Inductor and Inductance-II

Phy 108 course

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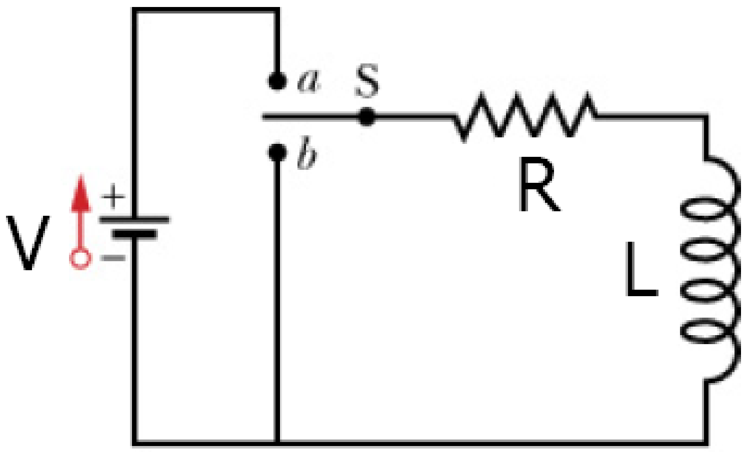
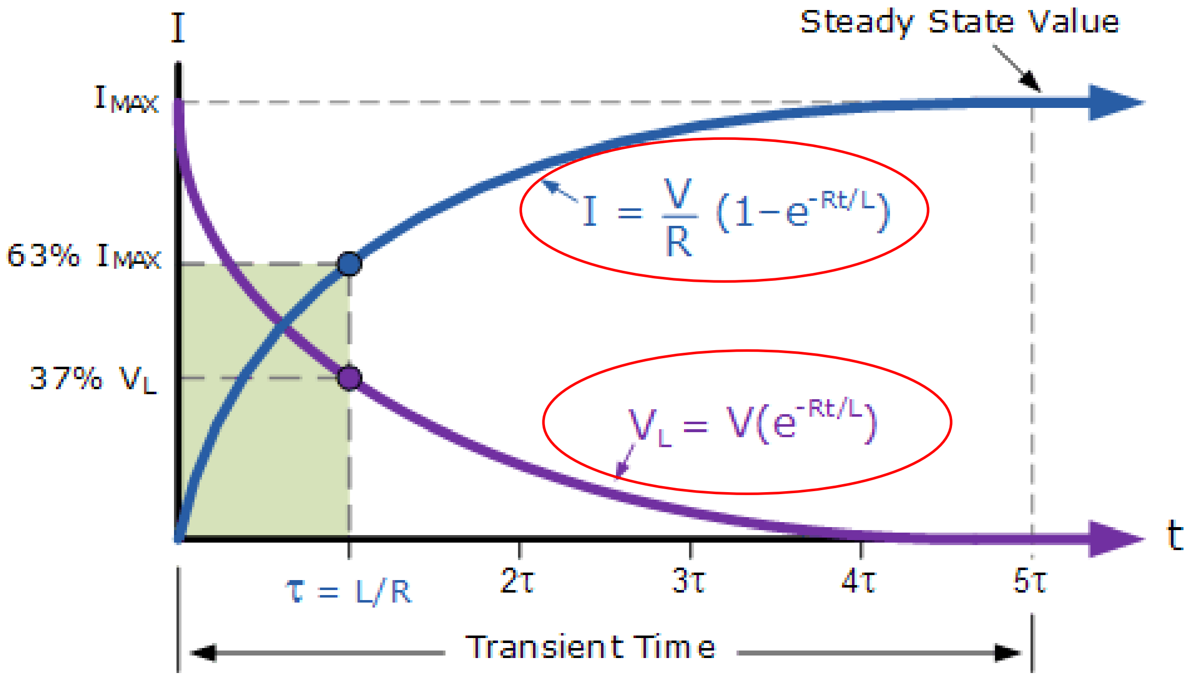
RL circuit



The induced emf ε is directly proportional to $-\frac{di}{dt}$, or the slope of the curve. Hence, while at its greatest immediately after the switches are thrown, the induced emf decreases to zero with time as the current approaches its final value of ε/R. The circuit then becomes equivalent to a resistor connected across a source of emf.



