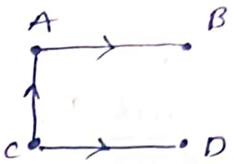
Graph Theory



This diagram consists of four vertices A, B, C, D and three edges. AB, CD, CA with directions attached to them, the directions being indicated by arrows

ie AB is edge, means the edge is from A to B (not B to A)

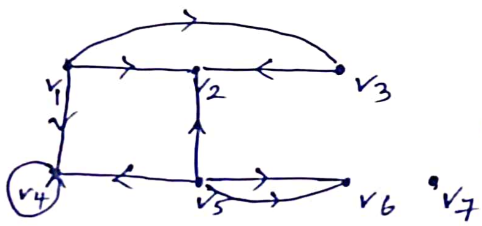
Hence the set of vertices $V = \{A, B, C, D\}$ and Edge set $E = \{(AB), (CA), (CD)\}$

such a diagram is called a Directed graph or digraph.

Def^n:. A directed graph is a pair (V, E), where 'V' is set of (non-empty) vertices and E is the set of directed edges. The directed graph (V, E) is also denoted by $D = (V, E)$ or $D = D(V, E)$

Terminology :-

1. In directed graphs, if AB is an edge the A is said to be initial vertex and 'B' is called terminal vertex..
2. Whenever, for an edge, initial and terminal vertices are same that edge is called a 'loop'
3. The directed edges having the same initial vertex and the same terminal vertex are called "parallel edges"
4. A vertex 'v' is called 'source', whenever edges are leaving from that vertex v, ~~that~~ but no vertex edge is terminating at 'v'
5. A vertex 'v' is called "sink", whenever all edges are terminating at 'v', but no edge is starting from v.
6. A vertex v, which is neither a terminal vertex nor a initial vertex for any edge is called 'isolated vertex'



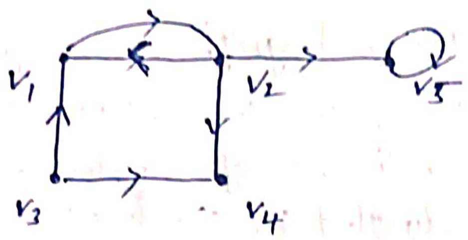
In this graph, (v_4, v_4) is a loop, (v_5, v_6) there are $\|^{el}$ edges. v_7 is a isolated vertex

In-degree and Out-degree:

The number of edges, which are leaving from a vertex 'v' is called "out-degree" of vertex v, denoted by $d^+(v)$

The number of edges, which are terminating at a vertex 'v' is called "In-degree" of vertex v, denoted by $d^-(v)$

- For a loop, $d^+(v) = 1$ and $d^-(v) = 1$



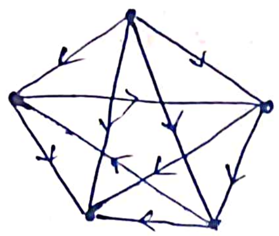
$d^+(v_1) = 1$	$d^-(v_1) = 2$
$d^+(v_2) = 3$	$d^-(v_2) = 1$
$d^+(v_3) = 2$	$d^-(v_3) = 0$
$d^+(v_4) = 0$	$d^-(v_4) = 2$
$d^+(v_5) = 1$	$d^-(v_5) = 2$

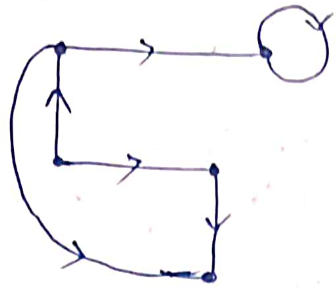
First theorem of the Diagraph theory

In every diagraph D, the sum of the out-degrees of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.

i.e $\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i)$

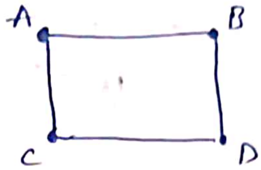
(P) Verify the first theorem of digraph theory for the digraphs (2)





Graphs:-

In this diagram, there are four vertices A, B, C, D and four edges connecting these vertices AB, AC, CD, BD. Here the edges are undirected. This diagram is called an 'undirected graph' (or) 'graph'



→ The edge AB is undirected, either we write that edge as AB or BA (order is not prefered)

Defn: A graph is a pair (V, E), where 'V' is a non-empty set of vertices and E is the set of undirected edges

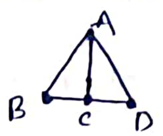
The graph (V, E) is also denoted by $G = (V, E)$ or $G = G(V, E)$

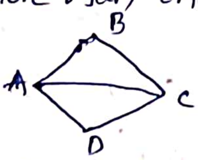
Terminology:

→ A graph containing no edges is called a "Null graph"

→ A graph with only one vertex is called 'Trivial graph'

→ There can be more than one diagram for the same graph





These graphs are same

→ A graph with finite no. of vertices and finite edges is called a 'finite graph'

→ The no. of vertices in a graph is called the "order of the graph" and the no. of edges in its is called 'size'

→ Suppose e_K is the edge joining vertices v_i and v_j Then v_i, v_j are called end vertices of e_K

i.e $e_K = v_i v_j$

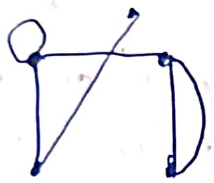
→ An edge whose end points are same, is called a 'Loop'

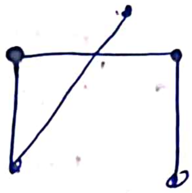
→ If there are two edges, whose end vertices are v_i, v_j then these ~~vertices~~ edges are called "parallel edges"

→ If there are two or more edges having the end points v_i and v_j i.e $e_1 = (v_i, v_j)$, $e_2 = (v_i, v_j)$, $e_3 = (v_i, v_j)$ then e_1, e_2, e_3 are called 'multiple edges'

Simple Graph :- A graph which does not contain loops and multiple edges is called a "simple Graph"

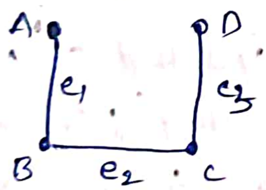
→ A graph which contains multiple edges, but not loops is called 'multigraphs'.


not simple


simple graph

Defn :- Two non-parallel edges are said to be "adjacent edges" if they are incident on a common vertex

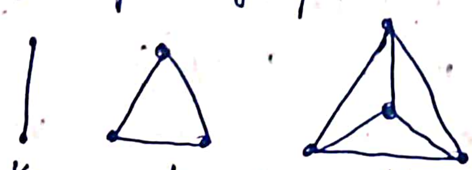
Defn :- Two vertices are said to be "adjacent vertices" if there is an edge joining them



Here A, B are adjacent vertices
A, C are not adjacent vertices
and e_1, e_2 are adjacent edges
e_1, e_3 are not adjacent edges.

Complete Graph :- A simple graph of more than two vertices is said to be "complete graph" if there is an edge b/w every pair of vertices

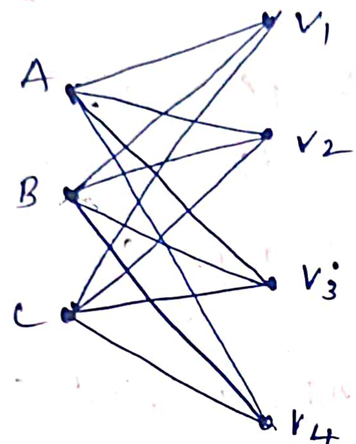
A complete graph with 'n' vertices is denoted by 'K_n'


K_2 K_3 K_4

Bipartite graph :-

A Simple graph G is such that its vertex set 'V' is the union of two mutually disjoint non-empty sets V_1 and V_2 which are such that every edge joins a vertex in V_1 and a vertex in V_2. Then G is called "Bipartite graph". If E is the edge set of this graph is denoted by $G = (V_1, V_2; E)$. The Sets V_1 and V_2 are called bipartites of the vertex set V.

Ex:



Here $V_1 = \{A, B, C\}$

$V_2 = \{v_1, v_2, v_3, v_4\}$

are bipartites. It is Bipartite graph.

