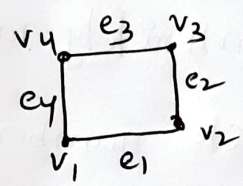
Graph theory

Basic concepts, isomorphism and sub-graphs, planar graphs, Euler's formula, multi-graphs and Euler's circuits, Hamiltonian graphs, chromatic numbers, the four-color problem.

Basic concepts:

(1) **Graph** A graph $G(V, E)$ consists of a set of objects $V = \{v_1, v_2, v_3 \cdots\}$ called vertices and another set $E = \{e_1, e_2, \cdots\}$ whose elements are called edges such that each edge is associated with an unordered pair of vertices. The vertex and edge sets, respectively are represented by $V(G)$ and $E(G)$.

The given below graph has four vertices and four edges

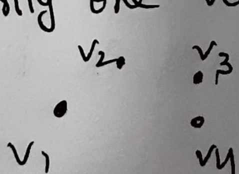


→ The most common representation of a graph is by means of a diagram, in which the vertices are represented by points and each edge as a line segment joining its end vertices. Often this diagram itself is reffered to as the graph.

→ According to the definition of a graph, the vertex set in a graph has to be non-empty. Thus, a graph must contain at least one vertex. But, the edge set can be empty. This means that a graph need not contain any edge.

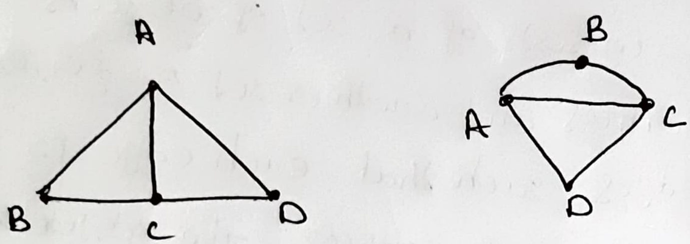
→ A graph containing no edges is called a null graph.

→ A null graph with only one vertex is called a trivial graph.



The null graph with 4 vertices

→ The way one draws a diagram of a graph is basically immaterial. There can be more than one diagram for the same graph. For example, the two diagrams in the following figure look different; yet they represent the same graph since each conveys the same information.



The two graphs has same vertices and edges but representations are different.

→ The definition of a graph does not impose any upper limit for the number of vertices and the number of edges. Thus, a graph can have infinitely many vertices & edges.

→ A graph with only a finite number of vertices as well as only a finite number of edges is called a finite graph. Otherwise it is called an infinite graph.

→ The number of vertices in a (finite) graph is called the order of the graph.

→ The number of edges in a graph is called its size.

→ A graph of order 'n' and size 'm' is called a (n, m) graph.

→ If v_i and v_j denote two vertices of a graph and if e_k denotes the edge joining v_i and v_j then v_i and v_j are called end vertices or adjacent vertices of e_k. This is symbolically written as $e_k = \{v_i, v_j\} = v_i v_j$

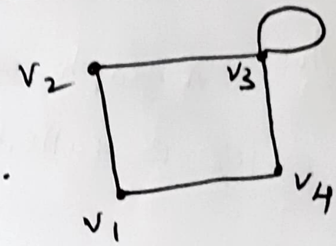
→ The vertices in a graph that are joined by an edge are known as adjacent vertices.

→ Neighbours are two vertices in a graph that are adjacent. The neighbouring set of v is the collection of all neighbour vertices of a fixed vertex v in G. It is represented by N(V).

→ Incident edges are the edges which have a common vertices.

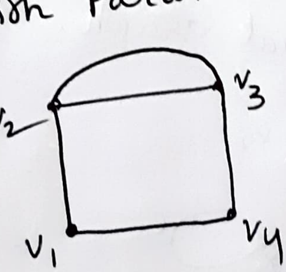
→ If end vertices of an edge are same such edge is known as self loop or loop.

A graph with self loop at v_3.



→ Two are more edges having common end points are known as parallel edges.

