Uppsala University

Quantum Information

Lecture Notes

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Contents

Chapter 1

Introduction

Quantum Theory

One can say, that quantum theory is a probability theory where events are assotiated with complex numbers α called probability amplitudes. There are three rules for these probability amplitudes:

(a) Born rule: $P = |\alpha|^2$ gives the probability of the event α

(b) For a sequence of events with amplitude $\alpha_1, \cdots, \alpha_n$, then the amplitude of the whole sequence is: $\prod_{i=1}^{n} \alpha_{N-i+1}$

(c) For two interconnected events with amplitude α_1 and α_2, then the probability of one event occuring if the other has occured, then the amplitude of this event is $\alpha = \alpha_1 + \alpha_2$. The probability of such event is given by:

$$P = |\alpha_1|^2 + |\alpha_2|^2 + 2\,\mathrm{Re}(\alpha_1^* \alpha_2) \tag{1.1}$$

where the mixed term is called the interference term. This is where the difference between classical information and quantum information theory lies.

Qubits

A qubit is a quantum mechanical system that can be described as a two dimensional Hilbert space. In general, this can be the polarisation of a photon, spin of an electron, of a neutron and so on. We can view a qubit as a abstract sense as $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$. An arbitrary qubit state as $\alpha_0|0\rangle + \alpha_1|1\rangle$ with $\alpha_i \in \mathbb{C}$, with the normalisation condition ${\alpha_0}^1 + |\alpha_1|^2 = 1$. From the normalisation we can write the probability amplitudes as

$$\begin{cases} \alpha_0 = \cos{}^{\theta}\!/_2 e^{i\varphi_0} \\ \alpha_1 = \sin{}^{\theta}\!/_2 e^{i\varphi_1} \end{cases} \tag{1.2}$$

This allows us to write a general state $|\phi\rangle$ as: #

$$\begin{aligned} |\phi\rangle &= \cos{}^{\theta}\!/_2 e^{i\varphi_0}|0\rangle + \sin{}^{\theta}\!/_2 e^{i\varphi_1}|0\rangle \\ &= e^{i\varphi_0}\left(\cos{}^{\theta}\!/_2|0\rangle + \sin{}^{\theta}\!/_2 e^{i(\varphi_1-\varphi_0)}|0\rangle\right) \\ &\sim \cos{}^{\theta}\!/_2|0\rangle + \sin{}^{\theta}\!/_2 e^{i\varphi}|1\rangle \end{aligned}$$

This can be represented in a sphere called the Bloch sphere

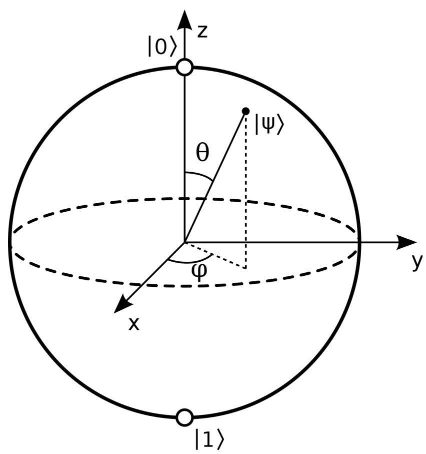

Fig. 1.1. Representation of a Bloch sphere

Chapter 2

Quantum Mechanics

2.1 Density operator

Suppose we have a preparation of quantum states (such as sending photons through a polariser), that are not perfect with some impurity. Say that each possible *pure* state $|\Psi_1\rangle, \cdots |\Psi_K\rangle$ with probabilities $P_1, \cdots P_K$. Suppose we measure an observable A throught the operator $\hat{A}$ on this set of states, and we want to calculate the expectation value of this observable:

$$A_{\text{av}} = \left\langle \hat{A} \right\rangle_{\{P_k, |\Psi_k\rangle\}} = \sum_{k=1}^{K} P_k \left\langle \Psi_k | \hat{A} | \Psi_k \right\rangle \tag{2.1}$$

where this $K \in \mathbb{N} \cup \{\infty\}$ *does not have to* be the same as dimension as the Hilbert space of the system. We can rewrite the equation 2.1 as:

$$
\begin{aligned}
A_{\text{av}} &= \sum_{k=1}^{K} P_k \left\langle \Psi_k | \hat{A} | \Psi_k \right\rangle \\
&= \sum_k \sum_n P_k \langle \Psi_k | n \rangle \left\langle n | \hat{A} | \Psi_k \right\rangle \\
&= \sum_k \sum_n P_k \left\langle n | \hat{A} | \Psi_k \right\rangle \langle \Psi_k | n \rangle \\
&= \sum_k p_k \operatorname{tr}(\hat{A} | \Psi_k \rangle \langle \Psi_k |) \\
&= \operatorname{tr}\left(\hat{A} \sum_k p_k | \Psi_k \rangle \langle \Psi_k | \right) \\
&= \operatorname{tr}(\hat{A} \hat{\rho})
\end{aligned}
$$

We define $\hat{\rho} := \sum_k p_k | \Psi_k \rangle \langle \Psi_k |$ as the density operator representing the ensemble $\{p_k, |\Psi_k\rangle\}$

2.1.1 Properties

a) The density operator is a hermitian semi-definit operator: $\langle \hat{\rho} \rangle \geq 0$.

