Mathematical Induction:

Mathematical Induction is a technique for proving a statment, theorem or formula which is thought to be true, for each and every natural number n. By generalizing this in form of a principle which we would use to prove any mathematical statment is "principle of Mathematical Induction".

Step(1): we prove that S(n) is true for some $n=1$

i.e S(1) is true.

step(2): we assume that S(n) is true for $n=K$

i.e S(K) is true.

step(3): we have to prove that S(n) is true for $n=K+1$

Then we say S(n) is true for all $n \in \mathbb{N}$

Q① prove that $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, for $\forall\, n \geq 1$ by Mathematical Induction.

proof: Let $S(n) = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Step(1): we need to prove S(n) is true for $n=1$

$L.H.S = 1^2 = 1$, $R.H.S = \frac{1(2)(3)}{6}$

$= 1$

$\therefore$ LHS = RHS

$\therefore$ S(n) is true for $n=1$

step(2): we will assume S(n) is true for $n=k$

$$S(K) = 1^2+2^2+3^2+\cdots+K^2 = \frac{K(K+1)(2K+1)}{6} \quad \text{—— ①}$$

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Step(3): we have to prove that $S(n)$ is true for $n = K+1$

i.e $S(K+1) = 1^2 + 2^2 + 3^2 + \cdots + K^2 + (K+1)^2 = \frac{(K+1)(K+2)(2(K+1)+1)}{6}$.

L.H.S $\underbrace{1^2 + 2^2 + 3^2 + \cdots + K^2}_{\text{from eq①}} + (K+1)^2$

$$= \frac{K(K+1)(2K+1)}{6} + (K+1)^2$$

$$= (K+1)\left[\frac{K(2K+1)}{6} + (K+1)\right]$$

$$= \frac{(K+1)}{6}\left[K(2K+1) + 6(K+1)\right]$$

$$= \frac{(K+1)}{6}\left[2K^2 + K + 6K + 6\right]$$

$$= \frac{(K+1)}{6}\left[2K^2 + 7K + 6\right]$$

$$= \frac{(K+1)}{6}\left[2K^2 + 4K + 3K + 6\right]$$

$$= \frac{(K+1)}{6}\left[2K(K+2) + 3(K+2)\right]$$

$$= \frac{(K+1)}{6}\left[(K+2)(2K+3)\right]$$

$$= \frac{(K+1)\left[(K+1)+1\right](2(K+1)+1)}{6}$$

$\therefore$ LHS = R.H.S

$S(n)$ is true for $n = K+1$

$\therefore$ By the principle of mathematical induction given statment is true for all $n \geq 1$.

prove the following by using Mathematical Induction for all +ve integers n.

1. $2.3+3.4+4.5+\cdots$ up to n terms $= \frac{n(n^2+6n+11)}{3}$ $\forall n\in N$
2. $1.3+3.5+5.7+\cdots$ up to n terms $= \frac{n(4n^2+6n-1)}{3}$ $\forall n\in N$
3. $1.2.3+2.3.4+3.4.5+\cdots$ up-to n terms $= \frac{n(n+1)(n+2)(n+3)}{4}$ $\forall n\in N$
4. $2+3.2+4.2^2+\cdots$ up-to n terms $= n.2^n$ $\forall n\in N$
5. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots$ up-to n terms $= \frac{n}{3n+1}$ $\forall n\in N$
6. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots$ up-to n terms $= \frac{n}{2n+1}$ $\forall n\in N$
7. $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$ up to n terms $= \frac{n(n+1)^2(n+2)}{12}$ $\forall n\in N$
8. $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$ up-to n terms $= \frac{n}{24}\left[2n^2+9n+13\right]$ $\forall n\in N$
9. $a+(a+d)+(a+2d)+\cdots$ up-to n terms $= \frac{n}{2}\left[2a+(n-1)d\right]$ $\forall n\in N$
10. $a+ar+ar^2+\cdots$ up-to n terms $= \frac{a(r^n-1)}{r-1}$, $r\neq 1$ $\forall n\in N$
11. $49^n+16n-1$ is divisible by 64 for all +ve integers n
12. $3.5^{2n+1}+2^{3n+1}$ is divisible by 17, $\forall n\in N$
13. (i) using Mathematical induction, S.T x^m+y^m is divisible by $x+y$, if 'm' is an odd natural Number and x, y are real Numbers
    (ii) if x, y are natural numbers and $x\neq y$, using mathematical induction, S.T x^n-y^n is divisible by $x-y$ $\forall n\in N$
14. $2n-3 < 2^{n-2}$ for all $n\geq 5, n\in N$
15. $(1+x)^n > 1+nx$ for $n\geq 2, x>-1, x\neq 0$.