A graph with parallel edges between v_2 & v_3 is



→ The number of edges incident with a vertex 'v' in a graph G determines its degree, taking 2 for a self loop, and it is denoted by $d(v)$ or $d_G(v)$ (or $deg(V)$). The degrees of the vertices of a graph arranged in non-decreasing order is called the degree sequence of the graph. Also, the minimum of the degrees of vertices of a graph is called the degree of the graph.

In the above diagram $d(v_1) = 2$

$d(v_2) = 3$

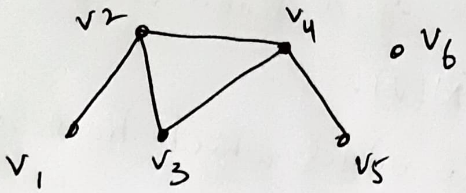
$d(v_3) = 3$

$d(v_4) = 2$

→ If the degree of a vertex is one is known as pendent vertex.

→ If the degree of a vertex is zero known as Isolated vertex

Ex:-

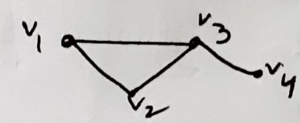


v_1, v_2 are pendent vertices

V_6 - Isolated vertex.

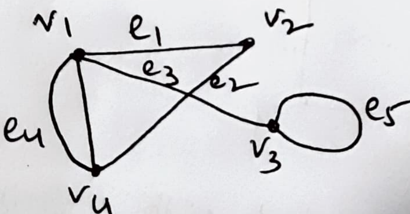
→ A graph with only one vertex is known as trivial graph.

Simple graph:- A graph which doesnot contain loops and multiple edges is called a simple graph. Ex:-

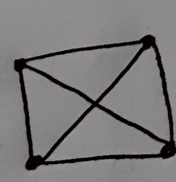


multigraph A graph which contains multiple edges but no loops is called a multi graph.

General graph:- A graph which contains multiple edges or loops or both is called a general graph.



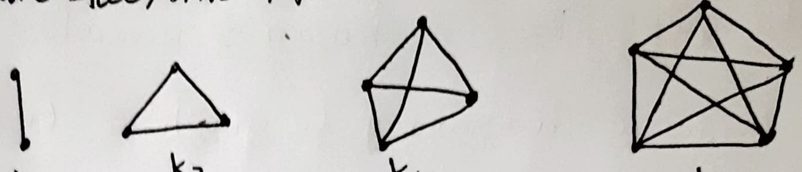
Regular graph:- A graph in which each vertex has same degree is known as regular graph



complete graph: A simple graph of order $n \geq 2$ in which there is an edge between every pair of vertices is called a complete graph

In other words, a complete graph is a simple graph in which every pair of distinct vertices are adjacent.

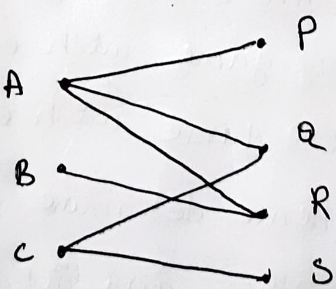
A complete graph with $n \geq 2$ vertices is denoted by K_n. complete graphs with two, three, four and five vertices are as follows :



K_2 K_3 K_4 K_5

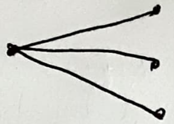
K_5 is also called the Kuratowski's first graph.

Bipartite graph Suppose a simple graph G is such that its vertex set V is the union of two mutually disjoint non-empty sets V_1 and V_2 which are such that every edge in G joins a vertex in V_1 and a vertex in V_2, then G is called a bipartite graph. If E is the edge set of this graph, the graph is denoted by $G(V_1, V_2; E)$. The sets V_1 and V_2 are called bipartites of the vertex set V.

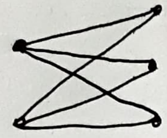


Complete Bipartite graph: A bipartite graph $G(V_1, V_2; E)$ is called a complete bipartite graph if there is an edge between every vertex in V_1 and every vertex in V_2.

