Uppsala University

Quantum Information

Lecture Notes

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Chapter 1

Introduction

Quantum Theory

One can say, that quantum theory is a probability theory where events are assotiated with complex numbers α called probability amplitudes. There are three rules for these probability amplitudes:

(a) Born rule: $P = |\alpha|^2$ gives the probability of the event α

(b) For a sequence of events with amplitude $\alpha_1, \cdots, \alpha_n$, then the amplitude of the whole sequence is: $\prod_{i=1}^{n} \alpha_{N-i+1}$

(c) For two interconnected events with amplitude α_1 and α_2, then the probability of one event occuring if the other has occured, then the amplitude of this event is $\alpha = \alpha_1 + \alpha_2$. The probability of such event is given by:

$$P = |\alpha_1|^2 + |\alpha_2|^2 + 2\,\mathrm{Re}(\alpha_1^* \alpha_2) \tag{1.1}$$

where the mixed term is called the interference term. This is where the difference between classical information and quantum information theory lies.

Qubits

A qubit is a quantum mechanical system that can be described as a two dimensional Hilbert space. In general, this can be the polarisation of a photon,

spin of an electron, of a neutron and so on. We can view a qubit as a abstract sense as $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$. An arbitrary qubit state as $\alpha_0|0\rangle + \alpha_1|1\rangle$ with $\alpha_i \in \mathbb{C}$, with the normalisation condition ${\alpha_0}^1 + |\alpha_1|^2 = 1$. From the normalisation we can write the probability amplitudes as

$$\begin{cases} \alpha_0 = \cos \theta/2 e^{i\varphi_0} \\ \alpha_1 = \sin \theta/2 e^{i\varphi_1} \end{cases} \tag{1.2}$$

This allows us to write a general state $|\phi\rangle$ as: #

$$\begin{aligned} |\phi\rangle &= \cos \theta/2 e^{i\varphi_0}|0\rangle + \sin \theta/2 e^{i\varphi_1}|0\rangle \\ &= e^{i\varphi_0}\left(\cos \theta/2|0\rangle + \sin \theta/2 e^{i(\varphi_1 - \varphi_0)}|0\rangle\right) \\ &\sim \cos \theta/2|0\rangle + \sin \theta/2 e^{i\varphi}|1\rangle \end{aligned}$$

This can be represented in a sphere called the Bloch sphere

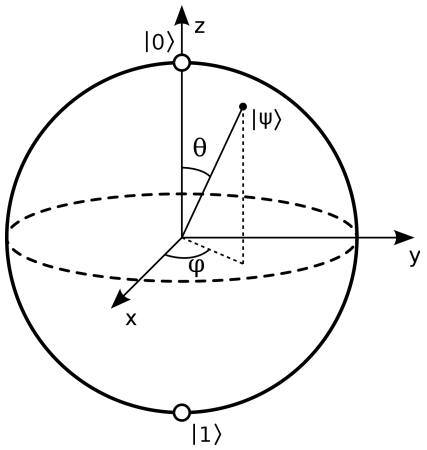

Fig. 1.1. Representation of a Bloch sphere

Chapter 2

Quantum Mechanics

2.1 Density operator

Suppose we have a preparation of quantum states (such as sending photons through a polariser), that are not perfect with some impurity. Say that each possible *pure* state $|\Psi_1\rangle, \cdots |\Psi_K\rangle$ with probabilities $P_1, \cdots P_K$. Suppose we measure an observable A throught the operator $\hat{A}$ on this set of states, and we want to calculate the expectation value of this observable:

$$A_{\text{av}} = \left\langle \hat{A} \right\rangle_{\{P_k, |\Psi_k\rangle\}} = \sum_{k=1}^{K} P_k \left\langle \Psi_k | \hat{A} | \Psi_k \right\rangle \tag{2.1}$$

where this $K \in \mathbb{N} \cup \{\infty\}$ *does not have to* be the same as dimension as the Hilbert space of the system. We can rewrite the equation 2.1 as:

$$\begin{aligned} A_{\text{av}} &= \sum_{k=1}^{K} P_k \left\langle \Psi_k | \hat{A} | \Psi_k \right\rangle \\ &= \sum_k \sum_n P_k \langle \Psi_k | n \rangle \left\langle n | \hat{A} | \Psi_k \right\rangle \\ &= \sum_k \sum_n P_k \left\langle n | \hat{A} | \Psi_k \right\rangle \langle \Psi_k | n \rangle \\ &= \sum_k p_k \operatorname{tr}(\hat{A} | \Psi_k \rangle \langle \Psi_k |) \\ &= \operatorname{tr}\left(\hat{A} \sum_k p_k | \Psi_k \rangle \langle \Psi_k | \right) \\ &= \operatorname{tr}(\hat{A} \hat{\rho}) \end{aligned}$$

We define $\hat{\rho} := \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ as the density operator representing the ensemble $\{p_k, |\Psi_k\rangle\}$

2.1.1 Properties

a) The density operator is a hermitian semi-definit operator: $\langle \hat{\rho} \rangle \geq 0$.

Proof:.

$$\langle \Psi | \hat{\rho} | \Psi \rangle = \sum_k p_k \left| \langle \Psi | k \rangle \right|^2 \geq 0 \forall \Psi \in \mathcal{H}$$

b) $\operatorname{tr} \hat{\rho} = 1$

c) $\hat{\rho}^2 \leq \hat{\rho}$ and $\hat{\rho}^2 = \hat{\rho} \iff \hat{\rho} = |\Psi_1\rangle\langle\Psi_1|$ called a pure state.

Example. Qubit density operator: In diagonal (spectral) form, we can write the density operator as:

$$\hat{\rho} = p_0 |\Psi_0\rangle\langle\Psi_0| + p_1 |\Psi_1\rangle\langle\Psi_1|$$

where $\langle\Psi_0|\Psi_1\rangle = 0$ and we can define the states Ψ_0, Ψ_1 as:

$$\Psi_0 = \cos\theta/2|0\rangle + \sin\theta/2 e^{i\varphi}|1\rangle$$
$$\Psi_1 = -\sin\theta/2 e^{-i\varphi}|0\rangle + \cos\theta/2|1\rangle$$

with $p_0 + p_1 = 1$, so we can write these as: $p_0 = \frac{1+r}{2}$, $p_1 = \frac{1-r}{2}$, so we can write the density operator as:

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\cdot\vec{\sigma}\right) \tag{2.2}$$

where $\vec{r}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a $3D$-vector and σ are the Pauli operators, and represent a point in the Bloch ball. For $r = 0$, the density operator is the identity operator and we have the maximal mixed state.

2.2 Mixing theorem

It was formulated by Hughstom et al. in PLA (1993)

Consider $\hat{\rho} = \sum_{k=1}^{\dim(\mathcal{H})} p_k|e_k\rangle\langle e_k|$, where $\langle e_k|e_\ell\rangle = \delta_{k,\ell}$ and $\hat{\eta} = \sum_{\ell=1}^{L} q_\ell|\phi_\ell\rangle\langle\phi_\ell|$. Then $\hat{\rho} = \hat{\eta}$ iff there exists an unitary matrix $V \in \mathrm{Mat}(\dim(\mathcal{L}) \times \dim(\mathcal{L}))$ such that

$$\sqrt{q_\ell}|\phi_\ell\rangle = \sum_{k=1}^{\dim\mathcal{H}} \sqrt{p_k}|e_k\rangle V_{kl} \forall l = 1, \cdots, \dim\mathcal{H} \tag{2.3}$$

2.3 Composite Systems

In quantum mechanics we use tennsor products to describe states of composite systems. Assume me have systems a, b and c with bases, then the wave function of such a composite system can be written as:

$$|\Psi_{ABC}\rangle = \sum_{k\ell m} a_{k\ell m}|a_k\rangle \otimes |b_\ell\rangle \otimes |c_m\rangle \tag{2.4}$$

where the dimension of the Hilbert space is given by

$$\dim \mathcal{H}_{ABC} = \dim \mathcal{H}_A \cdot \mathcal{H}_B \cdot \mathcal{H}_C \tag{2.5}$$

is the number of combinations of basis vectors of the subsystems. As a notation we usually write

$$|a_k\rangle \otimes |b_\ell\rangle \otimes |c_m\rangle \longrightarrow |a_k b_\ell c_m\rangle \tag{2.6}$$

As terminology: With N subsystems, we mean a N particle composite system.

