Graph Coloring

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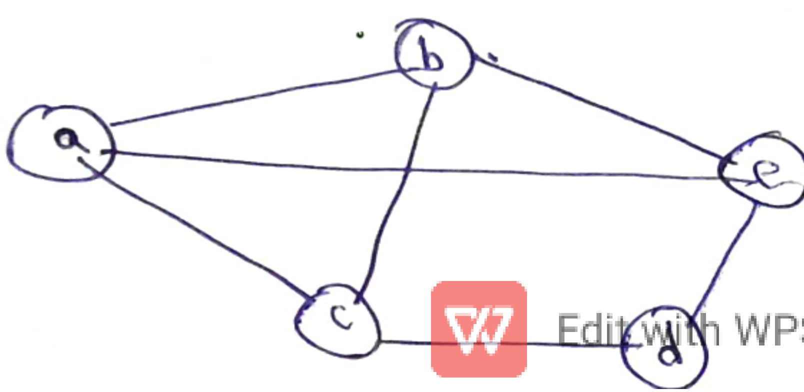
Let 'G' be a graph and m be a positive integer. we want to discover whether the nodes of G can be colored in such a way that no two adjacent nodes have the same color yet only m colors are used. This is termed the "m-colorability decision problem". Note that if d is the degree of the given graph then it can be colored with d+1 colors.

The m-colorability optimization problem ask for the smallest integer m for which the graph can be colored.

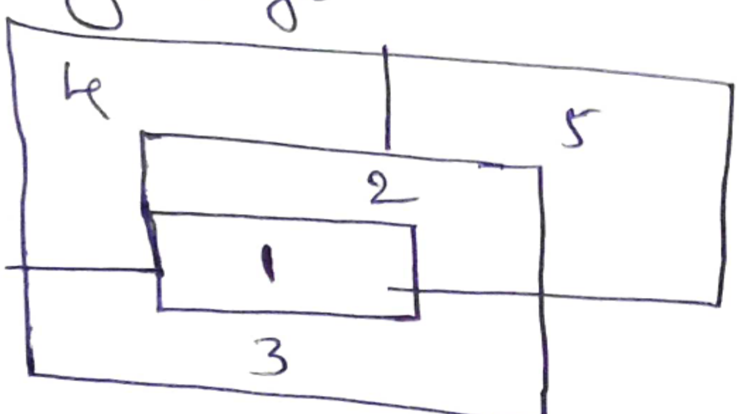


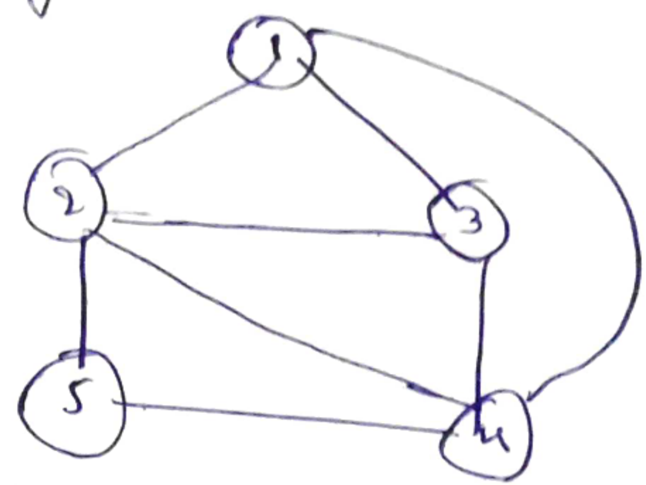
This graph can be colored with 3-colors. So the chromatic number is 3.

Graph Coloring

A graph is said to be planer iff it can be drawn in a plane in such a way that no two edges cross each other. A famous special case of m-coloring decision problem is 4-color problem for planar graph.

Given any map, can the regions be colored in such a way that no two adjacent regions have the same color yet only four colors are needed. Now convert map into graph as if two regions are adjacent then connect them using edge