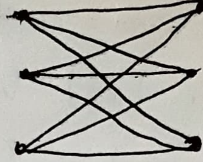
A complete bipartite graph in which the bipartites V_1 and V_2 contains m and n vertices respectively with $m \leq n$ is denoted by $K_{m,n}$. Thus $K_{m,n}$ has $m+n$ vertices and mn edges.



$K_{1,3}$



$K_{2,3}$

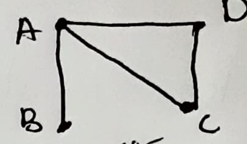


$K_{3,3}$

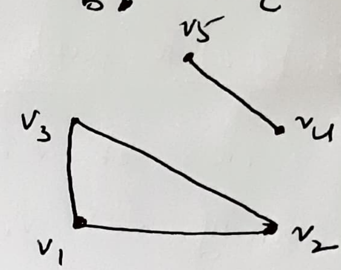
The graph $K_{3,3}$ is known as Kuratowski's second graph.

Pb ① Draw a diagram of the graph $G=(V,E)$ in each of the following cases:

i) $V=\{A,B,C,D\}$ $E=\{(A,B)\ (A,C)\ (A,D)\ (C,D)\}$



ii) $V=\{v_1\ v_2\ v_3\ v_4\ v_5\}$ $E=\{(v_1\ v_2)(v_1\ v_3)\ (v_2\ v_3)\ (v_4\ v_5)\}$

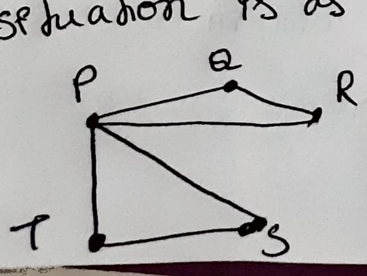


(iii) $V=\{P,Q,R,S,T\}$ $E=\{(P,S),(Q,R)\ (Q,S)\}$

(iv) $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ $E=\{(v_1\ v_4)(v_1\ v_6)(v_4\ v_6)(v_3 v_2)\ (v_3\ v_5)$
$(v_2\ v_5)\}$

Pb2 Let P, Q, R, S, T represent five cricket teams. Suppose that the teams P, Q, R have played one game with each other, and the teams P, S, T have played one game with each other. Represent this situation in a graph. Hence determine i) the teams that have not played with each other, and ii) the number of games played by each team.

Sol: Let the teams be represented by vertices and an edge represent the playing. Then the graph representing the given situation is as shown below:



i) We observe that there is no edge between (Q, S), (Q, T), (R, S), (R, T)

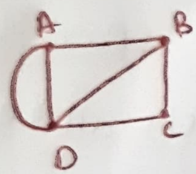
∴ The teams Q & S, Q & T, R & S, R & T have not played with each other.

ii) From the graph, we note that two edges are incident on each of the vertices Q, R, S, T and four edges are incident on P.

Thus, the teams Q, R, S, T have played two games each and the team P has played four games.

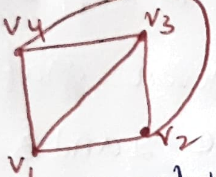
Pb Which of the following are complete graphs?

i)



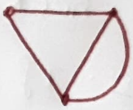
not complete

ii)



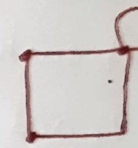
complete

iii)



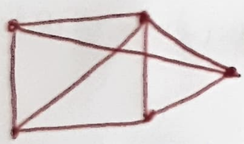
not complete

(iv)



not complete

(v)



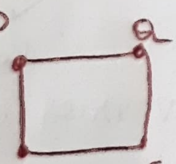
not complete

(vi)

not complete.

Pb Which of the following are bipartite graphs?

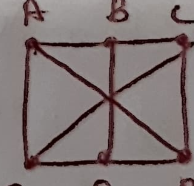
i)



Bipartite $V_1 = \{P, S\}$

$V_2 = \{Q, R\}$

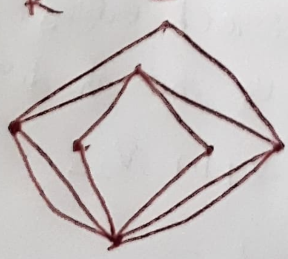
ii)



$V_1 = \{A, C, Q\}$

$V_2 = \{B, P, R\}$

iii)

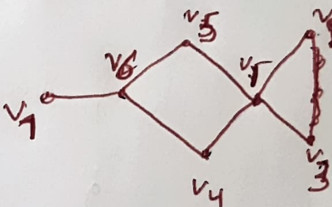


A Procedure to check if a graph G is a bipartite or not.