Assume we have system of N qubits, then the dimension of the composite system

$$\mathcal{H}_{2D}^{\otimes N} = \bigotimes_{i=1}^{N} \mathcal{H}_{2D}$$

with dimension

$$\dim \mathcal{H}_{2D}^{\otimes N} = 2^N \tag{2.7}$$

we can write the basis as binanry strings $x \in \{0,1\}^N$ of length N that generate the Hilbert space. A general wave function can then be written as

$$|\Psi_{N\text{-Qubit}}\rangle = \sum_{x=1}^{2^N} c_x |x\rangle \tag{2.8}$$

2.3.1 Bipartite case

In the bipartite case we set $N = 2$ with subsystems A and B, an arbitrary pure state is described by

$$|\Psi^{AB}\rangle = \sum_{k=1}^{n_A} \sum_{\ell=1}^{n_B} a_{k\ell} |k\rangle \otimes |\ell\rangle \tag{2.9}$$

with $n_i = \dim \mathcal{H}_i$.

Theorem. It states, that:

$$|\Psi^{AB}\rangle = \sum_{m=1}^{\min\{n_A, n_B\}} \sqrt{d_m} |A_m\rangle \otimes |B_m\rangle \tag{2.10}$$

where $|A_m\rangle \in \mathcal{H}_A$ and $\langle A_m | A_n \rangle = \delta_{mn}$ for A and B and $\sum_m d_m = 1$.

For the proof we need singular value decomposition: Assume $a \in \mathrm{Mat}(n_A \times n_B)$ matrix. Then we can write this matrix a as

$$a = U \cdot d \cdots V$$

where $U \in U(n_A)$, $V \in U(n_B)$ unitary matrices and d a "diagonal" $n_A \times n_B$ matrix with the singular values $\sqrt{d_i}$ on the diagonal.

Proof:. We view $a_{k\ell}$ as a $n_A \times n_B$ matrix. Then using singular value decomposition we get

$$a_{k\ell} = \sum_{m=1}^{\min\{n_A,n_B\}} U_{km}\sqrt{d_m}Vm\ell$$

For a general bipartite wave function we get

$$\begin{aligned}
|\Psi^{AB}\rangle &= \sum_{k=1}^{n_A}\sum_{\ell=1}^{n_B}\sum_{m=1}^{\min\{n_A,n_B\}} U_{km}\sqrt{d_m}Vm\ell\,|k\rangle \otimes |\ell\rangle \\
&= \sum_m \sqrt{d_m}\sum_k U_{km}|k\rangle \otimes \sum_\ell Vm\ell|l\rangle \\
&= \sum_{m=1}^{\min\{n_A,n_B\}} \sqrt{d_m}|A_m\rangle \otimes |B_m\rangle
\end{aligned}$$

It remains to prove, that orthonormality of the basis.

$$\begin{aligned}
\langle A_m|A_n\rangle &= \sum_{k,k'}^{n_A} U^*_{km}U_{k'm}\,\langle k|k'\rangle \\
&= \sum_{k=1}^{n_A} U^\dagger_{mk}U_{kn} \\
&= (U^\dagger U)_{m,n} = \delta_{m,n}
\end{aligned}$$

There exists no generalisation to N partite system. If it existsed it would take the form: $\Psi^{AB\cdots X} = \sum_m \sqrt{d_m}|A_m\rangle \cdots |X_m\rangle$

Example. Assume we have a 3-state qubit, we can write:

$$\begin{aligned}
\Psi^{ABC} &= |000\rangle + a\,(|011\rangle + |101\rangle + |110\rangle) \\
&= |+++\rangle + |---\rangle = |GHZ\rangle
\end{aligned}$$

where $|\pm\rangle = |0\rangle \pm |1\rangle$.

