$deg(r_2) = 3$ $deg(r_3) = 3$ $deg(r_4) = 3$ $deg(r_5) = 3$

$deg(r_6) = 3$.

Euler's formula: If G is a connected planar graph, then ~~they~~ any drawing of G in the plane as planar graph will always form $|R| = |E| - |V| + 2$ regions, including the exterior region, where $|R|$, $|E|$ and $|V|$ denote respectively, the number of regions edges and vertices of G.

Proof: we prove this result by induction on the number of regions 'K' determined by G.

It is obvious when $K = 1$

Assume the result for $K > 1$ and suppose that G is a connected plane graph having $(K+1)$ regions.

Delete an edge common to both the regions. The resulting graph G' has the same number of vertices, one fewer edge but also one fewer region since two previous regions have been combined by the removal of the edge.

i.e. $|E'| = |E| - 1$ & $|R'| = |R| - 1$; $|V'| = |V|$

where $|E'|$, $|R'|$ and $|V'|$ are number of edges, regions vertices of G'

Then $|V'| - |E'| + |R'| = |V| - |E| + 1 + |R| - 1$

$= |V| - |E| + |R|$

$= 2$ (By induction hypothesis)

Hence, the theorem is proved by mathematical induction.

Theorem: A complete graph K_n is planar iff $n \leq 4$

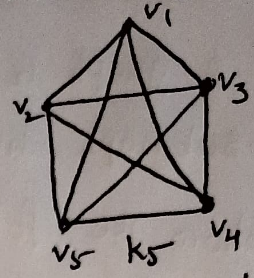
Proof: It is easy to see that K_n is planar for $n = 1, 2, 3, 4$

Now, we have to show that when $n \leq 4$, K_n is planar.

For this, it is sufficient to show K_n is non-planar when $n \geq 5$

In other words, we prove this by an indirect argument.

Now, Assume that K_5 is planar.



then $|R| = |E| - |V| + 2$

$= 10 - 5 + 2$

$= 7$

Since K_n is simple and loop free, we have $3|R| \leq 2|E|$

$3 \times 7 \leq 2 \times 10$

$21 \leq 20$ which is a contradiction.

$\therefore$ our assumption is wrong.

$\Rightarrow K_5$ is non-Planar.

Pb A complete graph $K_{m,n}$ is Planar iff $m \leq 2$ or $n \leq 2$

Sol: It is clear that $K_{m,n}$ is planar if $m \leq 2$ or $n \leq 2$

Now let $m \geq 3$ and $n \geq 3$. To prove that $K_{m,n}$ is non planar it sufficient to prove that $K_{3,3}$ is non-planar

Since $K_{3,3}$ has six vertices and nine edges, if $K_{3,3}$ is planar, By Euler's formula $|R| = |E| - |V| + 2$

$= 9 - 6 + 2 = 5$

Since $K_{3,3}$ is simple and loop free, we have $3|R| \leq 2|E|$

$\Rightarrow 3 \times 5 \leq 2 \times 9 \Rightarrow 15 \leq 18$.

which is a contradiction, our assumption is wrong.

$\therefore$ when $m \geq 3$ and $n \geq 3$, $K_{m,n}$ is non Planar.

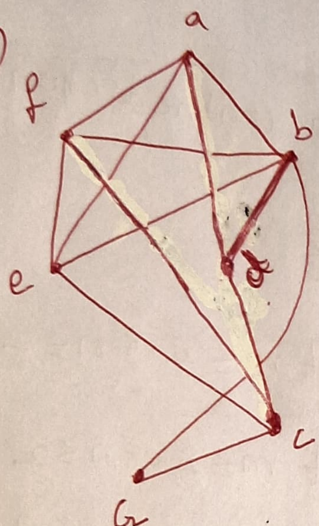
So, A complete bipartite graph $K_{m,n}$ is planar iff $m \leq 2$ or $n \leq 2$

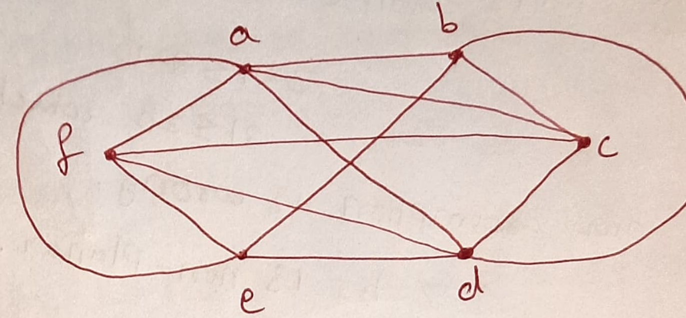
NOTE: A graph G is planar if and only if G does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

(OR) A graph G is non-planar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

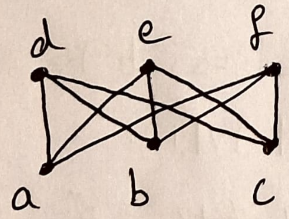
Pb Show the following graphs are not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$

i)

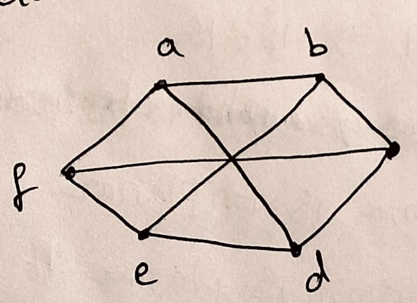




i) Delete the edges { {a,b}, {f,e} {a,g}, {g,h} {a,f} {a,c}



(ii) Delete the edges: {a,e}, {b,d}, {a,c}, {d,f} from G.



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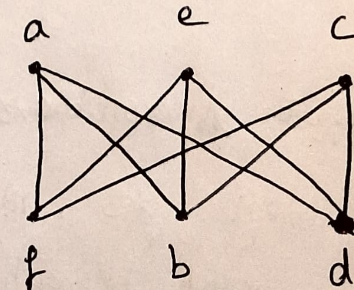


The given graph is not planar.