1. ~~Arbitrarily selected~~

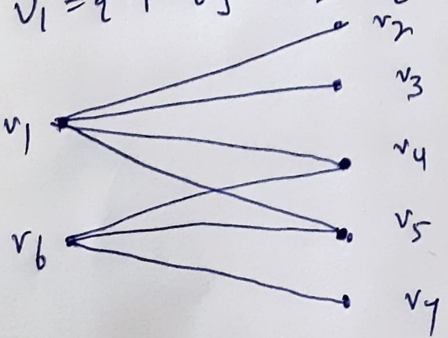
1. Arbitrarily select a vertex from G and include it into set V_1
2. Consider the edges directly connected to that vertex and put the other end vertices of these edges into the set V_2
3. Pick up one vertex from set V_2, and consider the edges directly connected to that vertex, and put the other end of these edges into the set V_1.
4. At each step, check if there is any edge among the vertices of set V_1 and set V_2. If so, the given graph is not bipartite graph and then return. Else continue 2 and 3 alternately all the vertices are included in the union sets V_1 and V_2.
5. If two computed sets are distinct, then the graph is bipartite.

Pb show that the following graph G is bipartite



Sol: select vertex v_1. The vertices joined to v_1 through direct edges are v_2, v_3, v_4 and v_5 take $V_1 = \{v_1\}$ $V_2 = \{v_2, v_3, v_4, v_5\}$ and vertices in V_2 are not connected among themselves by direct edges. Consider v_5 in V_2, the vertex which is connected to v_5 through direct edge is v_6. Then $V_1 = \{v_1, v_6\}$ and take v_7 in V_2

$\therefore V_1 = \{v_1, v_6\}$ $V_2 = \{v_2, v_3, v_4, v_5, v_7\}$



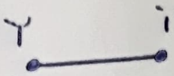
is the required bipartite graph.

Hand shaking Property (or) First theorem of Graph theory

Th 1: The sum of the degrees of all the vertices in a graph is an even number, and this number is equal to twice the number of edges in the graph. i.e., $\sum_{i=1}^{n} d(v_i) = 2|E|$

Proof: Let us consider a graph G with 'e' edges and n vertices $v_1, v_2, v_3, \ldots, v_n$.

Since each edge contributes two degrees, one at starting vertex and one at end vertex of the edge.



∴ The sum of the degrees of all vertices in G is twice the number of edges in G. that is $\sum_{i=1}^{n} d(v_i) = 2e$

Th 2. In every graph, the number of vertices of odd degrees is even.

(or)

The number of vertices of odd degree in a graph is always even.

Proof: Consider a graph with n vertices. Suppose k of these vertices are of odd degree so that the remaining n-k vertices are of even degree. Denote the vertices with odd degree by $v_1, v_2, v_3, \ldots, v_k$ and the vertices with even degree by $v_{k+1}, v_{k+2}, \ldots, v_n$. The sum of the degrees of the vertices is

$$\sum_{i=1}^{n} d(v_i) = \sum_{i=1}^{k} d(v_i) + \sum_{i=k+1}^{n} d(v_i) = 2|E|$$

$$\Rightarrow \sum_{i=1}^{k} d(v_i) = 2|E| - \sum_{i=k+1}^{n} d(v_i) = \text{even} \quad \to (1)$$

But each of $d(v_1), d(v_2) \ldots d(v_k)$ is odd. Therefore, the no. of terms in the left hand side of eq (1) must be even.

That is k is even.

This completes the proof of the theorem.

Th A simple graph with atleast two vertices has atleast two vertices of same degree.

