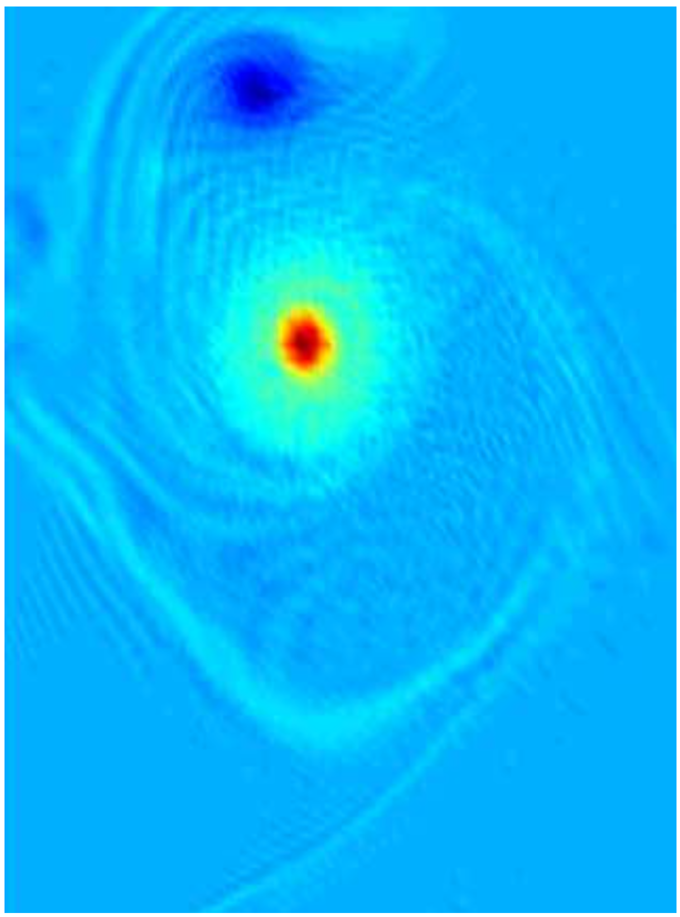
Mastered Learning Objectives from Partial Differential Equations

MATH 4900, Fall 2020 with Prof. Emma Zbarsky



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1 Mastery Check Planning

Standard	Suggested Date	Mastery Date	Writeup Completed	Productive Failure?
NM: Order of Error	Wk 2	9/18/20	9/26/20	☐
NM: Explicit Difference	Wk 3			☐
MC: Linear, constant speed	Wk 3	10/2/20	11/29/20	☑
MC: Linear, polynomial speed	Wk 4	10/2/20	11/30/20	☐
MC: Nonlinear	Wk 4			☐
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SV: Heat Equation	Wk 10	11/19/20	12/3/20	☐
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2 Numerical Methods of Solving Partial Differential Equations

Numerical Methods aim to solve PDE's by successively iterating a numerical scheme to generate points in a mesh that represent the solution to a PDE. Each point in the mesh represents the specific solution for a certain point in space and time. The numerical methods we use are formed from approximations of the derivatives in the PDE, like Euler's forward, backward, and centered difference formulas. We can plug in these approximations of the derivatives to solve for the equation that gives the value at the next time step and same point in space. Additionally, we need to choose a value for Δt and Δx - how much the method moves forward in space and time for each iteration. These are usually small, since we want an accurate approximation of the solution.

2.1 Order of Error

The Order of Error in a finite difference approximation tells us how fast the error term, that comes from approximating a function, changes when we increase or decease the step size.

The mastery problem I was given: Approximate $u'(x)$ with $u(x)$, $u(x-3h)$, and $u(x-2h)$. What is the optimal order of error?

To calculate the order of the error, we first use a Taylor Series to expand each function used to approximate the derivative except $u(x)$

$$\begin{aligned} u(x-3h) &= u(x) - u'(x)(3h) + \frac{u''(x)(3h)^2}{2!} - \frac{u'''(x)(3h)^3}{3!} + \ldots \\ u(x-2h) &= u(x) - u'(x)(2h) + \frac{u''(x)(2h)^2}{2!} - \frac{u'''(x)(2h)^3}{3!} + \ldots \end{aligned}$$

