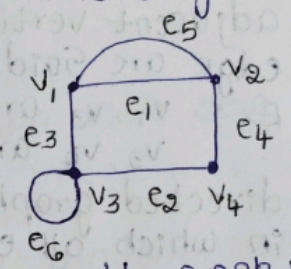
5. Graph Theory

Graph: A graph denoted by G consists of
a) a nonempty set $V = \{v_1, v_2 \ldots\}$ of elements v_i called vertices or nodes or points or junctions
b) a set $E = \{e_1, e_2 \ldots\}$ of elements of e_i called edges or arcs or lines or branches
c) A function ϕ that assigns to each edge e_k a subset $\{v_i, v_j\}$ of V. we write $G = (V, E, \phi)$

Representation of a graph: A graph is generally represented by a diagram on a surface with vertices represented by points and edge represented by line segment, (straight or curved and of any length). This diagram is called graph.



Ex: $\phi(e_5) = \{v_1, v_2\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

$V = \{v_1, v_2, v_3, v_4\}$

Self loop: It is an edge of the graph which has same vertex as both end vertices.

Ex: e_6 in a loop

Parallel edges: parallel edges are edges which are associated with same pair of vertices

Ex: e_1, e_5 are parallel edges

Incidence: An edge is said to be incident with each one of its end vertices

Ex: e_3 is incident with v_1 and v_3

adjacent edge: non-parallel edges that are incident with a common vertex are said to be adjacent to each other.

Ex: e_3, e_4 ; $e_1 e_4$ are adjacent edges

degree or valency of vertex: The number of edges that are incident with a vertex v_i with self loop counted twice is called degree of vertex v_i and written as $d(v_i)$.

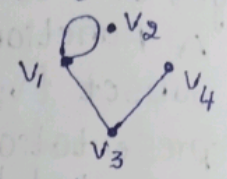
$d(v_1) = 3$, $d(v_2) = 3$, $d(v_3) = 4$, $d(v_4) = 2$

edges in Series: Two edges with a Common vertex in a graph are Said to be in Series if their Common vertex is of degree two

Ex: e_2, e_4 are edges in Series

Isolated vertex: An isolated vertex in a graph is a vertex having no (degree inc) edge incident on it, and is therefore a vertex whose degree is zero.

Ex: v_2 is isolated vertex



Pendant Vertex: A pendant vertex or end vertex in a graph is a vertex with degree 1.

Ex: v_4 is a pendant vertex

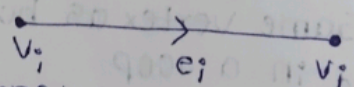
adjacent vertex: End vertices of the Same edge are Said to be adjacent vertices

Ex: v_1, v_3 are adjacent vertices

v_3, v_4 are adjacent Vertices

directed graph: directed graph is a graph in which all edges are directed.

Ex: $\phi(e_i) = (v_i, v_j)$ means e_i is an edge from v_i to v_j

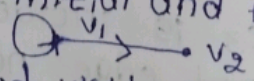


v_i is Called original or initial point (Source) predecessor;

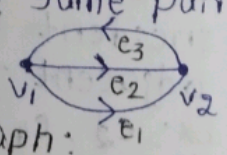
v_j is Called destination or terminal point or Successor.

e_i is incident out of v_i and into v_j

→ edge having Same initial and terminal points is Called loop.



→ Two edges associated with Same pair of vertices are parallel edges



Degree of a vertex in a diagraph:

1) Out degree or out valence or outward demi-degree of a vertex v_i in a graph:

It denotes
or incident
written as
2) Indegree
of a vertex
number of
v_i and is
out
$d^+(v_1) = 3$
$d^+(v_3) = 2$
$d^+(v_8) = 0$
v_8 is isola-
Source: A
is called a
Ex: $d^-(v_6)$
Sink: A ve
degree is
Ex: $d^+(v_8)$
Pendant ve
a digraph
a) Indegree
Ex: v_1
b) Indegre
Ex: v_6
Note:
d^+ (a pend
Vertex) = 1
walk: walk
Sequence of
ending with
incident vert
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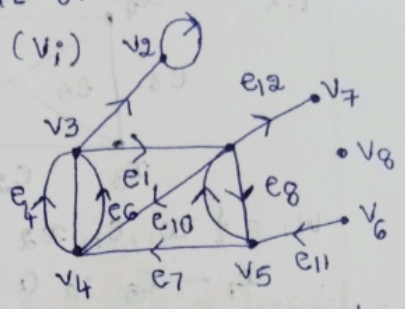
length of a
number of e
edges Coun

It denotes the number of edges beginning or incident out of the vertex v_i and is written as $d^+(v_i)$

2) Indegree or invalence or inward demi degree of a vertex v_i in a diagraph: It denotes the number of edges ending at or incident into v_i and is written as $d^{-1}(v_i)$

Out	In
$d^+(v_1) = 3$	$d^{-1}(v_1) = 2$
$d^+(v_3) = 2$	$d^-(v_3) = 3$
$d^+(v_8) = 0$	$d^{-1}(v_8) = 0$



v_8 is isolated vertex

Source: A vertex in digraph with zero indegree is called a Source

Ex: $d^{-1}(v_6) = 0$, v_6 is source

Sink: A vertex in a digraph with zero out-degree is called Sink

Ex: $d^+(v_7) = 0$, v_7 is sink

Pendant vertex in digraph: pendant vertex in a digraph is a vertex whose

a) Indegree is one and out degree is zero

Ex: v_1

b) Indegree is zero and out degree is one

Ex: v_6

Note:

d^+ (a pendant vertex) + d^+ (same pendant vertex) = 1

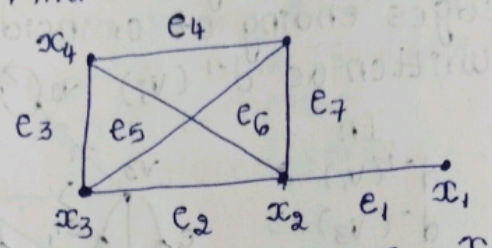
Walk: walk is defined as a finite alternating sequence of vertices and edges begining and ending with vertices, such that each edge is incident vertices, such that each edge is incident with vertices just preceding and following it. Vertices as well as edges may be repeated.

length of a walk: length of a walk is the number of edges it contains with repeated edges counted.

Open walk: A walk is open if its initial (starting) and final (ending) vertices are different.

Closed walk: A walk is closed if its initial and final vertices are same.

Ex:



$w_1 = \{x_1, e_1, x_2, e_6, x_4, e_3, x_3, e_5, x_5\}$

$\therefore$ $x_1 - x_5$ is a walk of length 4

$w_2 = \{x_1\ e_1\ x_2\ e_2\ x_3\ e_3\ x_4\ e_6\ x_2\ e_7\ x_5\ e_5\ x_3\ e_3\ x_4\ e_4\ x_5\}$

$\therefore$ $x_1 - x_5$ is a walk of length 8

w_1, w_2 are open walks. (Since initial and terminal points are not same)

$w_3 = \{x_2, e_7, x_5, e_5, x_3, e_2, x_2\}$ is a walk from $x_2 - x_2$ of length 3.

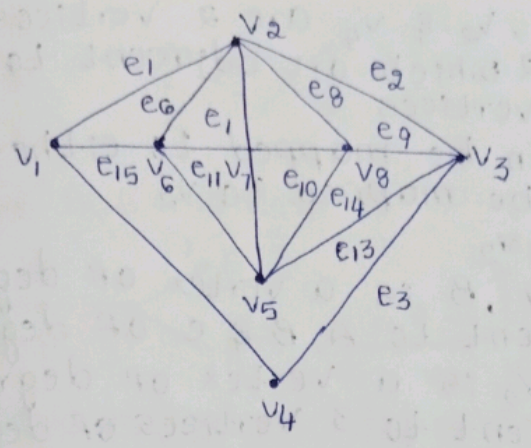
$\because$ Initial and terminal points are same the walk is a closed walk.

Trail: A trial is an open walk in which no edge is repeated. Vertices can be repeated in a trial.

Ex: $\{x_1, e_1, x_2, e_7, x_5, e_5, x_3, e_2, x_2\}$ is a trial in which x_2 has been repeated

Circuit: Circuit is a closed trial (walk) in which no edge is repeated. Vertices may however be repeated.

Ex: $\{v_6, e_6, v_2, e_7, v_7, e_{12}, v_5, e_{14}, v_8, e_{10}, v_7, e_{11}, v_6\}$ is a $v_6 - v_6$ circuit in which no edge is repeated but vertices v_6 & v_7 have been repeated



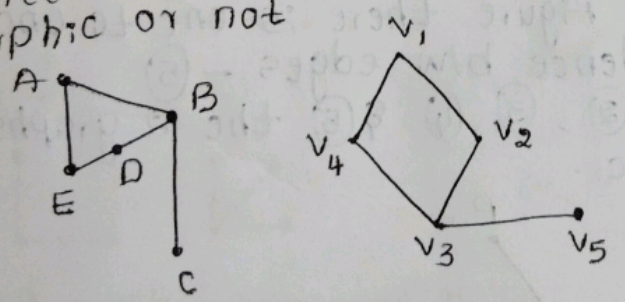
Write

Isomorphic graphs:

Let $G_1(V, E_1)$ & $G_2(V_2, E_2)$ are 2 graphs. The two graphs are said to be isomorphic if

1) they have same no. of vertices
2) same no. of edges
3) degree sequence is same
4) There exists a one to one correspondence b/w vertices & edges i.e one to one correspondence is preserved.

1. Check whether the 2 graphs are isomorphic or not



	G_1	G_2	
no. of vertices	5	5	Same — ①
no. of edges	5	5	Same — ②
degree sequence	2,3,1,2,2	2,2,3,2,1	Same — ③

One to one correspondence b/w vertices

A is a vertex of degree 2 which is adjacent to B & E of degree 3 & 2

In G_2, v_2 & v_4 are 2 vertices with degree 2 which are adjacent to 3 & 2 degree vertices.