In comparason a state:

$$|\Psi^{ABC}\rangle = |111\rangle + a\,(\cdots)$$

that connot be decomposed into a Schmidt form. A similar state is

$$|W\rangle := \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

that has no Schmidt decomposition.

2.4 Reduced denisty operator, partial trace

Assume we have a bipartite system with state $|\Psi^{AB}\rangle$. What does it mean to only have accoess to onu of the two subsystems, say A? Vhat does it do operationally. Suppose we measure observable O_A an A, then using Schmidt decomposition

$$\begin{aligned}
\left\langle \hat{O}_A \right\rangle_{\Psi^{AB}} &= \left\langle \Psi^{AB} | \hat{O}_A \otimes \hat{I}_B | \Psi^{AB} \right\rangle \\
&= \sum_{k\ell} \sqrt{d_k}\sqrt{d_\ell} \left\langle A_k | \hat{O}_A | A_\ell \right\rangle \left\langle B_k | \hat{I} | B_\ell \right\rangle \\
&= \sum_k d_k \left\langle A_k | \hat{O}_A | A_k \right\rangle \\
&= \mathrm{tr}\left(O_A \sum_k d_k |A_k\rangle\langle A_k| \right) \\
&= \mathrm{tr}\left(O_A \rho_B\right)
\end{aligned}$$

where ρ_A is the reduced density operator in spectral form. A reduced density operator can be computed as a partial trace:

$$\rho_A = \mathrm{tr}_A\, |\Psi^{AB}\rangle\langle\Psi^{AB}| = \sum_m \left\langle B_m | \Psi^{AB} \right\rangle \left\langle \Psi^{AB} | B_m \right\rangle \tag{2.11}$$

2.5 Completely positive maps (quantum channels)

Within the space of all possible states (Bloch sphere for a single qubit), we can have maps between different quantum states. If we restrict ourselves to

only using unitary transformation, that pure states can only stay pure states (called *closed system*) whereas completely positive maps that can transform a system more generally.

Definition. A completely positive map (CMP) $\rho \longrightarrow \mathcal{E}(\rho) \geq 0$ satisfies:

(a) $\mathrm{tr}[\mathcal{E}(\rho)]$ is the probability that $\mathcal{E}$ happens:

$$\Longrightarrow 0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1, \tag{2.12a}$$

with $\mathrm{tr}[\mathcal{E}(\rho)] = 1$ if the CPM is a completely positive trance perserving (CPTP) map.

(b) A CMP should be linear, so that: $\mathcal{E}\left(\sum_k p_k \rho_k\right) = \sum_k p_k \mathcal{E}(\rho_k)$.

(c) Let A, B be positive semi-definit operator, then the following inequality should hold

$$\mathcal{E}(A) \geq 0 \qquad \text{(positivity)} \tag{2.12b}$$

and, if B is an operator that acts on a langer Hilbert space, then

$$\mathcal{E} \otimes \mathcal{I}(B) \geq 0 \tag{2.12c}$$

for *any* extension of the system (complete positivity). This conditions guaranties that remote systems cannot influence the physical nature of $\mathcal{E}$

Definition. Transposition:

$$\left(|A\rangle\langle A^{\perp}|\right)^T = |A^{\perp}\rangle\langle A| \tag{2.13}$$

(partial transpostion)