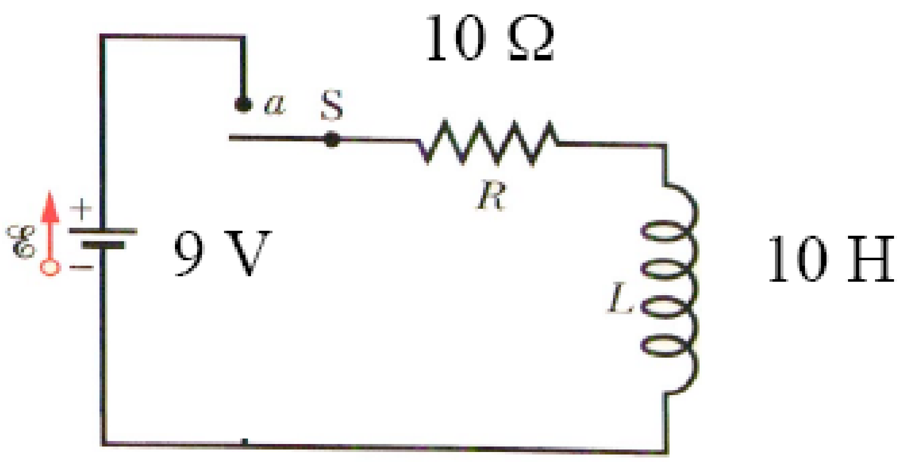
Example

Immediately after the switch is closed, what is the potential difference across the inductor?

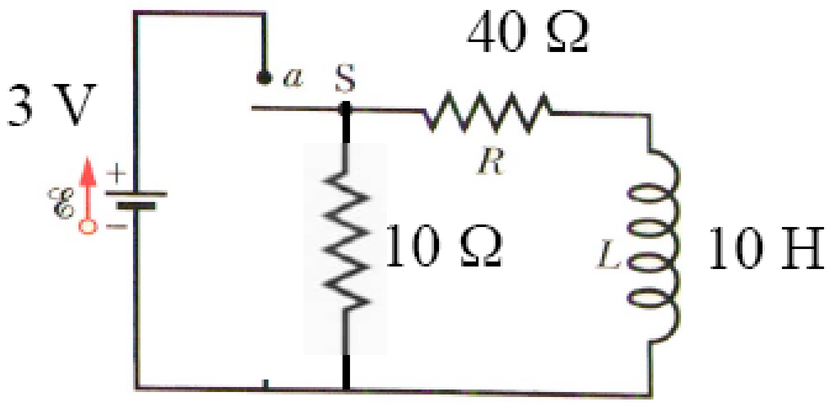
(a) 0 V

(b) 9 V

(c) 0.9 V



Example



- Immediately after the switch is closed, what is the current i through the 10 Ω resistor?

(a) 0.375 A

(b) 0.3 A

(c) 0

- Long after the switch has been closed, what is the current in the 40Ω resistor?

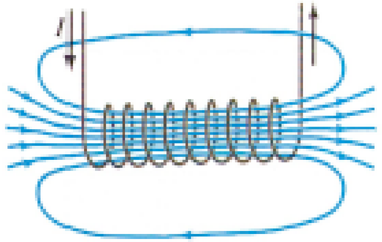
(a) 0.375 A

(b) 0.3 A

(c) 0.075 A

Capacitor and Inductor

Inductors are with respect to the magnetic field what capacitors are with respect to the electric field. They "pack a lot of field in a small region". Also, the higher the current, the higher the magnetic field they produce.



Capacitance → how much **potential** for a given charge: **Q=CV**

Inductance → how much **magnetic flux** for a given current: **$\Phi=Li$**

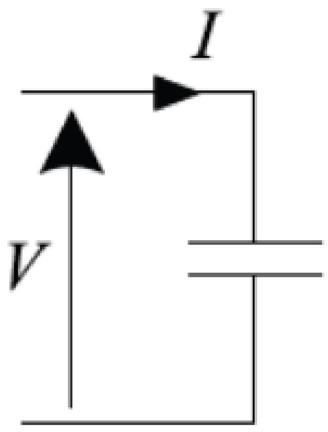
Capacitor and Inductor

Capacitor: $i = C\, dv / dt$

- For the voltage to change abruptly $dv / dt = \infty \Rightarrow i = \infty$.