Proof:.

$$\langle \Psi | \hat{\rho} | \Psi \rangle = \sum_k p_k \left| \langle \Psi | k \rangle \right|^2 \geq 0 \forall \Psi \in \mathcal{H}$$

b) $\operatorname{tr} \hat{\rho} = 1$

c) $\hat{\rho}^2 \leq \hat{\rho}$ and $\hat{\rho}^2 = \hat{\rho} \iff \hat{\rho} = |\Psi_1\rangle\langle\Psi_1|$ called a pure state.

Example. Qubit density operator: In diagonal (spectral) form, we can write the density operator as:

$$\hat{\rho} = p_0 |\Psi_0\rangle\langle\Psi_0| + p_1 |\Psi_1\rangle\langle\Psi_1|$$

where $\langle\Psi_0|\Psi_1\rangle = 0$ and we can define the states Ψ_0, Ψ_1 as:

$$\Psi_0 = \cos{}^{\theta}\!/\!_{2}|0\rangle + \sin{}^{\theta}\!/\!_{2}e^{i\varphi}|1\rangle$$
$$\Psi_1 = -\sin{}^{\theta}\!/\!_{2}e^{-i\varphi}|0\rangle + \cos{}^{\theta}\!/\!_{2}|1\rangle$$

with $p_0 + p_1 = 1$, so we can write these as: $p_0 = \frac{1+r}{2}$, $p_1 = \frac{1-r}{2}$, so we can write the density operator as:

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\cdot\vec{\sigma}\right) \tag{2.2}$$

where $\vec{r}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a $3D$-vector and σ are the Pauli operators, and represent a point in the Bloch ball. For $r = 0$, the density operator is the identity operator and we have the maximal mixed state.

2.2 Mixing theorem

It was formulated by Hughstom et al. in PLA (1993)

Consider $\hat{\rho} = \sum_{k=1}^{\dim(\mathcal{H})} p_k|e_k\rangle\langle e_k|$, where $\langle e_k|e_\ell\rangle = \delta_{k,\ell}$ and $\hat{\eta} = \sum_{\ell=1}^{L} q_\ell|\phi_\ell\rangle\langle\phi_\ell|$. Then $\hat{\rho} = \hat{\eta}$ iff there exists an unitary matrix $V \in \mathrm{Mat}(\dim(\mathcal{L})\times\dim(\mathcal{L}))$ such that

$$\sqrt{q_\ell}|\phi_\ell\rangle = \sum_{k=1}^{\dim\mathcal{H}} \sqrt{p_k}|e_k\rangle V_{kl} \forall l = 1,\cdots,\dim\mathcal{H} \tag{2.3}$$

2.3 Composite Systems

In quantum mechanics we use tennsor products to describe states of composite systems. Assume me have systems a, b and c with bases, then the wave function of such a composite system can be written as:

$$|\Psi_{ABC}\rangle = \sum_{k\ell m} a_{k\ell m}|a_k\rangle\otimes|b_\ell\rangle\otimes|c_m\rangle \tag{2.4}$$

where the dimension of the Hilbert space is given by

$$\dim \mathcal{H}_{ABC} = \dim \mathcal{H}_A \cdot \mathcal{H}_B \cdot \mathcal{H}_C \tag{2.5}$$

is the number of combinations of basis vectors of the subsystems. As a notation we usually write

$$|a_k\rangle \otimes |b_\ell\rangle \otimes |c_m\rangle \longrightarrow |a_k b_\ell c_m\rangle \tag{2.6}$$

As terminology: With N subsystems, we mean a N particle composite system.

Assume we have system of N qubits, then the dimension of the composite system

$$\mathcal{H}_{2D}^{\otimes N} = \bigotimes_{i=1}^{N} \mathcal{H}_{2D}$$

with dimension

$$\dim \mathcal{H}_{2D}^{\otimes N} = 2^N \tag{2.7}$$

we can write the basis as binanry strings $x \in \{0,1\}^N$ of length N that generate the Hilbert space. A general wave function can then be written as

$$|\Psi_{N\text{-Qubit}}\rangle = \sum_{x=1}^{2^N} c_x |x\rangle \tag{2.8}$$

2.3.1 Bipartite case

In the bipartite case we set $N = 2$ with subsystems A and B, an arbitrary pure state is described by

$$|\Psi^{AB}\rangle = \sum_{k=1}^{n_A} \sum_{\ell=1}^{n_B} a_{k\ell} |k\rangle \otimes |\ell\rangle \tag{2.9}$$

with $n_i = \dim \mathcal{H}_i$.

