedge (q \vee \sim p) \vee q$

$\Leftrightarrow (q \vee \sim p) \vee q$

$\Leftrightarrow q \vee \sim p \Leftrightarrow \sim p \vee q$

Taking dual in (*), $(p \vee q) \wedge (\sim p \wedge (\sim p \wedge q)) \Leftrightarrow \sim p \wedge q$.

4. Show that $(\sim p \wedge q) \rightarrow (\sim(q \rightarrow p))$ is a tautology.

$\equiv \sim(\sim p \wedge q) \vee (\sim(q \rightarrow p))$. , as $A \rightarrow B \equiv \sim A \vee B$

$\equiv (\sim\sim p \vee \sim q) \vee (\sim(q \rightarrow p))$, DeMorgan's law

$\equiv (p \vee \sim q) \vee (\sim(\sim q \vee p))$

$\equiv (p \vee \sim q) \vee (q \wedge \sim p)$

$\equiv p \vee (\sim q \vee (q \wedge \sim p))$, Associative law

$\equiv p \vee ((\sim q \vee q) \wedge (\sim q \cancel{\wedge} \vee \sim p))$

$\equiv p \vee (T \wedge (\sim q \cancel{\wedge} \vee \sim p))$

$\equiv p \vee (\sim q \cancel{\wedge} \vee \sim p)$

$\equiv (p \vee \sim p) \cancel{\wedge} \vee ([\text{struck}] \sim q) \equiv T \vee \sim q$

$\equiv T$

$\equiv \cancel{T \wedge (p \vee \sim q)}$

Functionally Complete Set of Connectives:

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives $(\vee, \wedge, \sim)$ is called a functionally comple set of connectives.

For this, we use $p \to q \Leftrightarrow \sim p \vee q$

$p \leftrightarrow q \Leftrightarrow \boxed{(p \to q) \wedge (q \to p)}$

$(\sim p \vee q) \wedge (\sim q \vee p)$

ex. Equivalent formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ is $((\sim p \wedge \sim q) \vee F) \vee (F \vee (q \wedge p))$

$\boxed{(\sim p \wedge \sim q) \vee (p \wedge q)}$

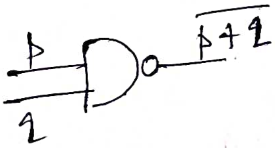
$p \wedge ((q \to r) \wedge (r \to q)) \vee ((r \to p) \wedge (p \to r))$

$\Leftrightarrow p \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \vee ((\sim r \vee p) \wedge (\sim p \vee r))$

Other Connectives

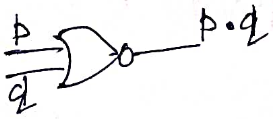
NAND (NOT AND): Symbol is '$\uparrow$'

$p \uparrow q \equiv \sim(p \wedge q)$

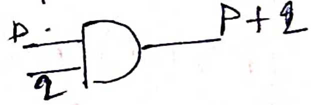


NOR (NOT OR): Symbol is '$\downarrow$'

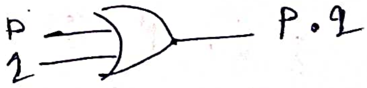
$p \downarrow q \equiv \sim(p \vee q)$



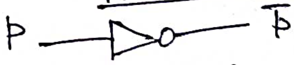
AND Gate



OR Gate



NOT Gate



XOR (exclusive OR): Symbol is '$\bar{\vee}$' or $\oplus$ ← in logic Gate used

$p \bar{\vee} q$ is true whenever either p or q is true not both.

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Th1 If $A \to B$ and A are tautologies, then B is a tautology

Th2 If $A \to B$ and $B \to C$ are tautologis, then $A \to C$ is tautology.

Principle of Substitution:

If a statement formula A is tautology. Containing statement letters $p_1, p_2, \dots p_n$ and statement B is also obtained from A by substituting statement formulas $A_1, A_2, \dots A_n$ for $P_1, P_2, \dots P_n$, then B is also a tautology.

Normal Forms.

The problem of finding in a finite number of steps whether a given expression is a tautology or a contradiction or at lest satisfiable is known as a decision problem.

We use the word 'sum' in place of disjunction ($\vee$) and 'product' in place of conjunction ($\wedge$).

A sum of variables and their negation is called elementary sum

such as $\sim P \vee Q$, $P \vee Q$, $P \vee \sim Q$.

A product of variables and their negation is called elementary product such as $P \wedge Q$, $\sim P \wedge Q$.

Ex. Disjunctive normal form (DNF)

Sum of elementary products.

ex. $P \wedge (P \to Q) \equiv P \wedge (\sim P \vee Q)$

$\equiv (P \wedge \sim P) \vee (P \wedge Q)$

ex. $\sim (P \vee Q) \leftrightarrow (P \wedge Q) \equiv (\sim P \wedge \sim Q) \leftrightarrow (P \wedge Q)$

$\equiv \boxed{((P \vee Q) \wedge \sim (P \wedge Q)) \vee ((\sim P \wedge \sim Q) \wedge (P \wedge Q))}$

$\equiv ((P \wedge \sim P) \vee (Q \wedge \sim P)) \vee ((P \wedge \sim Q) \vee (Q \wedge \sim Q))$

$\vee (\sim P \wedge \sim Q \wedge P \wedge Q)$

$\equiv (P \wedge \sim P) \vee (Q \wedge \sim P) \vee (P \wedge \sim Q) \vee (Q \wedge \sim Q) \vee (\sim P \wedge \sim Q \wedge P \wedge Q)$.

