Uppsala University

Quantum Information

Lecture Notes

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March 28, 2023

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Chapter 1

Introduction

Quantum Theory

One can say, that quantum theory is a probability theory where events are assotiated with complex numbers α called probability amplitudes. There are three rules for these probability amplitudes:

(a) Born rule: $P = |\alpha|^2$ gives the probability of the event α

(b) For a sequence of events with amplitude $\alpha_1, \cdots, \alpha_n$, then the amplitude of the whole sequence is: $\prod_{i=1}^{n} \alpha_{N-i+1}$

(c) For two interconnected events with amplitude α_1 and α_2, then the probability of one event occuring if the other has occured, then the amplitude of this event is $\alpha = \alpha_1 + \alpha_2$. The probability of such event is given by:

$$P = |\alpha_1|^2 + |\alpha_2|^2 + 2\,\mathrm{Re}(\alpha_1^* \alpha_2) \tag{1.1}$$

where the mixed term is called the interference term. This is where the difference between classical information and quantum information theory lies.

Qubits

A qubit is a quantum mechanical system that can be described as a two dimensional Hilbert space. In general, this can be the polarisation of a photon,

spin of an electron, of a neutron and so on. We can view a qubit as a abstract sense as $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$. An arbitrary qubit state as $\alpha_0|0\rangle + \alpha_1|1\rangle$ with $\alpha_i \in \mathbb{C}$, with the normalisation condition ${\alpha_0}^1 + |\alpha_1|^2 = 1$. From the normalisation we can write the probability amplitudes as

$$\begin{cases} \alpha_0 = \cos \theta/2 e^{i\varphi_0} \\ \alpha_1 = \sin \theta/2 e^{i\varphi_1} \end{cases} \tag{1.2}$$

This allows us to write a general state $|\phi\rangle$ as: #

$$\begin{aligned} |\phi\rangle &= \cos \theta/2 e^{i\varphi_0}|0\rangle + \sin \theta/2 e^{i\varphi_1}|0\rangle \\ &= e^{i\varphi_0}\left(\cos \theta/2|0\rangle + \sin \theta/2 e^{i(\varphi_1 - \varphi_0)}|0\rangle\right) \\ &\sim \cos \theta/2|0\rangle + \sin \theta/2 e^{i\varphi}|1\rangle \end{aligned}$$

This can be represented in a sphere called the Bloch sphere

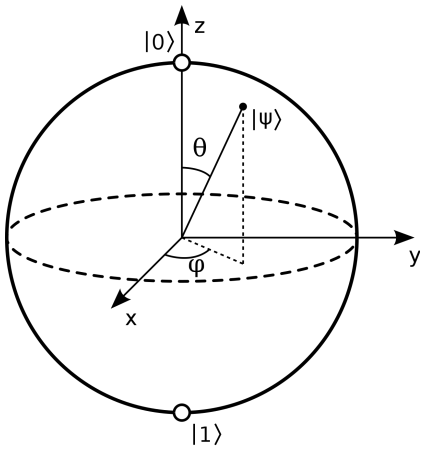

Fig. 1.1. Representation of a Bloch sphere

Chapter 2

Quantum Mechanics

2.1 Density operator

Suppose we have a preparation of quantum states (such as sending photons through a polariser), that are not perfect with some impurity. Say that each possible *pure* state $|\Psi_1\rangle, \cdots |\Psi_K\rangle$ with probabilities $P_1, \cdots P_K$. Suppose we measure an observable A throught the operator $\hat{A}$ on this set of states, and we want to calculate the expectation value of this observable:

$$A_{\text{av}} = \left\langle \hat{A} \right\rangle_{\{P_k, |\Psi_k\rangle\}} = \sum_{k=1}^{K} P_k \left\langle \Psi_k | \hat{A} | \Psi_k \right\rangle \tag{2.1}$$

where this $K \in \mathbb{N} \cup \{\infty\}$ *does not have to* be the same as dimension as the Hilbert space of the system. We can rewrite the equation 2.1 as:

$$\begin{aligned}
A_{\text{av}} &= \sum_{k=1}^{K} P_k \left\langle \Psi_k | \hat{A} | \Psi_k \right\rangle \\
&= \sum_k \sum_n P_k \langle \Psi_k | n \rangle \left\langle n | \hat{A} | \Psi_k \right\rangle \\
&= \sum_k \sum_n P_k \left\langle n | \hat{A} | \Psi_k \right\rangle \langle \Psi_k | n \rangle \\
&= \sum_k p_k \operatorname{tr}(\hat{A} | \Psi_k \rangle \langle \Psi_k |) \\
&= \operatorname{tr}\left(\hat{A} \sum_k p_k | \Psi_k \rangle \langle \Psi_k | \right) \\
&= \operatorname{tr}(\hat{A}\hat{\rho})
\end{aligned}$$

We define $\hat{\rho} := \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ as the density operator representing the ensemble $\{p_k, |\Psi_k\rangle\}$

2.1.1 Properties

a) The density operator is a hermitian semi-definit operator: $\langle\hat{\rho}\rangle \geq 0$.
Proof:

$$\langle \Psi | \hat{\rho} | \Psi \rangle = \sum_k p_k \left| \langle \Psi | k \rangle \right|^2 \geq 0 \forall \Psi \in \mathcal{H}$$

b) $\operatorname{tr} \hat{\rho} = 1$

c) $\hat{\rho}^2 \leq \hat{\rho}$ and $\hat{\rho}^2 = \hat{\rho} \iff \hat{\rho} = |\Psi_1\rangle\langle\Psi_1|$ called a pure state.

2.1.2 Example: Qubit density operator

In diagonal (spectral) form, we can write the density operator as:

$$\hat{\rho} = p_0 |\Psi_0\rangle\langle\Psi_0| + p_1 |\Psi_1\rangle\langle\Psi_1|$$

where $\langle\Psi_0|\Psi_1\rangle = 0$ and we can define the states Ψ_0, Ψ_1 as:

$$\begin{aligned}
\Psi_0 &= \cos \theta/2 |0\rangle + \sin \theta/2 e^{i\varphi} |1\rangle \\
\Psi_1 &= -\sin \theta/2 e^{-i\varphi} |0\rangle + \cos \theta/2 |1\rangle
\end{aligned}$$

with $p_0 + p_1 = 1$, so we can write these as: $p_0 = \frac{1+r}{2}$, $p_1 = \frac{1-r}{2}$, so we can write the density operator as:

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\cdot\vec{\sigma}\right) \tag{2.2}$$

where $\vec{r}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a $3D$-vector and σ are the Pauli operators, and represent a point in the Bloch ball. For $r = 0$, the density operator is the identity operator and we have the maximal mixed state.

2.2 Mixing theorem

It was formulated by Hughstom et al. in PLA (1993)

Consider $\hat{\rho} = \sum_{k=1}^{\dim(\mathcal{H})} p_k|e_k\rangle\langle e_k|$, where $\langle e_k|e_\ell\rangle = \delta_{k,\ell}$ and $\hat{\eta} = \sum_{\ell=1}^{L} q_\ell|\phi_\ell\rangle\langle\phi_\ell|$. Then $\hat{\rho} = \hat{\eta}$ iff there exists an unitary matrix $V \in \mathrm{Mat}(\dim(\mathcal{L}) \times \dim(\mathcal{L}))$ such that

$$\sqrt{q_\ell}|\phi_\ell\rangle = \sum_{k=1}^{\dim\mathcal{H}} \sqrt{p_k}|e_k\rangle V_{kl} \forall l = 1, \cdots, \dim\mathcal{H} \tag{2.3}$$

2.3 Composite Systems

In quantum mechanics we use tennsor products to describe states of composite systems. Assume me have systems a, b and c with bases, then the wave function of such a composite system can be written as:

$$|\Psi_{ABC}\rangle = \sum_{k\ell m} a_{k\ell m}|a_k\rangle \otimes |b_\ell\rangle \otimes |c_m\rangle \tag{2.4}$$

where the dimension of the Hilbert space is given by

$$\dim\mathcal{H}_{ABC} = \dim\mathcal{H}_A \cdot \mathcal{H}_B \cdot \mathcal{H}_C \tag{2.5}$$

is the number of combinations of basis vectors of the subsystems. As a notation we usually write

$$|a_k\rangle \otimes |b_\ell\rangle \otimes |c_m\rangle \longrightarrow |a_k b_\ell c_m\rangle \tag{2.6}$$

As terminology: With N subsystems, we mean a N particle composite system.