Theorem: A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof: Let G be a simple graph of order n. Let $G_1, G_2, \cdots G_k$ be the components of G. Let the no. of vertices in ith component be n_i i.e., $|V(G_i)| = |V_i| = n_i$, $1 \le i \le k$

Then $|V| = \sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k$ and $n_i \ge 1$

Now maximum possible edges in ith components cannot exceed

$$= \frac{n_i(n_i-1)}{2}$$

$$\Rightarrow \max|E(G_i)| = \max|E_i| \le \frac{n_i(n_i-1)}{2} \quad 1 \le i \le k$$

Hence $$|E(G)| \le \sum_{i=1}^{k} \max|E_i| = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$$

Now $$\sum_{i=1}^{k}(n_i-1) = (n_1-1) + (n_2-1) + \cdots + (n_k-1)$$

$$= (n_1+n_2+\cdots+n_k) - k$$

squaring on both sides $$\left[\sum_{i=1}^{k}(n_i-1)\right]^2 = (n-k)^2 = n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 + 2(\text{non negative terms}) = n^2+k^2-2nk \quad \longrightarrow (1)$$

$$\Rightarrow \sum_{i=1}^{k}(n_i-1)^2 = n^2+k^2-2nk-2(\text{non negative terms})$$

$$\le n^2+k^2-2nk$$

$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2\sum_{i=1}^{k} n_i \le n^2+k^2-2nk$$

$$\sum_{i=1}^{k} n_i^2 + k - 2n \le n^2+k^2-2nk$$

$$\Rightarrow \sum_{i=1}^{K} n_i^2 - n \le n^2 - nK + n - nK + K^2 - K$$
$$= n(n-K+1) - K(n-K+1)$$
$$= (n-K)(n-K+1)$$
$$\therefore \sum_{i=1}^{K} n_i^2 - n \le (n-K)(n-K+1) \rightarrow (2)$$

From (1) & (2) $|E(G)| \le \frac{1}{2}(n-K)(n-K+1)$

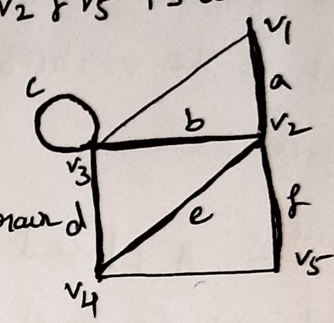
homeomorphic:- Two graphs G_1 and G_2 are homeomorphic if G_1 and G_2 can be reduced to isomorphic graphs by performing a sequence of series reductions

walks, paths and circuits

walk A walk is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it.

→ No edge appears more than once in a walk. A vertex, however may appear more than once.

Ex: For instance $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is a walk shown with heavy lines in the following fig



→ A walk is also referred to as an edge train or a chain.

→ vertices with which a walk begins and ends are called its terminal vertices. vertices v_1 and v_5 are the terminal vertices of the walk shown in the above fig. It is possible for a walk to begin and end at the same vertex. Such a walk is called a closed walk.

closed walk A walk that begins and ends at the same vertex is called a closed walk.

open walk: An open walk is a walk that begins and ends at two different vertices.

Trail: If in an open walk no edge appears more than once then the walk is called a trail.

Path: A Trail in which no vertex appears more than once is called a path.

Ex: $v_1 a v_2 b v_3 d v_4$ is a Path whereas $v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5$ is not a path.

→ A path does not intersect itself.

Length of a path: The number of edges in a path is called the length of a path.

→ Note that a self-loop can be included in a walk but not in a path.

→ The terminal vertices of a path are of degree one, and the rest of the vertices (called intermediate vertices) are of degree two.

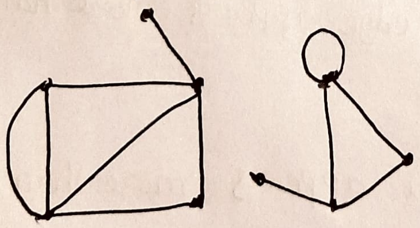
Circuit: A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit.

Ex: $v_2\, b\, v_3\, d\, v_4\, e\, v_2$ is a closed walk in which no vertex repeated implies it is a circuit.

→ A circuit is also called a cycle, elementary cycle, circular path, and polygon.

Connected graph: A graph G is said to be connected if there is atleast one path between every pair of vertices in G. Otherwise G is disconnected.

Ex:



A disconnected graph with two components.

→ It is easy to see that a disconnected graph consists of two or more connected graphs. Each of these connected subgraphs is called a component.

Trail and Circuit

As mentioned before, in a walk, vertices and/or edges may appear more than once. If in an open walk no edge appears more than once, then the walk is called a ***trail***. A closed walk in which no edge appears more than once is called a ***circuit***.

For example, in Figure 9.95, the open walk $v_1e_1v_2e_3v_5e_3v_2e_2v_3$ (shown separately in Figure 9.96(a)[†]) is not a trail (because, in this walk, the edge e_3 is repeated) whereas the open walk $v_1e_4v_5e_3v_2e_2v_3e_5v_5e_6v_4$ (shown separately in Figure 9.96(b)) is a trail.

Also, in the same Figure, the closed walk $v_1e_1v_2e_3v_5e_3v_2\ e_2v_3e_5v_5e_4v_1$ (shown separately in Figure 9.96(c)) is not a circuit (because e_3 is repeated) where as the closed walk $v_1e_1v_2e_3v_5e_5v_3$ $e_7v_4e_6v_5e_4v_1$ (shown separately in Figure 9.96(d)) is a circuit.