Proof: Let G be a simple graph with $n \geq 2$ vertices. The graph G has no loop and parallel edges. Hence the degree of each vertices is $\leq n-1$. Suppose all the vertices of G are of different degrees. Hence the following degrees $0, 1, 2, 3, \ldots, n-1$ are possible for n vertices of G. Let u be the vertex with degree 0. Then u is an isolated vertex. Let v be the vertex with degree $n-1$. Then v has $n-1$ adjacent vertices. Since v is not an adjacent vertex of itself, therefore every vertex of G other than u is an adjacent vertex of G. Hence u cannot be an isolated vertex, this contradiction ~~the~~ ~~proof~~ proves ~~that~~. That a simple graph contains two vertices of same degree.

NOTE: The converse of the above theorem is not true.

Th Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$

Proof: By Handshaking theorem $\sum_{i=1}^{n} d(v_i) = 2|E|$

where $|E|$ is the no. of edges with n vertices in the graph G.

$\Rightarrow d(v_1) + d(v_2) + \cdots + d(v_n) = 2|E| \longrightarrow ①$

Since the maximum (number of ~~(~~edges) ~~&~~degree of each vertex in the graph G can be $(n-1)$.

$\therefore$ eq ① becomes $(n-1) + (n-1) + \cdots + n \text{ times} = 2|E|$

$\Rightarrow n(n-1) = 2|E| \Rightarrow |E| = \frac{n(n-1)}{2}$

Hence, the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$.

Pb Is there a simple graph corresponding to the following degree sequences ?

i) (1,1,2,3) ii) (2,2,4,6) (iii) (1,1,1,1) (iv) (1,3,3,4,5,6,6)

(v) (1,1,3,3,3,4,6,7)

Sol: i) Since the sum of degrees of vertices is odd, there exist no graph corresponding to this degree sequence.

ii) Number of vertices in the graph sequence is four and the maximum degree of a vertex is 6, which is not possible as in a simple graph the maximum degree cannot exceed one less than the no. of vertices.

iii) The sum of the degrees of all vertices is 4, even. The number of odd vertices is 4, even. Hence a simple disconn-ected graph is possible which has 4 vertices of degree 1 each. The no. of edges is $\frac{4}{2}=2$

(iv) Here the sum of the degrees is 28, even. The no. of vertices having odd degree is 4, even. The maximum degree 6 does not exceed 7-1=6. But two vertices have degree 6, each of these two vertices is adjacent with every other vertex. Hence, the degree of each vertex is at least 2, so that the graph has no vertex of degree 1 which is a contradiction. Hence there does not exist a simple graph with the given degree sequence.

(v) Assume that there is such a graph, since the degrees of vertices are 8 in number, the graph should have 8 vertices say P, Q, R, S, T, U, V, W arranged in the order of degrees as given.

i,e $d(P)=1$ $d(Q)=1$ $d(R)=3$ $d(S)=3$ $d(T)=3$ $d(U)=4$ $d(V)=6$

$d(w)=7$.

Since $d(w)=7 \Rightarrow w$ is adjacent to P, Q, R, S, T, U, V.

In Particular w has an edge to both of the vertices P and Q which are of degree 1. Then P, Q are not joined to any other vertex in particular to the vertex V which is of degree 6.

which is a contradiction. ($\because$ G is simple).

Hence there is no simple graph for which the degrees of vertices are as given.

Pb show that a simple graph of order $n=4$ and size $m=7$ does not exists.

we know that the no. of edges in a simple graph (size)

$$= \frac{n(n-1)}{2} = \frac{4(3)}{2} = 6$$

but given size $m=7$.

There cannot be a simple graph of the given order and size.

Pb can there be a graph consisting of the vertices A, B, C, D with $\deg(A)=2$, $\deg(B)=3$, $\deg(C)=2$, $\deg(D)=2$?

NO, because sum of degrees $= 9$ which is not an even number

Pb Does there exists a graph with 12 vertices such that two of the vertices have degree 3 and the remaining vertices have degree 4 each)?

