Dynamic Programming :-

It is a method that can be used when the solution to the problem can be viewed as a result of sequence of decision.

In greedy method, one decision sequence is generated where as in dynamic programming many decision sequence are generated.

In DP, the principle of optimately states that an optimal sequence of decision has the property that whatever the initial state & decision are, the remaining decision are must constitute an optimal decision sequence with regard to the state resulting from the first decision.

The problems (applications) of DP are :-

→ Multi-stage graph
→ Optimal Binary Search Tree (OBST)
→ 0/1 Knapsack problem
→ All pairs - shortest path
→ Travelling Sales Person.
→ Reliability Design

All Pairs Shortest Path :-

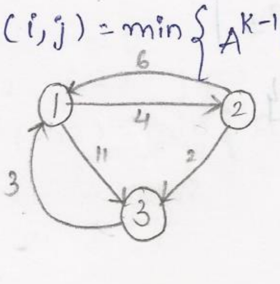
Let $G = \{V, E\}$ be a directed graph with 'n' vertices, Let c be a cost of adjacent matrix for 'G' such that $cost(i,i) = 0$ where $1 \le i \le n$. $Cost(i,j)$ is the length of the edge (i,j) if $(i,j) \in \underset{\text{Edge}}{E}(G)$. $Cost(i,j) = \infty$ if $i \neq j$ & $(i,j) \notin E(G)$. The all pairs shortest path problem is to determine a matrix 'A' such that $A(i,j)$ is the length of shortest path from i to j. The matrix 'A' can be obtained by solving 'n' single source problems using the algorithm shortest path.

Since each application of this procedure requires $O(n^2)$ time, the matrix 'A' can be obtained in $O(n^3)$

Formulae :-

$$A^0(i,j) = cost(i,j) \quad 1 \le i \le n,\ 1 \le j \le n.$$

$$A^k(i,j) = \min\{A^{k-1}(i,j),\ A^{k-1}(i,k) + A^{k-1}(k,j)\} \quad k \geq 1$$

1) 

Step 1 :-

$$A^0(i,j) = cost(i,j)$$

$A^0(1,1) = 0$ $A^0(2,1) = 6$ $A^0(3,1) = 3$

$A^0(1,2) = 4$ $A^0(2,2) = 0$ $A^0(3,2) = \infty$

$A^0(1,3) = 11$ $A^0(2,3) = 2$ $A^0(3,3) = 0$

Matrix :-

A^0	1	2	3
1	0	4	11
2	6	0	2
3	3	∞	0

Step 2 :-

$k=1, \; i=1, \; j=1,2,3$

$A^k(i,j) = \min\{A^k(i,j), A^{k-1}(i,k) + A^{k-1}(k,j)\}$

$A^1(1,1) = \min\{A^{1-1}(1,1), A^{1-1}(1,1) + A^{1-1}(1,1)\}$

$= \min\{A^0(1,1), \underbrace{A^0(1,1) + A^0(1,1)}_{0}\}$

$= \min\{0\}$

$A^1(1,1) = 0$

A^1	1	2	3
1	0		
2			
3			

$\Rightarrow k=1 \quad i=1, \; j=2$

$A^1(1,2) = \min\{A^{1-1}(1,2), A^{1-1}(1,1) + A^{1-1}(1,2)\}$

$= \min\{A^0(1,2), A^0(1,1) + A^0(1,2)\}$

$= \min\{4, \; 0+4\}$

$= \min\{4, 4\}$ $A^1(1,2) = 4$

A^0	1	2	3
1	0	4	
2			
3			

$\Rightarrow k=1 \quad i=1 \quad j=3$

$A^1(1,3) = \min\{A^{1-1}(1,3), A^{1-1}(1,1) + A^0(1,3)\}$

$= \min\{A^0(1,3), \; A^0(1,1) + A^0(1,3)\}$

$= \min\{11, 0+11\}$

$A^1(1,3) = 11$

A^0	1	2	3
1	0	4	11
2			
3			

$\Rightarrow k=2, \; i=2, \; j=1$

$A^2(2,1) = \min\{A^{1-1}(2,1), A^{1-1}(2,2) + A^{1-1}(1,1)\}$

$= \min\{A^0(2,1) + A^0(2,1) + A^0(1,1)\}$

$= \min\{6, 6+0\}$

$A^1(2,1) = \{6\}$

$\Rightarrow$ $K=1,\ i=2\ j=2$

$A^1(2,2) = \min\{A^{1-1}(2,2),\ A^{1-1}(2,1)+A^{1-1}(1,2)\} = \min\{0, 6+4\} = 0$