We want to solve for $u'(x)$ so lets begin by putting coefficients in front of each term in the expansions. This will allow us to combine the expansions and helps us cancel higher order terms. You need as many coefficients as you have expansions. We have 2 so we need A and B.
We now have:

$$\begin{aligned} Au(x-3h) &= Au(x) - Au'(x)(3h) + A\frac{u''(x)(3h)^2}{2!} - A\frac{u'''(x)(3h)^3}{3!} + \ldots \\ Bu(x-2h) &= Bu(x) - Bu'(x)(2h) + B\frac{u''(x)(2h)^2}{2!} - B\frac{u'''(x)(2h)^3}{3!} + \ldots \end{aligned}$$

Solve for A and B. We want the coefficients of each term in their expansions to add to 0. Don't include the term were solving for. In this case, don't include $u'(x)$.

$$\begin{aligned} u(x) &: A + B = 0 \\ u''(x) &: 9A + 4B = 0 \end{aligned}$$

The first statement is required to solve for $u'(x)$. In this case, A = -B. Let A = 1 and B = -1 Plug A and B back into the Taylor Series expansions:

$$\begin{aligned} u(x-3h) &= u(x) - u'(x)(3h) + \frac{u''(x)(3h)^2}{2!} - \frac{u'''(x)(3h)^3}{3!} + \ldots \\ -u(x-2h) &= -u(x) + u'(x)(2h) - \frac{u''(x)(2h)^2}{2!} + \frac{u'''(x)(2h)^3}{3!} + \ldots \end{aligned}$$

Combine terms by adding or subtracting and solve for $u'(x)$. Subtract to make sure $u'(x)$ doesn't cancel and then find the first term after $u'(x)$ that doesn't cancel - that becomes the error term

$$u(x-3h) - -u(x-2h) = -u'(x)(h) + O(h^2)$$
$$u'(x) = -\frac{u(x-3h) - -u(x-2h)}{h} + \frac{O(h^2)}{h}$$

The order of error is h

2.2 Neumann Stability

2.3 The CFL Condition

When using using a finite different method to approximate the solution to a PDE, we need to make sure that our scheme is accurate and remains stable as we go to later times. The CFL Condition is something we can use to help see if our scheme will be stable ahead of time. However, it is important to note that the CFL condition is necessary but not sufficient for stability. If the choices for Δt and Δx in our approximation don't pass the CFL condition, then it is not stable, but we can't guarantee stability even when we make good choices for Δt and Δx.

Define the domain of dependence as an area on the t, x plane, based on our scheme, which the characteristic curves of the solution will pass through to accurately approximate the solution. All curves will pass through the point $u_{j+1,m}$—the time step of the solution we are interested in—as well. By considering the domain of dependence, we will be able to find the CFL condition for our scheme.

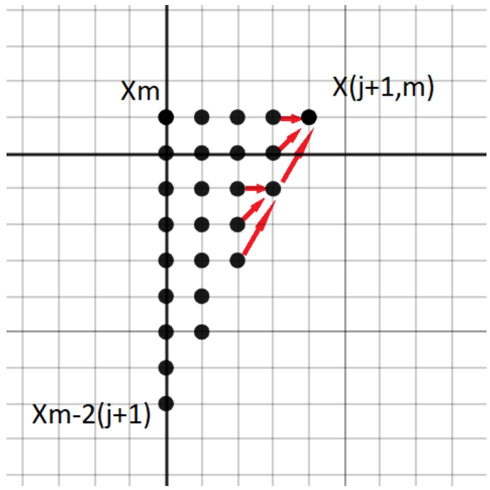
The mastery problem I was given:

Find the CFL condition for the first order wave equation (transport equation) $u_t + cu_x = 0$ given $u(0, x) = f(x)$ and using approximation:

$$u_{j+1,m} = (\sigma^2 - 1)u_{j,m} + 2u_{j,m} + \sigma u_{j,m-1} + (1-\sigma)u_{j,m-2}$$

We have a constant wave speed c, which tells us the level curves of our solution will be lines.

Let's set up a picture to make it easier to visualize the scheme and the domain of dependence:



$u_{j+1,m}$ depends on the point to the left of it, the point left and down, and the point left and 2 down. I only included arrows for 2 points to clearly demonstrate the pattern. The mesh is not exactly this size all the time. It could be bigger or smaller depending on which $x_{j+1,m}$ time step we are interested in.

From the graph we can see that:

$$x_{m-2(j+1)} \leq \xi \leq x_m$$

We know that the level curves of the solution, ξ, that give us accurate results as time moves forward need to pass through the point of interest $x_{j+1,m}$ and be inside the domain of dependence.