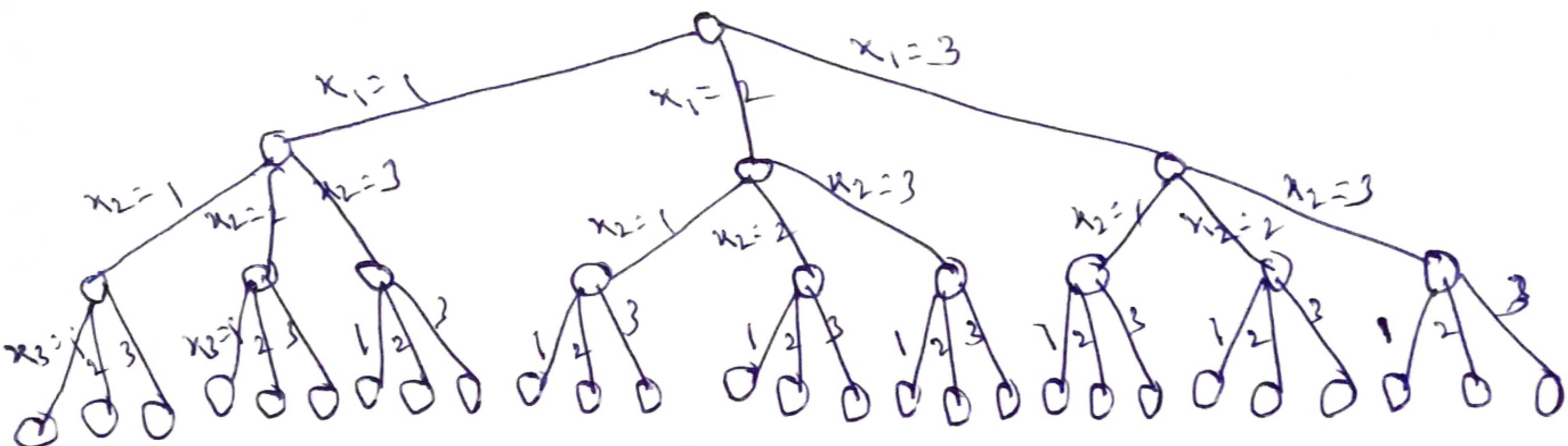




A map and its planar graph representation

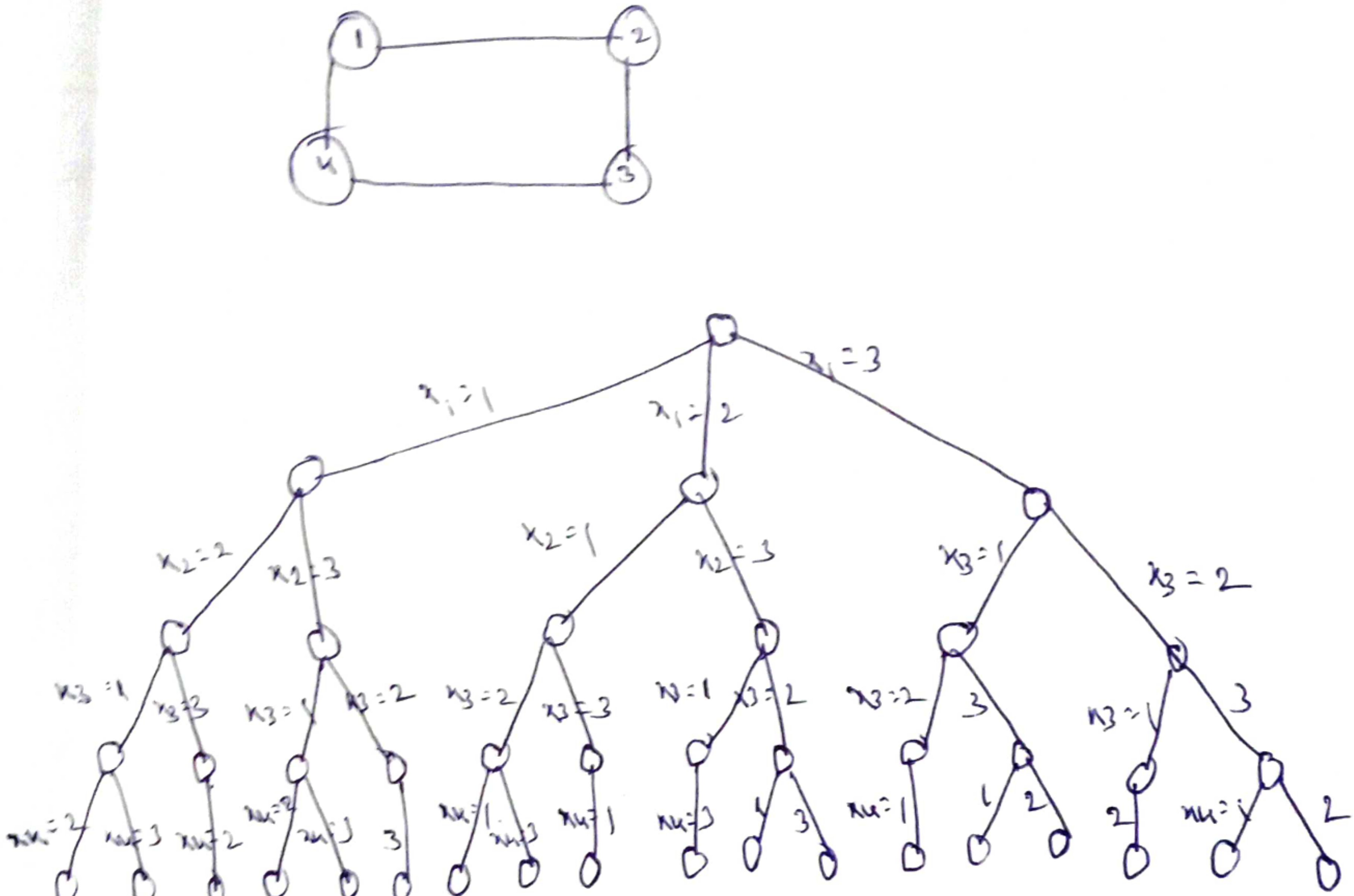
Graph Coloring

Suppose we represent graph by its adjacency matrix $G[1:n, 1:n]$, where $G[i,j] = 1$ if (i,j) is an edge of G and $G[i,j] = 0$ otherwise. The colors are represented by the integers $1, 2, \ldots, m$ and the solutions are given by the n-tuple $(x_1, x_2, \ldots x_k)$ where x_i is the color of node i.



State space tree for m-coloring with $m = 3$ and $n = 3$

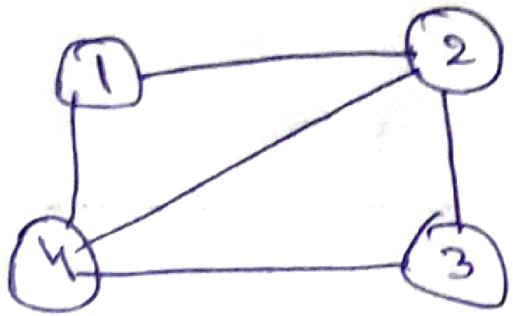
Graph Coloring



A 4-node graph and all possible 3-colorings.

Graph Coloring

Ex:



$$G: \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$n = 4$ and $m = 3$

Initially $x[i] = 0$, $1 \le i \le 4$

mcoloring(1):

$x[1] = (x[1]+1) \bmod 4$

$= (0+1) \bmod 4 = 1 \bmod 4$

$= 1$.

adjacent nodes of 1 are 2 and 4.

$x[2] = 0$ & $x[4] = 0$

So $x[1] \neq x[2]$ and $x[1] \neq x[4]$.

So $\boxed{x[1] = 1}$

Graph Coloring

mcoloring(2):

$$x[2] = (x[2]+1) \bmod 4$$
$$= (0+1) \bmod 4 = 1 \bmod 4$$
$$= 1$$

adjacent nodes of node 2 are 1, 3 and 4.

$x[1] = 1$, $x[3] = 0$, $x[4] = 0$

$x[2] = x[1]$.

- $x[2] = (x[2]+1) \bmod 4$

$$= (1+1) \bmod 4 = 2 \bmod 4 = 2$$

adjacent node colors are $x[1] = 1$, $x[3] = 0$ and $x[4] = 0$

$x[2] \neq x[1]$, $x[2] \neq x[3]$ and $x[2] \neq x[4]$

So $\boxed{x[2] = 2}$

Graph Coloring