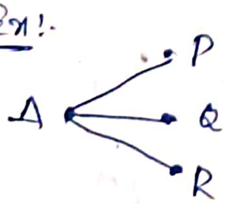
Complete Bipartite graph :-

A bipartite graph $G = (V_1; V_2; E)$ is called complete Bipartite graph", if there is an edge b/w every vertex in V_1 and every vertex in V_2

A complete bipartite graph $G(V_1, V_2; E)$ where V_1 is containing 'r' vertices and V_2 is containing 's' vertices with $r \leq s$ is denoted by '$K_{r,s}$'

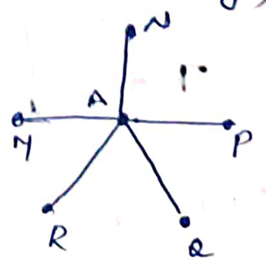
Thus $K_{r,s}$ has $r+s$ vertices & rs edges

Ex:-

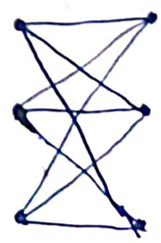


(i) $K_{1,3}$

(ii) $K_{2,3}$



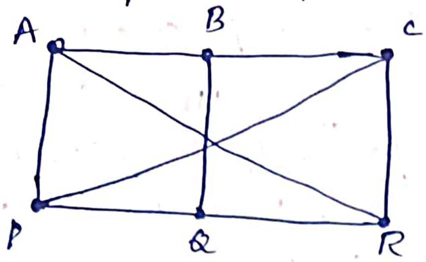
(iii) $K_{1,5}$



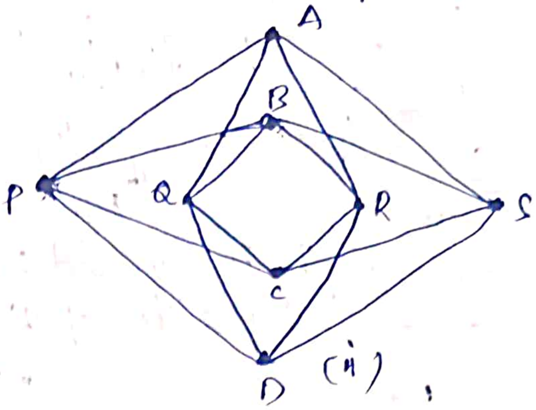
(iv) $K_{3,3}$

Here $K_{3,3}$ is called Kuratowski's second graph

Eg:- Verify that the following are bipartite graphs, what are their bipartites?



(i)



(ii)

Sol^n: (i) Bipartites are $V_1 = \{A, Q, C\}$, $V_2 = \{P, B, R\}$

(ii) Bipartites are $V_1 = \{P, Q, R, S\}$, $V_2 = \{A, B, C, D\}$

(P) Draw a diagram of the graph $G = (V, E)$ in each of the following cases

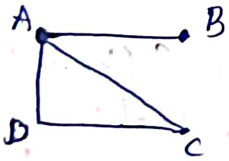
(i) $V = \{A, B, C, D\}$, $E = \{(A,B), (A,C), (A,D), (C,D)\}$

(ii) $V = \{V_1, V_2, V_3, V_4, V_5\}$, $E = \{(V_1,V_2), (V_1,V_3), (V_2,V_3), (V_4,V_5)\}$

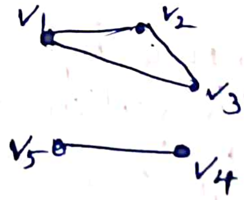
(iii) $V = \{P, Q, R, S, T\}$, $E = \{(P,S), (Q,R), (Q,S)\}$

(iv) $V = \{V_1, V_2, V_3, V_4, V_5, V_6\}$, $E = \{(V_1,V_4), (V_1,V_6), (V_4,V_6), (V_3,V_2), (V_3,V_5), (V_2,V_5)\}$

Sol^n:- (i)

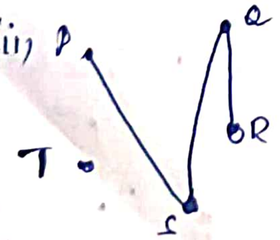


(ii)

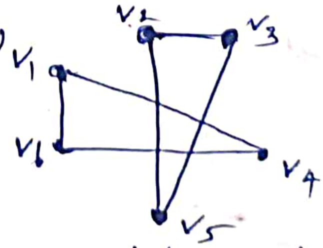


(iii)

(iii)

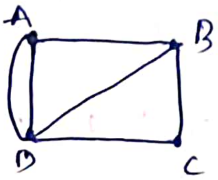


(iv)

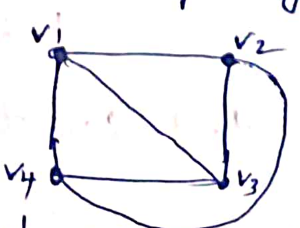


(P) Which of the following is a complete graph

(a)



(b)



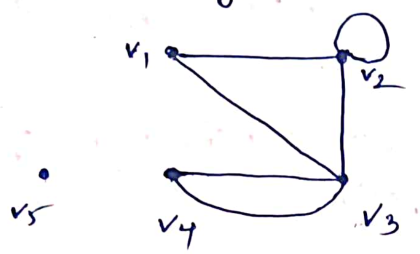
Sol^n: (a) Not complete graph. It is not simple and there is no edge b/w A and C

(b) Complete graph, It is simple and there is an edge b/w every two vertices

...of the degrees of vertices

Degree of a vertex: The no. of edges of the graph G, which are incident on a vertex G (the no. of edges that join v to the other vertices of G) with the loops counted twice is called the degree of the vertex 'v' and is denoted by deg(v) or d(v)

Also, the minimum of degrees of all vertices of a graph is called "Degree of the Graph"



$d(v_1) = 2$, $d(v_2) = 4$

$d(v_3) = 4$, $d(v_4) = 2$, $d(v_5) = 0$

∴ degree of the graph is '0'

→ A vertex, with degree '0' is isolated vertex

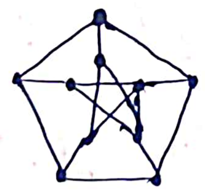
→ A vertex, with degree 1 is pendant vertex

An edge incident on a pendant vertex is called a pendant edge.

Regular graph: A graph in which all the vertices are of the same degree 'k' is called a regular graph of degree k or a k-regular graph

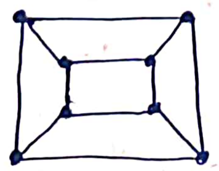
In particular, 3-regular graphs are called "Cubic graphs"

Ex:-



Here every vertex is of degree '3', so it is 3-regular graph (petersen graph)

Ex:-



Here every vertex is of degree 3, i.e 3-Regular graph

NOTE:- In this 3-regular graph, there are 2^3 vertices ∴ It is 3-dimensional hyper cube

In general, k-regular graph, with 2^k vertices is k-dimensional hyper cube

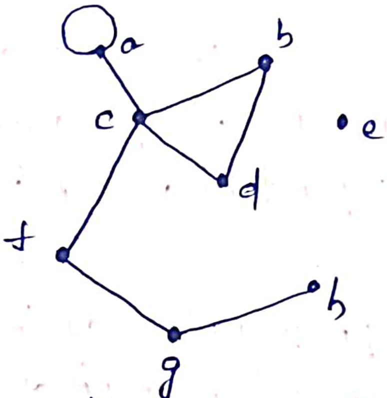
Hand shaking property:-

The sum of the degrees of all the vertices in a graph is an even number and this number is equal to the twice the no. of edges in the graph.

i.e for a graph $G = (V, E)$, we have $\sum_{v \in V} \deg(v) = 2|E|$

NOTE :- In every graph, the no. of vertices of odd degrees is even.