Assume we have system of N qubits, then the dimension of the composite system

$$\mathcal{H}_{2D}^{\otimes N} = \bigotimes_{i=1}^{N} \mathcal{H}_{2D}$$

with dimension

$$\dim \mathcal{H}_{2D}^{\otimes N} = 2^N \tag{2.7}$$

we can write the basis as binanry strings $x \in \{0,1\}^N$ of length N that generate the Hilbert space. A general wave function can then be written as

$$|\Psi_{N\text{-Qubit}}\rangle = \sum_{x=1}^{2^N} c_x |x\rangle \tag{2.8}$$

2.3.1 Bipartite case

In the bipartite case we set $N = 2$ with subsystems A and B, an arbitrary pure state is described by

$$|\Psi^{AB}\rangle = \sum_{k=1}^{n_A} \sum_{\ell=1}^{n_B} a_{k\ell} |k\rangle \otimes |\ell\rangle \tag{2.9}$$

with $n_i = \dim \mathcal{H}_i$.

Theorem. It states, that:

$$|\Psi^{AB}\rangle = \sum_{m=1}^{\min\{n_A, n_B\}} \sqrt{d_m} |A_m\rangle \otimes |B_m\rangle \tag{2.10}$$

where $|A_m\rangle \in \mathcal{H}_A$ and $\langle A_m | A_n \rangle = \delta_{mn}$ for A and B and $\sum_m d_m = 1$.

For the proof we need singular value decomposition: Assume $a \in \mathrm{Mat}(n_A \times n_B)$ matrix. Then we can write this matrix a as

$$a = U \cdot d \cdots V$$

where $U \in U(n_A)$, $V \in U(n_B)$ unitary matrices and d a "diagonal" $n_A \times n_B$ matrix with the singular values $\sqrt{d_i}$ on the diagonal.

Proof:. We view $a_{k\ell}$ as a $n_A \times n_B$ matrix. Then using singular value decomposition we get

$$a_{k\ell} = \sum_{m=1}^{\min\{n_A,n_B\}} U_{km}\sqrt{d_m}Vm\ell$$

For a general bipartite wave function we get

$$\begin{aligned}
|\Psi^{AB}\rangle &= \sum_{k=1}^{n_A}\sum_{\ell=1}^{n_B}\sum_{m=1}^{\min\{n_A,n_B\}} U_{km}\sqrt{d_m}Vm\ell\,|k\rangle \otimes |\ell\rangle \\
&= \sum_m \sqrt{d_m}\sum_k U_{km}|k\rangle \otimes \sum_\ell Vm\ell|l\rangle \\
&= \sum_{m=1}^{\min\{n_A,n_B\}} \sqrt{d_m}|A_m\rangle \otimes |B_m\rangle
\end{aligned}$$

It remains to prove, that orthonormality of the basis.

$$\begin{aligned}
\langle A_m|A_n\rangle &= \sum_{k,k'}^{n_A} U^*_{km}U_{k'm}\,\langle k|k'\rangle \\
&= \sum_{k=1}^{n_A} U^\dagger_{mk}U_{kn} \\
&= (U^\dagger U)_{m,n} = \delta_{m,n}
\end{aligned}$$

There exists no generalisation to N partite system. If it existsed it would take the form: $\Psi^{AB\cdots X} = \sum_m \sqrt{d_m}|A_m\rangle \cdots |X_m\rangle$

Example. Assume we have a 3-state qubit, we can write:

$$\begin{aligned}
\Psi^{ABC} &= |000\rangle + a\,(|011\rangle + |101\rangle + |110\rangle) \\
&= |+++\rangle + |---\rangle = |GHZ\rangle
\end{aligned}$$

where $|\pm\rangle = |0\rangle \pm |1\rangle$.