mcoloring(3):

$x[3] = (x[3]+1) \bmod 4 = (0+1) \bmod 4$
$= 1 \bmod 4 = 1$

adjacent nodes to 3 are 2 and 4.

$x[2] = 2$ and $x[4] = 0$

$x[3] \neq x[2]$ and $x[3] \neq x[4]$.

So $\boxed{x[3] = 1}$

mcoloring(4): $x[4] = (x[4]+1) \bmod 4 = (0+1) \bmod 4 =$
$= 1 \bmod 4 = 1.$

adjacent nodes to node 4 are nodes 1, 2 and 3.

$x[1] = 1$, $x[2] = 2$, $x[3] = 1$.

$x[4] = x[1] = x[3]$.

Graph Coloring

- $x[4] = (x[4]+1) \bmod 4 = (1+1) \bmod 4 = 2 \bmod 4$

$= \underline{\underline{2}}$

$x[1] = 1, \quad x[2] = 2, \quad x[3] = 1$

$x[4] = x[2].$

- $x[4] = (x[4]+1) \bmod 4 = (2+1) \bmod 4$

$= 3 \bmod 4 = \underline{\underline{3}}$

adjacent node colors are $x[1] = 1$, $x[2] = 2$ and $x[3] = 1$

$x[4] \neq x[1], \; x[4] \neq x[2]$ & $x[4] \neq x[3].$

So $\boxed{x[4] = 3}$

Chromatic number of the graph is $\underline{3}$

Graph Coloring

```
Algorithm mcoloring(m)
{
repeat
    {
    x[k] = Nextvalue(k); // Assign to x[k] a legal color
      if(x[k] = 0) then return; //no new color possible
      if (x[k] = n) Then write (x[1:n]);
else  mcoloring(k+1);
      } until (false);
}
```

```
Algorithm Nextvalue(k)
  {
   repeat
     {
      x[k] := (x[k] + 1) mod (k+1);
     if (x[k] = 0) then return
   for j := 1 to n do
          {
           if ((G[k,j] ≠ 0) and (x[k] = x[j])
      then break;
      }
  if (j = n+1) then return   // new color found
  }until (false)
}
```