Sol: sum of the degrees of vertices $= (3\times2)+(4\times10) = 46$

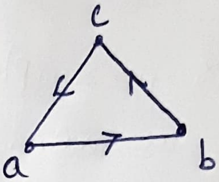
$\therefore$ By handshaking Property $\sum_{i=1}^{n} d(v_i) = 2|E|$

$\Rightarrow$ $46 = 2|E| \Rightarrow |E| = 23$

Hence, a graph of the desired type exists.

Directed graph: A graph in which every edge is directed is called a digraph or a directed graph. In other words, if each edge of the graph G has a direction then the graph is called directed graph.

→ In the diagram of a directed graph, each edge $e=(u,v)$ is represented by an arrow or directed curve from initial point u of e to the terminal point v.



Directed graph.

suppose $e=(u,v)$ is a directed edge in a digraph, then

i) u is called the initial vertex of e and v is the terminal vertex of e.

(ii) e is said to be initiating or originating in the node u and terminating or ending in the node v.

(iii) u is adjacent to v, and v is adjacent from u.

In-degree and out-degree

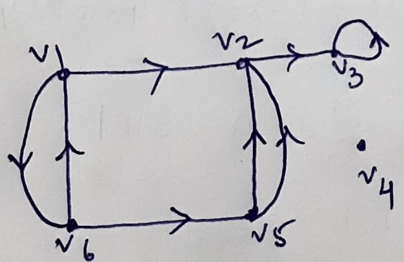
If v is a vertex of a digraph D, the number of edges for which v is the initial vertex is called the out-going degree or the out-degree and the number of edges for which v is the terminal vertex of v is called the incoming degree or the in-degree of v.

The out degree of v is denoted by $d^+(v)$ or $od(v)$ and the in-degree of v is denoted by $d^-(v)$ or $id(v)$.

It follows that i) $d^+(v)=0$, if v is a sink

(ii) $d^-(v)=0$ if v is a source

(iii) $d^+(v)=d^-(v)=0$ if v is an isolated vertex



$d^+(v_1)=2$	$d^-(v_1)=1$
$d^+(v_2)=1$	$d^-(v_2)=3$
$d^+(v_3)=1$	$d^-(v_3)=2$
$d^+(v_4)=0$	$d^-(v_4)=0$
$d^+(v_5)=2$	$d^-(v_5)=1$
$d^+(v_6)=2$	$d^-(v_6)=1$

we note that, in the above digraph, there is a loop at the vertex v_3 and this loop contributes a count 1 to each of $d^+(v_3)$ and $d^-(v_3)$.

Th In every digraph D, the sum of the out-degrees of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D

Proof: suppose D has n vertices $v_1\ v_2 \cdots v_n$ and m edges. Let r_1 be the no. of edges going out of v_1, r_2 be the no of edges going out of v_2, and so on. Then $d^+(v_1) = r_1,\ d^+(v_2) = r_2, \ldots, d^+(v_n) = r_n$

Since every edge terminates at some vertex and since there are m edges, we should have $r_1 + r_2 + \cdots + r_n = m$.

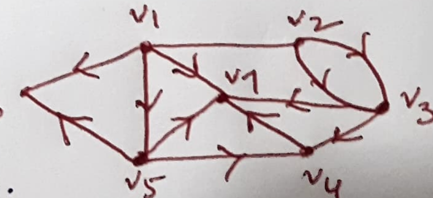
Accordingly, $d^+(v_1) + d^+(v_2) + \cdots + d^+(v_n) = r_1 + r_2 + \cdots + r_n = m$

Similarly, if s_1 is the number of edges coming into v_1, s_2 is the number of edges coming into v_2, and so on. we get

$$d^-(v_1) + d^-(v_2) + \cdots + d^-(v_n) = s_1 + s_2 + \cdots + s_n = m$$

Thus $\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = m$.

Pb Find the in-degrees and the out-degrees of the vertices of the digraph shown in Figure



Sol: The given digraph has 7 vertices and 12 directed edges.

Vertex	v_1	v_2	v_3	v_4	v_5	v_6	v_7
out-degree	4	2	2	1	3	0	0
In-degree	0	1	2	2	1	2	4

Pb write down the vertex set and the directed edge set of each of the following digraphs