For a given function to compute n inputs, the divide and conquer strategy splits input into K distinct subsets when K is between $1 < K \leq n$, yielding the K subproblems. These sub problems must be solved and a method must be found to combine the solution of sub-problems into a solution of a given function.

CONTROL ABSTRACTION OF DIVIDE AND CONQUER :-

It is nothing but the algorithm description of divide and conquer.

```
                 divide and conquer
Algorithm   D AND C (P)
                        └> problem
{
    if small (P) then return S(P);
    else {
        divide P into smaller instances P1, P2, ---- Pk  K>1
        Apply D AND C to each of these subproblems;
        return combine (D AND C, D AND C (P2) ---- DANDC(Pk));
    }
}
```

The complexity of many divide & conquer algorithm is given by Recurrence of the Form $T(n) = \begin{cases} T(1) & n=1 \\ aT(n/b) + f(n) & n>1 \end{cases}$ where a & b are constants and $n = b^k$,

$K = \log_b n$

In order to find the time complexities for any divide and conquer problems, we have two techniques

i) Substitution Method ii) Recurrence / Master Method.

i) SUBSTITUTION METHOD

In substitution method, the following formulas and a tabular column is used to solve any kind of problem.

(1) $h(n) = \dfrac{f(n)}{n^{\log_b a}}$

universal length value

(2) $T(n) = n^{\log_b a}\left[T(1) + \mu(n)\right]$

$h(n)$	$\mu(n)$
$O(n^r), r<0$	$O(1)$
$\theta((\log n)^i), i\geq 0$	$\theta((\log n)^{i+1}/i+1)$
$\Omega(n^r), r>0$	$\theta(h(n))$

1) $a=1 \quad b=2, f(n)=c$

$$h(n) = \frac{c}{n^{\log_2 1}} = \frac{c}{n^0} = c = c(\ )^0 = c$$

$$\mu(n) = \theta\left((\log n)^{0+1}/_{0+1}\right) \qquad T(n) = n^{\log_b a}[T(1)+\mu(n)]$$

$$\mu(n) = \theta(\log n) \qquad = n^{\log_2 1}[T(1) + \theta(\log n)]$$

$$= n^0[T(1) + \theta(\log n)]$$

$$T(n) = T(1) + \theta(\log n)$$

($\because$ according to asymptotic notations, $T(1)$ = constant)

$$\boxed{T(n) = \theta(\log n)}$$

2) $a=5 \quad, b=4 \quad f(n) = cn^2$

$$h(n) = \frac{cn^2}{n^{\log_4 5}} = \frac{cn^2}{n^{1.16}} = c\,n^{0.84}$$

$$\mu(n) = \theta(h(n))$$

$$= \theta(n^{0.84})$$

$$T(n) = n^{\log_b a}\left[T(1) + \mu(n)\right] = n^{\log_4 5}\left[T(1) + \theta(n^{0.84})\right]$$

$= n^{1.16}[T(1)+\theta(n^{0.84})]$

$= n^{1.16}[\theta(n^{0.84})] + T(1)(n^{1.16})$

$= \theta(n^{1.16(0.84)}) = \theta(n^2) + n^{1.16}$ $\boxed{T(n)=\theta(n^2)+n^{1.16}}$

3) $a=7$ $b=2$ $f(n)=18n^2$

$h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{18n^2}{n^{\log_2 7}} = \frac{18n^2}{n^{2.8}} = 18n^{2-2.8}$

$= 18n^{-0.8}$

$\mu(n) = O(1)$

$T(n) = n^{\log_b a}[T(1)+\mu(n)]$

$= n^{\log_2 7}[T(1)+O(1)]$ ($\because T(1)$ = constant according to asymptotic notations)

$\boxed{T(n) = n^{2.8}}$

4) $a=9$ $b=3$ $f(n)=4\times n^6$

$h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{4\times n^6}{n^{\log_3 9}} = \frac{4\times n^{\cancel{6}}}{\cancel{n^2}} = 4n^4 =$ $\mu(n) = \theta(h(n))$

$= \theta(n^4)$

$T[n] = n^{\log_b a}[T(1)+\mu(n)]$

$= n^2[T(1)+\theta(n^4)]$

$= n^2[\theta(n^4)]$ ($\because T(1)$ = constant according to the asymptotic notations)

$\boxed{T(n) = \theta(n^6)}$

5) $a=28$ $b=3$ $f(n)=cn^3$

6) $a=2$, $b=2$, $T(1)=2$, $f(n)=n$

7) $a=2$, $b=2$ $f(n)=cn$.

5) $a=28$ $b=3$ $f(n)=cn^3$

$\cancel{h(n) = \frac{cn^3}{n^{\log_3 28}}} = \frac{cn^3}{n^3} = c$ $\mu(n) = \theta(\log n)$

$$T(n) = n^3 \left[T(1) + \theta(\log n) \right]$$

$$= \theta(n^3 \log n)$$

6) $a=2 \quad b=2 \quad T(1)=2 \quad f(n)=n$

$$h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{n}{n^{\lg_2 2}} = \frac{\cancel{n}}{\cancel{n}} = 1 = 1 (\ )^0 = 1$$

$$h(n) = \theta((\log n)^1)$$

$$T(n) = n^{\log_b a} [T(1) + \mu(n)] \quad (\because T(1) = \text{constant according to asymptotic Notations})$$

$$= n^{\lg_2 2} [T(1) + \theta(\log n)]$$

$$= n [T(1) + \theta(\log n)] = n\theta(\log n) + 2n$$

$$\boxed{T(n) = \theta(n \log n)}$$

7) $f(n) = cn \quad a=2 \quad b=2$

$$* \ h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{cn}{n^{\log_2 2}} = \frac{c\cancel{n}}{\cancel{n}} = c (\ )^0 = c$$

$$\mu(n) = \theta(\log n) \quad (\because T(1) = \text{constant according to asymptotic Notations})$$

$$* \ T(n) = n^{\log_2 2} [T(1) + \mu(n)]$$

$$= n [T(1) + \theta(\log n)] \boxed{= \theta(n \log n)}$$

5) $a = 26 \quad b = 3 \quad f(n) = cn^3$

$$h(n) = \frac{f(n)}{n^{\log_b a}} = \frac{cn^3}{n^{\log_3 26}} = \frac{cn^3}{n^{3.033}} = cn^{-0.033} < 0$$

$$\mu(n) = O(1)$$

$$T(n) = n^{\log_3 26} [T(1) + O(1)] \quad (\because (T(1)), O(1) \text{ are constants according to asymptotic notations})$$

$$\boxed{T(n) = n^{3.033}}$$

ii) RECURSIVE METHOD

1) $a=2 \quad b=2 \quad f(n)=n$

$$T(n) = aT(n/b) + f(n)$$

$$T(n) = 2T(n/2) + n \quad - (1)$$

$$T(n/2) = 2T\left(\frac{n}{2}\cdot\frac{1}{2}\right) + \frac{n}{2}$$

$$T(n/2) = 2T(n/4) + n/2 \quad - (2)$$

Substitute (2) in (1)

$$T(n) = 2\left[2T(n/4) + n/2\right] + n$$

$$= 4T(n/4) + \cancel{2}\cdot n/\cancel{2} + n$$

$$T(n) = 4T(n/4) + 2n \quad - (3)$$

$$T(n/4) = 2T(n/4 \cdot 1/2) + n/4$$ (put $n = n/4$ in equation (1) and calculate $T(n/4)$)

$$T(n/4) = 2T(n/8) + n/4 \quad - (4)$$

Substitute (4) in (3)