Theorem. It states, that:

$$|\Psi^{AB}\rangle = \sum_{m=1}^{\min\{n_A,n_B\}} \sqrt{d_m} |A_m\rangle \otimes |B_m\rangle \tag{2.10}$$

where $|A_m\rangle \in \mathcal{H}_A$ and $\langle A_m|A_n\rangle = \delta_{mn}$ for A and B and $\sum_m d_m = 1$.

For the proof we need singular value decomposition: Assume $a \in \mathrm{Mat}(n_A \times n_B)$ matrix. Then we can write this matrix a as

$$a = U \cdot d \cdots V$$

where $U \in U(n_A)$, $V \in U(n_B)$ unitary matrices and d a "diagonal" $n_A \times n_B$ matrix with the singular values $\sqrt{d_i}$ on the diagonal.

Proof:. We view $a_{k\ell}$ as a $n_A \times n_B$ matrix. Then using singular value decomposition we get

$$a_{k\ell} = \sum_{m=1}^{\min\{n_A,n_B\}} U_{km} \sqrt{d_m} Vm\ell$$

For a general bipartite wave function we get

$$\begin{aligned}
|\Psi^{AB}\rangle &= \sum_{k=1}^{n_A} \sum_{\ell=1}^{n_B} \sum_{m=1}^{\min\{n_A,n_B\}} U_{km} \sqrt{d_m} Vm\ell \, |k\rangle \otimes |\ell\rangle \\
&= \sum_m \sqrt{d_m} \sum_k U_{km} |k\rangle \otimes \sum_\ell Vm\ell |l\rangle \\
&= \sum_{m=1}^{\min\{n_A,n_B\}} \sqrt{d_m} |A_m\rangle \otimes |B_m\rangle
\end{aligned}$$

It remains to prove, that orthonormality of the basis.

$$\begin{aligned}
\langle A_m | A_n \rangle &= \sum_{k,k'}^{n_A} U^*_{km} U_{k'm} \, \langle k | k' \rangle \\
&= \sum_{k=1}^{n_A} U^\dagger_{mk} U_{kn} \\
&= (U^\dagger U)_{m,n} = \delta_{m,n}
\end{aligned}$$

There exists no generalisation to N partite system. If it existsed it would take the form: $\Psi^{AB\cdots X} = \sum_m \sqrt{d_m} |A_m\rangle \cdots |X_m\rangle$

Example. Assume we have a 3-state qubit, we can write:

$$\begin{aligned}
\Psi^{ABC} &= |000\rangle + a \left(|011\rangle + |101\rangle + |110\rangle \right) \\
&= |+++\rangle + |---\rangle = |GHZ\rangle
\end{aligned}$$

where $|\pm\rangle = |0\rangle \pm |1\rangle$.

In comparason a state:

$$|\Psi^{ABC}\rangle = |111\rangle + a\,(\cdots)$$

that connot be decomposed into a Schmidt form. A similar state is

$$|W\rangle := \frac{1}{\sqrt{3}}\,(|001\rangle + |010\rangle + |100\rangle)$$

that has no Schmidt decomposition.

2.4 Reduced denisty operator, partial trace

Assume we have a bipartite system with state $|\Psi^{AB}\rangle$. What does it mean to only have accoess to onu of the two subsystems, say A? What does it do operationally. Suppose we measure observable O_A an A, then using Schmidt decomposition

$$\begin{aligned}
\left\langle \hat{O}_A \right\rangle_{\Psi^{AB}} &= \left\langle \Psi^{AB} | \hat{O}_A \otimes \hat{I}_B | \Psi^{AB} \right\rangle \\
&= \sum_{k\ell} \sqrt{d_k}\sqrt{d_\ell} \left\langle A_k | \hat{O}_A | A_\ell \right\rangle \left\langle B_k | \hat{I} | B_\ell \right\rangle \\
&= \sum_k d_k \left\langle A_k | \hat{O}_A | A_k \right\rangle \\
&= \operatorname{tr}\left(O_A \sum_k d_k |A_k\rangle\langle A_k| \right) \\
&= \operatorname{tr}\,(O_A \rho_B)
\end{aligned}$$

where ρ_A is the reduced density operator in spectral form. A reduced density operator can be computed as a partial trace:

$$\rho_A = \operatorname{tr}_A |\Psi^{AB}\rangle\langle\Psi^{AB}| = \sum_m \left\langle B_m | \Psi^{AB} \right\rangle \left\langle \Psi^{AB} | B_m \right\rangle \tag{2.11}$$

2.5 Completely positive maps (quantum channels)

Within the space of all possible states (Bloch sphere for a single qubit), we can have maps between different quantum states. If we restrict ourselves to only using unitary transformation, that pure states can only stay pure states (called *closed system*) whereas completely positive maps that can transform a system more generally.