$\Rightarrow$ $K=1,\ i=2,\ j=3$

$A^1(2,3) = \min\{A^0(2,3),\ A^0(2,1)+A^0(1,3)\} = \min\{2, 6+4\} = 2$

$\Rightarrow$ $K=1,\ i=3,\ j=1$

$A^1(3,1) = \min\{A^0(3,1),\ A^0(3,1)+A^0(1,1)\} = \min\{3, 3+0\} = 3$

$\Rightarrow$ $K=1\quad i=3\quad j=2$

$A^1(3,2) = \min\{A^0(3,2),\ A^0(3,1)+A^0(1,2)\} = \min\{0, 7\} = 7.$

$\Rightarrow$ $K=1\quad i=3\quad j=3$

$A^1(3,3) = \min\{A^0(3,3),\ A^0(3,1)+A^0(1,3)\} = \min\{0, 3+11\} = 0$

A^1	1	2	3
1	0	4	11
2	6	0	2
3	3	7	0

Step 3:-

$\Rightarrow$ $K=2,\ i=1,\ j=1$

$A^2(1,1) = \min\{A^{2-1}(1,1),\ A^{2-1}(1,2)+A^{2-1}(2,1)\}$

$= \min\{A^1(1,1),\ A^1(1,2)+A^1(2,1)\} = \min\{0, 4+6\} = 0$

$\Rightarrow$ $K=2\ \ i=1,\ j=2$

$A^2(1,2) = \min\{A^1(1,2),\ A^1(1,2)+A^1(2,2)\} = \min\{4, 4+0\} = 4$

$\Rightarrow$ $K=2\quad i=1\quad j=3$

$A^2(1,3) = \min\{A^1(1,3), A^1(1,2)+A^1(2,3)\} = \min\{11, 4+2\} = 6$

$\Rightarrow$ $K=2\quad i=2\quad j=1$

$A^2(2,1) = \min\{A^1(2,1),\ A^1(2,2)+A^1(2,1)\}$

$= \min\{6, 0+6\} = 6$

$\Rightarrow$ $K=2\quad i=2\quad j=2$

$A^2(2,2) = \min\{A^1(2,2),\ A^1(2,2)+A^1(2,2)\}$

$= \min\{0, 0\}$

$= 0$

$\Rightarrow$ $K=2\quad i=2\quad j=3$

$A^2(2,3) = \min\{A^1(2,3),\ A^1(2,2)+A^1(2,3)\} = \min\{2, 0+2\}$

$= 2$

$\Rightarrow$ $K=2,\ i=3,\ j=1$

$A^2(3,1) = \min\{A^1(3,1),\ A^1(3,2)+A^1(2,1)\}$

$= \min\{3,\ 7+6\} = \min\{3,\ 13\} = 3$

$\Rightarrow$ $K=2,\ i=3,\ j=2$

$A^2(3,2) = \min\{A^1(3,2),\ A^1(3,2)+A^1(2,2)\}$

$= \min\{7,\ 7+0\} = 7$

$\Rightarrow$ $K=2\ \ i=3,\ j=3$

$A^2(3,3) = \min\{A^1(3,3),\ A^1(3,2)+A^1(2,3)\} = \min\{0,\ 7+2\} = 0$

A^2	1	2	3
1	0	4	6
2	6	0	2
3	3	7	0

$\Rightarrow$ $K=3,\ i=1,\ j=1$

$A^3(1,1) = \min\{A^2(1,1),\ A^1(1,3)+A^1(3,1)\}$

$= \min\{0,\ 6+3\} = 0$

$\Rightarrow$ $K=3,\ i=1,\ j=2$

$A^3(1,2) = \min\{A^2(1,2),\ A^2(1,3)+A^2(3,2)\} = \min\{4,\ 6+7\} = 4$

$\Rightarrow$ $K=3\ \ i=1\ \ j=3$

$A^3(1,3) = \min\{A^2(1,3),\ A^2(1,3)+A^2(3,3)\} = \min\{6,\ 0+6\} = 6$