Conjunctive normal form (CNF)

Product of elementary sum.

ex. $\sim(p \vee q) \leftrightarrow (p \wedge q) \equiv \boxed{\left(\sim(p \vee q) \to (p \wedge q)\right) \wedge \left((p \wedge q) \to \sim(p \vee q)\right)}$

$$\equiv \left(\sim\{\sim(p \vee q) \vee (p \wedge q)\right) \wedge \left(\sim(p \wedge q) \vee \sim(p \vee q)\right)$$

$$\equiv \left((p \vee q) \vee (p \wedge q)\right) \wedge \left((\sim p \vee \sim q) \vee (\sim p \wedge \sim q)\right)$$

$$\equiv (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q).$$

Principal disjunctive normal form: (Sum of products
(PDNF) Cannonical form).

Consider the propositions p and q, then
$p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$ are called minterms
or Boolean conjunction of p and q.

Disjunction of minterms only ~~are~~ is called PDNF.

Process: i) Replace conditional and biconditional connectives by their equivalent ~~for~~ formulae containing $\vee, \wedge, \sim$ only

ii) Use De Morgan's laws, apply negation.

iii) Apply distributive law.

iv) Introduce the missing factors to obtain minterms in the disjunctions.

v) Delete identical minterms appearing in the disjunctions.

ex. $p \vee q \equiv \left(p \wedge \boxed{(q \vee \sim q)}\right) \vee \left(q \wedge \boxed{(p \vee \sim p)}\right)$

$$\equiv (p \wedge q) \vee (p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p).$$

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

$$\equiv \left(\sim p \wedge (q \vee \sim q)\right) \vee \left(\sim q \wedge (\sim p \vee p)\right)$$

$$\equiv (\sim p \wedge q) \vee \boxed{(\sim p \wedge \sim q) \vee (\sim q \wedge \sim p)} \vee (\sim q \wedge p)$$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (\sim q \wedge p).$$

Principal conjunctive normal form: (product of sum
(PCNF) Cannonical form).

For two propositions p & q, $p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$ are the maxterms or Boolean disjunction of p and q

Conjunction of ~~conjunctions~~ max terms called PCNF

ex. $P \to ((P \to q) \wedge \sim(\sim q \vee \sim P))$

$\sim P \wedge P = F$

$F \vee A = A$

$\equiv P \to ((\sim P \vee q) \wedge (P \wedge q))$

$\equiv \sim P \vee (((\sim P \wedge P) \vee \cancel{\sim P}(q \wedge P)) \wedge ((\sim P \wedge q) \vee (\sim q \wedge q)))$

$\equiv \sim P \vee ((q \wedge P) \wedge (\sim P \wedge q)) \equiv \sim P \vee q$

$\cancel{\equiv (\sim P \vee (q \wedge P))}$

ex. $(q \to P) \wedge (\sim P \wedge q) \equiv (\sim q \vee P) \wedge (\sim P \wedge q)$

Another form

$\equiv ((\sim q \vee P) \wedge \sim P) \wedge ((\sim q \vee P) \wedge q)$

$\equiv ((\sim q \wedge \sim P) \vee (P \wedge \sim P)) \wedge ((\sim q \wedge q) \vee (P \wedge q))$

$\equiv (\sim P \wedge \sim q) \wedge (P \wedge q)$

$\equiv F$

PCNF:

$(q \to P) \wedge (\sim P \wedge q) \equiv (\sim q \vee P) \wedge (\sim P \vee (q \wedge \sim q) \wedge q \vee (P \wedge \sim P))$

$\equiv (\sim q \vee P) \wedge (\sim P \vee q) \wedge (\sim P \vee \sim q) \wedge (q \vee P) \wedge (q \vee \sim P)$

$\equiv (\sim q \vee P) \wedge (\sim P \vee q) \wedge (\sim P \vee \sim q) \wedge (P \vee \sim q)$

$\equiv (P \vee q) \wedge (\sim P \vee q) \wedge (\sim P \vee \sim q) \wedge (P \vee \sim q)$.

Problem sets:

1. Show that $P \to (q \to r) \equiv P \to (\sim q \vee r) \equiv (P \wedge q) \to r$
2. " " $[(\sim P \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (P \wedge r)] \Leftrightarrow r$.
3. " " $(P \to r) \wedge (q \to r) \equiv (P \vee q) \to r$.
4. " " $P \to (q \vee r) \equiv (P \wedge \sim q) \to r$.
5. Show that $P \to q \equiv \sim q \to \sim P$ (contrapositive).

$q \to P$ (converse) $\equiv \sim P \to \sim q$ (inverse).

Validity of Arguments

A finite sequence $A_1, A_2, A_3, \cdots A_{n-1}, A_n$ of statement is called an argument. The final statement A_n is the conclusion, and the statements $A_1, A_2, \cdots A_{n-1}$ are called premises.

An argument $A_1, A_2, \cdots A_{n-1}, A_n$ is called logically valid if the statement formula

$$(A_1 \wedge A_2 \wedge \cdots \wedge A_{n-1}) \rightarrow A_n$$ is a tautology.