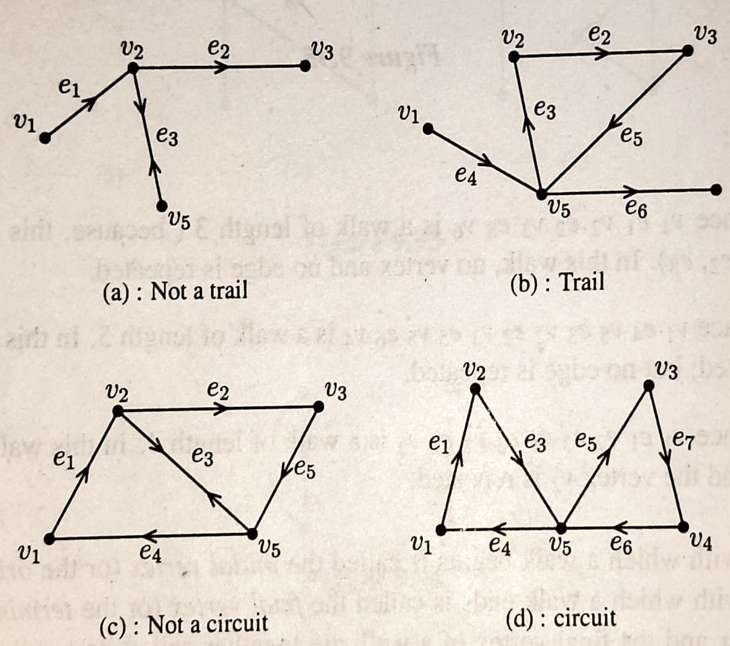


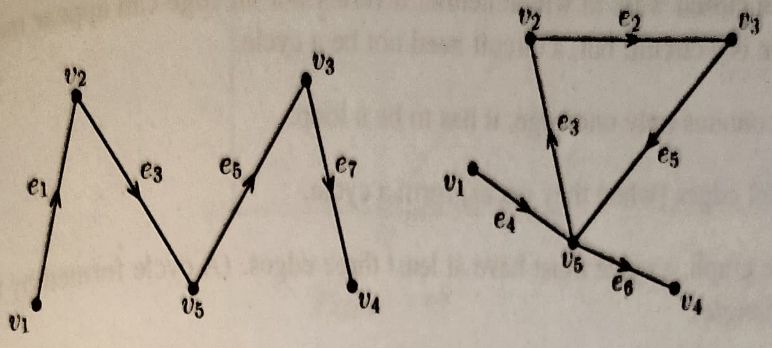
Figure 9.96

Path and Cycle

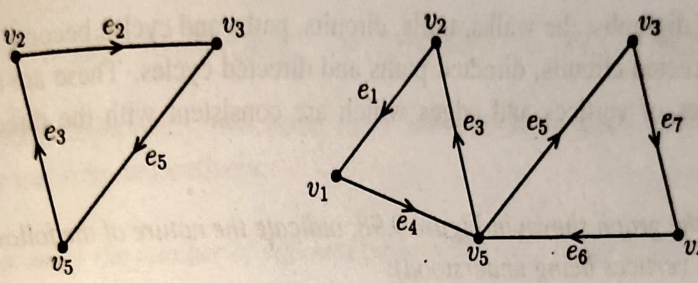
A trail in which no vertex appears more than once is called a ***path***.

A circuit in which the terminal vertex does not appear as an internal vertex (also) and no internal vertex is repeated is called a ***cycle***.

[†]In Figures 9.96 and 9.97, the arrows indicate the orders in which the vertices and edges in the corresponding sequences (walks) appear.



(a) : path (b) : Not a path



(c) : cycle (d) : Not a cycle

***Figure* 9.97**

For example, in Figure 9.95, the trail $v_1e_1v_2e_3v_5e_5v_3e_7v_4$ (shown separately in Figure 9.97(a)) is a path whereas the trail $v_1e_4v_5e_3v_2e_2v_3e_5v_5e_6v_4$ (shown separately in Figure 9.97(b)) is not a path (because in this trail, v_5 appears twice). Also, in the same Figure, the circuit $v_2e_2v_3e_5v_5e_3v_2$ (shown separately in Figure 9.97(c)) is a cycle whereas the circuit $v_2e_1v_1e_4v_5e_5v_3e_7v_4e_6v_5e_3v_2$ (shown separately in Figure 9.97(d)) is not a cycle (because, in this circuit, v_5 appears twice).

The following facts are to be emphasized.

1. A walk can be open or closed. In a walk (closed or open), a vertex and/or an edge *can* appear more than once.

2. A trail is an open walk in which a vertex *can* appear more than once but an edge *cannot* appear more than once.

3. A circuit is a closed walk in which a vertex *can* appear more than once but an edge *cannot* appear more than once.

4. A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trail; but a trail need not be a path.

5. A cycle is a closed walk in which neither a vertex nor an edge can appear more than once. Every cycle is a circuit; but, a circuit need not be a cycle.
6. If a cycle contains only one edge, it has to be a loop.
7. Two parallel edges (when they occur) form a cycle.
8. In a simple graph, a cycle must have at least three edges. (A cycle formed by three edges is called a *triangle*).

Case of Digraphs

In the case of digraphs, the walks, trails, circuits, paths and cycles become directed walks, directed trails, directed circuits, directed paths and directed cycles. These are defined by considering sequences of vertices and edges which are consistent with the directions of edges present.

Example 1. *For the graph shown in Figure 9.98, indicate the nature of the following walks (the edges in between vertices being understood):*

(i) $v_1v_2v_3v_2$ (ii) $v_4v_1v_2v_3v_4v_5$ (iii) $v_1v_2v_3v_4v_5$

(iv) $v_1v_2v_3v_4v_1$ (v) $v_6v_5v_4v_3v_2v_1v_4v_6$.

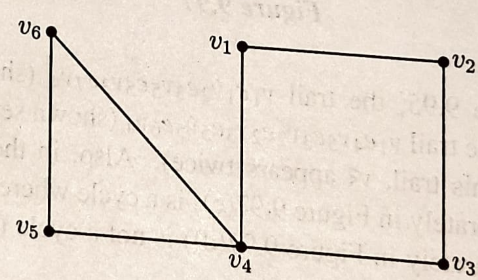


Figure 9.98

► (i) Open walk which is not a trail.

(ii) Trail which is not a path.

(iii) Trail which is a path.

(iv) Closed walk which is a cycle.