Definition. A completely positive map (CMP) $\rho \longrightarrow \mathcal{E}(\rho) \geq 0$ satisfies:

(a) $\mathrm{tr}[\mathcal{E}(\rho)]$ is the probability that $\mathcal{E}$ happens:

$$\Longrightarrow 0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1, \tag{2.12a}$$

with $\mathrm{tr}[\mathcal{E}(\rho)] = 1$ if the CPM is a completely positive trance perserving (CPTP) map.

(b) A CMP should be linear, so that: $\mathcal{E}\left(\sum_k p_k \rho_k\right) = \sum_k p_k \mathcal{E}(\rho_k)$.

(c) Let A, B be positive semi-definit operator, then the following inequality should hold

$$\mathcal{E}(A) \geq 0 \qquad \text{(positivity)} \tag{2.12b}$$

and, if B is an operator that acts on a langer Hilbert space, then

$$\mathcal{E} \otimes \mathcal{I}(B) \geq 0 \tag{2.12c}$$

for *any* extension of the system (complete positivity). This conditions guaranties that remote systems cannot influence the physical nature of $\mathcal{E}$

Definition. Transposition:

$$\left(|A\rangle\langle A^{\perp}|\right)^{T} = |A^{\perp}\rangle\langle A| \tag{2.13}$$

(partial transpostion)

Example. We have the state:

$$
\begin{aligned}
|\Psi^{AB}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |20\rangle\right) \quad \longrightarrow \quad \rho_{AB}^{T_A} &= |\Psi^{AB}\rangle\langle\Psi^A B|^{T_A} \\
&= \frac{1}{2}\left(|01\rangle - |10\rangle\right)\left(\langle 01| - \langle 10|\right)^{T_A} \\
&= \frac{1}{2}\left(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right)^{T_A} \\
&= \frac{1}{2}\left(|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| - |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\right) \\
&= \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & \\ 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

with eigenvalues $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}$

Theorem. $\mathcal{E}$ is a CPM $\iff \exists\{E_k\}$ such that

$$\sum_k E_k^\dagger E_k \leq \hat{1} \tag{2.14}$$

where the equality is given for CPTP maps in terms of which

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \tag{2.15}$$

called the Kraus representation or operator sum representation and the E_k are called Kraus operators.

Example. "Decay of an atom" Let $|e\rangle$ be the excited state of a atom and $|g\rangle$ be its ground state. Then the Kraus operators are

$$
\begin{cases}
E_0 &= |g\rangle\langle g| + \sqrt{1-\gamma}|e\rangle\langle e| \\
E_1 &= \sqrt{\gamma}|g\rangle\langle e|
\end{cases}
$$

with $\gamma \in [0,1]$. So

$$\begin{aligned}
\rho &\longrightarrow E_0\rho E_0^\dagger + E_1\rho E_1^\dagger \\
&= \left(|g\rangle\langle g| + \sqrt{1-\gamma}|e\rangle\langle e|\right)\rho\left(|g\rangle\langle g| + \sqrt{1-\gamma}|e\rangle\langle e|\right) + (\gamma|g\rangle\langle e|)\,\rho|e\rangle\langle g| \\
&\underset{\gamma\to 1}{\longrightarrow} |g\rangle\langle g|\rho|g\rangle\langle rho| + |g\rangle\langle e|\rho|e\rangle\langle rho| \\
&= |g\rangle\langle g|\underbrace{(\langle g|\rho|g\rangle + \langle e|\rho|e\rangle)}_{\operatorname{tr}\rho=1} \\
&= |g\rangle\langle g|
\end{aligned}$$

a decay with $\gamma = 1 - e^{-\lambda t}$.