Sometimes we write,

$$\begin{array}{l} A_1 \\ A_2 \\ \vdots \\ A_{n-1} \\ \therefore A_n \end{array}$$

To test the validity we check whether $A_1 \wedge A_2 \wedge \cdots \wedge A_{n-1}$ logically imply A_n or not.

Ex. Consider the following argument:

If sheila solved seven problems correctly, then she obtained grade A. Sheila solved seven problems correctly. Therefore, sheila obtained grade A.

Let the statements,

p : Sheila solved seven problems correctly

q : Sheila obtained grade A.

Argument is,

$$\left.\begin{array}{l} p \rightarrow q \\ p \\ \therefore q \end{array}\right\}$$

we need to show $r : (p \rightarrow q) \wedge p \rightarrow q$ is tautology. (we use truth table)

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	r
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Ex. If Peter solved seven problems correctly, Peter obtain grade A. Peter obtained grade A. Therefore, Peter solved seven problems correctly.

p: Peter solved seven problems correctly

q: Peter obtained grade A.

Argument,
$$\begin{array}{l} p \to q \\ q \\ \therefore p \end{array}$$

We need to show $r: (p \to q) \wedge q \to p$ is tautology

p	q	$p \to q$	$(p\to q)\wedge q$	r
T	T	T	T	T
T	F	F	F	T
$\boxed{F}$	T	T	$\boxed{T}$	F
F	F	T	F	T

Hence the argument is not valid.

Remark The statement formula $(A_1 \wedge A_2 \wedge \cdots \wedge A_{n-1}) \to A_n$ is not a tautology if the truth value of $A_1 \wedge A_2 \wedge \cdots \wedge A_{n-1}$ is T and A_n is F.

Some valid argument forms

1. Modus Ponens : (Method of affirming).

$$\begin{array}{l} p \to q \\ p \\ \therefore q \end{array}$$

i.e, $(p\to q)\wedge p \Rightarrow q$.

2. Modus tollens : (Method of denying)

$$\begin{array}{l} p \to q \\ \sim q \\ \therefore \sim p \end{array}$$

i.e $(p\to q)\wedge \sim q \Rightarrow \sim p$.

3. Disjunctive Syllogisms :

$$\begin{array}{l} p \vee q \\ \sim p \\ \therefore q \end{array} \qquad \begin{array}{l} p \vee q \\ \sim q \\ \therefore p \end{array}$$

4. Hypothetical Syllogism :

$$\begin{array}{l} p \to q \\ q \to r \\ \therefore p \to r \end{array}$$

5. Dilemma :

$p \vee q$
$p \to r$
$q \to r$

$\therefore r$

6. Conjunctive Simplifications:

$p \wedge q$
$\therefore p$

$p \wedge q$
$\therefore q$

7. Disjunctive additions:

p
$\therefore p \vee q$

q
$\therefore p \vee q$

8. Conjunctive addition

p
q
$\therefore p \wedge q$

Theorem : A statement formula A is said to be logically from the statement formulas $A_1, A_2, \dots A_{n-1}, A_n$, written as $A_1, A_2, \dots A_{n-1}, A_n \models A$ if there exist an argument $B_1, B_2, \dots, B_m$ satisfying the following conditions:

1) B_m is A

2) For $1 \le i \le m$, either

i) B_i is one of $A_1, A_2 \dots A_{n-1}, A_n$ (say B_i is a hypothesis),
or

ii) B_i is a tautology, or

iii) for $i \geq 2$, $\exists$ $B_{i_1}, B_{i_2}, \dots B_{i_t}$ s.t. $B_{i_1} \wedge B_{i_2} \wedge \dots \wedge B_{i_{t-1}} \to B_{i_t}$ is a tautology.

ex. Show that $p, q, p \to r, q \to s \models r \wedge s$

$B_1 : p \to r$ hypothesis

$B_2 : p$ hypothesis

$B_3 : r$ B_3 follows from B_1, B_2 by Modus Ponens.

$B_4 : q \to s$ hyp.

$B_5 : q$ hyp.

$B_6 : s$ B_6 follows from B_4, B_5 by Modus Ponens.

$B_7 : r \wedge s$ B_7 follows from B_3, B_6 by conjunctive addition.

Hence proved.

ex. Consider the statement,

i) If my checkbook in office table, then I paid my phone bills.

ii) I was looking at phone bill for payment at breakfast or I was looking at the phone bill for payment in my office.

iii) If I was looking at the phone bill at breakfast, then the checkbook is on breakfast table.

iv) I did not pay phone bill.

v) If I was looking phone bill in my office, then the checkbook is on my office table.

Where was my checkbook?

Let p : my check book in office table

q : I paid phone bill

r : I was looking at the phone bill for payment at breakfast

s : " in my office

t : The checkbook is on breakfast table.

Arguments: $p \to q$

$r \vee s$

$r \to t$

$\sim q$

$s \to p$

Now consider

B_1 :	$s \to p$	hypothesis
B_2 :	$p \to q$	hypothesis
B_3 :	$s \to q$	B_1, B_2 and hypothetical Syllogism
B_4 :	$\sim q$	hypothesis
B_5 :	$\sim s$	B_3, B_4 and Modus Tollens.
B_6 :	$r \vee s$	hypothesis
B_7 :	r	B_5, B_6 and disjunctive syllogism
B_8 :	$r \to t$	hypothesis
B_9 :	t	B_7, B_8 and Modus Ponens.