$\Rightarrow$ $K=3\ \ i=2\ \ j=1$

$A^3(2,1) = \min\{A^2(2,1),\ A^2(2,3)+A^2(3,1)\} = \min\{6,\ 2+3\} = 5$

$\Rightarrow$ $K=3\ \ i=2\ \ j=2$

$A^3(2,2) = \min\{A^2(2,2),\ A^2(2,3)+A^2(3,2)\} = \{0\}$

$\Rightarrow$ $K=3\ \ i=2\ \ j=3$

$A^3(2,3) = \min\{A^2(2,3),\ A^2(2,3)+A^2(3,3)\} = \min\{2,\ 2+0\} = 2$

$\Rightarrow$ $K=3,\ i=3,\ j=1$

$A^3(3,1) = \min\{A^2(3,1),\ A^2(3,3)+A^2(3,1)\} = \min\{3,\ 0+3\} = 3$

$\Rightarrow$ $K=3,\ i=3,\ j=2$

$A^3(3,2) = \min\{A^2(3,2),\ A^2(3,3)+A^2(3,2)\} = \min\{7,\ 0+7\} = 7$

$\Rightarrow$ $K=3\ \ i=3\ \ j=3$

$A^3(3,3) = \min\{A^2(3,3),\ A^2(3,3)+A^2(3,3)\} = 0$

A^3	1	2	3
1	0	4	6
2	5	0	2
3	3	7	0

Algorithm Allpaths (cost, A, n)
{

```
for i:1 to n do
for j:1 to n do
A[i,j]:=cost[i,j];
for K:=1 to n do
for i:= 1 to n do
for j:=1 to n do
  A[i,j]=min{ A[i,j], A[i,k]+A[k,j]};
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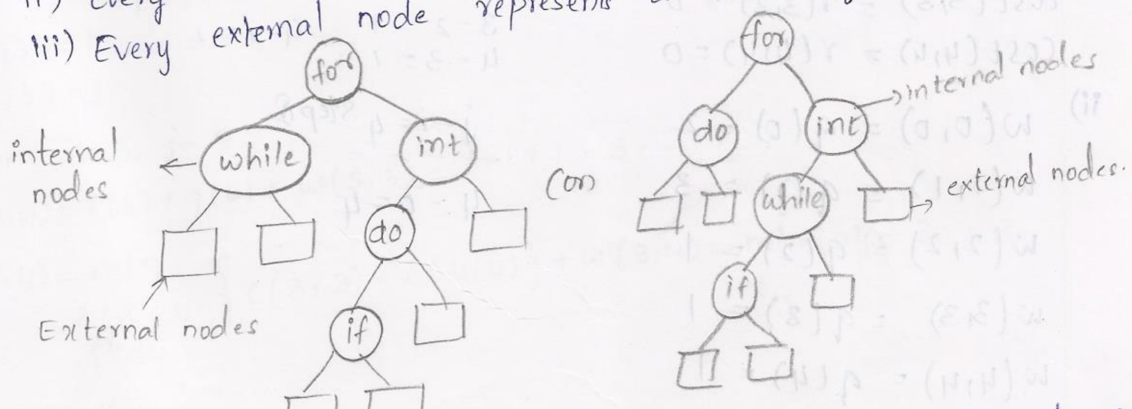
Optimal Binary Search Tree :

i) In a binary search tree for n identifiers, there will be n internal nodes and n+1 external nodes.

ii) Every internal node represents a successful search

iii) Every external node represents a unsuccessful search.



In above diagram the probability of successful nodes or successful BST, varies from 1 + 2 diagrams. In order to overcome this we concider optimal binary search tree where we are going to find maximum cost for the respective routes.

Formulae :-

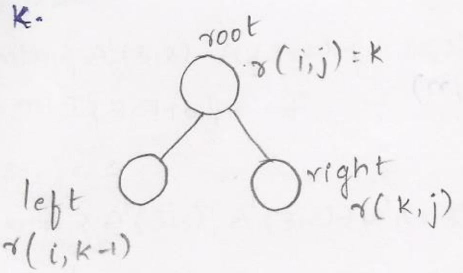
i) cost $(i,i) = r(i,i) = 0$

ii) $w(i,i) = q(i) \quad 0 \le i \le n$

3. $w(i,j) = p(j) + q(j) + w(i,j-1)$

4. $c(i,j) = \min\limits_{i<k\le j} \{c(i,k-1) + c(k,j)\} + w(i,j)$

5. $r(i,j) = k$.



Problem :-

1. $n = 4$, $(a_1, a_2, a_3, a_4) = (do, if, int, while)$

$P(1:4) = 3, 3, 1, 1$

$q(0:4) = 2, 3, 1, 1, 1$ q values are $0 \le i \le n$ ($i = 0$ to 4)

$i = 0, 1, 2, 3, 4$

i) $cost(0,0) = r(0,0) = 0$
$cost(1,1) = r(1,1) = 0$
$cost(2,2) = r(2,2) = 0$