Ex:- For the given graph below, indicate the degree of each vertex and verify the handshaking property



$\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 4$

$\deg(d) = 2$, $\deg(e) = 0$, $\deg(f) = 2$

$\deg(g) = 2$, $\deg(h) = 1$

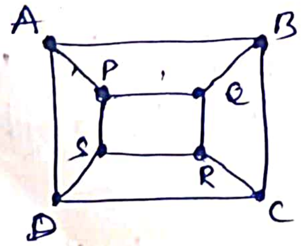
Here e is an isolated vertex and h is an pendant vertex

$\therefore \sum_{v \in V} \deg(v) = 3+2+4+2+0+2+2+1 = 16 = 2|E|$

Because $|E|$ = No. of Edges $= 8$

Ex:- Show that the hypercube Q_3 is a bipartite graph which is not a complete bipartite graph.

Soln: Hyper cube Q_3 is 3-regular graph, shown below



Suppose $V_1 = \{A, C, Q, S\}$

$V_2 = \{B, D, P, R\}$

Here every edge of the graph has one end in V_1 and the other end in V_2

Hence it is bipartite graph.

We observe that it is not complete graph, because no edge is joining A and R, where $A \in V_1$ and $R \in V_2$

=

Ex:- Prove that the hypercube Q_n has $n2^{n-1}$ edges. Hence determine the no. of edges in Q_8

Sol:- In the hypercube Q_n, the no. of vertices is 2^n and each vertex is of degree 'n'

$\therefore$ The sum of degrees of vertices of Q_n is $n \times 2^n$

By handshaking property, we have $n \times 2^n = 2|E|$, where $|E|$ is the size of Q_n.

Thus, $|E| = \frac{1}{2}(n \times 2^n) = n \times 2^{n-1}$

i.e Q_n has $n2^{n-1}$ edges

It follows that, the no. of edges in Q_8 is $8 \times 2^7 = 1024$ //

Ex:- What is the dimension of the hypercube with 524288 edges?

Soln:- If Q_n has 524288, we have $n2^{n-1} = 524288$

$= 2^{19} = 2^4 \times 2^{15} = 16 \times 2^{15}$

if $n = 16$

Thus, the dimension of the hypercube with 524288 edges is $n = 16$ //

Ex:- Determine the order $|V|$ of the graph $G=(V,E)$ in the following cases

(1) G is a cubic graph with 9 edges

(2) G is regular with 15 edges

(3) G has 10 edges with 2 vertices of degree 4 and all others of degree '3'

Soln:- (1) Suppose the order of G is n.

Since 'G' is a cubic graph, all vertices of G have degree '3'

$\therefore$ The sum of the degrees of vertices is $3n$

Since 'G' has 9 edges, we have $3n = 2 \times 9$ (by handshaking property)

$\Rightarrow n = 6$

$\therefore$ The order of G is '6'

(2) Since G is regular, all vertices of G must be the same degree, say k.

If 'G' is of order 'n', then the sum of the degrees of vertices is kn.

Since G has 15 edges, we have $kn = 2 \times 15$

$\Rightarrow k = 30/n$

Since k has to be a +ve integer, it follows that n must be a divisor of 30

Thus, the possible orders of G are 1, 2, 3, 5, 6, 10, 15 and 30

(3) Suppose the order of G is n.

Since two vertices of G are of degree 4 and all others are of degree 3, the sum of the degrees of vertices of G is

$2 \times 4 + (n-2) \times 3$.

Since G has 10 edges, we have $2 \times 4 + (n-2) \times 3 = 2 \times 10$

$\Rightarrow n = 6$.

$\therefore$ The order of G is 6 //

Isomorphism :-

Consider two graphs $G = (V, E)$ and $G' = (V', E')$.

Suppose there exist a function $f : V \to V'$ such that

(i) f is one-one and onto

(ii) for all vertices A, B of G, the edge $\{A, B\} \in E$ if and only if the edge $\{f(A), f(B)\} \in E'$

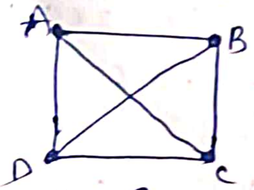
Then f is called an isomorphism b/w G and G' and we say that G and G' are isomorphic graphs ($G \cong G'$)

(OR)

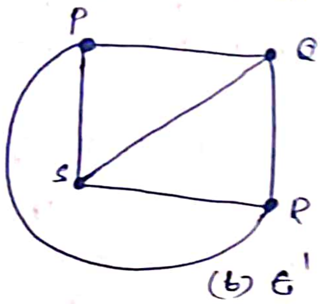
Two graphs are said to be isomorphic, if there is one-one correspondence b/w their vertices and b/w their edges such that the adjacency of vertices is preserved

i.e for two vertices u, v in G which are adjacent in G The corresponding vertices $u', v' \in G'$ are also adjacent in G'

Ex:-



(a) G

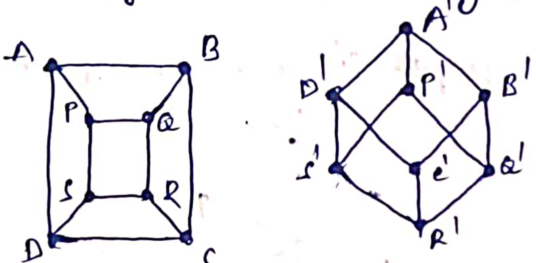


(b) G'

Consider the one-one Correspondence b/w the vertices
$A \leftrightarrow P$, $B \leftrightarrow Q$, $C \leftrightarrow R$, $D \leftrightarrow S$
under this one-one Correspondence b/w the edges
$AB \leftrightarrow PQ$, $AC \leftrightarrow PR$, $AD \leftrightarrow PS$, $BC \leftrightarrow QR$, $BD \leftrightarrow QS$, $CD \leftrightarrow RS$
The above indicated the two graphs preserves the adjacency of vertices
$\therefore G \cong G'$

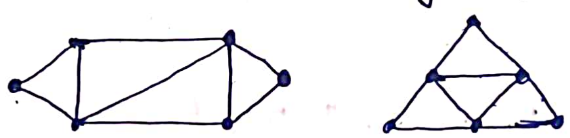
NOTE: If two graphs are isomorphic, then they must have
(i) The same no. of vertices
(2) The same no. of edges
(3) An equal no. of vertices with a given degree.

Ex:- Verify the the following graphs are isomorphic



Sol^n:- Consider the one-one correspondence b/w vertices
$A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, $D \leftrightarrow D'$, $P \leftrightarrow P'$, $Q \leftrightarrow Q'$, $R \leftrightarrow R'$, $S \leftrightarrow S'$
and one-one correspondence b/w edges
$AP \leftrightarrow A'P'$, $BQ \leftrightarrow B'Q'$, $CR \leftrightarrow C'R'$, $DS \leftrightarrow D'S'$
$PQ \leftrightarrow P'Q'$, $QR \leftrightarrow Q'R'$, $RS \leftrightarrow R'S'$, $SP \leftrightarrow S'P'$
$BC \leftrightarrow B'C'$, $CD \leftrightarrow C'D'$, $DA \leftrightarrow D'A'$, $AB \leftrightarrow A'B'$)
also adjacency is preserved
$\therefore$ Two graphs are isomorphic//

Ex:- Show that the following graphs are not isomorphic



(a) (b)

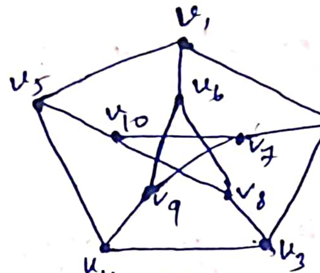
Sol^n:- Here graph (a) has 6 vertices and 9 edges and graph (b) also has 6 vertices and 9 edges

But graph (a) has 2 vertices of degree 4 whereas graph (b) has 3 vertices of degree 4.