This never happens so ...

- **The voltage across a capacitor never changes instantaneously.**
- Informal version: A capacitor “tries” to keep its voltage constant.

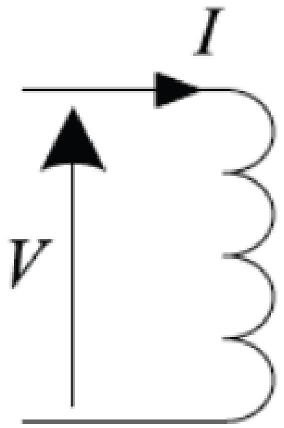


Inductor: $v = L di / dt$

- For the current to change abruptly $di / dt = \infty \Rightarrow v = \infty$.

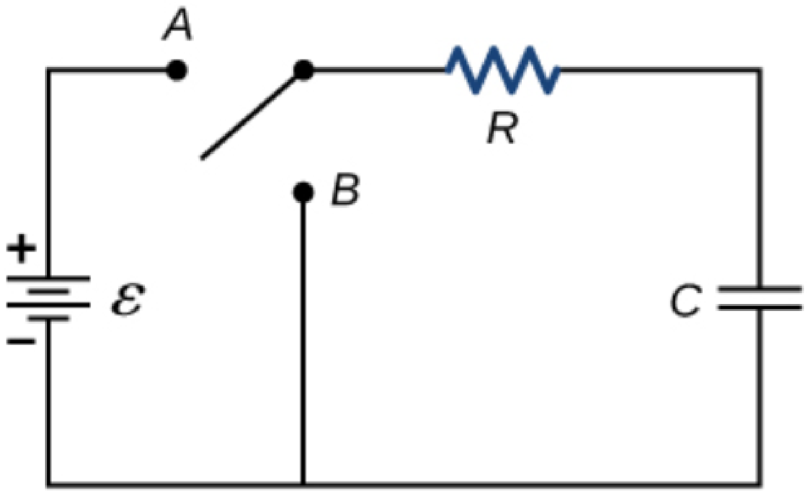
This never happens so ...

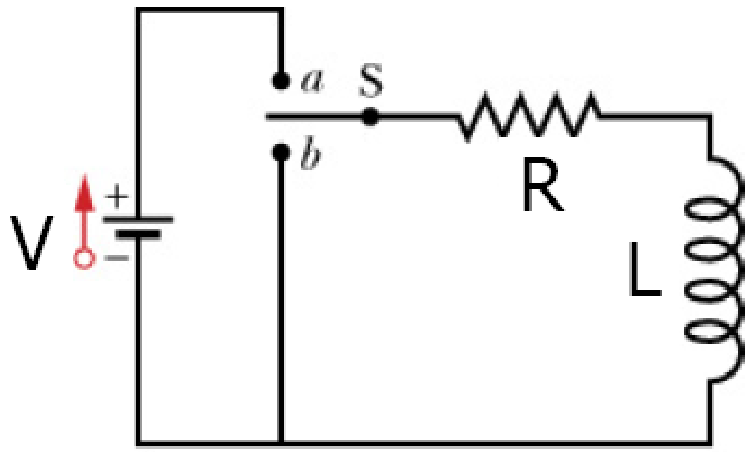
- **The current through an inductor never changes instantaneously.**
- Informal version: An inductor “tries” to keep its current constant.



Capacitor tries to keep its voltage constant. Inductor tries to keep its current constant.