2.5.1 Stine spring dilation

Assume we have a system s in an environement e and we want to determine the effect of the environement on the system with the time evolution operator U_{se}, then we can decompose the state:

$$\rho_{se} = \rho \otimes |e_0\rangle\langle e_0| \tag{2.16}$$

With a unitary map:

$$\rho_{se} \longmapsto U_{se}\rho \otimes |e_0\rangle\langle e_0|U_{se}^\dagger$$

The map of the system gives us:

$$\begin{aligned}
\rho \longmapsto \operatorname{tr}_e\left(U_{se}\rho_{se}U_{se}^\dagger\right) &= \sum_{n=0}^{\dim\mathcal{H}_e-1}\langle e_n|U_{se}\rho\otimes|e_0\rangle\left\langle e_0|U_{se}^\dagger|e_n\right\rangle \\
= \sum_n \langle e_n|U_{se}|e_0\rangle\,\rho\left\langle e_0|U_{se}^\dagger|e_n\right\rangle \\
= \sum_n E_n\rho E_n^\dagger
\end{aligned}$$

To prove that $(\langle e_n|U_{se}|e_0\rangle)^\dagger = \left\langle e_n|U_{se}^\dagger|e_0\right\rangle$ use:

$$\sum_{k\ell,pq}|s_k\rangle|s_\ell\rangle(U_{se})_{k\ell,pq}\langle s_q|\langle p|$$

This is a CPTP map:

$$\begin{aligned}
\sum_n E_n^\dagger E_n &= \sum_n \left(\langle e_n|U_{se}|e_0\rangle\right)^\dagger \langle e_n|U_{se}|e_0\rangle \\
&= \sum_n \left\langle e_0|U_{se}^\dagger|e_n\right\rangle \langle e_n|U_{se}|e_0\rangle \\
&= \left\langle e_0|U_{se}^\dagger U_{se}|e_0\right\rangle \\
&= \hat{I}_s \left\langle e_0|\hat{I}_e|e_0\right\rangle = \hat{I}_s
\end{aligned}$$

One can use Stinespring dilation to prove that the Kraus representation is not unique. Using a basis transformation: $|e_n\rangle \longmapsto |f_n\rangle = \sum_m |e_m\rangle U_{mn}^\dagger$ with a unitary basis transformation U. As U only is unitary transformation should not affect the map. Kraus operators $\{F_n = \langle f_n|U_{se}|e_0\rangle\}$ define the same map as $\{E_n = \langle e_n|U_{se}|e_0\rangle\}$, one finds

$$F_n = \sum_m U_{mn} E_m$$

2.6 Application of the CPM formalism:

2.6.1 Measurement

A generalized measurement is a CPTP map with the following interpretation:

(i) Each measurement outcome m_k is associated with a Kraus operator E_k.

(ii) The **Born rule**: the probability $P(m_k|\rho)$ is given by:

$$P(m_k|\rho) = \mathrm{tr}(E_k^\dagger E_k \rho) \tag{2.17}$$

(iii) Update the probability distribution (i.e. the state) usnig the following rule:

$$\rho \underset{m_k}{\longrightarrow} E_k \rho E_k^\dagger \tag{2.18}$$

The special case called von Neumann or projective measurment we have $E_k = |k\rangle\langle k|$ and the measurement with probability yields:

$$\begin{aligned}
P(m_k|\rho) &= \mathrm{tr}(|k\rangle\langle k||k\rangle\langle k|\rho) \\
&= \mathrm{tr}(|k\rangle\langle k|\rho) \\
&= \langle k|\rho|k\rangle
\end{aligned}$$

if $\rho = |\Psi\rangle\langle\Psi|$, then:

$$\begin{aligned}\implies P(m_k|\rho) &= \langle k|\Psi\rangle \langle\Psi|k\rangle \\ &= |\langle k|\Psi\rangle|^2\end{aligned}$$

Note. The sum of the probabilty of all outcomes:

$$\begin{aligned}\sum_k P(m_k|\rho) &= \sum_k \operatorname{tr}(E_k^\dagger E_k \rho) \\ &= \operatorname{tr}\left(\sum_k \underbrace{E_k^k E_k}_{\hat{I}} \rho\right) \\ &= \operatorname{tr}\rho = 1\end{aligned}$$

Notation. We define $\{\Pi_k := E_k^\dagger E_k\}$ as a positive operator valued measure (POVM).

Theorem. Non-orthogonal states can not be distinguished in a general measurement

Proof:. Assume $|\Phi\rangle$ and $|\Psi\rangle$ are non-orthogonal and *can* be distinguished. If this is possible then there exists Π_1 and P_2 such that $\langle\Psi|\Pi_1|\Psi\rangle = 1$ and $\langle\Phi|\Pi_2|\Phi\rangle = 1$. Because probabilities must add up to 1, we must have $\langle\Psi|\Pi_2|\Psi\rangle = 0$ where $\Pi_k \geq 0$, so $\sqrt{\Pi_k} \geq 0$. We get:

$$0 = \langle\Psi|\Pi_2|\Psi\rangle = \left\|\sqrt{\Pi_2}|\Psi\rangle\right\|^2$$

so $\sqrt{\Pi_2}|\Psi\rangle = 0$ (0-vector).
We can write $|\Phi\rangle = a|\Psi^\perp\rangle + b|\Psi\rangle$ where $b \neq 0$ by assumption. This implies also that $|a|^2 + |b|^2 = 1$, so $|a|^2 < 1$. Thus

$$\begin{aligned}\sqrt{\Pi_2}|\Phi\rangle &= a\sqrt{\Pi_2}|\Psi^\perp\rangle + b\sqrt{\Pi_2}|\Psi\rangle \\ &= a\sqrt{\Pi_2}|\Psi^\perp\rangle\end{aligned}$$