Example. We have the state:

$$\begin{aligned}
|\Psi^{AB}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle - |20\rangle\right) \quad \longrightarrow \quad \rho_{AB}^{T_A} = |\Psi^{AB}\rangle\langle\Psi^A B|^{T_A} \\
&= \frac{1}{2}\left(|01\rangle - |10\rangle\right)\left(\langle 01| - \langle 10|\right)^{T_A} \\
&= \frac{1}{2}\left(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right)^{T_A} \\
&= \frac{1}{2}\left(|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| - |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\right) \\
&= \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & \\ 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

with eigenvalues $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}$

Theorem. $\mathcal{E}$ is a CPM $\iff$ $\exists\{E_k\}$ such that

$$\sum_k E_k^\dagger E_k \leq \hat{1} \tag{2.14}$$

where the equality is given for CPTP maps in terms of which

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \tag{2.15}$$

called the Kraus representation or operator sum representation and the E_k are called Kraus operators.

Example. "Decay of an atom" Let $|e\rangle$ be the excited state of a atom and $|g\rangle$ be its ground state. Then the Kraus operators are

$$\begin{cases} E_0 &= |g\rangle\langle g| + \sqrt{1-\gamma}|e\rangle\langle e| \\ E_1 &= \sqrt{\gamma}|g\rangle\langle e| \end{cases}$$

with $\gamma \in [0,1]$. So

$$\begin{aligned}
\rho &\longrightarrow E_0\rho E_0^\dagger + E_1\rho E_1^\dagger \\
&= \left(|g\rangle\langle g| + \sqrt{1-\gamma}|e\rangle\langle e|\right)\rho\left(|g\rangle\langle g| + \sqrt{1-\gamma}|e\rangle\langle e|\right) + (\gamma|g\rangle\langle e|)\,\rho|e\rangle\langle g| \\
&\underset{\gamma\to 1}{\longrightarrow} |g\rangle\,\langle g|\rho|g\rangle\,\langle rho| + |g\rangle\,\langle e|\rho|e\rangle\,\langle rho| \\
&= |g\rangle\langle g|\underbrace{(\langle g|\rho|g\rangle + \langle e|\rho|e\rangle)}_{\mathrm{tr}\,\rho=1} \\
&= |g\rangle\langle g|
\end{aligned}$$

a decay with $\gamma = 1 - e^{-\lambda t}$.

2.5.1 Stine spring dilation

Assume we have a system s in an environement e and we want to determine the effect of the environement on the system with the time evolution operator U_{se}, then we can decompose the state:

$$\rho_{se} = \rho \otimes |e_0\rangle\langle e_0| \tag{2.16}$$

With a unitary map:

$$\rho_{se} \longmapsto U_{se}\rho \otimes |e_0\rangle\langle e_0|U_{se}^\dagger$$

The map of the system gives us:

$$\begin{aligned}
\rho \longmapsto \mathrm{tr}_e\left(U_{se}\rho_{se}U_{se}^\dagger\right) &= \sum_{n=0}^{\dim\mathcal{H}_e-1} \langle e_n|U_{se}\rho\otimes|e_0\rangle\left\langle e_0|U_{se}^\dagger|e_n\right\rangle \\
&= \sum_n \langle e_n|U_{se}|e_0\rangle\,\rho\left\langle e_0|U_{se}^\dagger|e_n\right\rangle \\
&= \sum_n E_n\rho E_n^\dagger
\end{aligned}$$

To prove that $(\langle e_n|U_{se}|e_0\rangle)^\dagger = \left\langle e_n|U_{se}^\dagger|e_0\right\rangle$ use:

$$\sum_{k\ell,pq} |s_k\rangle|s_\ell\rangle(U_{se})_{k\ell,pq}\langle s_q|\langle p|$$

This is a CPTP map:

$$\begin{aligned}\sum_n E_n^\dagger E_n &= \sum_n \left(\langle e_n|U_{se}|e_0\rangle\right)^\dagger \langle e_n|U_{se}|e_0\rangle \\ &= \sum_n \left\langle e_0|U_{se}^\dagger|e_n\right\rangle \langle e_n|U_{se}|e_0\rangle \\ &= \left\langle e_0|U_{se}^\dagger U_{se}|e_0\right\rangle \\ &= \hat{I}_s \left\langle e_0|\hat{I}_e|e_0\right\rangle = \hat{I}_s\end{aligned}$$