$cost(3,3) = r(3,3) = 0$
$cost(4,4) = r(4,4) = 0$

iii)

Step I	Step II	Step III	Step IV
$j-i=1$	$j-i=2$	$j-i=3$	$j-i=4$
$1-0=1$	$2-0=2$	$3-0=3$	$4-0=4$
$2-1=1$	$3-1=2$	$4-1=3$	
$3-2=1$	$4-2=2$		
$4-3=1$			

ii) $w(0,0) = q(0) = 2$
$w(1,1) = q(1) = 3$
$w(2,2) = q(2) = 1$
$w(3,3) = q(3) = 1$
$w(4,4) = q(4) = 1$

⇒ For $j-i=1$ (Step I)

3) $w(i,j) = P(j) + q(j) + w(i,j-1)$

* $i=0,\ j=1$ ($\because 1-0=1$)

$w(0,1) = P(1) + q(1) + w(0,0) = 3+3+2 = 8$ $w(0,1) = 8$

4) $c(i,j) = \min\limits_{i<k\le j} \{c(i,k-1) + c(k,j)\} + w(i,j)$

$$C(0,1) = \min_{\substack{0 < k \le 1 \\ k=1}} \{ C(0,0) + C(1,1) \} + w(0,1) = \min\{0+0\} + 8$$

$$= 8 + 0$$

$$C(0,1) = 8$$

5) $r(i,j) = k$

$r(0,1) = 1$

* when $i=1$ $j=2$

$w(1,2) = P(2) + q(2) + w(1,1)$

$= 3 + 1 + 3 = 7.$

$$C(1,2) = \min_{\substack{1 < k \le 2 \\ k=2}} \{ C(1,1) + C(2,2) \} + w(1,2) = \min\{0+0\} + w(1,2)$$

$C(1,2) = 7.$

$r(1,2) = k$

$r(1,2) = 2.$

* when $i=2$ $j=3$

$w(2,3) = P(3) + q(3) + w(2,2) = 1+1+1 = 3$

$$C(2,3) = \min_{\substack{2 < k \le 3 \\ k=3}} \{ C(2,2) + C(3,3) \} + w(2,3) = \min\{0+0\} + 3 = 3$$

$r(2,3) = k$

$r(2,3) = 3$

* when $i=3$, $j=4$

$w(3,4) = P(4) + q(4) + w(3,3) = 1+1+1 = 3.$

$$C(3,4) = \min_{\substack{3 < k \le 4 \\ k=4}} \{ C(3,3) + C(4,4) \} + w(3,4) = \min\{0\} + 3 = 3$$

$r(3,4) = k$

$r(3,4) = \cancel{3}4$

Step II :- For $j-i = 2$

* $i=0$, $j=2$

$w(0,2) = P(2) + q(2) + w(0,1)$

$= 3 + 1 + 8 = 12$

$$C(0,2) = \min_{\substack{0 < k \le 2 \\ k=\cancel{1}, k=2}} \{ C(0,0) + C(\cancel{1},2), C(0,1) + C(2,2) \} + w(0,2) = \min\{7+0, 8+0\} = \cancel{12}\min\{7, 8\} + 12$$

$= 7 + 12 = 19$

$r(0,2) = 1$ ($\because k=1$ we got minimum value.

* when $i=1 \quad j=3$

$w(1,3) = p(3) + q(3) + w(1,2)$

$= 1+1+7 = 9.$

$c(1,3) = \min\limits_{\substack{1<k\le 3 \\ k=2 \\ k=3}} \{c(1,1)+c(2,3),\ c(1,2)+c(3,3)\} + w(1,3)$

$= \min\{0+3,\ 7+0\} + 9$

$= 3+9 = 12$

$r(1,3) = k$

$r(1,3) = 2$