RC circuit and RL circuit





In an RC circuit, while charging, Q = CV and the loop rule mean:

- charge increases from 0 to CE
- current decreases from E/R to 0
- voltage across capacitor increases from 0 to E

In an RL circuit, while "charging" (rising current), emf = L*di*/dt and the loop rule mean:

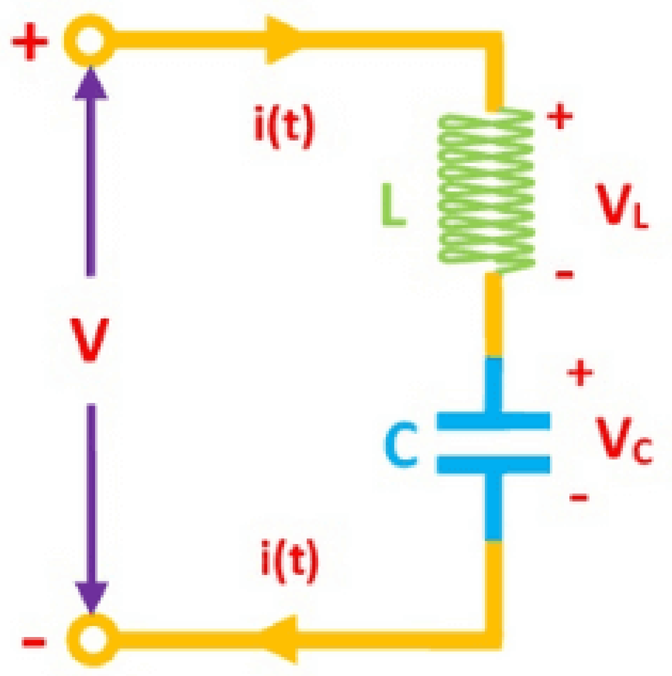
- magnetic field increases from 0 to B
- current increases from 0 to E/R
- voltage across inductor decreases from –E to 0

LC circuit

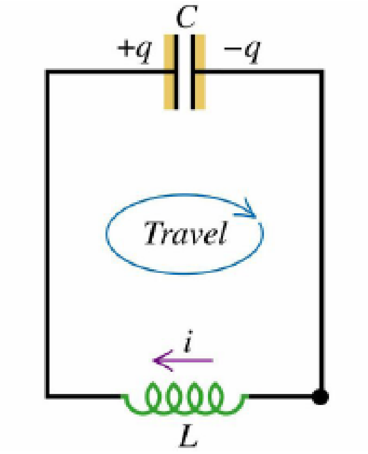
Suppose that we are now given a fully charged capacitor and an inductor that are hooked together in a circuit

Since the capacitor is fully charged there is a potential difference across it given by $\boldsymbol{Vc = Q / C}$

The capacitor will begin to discharge as soon as the switch is closed



L-C Circuit



We apply Kirchoff's Law to this circuit

$$-L\frac{di}{dt}-\frac{q}{C}=0$$

Remembering that $i = {}^{dq}\!/_{dt}$

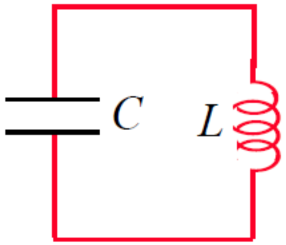
We then have that $\frac{di}{dt}=\frac{d^2q}{dt^2}$

The circuit equation then becomes $\frac{d^2q}{dt^2}+\frac{1}{LC}q=0$

LC Oscillations

→Work out equation for LC circuit (loop rule)

$$-\frac{q}{C} - L\frac{di}{dt} = 0$$



→Rewrite using i = dq/dt

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \Rightarrow \frac{d^2q}{dt^2} + \omega^2 q = 0$$

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

- ω (angular frequency) has dimensions of 1/t

→Identical to equation of mass on spring

$$m\frac{d^2x}{dt^2} + kx = 0 \Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

→Solution is same as mass on spring ⇒ oscillations

$$\boxed{q = q_{\max}\cos(\omega t + \theta)} \qquad \omega = \sqrt{\frac{k}{m}}$$