This can be rewritten so we can solve for what the CFL condition is:

$$x_m - 2(j+1)\Delta x \leq x_m - c(j+1)\Delta t \leq x_m + (j+1)\Delta t$$

We get from x_m to $x_m - 2(j+1)$ by (j+1) steps of whatever step size Δx we choose. ξ can be written in terms of how time is represented in the scheme - so t (from x - ct) changes to (j+1) steps of whatever step size Δt we choose.

Let's solve for the CFL condition:
Subtract x_m from all sides:

$$-2(j+1)\Delta x \leq -c(j+1)\Delta t \leq (j+1)\Delta t$$

Divide by $(j+1)\Delta x$:

$$-2 \quad \leq \quad -c\frac{\Delta t}{\Delta x} \quad \leq 1$$

c is still the wave speed constant and slope of the level curve. C can be anything, and to pass the CFL condition, we just need need to adjust Δt and Δx to make the value of $-c\dfrac{\Delta t}{\Delta x}$ in between -2 and 1.

In cases where c is a non-constant wave speed, the level curves become curves and not lines. To pass the CFL condition with non-constant wave speed, the characteristic curves passing through $x_{j+1,m}$ need to stay inside the domain of dependence the whole time they pass through those time steps. Therefore, these curves follow the same idea, that they are using only information relevant to the solution, as lines do because lines automatically stay inside the domain of dependence for all relevant time steps.

2.4 Implementing an Explicit Difference Method

2.5 Implementing an Implicit Difference Method

3 The Method of Characteristics

The method of characteristics is used to solve first order linear and non-linear PDEs of the form $u_t + cu_x = d$. C can take the form of a constant or a polynomial and the equation could be homogeneous or non-homogeneous. The method of characteristics can be used to solve first order nonlinear PDEs as well, where c is a function of u. Using the method of characteristics requires an initial condition $u(0, x) = f(x)$ and that we consider what is happening with the level curves of our solution on the t, x plane. We will see what happens when c is a constant, a polynomial, and non-linear.

3.1 Solving $u_t + cu_x + du = g(x)$ with $u(0, x) = f(x)$

First, let's consider what we know about $u(t, x)$ and $u(0, x) = f(x)$.

The level curves of the solution u(t,x) on the t, x plane are curves of constant value for u(t,x). No matter which point we choose on the curve, we get the same value for u(t,x). A characteristic variable ξ can be introduced that we choose to define as $\xi = x - ct$. ξ represents a level curve of the solution on the t, x plane where c is a constant slope and ξ is the x-intercept.

Since we know the initial condition $u(0, x) = f(x)$, we know what the solution looks like at $t = 0$. If every point on our level curve gives the same value for $u(t, x)$, we can go to where $t = 0$ on our level curve, which is equal to ξ, and know the value of the function. ξ will be exactly f at time 0. Therefore, $f(\xi) = f(x)$ and we can plug in ξ for x in the initial condition to get $u(t, x)$

The mastery problem I was given: Find $u(t, x)$ given $u_t - 5u_x = 12$ with $u(0, x) = \dfrac{\arctan(x)}{(x^2 + 1)}$

First, solve the homogeneous case for u(t,x). That means to solve $u_t - 5u_x = 0$.

The level sets of the solution look like:

$$h = u(t, x(t)) = \text{constant}$$

Take the normal derivative of both sides.

$$\frac{dh}{dt} = 0 = \frac{du}{dt}\frac{dt}{dt} + \frac{dx}{dt}\frac{du}{dx}$$

This looks like the given equation when:

$$\begin{aligned} c &= \frac{dx}{dt} = -5 \\ \xi &= x - ct \\ \xi &= x + 5t \\ u(t, x) &= \frac{\arctan(x + 5t)}{((x + 5t)^2 + 1)} \end{aligned}$$

Now, let's see what we can add on to $u(t,x)$ by considering the constant on the right side of the PDE.

$$\begin{aligned}
\frac{dh}{dt} &= 12 \\
h &= 12t + g(\xi)
\end{aligned}$$

We know the form of $g(\xi)$. It is the homogeneous solution we just calculated above.

$$u(t,x) = \frac{\arctan(x+5t)}{((x+5t)^2+1)} + 12t$$

3.2 Solving the linear transport equation with polynomial coefficients

We can solve linear transport equations with polynomial coefficients using the same idea as when they had constant coefficients. However, because the wave speed c is not a constant, we can't find ξ right away and will need to solve for it with separation of variables and integration.