Conclusion is t: Checkbook on breakfast table ex. Determine, using truth table, the following arguments are valid or not.

i) $p \to q$

$p \to r$

$\therefore p \to (q \vee r)$

To show,

$(p \to q) \wedge (p \to r) \Rightarrow p \to (q \vee r)$. is tautology

$\equiv (\sim p \vee q) \wedge (\sim p \vee r) \to (\sim p \vee (q \vee r))$

$\equiv (\sim p \vee (q \wedge r)) \to (\sim p \vee (q \vee r))$

$\equiv (p \wedge \sim (q \wedge r)) \vee (\sim p \vee (q \vee r))$

$\equiv^{T} (p \vee \sim p) \wedge (\sim (q \wedge r) \vee \sim p) \vee (q \vee r) \equiv T \vee \sim p \equiv T$

$\equiv$

$(p \to q) \wedge (p \to r)$

$\equiv \sim p \vee (q \wedge r)$

p	q	r	$p \to q$	$p \to r$	$\overset{a}{(p \to q) \wedge (p \to r)}$	$q \vee r$	$\overset{b}{p \to (q \vee r)}$	$a \to b$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T	T
T	F	T	F	T	F	T	T	T
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	F	T	T

ex. Test the validity, for a particular real number x: x is +ve or x is −ve. If x is +ve, then $x^2 > 0$. If x is −ve, then $x^2 > 0$. Therefore $x^2 > 0$.

p : x is +ve

q : x is −ve

r : $x^2 > 0$

Argument: B1: $p \vee q$, hyp.

B2: $p \to r$, hyp.

B3: $q \to r$, hyp

B4: $\therefore r$, B_1, B_2, B_3 and dilemma.

hence the argument is valid.

$(p \vee q) \wedge ((\sim p \wedge \sim q) \vee r) \equiv (p \vee q) \wedge (\sim (p \vee q) \vee r)$

$\equiv ((p \vee q) \wedge \sim (p \vee q)) \vee ((p \vee q) \wedge r)$

$\equiv F \vee ((p \vee q) \wedge r)$

$\equiv (p \vee q) \wedge r \equiv r$, conjunctive simplification ex. Show that R is valid inference from the premises $P \to Q$, $Q \to R$ and P.

Arguments: B1 : $P \to Q$ hyp.

B2 : P hyp.

B3 : Q , B1, B2 and Modus Ponens.

B4 : $Q \to R$, hyp

B5 : R , B4, B3 and Modus Ponens.

ex. $R \vee S$ follows logically from premises: $C \vee D$, $(C \vee D) \to \sim H$, $\sim H \to (A \wedge \sim B)$, and $(A \wedge \sim B) \to (R \vee S)$.

Arguments: B1 : $C \vee D$, hyp

B2 : $C \vee D \to \sim H$, hyp

B3 : $\sim H$, B1, B2, Modus Ponens

B4 : $\sim H \to A \wedge \sim B$, hyp.

B5 : $A \wedge \sim B$, B3, B4, Modus Ponens.

B6 : $(A \wedge \sim B) \to (R \vee S)$, hyp.

B7 : $R \vee S$, B6, B5, Modus Ponens.

ex. $r \wedge (p \vee q)$ is valid conclusion from the premises $p \vee q$, $q \to r$, $p \to m$, and $\sim m$.

Arguments: B1 : $p \to m$, Hyp

B2 : $\sim m$, Hyp.

B3 : $\sim p$, B1, B2 and Modus Tollens.

B4 : $p \vee q$, Hyp.

B5 : q , B3, B4 and Modus Tollens.

B6 : $q \to r$, Hyp.

B7 : r , B5, B6 and Modus Ponens.

B8 : $r \wedge (p \vee q)$, B4, B7 and Conjunctive addition.

Deduction Theorem : (CP rule).

If the conclusion is of the form $R \to S$, then R is taken as the additional premises and S is derived from given premises and R.

ex. $R \to S$ can be derived from $P \to (Q \to S)$, $\sim R \vee P$ and Q.

Argument :

$B1$: $\sim R \vee P$, hyp.

$B2$: R , Additional hyp.

$B3$: P , $B1$, $B2$ and disjunctive syllogism

$B4$: $P \to (Q \to S)$, hyp.

$B5$: $Q \to S$, $B3$, $B4$ and Modus Ponens.

$B6$: Q , hyp.

$B7$: S , $B5$, $B6$ and Modus Ponens.

$B8$: $R \to S$, $B2$, $B7$ and deduction rule.

Consistency and Indirect method :

A set of formulas $H_1, H_2, \dots, H_m$ is inconsistent if

$$H_1 \wedge H_2 \wedge \dots \wedge H_n \Rightarrow R \wedge \sim R \text{ (contradiction)}.$$

Rules for indirect method :

To show a conclusion C follows logically from the premises $H_1, H_2, \dots H_m$, we assume C is false and $\sim C$ is the additional premise.