In comparason a state:

$$|\Psi^{ABC}\rangle = |111\rangle + a\,(\cdots)$$

that connot be decomposed into a Schmidt form. A similar state is

$$|W\rangle := \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

that has no Schmidt decomposition.

2.4 Reduced denisty operator, partial trace

Assume we have a bipartite system with state $|\Psi^{AB}\rangle$. What does it mean to only have accoess to onu of the two subsystems, say A? Vhat does it do operationally. Suppose we measure observable O_A an A, then using Schmidt decomposition

$$\begin{aligned}
\left\langle \hat{O}_A \right\rangle_{\Psi^{AB}} &= \left\langle \Psi^{AB} | \hat{O}_A \otimes \hat{I}_B | \Psi^{AB} \right\rangle \\
&= \sum_{k\ell} \sqrt{d_k}\sqrt{d_\ell} \left\langle A_k | \hat{O}_A | A_\ell \right\rangle \left\langle B_k | \hat{I} | B_\ell \right\rangle \\
&= \sum_k d_k \left\langle A_k | \hat{O}_A | A_k \right\rangle \\
&= \mathrm{tr}\left(O_A \sum_k d_k |A_k\rangle\langle A_k| \right) \\
&= \mathrm{tr}\left(O_A \rho_B\right)
\end{aligned}$$

where ρ_A is the reduced density operator in spectral form. A reduced density operator can be computed as a partial trace:

$$\rho_A = \mathrm{tr}_A\, |\Psi^{AB}\rangle\langle\Psi^{AB}| = \sum_m \left\langle B_m | \Psi^{AB} \right\rangle \left\langle \Psi^{AB} | B_m \right\rangle \tag{2.11}$$

2.5 Completely positive maps (quantum channels)

Within the space of all possible states (Bloch sphere for a single qubit), we can have maps between different quantum states. If we restrict ourselves to

only using unitary transformation, that pure states can only stay pure states (called *closed system*) whereas completely positive maps that can transform a system more generally.

Definition. A completely positive map (CMP) $\rho \longrightarrow \mathcal{E}(\rho) \geq 0$ satisfies:

(a) $\mathrm{tr}[\mathcal{E}(\rho)]$ is the probability that $\mathcal{E}$ happens:

$$\Longrightarrow 0 \leq \mathrm{tr}[\mathcal{E}(\rho)] \leq 1, \tag{2.12a}$$

with $\mathrm{tr}[\mathcal{E}(\rho)] = 1$ if the CPM is a completely positive trance perserving (CPTP) map.

(b) A CMP should be linear, so that: $\mathcal{E}\left(\sum_k p_k \rho_k\right) = \sum_k p_k \mathcal{E}(\rho_k)$.

(c) Let A, B be positive semi-definit operator, then the following inequality should hold

$$\mathcal{E}(A) \geq 0 \qquad \text{(positivity)} \tag{2.12b}$$

and, if B is an operator that acts on a langer Hilbert space, then

$$\mathcal{E} \otimes \mathcal{I}(B) \geq 0 \tag{2.12c}$$

for *any* extension of the system (complete positivity). This conditions guaranties that remote systems cannot influence the physical nature of $\mathcal{E}$

Definition. Transposition:

$$\left(|A\rangle\langle A^{\perp}|\right)^T = |A^{\perp}\rangle\langle A| \tag{2.13}$$

(partial transpostion)

Example. We have the state:

$$
\begin{aligned}
|\Psi^{AB}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |20\rangle\right) \quad \longrightarrow \quad \rho_{AB}^{T_A} &= |\Psi^{AB}\rangle\langle\Psi^{A}B|^{T_A} \\
&= \frac{1}{2}\left(|01\rangle - |10\rangle\right)\left(\langle 01| - \langle 10|\right)^{T_A} \\
&= \frac{1}{2}\left(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|\right)^{T_A} \\
&= \frac{1}{2}\left(|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| - |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|\right) \\
&= \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & \\ 0 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

with eigenvalues $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\}$