$$T(n) = 4\left[2T(n/8) + n/4\right] + 2n$$

$$= 8T(n/8) + 4\cdot n/4 + 2n$$

$$T(n) = 8T(n/8) + 3n \quad - (5)$$

$$\vdots$$

$$T(n) = 2^k T(n/2^k) + k\cdot n$$

We know that

$$n = b^k$$

$$k = \log_b n$$

$$n = 2^k \Rightarrow k = \log_2 n$$

$$T(n) = 2^{\log_2 n}\, T\left(\frac{2^k}{2^k}\right) + n.\log_2 n$$

$$= n^{\log_2 2}\, T(1) + n\log_2 n \qquad \boxed{\therefore T(n) = \theta(n \log_2 n)}$$

$$T(n) = n + n\log_2 n$$

2) $a=2 \quad b=2 \quad f(n) = cn.$

$$T(n) = aT(n/b) + f(n)$$

$$\boxed{T(n) = 2T(n/2) + cn} \quad - ①$$

$$T(n/2) = 2T(n/2 \cdot 1/2) + \frac{cn}{2}$$

$$T(n/2) = 2T(n/4) + \frac{cn}{2}$$

$$T(n) = 2\left[2T(n/4) + \frac{cn}{2}\right] + cn$$

$$= 4T(n/4) + cn + cn \Rightarrow T(n) = 4T(n/4) + 2cn \quad - ②$$

$$T(n/4) = 2T(n/4 \cdot 1/2) + c.n/4$$

$$T(n/4) = 2T(n/8) + \frac{cn}{4}$$

$$T(n) = 4\left[2T(n/8) + \frac{cn}{4}\right] + 2cn$$

$$T(n) = 8T(n/8) + 3cn \quad — ③$$

$$\boxed{T(n) = 2^k T\left(\frac{n}{2^k}\right) + k(cn)}$$

$n = b^k \quad k = \log_b n$

$n = 2^k \quad k = \log_2 n$

$$T(n) = 2^{\log_2 n} T\left(\frac{2^k}{2^k}\right) + (cn)(\log_2 n)$$

$$T(n) = n^{\log_2 2} T(1) + (cn)(\log_2 n)$$

$$= nT(1) + cn\log_2 n$$

$$T(n) = n + cn\log_2 n$$

$$T(n) = cn\log_2 n \Rightarrow \boxed{T(n) = \Theta(n\log_2 n)}$$

3) $a=1 \quad b=2 \quad f(n)=cn$

$1+ar+ar^2+\cdots$ and $r<1$ $=\frac{a(1-r^k)}{1-r}$

if $r>1$ $\quad a\frac{(r^k-1)}{r-1}$

$T(n)=aT(n/b)+f(n)$

$\boxed{T(n)=T(n/2)+cn}$

$T(n/2)=T(n/4)+\frac{cn}{2}$

$T(n)=\left[T(n/4)+\frac{cn}{2}\right]+cn$

$T(n)=T(n/4)+\frac{3cn}{2}$ — (1)

$T(n/4)=T(n/4 \cdot 1/2)+c \cdot n/4$

$T(n/4)=T(n/8)+\frac{cn}{4}$

$T(n)=T(n/8)+\frac{cn}{4}+\frac{3cn}{2}$

$T(n)=T(n/8)+\frac{7cn}{4}$

$\vdots$

$T(\frac{n}{k})=T(n/8)+cn/4+\frac{cn}{2}+n$

$=T(n/8)+c\left[n/4+n/2+n+\cdots\right]$

$=T(n/8)+cn\left[1+1/2+1/4+\cdots\right]$

$=T(n/8)+nc\left[\frac{1-(1/2)^k}{1-1/2}\right]$

$=T(n/8)+2nc\left[\frac{2^k-1}{2^k}\right]$

$=T(n/2^k)+2nc\left[\frac{2^k-1}{2^k}\right]$

$n=b^k$

$n=2^k$

$k=\log_2 n$

$\boxed{n=2^k}$

$k=\log_2 n$

$=T\left(\frac{2^k}{2^k}\right)+2\cdot 2^k \cdot c\left[\frac{2^k-1}{2^k}\right]$

$=T(1)+2c\left[2^k-1\right]$

$=T(1)+c\left[2^k-1\right]$

$=T(1)+c\cdot 2^k-c$

$=c\cdot 2^k - c \cdot 2^{\log_2 n} = n^{\log_2 2}$ $\quad \boxed{T(n)=\Theta(n)}$

4) $a=2 \quad b=3 \quad f(n)=n^3$

$$T(n)=aT(n/b)+f(n)$$

$$\boxed{T(n)=2T(n/3)+n^3}$$

$$T(n/3)=2T(n/3\cdot 1/3)+(n/3)^3$$

$$T(n/3)=2T(n/9)+(n/3)^3$$

$$T(n)=2\left[2T(n/9)+\frac{n^3}{3^3}\right]+n^3$$

$$T(n)=4T(n/9)+2(n/3)^3+n^3 \quad -①$$

$$T(n/9)=2T(n/9\cdot 1/3)+\left(\frac{n}{9}\right)^3$$

$$=2T(n/27)+\left(\frac{n}{3^2}\right)^3 \quad -②$$

② in ①

$$T(n)=4\left[2T(n/27)+\left(\frac{n}{3^2}\right)^3\right]+2\left(\frac{n}{3^1}\right)^3+n^3$$

$$=8T(n/27)+2^2\cdot\left(\frac{n}{3^2}\right)^3+2^1\left(\frac{n}{3^1}\right)^3+n^3$$

$$T(n)=2^kT\left(\frac{n}{3^k}\right)+n^3\Bigg[\underbrace{2^2\cdot\left(\frac{1}{3^2}\right)^3+2^1\left(\frac{1}{3}\right)^3+2^0\left(\frac{1}{3}\right)^0+\dots}_{\text{constant}}\Bigg]$$

$$=2^kT(n/3^k)+n^3[\ , \quad \text{can be neglected.}$$

$$=2^kT(n/3^k)+n^3 \qquad n=b^k$$

$$n=3^k$$

$$T(n)=2^kT(n/3^k)+n^3 \qquad k=\log_3 n$$

$$T(n)=2^{\log_3 n}T\left(\frac{3^k}{3^k}\right)+n^3$$

$$\boxed{T(n)=\Theta(n^3)} \qquad (\because \text{According to order of growth } \log_3 n < n^3)$$

4) $a=4 \quad b=2 \quad f(n)=n^2$

$$T(n)=aT(n/b)+f(n)$$

$$\boxed{T(n) = 4T(n/2) + n^2}$$

$$T(n/2) = 4T(n/2 \cdot 1/2) + \frac{n^2}{2^2}$$

$$T(n/2) = 4T(n/4) + \frac{n^2}{2^2} \rightarrow ② \quad T(n) = 4^2T(n/4) + 2n^2 \text{ — } ①$$

$$T(n/4) = 4T(n/4 \cdot 1/2) + \frac{n^2}{4^2}$$

$$T(n/4) = 4T(n/8) + \left(\frac{n}{4}\right)^2 \text{ — } ②$$

② in ①

$$T(n) = 4^2\left[4T(n/8) + (n/4)^2\right] + 2n^2$$

$$= 4^3T(n/8) + n^2 + 2n^2$$

$$= 4^3T(n/8) + 3n^2$$

$$T(n) = 4^kT(n/2^k) + kn^2.$$

$$n = b^k \Rightarrow n = 2^k \Rightarrow k = \log_2 n$$

$$T(n) = 4^kT\left(\frac{2^k}{2^k}\right) + \log_2 n \cdot n^2$$

$$= 4^kT(1) + n^2\log_2 n$$

$$T(n) = 4^kT(1) + n^2\log_2 n = 4n^{\log_2 4}T(1) + n^2\log_2 n$$

$$T(n) = n^2\log_2 n + 2n \quad (\because \text{According to order of growth } 2n < n^2\log_2 n)$$