If the new set of pemises is inconsisten, then C is true whenever $H_1 \wedge H_2 \wedge \dots \wedge H_m$ is true.

ie, $H_1 \wedge H_2 \wedge \dots \wedge H_m \Rightarrow C$.

ex. Show that $\sim P \wedge \sim Q \Rightarrow \sim (P \wedge Q)$.

Consider arguments,

$B_1 : \sim(\sim(P \wedge q))$, additional premise

$B_2 : P$, B_1 and Conjunctive Simplification

$B_3 : \sim P \wedge \sim q$, hyp.

$B_4 : \sim P$, B_3 and conjunctive Simplification

$B_5 : P \wedge \sim P$, B_2, B_4 and conjunctive addition

B_5 leads to contradiction.

Ex. Show that the following premises are inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

p : Jack misses many classes. $\quad p \to q$

q : Jack fails high school. $\quad q \to s$

r : Jack reads a lot of books $\quad r \to \sim s$

s : Jack is uneducated. $\quad p \wedge r$

$B1 : P \to q$, hyp

$B2 : q \to s$, hyp

$B3 : P \to S$, B1, B2 and hyp. Syllogism

$B4 : r \to \sim S$, hyp

$B5 : S \to \sim r$, B4 and equivalence $\boxed{a \to b \equiv \sim b \to \sim a}$

$B6 : P \to \sim r$, B3, B5 and hyp. Syllogism.

$B7 : \sim P \vee \sim r$, B_6 and equivalence. $a \to b \equiv \sim a \vee b$.

$B_8 : \sim(P \wedge r)$, B7 and Demorgan's Law

$B9 : P \wedge r$, hyp

$B10 : F$, B8, B9 and conjunctive addition.

Quantifiers

Consider the argument,

p: Every integer is a rational number.
q: 3 is an integer
r: Therefore, 3 is a rational number.

In mathematics, it is justified argument.

The argument takes the form,

$$\begin{array}{l} p \\ q \\ \therefore r \end{array}$$

The argument is valid if $(p \wedge q) \rightarrow r$ is tautology, which is not, hence the argument is not valid.

Therefore, validity of argument depends only on the structure of the components, not analysis of the sentence structure and the subject predicate lines.

Conside, if we analyze the sentence,

"Every integer is a rational number"

Which is equivalent to,

"For all x, if x is an integer, then x is a rational number"

The sentences "$\underbrace{x \text{ is an integer}}$" & "$x$ is a rational number" are declarative

↳ not a statement in propositional logic.

Since, if $x = 2$, it is T.
$x = 2.5$, it is F.

Predicates and quantifiers can verify the validity of argument.

Predicate or Propositional function:

Let x be a variable and D be a set ; $P(x)$ be a sentence. Then $P(x)$ is called predicate w.r.t. the set D if for each value of x in D, $P(x)$ is a ~~sta~~ statement i.e., $P(x)$ is T or F.

D is called domain and x is free variable.

Ex. $P(x)$: x is an even integer, where domain of discourse is the set of integers.

Then $P(4)$ i.e. 4 is an even integer is T
$P(3)$ is F.

Ex. Predicate involving two variables:

$P(x,y)$: x equals to $y+1$

$P(2,1)$ is T, $P(6,4)$ is F.

$Q(x,y)$: x^2 is greater than or equal to y, domain is $\mathbb{Z}$.

n-place predicate:

Let x_i, $i=1,2,\ldots n$ be n variables, an n-place predicate is a sentence $P(x_1, x_2, \ldots x_n)$ containing $x_1, x_2, \ldots x_n$ s.t. on assignment of values to the variables $x_1, x_2, \ldots x_n$ form appropriate domains.

Quantifier → Universal
→ Existential

Definition:

Let $P(x)$ be a predicate and D be the domain, then Universal quantification of $P(x)$ is the statement,

for all x, $P(x)$ or for every x, $P(x)$

i.e. $\forall x\, P(x)$, $\forall$ is universal quantifier.

Ex. $P(x)$: $x^2 \geq x$, D is set of integers.

$\forall x$, $P(x)$ is True, hence the value of the universal quantification is true.

$P(x)$: $x \geq 3$, here $\forall x$, $P(x)$ is false.

For two place predicate $P(x,y)$, the universal quantification is $\forall x \forall y\, P(x,y)$ which is true for all x and for all y.

Ex. $P(x,y)$: $xy > 0$; D is set of non-negative integers, then $\forall x \forall y\, P(x,y)$ is true.

Definition: Let $P(x)$ be a predicate and D be the domain of discourse. The existential quantification of $P(x)$ is the statement, there exists x, $P(x)$.

In notation, $\exists x P(x)$
$\downarrow$
existential quantifiers

$\exists x P(x)$ is a statement, hence it has truthvalue T or F.

eg. $P(x): x^2 > x$, D is set of real numbers.
$P(2)$ is true but $P(\frac{1}{2})$ is false.
Hence we conclude $\exists x P(x)$ is T.
$\forall x P(x)$ is F.

eg. $P(x): x^2 < x$, D is set of integers.
There is no integer for which the predicate is true.
Hence, $\forall x P(x)$ is F
or, $\exists x P(x)$ is F $\Big\}$ The variable x is apeaping in both, it is called bounded variable which is considered bounded by the both quantifiers $\forall$ and $\exists$.

If $P(x,y)$ is a sentence, then in $\forall x P(x,y)$, only x is bounded.