$\therefore$ A can be mapped to either v_2 or v_4

Let A be mapped to v_2

$A \rightarrow v_2$

Similarly, B is a vertex of degree 3 which is adjacent to A, B & C of degrees 2, 2 & 1

In G_2 v_3 is a vertex of degree 3 which is adjacent to 3 vertices of degree 2, 2 & 1

$\therefore$ $B \rightarrow v_3$

$C \rightarrow v_5$

$D \rightarrow v_4$

$E \rightarrow v_1$

$\therefore$ There is a one to one correspondence b/w the vertices — 4

One to one correspondence b/w edges

$A \rightarrow v_2$ { From A to B there is an edge in G_1, lly v_2 to v_3 also there is an edge in G_2 }

$B \rightarrow v_3$

$C \rightarrow v_5$ { A to C there is no edge in G_1, lly v_2 to v_5 there is no edge in G_2 }

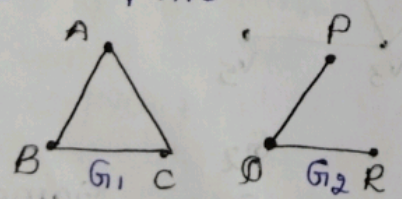
$D \rightarrow v_4$

$E \rightarrow v_1$

From the figure there is one to once correspondence b/w edges — 5

From 1, 2, 3, 4 & 5 the 2 graphs are isomorphic.

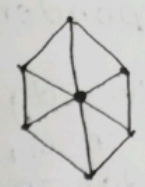
2)



Since the 2 graphs are not having same no. of edges

$\therefore$ The 2 graphs are not isomorphic

3)



They are not is

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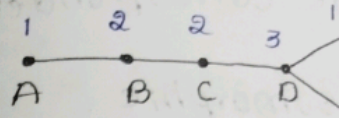


Same vertices,

D is a vertex

nt to C, E, F

But v_3 is a

is adjacent to

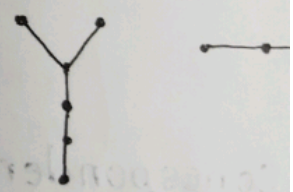
$\therefore$ There is no

b/w vertices

$\therefore$ The graphs

5)

7)



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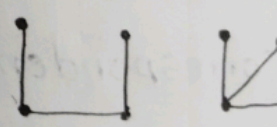

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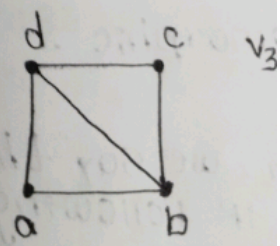


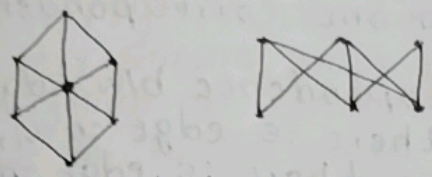
Same vert

= 2, 2, 3, 3

$d \rightarrow v$

$c \rightarrow v$

3)



They are not isomorphic

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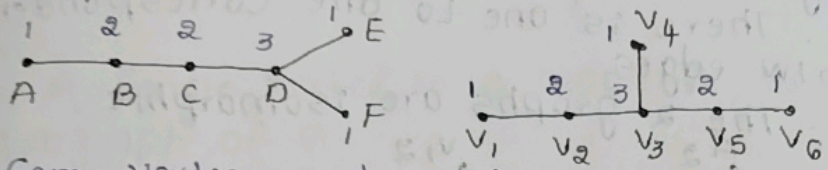


Same vertices, edges, degree sequence

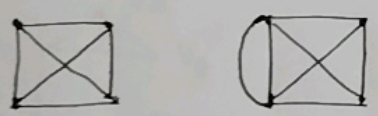
D is a vertex of degree 3 which is adjacent to C, E, F of degrees 2, 1, 1.

But v_3 is a 3rd degree vertex in G_2 which is adjacent to v_2, v_4, v_5 of degree 2, 1, 2

∴ There is no one to one correspondence b/w vertices

∴ The graphs are not isomorphic

5)



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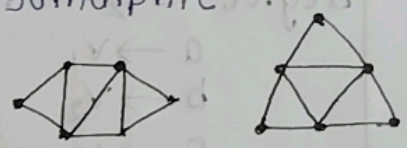

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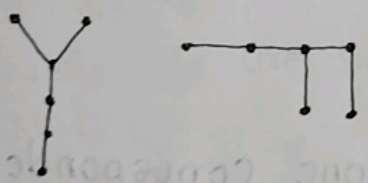

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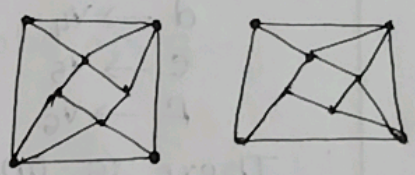

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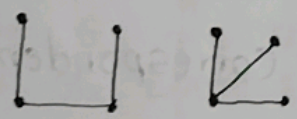

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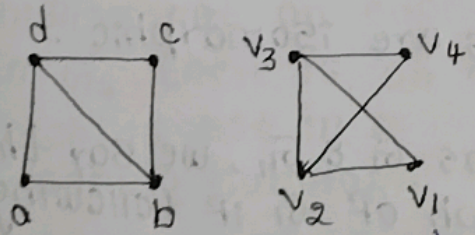


Same vertices, edges, degree sequence = 2, 2, 3, 3

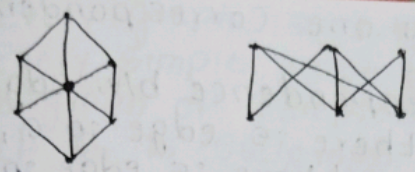
$d \to v_2$

$c \to v_1$

$b \to v_3$

$a \to v_4$

3)



They are not isomorphic

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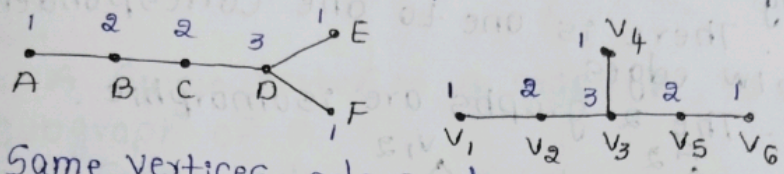

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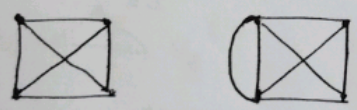

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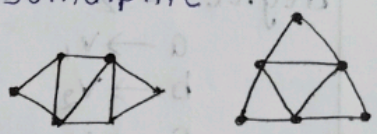

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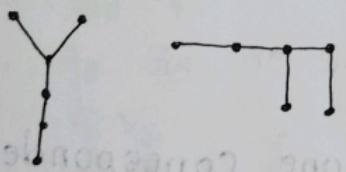

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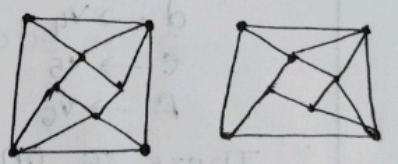

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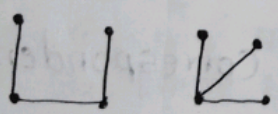

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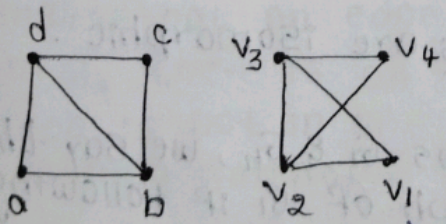


Same vertices, edges, degree sequence = 2, 2, 3, 3

$d \rightarrow v_2$

$c \rightarrow v_1$

$b \rightarrow v_3$

$a \rightarrow v_4$

There is one to one correspondence b/w vertices

One to one correspondence b/w edges

From d to c there is edge in G_1,

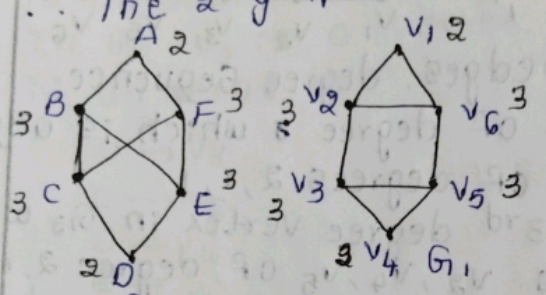
From v_3 to v_2 there is edge in G_2

By

∴ There is one to one correspondence b/w edges.

∴ The 2 graphs are isomorphic.

11)



	G_1	G_2
No. of vertices	6	6
No. of edges	8	8
degree sequence	2,3,3,3,3,2	2,3,3,3,3,2

$a \rightarrow v_1$

$b \rightarrow v_2$

$c \rightarrow v_3$

$d \rightarrow v_4$

$e \rightarrow v_5$

$f \rightarrow v_6$

∴ There is one to one correspondence b/w vertices

∴ There is one to one correspondence b/w edges

∴ The 2 graphs are isomorphic.