$$\boxed{T(n) = \theta(n^2\log_2 n)}$$

5. $$T(n) = \begin{cases} K & n = 1 \\ 3T(n/2) + Kn & n > 1 \end{cases}$$

$$T(n) = 3T(n/2) + Kn$$

$$T(n/2) = 3T(n/2 \cdot 1/2) + K \cdot n/2$$

$$T(n/2) = 3T(n/4) + K \cdot n/2$$

$$T(n) = 3\left[3T(n/4) + K\frac{n}{2}\right] + Kn$$

$$= 9T(n/4) + \frac{3Kn}{2} + Kn$$

$$T(n/4) = 3T(n/4 \cdot 1/2) + K \cdot \frac{n}{4} = 3T(n/8) + \frac{Kn}{4}$$

$T(n) = 9\left[3T(n/8) + \frac{kn}{4}\right] + \frac{3kn}{2} + kn$

$= 3^3 T(n/2^3) + \frac{9}{4}kn + \frac{3}{2}kn + kn$

$T(n) = 3^k T\left(\frac{n}{2^k}\right) + kn\left[1 + 3/2 + (3/2)^2 + \dots\right]$

$= 3^k T(n/2^k) + kn\left[\frac{(3/2)^k - 1}{3/2 - 1}\right]$

$= 3^k T(n/2^k) + 2kn\left[(3/2)^k - 1\right]$ $\quad n = b^k$, $n = 2^k$, $k = \log_2^n$

$= 3^k T\left(\frac{2^k}{2^k}\right) + 2kn(3/2)^k - 2kn$ $\quad 3^{\log_2^n}$

$= 3^k T(1) + 2kn(3/2)^k - 2kn$ $\quad n^{\log_2^3}$

$= k \cdot 3^k + 2kn(3/2)^k - 2kn$ $\quad n = 2^k$, $k = \log_2^n$

$= k \cdot 3^{\log_2 n} + 2kn(3/2)^{\log_2 n} - 2kn$

$= k \cdot 3^{\log_2 n} + \frac{2k\cancel{n}\, 3^{\log_2 n}}{2^{\cancel{\log_2 n}}}$ $\quad a^{\log_b n} = n^{\log_b a}$

$= 3^{\log_2 n}\left(k + \frac{2k\cancel{n}}{\cancel{n}}\right) - 2kn$

$= 3k \cdot n^{\log_2 3} - 2kn$

6 $T(n) = mT(n/2) + an^2$ P.T $T(n) = O(n^{\log m})$

$\boxed{T(n) = mT(n/2) + an^2}$

$T(n/2) = mT(n/2 \cdot 1/2) + a(n/2)^2$

$T(n/2) = mT(n/4) + a(n/2)^2$

$T(n) = m\left[mT(n/4) + a(n/2)^2\right] + an^2$

$$T(n) = m^2 T(n/4) + ma(n/2)^2 + an^2 \quad (1)$$

$$T(n/4) = mT(n/4 \cdot 1/2) + a(n/4)^2$$

$$T(n/4) = mT\left(\frac{n}{8}\right) + a(n/4)^2 \quad (2)$$

$$T(n) = m^2\left[mT(n/8) + a(n/4)^2\right] + ma(n/2)^2 + an^2$$

$$= m^3 T(n/8) + m^2 a(n/4)^2 + ma(n/2)^2 + an^2$$

$$= m^k T(n/2^k) + a\left[(n/2^0)^2 + m(n/2^1)^2 + m^2\left(\frac{n}{2^2}\right)^2 + \cdots\right]$$

$$= m^k T(n/2^k) + an^2\left[1 + \left(\frac{m}{4}\right) + \left(\frac{m}{4}\right)^2 + \cdots\right]$$

$$= m^k T\left(\frac{n}{2^k}\right) + \cancel{an^2} \quad an^2 \cdot m\left(\frac{1-(m/4)^k}{1-1/4}\right)$$

let $m<1$

$\left(\frac{1}{4}\right)$

$m>1$

$\frac{5}{4}$

BINARY SEARCH

→ Divide and conquer solves the binary search problem in the following way:

If p is a problem which contains more than one element, it can be divided into new sub problems as

→ Pick the index q in the range i, l and compare x with $a[q]$.

there are 3 possibilities

= $x = a[q]$ the p is immediately solved.

= If $x < a[q]$ x has to be searched for only sublist a_i can $(a_i, a_{i+1}, \ldots a_{q-1})$ and p reduces to $(a_{q-i}, a_i \ldots a_{q-1}, x)$ and.

= If $x > a[q]$ the subset to be searched $a_{q+1}, \ldots a_l$ and p reduces to $p(l-q, a_{q+1}, \ldots a_l, x)$

The time taken for the above operations is $O(1)$

The Q is always choosen such that a_q is middle element i.e., $q = \frac{n+1}{2}$, the resultant search algorithm is called binary search.

ALGORITHM FOR BINARY SEARCH USING RECURSIVE AND NON RECURSIVE

i) Recursive Binary Search

```
Algorithm BinSrch(a, i, l, x)
{
    if(l = i) then {    // If p is small
        if(x = a[i]) then return i;
        else        return 0;
    }
    else {
        mid := [(i+l)/2]; //reduce p into sub problems
        if(x = a[mid]) then return mid;
        else if(x < a[mid]) then
            return Binsrch(a, i, mid-1, x);
        else return Binsrch(a, mid+1, l, x);
    }
```

ii) Iterative / Non Recursive Binary Search

```
Algorithm BinSearch(a, n, x)
{
    low := 1; high := n;
    while (low ≤ high) do {
        mid := (low+high)/2;
        if(x < a[mid]) then high := mid-1;
        else if (x > a[mid]) then
            low := mid+1;
        else return mid;
    }
```

1) $\overset{1}{-15}, \overset{2}{-6}, \overset{3}{0}, \overset{4}{7}, \overset{5}{9}, \overset{6}{23}, \overset{7}{54}, \overset{8}{82}, \overset{9}{101}, \overset{10}{112}, \overset{11}{125}, \overset{12}{131}, \overset{13}{142}, \overset{14}{151}$

$x = 151$

low	high	mid		
1	14	$\frac{15}{2} = 7$	$x = 151$ mid[7] = 54	151 > 54 low: = mid+1
8	14	$\frac{22}{2} = 11$	$x = 151$ mid[11] = 125	151 > 125 low: = mid+1
12	14	$\frac{26}{2} = 13$	$x = 151$ mid[13] = 142	151 > 142 low: = mid+1
14	14	$\frac{28}{2} = 14$	$x = 151$ mid[14] = 151	151 = 15

element found.

$x = 9$ $x = -14$

low	high	mid		
1	14	$\frac{15}{2} = 7$	$x = 9$ mid[7] = 54 9 < 54	high: = mid-1;
1	~~6~~	$\frac{7}{2} = 3$	$x = 9$ mid[3] = 0 9 > 0	low: = mid+1;
4	6	5	$x = 9$ mid[5] = 9	9 = 9

↳ element found.

$x = -14$

low	high	mid		
1	14	$\frac{15}{2} = 7$	$x = -14$ mid[7] = 54 -14 < 54	high: = mid-1;
1	~~6~~	3	$x = -14$ mid[3] = 0 -14 < 0	high = mid-1;
1	2	1	$x = -14$ mid[1] = -15 ~~-14~~ -15	~~high: = mid-1;~~ low: = md+1;
2	2	2	$x = -14$ mid[2] = -6 -14 < -6	high: = mid-1;

2 1 1 $x = -14$

mid[1] = -15 low = mid+1

$-14 > -15$

2 1 1 $x = -14$

mid[1] = -15 low: = mid+1

$-14 > -15$

* Note:

→ If n is in the range $(2^{k}-1, 2^{k})$ then binary search makes atmost k element comparisions for a successful search

→ (k−1, k) comparision for an unsuccessful search.