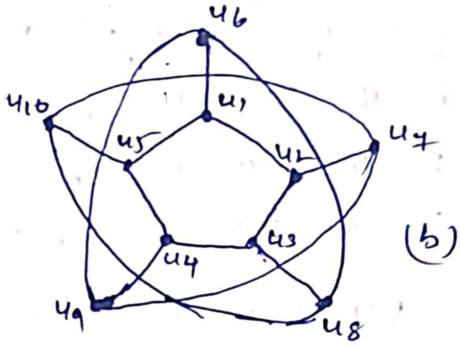
Therefore there cannot be any one-to-one correspondence b/w the vertices and b/w the edges of the two graphs

Hence the two graphs are not isomorphic //

Ex:- Show that the following two graphs are isomorphic

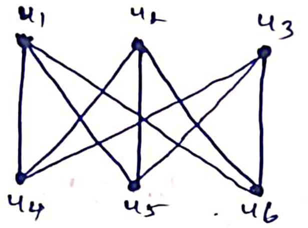


(a)

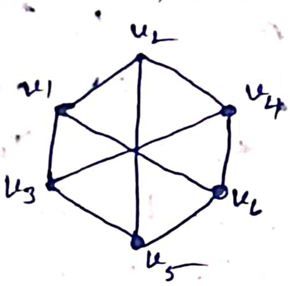


(b)

Ex:- Show that the two graphs shown below are isomorphic

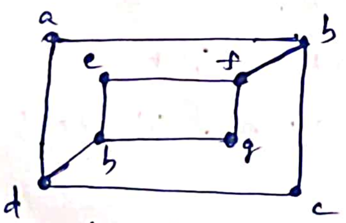


(a)

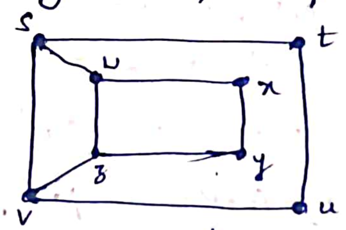


(b)

Ex:- Find whether the following are isomorphic



(a) G



(b) G'

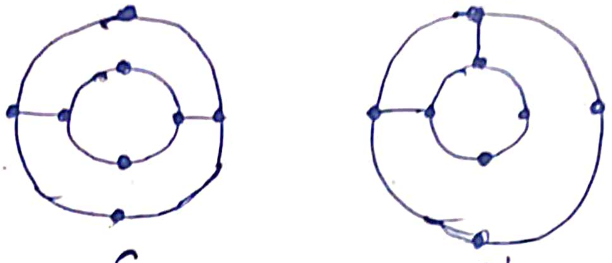
Sol^n:- The graphs G, G' both have 8 vertices and 10 edges. They also have four vertices of degree 2 and four vertices of degree 3.

But, deg (a) = 2, in G, a must correspond to either of t, u, x or y in G', because these are vertices of degree 2.

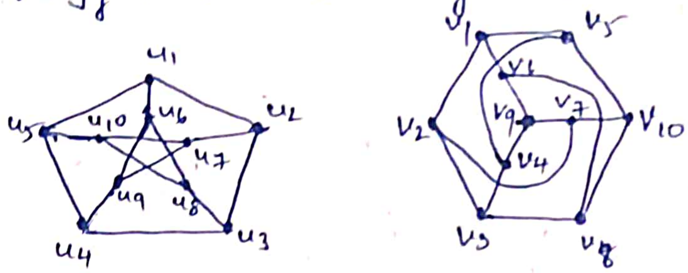
However, each of these vertices in G' is adjacent to another vertex of degree 2 in G', which is not happening for a in G

∴ G, G' are not isomorphic //

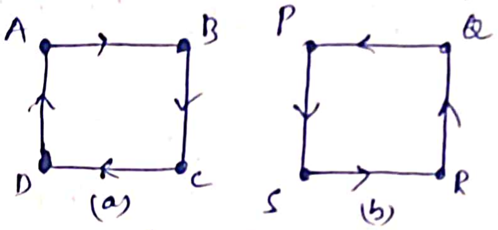
Ex:- Show that the following graphs are not isomorphic



Ex:- Verify that the following graphs are isomorphic



Ex:- Show that the following digraphs are isomorphic



Sol^n: Consider the one-to-one correspondence b/w the vertices

A ⟷ Q, B ⟷ P, C ⟷ S, D ⟷ R

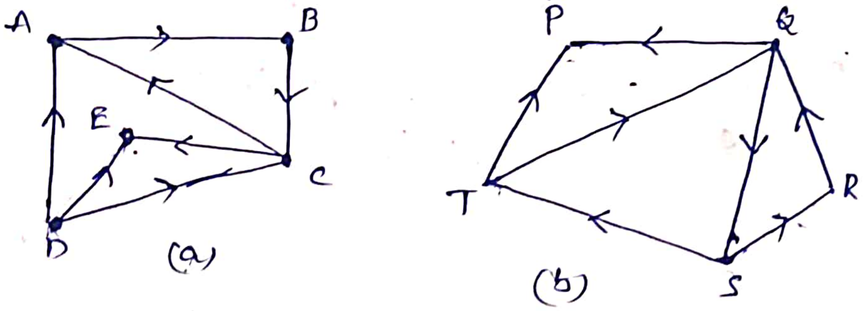
under this one-to-one correspondence b/w the edges

AB ⟷ QP, BC ⟷ PS, CD ⟷ (SR), DA ⟷ RQ

Evidently, under this correspondence, the adjacency of vertices, including directions of the edges is preserved.

Hence The given graphs are isomorphic.

Ex:- Show that the following digraphs are not isomorphic



Sol^n:- The two graphs have same no. of vertices and same no. of directed edges (7)

We observe that the vertex A of the first graph has 1 as its out-degree and 2 as its in-degree.

There is no such vertex in the second graph.

Therefore, there cannot be any one-to-one correspondence b/w the vertices of the two graphs which preserves the direction of edges.

∴ The two graphs are not isomorphic //

Subgraphs:-

A graph G' is said to be subgraph of G if

(1) All the vertices and all the edges of G' are in G

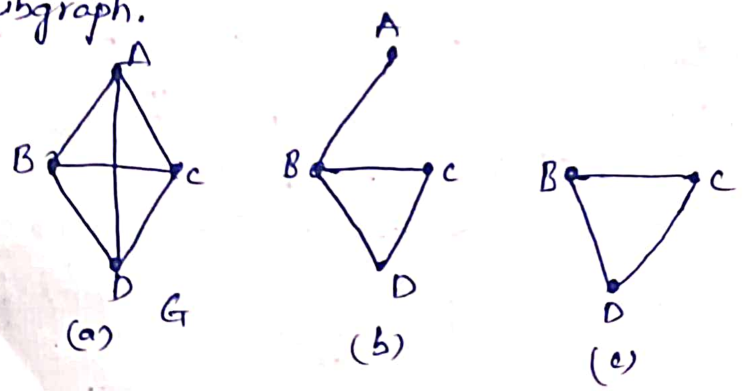
(2) Each edge of G' has the same end vertices in G as in G'

Essentially, a subgraph is a graph which is a part of another graph.

Spanning Subgraph:- Given a graph $G = (V, E)$, if there is a subgraph $G' = (V', E')$ of G such that $V' = V$ then G' is called a spanning subgraph of G.

(OR)

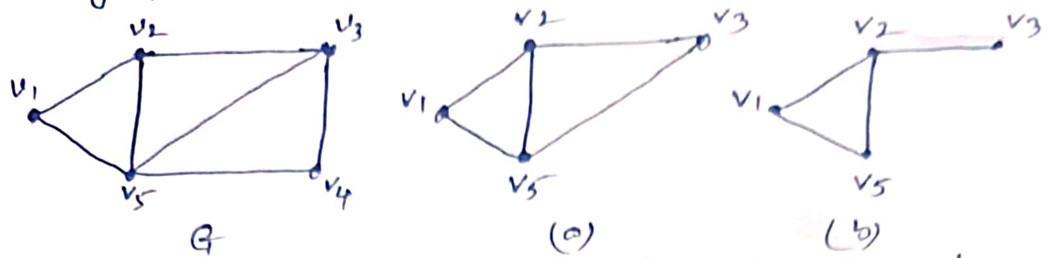
A subgraph whose vertex set is same as of G is spanning subgraph.