Negation of Predicate:
Let $P(x)$: x has taken programming course.
Where domain of discourse is set of all students in discrete structure course.
Universal quantification of $P(x)$ is $\forall x P(x)$ i.e.,
every student in discrete structure taken programming course.
$\sim \forall x P(x)$: It is not the case, every student in discrete structure taken programming course i.e.,

There exist at least one student in discrete structure who has not taken programming course.

i.e, $\exists x \sim P(x)$.

Hence, $\boxed{\sim \forall x\, P(x) \equiv \exists x \sim P(x)}$.

Now, existential quantification of $P(x)$ is $\exists x\, P(x)$.

i.e, There exist a student in discrete structure who has taken programming course.

$\sim \exists x\, P(x)$: It is not the case, there exist a student in discrete structure who has taken programming course.

i.e, No student in discrete structures has taken the programming course.

i.e, For all students x in discrete structures, x has not taken the programming course i.e,

$\forall x \sim P(x)$.

Hence, $\boxed{\sim \exists x\, P(x) \equiv \forall x \sim P(x)}$

DeMorgan's Law :

i) $\sim \forall x\, P(x) \equiv \exists x \sim P(x)$

ii) $\sim \exists x\, P(x) \equiv \forall x \sim P(x)$.

Additional Rule of Inference :

1. If $\forall x\, P(x)$ is true, then $P(a)$ is also true where, a is arbitrary member of the domain of discourse. The rule is called the universal specification (US).
2. If $P(a)$ is true, where a is an arbitrary member of the domain of the discourse, then $\forall x\, P(x)$ is true. This rule is called universal generalization (UG).
3. If $\exists x\, P(x)$ is true, then $P(a)$ is true, for some member of the domain of the discourse. This rule is called existential specification (ES).

4. If $P(a)$ is true for some member a of the domain of the discourse, then $\exists x\, P(x)$ is also true. This rule is called existential generalization (EG)

Problem:

1. Suppose the universe consists of all integers.

$P(x): x \le 3$, $q(x): x+1$ is odd , $r(x): x > 0$

Write truthvalues of the following:

i) $P(2)$, $P(2): 2 \le 3$ is true.

ii) $\sim q(4)$, $q(4): 4+1$ is odd is true, $\sim q(4)$ is false.

iii) $P(-1) \wedge q(1)$, $P(-1): -1 \le 3$ is true. $q(1): 1+1$ is odd is false $\Big\}$ $T \wedge F \equiv F$

iv) $P(4) \vee (q(1) \wedge r(2))$, $P(4): 4 \le 3$ is False, $q(1): 1+1$ is odd is False, $r(2): 2 > 0$ is True $\Big\}$ $F \vee (F \wedge T) \equiv F \vee F \equiv F$

v) $P(2) \wedge (q(0) \vee \sim r(2))$, $P(2): 2 \le 3$ is True, $q(0): 0+1$ is odd is True, $r(2): 2 > 0$ is True, $\sim r(2): 2 < 0$ is False $\Big\}$ $T \wedge (T \vee F) \equiv T \wedge T \equiv T$

2. For the universe of all integers,

$P(x): x > 0$, $q(x): x$ is even, $r(x): x$ is perfect square,
$s(x): x$ is divisible by 3 , $t(x): x$ is divisibly by 7.

Write the following in symbolic form,

i) Atleast one integer is even : $\exists x\, q(x)$.

ii) There exist a positive integer that is even : $\exists x\, (P(x) \wedge q(x))$.

iii) Every integer either even or odd : $\forall x\, (q(x) \veebar q(x))$

iv) If x is even and perfect square, then x is not divisible by 3 : $\forall x, (q(x) \wedge r(x)) \rightarrow \sim s(x)$

v) If x is odd or not divisible by 7, then x is divisible by 3.
$\forall x\, (\sim q(x) \vee \sim (t(x))) \rightarrow s(x)$.

Indicate truth value:

i) $\forall x\ r(x) \to p(x)$, for any integer, if x is perfect square then $x > 0$ - False $(x=0)$.

ii) $\exists x, (S(x) \wedge \sim q(x))$, True.

iii) $\forall x, \sim r(x)$, False $(x=9)$

iv) $\forall x, (r(x) \vee t(x))$, False $(x=8)$

3. Consider $P(x): |x| > 3$, $q(x): x > 3$, Universe is of real nos.
Find truth value of $\forall x\ P(x) \to q(x)$ (conditional).

For each x, if $|x| > 3$ then $x > 3$, False.

Contrapositive, $\forall x (\sim q(x) \to \sim p(x))$, False.

i.e. For each x, if ~~min~~ $x \leq 3$ then its magnitude less than or equal to 3.

converse, $\forall x, q(x) \to P(x)$ i.e. for each x, if $x > 3$ then $|x| > 3$, True

inverse, $\forall x, \sim P(x) \to \sim p(x)$, true.

4. Let $P(x): x^2 - 7x + 10 = 0$, $q(x): x^2 - 2x - 3 = 0$, $r(x): x < 0$.
Universe contain only 2 and 5. Determin truth value.

i) $\forall x\ P(x) \to \sim r(x)$.

$x^2 - 7x + 10 = 0 \Rightarrow (x-5)(x-2) = 0 \Rightarrow x = 2, 5$.