So

$$\begin{cases}\left\|\sqrt{\Pi_2}|\Psi^\perp\rangle\right\| & = \langle\Phi|\Pi_2|\Phi\rangle = 1 \\ \left\|\sqrt{\Pi_2}|\Psi^\perp\rangle\right\| & = |a|^2\left\langle\Psi^\perp|\Pi_2|\Psi^2\right\rangle \leq |a|^2 < 1\end{cases}$$

2.6.2 Lindblad equation

So far we have only been looking at discreet maps through CPMs. How would the dynamics of an open system be described, from a time ρ_0 to a time T ρ_T. We can model such case throught a series of small CPMs for timesteps $\Delta t \longrightarrow 0$, where the environemental state is measured at each timesteps t_i, and the system always can be described as a product state.
We can map the density operator as a sequence of CPM (Markonvian approximation)

$$\rho \longmapsto \mathcal{E}_{k\delta t,(k-1)\delta t} \circ \cdots \circ \mathcal{E}_{2\delta t,\delta t} \circ \mathcal{E}_{\delta t,0}(\rho_0)$$

We do the simplifying assumption, that $\mathcal{E}_{t+\delta t,\delta t} \equiv \mathcal{E}_{\delta t}$ (which implies that the Hamiltonian of the system is time independent). We consider the map

$$\begin{aligned}
\rho_t \longmapsto \rho_{t+\delta t} &= \sum_k E_k(\delta t)\rho_t E_k^\dagger(\delta t) \\
&= \mathcal{E}_{\delta t}(\rho_t) \\
\mathcal{E}_{\delta t} &\underset{\delta t \to 0}{\longrightarrow} \mathcal{I}
\end{aligned}$$

with $\mathcal{I}$ being the identity map.
This allows us to make an ansatz:

$$E_0(\delta t) = 1 - i(H - iG)\delta t$$

With H and G hermitian operators.

$$E_{k>0}(\delta t) = \sqrt{\delta t} L_k$$

We assume that $\mathcal{E}_{\delta t}$ is a CPTP since we do not exctact information of the system during the evolution, in other words $\operatorname{tr}\rho_t = 1$ for all t. This implies for our ansatz:

$$\begin{aligned}
\hat{I} &= \sum_k E_k^\dagger(\delta t)E_k(\delta t) \\
&= E_0^\dagger(\delta t)E_0(\delta t) + \sum_{k=0} E_k^\dagger(\delta t)E_k(\delta t) \\
&= \hat{I} + 2G\delta t + \delta t \sum_{k=1} L_k^\dagger L_k + O(\delta t^2)
\end{aligned}$$

So we get:

$$G = \frac{1}{2}\sum_{k=1} L_k^\dagger L_k \tag{2.19}$$

This gives us the Lindblad equation:

$$\begin{aligned}\frac{\rho_{t+\delta t} - \rho_t}{\delta t} &= -i[H, \rho_t] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right) \\ \delta t \longrightarrow 0 \Longrightarrow \dot{\rho}_t &= -i[H, \rho_t] + \sum_k \left(L_k \rho_t L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\} \right) \end{aligned} \tag{2.20}$$

where $\{A, B\} = AB + BA$ is the anticommutator, and H should be interpreted as the Hamiltonian, thus the first term of the equation is called the "Schrödinger term", the L_k are called Lindblad operators and the anticommutator term give non hermitian dynamics.

Example. Assume a qubit system with $L = \sqrt{\gamma}\sigma_z$ with Hamiltonian $H = \omega\sigma_z$, would give a density operator:

$$\rho_t = \frac{1}{2}(\hat{I} + \vec{r}_t \cdot \vec{\sigma})$$

with the solution in the rotating frame.

$$\begin{aligned}\dot{\vec{r}} &= -2\gamma(x_t, y_t, 0) \\ \vec{r} &= (x_0 e^{-2\gamma t}, y_0 e^{-2\gamma t}, z_0)\end{aligned}$$

which for $t \longrightarrow \infty$, we end up with a state lying on the z-axis, and the exponential term in the x and y coordinates are called decoherence terms.

Appendix A

Exercices

A.1 Qubit

A.2 Quantum ensemble: Density operator

A.3 Composite system

A.4 Completely positive maps

18.