① Sol

Let $S(n) = 2.3 + 3.4 + 4.5 + \cdots$ (up to n terms) $= \frac{n(n^2+6n+11)}{3}$ $\forall n \in N$.

is not given so find n^{th} term.

first term, common difference

$2, 3, 4, \ldots$ are in A-P then n^{th} term $= a + (n-1)d$

$= 2 + (n-1).1$

$= n+1$

$3, 4, 5, \ldots$ are in A.P then n^{th} term $= a + (n-1)d$

$= 3 + (n-1)1$

$= n+2$

$$S(n) = 2.3 + 3.4 + 4.5 + \cdots + (n+1)(n+2) = \frac{n(n^2+6n+11)}{3} \quad \forall n \in N$$

Step(1): we need to prove $S(n)$ is true for $n=1$

L.H.S $= (1+1)(1+2)$

$= (2)(3)$

$= 6$

R.H.S $= \frac{1(1^2+6.1+11)}{3}$

$= \frac{18}{3}$

$= 6$

$\therefore$ L.H.S = R.H.S

$\therefore$ $S(n)$ is true for $n=1$

Step(2): we will assume $S(n)$ is true for $n=K$

$$S(K) = 2.3 + 3.4 + 4.5 + \cdots + (K+1)(K+2) = \frac{K(K^2+6K+11)}{3} \longrightarrow (1)$$

Step(3) we have to prove that $S(n)$ is true for $n = K+1$

i.e $S(K+1) = 2.3 + 3.4 + 4.5 + \cdots + (K+1)(K+2) + (K+1+1)(K+1+2) =$

$$\frac{(K+1)\left((K+1)^2 + 6(K+1) + 11\right)}{3}$$

L.H.S: $\underbrace{2.3 + 3.4 + 4.5 + \cdots + (K+1)(K+2)}_{\text{from eq}^n\ (1)} + (K+2) * (K+3)$

$$\frac{K(K^2+6K+11)}{3} + (K+2)(K+3)$$

$$= \frac{K(K^2+6K+11)}{3} + (K+2)(K+3)$$

$$= \frac{K(K^2+6K+11) + 3(K+2)(K+3)}{3}$$

$$= \frac{K^3+6K^2+11K + 3(K^2+5K+6)}{3}$$

$$= \frac{K^3+6K^2+11K+3K^2+15K+18}{3}$$

$$= \frac{K^3+6K^2+11K+3K^2+15K+1+11+6}{3}$$

$$= \frac{K^3+1^3+3K^2+3K+12K+6K^2+11K+11+6}{3}$$

$$= \frac{K^3+1^3+3.K^2.1+3.K.1^2+(6K^2+12K+6)+11K+11)}{3}$$

$$= \frac{(K+1)^3+6(K^2+2K+1)+11(K+1)}{3}$$

$$= \frac{(K+1)^3+6(K+1)^2+11(K+1)}{3}$$

$$= \frac{(K+1)[(K+1)^2+6(K+1)+11]}{3}.$$

Hence by the principle of Mathematical induction given statment is true for $\forall n \in N$

2.Q $\quad 1.3 + 3.5 + 5.7 + \cdots$ upto n terms $= \frac{n(4n^2+6n-1)}{3} \quad \forall n \in N$

H.W

Sol

(3) Prove that
$1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$ up to n terms $=\frac{n(n+1)(n+2)(n+3)}{4}$ $\forall n\in N$
by using Mathematical induction.

Sol

Let $S(n)=1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots$ up to n terms $=\frac{n(n+1)(n+2)(n+3)}{4}$

n^{th} term is not given
so find n^{th} term.

$1,2,3,\ldots$ are in A.P then n^{th} term $=n$

$2,3,4,\ldots$ " " $=2+(n-1)\cdot1$
$=n+1$

$3,4,5,\ldots$ " " $=3+(n-1)\cdot1$
$=n+2.$

$\therefore$

$S(n)=1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}.$

Step(1): we need to prove $S(n)$ is true for $n=1$

L.H.S $=1(1+1)(1+2)$
$=(2)(3)$
$=6$

R.H.S $=\frac{1(1+1)(1+2)(1+3)}{4}$
$=\frac{24}{4}$
$=6$

$\therefore$ L.H.S = R.H.S

$\therefore$ $S(n)$ is true for $n=1$

Step(2): we will assume $S(n)$ is true for $n=k$

$S(k)=1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots+k(k+1)(k+2)=\frac{k(k+1)(k+2)(k+3)}{4}.$ —— (1)