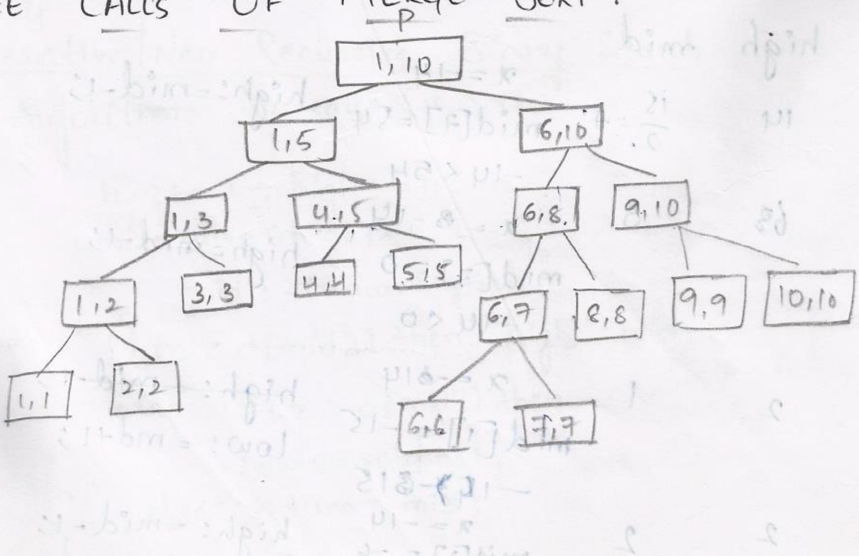
→ Time taken for ~~un~~successful search is $O(\log n)$

→ Time taken for unsuccessful search is $\theta(\log n)$

→ MERGE SORT:

For a given sequence of n elements a[1], --- a[n], the general idea is to split it into 2 sets i.e., a[1] to a[n/2] and a[n/2 + 1] - --- a[n]. Each set is individually sorted and the resulting sorted sequences are merged to produce a single sorted sequence of n elements. used in divide & con strategy

TREE CALLS OF MERGE SORT:



1) 310, 285, 179, 652, 351, 423, 861, 254, 450, 520

a[1] a[2] a[3]

The given array size elements are 1 to 10. So divide the n size by using mid = low+high/2 ; i.e., l=1 and h=10 and m=5. a[1] --- a[5] is one set. a[6] to a[10] is another set. first Sort the a[1] to a[5].

310 285 179 652 351

a[1] a[2] a[3] a[4] a[5]

Compare a[0] with a[1]

a[1]>a[2]. So merge a[1] & a[2].

$\underbrace{285 \quad 310}$ a[1,2]

a[1,2] is compared with a[3]

So a[1,2] is merged with a[3] because a[1,2]>a[3]

$\underbrace{179 \quad 285 \quad 310}$ a[1,3]

a[1,3] is compared with a[4,5]

652 351

a[4] a[5]

They are merged [a[4]>a[5]]

$\underbrace{351 \quad 652}_{a[4,5]}$

$\boxed{179 \quad 285 \quad 310 \quad 351 \quad 652}$

↳ set① with Soreted List

423 861 254 450 520

a[6] a[7] a[8] a[9] a[10]

Compare a[6] with a[7]. a[6]<a[7]

So no need to sort the elements

a[6,7] = 423 861

Now, a[6,7] is compared with a[8]

a[6,7] > a[8]. So merge, the List

$\underbrace{254 \quad 423 \quad 861}_{a[6,8]}$

a[9] is merged with a[10].

a[9] < a[10]. No merge.

a[9,10] = 450 520

Now a[6,8] is merged with a[9,10]

254 423 861 | 450 520

$\boxed{254 \quad 423 \quad 450 \quad 520 \quad 861}$

↳ set② with sorted list

Combine set① & set②

$\boxed{179, 254, 285, 3\ldots \quad 520, 652}$

2. 33, 29, 17, 66, 37, 44, 78, 26, 51, 56

a[1] a[2] a[3] a[4] a[5] a[6] a[7] a[8] a[9] a[10]

$mid = \frac{low+high}{2} = \frac{1+10}{2} = 11/2 = 5$

33 29 17 66 37 | 44 78 26 51 56

a[1] a[2] a[3] a[4] a[5] | a[6] a[7] a[8] a[9] a[10]

a[1] > a[2] So merge a[1] and a[2]

$\underbrace{29 \quad 33}$ → a[1,2] is merged with a[3]

$\underbrace{17 \quad 29 \quad 33}$ a[1,3]

a[4] > a[5] So merge a[4] & a[5]

$\underbrace{37 \quad 66}$ → a[4,5]

a[1,3] is merged with a[4,5] to get a[1,5]

17	29	33	37	66

↳ set ① with the sorted list.

a[6] < a[7]. No need of Merging.

a[6,7] = 44 78

a[6,7] is compared with a[8]

$\underbrace{26 \quad 44 \quad 78}_{a[6,8]}$

a[9] < a[10]. So no need to merge

a[9,10] = 51 56

a[6,8] is merged with a[9,10] to get a[6,10]

26	44	51	56	78

↳ set ② with the sorted list.

set ① & set ② are combined to give the required sorted elements.

$\boxed{17, 26, 29, 33, 37, 44, 51, 56, 66, 78}$ → are the required sorted elements.

The recurrence relation for merge sort is given as

$$T(n) = \begin{cases} a & \text{when } n=1 \\ 2T(n/2)+cn & n>1 \end{cases} \quad c \text{ is constant.}$$

$a=2 \quad b=2 \quad f(n)=cn$

$$h(n) = \frac{f(n)}{n^{\log_b^a}} = \frac{cn}{n^{\log_2^2}} = \frac{c\cancel{n}}{\cancel{n}} = c = c(.)^0 = c$$

$$\mu(n) = \theta\left(\frac{(\log n)^{0+1}}{0+1}\right) = \theta(\log n)$$

$$T(n) = n^{\log_b^a}[T(1)+\mu(n)]$$

$$= n^{\log_2^2}[a+\theta(\log n)]$$

$$= n[\theta(\log n)] = \underline{\theta(n\log n)}$$

$$\boxed{T(n)=\theta(n\log n)}$$

The time complexity of a Merge Sort in best case, worst and average analysis is $O(n\log n)$

The recurrence relation for binary search is given as

$$T(n) = \begin{cases} a, & n=1 \\ T(n/2)+1 & n>1 \end{cases}$$

<u>Algorithm for Merge Sort</u> :-

```
Algorithm Mergesort(low,high)
{
    if (low < high) then
    {
        mid : = low+high/2;
        MergeSort(low, mid);
        MergeSort(mid+1,high);
        Mergesort(low, mid+1,high);
    }
}
Algorithm Mergesort (low, mid, high)
{
    h: =low, i: =low, j: =mid+1;
    while ((h≤mid) and (j≤high)) do
    {
        if(a[h]≤a[j])then
        {
            b[i]: =a[h];
            h=h+1;
        }
        else {
```

```
        b[i] = a[j];
        j: = j+1;
      }
      i: = i+1;
    }
  if (h > mid) then
    for (k: = j to high) do
    {
        b[i]: = a[k];
        i = i+1;
    }
  else {
    for k: = h to mid do
      b[i]: = a[k];
      i: = i+1;
    }
    for (k: = low to high) do
        a[k]: = b[k];
  }
}
```