Here (b) is spanning subgraph of G

(c) is subgraph of G

Induced Subgraph :- Given $G=(V,E)$. Suppose there is a subgraph $G'=(V',E')$ of G such that every edge $\{A,B\}$ of G where $A,B \in V'$ is an edge of G' also. Then G' is called a subgraph of G induced by V' and is denoted by $\langle V' \rangle$



Here (a) is induced subgraph, induced by $V'=\{v_1,v_2,v_3,v_5\}$

(b) is not induced subgraph of G

Walks and their Classification :-

We consider five important subgraphs of a graph, called a walk, a trial, a Circuit, a path and a cycle.

Walk :- A walk in a graph $G(V,E)$ is a sequence, $v_1, v_2 \ldots v_k$ of vertices each adjacent to the next and a choice of an edge b/w each v_r and v_{r+1}, so that no edge is chosen more than once.

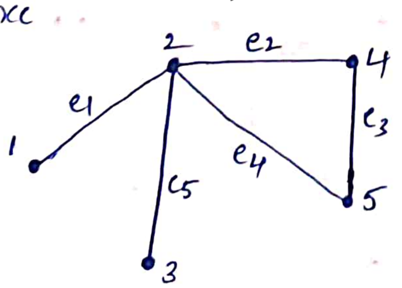
(OR)

A walk is a sequence of vertices and edges that begins at v_1 and travel along edges to v_K so that no edge appears more than once. However, a vertex may appear more than once.

→ The no. of edges present in a walk is called its length

→ In a walk, a vertex or an edge (or both) can appear more than once.

Ex :-

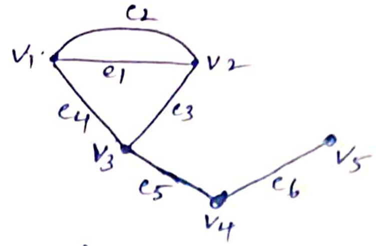


In this graph, the sequence $1-2-4-5-2-3$ is a walk of length '5'. In this walk, no vertex and no edge is repeated

In a walk, first and last vertices are called terminal vertices

Closed walk :- A walk is said to be closed walk if it is possible that a walk begins and end at the same vertices

Open walk :- A walk is said to be open walk if it is not closed or a walk in which the terminal vertices are different.



In this graph, The sequence

(a) $v_1\, e_1\, v_2\, e_3\, v_3\, e_4\, v_1$ — closed walk

(b) $v_3\, e_3\, v_2\, e_2\, v_1\, e_1\, v_2$ — open walk.

Trail:- An open walk in which no edge appears more than once is called a 'Trail'.

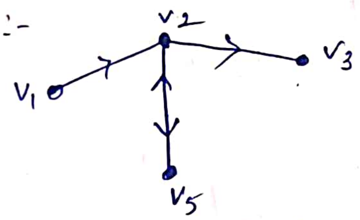
Circuit:- A closed walk in which no edge appears more than once is called a 'Circuit'

path:- A trail in which no vertex appears more than once is called a 'path'

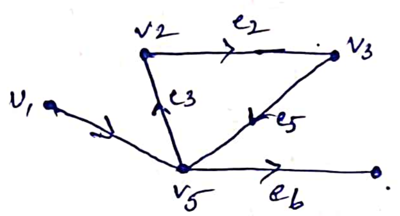
Cycle:- A Circuit in which the terminal vertex does not appear as an internal vertex (also) and no internal vertex is repeated is called a cycle

i.e A cycle is a closed walk in which neither a vertex nor an edge can appear more than once.

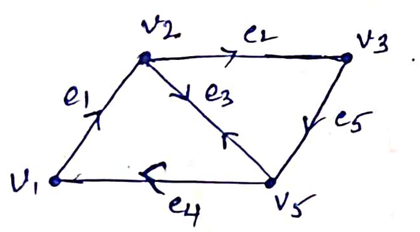
Ex:-



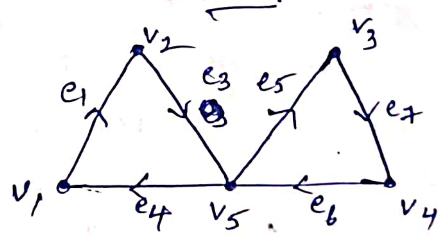
Not a trial



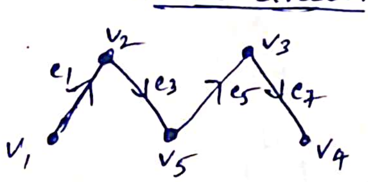
Trail



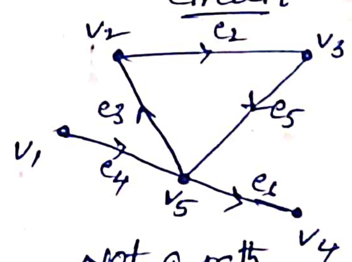
Not a Circuit



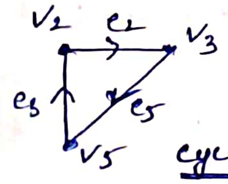
Circuit



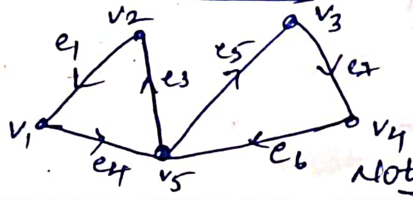
path



Not a path



cycle

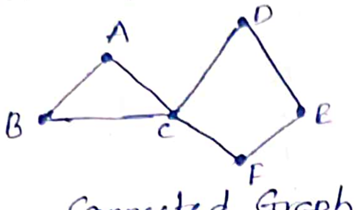


Not a cycle

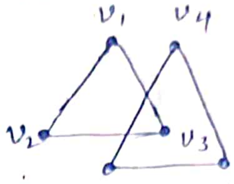
Connected graphs :-

A graph G is said to be connected if there exists atleast one path b/w every pair of its vertices, otherwise it is disconnected.

Intuitively, A graph G is connected if we can reach any vertex of G from any other vertex of G by travelling along the edges and disconnected otherwise.



Connected Graph



Disconnected Graph

There is no path from v_1 to v_4

→ All walks, all trails, all circuits, all paths and all cycles in a graph G (when they exist) are connected subgraphs of G

→ Every graph G consists of one or more connected graphs. Each such connected graph is a subgraph of G and is called a 'component' of G

Theorem :- A connected graph with 'n' vertices has atleast $(n-1)$ edges

proof :- Let 'n' be the vertices, m be the no. of edges of a connected graph and $n \geq 2$

We prove the result by mathematical induction i.e $m \geq n-1$

suppose $n=2$, Then there are two vertices and the graph is connected, there must be atleast one edge joining them

$\therefore m \geq 1$ means $m \geq (n-1)$

The result is true for $n=2$

Assume that the result is true for $n=k$ (≥ 2)

i.e $m \geq (n-1)$ holds, for $n=k$ graphs

$m \geq (k-1) \longrightarrow (1)$

Let $n=k+1$.

choose a vertex v, of this graph and consider the graph G_k obtained by deleting an edge from G_{k+1} for which v is an end vertex

Then G_k is a connected graph with k vertices

Let m_K be the no. of edges in G_K

so that our graph with $(k+1)$ vertices, will have the edges

$$m \geq (k+1) - 1$$

$$\Rightarrow m \geq k$$

$\therefore$ This result holds for $n = k+1$

Hence by mathematical induction, $m \geq (n-1) \ \forall \ n \geq 2$ //

Euler Circuits and Euler trails :-

Let 'G' be a connected graph. If there is a circuit in G that contains all the edges of G, then that circuit is called an Euler circuit in G

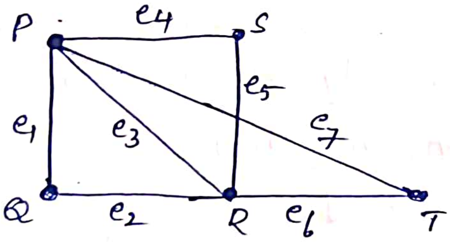
If there is a trail in G that contains all the edges of G then that trial is called an Euler trail in G.

i.e In a trial and a circuit no edge can appear more than once but a vertex can apper more than once. This property carried to Euler trails and Euler circuits also.