Step(3): we have to prove that $S(n)$ is true for $n=k+1$

i.e $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\cdots+k(k+1)(k+2)+(k+1)(k+1+1)+(k+1+2)$

$=\frac{(k+1)((k+1)+1)(k+1+2)(k+1+3)}{4}.$

L.H.S: $\underbrace{1.2.3 + 2.3.4 + 3.4.5 + \cdots + K(K+1)(K+2)}_{\text{from eq}^n (1)} + (K+1)(K+2)(K+3)$

$$= \frac{K(K+1)(K+2)(K+3)}{4} + (K+1)(K+2)(K+3)$$

$$= (K+1)(K+2)(K+3)\left[\frac{K}{4}+1\right]$$

$$= \frac{(K+1)(K+2)(K+3)(K+4)}{4}$$

$$= \frac{(K+1)(K+1+1)(K+1+2)(K+1+3)}{4}$$

$\therefore$ $S(n)$ is true for $n = K+1$

Hence by the principle of Mathematical induction given statment is true for $\forall n \in N$

(4) $2 + 3.2 + 4.2^2 + \cdots$ up to n terms $= n.2^n$ $\forall n \in N$

Sol $2.2^0 + 3.2^1 + 4.2^2 + \cdots$ up to n terms $= n.2^n$

not given find out n^{th} term.

$\underbrace{2.2^0 + 3.2^1 + 4.2^2}_{} + \cdots (n+1)2^{n-1} = n.2^n$

let $S(n) = 2 + 3.2 + 4.2^2 + \cdots + (n+1)2^{n-1} = n2^n$

step(1): we need to prove $S(n)$ for $n=1$

$$L.H.S = (1+1)2^{1-1} \qquad R.H.S = 1.2^1$$

$$= 2.2^0 \qquad = 2$$

$$= 2.1$$

$$= 2$$

LHS = RHS

$\therefore$ $S(n)$ is true for $n=1$

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Step(2): we will assume $S(n)$ is true for $n=k$

$$S(k) = 2+3.2+4.2^2+\cdots+(k+1)2^{k-1} = k.2^k \quad \text{--- (1)}$$

step(3): we have to P.T $S(n)$ is true for $n=k+1$

i.e $S(k+1) = 2+3.2+4.2^2+\cdots+(k+1)2^{k-1}+(k+1+1)2^{k+1-1} = (k+1)2^{k+1}$

L.H.S: $\underbrace{2+3.2+4.2^2+\cdots+(k+1)2^{k+1}}_{\text{from (1)}}+(k+2)2^k$

$$= k.2^k+(k+2)2^k$$

$$= (k+k+2)2^k$$

$$= (2k+2)2^k$$

$$= (k+1)2.2^k$$

$$= (k+1)2^{k+1}$$

$\therefore$ $S(n)$ is true for $n=k+1$

By the principle of mathematical induction given statment is true ~~for~~ $\forall n \in N$

(8) $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$ up to n terms $= \frac{n}{24}[2n^2+9n+13]$ $\forall n \in N$

Let $S(n) = \frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots+\frac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+\cdots(2n-1)} = \frac{n}{24}(2n^2+9n+13)$

~~Step(1)~~:

$S(n) = \frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots+\frac{\frac{\cancel{n^2}(n+1)^2}{4}}{\cancel{n^2}} = \frac{n}{24}(2n^2+9n+13)$

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$\because$ sum of integers formula $(S) = \frac{n(a+l)}{2}$

where $n \to$ no. of integers

$a \to$ first term

$l \to$ last term

$\because$ sum of cubes of natural numbers $(S) = \frac{n^2(n+1)^2}{4}$

$$S(n) = \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots + \frac{(n+1)^2}{4} = \frac{n}{24}(2n^2+9n+13)$$

Step(1): we need to prove $S(n)$ is true for $n=1$

$$LHS = \frac{(1+1)^2}{4} = \frac{2^2}{4} = 1 \qquad R.H.S = \frac{1}{24}(2\cdot 1^2+9\cdot 1+13) = 1$$

$LHS = RHS$

$\therefore$ $S(n)$ is true for $n=1$

Step(2): Assume $S(n)$ is true for $n=K$

$$S(K) = \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \dots + \frac{(K+1)^2}{4} = \frac{K}{24}[2K^2+9K+13] \quad \text{—— (1)}$$