(v) Closed walk which is a circuit but not a cycle. ■

Example 2. *Consider the graph shown in Figure 9.99. Find all paths from vertex A to vertex R. Also, indicate their lengths.*

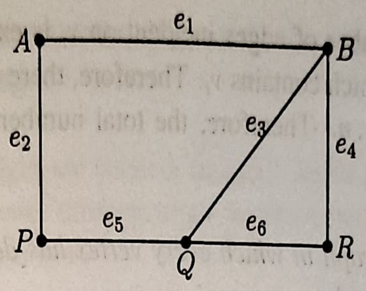


Figure 9.99

▸ There are four paths from A to R. These are

$$Ae_1Be_4R, \quad Ae_1Be_3Qe_6R, \quad Ae_2Pe_5Qe_6R, \quad Ae_2Pe_5Qe_3Be_4R.$$

These paths contain, respectively, two, three, three and four edges. Their lengths are, therefore, two, three, three and four, respectively. ■

Example 3. *Determine the number of different paths of length 2 in the graph shown below:*

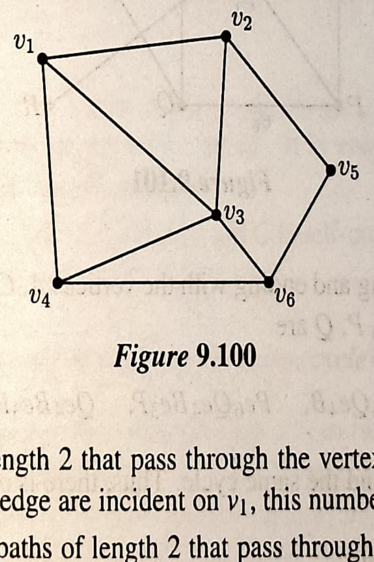


Figure 9.100

▸ The number of paths of length 2 that pass through the vertex v_1 is the number of pairs of edges incident on v_1. Since 3 edge are incident on v_1, this number is $C(3,2) = 3$.

Similarly, the number of paths of length 2 that pass through the vertices v_2, v_3, v_4, v_5 and v_6 are, respectively,

$$C(3,2) = 3, \; C(4,2) = 6, \; C(3,2) = 3, \; C(2,2) = 1, \; C(3,2) = 3.$$

Accordingly, the total number of paths of length 2 in the given graph is $3+3+6+3+1+3 = 19$. ■

Example 4. *If G is a simple graph of order n with d_i as the degree of a vertex v_i of G for $i = 1, 2, \ldots n$, find the number of paths of length 2 in G.*

9.8 Euler circuits and Euler trails

Consider a connected graph G. If there is a *circuit* in G that contains *all the edges* of G, then that circuit is called an ***Euler circuit*** (or *Eulerian line, or Euler tour*) in G. If there is a *trail* in G that contains *all the edges* of G, then that trail is called an ***Euler trail*** (or *unicursal line*) in G.

Recall that in a trail and a circuit no edge can appear more than once but a vertex can appear more than once. This property is carried to Euler trails and Euler circuits also.

Since Euler circuits and Euler trails include all the edges, they automatically should include all vertices as well.

A connected graph that contains an Euler circuit is called an ***Euler graph or Eulerian graph***. A connected graph that contains an Euler trail is called a ***semi-Euler graph*** (or a *semi-Eulerian graph* or *unicursal graph*).

For example, in the graph shown in Figure 9.112, the closed walk

$$Pe_1Qe_2Re_3Pe_4Se_5Re_6Te_7P$$

is an Euler circuit. Therefore, this graph is an Euler graph.

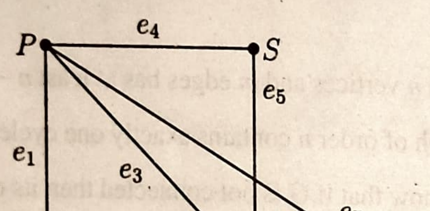


***Figure* 9.112**

Consider the graph shown in Figure 9.113. We observe that, in this graph, every sequence of edges which starts and ends with the same vertex and which includes all edges will contain at least one repeated edge. Thus, the graph has no Euler circuits. Hence this graph is *not* an Euler graph.

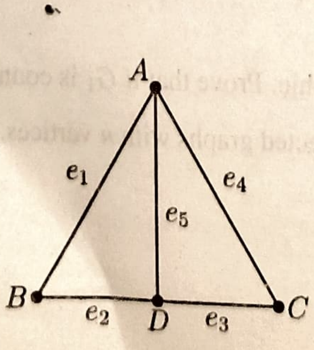


***Figure* 9.113**

It may be seen that the trail $Ae_1Be_2De_3Ce_4Ae_5D$ in the graph in Figure 9.113 is an Euler trail. This graph is therefore a semi-Euler graph.

The following Theorems contain some basic properties of Euler graphs.

Theorem 1. *A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if all vertices of G are of even degree.*

Proof: First, suppose that G has an Euler circuit. While tracing this circuit we observe that every time the circuit meets a vertex v it goes through two edges incident on v (— with the one through which we enter v and the other through which we depart from v). This is true for all vertices that belong to the circuit. Since the circuit contains all edges, it meets all the vertices at least once. Therefore, the degree of every vertex is a multiple of two (i.e. every vertex is of even degree).

Conversely, suppose that all the vertices of G are of even degree. Now, we construct a circuit starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. Since every vertex is of even degree, we can depart from every vertex we enter, and the tracing cannot stop at any vertex other than v. In this way, we obtain a circuit q having v as the initial and final vertex. If this circuit contains all the edges in G, then the circuit is an Euler circuit. If not, let us consider the subgraph H obtained by removing from G all edges that belong to q. The degrees of vertices in this subgraph are also even. Since G is connected, the circuit q and the subgraph H must have at least one vertex, say v', in common. Starting from v', we can construct a circuit q' in H as was done in G. The two circuits q and q' together constitute a circuit which starts and ends at the vertex v and has more edges than q. If this circuit contains all the edges in G, then the circuit is an Euler circuit. Otherwise, we repeat the process until we get a circuit that starts from v and ends at v and which contains all edges in G. In this way, we obtain an Euler circuit in G.

This completes the proof of the theorem. •

Theorem 2. *A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if G can be decomposed into edge-disjoint cycles.*

Proof: First, suppose that G can be decomposed (partitioned) into edge-disjoint cycles. Since the degree of every vertex in a cycle is two, it follows that every vertex in G is of even degree. Therefore, by Theorem 1, G has an Euler circuit.

Conversely, suppose G has an Euler circuit. Then, by Theorem 1, every vertex in G is of even degree. Now, consider a vertex v_1 in G. Since v_1 is of even degree, there are at least two edges incident on v_1. Choose one of these edges, and let v_2 be the other end vertex of this (chosen) edge.