→ A connected graph that contains an Euler circuit is called an Euler graph

→ A connected graph that contains an Euler trail is called a semi-Euler graph

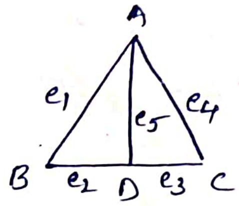
Ex:-



In this graph, $P e_1 Q e_2 R e_3 P e_4 S e_5 R e_6 T e_7 P$ is an Euler circuit.

$\therefore$ Graph is Euler graph

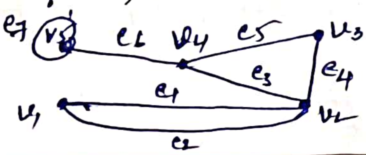
Ex:

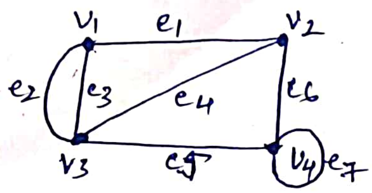


Not an Euler graph

It is semi-Euler graph.

Multigraph: A diagram which contains multiple edges as well as loops is called a multigraph





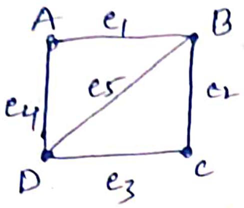
Hamilton Cycles and Hamilton paths:

Let G be a connected graph. If there is a cycle in G that contains all the vertices of G, then that cycle is called a Hamilton cycle in G.

A Hamilton cycle in a graph of 'n' vertices consists of exactly 'n' edges. Because, a cycle with 'n' vertices has 'n' edges.

Defn: A graph that contain a Hamilton cycle is called a Hamilton graph

Ex:-



In this, $Ae_1Be_2Ce_3De_4A$ is Hamilton cycle

Defn: A path in a connected graph which includes every vertex of the graph is called a Hamilton path

$Ae_1Be_2Ce_3D$ is Hamilton path in above graph

NOTE: ① Hamilton path with 'n' vertices has 'n-1' edges

② A simple connected graph with n vertices $(n \geq 3)$ is Hamiltonian if the degree of every vertex is greater than or equal to $n/2$

Ex:- prove that the complete graph K_n, $n \geq 3$, is a Hamilton graph

Soln: In K_n, the degree of every vertex is $n-1$

If $n \geq 3$, we have $n-2 > 0$

$$\Rightarrow n+n-2 > 0+n$$

$$\Rightarrow 2n-2 > n$$

$$\Rightarrow (n-1) > n/2$$

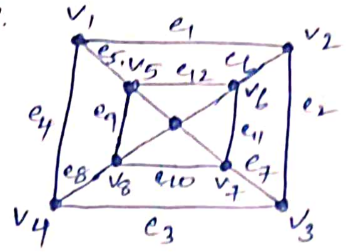
i.e In K_n, when $n \geq 3$, the degree of every vertex is greater than $n/2$

Hence K_n is Hamilton graph //

NOTE!. A connected graph G has an Euler ~~gra~~ Circuit if and only if all vertices of G are of even degree.

(P) Show that the following graph is Hamilton, but not Euler graph

Sol:



Consider the Hamilton cycle, (cycle contains all vertices)

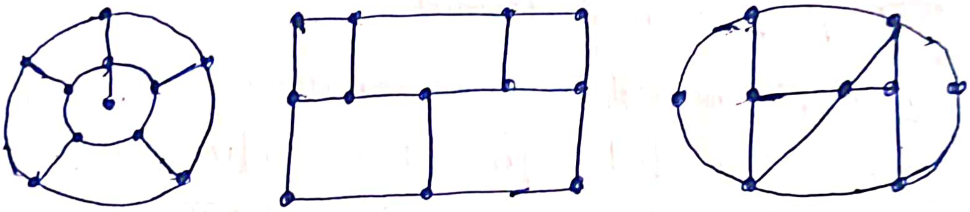
$v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_8\, v_8\, e_{10}\, v_7\, e_{11}\, v_6\, e_{12}\, v_5\, e_5\, v_1$

$\therefore$ G is Hamilton graph

There doesn't exist a circuit containing all edges, because degree of all vertices is not even

$\therefore$ G is Not Euler graph //

(P) Show that the following graphs are Hamiltonian



(a) (b) (c)

planar and Non-planar graphs

A graph which can be represented by atleast one plane drawing in which the edges meet only at the vertices is called a plannar graph.

A graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a non-planar graph.

In other words, a non-planar graph is a graph whose every possible plane drawing contains atleast two edges which intersect each other at points other than vertices..

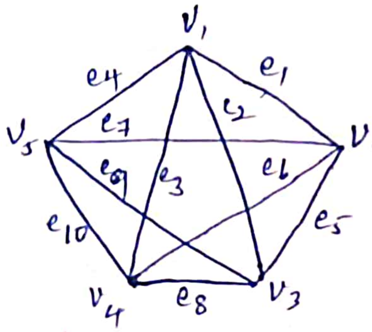
Ex:-

K_2 K_3 K_4

Complete graphs, K_2, K_3 and K_4 are clearly planar graphs

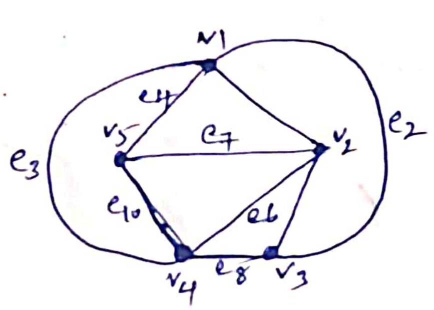
Ex:- Show that the complete graph K_5 (Kuratowski's first graph) is a non-planar graph.

Sol:. The graph K_5 (having 5 vertices and b/w any pair there is an edge) is shown below



In this graph, all the edges which are inside the cycle are intersecting at the points which are not vertices

So, firstly draw the cycle, and try to draw the inside edges without intersection and then few edges outside the cycle

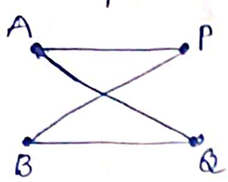


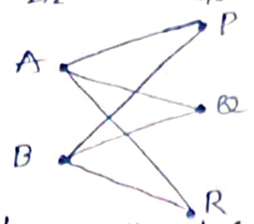
We cannot draw the edge e_9 without intersecting the remaining edges either inside or outside the cycle

Hence K_5 is non-planner graph.

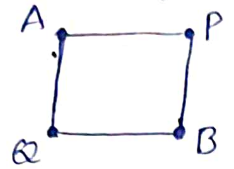
Ex:- Show that the complete bipartite graphs $K_{2,2}$ and $K_{2,3}$ are planar.

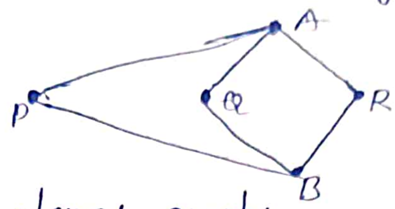
Soln. Bipartite graphs $K_{2,2}$ and $K_{2,3}$ are





Redrawing the graphs, without intersecting edges





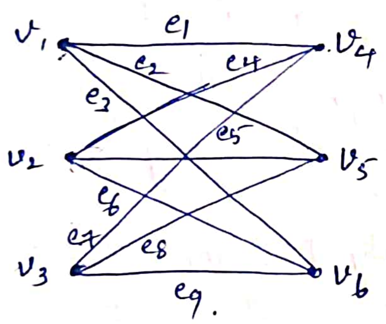
∴ These are planar graphs.

Ex:- Show that the complete Bipartite graph $K_{3,3}$ is non-planar

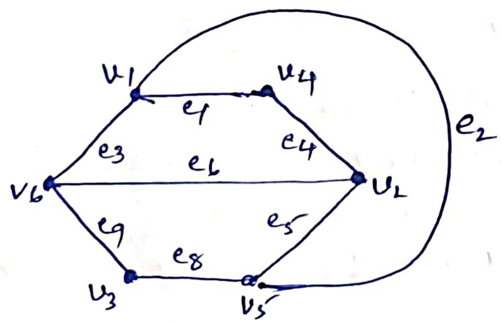
Soln: Complete bipartite graph $K_{3,3}$ with vertices sets

$V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4, v_5, v_6\}$ is shown below.