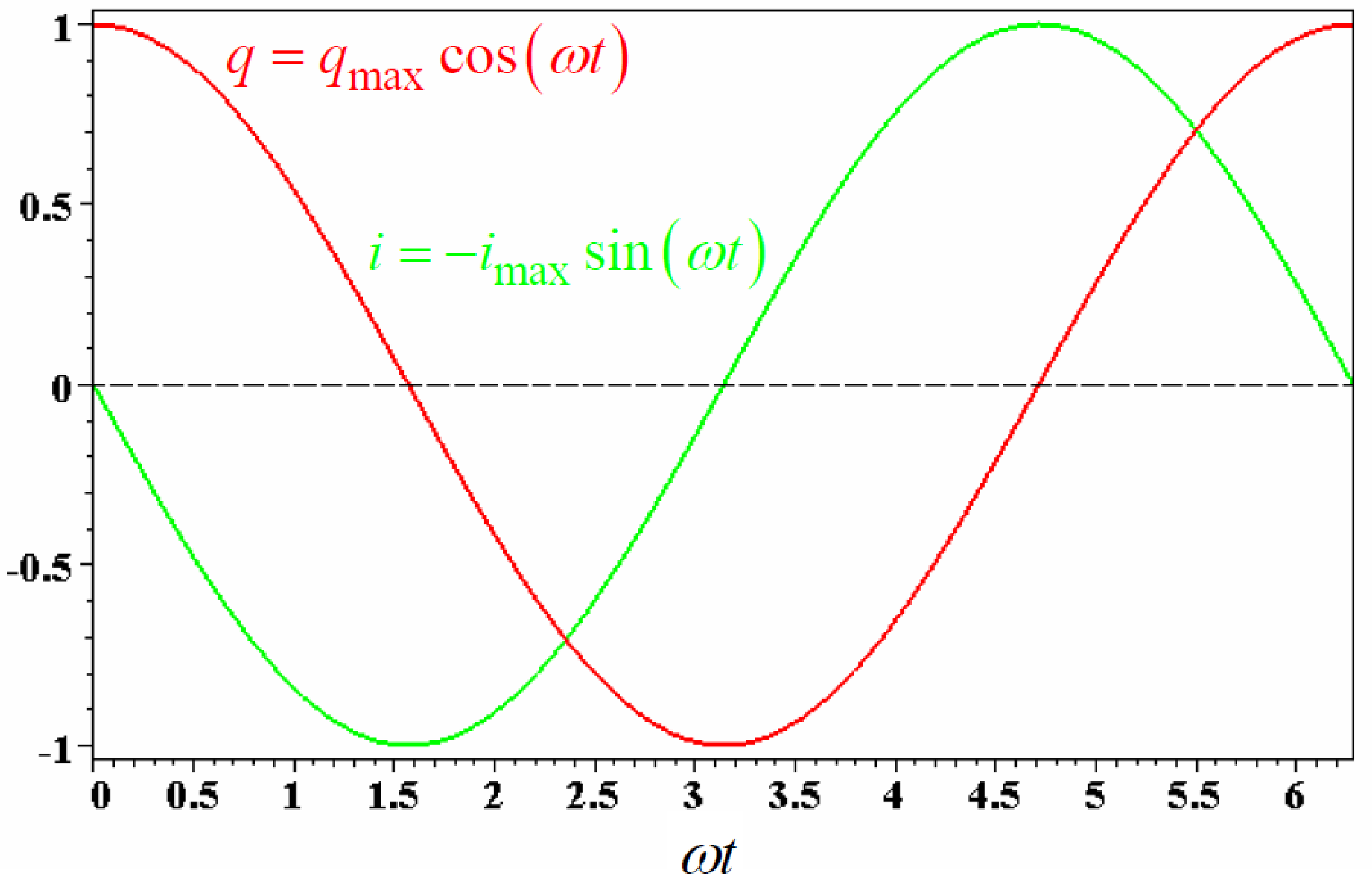
- q_{max} is the maximum charge on capacitor
- θ is an unknown phase (depends on initial conditions)

→Calculate current: i = dq/dt

$$\boxed{i = -\omega q_{\max}\sin(\omega t + \theta) = -i_{\max}\sin(\omega t + \theta)}$$