One can use Stinespring dilation to prove that the Kraus representation is not unique. Using a basis transformation: $|e_n\rangle \longmapsto |f_n\rangle = \sum_m |e_m\rangle U_{mn}^\dagger$ with a unitary basis transformation U. As U only is unitary transformation should not affect the map. Kraus operators $\{F_n = \langle f_n|U_{se}|e_0\rangle\}$ define the same map as $\{E_n = \langle e_n|U_{se}|e_0\rangle\}$, one finds

$$F_n = \sum_m U_{mn} E_m$$

2.6 Application of the CPM formalism:

2.6.1 Measurement

A generalized measurement is a CPTP map with the following interpretation:

(i) Each measurement outcome m_k is associated with a Kraus operator E_k.

(ii) The **Born rule**: the probability $P(m_k|\rho)$ is given by:

$$P(m_k|\rho) = \mathrm{tr}(E_k^\dagger E_k \rho) \tag{2.17}$$

(iii) Update the probability distribution (i.e. the state) usnig the following rule:

$$\rho \underset{m_k}{\longrightarrow} E_k \rho E_k^\dagger \tag{2.18}$$

The special case called von Neumann or projective measurment we have $E_k = |k\rangle\langle k|$ and the measurement with probability yields:

$$\begin{aligned}P(m_k|\rho) &= \mathrm{tr}(|k\rangle\langle k||k\rangle\langle k|\rho) \\ &= \mathrm{tr}(|k\rangle\langle k|\rho) \\ &= \langle k|\rho|k\rangle\end{aligned}$$

if $\rho = |\Psi\rangle\langle\Psi|$, then:

$$\begin{aligned} \implies P(m_k|\rho) &= \langle k|\Psi\rangle \langle\Psi|k\rangle \\ &= |\langle k|\Psi\rangle|^2 \end{aligned}$$

Note. The sum of the probabilty of all outcomes:

$$\begin{aligned} \sum_k P(m_k|\rho) &= \sum_k \mathrm{tr}(E_k^\dagger E_k \rho) \\ &= \mathrm{tr}\left(\sum_k \underbrace{E_k^k E_k}_{\hat{I}} \rho\right) \\ &= \mathrm{tr}\,\rho = 1 \end{aligned}$$

Notation. We define $\{\Pi_k := E_k^\dagger E_k\}$ as a positive operator valued measure (POVM).

Theorem. Non-orthogonal states can not be distinguished in a general measurement

Proof:. Assume $|\Phi\rangle$ and $|\Psi\rangle$ are non-orthogonal and *can* be distinguished. If this is possible then there exists Π_1 and P_2 such that $\langle\Psi|\Pi_1|\Psi\rangle = 1$ and $\langle\Phi|\Pi_2|\Phi\rangle = 1$. Because probabilities must add up to 1, we must have $\langle\Psi|\Pi_2|\Psi\rangle = 0$ where $\Pi_k \geq 0$, so $\sqrt{\Pi_k} \geq 0$. We get:

$$0 = \langle\Psi|\Pi_2|\Psi\rangle = \left\| \sqrt{\Pi_2}|\Psi\rangle \right\|^2$$

so $\sqrt{\Pi_2}|\Psi\rangle = 0$ (0-vector).
We can write $|\Phi\rangle = a|\Psi^\perp\rangle + b|\Psi\rangle$ where $b \neq 0$ by assumption. This implies also that $|a|^2 + |b|^2 = 1$, so $|a|^2 < 1$. Thus

$$\begin{aligned} \sqrt{\Pi_2}|\Phi\rangle &= a\sqrt{\Pi_2}|\Psi^\perp\rangle + b\sqrt{\Pi_2}|\Psi\rangle \\ &= a\sqrt{\Pi_2}|\Psi^\perp\rangle \end{aligned}$$