The graph $K_{3,3}$ with 6 vertices and 9 edges



The six edges $e_1, e_4, e_5, e_8, e_9, e_3$ will form a cycle and remaining edges e_2, e_6, e_7 intersect with these among themselves. So let us draw the cycle first, and draw remaining edges either inside or outside the cycle if possible.



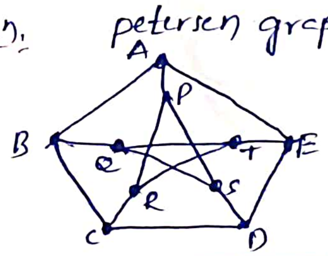
Here we can draw the edge e_7, without intersecting the remaining edges

∴ $K_{3,3}$ is non-planar graph //

(Kuratowski's second graph)

Ex:- Prove that the petersen graph is non-coplanar

Soln. petersen graph is a 3-regular graph of order 10 and size 15

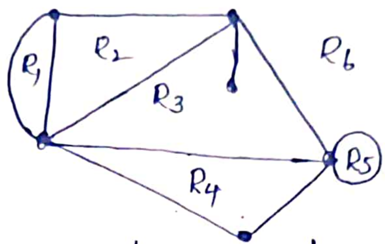


Euler's formula:-

If G is a planar graph, then G can be represented by a diagram in a plane in which the edges meets only at the vertices. Such a diagram divides the plane into a number parts, called regions, of which exactly one part is unbounded.

The no. of edges that form the boundary of a region is called the degree of that region.

for example,



In the diagram of a planar graph, the diagram divides the plane into 6 regions $R_1, R_2, R_3, R_4, R_5, R_6$. We observe that each of regions R_1 to R_5 is bounded and the region R_6 is unbounded i.e R_1 to R_5 are in the interior of the graph while R_6 is in the exterior.

Here $d(R_1)=2$, $d(R_2)=d(R_4)=3$, $d(R_3)=5$ (i.e R_3 consist of 4 edges of which one is a pendant edge), $d(R_5)=1$, $d(R_6)=6$, i.e exterior region R_6 consist of six edges

$$\therefore d(R_1)+d(R_2)+d(R_3)+d(R_4)+d(R_5)+d(R_6) = 2+3+5+3+1+6 = 20$$

which is twice the no. of edges in the graph.

Euler theorem on planar graph:-

A connected planar graph G with n vertices and m edges has exactly m−n+2 regions in all of its diagrams.

proof:- Let 'r' denotes the no. of regions in G.

By Euler's formula, $r=m-n+2$ (or) $n-m+r=2 \rightarrow (1)$

We give the proof by induction on 'm'

If $m=0$, n must be equal to 1. ~~Because~~ and also $r=1$

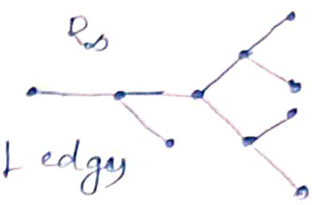
so that $n-m+r=1-0+1=2$

(1) is true for $m=0$

Assume that the theorem hold for graphs with $m=k$ no. of edges

Consider a graph G_{k+1} with $k+1$ edges and n vertices

Case (i):. G_{k+1} has no cycles in it



G_{k+1} has only one region, $r=1$ and no. of vertices $n=$ one more than no. of edges

$= (k+1)+1$

So that $n-m+r = (k+2)-(k+1)+1 = 2$

① is true for $m=k+1$

Case (ii): Suppose G_{k+1} contains atleast one cycle

Let 'r' be the no. of regions

Consider an edge e in a cycle and remove it from G_{k+1}

The resulting graph will contain 'n' vertices and $(k+1)-1=k$ edges and $(r-1)$ regions.

$\therefore n-m+r = n-k+(r-1) = 2$

$\Rightarrow n-(k+1)+r = 2$

$\therefore$ for $m=k+1$, ① is true

Hence by induction, ① is true for all m //

NOTE:- If G is a connected simple planar graph with $n(\geq 3)$ vertices, $m(>2)$ edges and 'r' regions then

(i) $m \geq \frac{3}{2} r$ (ii) $m \leq 3n-6$

Ex:- Kuratowski's first graph K_5 is non-planar

Soln: In K_5, $n=5$ vertices and $m=10$ edges

Assume that K_5 is a planar graph

So, $m \leq 3n-6$

i.e $10 \leq 3(5)-6$

$\Rightarrow 10 \leq 9$ is wrong

$\therefore K_5$ is non-planar graph.

Ex:- Kuratowski's Second graph $K_{3,3}$ is non-planar

Soln: In $K_{3,3}$ graph will have $n=6$ vertices, $m=9$ edges and $K_{3,3}$ has no triangles

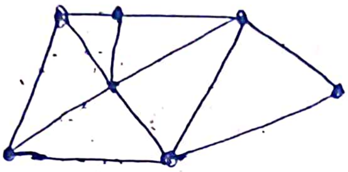
So, if $K_{3,3}$ is planar, it must satisfy $m \leq 2n-4$

$\Rightarrow 9 \leq 2(6)-4 \Rightarrow 9 \leq 8$ is wrong

$\therefore K_{3,3}$ is non-planar //

Ex:- Verify Euler's formula for the following graph

Soln:



G has $r=7$ regions

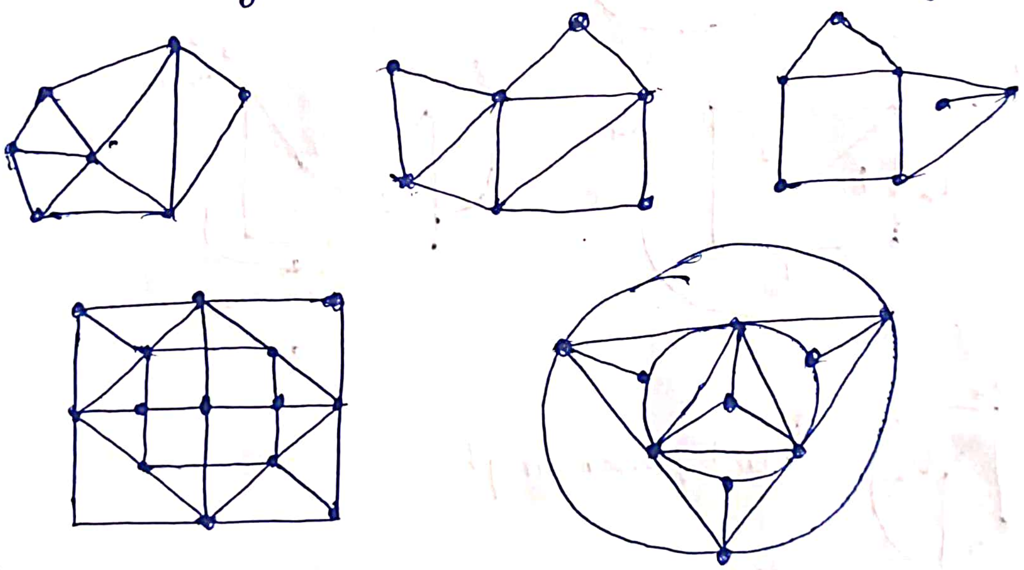
$n=7$ vertices

$m=12$ edges

By Euler's formula $n-m+r = 7-12+7 = 2$

$\therefore$ Euler's formula is verified

Ex:- Verify Euler's formula for the following planar graph



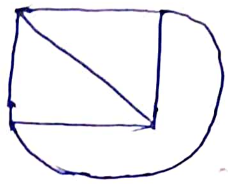
Dual of a planar graph

Consider a Dual of a planar graph G is denoted by G^* Can be obtained by using the following steps

1) If two regions R_i and R_j are adjacent (having a common edge), draw a line joining the points v_i and v_j that intersects the common edge b/w R_i and R_j exactly once.
2) If there is more than one edge common to R_i and R_j draw one line b/w the points v_i and v_j for each common edge
3) For an edge e_i lying entirely in one region, say R_i, draw a loop e_i^* at the point v_i intersecting e_i exactly once.
4) The vertices of G^* are corresponding to the faces or region of G.