Then v_2 is also of even degree, and therefore there must be at least one other edge incident on v_2. Choose one of such edges, and let v_3 be the other end vertex of the edge. Proceeding like this, we eventually arrive at a vertex that has previously been traversed, thus forming a cycle

C_1. Let us remove C_1 from G. All vertices in the resulting graph must also be of even degree, and in this graph we can construct a cycle C_2 as was done in G. Remove this cycle C_2 and proceed as above. The process ends when no edges are left. In this way we get a sequence of cycles whose union is G and whose intersection is a null graph. Thus, G has been decomposed into edge-disjoint cycles.

This completes the proof of the theorem. ●

Example 1. *Find all positive integers n (≥ 2) for which the complete graph K_n contains an Euler circuit. For which n does K_n have an Euler trail but not an Euler circuit?*

► For $n = 2$, the graph K_n contains exactly one edge. This edge together with its end vertices constitutes an Euler trail. In this case, K_n cannot have an Euler circuit. For $n \geq 3$, K_n contains an Euler circuit if and only if $n - 1$ (which is the degree of every vertex in K_n) is even; that is if and only if n is odd. ■

Example 2. (a) *Is there a graph with even number of vertices and odd number of edges that contains an Euler circuit?*

(b) *Is there a graph with odd number of vertices and even number of edges that contains an Euler circuit?*

► (a) Yes. Suppose C is a circuit with even number of vertices. Let v be one of these vertices. Consider a circuit C' with odd number of vertices passing through v such that C and C' have no edge in common. The circuit q that consists of the edges of C and C' is an Eulerian graph of the desired type.

(b) Yes. In (a), suppose C and C' are circuits with odd number of vertices. Then q is a graph of the desired type. ■

Example 3. *Show that a connected graph with exactly two vertices of odd degree has an Euler trail.*

► Let A and B be the only two vertices of odd degree in a connected graph G. Join these vertices by an edge e (even if there is already an edge between them). Then A and B become vertices of even degree. Since all other vertices in G are of even degree, the graph $G_1 = G \cup e$ is connected and has all vertices of even degree. Therefore, G_1 contains an Euler circuit which must include e. The trail got by deleting e from this Euler circuit is an Euler trail in G. ■

Exercises

1. Show that the following graph contains an Euler circuit.

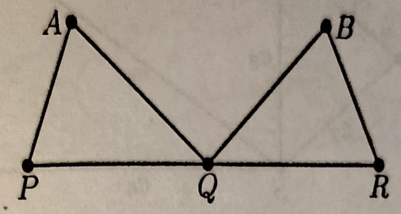


Figure 9.114

2. Find an Euler circuit in the graph shown below:

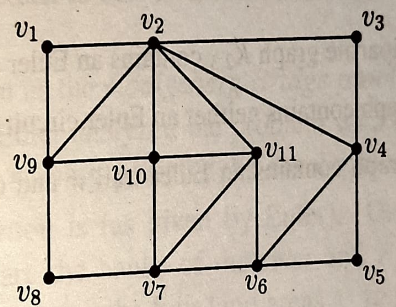


Figure 9.115

3. Show that the following graph does not contain an Euler circuit.

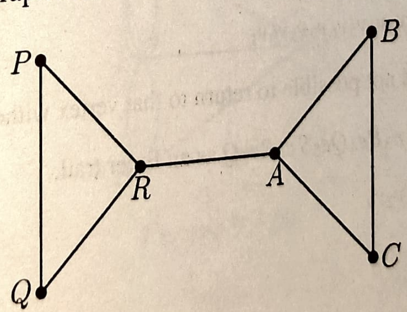


Figure 9.116

4. Show that the following graph contains an Euler trail.

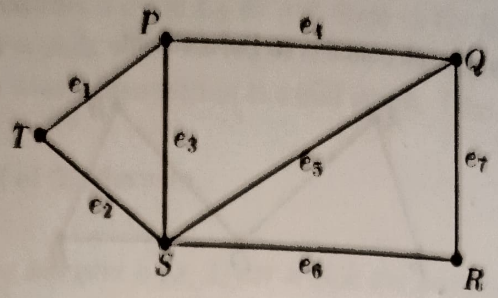


Figure **9.117**

5. If the edge $\{v_9, v_{10}\}$ is removed from the graph shown in Figure 9.115, find an Euler trail in the resulting subgraph.

6. Prove that the complete bipartite graph $K_{2,3}$ contains an Euler trail.

7. Prove that the Petersen graph contains neither an Euler circuit nor an Euler trail.

8. Prove that a connected graph contains an Euler trail if and only if it has exactly zero or two vertices of odd degree.

Answers

1. The graph contains as an Euler circuit : $PAQBRQP$.

2. $v_1 v_2 v_9 v_{10} v_2 v_{11} v_7 v_{10} v_{11} v_6 v_4 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_1$

3. Starting with any vertex, it is not possible to return to that vertex without traversing the edge RA twice.

4. The graph contains $Pe_1Te_2Se_3Pe_4Qe_5Se_6Re_7Q$ as an Euler trail.

5. $v_{10} v_{11} v_6 v_4 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_9 v_1 v_2 v_9$

9.8.1 The Königsberg Bridge Problem

In the eighteenth century city named Königsberg in East Prussia (Europe), there flowed a river named Pregel River which divided the city into four parts. Two of these parts were the banks of the river and two were islands. These parts were connected with each other through seven bridges as depicted in Figure 9.118.

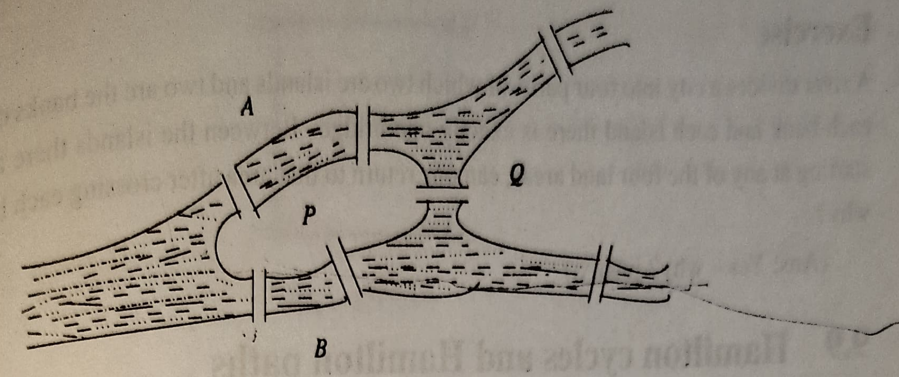


Figure 9.118

The citizens of the city seemed to have posed the following problem: By starting at any of the four land areas, can we return to that area after crossing *each* of the seven bridges *exactly once?*

This problem, now known as the *Königsberg Bridge problem*, remained unsolved for several years. In the year 1736, Euler analyzed the problem with the help of a graph and gave the solution. This was indeed the starting point for the development of graph theory.