Step(3): we have to prove that $S(n)$ is true for $n=K+1$

i.e $$S(K+1) = \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \dots + \frac{(K+1)^2}{4} + \frac{(K+1+1)^2}{4} = \frac{K+1}{24}\left(2(K+1)^2+9(K+1)+13\right)$$

L.HS $$\underbrace{\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \dots + \frac{(K+1)^2}{4}}_{\text{from (1)}} + \frac{(K+2)^2}{4}$$

$$= \frac{K}{24}(2K^2+9K+13) + \frac{(K+2)^2}{4}$$

$$= \frac{K(2K^2+9K+13) + 6(K+2)^2}{24}$$

$$= \frac{2K^3+9K^2+13K+6(K^2+4K+4)}{24}$$

$$= \frac{2K^3+9K^2+13K+6K^2+24K+24}{24}$$

$$= \frac{2K^3+15K^2+37K+24}{24}$$

$\because$ synthetic division

$\because K=-1$

2	15	37	24
0	-2	-13	-24
2	13	24	0

$2K^2+13K+24$

$$= \frac{(K+1)(2K^2+13K+24)}{24}$$

$$= \frac{(K+1)}{24}[2K^2+4K+9K+2+9+13]$$

$$= \frac{K+1}{24}[2K^2+4K+2+9K+9+13]$$

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$$= \frac{(K+1)}{24}\left[2(K^2+2K+1)+9(K+1)+13\right]$$

$$= \frac{(K+1)}{24}\left[2(K+1)^2+9(K+1)+13\right]$$

$\therefore$ $S(K+1)$ is true

Hence, by the principle of mathematical induction the given statment is true for all $n \in N$.

(5) $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots$ upto n terms $= \frac{n}{3n+1}$ for all $n \in N$

let $S(n) =$

Sol $\frac{1}{1\cdot4}+\frac{1}{4\cdot7}+\frac{1}{7\cdot10}+\ldots+\frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

$\because$ 1, 4, 7, ... are in A.P nth term $= a+(n-1)d$

$= 1+(n-1)3$

$= 1+3n-3$

$= 3n-2$

4, 7, 10, .. " " $= a+(n-1)d$

$= 4+(n-1)3$

$= 3n+1$

step(1): we need to prove $n=1$ is true.

$LHS = \frac{1}{(3\cdot1-2)\cdot(3\cdot1+1)}$ $RHS = \frac{1}{3\cdot1+1}$

$= \frac{1}{1\cdot4}$ $= \frac{1}{4}$

$= \frac{1}{4}$

$LHS = RHS$

$\therefore$ $S(1)$ is true.

Step(2): Assume S(n) is true for n = K

$$S(K) = \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{(3K-2)(3K+1)} = \frac{K}{3K+1} \quad (1)$$

Step(3): we have to S.T S(n) is true for n = K+1

$$S(K+1) = \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{(3K-2)(3K+1)} + \frac{1}{(3(K+1)-2)(3(K+1)+1)} = \frac{K+1}{3(K+1)+1}$$

LHS

$$\underbrace{\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{(3K-2)(3K+1)}}_{\text{from } (1)} + \frac{1}{(3K+1)(3K+4)}$$

$$= \frac{K}{3K+1} + \frac{1}{(3K+1)(3K+4)}$$

$$= \frac{K(3K+4)+1}{(3K+1)(3K+4)}$$

$$= \frac{3K^2+4K+1}{(3K+1)(3K+4)}$$

$$= \frac{3K^2+3K+K+1}{(3K+1)(3K+4)}$$

$$= \frac{3K(K+1)+1(K+1)}{(3K+1)(3K+4)}$$

$$= \frac{\cancel{(3K+1)}(K+1)}{\cancel{(3K+1)}(3K+4)}$$

$$= \frac{K+1}{(3K+3+1)}$$

$$= \frac{K+1}{3(K+1)+1}$$

$\therefore$ S(K+1) is true

Hence, by mathematical induction the given statment is true for all $n \in N$.

(11) Show that $49^n + 16n - 1$ is divisible by 64 for all positive integers n

Sol: Let $S(n) = 49^n + 16n - 1$

Step(1): we need to show $S(n)$ is true for $n=1$

$S(1) = 49 + 16 - 1$
$= 64(1)$

Thus the statment is true for $n=1$

Step(2): Assume $S(n)$ is true for $n=k$
i.e $49^k + 16k - 1$ is divisible by 64
$49^k + 16k - 1 = 64m$ for some $m \in N$
$49^k = 64m - 16k + 1$ —— (1)

Step(3): we have to show that $S(n)$ is true for $n=k+1$

consider $49^{k+1} + 16(k+1) - 1 = 49 \cdot 49^k + 16k + 16 - 1$
$= 49[64m - 16k + 1] + 16k + 15$
$= 49 \cdot 64m - 49 \cdot 16k + 49 + 16k + 15$
$= 49 \cdot 64m + 16k(49-1) \cancel{+48k} + 64$
$= 49 \cdot 64m - 16k \cdot 48 \cancel{+16k} + 64$
$= 49 \cdot 64m - 16k \cdot 4 \cdot 12 + 64$
$= 64[49m - 12k + 1]$ is divisible by 64.