$\forall x\ P(x)$ True, $\forall x \sim r(x)$ true.

i.e. $\forall x\ P(x) \to \sim r(x)$ is true.

ii) $\forall x\ q(x) \to r(x)$

$q(x): x^2 - 2x - 3 = 0 \Rightarrow x = 3, -1$

$\forall x\ q(x)$ is False, $\forall x\ r(x)$ is False.

$\forall x\ q(x) \to r(x)$ is true.

iii) $\exists x, q(x) \to r(x)$

$\exists x\ q(x)$ is False, $\exists x\ r(x)$ is False.

$\therefore \exists x\ q(x) \to r(x)$ ~~Fals~~. True.

iv) $\exists x\ P(x) \to r(x)$.

$\exists x\ P(x)$ is True, $\exists x\ r(x)$ is False.

$\therefore \exists x\ P(x) \to r(x)$ is False.

5. Write in symbolic form and its negation:

"If all triangles are right-angled, then no triangle is equiangular"

$P(x)$: x is right angled, $q(x)$: x is equiangular.

T: set of Triangle.

Statement $(\forall x \in T\ P(x)) \to (\forall x \in T \sim q(x))$

Negation: $\sim(\sim(\forall x \in T\ P(x)) \vee (\forall x \in T \sim q(x)))$

$\Leftrightarrow \forall x \in T\ P(x) \wedge \sim(\forall x \in T \sim q(x))$

$\Leftrightarrow (\forall x \in T\ P(x)) \wedge (\exists x \in T \sim\sim q(x))$

$\Leftrightarrow (\forall x \in T\ P(x)) \wedge (\exists x \in T\ q(x))$.

ie, All triangles are right angled and some are equiangular.

<u>Logical implication involving quantifiers</u>

1. $\forall x\ P(x) \Rightarrow \exists x\ P(x)$, S is the Universe.

$\forall x\ P(x) \Rightarrow$ For every $x \in S$, $P(x)$ is true

$\Rightarrow$ For ~~some~~ $a \in S$, $P(a)$ is true

$\Rightarrow$ For some $x \in S$, $P(x)$ is true.

$\Rightarrow \exists x\ P(x)$.

2. $\forall x\ P(x) \vee q(x) \Rightarrow$ For all $x \in S$ $P(x)$ is true or $q(x)$ is true

$\Rightarrow$ For all $x \in S$ $P(x)$ is true or $q(x)$ is true for $a \in S$

$\Rightarrow \forall x\ P(x) \vee \exists x\ q(x)$

3. $\exists x\, P(x) \wedge Q(x) \Rightarrow \exists x\, P(x) \wedge \exists\, Q(x)$, converse not true.

4. $(\forall x\, P(x)) \vee (\forall x\, Q(x)) \Rightarrow \forall x\, (P(x) \vee Q(x))$.

5. Show that $\exists x\, Q(x)$ is logically from the premises

$\forall x\, P(x) \to Q(x)$ and $\exists x\, P(x)$.

B1: $\forall x\, P(x) \to Q(x)$, hyp.

B2: $P(a) \to Q(a)$, by rule of US.

B3: $\exists x\, P(x)$, hyp.

B4: $P(a)$, by rule of ES.

B5: $Q(a)$, B2, B4, Modus Ponen.

B6: $\exists x\, Q(x)$, By rule of EG.

6. All men are mortal — $\forall x\, P(x)$

Randy is a man — $a \in S$

Therefore, Randy is mortal — $\therefore P(a)$.

B1: $\forall x\, P(x)$, hyp.

B2: $a \in S$, hyp

B3: $P(a)$, by rule of US.

7. No engg. is bad in Maths.

a. Anil is not bad in Math

Therefore Anil is an engg.

$P(x)$: x is engg.

$Q(x)$: x is bad in Maths.

Argument:

$\forall x\, P(x) \to \sim Q(x)$.

$\sim Q(a)$

$\therefore P(a)$

Not valid argument.

$\forall x\, P(x) \to \sim Q(x) \Rightarrow P(a) \to \sim Q(a)$

$(P(a) \to \sim Q(a)) \wedge (\sim Q(a)) \not\Rightarrow P(a)$.

$\cancel{\sim(P \to Q) \equiv \sim(\sim P \vee Q) \equiv P \wedge \sim Q}$

$P \to \sim Q \equiv \sim P \vee \sim Q = \sim(P \wedge Q)$

8. No graduate in commerce or literate studies physics.
a. $\boxed{\text{Anil}}$ is graduate who studies physics.
∴ Anil is not graduate of literature.

$\forall x \; P(x) \vee q(x) \rightarrow \sim r(x)$

$r(a)$

$\therefore \sim q(a)$.

$\forall x \; P(x) \vee q(x) \rightarrow \sim r(x) \Rightarrow P(a) \vee q(a) \rightarrow \sim r(a)$

$\equiv \sim(P(a) \vee q(a)) \vee \sim r(a)$

B1: $\sim(P(a) \vee q(a)) \vee \sim r(a)$, hyp

B2: $\sim(\sim r(a))$, hyp.

B3: $\sim(P(a) \vee q(a))$, B1, B2, disjunctive syllogisms.

B4: $\sim P(a) \wedge \sim q(a)$, De morgan's.