Let us see what the solution is (as given by Euler). Denote the land areas of the city by A, B, P, Q, where A, B are the banks of the river and P, Q are the islands (See Figure 9.118). Construct a graph by treating the four land areas as four vertices and the seven bridges connecting them as seven edges. The graph is as shown in Figure 9.119.

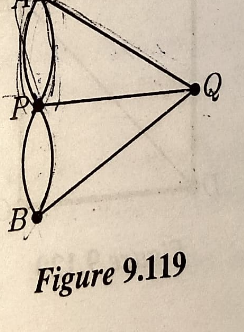


Figure 9.119

We note that, in this graph,

$$\deg(A) = \deg(B) = \deg(Q) = 3, \ \deg(P) = 5.$$

which are not even. Therefore, the graph does not have an Euler circuit[††]. This means that there does not exist a closed walk that contains all the edges exactly once. This amounts to saying that *it is not possible* to walk over each of the seven bridges exactly once and return to the starting point.

Thus, the solution to the Königsberg problem is in the *negative*.

[††] See Theorem 1.

Exercise

A river divides a city into four parts, of which two are islands and two are the banks of the river. Between each bank and each island there is exactly one bridge. Between the islands there are two bridges. By starting at any of the four land areas, can one return to that area after crossing each bridge exactly once? why?

(Ans: Yes – why? justify.)

9.9 Hamilton cycles and Hamilton paths

Let G be a connected graph. If there is a *cycle* in G that contains *all the vertices* of G, then that cycle is called a ***Hamilton cycle***[‡‡] in G.

A Hamilton cycle (when it exists) in a graph of n vertices consists of exactly n edges. Because, a cycle with n vertices has n edges.[§§]

By definition, a Hamilton cycle (when it exists) in a graph G must include all vertices in G. This *does not mean* that it should include all edges of G.

A graph that contains a Hamilton cycle is called a **Hamilton graph** (or *Hamiltonian graph*).

For example, in the graph shown in Figure 9.120, the cycle shown in thick lines is a Hamilton cycle. (Observe that this cycle does not include the edge BD). The graph is therefore a Hamilton graph.

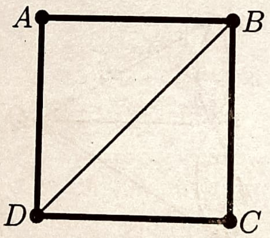


***Figure* 9.120**

It is easy to see that in the hyper-cube Q_3 shown in Figure 9.40, the cycle $ABCDSRQPA$ is a Hamilton cycle. (Observe that this cycle does not include all the edges). Therefore, Q_3 is a Hamilton graph.

A path (if any) in a connected graph which includes every vertex (but not necessarily every edge) of the graph is called a *Hamilton/Hamiltonian path* in the graph.

For example, in the graph shown in Figure 9.121, the path shown in thick lines is a Hamilton path.

[‡‡] Named after the famous Irish mathematician Sir William Hamilton (1805-1865).

[§§] See Section 9.6, Example 4.

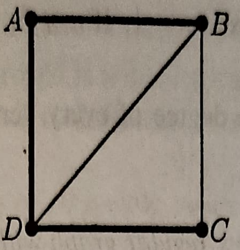


Figure 9.121

In the graph shown in Figure 9.122, the path $ABCFEDGHI$ is a Hamilton path. We check that this graph *does not contain* a Hamilton cycle.

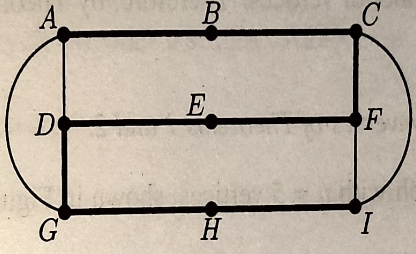


Figure 9.122

Since a Hamilton path in a graph G meets every vertex of G, the length of a Hamilton path (if any) in a connected graph of n vertices is $n - 1$. (Recall that a path with n vertices has $n - 1$ edges). [¶]

The following theorem, known as *Ore's theorem*, whose proof is beyond the scope of this text, provides a sufficient condition for a simple graph to be Hamiltonian.

Theorem 1. *A simple connected graph with n vertices (where $n \geq 3$) is Hamiltonian if the sum of the degrees of every pair of non-adjacent vertices is greater than or equal to n.*

The following theorem known as *Dirac's theorem* is an immediate consequence of the above theorem.

Theorem 2. *A simple connected graph with n vertices (where $n \geq 3$) is Hamiltonian if the degree of every vertex is greater than or equal to $n/2$.*

Proof: If in a simple connected graph with n vertices, the degree of each vertex is greater than or equal to $n/2$, then the sum of the degrees of every pair of adjacent or non-adjacent vertices is greater than or equal to n. Therefore, the graph is Hamiltonian (by Theorem 1). •

Example 1. *Prove that the complete graph K_n, $n \geq 3$, is a Hamilton graph.*

[¶] See Section 9.6, Example 4.

► In K_n, the degree of every vertex is $n - 1$. If $n \geq 3$, we have $n - 2 > 0$, or $2n - 2 > n$, or $(n - 1) > n/2$.

Thus, in K_n, where $n \geq 3$, the degree of every vertex is greater than $n/2$. Hence K_n is Hamiltonian by Theorem 2. ∎

Example 2. *Show that every simple k-regular graph with $2k - 1$ vertices is Hamiltonian.*

► In a k-regular graph, the degree of every vertex is k, and

$$k > k - \frac{1}{2} = \frac{1}{2}(2k - 1) = \frac{1}{2}$$

where $n = 2k - 1$ is the number of vertices. Therefore, by Theorem 2, the graph considered is Hamiltonian if it is simple. ∎

Example 3. *Disprove the converses of Theorems 1 and 2.*

► Consider a 2-regular graph with $n = 5$ vertices, shown in Figure 9.123.