→ **STRASSEN'S MATRIX MULTIPLICATION :**

let A, B be two matrix of $n \times n$, the product matrix $C = A \times B$ is also an $n \times n$ matrix whose ith & jth element is found by taking elements in ith row of a and jth row of b by multiplying them to get $C[i,j] = \sum A(i,k) * B(k,j) \quad i \le k \le n$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C = A * B$$

$$C = \begin{bmatrix} \underbrace{A_{11}B_{11} + A_{12}B_{21}}_{C_{11}} & \underbrace{A_{11}B_{12} + A_{12}B_{22}}_{C_{12}} \\ \underbrace{A_{21}B_{11} + A_{22}B_{21}}_{C_{21}} & \underbrace{A_{21}B_{12} + A_{22}B_{22}}_{C_{22}} \end{bmatrix}$$

$$T(n) = 8T(n/2) + an^2 \qquad n > 2$$

H.W (Recurrence)

The computing time of above matrix is $\boxed{T(n) = 8T(n/2) + an^2}$, $n>2$ i,e., $T(n) = \Theta(n^3)$. The total computations it requires is 8 multiplication and 4 additions. In order to reduce the above computing time using divide & conquer policy Volker Strassen has introduced a method where you require 7 multiplications and 18 add/subtraction., i.e., $\boxed{T(n) = 7T(n/2) + an^2 \quad n>2}$ H.W, $T(n) = \Theta(n^{\sim 2.81})$. This method involves computing of 7 $n/2 * n/2$ matrixes i,e.,

$$P = (A_{11} + A_{12})(B_{11} + B_{22})$$
$$Q = (A_{21} + A_{22}) B_{11}$$
$$R = A_{11}(B_{12} - B_{22})$$
$$S = A_{22}(B_{21} - B_{11})$$
$$T = (A_{11} + A_{12}) B_{22}$$
$$U = (A_{21} - A_{11})(B_{11} + B_{12})$$
$$V = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$C_{11} = P + S - T + V$$
$$C_{12} = R + T$$
$$C_{21} = Q + S$$
$$C_{22} = P + R - Q + U.$$

1) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$; $A_{11} = 1$, $A_{12} = 1$, $A_{21} = 1$, $A_{22} = 1$
$B_{11} = 0$, $B_{12} = 0$, $B_{21} = 1$, $B_{22} = 1$

$P = (1+1)(0+1)$
$P = 2+0 = 2$
$Q = (2)(0) = 0$
$R = 1(-1) = -1$
$S = 1(1) = 1$
$T = (2)(1) = 2$
$U = (1-1)(0+0) = 0$
$V = (1-1)(1+1) = 0$

$C_{11} = 2 + 1 - 2 + 0$
$C_{11} = +1$
$C_{12} = -1 + 2 = 1$
$C_{21} = 0 + 1 = 1$
$C_{22} = 2 - 1 - 0 + 0$
$= +1$

$C = \begin{bmatrix} +1 & 1 \\ 1 & +1 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 0 & 3 \\ 4 & 1 & 1 & 2 \\ 0 & 3 & 5 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 & 2 & 7 \\ 3 & 1 & 3 & 5 \\ 2 & 0 & 1 & 3 \\ 1 & 4 & 5 & 1 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$ $A_{12} = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}$; $B_{11} = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$ $B_{12} = \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$ $A_{22} = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$; $B_{21} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$ $B_{22} = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}$

$P = \left(\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\right)\left(\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}\right)$

$P = \begin{array}{cc} a_{11} & a_{12} \\ \end{array} \begin{array}{cc} b_{11} & b_{12} \end{array}$

$P = \begin{pmatrix} 2 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ 8 & 2 \end{pmatrix} = \cancel{\begin{bmatrix} 16 & 22 \\ 58 & 47 \end{bmatrix}}$

$\begin{array}{cc} a_{21} & a_{22} \end{array} \begin{array}{cc} b_{21} & b_{22} \end{array}$

$Q = \cancel{\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}\left[\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\right] = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 15 \\ 20 & 12 \end{bmatrix}}$

$R = \cancel{\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\left[\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}\right] = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -2 & 4 \end{bmatrix} = [-3}$

$p = (2+6)(4) = 8(4) = 32$

$q = 2(\cancel{30}11) = 22$

$r = 2(5) = 10$

$s = 6(6) = 36$

$t = 6(2) = 12$

$u = 3(9) = 27.$

$v = (-2)(10) = -20$

$C_{11} = P+S-T+V$

$= 32+36-12-20$

$= 68-32$

$C_{11} = 36$

$C_{12} = R+T = 22$

$C_{21} = Q+S = 22+36 = 58$

$C_{22} = P+R-Q+U = 32+10-22+27$

$C_{22} = 69-22 = 47$

$$P = \begin{bmatrix} 36 & 22 \\ 58 & \cancel{47} \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}\left[\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\right] = \begin{array}{cc} b_{11}\ b_{12} & a_{11}\ a_{12} \\ \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} & \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} \\ b_{21}\ b_{22} & a_{21}\ a_{22} \end{array}$$

$p = (A_{11}+A_{22})(B_{11}+B_{22}) = (2)(8) = 16$

$q = B_{11}(A_{21}+A_{22}) = \cancel{5(3+1)} = \cancel{5.4} = \cancel{20} \quad 5(3) = 8$

$r = A_{11}(B_{12}-B_{22}) = 1(3-3) = 0 \quad 5(3) = 15$

$s = A_{22}(B_{21}-B_{11}) = 1(5-3) = 2 \quad 3(2) = 6$

$t = (A_{11}+A_{12})B_{22} = 3(5) = 15 \quad 1(8) = 8$

$u = (A_{21}-A_{11})(B_{11}+B_{12}) = (8)(2) = 16 = 0$

$v = (A_{12}-A_{22})(B_{21}+B_{22}) = 3(8) = 24 = 0$

$C_{11} = P+S-T+V = 16+6-8-0 = 14$

$C_{12} = R+T = 15+8 = 23$

$C_{21} = Q+S = 8+6 = 14$

$C_{22} = P+R-Q+U = 16+15-8+0 = 23$

$$R = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\left[\begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}\right] = \begin{array}{cc} a_{11}\ a_{12} & b_{11}\ b_{12} \\ \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix} & \begin{bmatrix} 1 & 4 \\ -2 & 4 \end{bmatrix} \\ a_{21}\ a_{22} & b_{21}\ b_{22} \end{array}$$

$p = (7)(5) = 35$ $\quad C_{11} =$

$q = (1)(6) = 6$

$r = 1(0) = 0$

$s = 6(-3) = -18$

$t = (3)(4) = 12$

$u = (-1)(4) = \cancel{-4}\ 5$

$v = (-4)(2) = -8$

2) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 6 & 0 & 3 \\ 4 & 1 & 1 & 2 \\ 0 & 3 & 5 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 4 & 2 & 7 \\ 3 & 1 & 3 & 5 \\ 2 & 0 & 1 & 3 \\ 1 & 4 & 5 & 1 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$ $\quad A_{21} = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad B_{11} = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$ $\quad B_{12} = \begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix}$

$A_{12} = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}$ $\quad A_{22} = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$ $\quad B_{21} = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$ $\quad B_{22} = \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}$

$P = (A_{11} + A_{22})(B_{11} + B_{12})$

$= \left[\begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}\right]$ $\quad n = 2\times2$

$$P = \begin{array}{cc} a_{11} & a_{12} \\ \end{array} \quad \begin{array}{cc} b_{11} & b_{12} \end{array}$$
$$P = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 8 & 2 \end{bmatrix} \quad a * b$$
$$\begin{array}{cc} a_{21} & a_{22} \end{array} \quad \begin{array}{cc} b_{21} & b_{22} \end{array}$$

$p = 4(2+6) = 4(8) = 32$

$q = 2(11) = 22$

$r = 2(5) = 10$

$s = 6(6) = 36$

$t = 6(2) = 12$

$u = 3(9) = 27$

$v = (-2)(10) = -20$

$C_{11} = P + S - T + V$

$C_{11} = 32 + 36 - 12 - 20 = 36$

$C_{12} = R + T$

$C_{12} = 10 + 12 = 22$

$C_{21} = Q + S$

$C_{21} = 22 + 36 = 58$

$C_{22} = P + R - Q + U = 32 + 10 - 22 + 27 = 47$

$P = \begin{bmatrix} 36 & 22 \\ 58 & 47 \end{bmatrix}$

$Q = (A_{21} + A_{22}) B_{11}$.