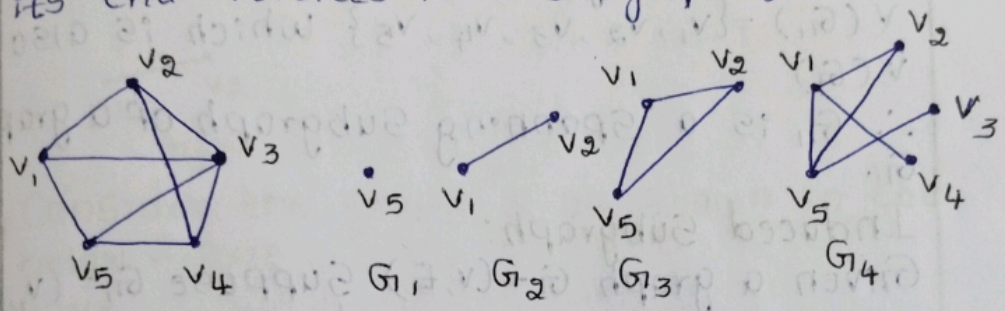
Subgraphs:

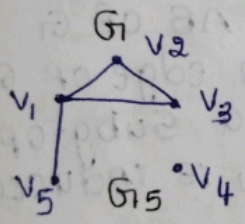
Given two graphs G & G_1, we say that G_1 is a subgraph of G if following conditions hold.

1) All vertices or all edges of G_1 are in G

2) Each edge of G has has same end vertices in G as in G_1.

1) Every graph is subgraph of itself
2) Every simple graph of n vertices is a sub graph of complete graph K_n.
3) If G_1 is subgraph of G_2 & G_2 is subgraph of G then G_1 is also a subgraph of G.
4) A single vertex in a graph G, is a subgraph of G
5) A single edge in a graph together with its end vertices is a subgraph of G.





G_1, G_2, G_5 are subgraphs of G

$\because V \subseteq V_G$

$E \subseteq E_G$

G_3, G_4 are not subgraphs

In G_3 there is an edge from v_2 to v_5 which is not an edge in G

In G_4 there is an edge from v_1 to v_4 which is not in G

Spanning subgraph

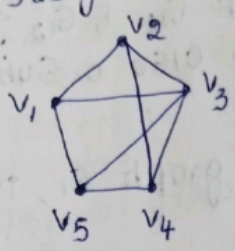
Given a graph $G = (V, E)$, if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$ then G_1 is called spanning subgraph of G.

In other words a subgraph G_1 of a graph G is said to be spanning subgraph of G

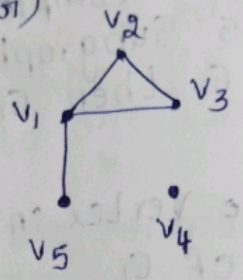
whenever the vertex set of G_1 contains all vertices of G.

$V(\text{Subgraph}) = V(G)$

Ex:



G



G_1

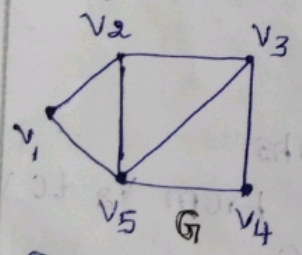
G_1 is a subgraph of G.

$V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$ which is also $V(G)$

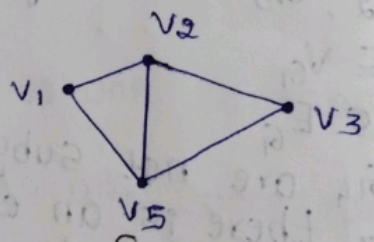
$\therefore$ G_1 is a spanning subgraph of a graph G.

Induced subgraph:

Given a graph $G = (V, E)$ suppose $G_1 = (V_1, E_1)$ of G such that every edge AB of G whenever $AB \in V_1$ is an edge of G_1 (Such that ever) then G_1 is a subgraph of G induced by V_1 i.e G_1 is induced subgraph



G

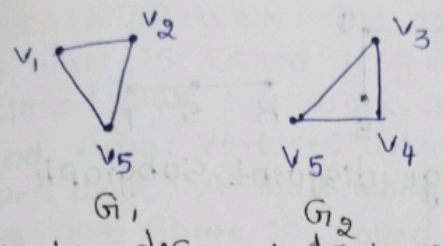


G_1

G_1 is induced subgraph of G

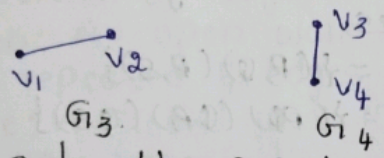
Edge disjoint subgraph:

Let G be a graph, G_1 & G_2 be 2 subgraphs of the graph G then G_1 & G_2 are said to be edge disjoint subgraphs if they do not have any edge in common

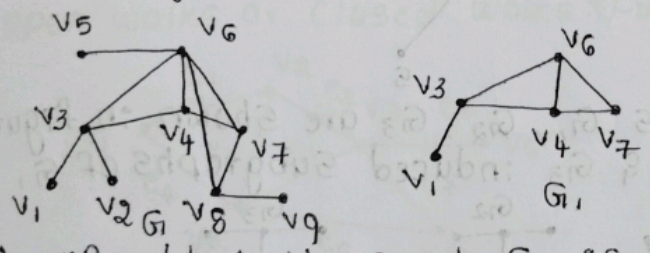


Vertex disjoint subgraph

Let G be a graph & G_1 & G_2 be 2 subgraphs of G. G_1 & G_2 are said to be vertex disjoint if they do not have any common edge & any common vertex



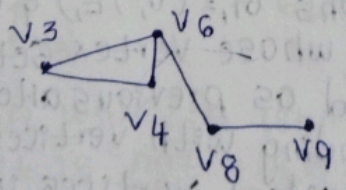
Consider the graph G as shown in the below figure.



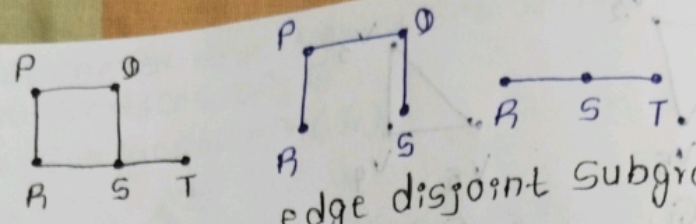
Verify that the graph G_1 is an induced subgraph of a graph G. Is this a spanning subgraph of a graph G

Draw the subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$

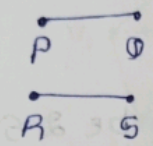
The given graph G_1 is a subgraph which is an induced subgraph but not a spanning subgraph.



For the graph shown below find 2 edge disjoint subgraph & 2 vertex disjoint subgraph



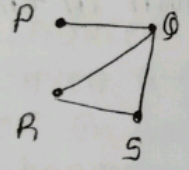
edge disjoint subgraph

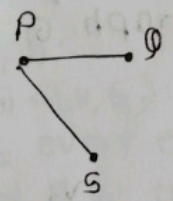


vertex disjoint subgraph

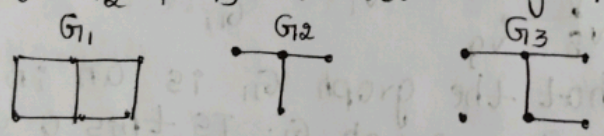
Verify $G_1 = (V_1, E_1)$ is a subgraph of G in the following cases

1) $V_1 = \{P, Q, S\}$ $E_1 = \{(P,Q)(P,S)\}$
2) $V_1 = \{P, Q, R\}$ $E_1 = \{(PQ)(QR)(QS)\}$

1) 

2) 

Q: 3 graphs G_1, G_2, G_3 are shown in figure. Are G_2 & G_3 induced subgraphs of G_1



Are they spanning subgraphs

G_2 & G_3 are induced subgraphs
↓
not induced

G_3 is spanning subgraph

G_2 is not

Operations on graphs:

Consider 2 graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$

Then the graph whose vertex set is

* Walk is defined as previous alternating sequence beginning and ending with vertices such that each edge is incident vertices just preceeding and following it.

Vertices & edges may be repeated

length of walk: The no. of edges present in a walk is called length of walk.

Closed walk: A walk is closed if its initial and final vertices are same.

Open walk: A walk whose initial & final vertices are not same is called open walk.

Trail:

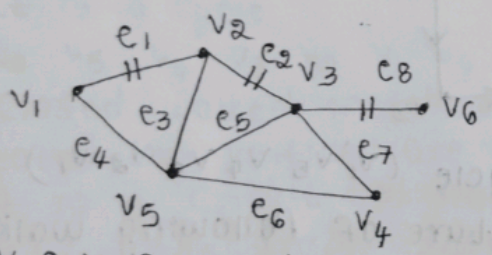
An open walk in which there is no repetion of edges is called trail

Circuit: A closed walk in which there is no repetition of edges is called circuit

Path: An open walk in which vertices are not repeated & edges are not repeated.

Cycle: A closed walk in which edges & vertices are not repeated.

1. Check whether the following walks are open walks or closed walks. Find then lengths

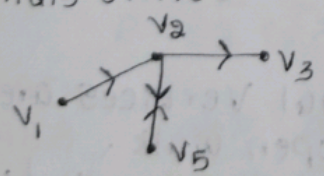


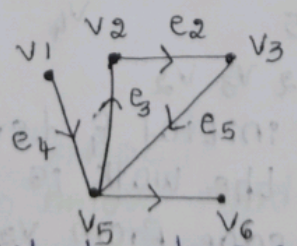
1) $v_1 e_1 v_2 e_2 v_3 e_8 v_6$ - open (3)

2) $v_1 e_1 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4$ - open (5)

3) $v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_2 e_2 v_3 e_5 v_5 e_4 v_1$ - closed (7)

2. Check whether the following graphs are trials or not



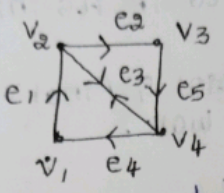


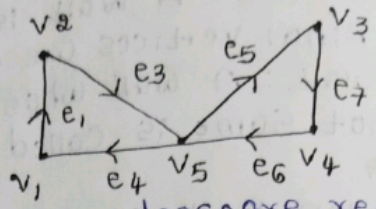
1) An open walk in which edges are repeated. Hence it is not a trial $v_1 v_2 v_5 v_2 v_3$.