The mastery problem I was given: Find $u(t,x)$ given $u_t + (x^2+1)\sin(t)u_x = 0$ with $u(0,x) = e^{-x^2}\cos(x)$

The level sets of the solution look like:

$$h(t) = u(t, x(t)) = \text{constant}$$

Take the normal derivative of both sides:

$$\frac{dh}{dt} = \frac{du}{dt}\frac{dt}{dt} + \frac{dx}{dt}\frac{du}{dx} = 0$$

This looks like the given equation when:

$$\frac{dx}{dt} = (x^2+1)\sin(t)$$

Separation of variables:

$$\frac{dx}{(x^2+1)} = \sin(t)dt$$

Integrate and solve for ξ:

$$\begin{aligned}
\int_\xi^x \frac{1}{(x^2+1)}dx &= \int_0^t \sin(t)dt \\
\arctan(x)\Big|_\xi^x &= -\cos(t)\Big|_0^t \\
\tan^{-1}(x) - \tan^{-1}(\xi) &= -\cos(t) + 1 \\
\tan^{-1}(\xi) &= \tan^{-1}(x) + \cos(t) - 1 \\
\xi &= \tan(\tan^{-1}(x) + \cos(t) - 1)
\end{aligned}$$

Plug back in to the initial condition $u(0,x) = e^{-x^2}\cos(x)$:

$$u(t,x) = e^{-(\tan(\tan^{-1}(x)+\cos(t)-1))^2}\cos(\tan(\tan^{-1}(x)+\cos(t)-1))$$

In this case, there is nothing more to add on at the end because the given function was homogeneous.

3.3 Solving first order nonlinear PDEs using the method of characteristics

3.4 Shocks

3.5 Rarefaction Waves

4 Fourier Series

Fourier series approximate functions with an infinite series of sines and cosines. They can be used to approximate real functions and complex functions. A complex Fourier Series can be written in terms of exponentials or sines and cosines because of their relationship for complex functions but will still be an infinite series.
A Fourier Series will only match the function it's approximating on the interval its Fourier Coefficients were integrated on because Fourier Series are periodic on intervals of $2L$, where $-L$ to L is the interval the function is being approximated on.

4.1 Real Fourier Series

A real Fourier Series has the form:

$$f(x) \quad \approx \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx)$$

Where a_k and b_k are the L_2 inner product of the function $f(x)$ and $\cos(kx)$ and $\sin(kx)$:

$$\begin{aligned} a_k &= < f(x), \cos(kx) >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx, \qquad k \geq 0 \quad \textit{integers} \\ b_k &= < f(x), \sin(kx) >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)dx, \qquad k \geq 1 \quad \textit{integers} \end{aligned}$$

We can extend a Fourier Series to an interval $-L$ to L with the form:

$$f(x) \quad \approx \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\left(\frac{k\pi x}{L}\right) + b_k\sin\left(\frac{k\pi x}{L}\right)$$

where:

$$\begin{aligned} a_k &= < f(x), \cos(kx) >= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx, \qquad k \geq 0 \quad \textit{integers} \\ b_k &= < f(x), \sin(kx) >= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx, \qquad k \geq 1 \quad \textit{integers} \end{aligned}$$

The mastery problem I was given:

Find the Fourier Series for $f(x) = |x|$ on [-3,3]

Since we are on $[-L, L]$, the Fourier Series will have the form:

$$f(x) \quad \approx \quad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)$$

Solve for a_k:

$$\begin{aligned} a_k &= \frac{1}{3}\int_{-3}^{3} |x| \cos\left(\frac{k\pi x}{3}\right) dx \\ a_k &= \frac{6(\pi k \sin(\pi k) + \cos(\pi k) - 1)}{\pi^2 k^2} \end{aligned}$$

a_k can be simplified to the following because $sin(\pi k)$ is always 0 and $cos(\pi k)$ is $(-1)^k$:

$$a_k \quad = \quad \frac{6((-1)^k - 1)}{\pi^2 k^2}$$

Solve for b_k:

$$\begin{aligned} b_k &= \frac{1}{3}\int_{-3}^{3} |x| \sin\left(\frac{k\pi x}{3}\right) dx \\ b_k &= 0 \end{aligned}$$

$b_k = 0$ because the inner product of an even and an odd function is always 0.