$Q = \left[\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\right]\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$

$a_{11} \; a_{12} \; b_{11} \; b_{12}$

$a_{21} \; a_{22} \; b_{21} \; b_{22}$

$p = (8)2 = 16$

$q = 1(8) = 8$

$r = 5(3) = 15$

$S = 3(2) = 6$

$t = 1(8) = 8$

$u = 0$

$v = 0$

$C_{11} = P + S - T + V = 16 + 6 - 8 + 0 = 14$

$C_{12} = R + T = 15 + 8 = 23$

$C_{21} = Q + S = 8 + 6 = 14$

$C_{22} = P + R - Q + U = 16 + 15 - 8 + 0 = 23$

$Q = \begin{bmatrix} 14 & 23 \\ 14 & 23 \end{bmatrix}$

$R = A_{11}(B_{12} - B_{22})$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\left(\begin{bmatrix} 2 & 7 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix}\right) = \begin{array}{cc} a_{11} & a_{12} \\ \end{array}\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -2 & 4 \end{bmatrix}$$

$a_{11}\ a_{12}\ b_{11}\ b_{12}$, $a_{21}\ a_{22}\ b_{21}\ b_{22}$

$p = 7(5) = 35$ $t = 3(4) = 12$ $C_{11} = 35 - 18 - 12 - 8 = -3$

$q = 6(1) = 6$ $u = (-1)5 = -5$ $C_{12} = 0 + 12 = 12$

$r = 1(0) = 0$ $v = (-4)2 = -8$ $C_{21} = 6 - 18 = -12$

$s = 6(-3) = -18$ $C_{22} = 35 + 0 - 6 - 5 = 35 - 11 = 24$

$$R = \begin{bmatrix} -3 & 12 \\ -12 & 24 \end{bmatrix}$$

$S = A_{22}(B_{21} - B_{11})$

$$\begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}\left(\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$$

$a_{11}\ a_{12}\ b_{11}\ b_{12}$, $a_{21}\ a_{22}\ b_{21}\ b_{22}$

$p = (1+0)(1+3) = 4$ $C_{11} = P + S - T + V = 4 + 0 - 9 + 2 = -3$

$q = 5(1) = 5$ $C_{12} = R + T = -7 + 9 = 2$

$r = 1(-4-3) = -7$ $C_{21} = Q + S = 0 + 5 = 5$

$s = 0$ $C_{22} = \cancel{R+T}\ P + R - Q + U$

$t = (1+2)3 = 9$ $= 4 - 7 - 5 - 12 = 4 - 24 = -20$

$u = 4(-3) = -12$

$v = 2(1) = 2.$ $S = \begin{bmatrix} -3 & 2 \\ 5 & -20 \end{bmatrix}$

$T = (A_{11} + A_{12})B_{22}$

$$\left[\begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix} + \begin{pmatrix} 4 & 4 \\ 0 & 3 \end{pmatrix}\right]\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 1 \end{bmatrix} =$$

$a_{11}\ a_{12}\ b_{11}\ b_{12}$, $a_{21}\ a_{22}\ b_{21}\ b_{22}$

$p = (5+9)(1+1) = 2(14) = 28$ $C_{11} = P + S - T + V$

$q = (9)(1) = 9$ $= 28 + 36 - 8 - 36$

$r = 5(3-1) = 10$ $C_{11} = 20$

$s = 9(4) = 36$ $C_{12} = R + T = 10 + 8 = 18$

$t = (5+3)(1) = 8$ $C_{21} = Q + S = 9 + 36 = 45$

$u = 4(-5) = -20$ $C_{22} = P + R - Q + U = 28 + 10 - 9 - 20$

$v = 6(-6) = -36$ $= 38 - 20 - 9 = 18 - 9$

$C_{22} = 9$

$$T = \begin{bmatrix} \overset{34}{20} & 18 \\ 45 & 9 \end{bmatrix}$$

$$U = (a_{21} - a_{11})(b_{11} + b_{12})$$

$$= \left[\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 6 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 3 & 5 \end{pmatrix}\right]$$

$$\begin{array}{cc} a_{11} & a_{12} \\ & \end{array} \quad \begin{array}{cc} b_{11} & b_{12} \end{array}$$

$$\cdot \begin{bmatrix} 3 & -1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 3 & 11 \\ 6 & 6 \end{bmatrix}$$

$$\begin{array}{cc} a_{21} & a_{22} \end{array} \quad \begin{array}{cc} b_{21} & b_{22} \end{array}$$

$p = (3-3)(9) = 0$

$q = 3(0-3) = -9$

$r = 3(11-6) = 15$

$s = (-3)(3) = -9$

$t = (3-1)6 = 12$

$u = (0-3)(14) = -3(14) = -42$

$v = (-1+3)(12) = 2(12) = 24$

$C_{11} = P+S-T+V = 24-21 = 3$

$C_{12} = R+T = 15+12 = 28$

$C_{21} = Q+S = -9-9 = -18$

$C_{22} = P+R-Q+U$

$= 0+15+9-42 = -18$

$$U = \begin{bmatrix} 3 & 28 \\ -18 & -18 \end{bmatrix}$$

$$V = (a_{12} - a_{22})(b_{21} + b_{22})$$

$$V = \left[\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\right]\left[\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}\right]$$

$$\begin{array}{cc} a_{11} & a_{12} \end{array} \quad \begin{array}{cc} b_{11} & b_{12} \end{array}$$

$$V = \begin{bmatrix} 2 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 6 & 5 \end{bmatrix}$$

$$\begin{array}{cc} a_{21} & a_{22} \end{array} \quad \begin{array}{cc} b_{21} & b_{22} \end{array}$$

$p = 5(8) = 40$

$q = (-5+3)(3) = 3(-2) = -6$

$r = 2(3-5) = 2(-2) = -4$

$s = 3(3) = 9$

$t = 4(5) = 20$

$u = 6(-7) = -42$

$v = (-1)(11) = -11$

$C_{11} = P+S-T+V = 18$

$C_{12} = R+T = -4+20 = 16$

$C_{21} = Q+S = 3$

$C_{22} = P+R-Q+U$

$= 40-4+6-42$

$= 46-46 = 0$

$$V = \begin{bmatrix} 18 & 16 \\ 3 & 0 \end{bmatrix}$$

$$C_{11} = P+S-T+V = \begin{bmatrix} 36 & 22 \\ 58 & 47 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 5 & -20 \end{bmatrix} - \begin{bmatrix} \overset{4}{\cancel{30}} & 18 \\ 45 & 9 \end{bmatrix} + \begin{bmatrix} 18 & 16 \\ 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{17}{\cancel{21}} & 22 \\ 21 & 18 \end{bmatrix}$$

$$\begin{array}{r} 36 \\ -3 \\ \hline 33 \\ -20 \\ \hline 13 \\ 18 \\ \hline \end{array}$$

$$C_{12} = R + T$$
$$= \begin{bmatrix} -3 & 12 \\ -12 & 24 \end{bmatrix} + \begin{bmatrix} 34 & 18 \\ 45 & 9 \end{bmatrix} = \begin{bmatrix} 31 & 30 \\ 33 & 33 \end{bmatrix}$$

$$C_{21} = Q + S$$
$$= \begin{bmatrix} 14 & 23 \\ 14 & 23 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 5 & -20 \end{bmatrix} = \begin{bmatrix} 11 & 25 \\ 19 & 3 \end{bmatrix}$$

$$C_{22} = P + R - Q + U$$
$$= \begin{bmatrix} 36 & 22 \\ 58 & 47 \end{bmatrix} + \begin{bmatrix} -3 & 12 \\ -12 & 24 \end{bmatrix} - \begin{bmatrix} 14 & 23 \\ 14 & 23 \end{bmatrix} + \begin{bmatrix} 3 & 27 \\ -18 & -18 \end{bmatrix} = \begin{bmatrix} 22 & 39 \\ 14 & 30 \end{bmatrix}$$