(a) The phase damping completely positive map: $\mathcal{E}(\rho) = (1-p)\rho + p\sigma_z\rho\sigma_z$ for $p \in [0,1]$. The Kraus operators are

$$\begin{cases} E_0 &= \sqrt{1-p}\hat{I} \\ E_1 &= \sqrt{p}\hat{\sigma}_z \end{cases}$$

(b) The bit flip map $\mathcal{E}(\rho) = (1-p)\rho + p\sigma_x\rho\sigma_x$ for $p \in [0,1]$. The Kraus operators are

$$\begin{cases} E_0 &= \sqrt{1-p}\hat{I} \\ E_1 &= \sqrt{p}\hat{\sigma}_x \end{cases}$$

(c) The phase shape and bit shift map $\mathcal{E}(\rho) = (1-p)\rho + p\sigma_y\rho\sigma_y$ for $p \in [0,1]$.

The Kraus operators are

$$\begin{cases} E_0 & = \sqrt{1-p}\hat{I} \\ E_1 & = \sqrt{p}\hat{\sigma}_y \end{cases}$$

22. Consider a two lever atom with ground state $|g\rangle$ and excited state $|e\rangle$. The atom is placed in a cavity, which includes decay of an atom modeled by the unitary transformation

$$\begin{aligned} |e\rangle \otimes |n\rangle &\longmapsto \sqrt{1-p}|e\rangle \otimes |n\rangle + \sqrt{p}|g\rangle \otimes |n+1\rangle \\ |g\rangle \otimes |n\rangle &\longmapsto |g\rangle \otimes |n\rangle \end{aligned}$$

where $|n\rangle$ and $|n+1\rangle$ are the states of the quantizised field corresponding to the n and $n+1$ photons in the cavity. Suppose the atom+cavity field start in the pure states

$$|\Psi\rangle = (\cos{}^{\theta}\!/\!{}_{2}|g\rangle + \sin{}^{\theta}\!/\!{}_{2}|g\rangle) \otimes |n\rangle$$

and undergoes a decay. Compute the average number of photons in the cavity as a function of the decay probability p.
The expectation value is $\langle N\rangle = \left\langle a^{\dagger}a\right\rangle = \mathrm{tr}(\mathcal{E}(\rho_{\gamma})N)$. The density of the composite system is:

$$\rho_{a\gamma} = |\Psi\rangle\langle\Psi| = |\Psi_0\rangle\langle\Psi_0| \otimes |n\rangle\langle n| \tag{A.1}$$

Now appling the positivy map and calculate the partial trace.

$$\begin{aligned} \mathcal{E}(\rho_{a\gamma} &= [\mathcal{E}(|\Psi\rangle)]\,[\mathcal{E}(|\Psi\rangle)]^{\dagger} \\ &= \mathrm{tr}_a(\mathcal{E}(\rho_{a\gamma})) = ... \end{aligned}$$

An alternative: From the lecture, we know, that a composite system evolves as $\rho_{se} \longrightarrow U_{se}\rho_{se}U_{se}^{\dagger}$, then the system has a Kraus decomposition

$$\rho \longrightarrow \sum E_n \rho E_n^{\dagger}$$

where $E_n = \langle E_n|U_{se}|E_0\rangle$ with E_n an orthogonal basis for the environement. On out atom system, we need to determine the basis for the composite system

$\{|g\rangle|0\rangle, |g\rangle|0\rangle, \cdots, |g\rangle|n\rangle, \cdots, |e\rangle|0\rangle, \cdots |e\rangle|n\rangle, \cdots\}$. As a matrix, the time evolution operator looks like:

$$U = \begin{pmatrix}
1 & & & & & 0 & & & \\
& 1 & & & & \sqrt{p} & \ddots & & \\
& & \ddots & & & & \ddots & & \\
& & & 1 & & & & 0 & \\
& & & & \ddots & & & \sqrt{p} & \ddots \\
& & & & & \sqrt{1-p} & & & \ddots \\
& & & & & & \ddots & & \\
& & & & & & & \sqrt{1-p} & \\
& & & & & & & & \ddots
\end{pmatrix} = \begin{pmatrix} I & B \\ 0 & C \end{pmatrix}$$

We can calculate the Kraus coeffitions:

$$\begin{cases} E_g & = \langle g|U_{ac}|\Psi_0\rangle \\ E_e & = \langle e|U_{ac}|\Psi_0\rangle \end{cases}$$

with $|\Phi_0\rangle = \cos{}^{\theta}\!/_2|g\rangle + \sin{}^{\theta}\!/_2|e\rangle$ and get:

$$\begin{aligned}
E_g &= \cos{}^{\theta}\!/_2 \,\langle g|U_{ac}|g\rangle + \sin{}^{\theta}\!/_2 \,\langle g|U_{ac}|e\rangle \\
&= \cos{}^{\theta}\!/_2 I + \sin{}^{\theta}\!/_2 B \\
E_e &= \cos{}^{\theta}\!/_2 \,\langle e|U_{ac}|g\rangle + \sin{}^{\theta}\!/_2 \,\langle e|U_{ac}|e\rangle \\
&= \sin{}^{\theta}\!/_2 C
\end{aligned}$$