Now that we have a_k and b_k did we make any assumptions that are easy to overlook?

When solving for a_k we assumed that k was not equal to 0. If k = 0 then we are left with:

$$\begin{aligned} a_0 &= \frac{1}{3}\int_{-3}^{3} |x| dx \\ a_0 &= 3 \end{aligned}$$

The Fourier Series for $f(x) = |x|$ on [-3,3] has the form:

$$f(x) \quad \approx \quad \frac{3}{2} + \sum_{k=1}^{\infty} \frac{6((-1)^k - 1)}{\pi^2 k^2} \cos\left(\frac{k\pi x}{3}\right)$$

4.2 Complex Fourier Series

A Complex Fourier Series has the form:

$$f(x) \quad \approx \quad \sum_{k=1}^{\infty} c_k e^{ikx}$$

where c_k is the Hermitian inner product of $f(x)$ and e^{ikx}. This means we need the conjugate of e^{ikx} when doing the integral for the inner product.

$$c_k \quad = \quad < f(x), e^{ikx} >= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

The mastery problem I was given:

Find the Complex Fourier Series for $f(x) = sin^3(x) + cos^2(5x)$ on $[-\pi, \pi]$
Solve for c_k:

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(sin^3(x) + cos^2(5x))e^{-ikx}dx \\ c_k &= \left(\frac{1}{2\pi}\right)\left(\frac{2(k^6 - 60k^4 - 6ik^3 + 509k^2 + 600ik - 450)\sin(\pi k)}{k(k^6 - 110k^4 + 1009k^2 - 900)}\right) \end{aligned}$$

c_k will equal 0 for all k's except when the denominator is equal to 0. This is because $sin(\pi k)$ is always equal to 0.

If we solve for k when the denominator is equal to 0, we can find the values for c_k that need to be computed separately.

$k(k^6 - 110k^4 + 1009k^2 - 900) = 0$ when k = -10, -3, -1, 0, 1

Solve for c_k when k = -10,-3,-1,0,1 - don't forget to divide by 2π for each integral:

$$\begin{aligned} c_{-10} &= \frac{1}{4} \\ c_{-3} &= \frac{-i}{8} \\ c_{-1} &= \frac{3i}{8} \\ c_0 &= \pi \\ c_1 &= \frac{-3i}{8} \end{aligned}$$

The complex Fourier Series for $f(x) = sin^3(x) + cos^2(5x)$ on $[-\pi, \pi]$ looks like:

$$\begin{aligned} f(x) &\approx \sum_{k=1}^{\infty} c_{-10}e^{ikx} + c_{-3}e^{ikx} + c_{-1}e^{ikx} + c_0e^{ikx} + c_1e^{ikx} \\ f(x) &\approx \sum_{k=1}^{\infty} \frac{1}{4}e^{ikx} + \frac{-i}{8}e^{ikx} + \frac{3i}{8}e^{ikx} + \pi e^{ikx} + \frac{-3i}{8}e^{ikx} \\ f(x) &\approx \sum_{k=1}^{\infty} \frac{1}{4}e^{ikx} + \frac{-i}{8}e^{ikx} + \pi e^{ikx} \end{aligned}$$

4.3 Convergence of Fourier Series

4.4 Integrability and Differentiability of Fourier Series

4.5 Boundary Conditions

5 Separation of Variables

Separation of Variables assumes the solution to a PDE can take the form of the product of two functions w(t) and v(x). Therefore, u(t,x) = w(t)v(x). We will use separation of variables to solve heat equations, wave equations, and Laplace equations and see what the solutions for w(t), v(x), and u(t,x) look like in each case. We can also use this idea to see what the equilibrium solution for a u(t,x) looks like.

5.1 Equilibrium behavior of a solution

Equilibrium behavior for a solution u(t,x) is what is happening with the solution at $t = \infty$. Equilibrium solutions are independent of t at the end so we will only have a function of x.
The Mastery problem I was given:

Find the equilibrium solution for the heat equation $u_t = .005u_{xx}$ given the following initial and boundary conditions:

$$\begin{aligned} u(0,x) &= f(x) \\ u(t,0) &= 6 \\ u(t,5) &= -3 \end{aligned}$$

Note: An equilibrium solution for a heat equation would be a temperature. For it to be an equilibrium solution, the behavior would have to approach an actual value.