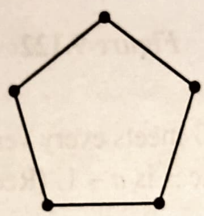


Figure **9.123**

Evidently, this graph is Hamiltonian. But the degree of every vertex is 2 which is less than $n/2$ and the sum of the degrees of every pair of vertices is 4 which is less than n.

Thus, the converses of Theorems 1 and 2 are not necessarily true. ∎

Example 4. *Let G be a simple graph with n vertices and m edges where m is at least 3. If $m \geq \frac{1}{2}(n - 1)(n - 2) + 2$, prove that G is Hamiltonian. Is the converse true?*

► Let u and v be any two non-adjacent vertices in G. Let x and y be their respective degrees. If we delete u, v from G, we get a subgraph with $n - 2$ vertices. If this subgraph has q edges, then $q \leq \frac{1}{2}(n - 2)(n - 3)$.[||||]

Since u and v are nonadjacent, $m = q + x + y$. Thus,

$$\begin{aligned} x + y = m - q &\geq \left\{\frac{1}{2}(n - 1)(n - 2) + 2\right\} - \left\{\frac{1}{2}(n - 2)(n - 3)\right\} \\ &= n. \end{aligned}$$

[||||]Recall that in a simple graph of order n, the number of edges is $\leq \frac{1}{2}n(n - 1)$.

Therefore, by Theorem 1, the graph is Hamiltonian.

The converse of the result just proved is not always true. Because, a 2-regular graph with five vertices (shown in Figure 9.123) is Hamiltonian but the inequality does not hold. ■

Example 5. *At a committee meeting of 10 people, every member of the committee has previously sat next to at most four other members. Show that the members may be seated round a circular table in such a way that no one is next to some one they have previously sat beside.*

▸ Consider a graph with 10 vertices representing the 10 members. Let two vertices be joined by an edge if the corresponding members have *not* previously sat next to each other. Since any member has not previously sat next to at least five members, the degree of every vertex is at least five (which is one-half of the number of vertices). Therefore, by Theorem 2, the graph has a Hamilton cycle. This cycle provides a seating arrangement of the desired type. ■

Example 6. *Exhibit the following:*

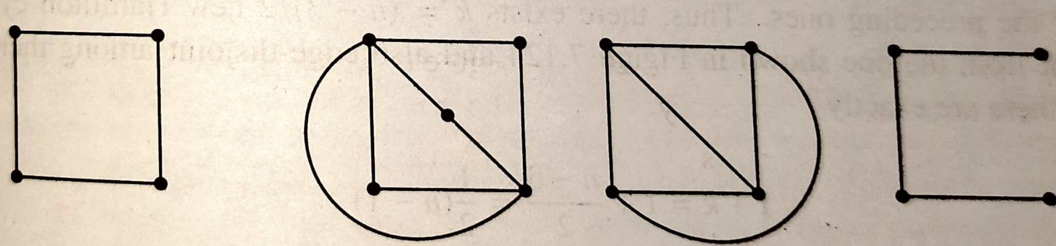
(a) *A graph which has both an Euler circuit and a Hamilton cycle.*

(b) *A graph which has an Euler circuit but no Hamilton cycle.*

(c) *A graph which has a Hamilton cycle but no Euler circuit.*

(d) *A graph which has neither a Hamilton cycle nor an Euler circuit.*

▸ The graphs (a) - (d) shown below are the required graphs in the desired order.



(a) (b) (c) (d)

Figure 9.124

■

The following theorem contains useful information on the existence of Hamilton cycles in K_n.

Theorem 3. *In the complete graph with n vertices, where n is an odd number ≥ 3, there are $(n-1)/2$ edge-disjoint Hamiltonian cycles.*

Proof: Let G be a complete graph with n vertices, where n is odd and ≥ 3. Denote the vertices of G by $1, 2, 3, \ldots, n$ and represent them as points as shown in Figure 9.125.***

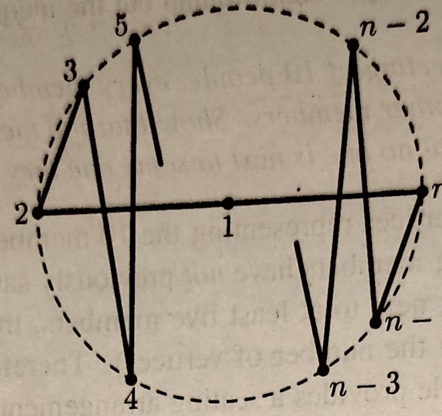


Figure **9.125**

We note that the polygonal pattern of edges from vertex 1 to vertex n as depicted in the Figure is a cycle that includes all the vertices of G. This cycle is therefore a Hamilton cycle. This representation demonstrates that G has at least one Hamilton cycle.

Now, rotate the polygonal pattern clockwise by $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ degrees, where

$$\alpha_1 = \frac{360}{n-1}, \quad \alpha_2 = 2.\frac{360}{n-1}, \quad \alpha_3 = 3.\frac{360}{n-1}, \quad \ldots \alpha_k = \frac{n-3}{2}.\frac{360}{n-1}.$$

Each of these $k = (n-3)/2$ rotations gives a Hamilton cycle that has no edge in common with any of the preceding ones. Thus, there exists $k = (n-3)/2$ new Hamilton cycles, all edge-disjoint from the one shown in Figure 7.124 and also edge-disjoint among themselves. Thus, in G, there are exactly

$$1 + k = 1 + \frac{n-3}{2} = \frac{1}{2}(n-1)$$

mutually edge-disjoint Hamilton cycles. This completes the proof of the theorem. •

Example 7. *How many edge-disjoint Hamilton cycles exist in the complete graph with seven vertices? Also, draw the graph to show these Hamilton cycles.*

► According to Theorem 3, the complete graph K_n has $(n-1)/2$ edge-disjoint Hamilton cycles when $n \geq 3$ and n is odd. When $n = 7$, their number is $(7-1)/2 = 3$. As indicated in the proof of Theorem 3, one of these Hamilton cycles appears as shown in Figure 9.126.

