7.4 PIGEONHOLE PRINCIPLE

Pigeonhole principle is an important concept that is required for solving counting problems. The statement and proof of this principle are given below.

Statement If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.

or

If n objects are placed into m boxes and $n > m$, then there is at least one box that contains two or more objects.

Proof Suppose that none of the m boxes will contain more than one object. Then, the total number of objects is at most m. This is a contradiction since $n > m$. That is, the number of objects is greater than the number of pigeonholes.

Hence, our assumption is wrong.

Therefore, at least one box will contain two or more objects.

Generalized pigeonhole principle If N objects are placed into K boxes, then there is at least one box containing at least $\left\lceil \frac{N}{K} \right\rceil$ objects.

(or)

If N objects are placed into K boxes and $N > K$, then at least one of the pigeonholes must contain $\left\lfloor \frac{(N-1)}{K} \right\rfloor + 1$ objects, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, which is a real number.

Proof We assume that each box contains at most $\left\lfloor \frac{(N-1)}{K} \right\rfloor$ objects. Then, the maximum number of objects in all the boxes $= K\left\lfloor \frac{(N-1)}{K} \right\rfloor \le K\frac{(N-1)}{K}$, since $\left\lfloor \frac{N-1}{K} \right\rfloor \le \frac{(N-1)}{K}$

That is, the maximum number of objects in all the boxes is $\le N-1$ which is a contradiction, since the total number of objects is N.

Thus, one of the boxes must contain at least $\left\lfloor \frac{(N-1)}{K} \right\rfloor + 1$ objects.

Example 7.42 Show that in any set of six classes, there must be two that meet on the same day, assuming that no classes are held on weekends.

Solution Since there are six classes, but only five weekdays, by the pigeonhole principle, at least two classes must be held on the same day.

Example 7.43 If an examination is graded on a scale of 0 to 100 points, how many students in a class must be there to guarantee that at least two students receive the same score on the final examination?

Solution If the examination is graded on a scale of 0 to 100 points, there are 101 possible scores. Thus, by the pigeonhole principle, among 102 students there must be at least two students who receive the same score.

Hence, the required number of students = 102. ■

Example 7.44 A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

(i) How many socks must he take out to be sure that he has at least two socks of the same color?

(ii) How many socks must he take out to be sure that he has at least two black socks?

Solution A drawer contains unmatched 12 brown socks and 12 black socks.

(i) We need at least two socks of the same color. Thus, by the pigeonhole principle, he has to take out at least 3 socks so that at least two socks are of the same color. Hence, the required number of socks = 3.

(ii) By the pigeonhole principle, the number of socks he must take out is 12 + 2, so that at least two socks must be black.

Hence, the required number of socks = 14. ■

Example 7.45 Among 200 people, how many of them were born on the same month?

Solution Since there are 12 months in a year, the number of people born on the same month

$$= \left\lceil \frac{200-1}{12} \right\rceil + 1 \quad \text{[by generalized pigeonhole principle]}$$

$$= 17 + 1 = 18$$ ■

Example 7.46 If there are six possible grades A, B, C, D, E and F, what is the minimum number of students required in a class to be sure that at least seven will receive the same grade?

Solution The minimum number of students required in a class to ensure that at least seven students receive the same grade is the smallest integer N such that $\left\lceil \frac{N}{6} \right\rceil = 7$

$$N = 6 \times 6 + 1 = 37$$

Hence, the required minimum number of students = 37. ■

Example 5 *Prove that in any set of 29 persons at least five persons must have been born on the same day of the week.*

► Treating the seven days of a week as 7 pigeonholes and 29 persons as pigeons, we find by using the generalized pigeonhole principle that at least one day of the week is assigned to $\left(\frac{29-1}{7}\right) + 1 = 5$ or more persons. In other words, at least 5 of any 29 persons must have been born on the same day of the week.

Example 6 *How many persons must be chosen in order that at least five of them will have birth days in the same calendar month?*

► Let n be the required number of persons. Since the number of months over which the birthdays are distributed is 12, the least number of persons who have their birthdays in the same month is, by the generalized pigeonhole principle, equal to $\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1$. This number is 5, if

$$\left\lfloor \frac{(n-1)}{12} \right\rfloor + 1 = 5, \text{ or } n = 49.$$

Thus, the number of persons is 49 (at the least).

Example 7 *Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.*

► From the numbers from 1 to 10, we can choose three different numbers in $C(10, 3) = 120$ ways.

The smallest possible sum that we get from a choice is 1+2+3=6 and the largest sum is 8+9+10=27. Thus, the sums vary from 6 to 27 (both inclusive), and these sums are 22 in number.

Accordingly, here, there are 120 choices (pigeons) and 22 sums (pigeonholes). Therefore, the least number of choices assigned to the same sum is, by the generalized pigeonhole principle,

$$\left\lfloor \frac{120-1}{22} \right\rfloor + 1 = \lfloor 6.4 \rfloor \approx 6.$$

Example 8 Sh...

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