If an equilibrium solution exists, it will take the form:

$$u^\star(x) = \lim_{t\to\infty} u(t,x)$$

Since our heat equation has both u_t and u_{xx}, lets find the PDE that $u^\star(x)$ satisfies:

$$\frac{\partial u^\star}{\partial t} = 0$$

because $u^\star$ doesn't depend on t at all, and

$$\frac{\partial u^\star}{\partial x} = (u^\star)''$$

because $u^\star(x)$ is a function of x.

Plug in to heat equation to get the form of it:

$$0 = .005(u^\star)''$$

Tells us that $u^\star(x)$ looks like:

$$u^\star(x) = Ax + B$$

$u^\star(x)$ looks like this because we want the second derivative of $u^\star$ to equal 0 so $u^\star(x)$ needs to have this form.

We can find A and B using initial conditions:

We know:

$$\begin{aligned} u^\star(0) &= 6 \\ u^\star(5) &= -3 \end{aligned}$$

Solve for A and B:

$$\begin{aligned} A(0)+B &= 6 \\ B &= 6 \\ A(5)+6 &= -3 \\ A &= -\frac{9}{5} \end{aligned}$$

The equilibrium solution for the given heat equation looks like:

$$u^\star(x) = -\frac{9}{5}(x) + 6$$

Note: If we the equilibrium solution to give us an actual value for a certain x, it limits what the 2nd order diff eq's can be to only ones where $u^\star$ works out nicely. The equilibrium solution could be a cycle or something but we are not interested in when that would be the case.

5.2 Solving the 1D heat equation

The heat equation is $u_t = \gamma u_{xx}$

Assume the solution looks like $u(t,x) = w(t)v(x)$

We found that:

$$\begin{aligned} \frac{\partial u}{\partial t} &= w'(t)v(x) \\ \frac{\partial^2 u}{\partial x^2} &= \gamma w(t)v''(x) \\ w'(t)v(x) &= \gamma w(t)v''(x) \\ \frac{w'(t)}{w(t)} &= \gamma\frac{v''(x)}{v(x)} = \text{constant} = -\lambda \end{aligned}$$

This is constant because the only thing that can be both a function of t a function of x is a constant. We don't have to call it -λ, it could have been λ if we wanted but it would change the way that the different cases for λ look.

We now have a system or first order normal differential equations that we can use to find the solution.

$$\begin{aligned} w'(t) &= -\lambda w(t) \\ v''(x) &= \frac{-\lambda}{\gamma}v(x) \end{aligned}$$

We know that $u(t,x) = w(t)v(x)$ so we are interested in cases where both $w(t)$ and $v(x)$ are not equal 0 because if either $w(t) = 0$ or $v(x) = 0$, then we get the trivial solution $u(t,x) = 0$.

The mastery problem I was given:

Find $u(t,x)$ for $u_t = .3u_{xx}$ given the following intial and boundary conditions:

$$\begin{aligned} u(0,x) &= x^2 - 8x \\ u_x(t,0) &= 0 \\ u_x(t,4) &= 0 \end{aligned}$$

Note: $\gamma = .3$ - that's gamma
Note: This problem has Neumann (not Dirichlet) boundary conditions because they prescribe the derivative (not the function value) at the boundary. When doing the 3 cases for λ we need to solve $v'(0) = 0$ and $v'(0) = 4$ not $v(0) = 0$ and $v(0) = 4$

We know the solutions for $w(t) = Ce^{-\lambda t}$

We know the solutions for $v(x)$ will fall into 3 cases depending on if $\lambda > 0$, $\lambda < 0$, or $\lambda = 0$,
Case 1: $\lambda = 0$

$$\begin{aligned} v(x) &= Ax + B \\ v'(x) &= A \end{aligned}$$

Use 1st boundary condition:

$$\begin{aligned} v'(0) &= 0 = A \\ A &= 0 \end{aligned}$$

Use 2nd boundary condition:

$$\begin{aligned} v'(4) &= 0 = A \\ A &= 0 \end{aligned}$$

Tells us that A = 0 and B is unconstrained.
Case 2: $\lambda > 0$ This will mean $\omega = \sqrt{\frac{\lambda}{\gamma}}$

$$\begin{aligned} v(x) &= A\cos(\omega x) + B\sin(\omega x) \\ v'(x) &= -A\omega\sin(\omega x) + B\omega\cos(\omega x) \end{aligned}$$