$\therefore$ $S(k+1)$ is true

Hence by the principle of mathematical induction $49^n + 16n - 1$ is divisible by 64 $\forall n \in N$.

(12) Show that $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all $n \in N$.

Proof: Let $S(n) = 3.5^{2n+1} + 2^{3n+1}$

Step(1): We need to show that $S(n)$ is true for $n=1$

$S(1) = 3.5^{2.1+1} + 2^{3.1+1}$

$= 3.5^3 + 2^4$

$= 3(125) + 16$

$= 391$

$= 17(23)$ is divisible by 17

$\therefore$ $S(n)$ is true for $n=1$

Step(2): We assume $S(n)$ is true for $n=k$

$S(k) = 3.5^{2k+1} + 2^{3k+1}$ is divisible by 17

$\Rightarrow 3.5^{2k+1} + 2^{3k+1} = 17m$ for some $m \in N$

$3.5^{2k+1} = 17m - 2^{3k+1}$ —— (1)

Step(3): We have to show that $S(n)$ is true for $n=k+1$

consider $3.5^{2(k+1)+1} + 2^{3(k+1)+1} = 3.5^{2k+2+1} + 2^{3k+3+1}$

$= 3.5^{2k+1}.5^2 + 2^{3k+1}.2^3$

$= 25(3.5^{2k+1}) + 8.2^{3k+1}$

$= 25(17m - 2^{3k+1}) + 8.2^{3k+1}$

$= 25.17m - 25.2^{3k+1} + 8.2^{3k+1}$

$= 25.17m - 2^{3k+1}(25-8)$

$= 25.17m - 2^{3k+1}.17$

$= 17(25m - 2^{3k+1})$

$\therefore$ $S(k+1)$ is true

Hence, by mathematical induction $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17

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(13) ii) Show that $x^n - y^n$ is divisible by $x-y$ for all $n \in N$ by using mathematical induction.

Proof: let $S(n) = x^n - y^n$

Step(1): we need to show that $S(n)$ is true for $n=1$

$$S(1) = x^1 - y^1$$
$$= x-y$$
$$= 1(x-y) \text{ is divisible by } x-y$$

$\therefore S(n)$ is true for $n=1$

Step(2). Assume $S(n)$ is true for $n=K$

$S(K) = x^K - y^K$ is divisible by $x-y$

$\Rightarrow x^K - y^K = (x-y)f(x,y)$ where f is some function in x,y — (1)

Step(3): we have to S.T $S(n)$ is true for $n=K+1$

consider $x^{K+1} - y^{K+1} = x^{K+1} - x^K \cdot y + x^K \cdot y - y^{K+1}$

$$= x^K(x-y) + y(x^K - y^K)$$
$$= x^K(x-y) + y(x-y)f(x,y) \quad \because (1)$$
$$= (x-y)[x^K + y f(x,y)]$$

$\therefore$ The statment is true for $n=(K+1)$

Hence, by the principle of mathematical induction $x^n - y^n$ is divisible by $x-y$ for all $n \in N$.

(15) Use mathematical induction to prove that $(1+x)^n > 1+nx$ for $n \geq 2$, $x > -1$, $x \neq 0$.

proof: Step(1): we need to prove $n=2$ is true

here donot take $n=1$ becaus given $n \geq 2$

$$(1+x)^2 = 1+2x+x^2$$
$$> 1+2x \quad \because x \neq 0, x > -1$$

$\therefore S(2)$ is true.

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Step(2): Assume $S(K)$ is true for $K \geq 2$.

$\therefore (1+x)^K > 1+Kx$ for $K \geq 2$

Step(3): we have to prove that $S(n)$ is true for $n=K+1$

Now $(1+x)^{K+1} = (1+x)^K (1+x)$

$> (1+Kx)(1+x)$

$> 1+Kx(1+x)+x$

$> 1+Kx+Kx^2+x$

$> 1+(K+1)x$

Thus the statment is true for $n=K+1$

Hence by the principle of mathematical induction

$S(n)$ is true for all $n \geq 2$, $n \in N$