2) An open walk in which edges are not repeated. Hence it is a trial

$v_1 v_5 v_2 v_3 v_5 v_6$

3. Check whether the given walks is a circuit or cycle.

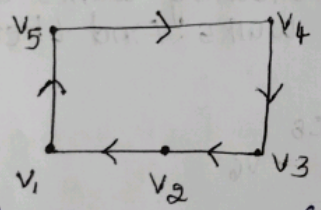




1) A closed walk but edges are repeated so it is not a circuit ($v_4\ v_1\ v_2\ v_4\ v_2\ v_3\ v_4$)

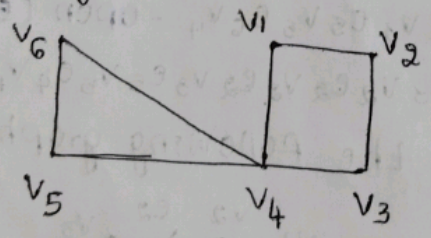
2) A closed walk & edges are not repeated. Hence it is a circuit but not cycle (∵ vertices are repeated) ($v_1\ v_2\ v_5\ v_3\ v_4\ v_5\ v_1$)

4. Check whether the following figures are cycles or not.



It is a cycle ($v_1\ v_5\ v_4\ v_3\ v_2\ v_1$)

5. Indicate nature of following walks (the edges b/w vertices being understood)



1) $v_1\ v_2\ v_3\ v_2$

Since initial & terminal vertices are not same, the walk is an open walk.

∵ The edge from v_2 to v_3 is repeated twice

∴ The walk is not a trial

2) $v_4\ v_1\ v_2\ v_3\ v_4\ v_5$

∵ Initial & terminal vertices are not same

It is open walk

∵ edges are not repeated

∴ It is a trial

∵ vertices are repeated

∴ It is not a path.

∴ The given walk is a trial but not a path

3) $v_1\ v_2\ v_3\ v_4\ v_5$

It is an open walk in which no edges & vertices are repeated

∴ It is a path

4) $v_1\ v_2\ v_3\ v_4\ v_1$

It is a walk (closed) in which edges & vertices are not repeated

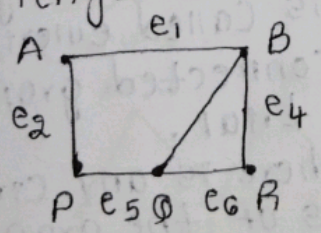
∴ It is a cycle

5) $v_6\ v_5\ v_4\ v_3\ v_2\ v_1\ v_4\ v_6$

A closed walked in which edges are not repeated but vertices are repeated

∴ It is a circuit but not cycle.

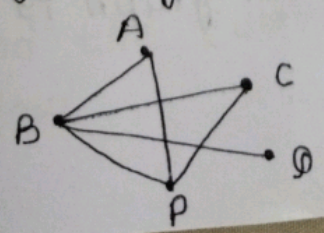
6. Consider the graph shown below. Find all paths from vertex A to vertex R also indicate their lengths



PATH:

ABR, APQR, ABQR, APQBR

7. For the graph shown below. Find the nature of following graphs



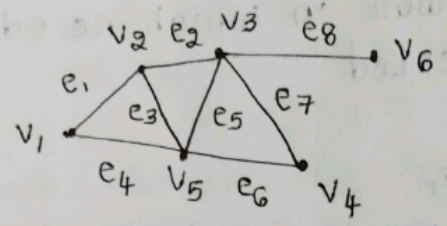
1) BAPCB
2) PABQ
3) CBAPBQ

1) BAPCB is a closed walk in which edges & vertices are not repeated
Hence it is a cycle

2) PABQ is a open walk in which edges & vertices are not repeated
Hence it is a path

3) CBAPBQ is an open walk in which edges are not repeated but vertices are repeated
Hence is a trial not path

8 Find nature of following walks



1) $v_1 v_2 v_5 v_3 v_4 v_5 v_1$
It is a closed walk in which edges are not repeated
∴ It is a Circuit

2) $v_1 v_2 v_3 v_5 v_1$
It is a closed walk in which edges & vertices are not repeated
∴ It is a cycle.

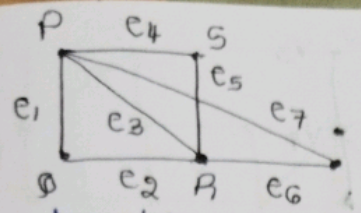
Eulerian trial: A trial consisting of all edges of the graph is called eulerian trial

Semi Euler Graph: A connected graph that contains an Eulerian trial.

Euler's Circuit: If there is any circuit that contains all edges of the graph then that circuit is called Euler Circuit.

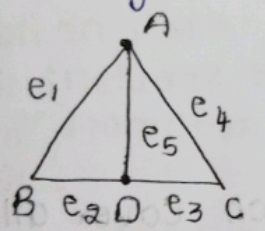
Euler Graph: A connected graph which contains Euler's Circuit is called Euler Graph

1. Check whether the given graph is an Euler graph or not.



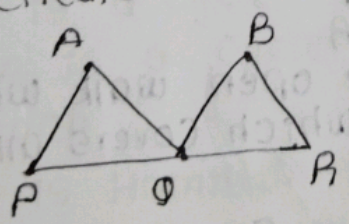
Consider the walk RQPSRTPR is a closed walk in the graph in which edges are not repeated. Hence it is a Circuit.
This Circuit Covers all the edges in the graph.
Hence the circuit is Euler Circuit.
∴ The graph Contains Euler Circuit it is an Euler graph.

2.



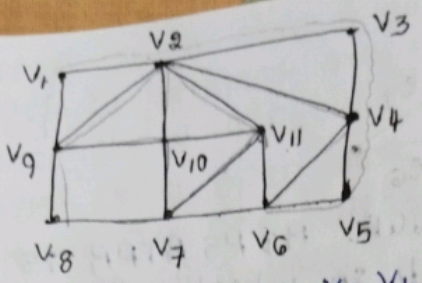
In the given graph if we try to write closed walk which covers all edges, there is repetition of edges.
∴ We Cannot form a Circuit which Covers all the edges of graph
Hence the graph does not Contain Euler Circuit.
∴ The graph is not Euler graph.

3. Show that the following graph Contains an Euler Circuit.



PAQBRQP
∵ Graph Contains Euler Circuit
It is an Euler Graph.

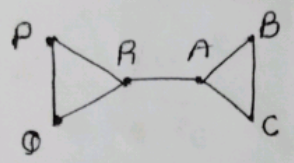
4. Find the Euler Circuit in graph shown below



$v_9\ v_1\ v_2\ v_3\ v_4\ v_5\ v_6\ v_4\ v_2\ v_9\ v_{10}\ v_7\ v_{11}\ v_6\ v_7$

v_9 is a circuit which covers all edges of graph. ∴ It is Euler graph.

Check whether the following graph contains Euler circuit



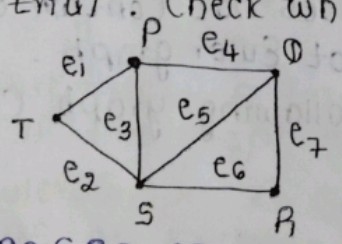
There is

To form a circuit which cover all edges of graph. RA should be repeated twice.

∴ There is no such circuit which will cover all edges.

Hence the graph is not an Euler graph.

Show that the following graph contains an Euler trial. Check whether it is Semi-Eulerian graph.

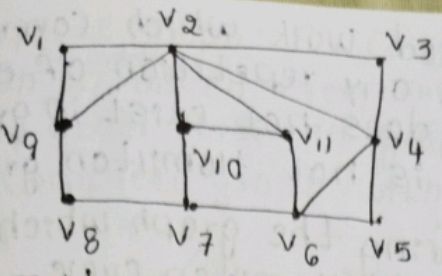


PTSPQSRQ is an open walk where edges are not repeated, which covers all edges of graph.

Hence the walk is an Eulerian trial

∴ It is Semi-Eulerian graph.

Find an Euler trial in the graph

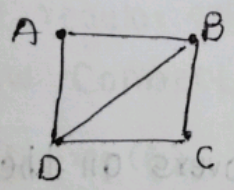


Hamilton Cycle: Let G be a Connected graph if there is a Cycle in G that Contains all vertices of G. Then the Cycle is called Hamilton Cycle.

Hamilton graph: A graph which Contains Hamilton Cycle is called Hamilton graph.

Hamilton path: A path in a Connected graph which includes every vertex of the graph is Called Hamilton / Hamiltonian path in the graph.

Check whether the graph is Hamilton graph or not.



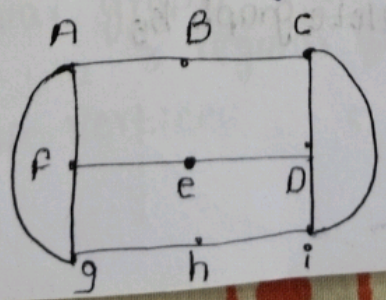
ABCDA is a closed walk in the graph where edges are not repeated vertices are not repeated expect initial & final

∴ It is a cycle

ABCDA is a Cycle which Covers all vertices of graph

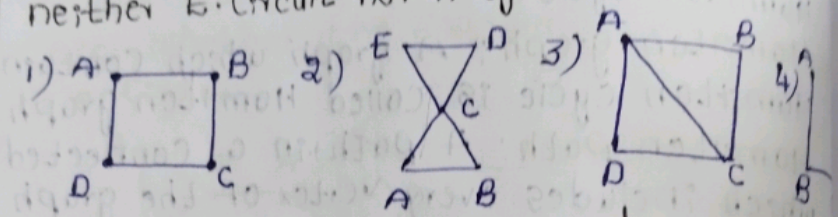
∴ It is Hamilton cycle

Hence graph is Hamilton graph ∵ it Contains Hamilton Cycle.