B5: $\sim q(a)$, B4, Conjunctive simplification.

9. All athlets are healthy.
All healthy people take vitamins.
Grant is an athlete.
∴ Grant takes vitamin.

$\forall x \; P(x) \rightarrow q(x)$

$\forall x \; q(x) \rightarrow r(x)$

$P(a)$

$\therefore r(a)$

B1: $\forall x \; P(x) \rightarrow q(x)$, hyp

B2: $P(a) \rightarrow q(a)$, by US

B3: $\forall x \; q(x) \rightarrow r(x)$, hyp

B4: $q(a) \rightarrow r(a)$, by US

B5: $P(a) \rightarrow r(a)$, hypothetical Syllogism

B6: $P(a)$, hyp.

B7: $r(a)$, Modus ponen.

Method of Proof (Proof Techniques)

We discuss the proofs of which can be expressed as $P(x) \to Q(x)$.

Step 1 chose arbitrarily a from domain D.
Step 2 Assume $P(a)$ is true.
Step 3 show that $Q(a)$ is true.
Step 4 Then by UG, $\forall x\, P(x) \to Q(x)$

} Direct proof

1. Give a direct proof of the statement:
" The square of an odd integer is an odd integers.

$P(x)$: x is odd integer, $Q(x)$: x^2 is odd integers.

$\forall x \; P(x) \to Q(x)$. , D = domain of integers

Consider $a \in D$, $P(a)$ is true then

$$a = 2n+1$$
$$a^2 = (2n+1)^2 = 4n^2 + 4n + 1$$
$$= 2(2n^2 + 2n) + 1$$
$$= 2m+1, \text{ odd integers.}$$

Hence the proof.

2. Show that the product of two odd integers is an odd integers

ie, " For all integers x, y, if x and y are odd, then product xy is odd "

ie, $\forall x \forall y \, (P(x) \wedge Q(x)) \to R(x, y)$, $P(x)$: x is odd
$Q(y)$: y is odd
$R(x, y)$: xy is odd.
D: domain of integers.

Consider, $a, b \in D$.

$$a = 2n+1, \; b = 2m+1$$
$$ab = (2n+1)(2m+1) = 4mn + 2m + 2n + 1,$$
$$= 2(2mn + m + n) + 1$$
$$= 2t + 1, \; t \in D.$$

$\therefore$ ab is odd.

Hence the proof.

Indirect Proof: Consider $p \to q \equiv \sim q \to \sim p$.

Step 1: $\sim q$ is true, assume.

Step 2: Show that $\sim p$ is true.

Step 3: Then by UG, $\forall x \; P(x) \to Q(x)$

1. Show that for all integers, if n^2+3 is odd, then n is even. i.e. $\forall n\ P(n) \to q(n)$.

i.e. $\forall n \sim q(n) \to \sim P(n)$.

Consider, $\sim q(n)$ is true i.e. n is odd,

$$n = 2m+1$$

$$n^2+3 = (2m+1)^2+3 = 4m^2+4m+4$$

$$= 2(2m^2+2m+4)$$

$$= 2t, \text{ even integer.}$$

i.e. $P(n)$ is not odd

i.e. $\sim P(n)$ is true.

Hence the proof.

2. Show that for all positive real numbers x and y, if $xy > 25$ then $x > 5$ or $y > 5$.

$p(x): x > 5$, $q(x): y > 5$, $r(x,y): xy > 25$.

To show, $\cancel{\forall x \forall y\ P(x) \vee q(y) \to r(x,y)}$

$\cancel{\text{i.e. } \forall x \forall y \sim r(x,y) \to \sim p(x) \wedge \sim q(y)}$

$\forall x \forall y\ r(x,y) \to P(x) \vee q(y)$

i.e. $\forall x \forall y \sim p \wedge \sim q \to \sim r$.

Consider $\sim p \wedge \sim q$ is true i.e. $x < 5$ and $y < 5$, then $xy < 25$.

i.e. r is not true i.e. $\sim r$ is true.

Hence the proof.

3. Let m and n be integers, Prove that $\boxed{n^2 = m^2}_{p}$ iff $\boxed{m=n}_{q}$ or $\boxed{m=-n}_{r}$

We have to show $p \leftrightarrow (q \vee r)$

i.e. $p \to (q \vee r)$ and $(q \vee r) \to p$.

i.e. $\sim q \wedge \sim r \to p$ and $(q \vee r) \to p$.

$m \neq n$, $m \neq -n$ $\quad\quad$ $m = n$ ~~or~~ $m = -n$

i.e. $m^2 \neq n^2$ $\quad\quad$ then $m^2 = n^2$.

Proof By Contradiction:

Assume $p \to q$ is false ie p is true and q is false.

Step 1: Assume q is false

Step 2: Show that p is false, a contradiction.

Step 3: $p \to q$ is true.

1. Show that $\sqrt{2}$ is an irrational number.

Assume $\sqrt{2}$ is not irrational, ie, rational number

then $\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2 =$ even.

ie, a is even.

Consider $a = 2n$

then, $4n^2 = 2b^2 \Rightarrow b^2 = 2n^2$ ie, b^2 is even.

ie, b is even.

hence a & b has common factor 2, which contradicts $\sqrt{2}$ is rational.

Hence the proof.