47
24
71

$$C = \begin{bmatrix} 17 & 22 & 31 & 30 \\ 21 & 18 & 33 & 33 \\ 11 & 25 & 22 & 39 \\ 19 & 3 & 14 & 30 \end{bmatrix}$$

1) $T(n) = 8T(n/2) + an^2 \quad , n > 2$ by Recurrence Method.

$a = 8 \quad b = 2 \quad f(n) = an^2$

$T(n/2) = 8T(n/2 \cdot 1/2) + a(n/2)^2$

$T(n/2) = 8T(n/4) + \frac{an^2}{4}$

$T(n) = 8\left[8T(n/4) + \frac{an^2}{4}\right] + an^2$

$= 8^2 T(n/4) + 8\frac{an^2}{4} + an^2$

$T(n) = 64T(n/4) + 3an^2$

$T(n/4) = 8T(n/4 \cdot 1/2) + a\left(\frac{n}{4}\right)^2$

$T(n/4) = 8T(n/8) + \frac{an^2}{16}$

$$T(n) = 64\left[8T(n/8) + \frac{an^2}{\cancel{16}_{4}}\right] + 3an^2$$

$$= 8^3 T(n/8) + \frac{\cancel{64}}{\cancel{16}} an^2 + 3an^2$$

$$T(n) = 8^3 T(n/8) + 4an^2 + 3an^2$$

$$T(n) = 8^3 T(n/2^3) + 3an^2 + 4an^2 = 8^3 T(n/2^3) + 7an^2$$

$$\vdots$$

$$T(n) = 8^K T(n/2^K) + [\underbrace{an^2 + 3an^2 + 5an^2 + \cdots}] \text{ (or)}$$

$$T(n) = 8^K T(n/2^K) + an^2[1 + 3 + 5 + 7 + \cdots]$$

$$an^2 + 2an^2 + 3an^2 + 4an^2$$

$$= an^2[1 + 2 + 3 + \cdots]$$

$$= \frac{an^2 . n(n+1)}{2}$$

$$= \frac{an^3(n+1)}{2}$$

$$= \underbrace{}_{\text{constant}}$$

$$n = b^K$$

$$n = 2^K \Rightarrow K = \log_b n \Rightarrow K = \log_2 n$$

$$= 8^{\log_2 n} T\left(\frac{2^K}{2^K}\right) + \underbrace{an^2[1+3+5+\cdots]}_{\text{constant}}$$

$$= n^3 + \text{constant}$$

$$\boxed{T(n) = \Theta(n^3)}$$

2) $T(n) = 7T(n/2) + an^2$, $n > 2$

$a = 7$ $\quad b = 2$ $\quad f(n) = an^2$

$$T(n/2) = 7T(n/2 \cdot 1/2) + a(n/2)^2$$

$$T(n/2) = 7T(n/4) + \frac{an^2}{4}$$

$$T(n) = 7\left[7T(n/4) + \frac{an^2}{4}\right] + an^2$$

$$T(n) = 7^2 T(n/4) + \frac{7an^2}{4} + an^2$$

$$T(n/4) = 7T(n/4 \cdot 1/2) + a(n/4)^2$$

$$T(n/4) = 7T(n/8) + \frac{an^2}{16}$$

$$T(n) = 7^2\left[7T(n/8) + \frac{an^2}{16}\right] + 7/4\, an^2 + an^2$$

$$T(n) = 7^3 T(n/8) + \left(\tfrac{7}{4}\right)^2 an^2 + (7/4)\, an^2 + an^2$$

$$\vdots$$

$$T(n) = 7^k (n/2^k) + an^2\left[1 + 7/4 + (7/4)^2 + \cdots\right]$$

$n = b^k$

$n = 2^k \Rightarrow k = \log_2 n$

$$= 7^{\log_2 n}\, T\left(\frac{2^k}{2^k}\right) + an^2\left[\frac{(7/4)^k - 1}{7/4 - 1}\right]$$

$$= 7^{\log_2 n}\, T(1) + an^2\left[\frac{(7/4)^k - 1}{3/4}\right]$$

$$= n^{\log_2 7}\, T(1) + \frac{4an^2}{3}\underbrace{\left[\frac{7^k - 4^k}{4^k}\right]}_{\text{constant}}$$

$$= n^{2.80}\, T(1)$$

$$\boxed{T(n) = \Theta(n^{2.8})}$$

→ QUICK SORT :-

i) In merge sort sub arrays or sub problems are sorted independently and later merged.
Whereas in quicksort the division of sub problems is made so that the sorted sub problems need not to be merged.

ii) using divide & conquer strategy the quick sort uses partition of the elements and finding the solutions of element.

iii) let $A[p \ldots r]$ be an allay which is divided into sub problems as $A[p] \ldots a[q-1]$ $A[q]$ $A[q+1] \ldots A[r]$ where $A[q]$ is pivot element

$$\underbrace{A[p] \ldots a[q-1]}_{\leq a[q]} \quad \underset{\text{pivot}}{A[q]} \quad \underbrace{A[q+1] \ldots A[r]}_{\geq A[q]}$$

Conditions for quick sort :-

i) L ←—— R
↑
pivot

p < r ?
if no → then swap pivot & right
p < r ?
if yes → then move right one position towards left.

ii) L $\longrightarrow$ R

↑ pivot

If. p > left?

if NO → then swap right & left.

If p > left?

If yes → then move left one position towards right.

*. Show how quick sort sorts the following sequence of keys.

1) 1, 1, 1, 1, 1, 1, 1   2) 5, 5, 8, 3, 4, 3, 2

1, 1, 1, 1, 1, 1, 1

↑ P, L   ↑ R

P < r ? no then swap

1, 1, 1, 1, 1, 1

↑ L   ↑ P, R

1, 1, 1, 1, 1, 1,

↑ P, L   ↑ R

1, 1, 1

ALGORITHM :-

```
Algorithm partition (a,m,p)
{
    v := a[m] ; i=m; j := p;
    repeat
    {
        repeat
            i := i+1;
        until (a[i] >= v);
        repeat
            j := j-1;
        until (a[j] <= v);
        if (i<j) then interchange (a,i,j);
    }
    until ( i >= j);
    a[m] := a[j];
    a[j] := v;
    return j;
}
Algorithm Interchange(a,i,j)
{
    p := a[i];
    a[i] := a[j];
    a[j] := p;
}
Algorithm Quicksort (p,q)
{
  if (p< q) then //if there are more than one element.
  {
      j := partition (a,p,q+1);   j = pivot element
    //divide p into sub problems
    // j is the position of partition element.
    Quicksort (p,j-1);
    Quicksort (j+1, q);
  }
}
```

When you go for analysis of quick sort we count only number of element comparision $C(n)$.