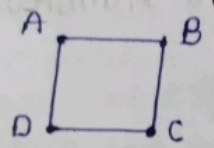


There is no closed walk which covers all vertices without any repetition of edges
∴ Hamilton cycle does not exist in graph
Hence the graph is not Hamilton graph

Exhibit the following the graph which has
1) both Euler circuit & Hamilton cycle
2) has Euler circuit but not Hamilton cycle
3) has H.C but not E.circuit
4) neither E.circuit nor H.cycle

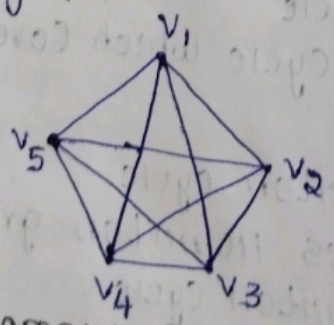


Verify that hyper cube Q_2 and complete graph K_5 are hamilton graphs by find hamilton cycle in each of them.



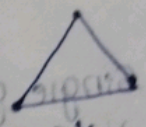
hyper cube Q_2

ABCDA is a cycle which covers all the vertices of the graph.
Hence it is Hamilton cycle
∵ Graph contains Hamilton cycle, it is a hamilton graph.



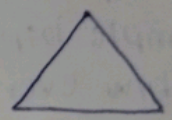
Complete graph K_5

Complete graph: A graph of order > 2 in which every every other vertex is
A Complete graph
→ A



K_3

→ A Complete g
Kuratowski's Pr
Regular graph: A
has same degree
If every vertex
is r-regular grap
Ex:



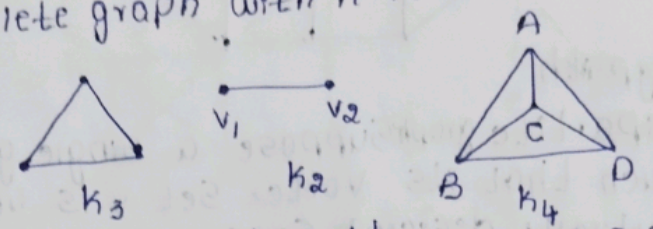
2-regular grap
Draw connected
1, 2 Degree (0)
v_1 v_1
Draw two 3 r

3-regu
Draw 3 reg
A 3 regular
possible ∵ a
even no. of v

Complete graph: A Complete graph is a simple graph of order >= 3 in which every vertex is connected to every other vertex in graph.

A Complete graph with n vertices denoted by K_n

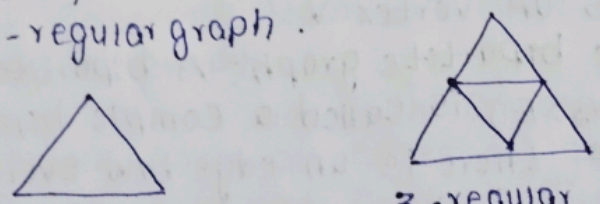
→ A



→ A Complete graph with 5 vertices is called Kuratowskis first graph.

Regular graph: A graph in which every vertex has same degree is called a regular graph. If every vertex has degree 'r' then the graph is r-regular graph.

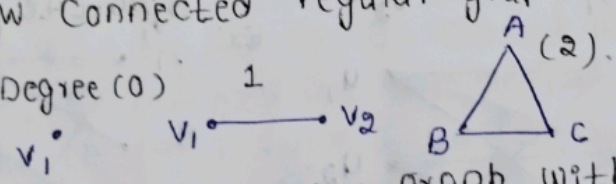
Ex:



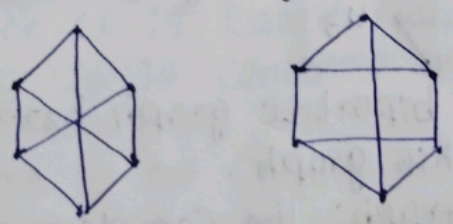
2-regular graph 3-regular graph

Draw Connected regular graphs of degree 0, 1, 2

Degree (0) 1 (2).



Draw two 3 regular graph with 6 vertices

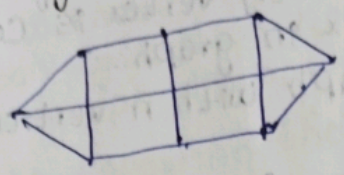


3-regular graph with 6 vertices

Draw 3 regular graph with 7 vertices

A 3 regular graph with 7 vertices is not possible ∵ a 3 regular graph must have even no. of vertices.

Draw 3 regular graph with 8 vertices



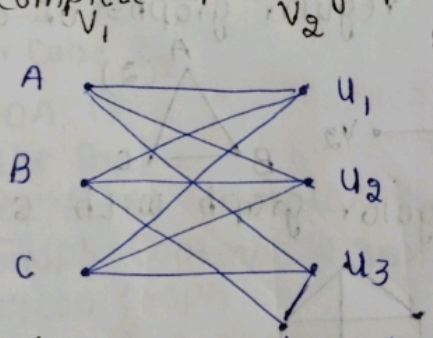
Bypath

Bipartite graph: Suppose a single graph G is such that its vertex set V is union of 2 mutually disjoint sets V_1 & V_2 which are such that every edge in G joins a vertex in V_1 & a vertex in V_2 then G is called a bipartite graph.

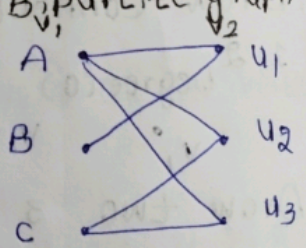
The sets V_1 & V_2 are called bipartites / partitions of vertex V.

Complete bipartite graph: A bipartite graph $G = (V_1, V_2; E)$ is called a comple bipartite graph if there is an edge b/w every vertex in V_1 & every vertex in V_2

Ex: Complete bipartite graph

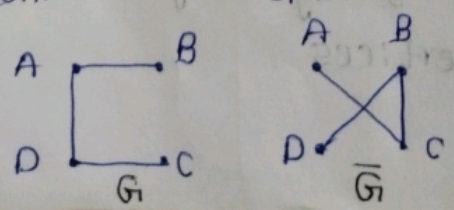


Ex: Bipartite graph



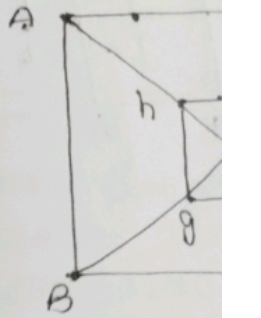
Note: Complete bipartite graph $K_{3,3}$ is called Kuratowskis graph.

Complementary graph: The complement of a graph $G = (V, E)$ denoted as $\bar{G}$ is the one which as same no. of vertices in which 2 of them are connected if & only if, they are not connected in G.



Show that the
Hamiltanion bu

1)



A B C D e f g i
Vertices are no
not repeated
Hence walk is
∵ It Contain
Hamiltan gra
∵ There exist
which will co
∴ Graph do
Hence it is

2) A B C D h g
edges & ver
Hence it is
Covered once
Hence it is
Hence it is
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There is n
Covering al
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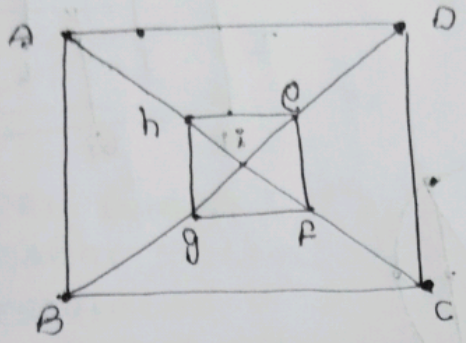
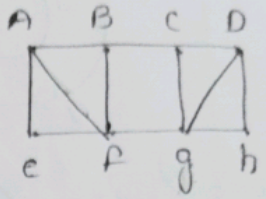
Show tha
Hamiltania

Show that the following graphs are Hamiltanian but not Eularian.

1)



2)

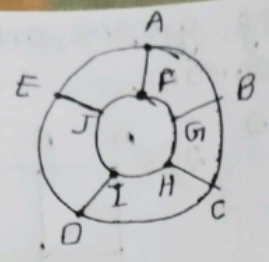


ABCDefgihA is a closed walk in which vertices are not repeated and edges are not repeated & all vertices are covered
Hence walk is Hamilton cycle
∵ It contains Hamilton cycle it is a Hamilton graph
∵ There exists no such circuit in graph which will cover all edges of graph
∴ Graph does not contain Euler circuit
Hence it is not Eulerian.

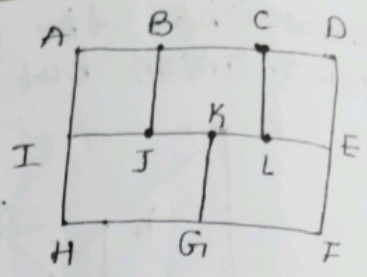
2) ABCDhgfa is a closed walk in which edges & vertices are not repeated
Hence it is a cycle & all the vertices are covered once in this cycle
Hence it is called Hamilton cycle
Hence it is contains Hamilton cycle so it is Hamilton graph
There is no circuit in the graph formed by covering all edges without repetition
As Euler circuit is not present it is not a Euler Graph.

Show that the following graphs are Hamiltanian.