***In the Figure, the vertex 1 is at the centre of a circle and the other vertices are on its circumference. The circle is dotted; it is not a part of the graph.

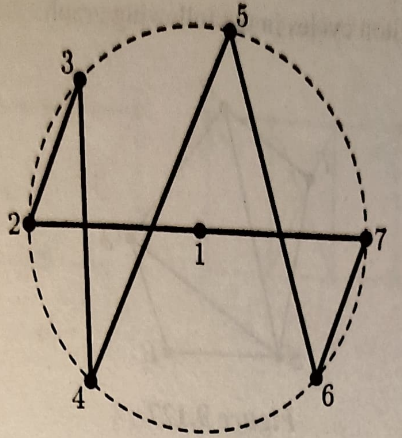


Figure 9.126

The other two cycles are got by rotating the above shown cycle clockwise through angles

$$\alpha_1 = \frac{360^\circ}{7-1} = 60^\circ \quad \text{and} \quad \alpha_2 = 2 \times \frac{360^\circ}{7-1} = 120^\circ$$ ■

Example 8. *Suppose a new club has three or more odd number of members, say $n = 2k + 1$, where k is a positive integer. These members meet each day for lunch at a round table. They decide to sit in such a way that every member has different neighbors at each lunch. How many days can this arrangement last?*

▸ Let us consider a graph G in which a member x is represented by a vertex and the possibility of his sitting next to another member y is represented by an edge between x and y. Since every member is allowed to sit next to any other member, G is a complete graph. Since there are n members, there are n vertices. Every sitting arrangement around the table is a Hamilton cycle.

On the first day of their meeting, they can sit in any order; this will be a Hamilton cycle, say C_1. On the second day, if they are to sit such that every member has different neighbors, we must find a Hamilton cycle C_2 which is edge-disjoint with C_1. If the same arrangement has to be there on subsequent days, then for each day we have to find a Hamilton cycle which is edge-disjoint with the Hamilton cycles found earlier. By Theorem 3, the number of such cycles is exactly equal to $(n-1)/2 = (2k+1-1)/2 = k$. Therefore, the seating arrangement of the desired type can last only for k days. ■

Exercises

1. Verify that the hyper-cube Q_2 and the complete graph K_5 are Hamilton graphs by finding a Hamilton cycle in each of them.

2. Identify five different Hamilton cycles in the following graph:

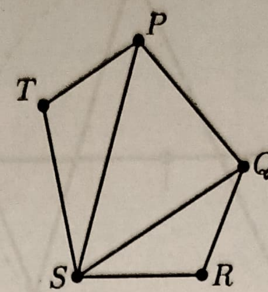
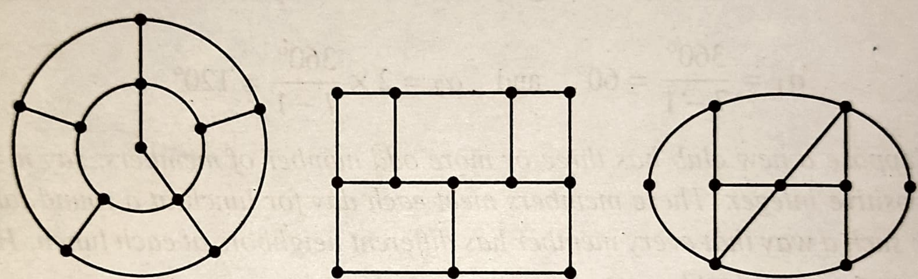


Figure 9.127

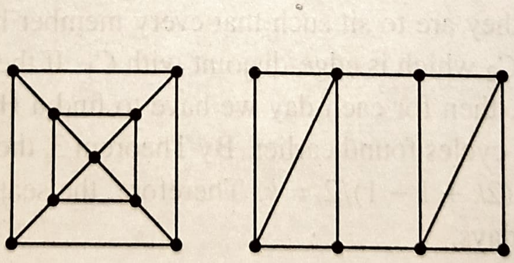
3. Show that the following graphs are Hamiltonian:



(a) (b) (c)

Figure 9.128

4. Show that the following graphs are Hamiltonian but not Eulerian.



(a) (b)

Figure 9.129

5. Which of the following are Euler graphs? Hamilton graphs?

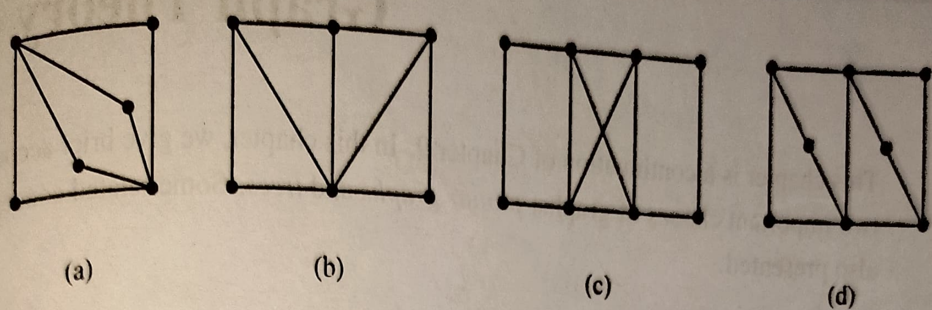


Figure 9.130

6. Prove that the complete bipartite graph $K_{3,3}$ is Hamiltonian but not Eulerian.

7. Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.

8. If the degree of each vertex of a simple graph is at least $(n - 1)/2$, where n is the number of vertices, show that the graph has a Hamilton path.

9. Show that the complete graph K_n contains $\frac{1}{2}(n-1)!$ different Hamilton cycles.

10. In Example 8, if the number of members is 9, find the possible seating arrangements where every member has different new neighbors at each lunch.

Answers

5. (a) Eulerian but not Hamiltonian (b) Hamiltonian but not Eulerian
   (c) Both Eulerian and Hamiltonian (d) Neither Eulerian nor Hamiltonian

10. There are four possible arrangements:

1	2	3	4	5	6	7	8	9	1
1	3	5	2	7	4	9	6	8	1
1	5	7	3	9	2	8	4	6	1
1	7	9	5	8	3	6	2	4	1