Use 1st boundary condition:

$$\begin{aligned} v'(0) &= 0 = B\omega \\ B &= 0 \end{aligned}$$

Use 2nd boundary condition: B = 0 so it's not in $v'(x)$ anymore.

$$v'(4) = 0 = -A\sin(\omega 4)$$

$-A\sin(\omega 4)$ equals 0 when $\omega = \frac{k\pi}{4}$
Case 3: $\lambda < 0$ This will mean $\omega = \sqrt{\frac{-\lambda}{\gamma}}$

$$\begin{aligned} v(x) &= A\cosh(\omega x) + B\sinh(\omega x) \\ v'(x) &= B\omega\cosh(\omega x) + A\omega\sinh(\omega x) \end{aligned}$$

Use 1st boundary condition:
Note: $\sinh(0) = 0$ and $\cosh(0) = 1$

$$\begin{aligned} v'(0) &= 0 = B\omega \\ B &= 0 \end{aligned}$$

Use 2nd boundary condition: B = 0 so it's not in $v'(x)$ anymore.

$$v'(4) = 0 = -A\sinh(\omega 4)$$

Either:

$$\begin{aligned} A &= 0 \\ \textit{or} & \\ \omega &= 0 \\ \textit{or} & \\ \sinh(\omega 4) &= 0 \end{aligned}$$

We know ω is positive so it can't equal 0

$\sinh(\omega 4)$ equals 0 when evaluated at 0. However, we know that ω is positive so this can't happen.

So A = 0 which makes this case trivial because both A = 0 and B = 0.

Put everything back together:
$v(x) = B$ with $\lambda = 0$

$v(x) = A\cos\left(\frac{k\pi x}{L}\right)$ with $\lambda > 0$. This is because we found that B = 0 when we solved and that $v(x)$ still equals $A\cos(\omega x) + B\sin(\omega x)$.

We have just found a whole set of solutions for $v(x)$: $v_0(x) = B$ and $v_k(x) = A_k\cos\left(\frac{k\pi x}{L}\right)$

w(t)=$Ce^{-\lambda t}$:
w(t) = constant with $\lambda = 0$
w(t) =$Ce^{\frac{-k^2\pi^2\gamma}{L^2}t}$ with $\lambda > 0$
Note:

$$\begin{aligned} \omega = \frac{k\pi}{L} &= \sqrt{\frac{\lambda}{\gamma}} \\ \lambda &= \frac{k^2\pi^2\gamma}{L^2} \end{aligned}$$

$u(t,x) = w(t)v(x)$
$u(t,x) = a_0 + \sum_{k=1}^{\infty} a_k e^{\frac{-k^2\pi^2\gamma}{L^2}t}\cos\left(\frac{k\pi x}{4}\right)$

Replaced B with a_0 and the C*A inside the sum with a_k to make it look more like a Fourier Series.

We can still use $u(0,x) = x^2 - 8x$ as piece of information we haven't used.

$u(0,x) = x^2 - 8x = a_0 + \sum_{k=1}^{\infty} a_k\cos\left(\frac{k\pi x}{4}\right)$

This looks like a Fourier Cosine Series for f(x) on [0,4]. We want a Cosine Series on [-L,L]. So use an even extension (because cosine is an even function) to get a_0 and a_k:

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx \\
a_k &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx \\
\text{where :} & \\
L &= 4 \\
f(x) &= x^2 - 8x
\end{aligned}
$$

$a_0 = -\dfrac{32}{3}$

$a_k = -\dfrac{32((\pi^2 k^2 + 2)sin(k\pi) - 2\pi k)}{\pi^3 k^3}$

$a_k = \dfrac{64\pi k}{\pi^3 k^3}$ because $sin(k\pi) = 0$

$a_k = \dfrac{64}{\pi^2 k^2}$

$u(t,x) = -\dfrac{32}{3} + \sum_{k=1}^{\infty}\left(\dfrac{64}{\pi^2 k^2}\right) e^{\dfrac{-k^2\pi^2.3}{4^2}t} \cos\left(\dfrac{k\pi x}{4}\right)$

5.3 Solving the Laplace equation

5.4 Solving the 1D wave equation

5.5 Solving the 1D wave equation using d'Alembert's formula

References

References