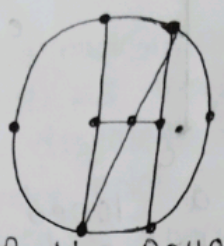
1)



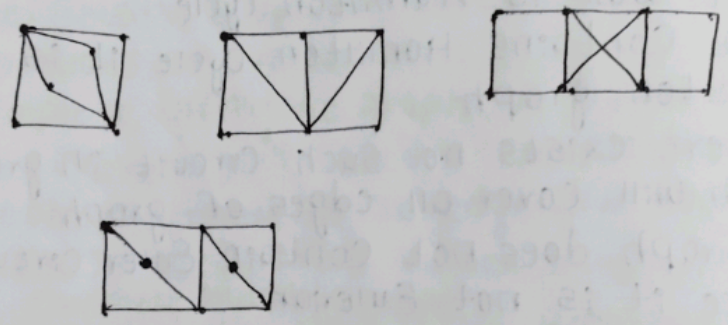
2)



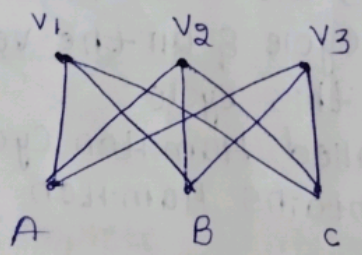
3)



Which of the following are Euler graphs are not, Hamilton graphs are or not



Prove that the Complete bipartite graph K_{33} is Hamiltanian but not Eulerian.



K_{33}

K_{33} is a Complete bipartite graph

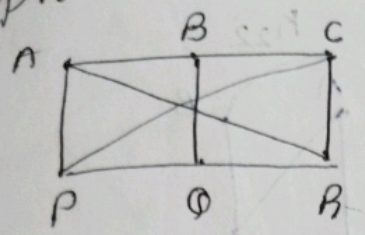
The Close walk $V_1 A V_2 B V_3 C V_1$ Covers all Vertices of the graph

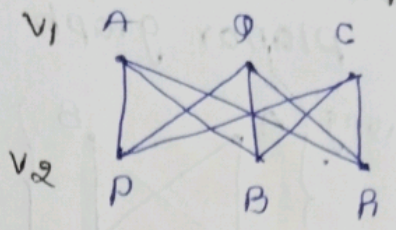
Hence graph is Hamilton graph

∴ We Cannot find a Circuit which Covers all edges of the graph

∴ The graph is not Eularian

Procedure to check Bipartite Graphs



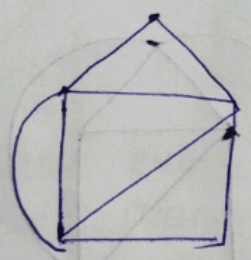


Planar Graph : A Graph G is said to be planar if there exists some geometric representation of G. which can be drawn on a plane such that no two of its edges intersect expect at the common vertex. The point of intersection are called Gossover

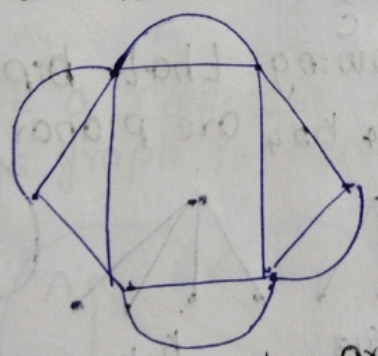
Non - planar Graph : A Graph that cannot be drawn on a plane without any crossover

Check whether following graphs are planar graphs or not.

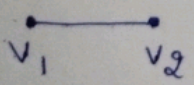
1) Graph of order 5, Size 8



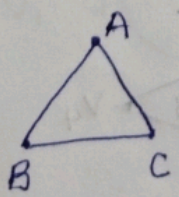
2) Order 6 Size 12



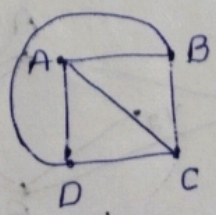
Show that k_2, k_3, k_4 are planar graph



k_1



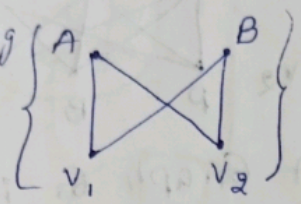
k_2



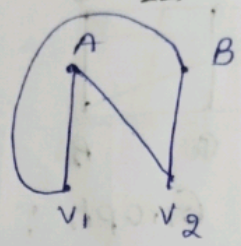
k_3

Show that the bipartite graph K_{22}, K_{23} are planar graph

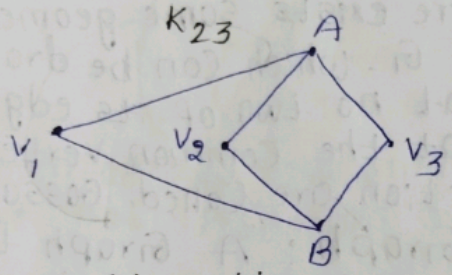
wrong



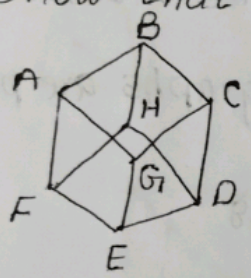
K_{22}

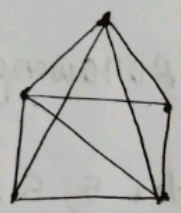


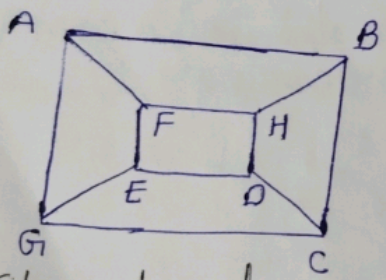
K_{23}

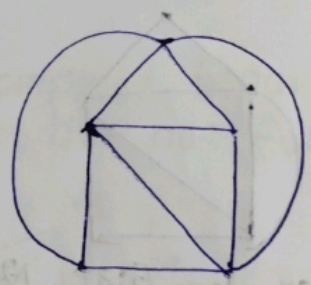


Show that the following graphs are planar



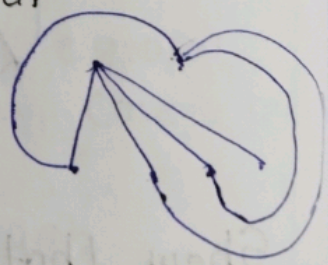


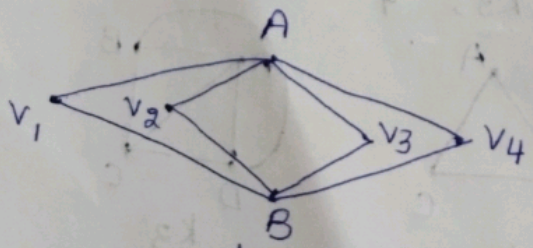




Show by drawing that bipartite graph $K_{11}, K_{12}, K_{13}, K_{14}$ K_{24} are planar

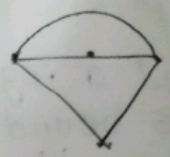
K_{11} K_{12} K_{13} K_{14}





K_{24}

Identify ea
K_4, K_{14}, K_{22}



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graph of ord

Show that
graphs by re

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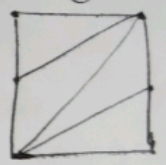
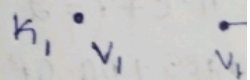
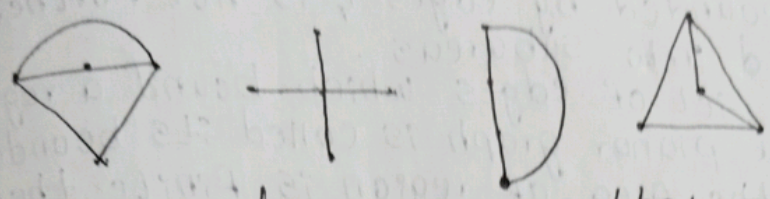


prove that
vertices is

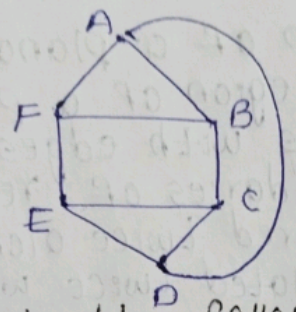


Show that
kuratowskis

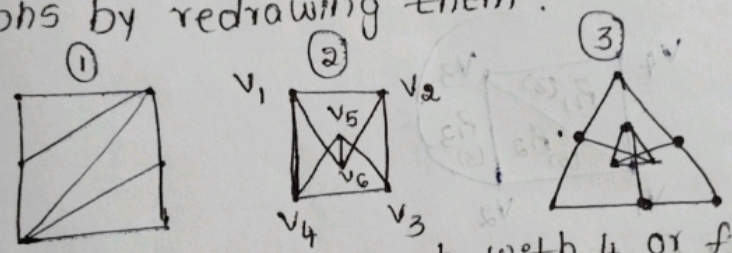
Identify each of the following graph as K_4, K_{14}, K_{22}, K_{23}



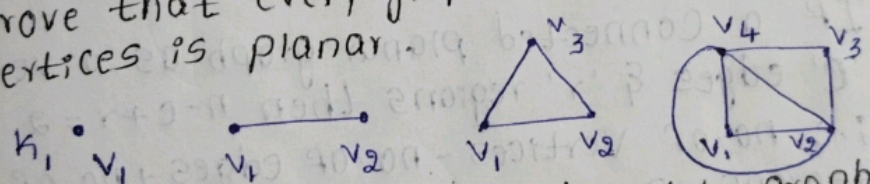
Through a diagram, prove that 3 regular graph of order 6 & size 9 is planar



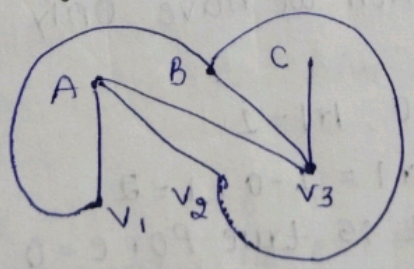
Show that the following graphs are planar graphs by redrawing them.



prove that every graph with 4 or fewer vertices is planar.