Best Case :- Quick sort works best if each array is divided into two equal sub arrays of size $n/2$. This generates $\log(n)$ levels in the recursion tree. The recurrence relation for Best case is given by

$T(n) = 2T(n/2) + cn$ when $n > 2$.

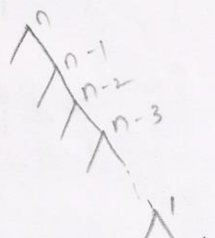
$$h(n) = \frac{cn}{n^{\log_2 2}} = c(\,)^0 = c$$

$$\mu(n) = \theta(\log n)$$

$$T(n) = n^{\log_2 2}[T(1) + \mu(n)] = n^1[\theta(\log n) + T(1)]$$

$$\boxed{T(n) = \theta(n\log n)}$$

Worst Case ~~Analysis~~ :- If the choosen pivot element is either smallest or largest in an array. In this case one part will be empty and other contains the remaining elements which generates $n-1$ levels in recursion tree.



If pivot is smallest element then the recursive relation is given as

$T(n) = T(n-1) + cn$ where $n > 1$

$$T(n) = T(n-1) + cn$$
$$T(n-1) = T(n-2) + c(n-1)$$
$$T(n-2) = T(n-3) + c(n-2)$$
$$T(n-3) = T(n-4) + c(n-3)$$
$$\vdots$$
$$T(n) = T(n-1) + T(n-2) + T(n-3) + \cdots + T(2) + T(1) + c(n-1) + c(n-2) + c(n-3) \cdots + c(2) + c(1)$$

$$T(n) = T(1) + c[1+2+3+\cdots+n]$$

$$T(n) = T(1) + c \cdot \frac{n(n+1)}{2}$$

$$T(n) = c \cdot \frac{n(n+1)}{2} \quad (\because T(1) = \text{constant})$$

$T(n) = n(n+1)$ ($\because$ 1/2 is constant)

$T(n) = n^2$

$\boxed{T(n) = \theta(n^2)}$

2) 5, 5, 8, 3, 4, 3, 2
↑P, L ↑R

P < R ? NO

2, 5, 8, 3, 4, 3, 5
↑L ↓P ↑R

2, 5, 8, 3, 4, 3, 5
↑L ↓P ↑R

2, 5, 8, 3, 4, 3, 5
↑L ↑P, R

2, 5, 5, 3, 4, 3, 8
↓P, L ↑R

2, 5, 5, 3, 4, 3, 8
↓P, L ↑R

2, 5, 3, 3, 4, 5, 8
↑L ↑P, R

2, 5, 3, 3, 4, 5, 8
↑L ↑P, R

2, 5, 3, 3, 4, 5, 8
↑L ↑P, R

P, L ↓ 2, 5, 3, 3, 4 | 5, 8
↑R

P, L ↓ 2, 5, 3, 3, 4 | 5, 8
↑R

P, L → 2, 5, 3, 3, 4 | 5, 8
↑R

P, L → 2, 5, 3, 3, 4 | 5, 8
↑R

P, L → 2, 5, 3, 3, 4 | 5, 8
↑R ↑

2 | 5, 3, 3, 4 | 5, 8
↓P, L ↑R

2 | 4, 3, 3, 5 | 5, 8
↑L ↓P ↑R

2 | 4, 3, 3, 5 | 5, 8
↑L ↑P, R

$2 \mid 4, 3, 3, 5 \mid 5, 8$

$\uparrow L \quad \uparrow P, R$

$2 \mid 4, 3, 3, \mid 5, 5, 8$

$\uparrow P, L \quad \uparrow R$

$2 \mid 3, 3, 4 \mid 5, 5, 8$

$\uparrow L \quad \uparrow P, R$

$2, 3, 3, 4, 5, 5, 8$

*** Imp :- Average Case Analysis of Quick Sort :-**

$$T_A(n) = (n+1) + \frac{1}{n} \sum_{1 \le k \le n} \left[T_A(k-1) + T_A(n-k) \right]$$

$n+1$ = no. of repetitions.

A - Average case

Consider

$$\sum_{1 \le k \le n} \left[T_A(k-1) + T_A(n-k) \right] \Rightarrow$$

if $k=1 \Rightarrow T_A(0) + T_A(n-1)$

if $k=2 \Rightarrow T_A(1) + T_A(n-2)$

if $k=3 \Rightarrow T_A(2) + T_A(n-3)$

⋮

$k=n \Rightarrow T_A(n-1) + T_A(0)$

$* \; n$ - ①

$n = n-1$ - ②

① - ② - ④

$\div \; n(n+1)$ - ⓐ

$n=n+1$ - ⓑ

$n=n-2$ - ⓒ

$$\Rightarrow T_A(n) = n+1+\frac{1}{n}\left[2T_A(0) + 2T_A(1) + 2T_A(2) + \cdots + 2T_A(n-2) + 2T_A(n-1)\right]$$

$$\Rightarrow T_A(n) = (n+1) + \frac{1}{n} \cdot 2\left[T_A(0) + T_A(1) + T_A(2) + \cdots + T_A(n-2) + T_A(n-1)\right]$$

Multiply LHS & RHS by 'n'.

$$\Rightarrow nT_A(n) = n(n+1) + \frac{\cancel{n}}{\cancel{n}} \cdot 2\left[T_A(0) + T_A(1) + \cdots + T_A(n-2) + T_A(n-1)\right]$$

$$\Rightarrow nT_A(n) = n(n+1) + 2\left[T_A(0) + T_A(1) + \cdots + T_A(n-2) + T_A(n-1)\right] \quad \text{—①}$$

Put $n = n-1$ in equation ①

$$\Rightarrow (n-1)T_A(n-1) = (n-1)(n-\cancel{1}+\cancel{1}) + 2\left[T_A(0) + T_A(1) + \cdots + T_A(n-3) + T_A(n-2)\right]$$

$$\Rightarrow (n-1)T_A(n-1) = n(n-1) + 2\left[T_A(0) + T_A(1) + \cdots + T_A(n-3) + T_A(n-2)\right] \quad \text{—②}$$

$$\frac{T_A(n)}{n+1} = \frac{T_A(n-1)}{n} + \frac{T_A(n-2)}{n-1} + \frac{T_A(n-3)}{n-2} + \cdots + \frac{T_A(1)}{2} +$$

$$\frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + \cdots$$

$$\frac{T_A(n)}{n+1} = \frac{T(1)}{2} + 2\left[\frac{1}{n+1} + \frac{1}{n} + \frac{1}{n-1} + \cdots\right]$$

$$= 2 \sum_{2 \le K \le n+1} 1/K \qquad (\because T(1) \text{ is constant})$$

$$\frac{T_A(n)}{n+1} = 2 \int_3^{n+1} 1/K \, dx$$

$$\frac{T_A(n)}{n+1} = 2\left[\log K\right]_3^{n+1}$$

$$\frac{T_A(n)}{n+1} = 2\left[\log(n+1) - \log 3\right] = 2\log n + 2\log 1 - 2\log 3$$

or

$$\frac{T_A(n)}{n+1} = 2\left[\log(n+1)\right] \qquad (\because \log 3 \text{ is constant})$$

$$T_A(n) = 2(n+1)\log(n+1)$$

$$T_A(n) = n\log(n+1) + \log(n+1)$$

$$T_A(n) = n\log(n+1) \qquad \text{neglecting}$$

$$T_A(n) = n\log(n) \qquad (\because 1 \text{ is constant and we can neglect it})$$

$$\therefore \boxed{T_A(n) = O(n\log n)}$$