So we get:

$$\begin{aligned}
\mathcal{E}(\rho) &= \mathcal{E}(|n\rangle\langle n|) \\
&= E_g|n\rangle\langle n|E_g^\dagger + E_e|n\rangle\langle n|E_e^\dagger \\
&= \cos^2{}^{\theta}\!/_2|n\rangle\langle n| + \cos{}^{\theta}\!/_2 \sin{}^{\theta}\!/_2 \left(|n\rangle\langle n|B^\dagger + B|n\rangle\langle n|\right) \\
&\quad + \sin^2{}^{\theta}\!/_2 B|n\rangle\langle n|B^\dagger + \sin^2{}^{\theta}\!/_2 \sin C|n\rangle\langle n|C^\dagger \\
&= \left[\cos^2{}^{\theta}\!/_2 + (1-p)\sin^2{}^{\theta}\!/_2\right]|n\rangle\langle n| + p\sin^2{}^{\theta}\!/_2|n+1\rangle\langle n+1| \\
&\quad + \sqrt{p}\sin{}^{\theta}\!/_2 \cos{}^{\theta}\!/_2 \left[|n\rangle\langle n+1| + |n+1\rangle\langle n|\right]
\end{aligned}$$

We can now calculate the expectation value of N with $N|n\rangle = a^\dagger a|n\rangle = n$

$$\begin{aligned}\langle N\rangle &= \mathrm{tr}(\mathcal{E}(\rho_c)N)\\ &= n\left[\cos^2 \theta/2 + (1-p)\sin^2 \theta/2\right] + (n+1)p\sin^2 \theta/2\\ &= n + p\sin^2 \theta/2\end{aligned}$$

A.5 Lindblad equation

26. Let H be the Hamiltonian:

$$H = \sum E_k|\Phi_k\rangle\langle\Phi_k|$$

with an error $L = \sqrt{\gamma}H$. Let $\rho(t)$ be the state of the system at time t, then:

$$\dot{\rho}(t) = -i[H,\rho] + \sum\left(L_n\rho L_n^\dagger - \frac{1}{2}\{L_n^\dagger L_n,\rho\}\right)$$

We want to show, that it reduced to

$$\dot{\rho}(t) = -[H,\rho] - \frac{\gamma}{2}[H,[H,\rho]]$$

We only need to show, that:

$$\begin{aligned}L\rho L^\dagger - \frac{1}{2}\{LL^\dagger,\rho\} &= \gamma\left(H\rho H - \frac{1}{2}(H^2\rho + \rho H^2)\right)\\ -\gamma[H,[H,\rho]] &= -\gamma\left(H, H\rho - \rho H\right)\\ &= -\gamma\left(H^2\rho - H\rho H - H\rho H + \rho H^2\right)\\ &= \frac{\gamma}{2}\left(H\rho H - \frac{1}{2}(H^2\rho + H^2\rho)\right)\end{aligned}$$

This is the damping equation used in magnetism. We can write the density operator as

$$\rho = \sum_{\ell k} = \rho_{k\ell}(t)|\Phi_k\rangle\langle\Phi_\ell|$$

that, because of linearity, we only need to solve them for one of the $\rho_{k\ell}$

$$\begin{aligned}\dot{\rho}_{k\ell}(t) &= -\rho_{k\ell}[H,|\Phi_k\rangle\langle\Phi_\ell|] - \frac{\gamma}{2}\rho_{k\ell}[H,[H,|\Phi_k\rangle\langle\Phi_\ell|]]\\ &= \rho_{k\ell}(t)\left(-i(E_k - E_\ell) - \frac{\gamma}{2}(E_k - E_\ell)^2\right)|\Phi_k\rangle\langle\Phi_\ell|\end{aligned}$$

And we get

$$\rho_{k\ell}(t)e^{t(-i(E_k-E_\ell))}e^{-\frac{\gamma t}{2}(E_k-E_\ell)^2}$$

For $\gamma t \to \infty$, $\rho_{k\ell}$ goes to 0.

A.6 Measurements

A.7 Quantum Entanglement

A.8 Detecting quantum entanglement

A.9 Quantifying quantum entanglement

A.10 Quantum teleportation

A.11 Bell's inequality

A.12 Quantum copying

A.13 von Neumann and relative entropy