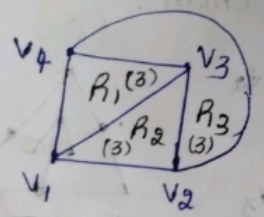


Show that a complete bipartite graph K_{33} kuratowskis graph is non-planar



Region of a graph: A region of a planar graph is defined to be an area of the plane that is bounded by edges & is not further divided into subareas.
The set of edges which bound a region of a planar graph is called its boundaries.
If the area of region is finite, then is finite region. If the area of region is infinite, then it is infinite region.

Degree of region of a planar graph:
The degree of region of a planar graph is no. of encounters with edges in a walk round the boundaries of region. If an edge is encountered twice along boundary then it is calculated twice when counting degree of region



Euler Formula:
If a connected planar graph as 'n' vertices 'e' edges & 'r' regions then $n-e+r=2$
i.e no. of vertices - no. of edges + no. of regions = 2

Proof:
Case 1:
If $e=0$, then we have only one vertex & one region
$|V|=1$, $|E|=0$, $|r|=1$
$\therefore |V|-|E|+|r| = 1-0+1 = 2$
$\therefore$ The result is true for $e=0$ —— (1)

Case 2:
If $e=1$, one edge can be a loop or an edge b/w 2 distinct vertices

a) If the edge is a loop, the no. of vertices are 1 & $|e| = 1$ & $|r| = 2$

$\therefore |V| - |e| + |r| = 1 - 1 + 2 = 2$

b) If it is an edge b/w 2 distinct vertices then $|V| = 2$, $|e| = 1$, $|r| = 1$

$\therefore |V| - |e| + |r| = 2 - 1 + 1 = 2$

$\therefore$ The result is true for $e = 1$ — (2)

Assume that the result is true for 'e' no. of edges

Let no. of vertices $= m$

no. of edges $= e$

no. of region $= r$

$\therefore |V| - |e| + |r| = 2$ i.e $m - e + r = 2$ — (3).

we prove that the result is true for $e = e+1$

when 1 edge is added to the graph, 3 cases may arise

1) If the edge is added at vertex of a graph G in the form of a loop then $|V|$ remains same, no. of edges is increased by 1 & no. of regions is also increased by 1

$\therefore |V_1| = m$ $|e_1| = e + 1$ $|r_1| = r + 1$

$\therefore |V_1| - |e_1| + |r_1| = m - e - 1 + r + 1 = m - e + r = 2$

from eq (3)

$\therefore$ Eulers formula is satisfied for new graph where the edge is added in form of loop at vertex G — (4)

2) An edge is added b/w 2 distinct vertices of G

$\therefore |V_1| = m$, $|e_1| = e + 1$, $|r_1| = r + 1$

$\therefore |V_1| - |e_1| + |r_1| = m - e - 1 + r + 1 = m - e + r = 2$

$\therefore$ Eulers formula is satisfied in new graph where edge is added b/w any 2 distinct vertices of G — (5)

3) Let an edge be added at only 1 vertex of G

For this edge, to be an edge an external vertex should be added to graph.

$|V_1| = m+1 \quad |e_1| = e+1 \quad |r_1| = r$

$\therefore |V_1| - |e_1| + |r_1| = m+1-e-1+r = m-e+r = 2$

From eq ③

$\therefore$ Eulers formula is verified when an edge is added at only one vertex of G — ⑥

From ④, ⑤ & ⑥ Eulers formula is true when $e = e+1$ — ⑦

From ①, ②, ③ & ⑦ By mathematical induc

The result is true for all m (no. of vertices)

$\therefore |V| - |e| + |r| = 2$ (or)

Dual of a planar graph :

Consider planar graph G & a plane drawing

Suppose $R_1, R_2, R_3 \ldots$ be the regions includ interior and exterior regions.

Construct a graph G^* which with the following procedure.

1) Choose one point inside regions $R_1, R_2, R_3 \ldots$ denote these points by $v_1^*, v_2^* \ldots$

These points will be the vertices of G^*.

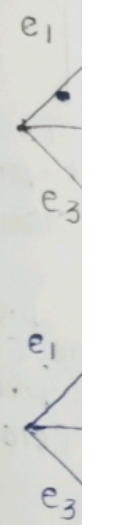
If 2 regions R_i & R_j are adjacent (they have an edges in common) draw a line e_k^* joining v_i^* & v_j^* that intersects the common edge exactly once.

If there is more than one edge common to R_i & R_j draw one line e_p^* b/w the points v_i^* & v_j^* for each common e_p, intersecting e_p exactly once.

For an edge e_i lying entirely in one region draw a loop e_i^* at the point e_i^* intersecting e_i exactly at once.

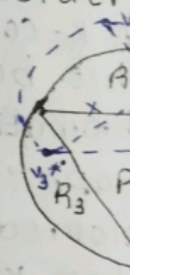
The grap
tric dual
Draw the

4)

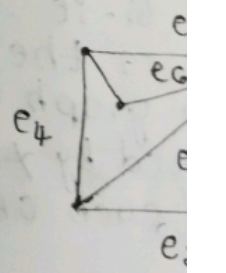


Consider

a)



Verify t
of each



Graph

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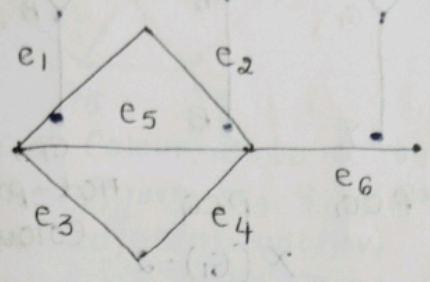
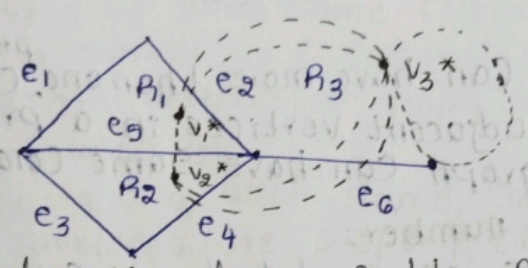
The graph G^* constructed is called geometric dual (or) dual of G.

Draw the dual of G which is

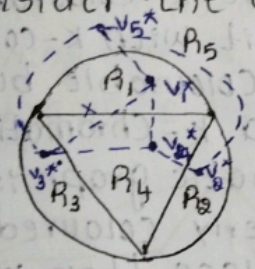




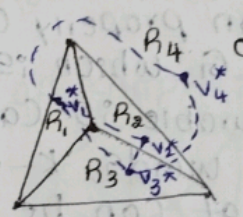
planar

Consider the duals of the following graphs

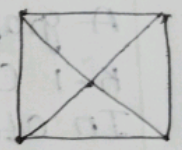
a)



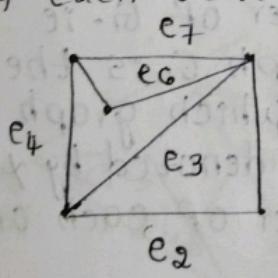
b)

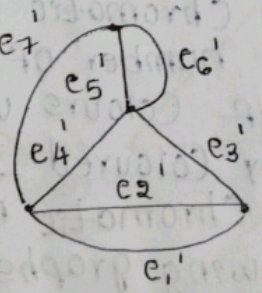


c)



Verify that the 2 planar graphs are duals of each other

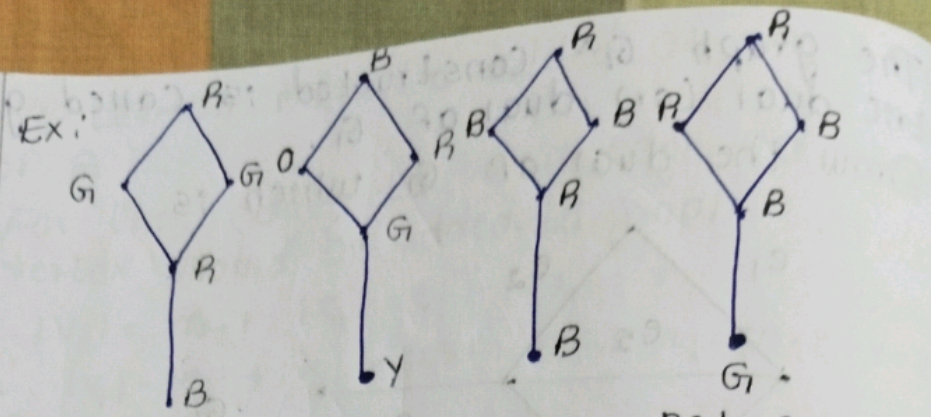




Graph Colouring

Given a planar graph G, if we assign colours to its vertices in such a way that no 2 vertices have same colour we say that the graph is properly coloured. i.e Assigning colours to its vertices such that the adjacent vertices have different colors.

Ex:



properly Coloured | PCG | PCG | not-properly Coloured

$\chi(G) = 2$

Note:

A graph can have more than one proper Colouring

Two non-adjacent vertices in a properly Coloured graph can have same colour

Chromatic number:

A graph G is said to be k-Colourable if we can properly colour it with k-colours.

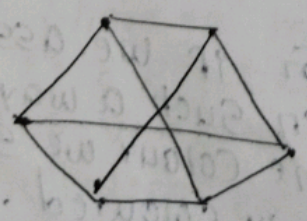
A graph G, which is k-Colourable but not k-1 Colourable is called a k-Chromatic graph

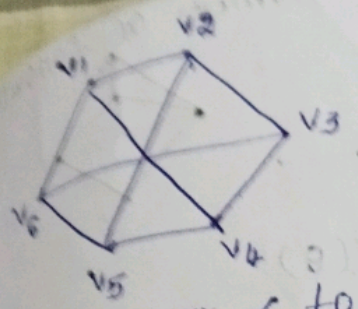
In other words, k-Chromatic graph is a graph that can be properly coloured with k-colours but not with less than k-colors.

If a graph G is k-Chromatic, then k is called Chromatic number of G. ie Chromatic number of Graph G is the min no. of Colours with which graph can be properly Coloured. It is denoted by $\chi(G)$.