So

$$\begin{cases} \left\|\sqrt{\Pi_2}|\Psi^\perp\rangle\right\| &= \langle\Phi|\Pi_2|\Phi\rangle = 1 \\ \left\|\sqrt{\Pi_2}|\Psi^\perp\rangle\right\| &= |a|^2 \left\langle\Psi^\perp|\Pi_2|\Psi^2\right\rangle \leq |a|^2 < 1 \end{cases}$$

2.6.2 Lindblad equation

So far we have only been looking at discreet maps through CPMs. How would the dynamics of an open system be described, from a time ρ_0 to a time T ρ_T. We can model such case throught a series of small CPMs for timesteps $\Delta t \longrightarrow 0$, where the environemental state is measured at each timesteps t_i, and the system always can be described as a product state.
We can map the density operator as a sequence of CPM (Markonvian approximation)

$$\rho \longmapsto \mathcal{E}_{k\delta t,(k-1)\delta t} \circ \cdots \circ \mathcal{E}_{2\delta t,\delta t} \circ \mathcal{E}_{\delta t,0}(\rho_0)$$

We do the simplifying assumption, that $\mathcal{E}_{t+\delta t,\delta t} \equiv \mathcal{E}_{\delta t}$ (which implies that the Hamiltonian of the system is time independent). We consider the map

$$\begin{aligned} \rho_t \longmapsto \rho_{t+\delta t} &= \sum_k E_k(\delta t)\rho_t E_k^\dagger(\delta t) \\ &= \mathcal{E}_{\delta t}(\rho_t) \\ \mathcal{E}_{\delta t} &\underset{\delta t\to 0}{\longrightarrow} \mathcal{I} \end{aligned}$$

with $\mathcal{I}$ being the identity map.
This allows us to make an ansatz:

$$E_0(\delta t) = 1 - i(H - iG)\delta t$$

With H and G hermitian operators.

$$E_{k>0}(\delta t) = \sqrt{\delta t} L_k$$

We assume that $\mathcal{E}_{\delta t}$ is a CPTP since we do not exctact information of the system during the evolution, in other words $\operatorname{tr}\rho_t = 1$ for all t. This implies for our ansatz:

$$\begin{aligned} \hat{I} &= \sum_k E_k^\dagger(\delta t)E_k(\delta t) \\ &= E_0^\dagger(\delta t)E_0(\delta t) + \sum_{k=0} E_k^\dagger(\delta t)E_k(\delta t) \\ &= \hat{I} + 2G\delta t + \delta t \sum_{k=1} L_k^\dagger L_k + O(\delta t^2) \end{aligned}$$

So we get:

$$G = \frac{1}{2}\sum_{k=1} L_k^\dagger L_k \tag{2.19}$$

This gives us the Lindblad equation:

$$\frac{\rho_{t+\delta t} - \rho_t}{\delta t} = -i[H, \rho_t] + \sum_k \left(L_k \rho L_k \dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right)$$
$$\delta t \longrightarrow 0 \Longrightarrow \dot{\rho}_t = -i[H, \rho_t] + \sum_k \left(L_k \rho_t L_k \dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho_t\} \right)$$

where $\{A, B\} = AB + BA$ is the anticommutator, and H should be interpreted as the Hamiltonian, thus the first term of the equation is called the "Schrödinger term", the L_k are called Lindblad operators and the anticommutator term give non hermitian dynamics.

Example. Assume a qubit system with $L = \sqrt{\gamma}\sigma_z$ with Hamiltonian $H = \omega\sigma_z$, would give a density operator:

$$\rho_t = \frac{1}{2}(\hat{I} + \vec{r}_t \cdot \vec{\sigma})$$

with the solution in the rotating frame.

$$\dot{\vec{r}} = -2\gamma(x_t, y_t, 0)$$
$$\vec{r} = (x_0 e^{-2\gamma t}, y_0 e^{-2\gamma t}, z_0)$$

which for $t \longrightarrow \infty$, we end up with a state lying on the z-axis, and the exponential term in the x and y coordinates are called decoherence terms.