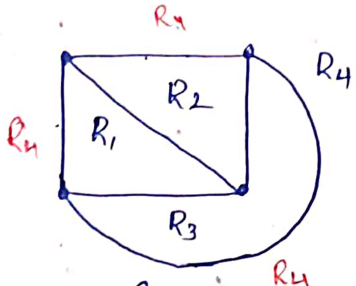
Ex:- Draw the dual graph G^* for the following planar graph G.

Soln

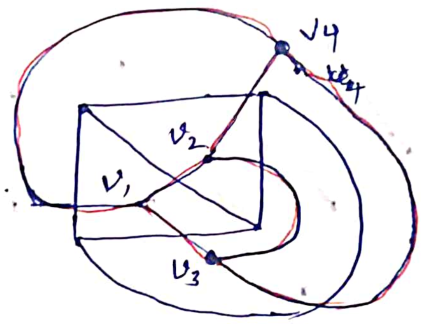


G

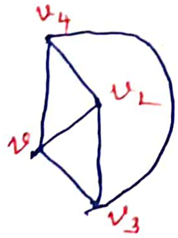
Soln:-



G

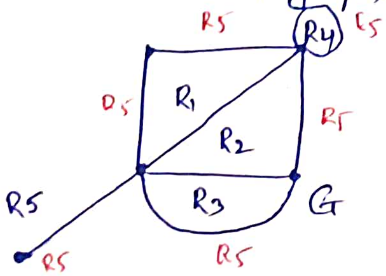


G^*



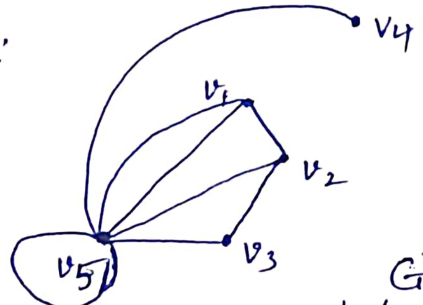
Dual planar graph G'

(P) Draw the dual graph G^* for the following planar graph G



planar graph

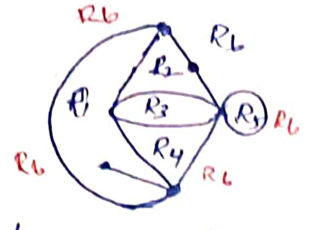
Soln:



G^*

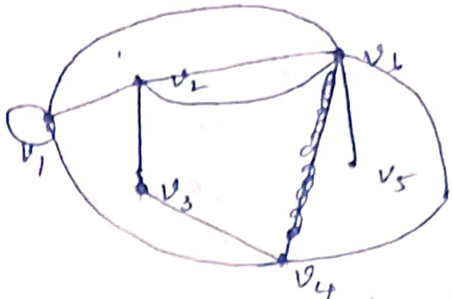
Dual planar graph

(P) Draw the Dual Graph G^* for the following planar graph G



planar Graph G

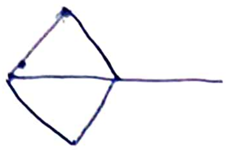
Sol^n:



Dual planar graph G'

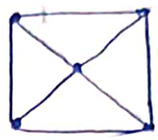
(P) Draw the dual graph G^* for the following planar graphs G

(i)



(ii)

(iii)



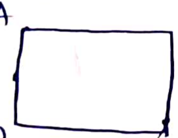
(iv)

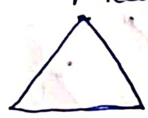
Graph Coloring:

Given a planar or non-planar graph. If we assign colors to its vertices in such a way that no two adjacent vertices have the same colour, then we can say that the graph G is properly colored.

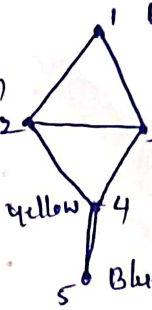
proper coloring of a graph means assigning colors to its vertices such that adjacent ~~colors~~ ~~to its~~ vertices have different Colors.

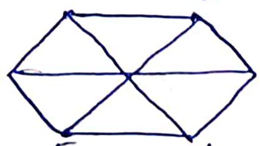
Ex:-





It is 2-colourable graph   It is 3-colourable graph





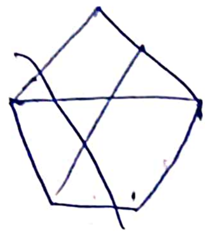
chromatic number :-

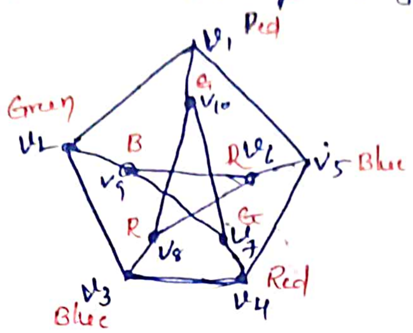
The minimum no. of colours required to colour all the vertices of a given graph is called a chromatic number of a given graph.

A chromatic number of a graph is usually denoted by $\chi(G)$

A graph G is said to be k-colorable if we can properly color it with k colors

(P) find the chromatic number of the following graphs





v_1 – Red $\quad$ v_6 – Red

v_2 → Green $\quad$ v_7 – Green

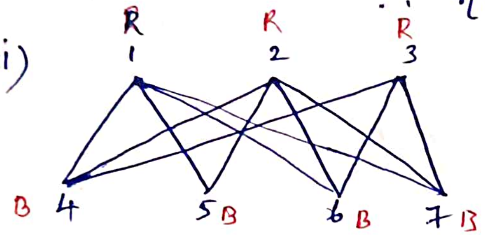
v_3 – Blue $\quad$ v_8 – Red

v_4 – Red $\quad$ v_9 – Blue

v_5 – Blue $\quad$ v_{10} – Green

$\therefore \chi(G) = 3$

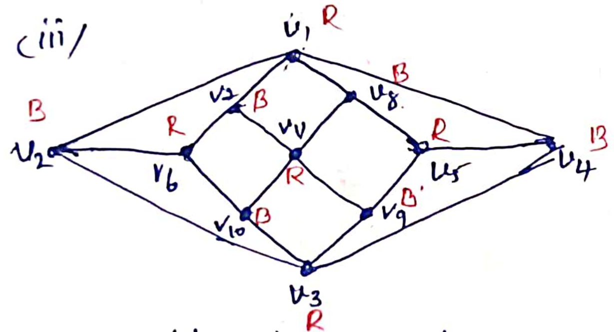
(ii)



$\therefore \chi(K_{3,4}) = 2$

Complete Bipartite graph $K_{3,4}$

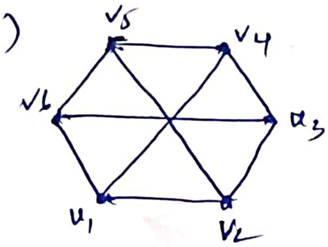
(iii)



$\chi(G) = 2$

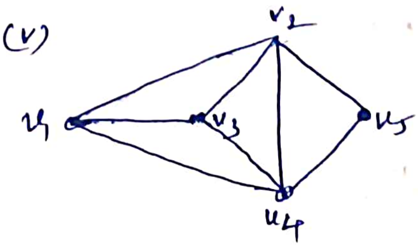
Herschel graph

(iv)



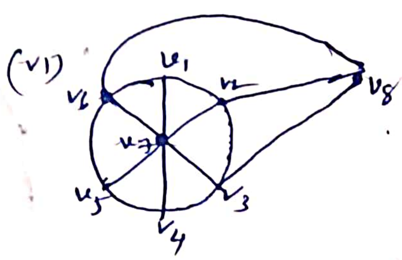
$\chi(G) = 2$

(v)



$\chi(G) = 4$

(vi)



$\chi(G) = 3$