Find the Chromatic number of each of the following graphs

1)





Assign a colour α to a vertex v_1

From the figure v_3 & v_5 are vertices which are not adjacent with v_1

Color v_3 & v_5 with same colour 'α'

Select a vertex v_2 & colour it with color 'B'

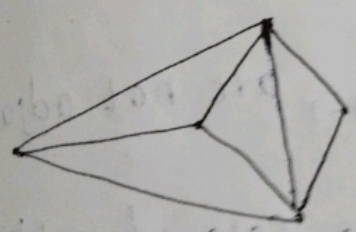
v_4 & v_6 are vertices of graph which are not adjacent with v_2.

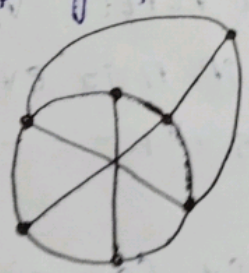
Color v_4 & v_6 with same color β

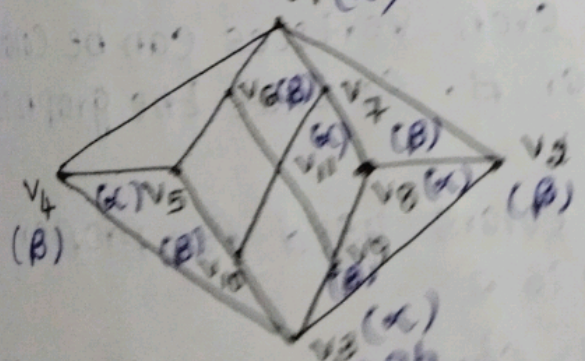
∴ All vertices are properly coloured

∴ The min no. of colours used for proper colouring are 2

∴ Chromatic number of graph is 2.

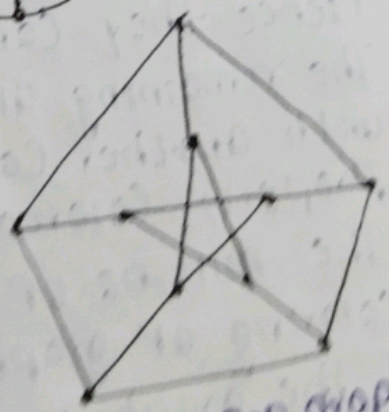






Herschel graph

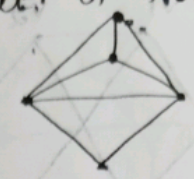
$\chi(G) = 2$

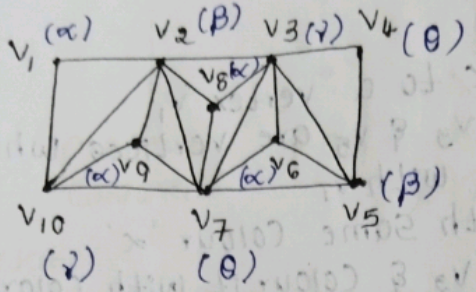


Petersen graph

$\chi(G) = 3$

Find the Chromatic number of following





$\chi(G) = 4$

Prove that a graph of order $n \geq 2$, Consisting of a Single Cycle 2-Chromatic if n is even, 3-chromatic if n is odd.

The graph consists of a Single Cycle.

Let us draw a Cycle with 'n' vertices

Colour a vertex v_1, with Color α

Case1: if n is even

$\therefore$ $v_3, v_5, v_7 - - - - v_{n-1}$ are not adjacent with v_1

Hence they can also be Colored with α.

The remaining all even vertices can be Coloured with another Color β. So that the graph is properly Colored.

The min no. of Colors used for proper Coloring of graph is 2

$\therefore$ The graph is 2-Chromatic graph if n is even

Case 2: If n is odd

$v_3, v_5 - - - v_{n-2}$ Can be coloured with same Color α whereas v_1 & v_n are odd which are adjacent

$v_2, v_4 - - - - v_{n-1}$ are Coloured with Color β

∴ v_n is adjacent with v_1 we cannot color v_n with α

∴ v_n is adjacent with v_{n-1} we cannot color v_n with β.

∴ We color v_n with a new color 'γ'

Hence min no. of colors used for proper coloring of graph is 3.

∴ The graph is 3-chromatic if n is odd.

Map colouring:

A proper colouring of regions is called map colouring.

4-Colour problem

Every simple connected planar graph is 4-colourable.

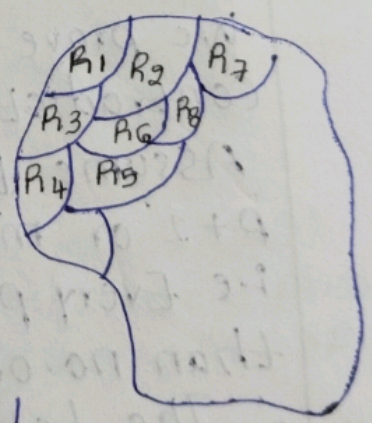
Let G be a simple connected planar graph with regions R_1, R_2, R_3 - - - -

Now color the region R_1 with color α

R_2 & R_3 are adjacent with R_1

∴ R_2 & R_3 cannot be colored with color α

∴ Let β & γ be 2 colors for region R_2 & region R_3.



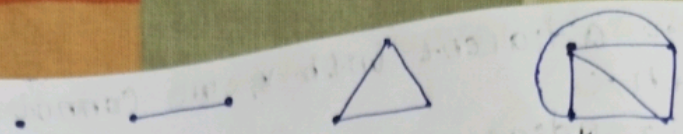
From the figure region R_6 is adjacent R_2, R_5, R_3, R_7

Hence we cannot assign α, β, γ to R_6

A new color is required to color R_6. Let it be δ.

If this coloring is continued for all regions in graph. The min no. of colors required for coloring of graph is 4.

∴ Every simple connected planar graph is 4-colourable.



K_1 K_2 K_3 K_4

1-colorable (2) (3) (4)

Pigeon-Hole problem

If m pegions occupy 'n' pigeon holes where $m>n$ then 2 or more pigeons should occupy same pigeon hole (or)

If m pigeons occupy 'n' pigeon holes where $m>n$ then atleast 1 pigeon hole must contain 2 or more pigeons in it (or)

If m pigeon holes are occupied by $kn+1$ pigeons then atleast one hole is occupied by $k+1$ (or) more pigeons.

Generalization:

If m pigeons occupy 'n' pigeon holes then atleast one pigeon hole must contain $p+1$ or more pigeons where $p = \left[\frac{m-1}{n}\right]$

proof:

We prove this by the method of contradiction

Assume that no pigeon hole contains $p+1$ or more ~~regions~~ pigeons

i.e Every pigeon hole contains p (or) less than no. of pigeons

$\therefore$ The total no. of pigeons in 'n' pigeon holes is $\leq np = \leq \cancel{n}\left(\frac{m-1}{\cancel{n}}\right)$.

$\therefore$ Total no. of pigeons in 'n' pigeon hole is $\leq m$

$\therefore$ Is a contradiction to given statement

This contradiction arises due to our assumption that every hole contains p or lesser no. of pigeons

∴ There exists atleast 1 pigeon hole which contains p+1 or more pigeons.

1. Find the min no. of students in a class so that 3 of them are born in the same month.

Let $n = 12$

Total no. of students $k+1 = 3$

$\therefore k = 2$ (3 students)

min no. of students in a class is $kn+1$

$= 2(12)+1 = 25$

2. ~~wha~~ If a bag contains many white, red & green balls then find no. of balls to be choosen to be sure of getting a pair of balls with same color.

$n = 3$

$k+1 = 2 \Rightarrow k = 1$

$kn+1 = 1(3)+1 = 4$

3. Several black, white & green balls in a bag then find min no. of balls that must be choosen to be sure of getting with same color.

Answer is 4

4. A bag contains several red, white, black shoes. Find min no. of shoes that a person needs to choose inorder to get 2 pairs of shoes of same color.

Answer is 10

5. Find the min no. of elements that must be chosen $A = \{1,2,3,4,5,6,7,8\}$ to be sure that 2 of them add up to 9.

If 5 elements are selected from 8 elements come from same subset & they will add upto 9.

6. What shall be the min no. of words that must begin with same alphabet at 27 english words

Answer is 1.

7. Show that if we choose 9 single shoes out of 8 distinct pair of shoes then are we sure to have a pair of shoes

$n = 8$

$kn + 1 = 9$

$k(8) + 1 = 9$

$8k = 8$

$\boxed{k = 1}$

∴ We are sure to have atleast one pair of shoes.

8. There are 13 boys in a class. Then find min no. of boys born in same month.

Answer is 2

9. If there are

Find the min no. of students to be sure that 5 of them belong to same caste out of 4 castes.

$n = 4$

$k + 1 = 5$

$k = 4$

$kn + 1 = 4(4) + 1 = 17$

0. A, B C is an

Combinations with repetitions

Suppose we wish to select with repetition a combination of 'r' objects from a set of 'n' distinct objects. The no. of such selections is given by ${}^{n+r-1}C_r$ (or) ${}^{n+r-1}C_{n-1}$

In other words ${}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$ represents the no. of combinations of n distinct objects taken 'r' at a time with repetitions allowed.

1. A bag contains 7 different denominations with atleas one dozen coins in each denomination. In how many ways can we select a dozen coins from the bag.

The no. of ways in which we can select a dozen coins from a bag

$$ {}^{n+r-1}C_r = {}^{7+12-1}C_{12} = {}^{18}C_{12} $$

$$ = \frac{18!}{12!(18-12)!} $$

2. In how many ways can we distribute 10 identical marbles among 6 distinct containers

$n = 6$, $r = 10$

$$ {}^{n+r-1}C_r = {}^{6+10-1}C_{10} = {}^{15}C_{10} $$

$$ = \frac{15!}{10!(15-10)!} $$