* when $i=2 \quad j=4$

$w(2,4) = p(4)+q(4)+w(2,3)$

$= 1+1+3 = 5$

$c(2,4) = \min\limits_{\substack{2<k\le 4 \\ k=3 \\ k=4}} \{c(2,2)+c(3,4),\ c(2,3)+c(4,4)\} + w(2,4)$

$= \min\{0+3,\ 3+0\} + 5$

$= 3+5 = 8$

$r(2,4) = k$

$r(2,4) = 3$ (first 3 for $k=3$)

* when i :

Step III :-

* when $i=0 \quad j=3$

$w(0,3) = p(3)+q(3)+w(0,2)$

$= 1+1+12 = 14$

$c(0,3) = \min\limits_{\substack{0<k\le 3 \\ k=1,2,3}} \{\underbrace{c(0,0)+c(1,3)}_{k=1},\ \underbrace{c(0,1)+c(2,3)}_{k=2},\ c(0,2)+c(3,3)\} + w(0,3)$

$= \min\{\underbrace{0+12}_{12},\ \underbrace{8+3}_{11},\ \underbrace{19+0}_{19}\} + 14 = \min\{12, 11, 19\} + 14 = 11+14$

$= 25$

$r(0,3) = k \quad r(0,3) = 2$

* when $i = 1 \quad j = 4$

$w(1,4) = P(4) + q(4) + w(1,3)$

$= 1 + 1 + 9 = 11$

$$C(1,4) = \min_{1 < k \le 4} \{ \overset{k=2}{c(1,1) + c(2,4)},\ \overset{k=3}{C(1,2) + C(3,4)},\ \overset{k=4}{C(1,3) + C(4,4)} \} + w(1,4)$$

$= \min\{ 8 + 0,\ 7 + \overset{3}{\cancel{8}},\ \overset{12+3}{\cancel{25+0}} \} + 11$

$= 8 + 11 = 19.$

$r(1,4) = 2$

Step IV :-

* when $i = 0 \quad j = 4$

$w(0,4) = P(4) + q(4) + w(0,3)$

$= 1 + 1 + 14 = 16$

$$c(0,4) = \min_{0 < k \le 4} \{ c(0,0) + c(1,4),\ c(0,1) + c(2,4),\ c(0,2) + c(3,4),\ c(0,3) + c(4,4) \} + w(0,4)$$

$k = 1$
$k = 2$
$k = 3$
$k = 4$

$= \min\{0 + 19,\ 8 + 8,\ 19 + 3,\ 25 + 0\} + 16$

$= \min\{19, 16, 22, 25\} = 16 + 16 = 32$

$r(0,4) = k$

$r(0,4) = 2$

	0	1	2	3	4
$j-i=0$	$w(0,0)=2$ $c(0,0)=0$ $r(0,0)=0$	$w(1,1)=3$ $c(1,1)=0$ $r(1,1)=0$	$w(2,2)=1$ $c(2,2)=0$ $r(2,2)=0$	$w(3,3)=1$ $c(3,3)=0$ $r(3,3)=0$	$w(4,4)=1$ $c(4,4)=0$ $r(4,4)=0$
$j-i=1$	$w(0,1)=8$ $c(0,1)=8$ $r(0,1)=1$	$w(1,2)=7$ $c(1,2)=7$ $r(1,2)=2$	$w(2,3)=3$ $c(2,3)=3$ $r(2,3)=3$	$w(3,4)=3$ $c(3,4)=3$ $r(3,4)=4$	×
$j-i=2$	$w(0,2)=12$ $c(0,2)=19$ $r(0,2)=1$	$w(1,3)\ 9$ $c(1,3)=12$ $r(1,3)=2$	$w(2,4)=5$ $c(2,4)=8$ $r(2,4)=3$	×	×
$j-i=3$	$w(0,3)=14$ $c(0,3)=25$ $r(0,3)=2$	$w(1,4)=11$ $c(1,4)=19$ $r(1,4)=2$	×	×	×