→Thus both charge and current oscillate

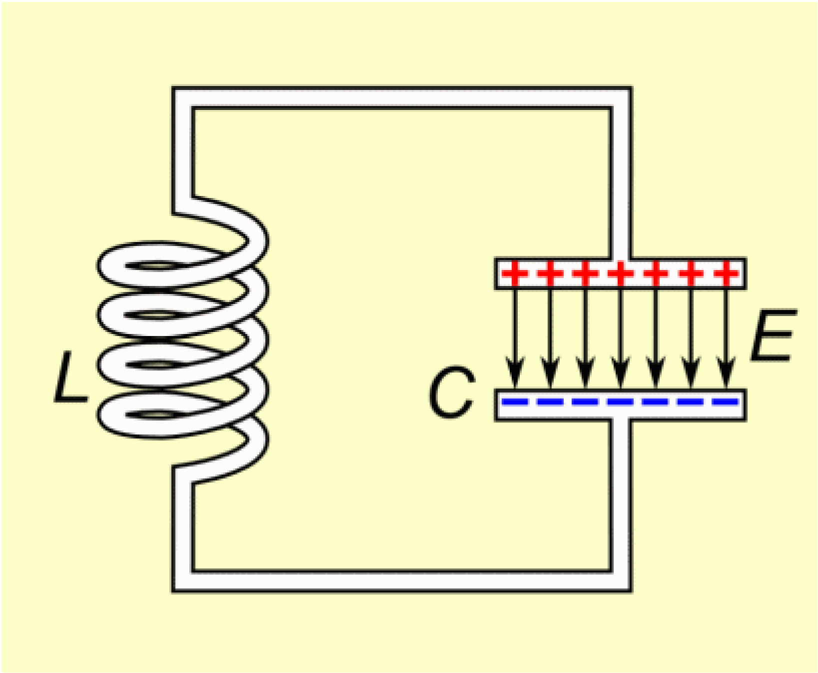
- Angular frequency ω, frequency $f = \omega/2\pi$
- Period: $T = 2\pi/\omega$
- Current and charge differ in phase by 90°

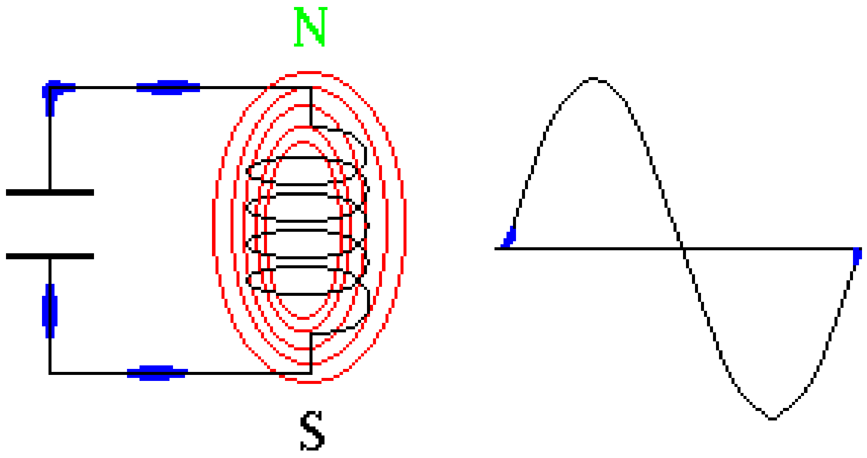


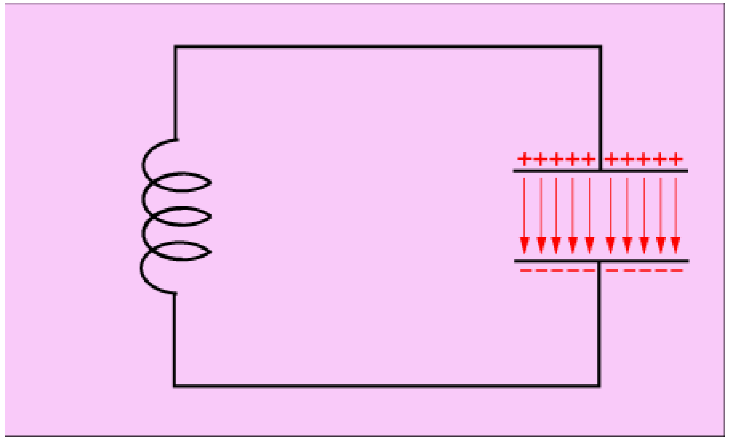
$q = q_{\max} \cos(\omega t)$
$i = -i_{\max} \sin(\omega t)$
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
ωt

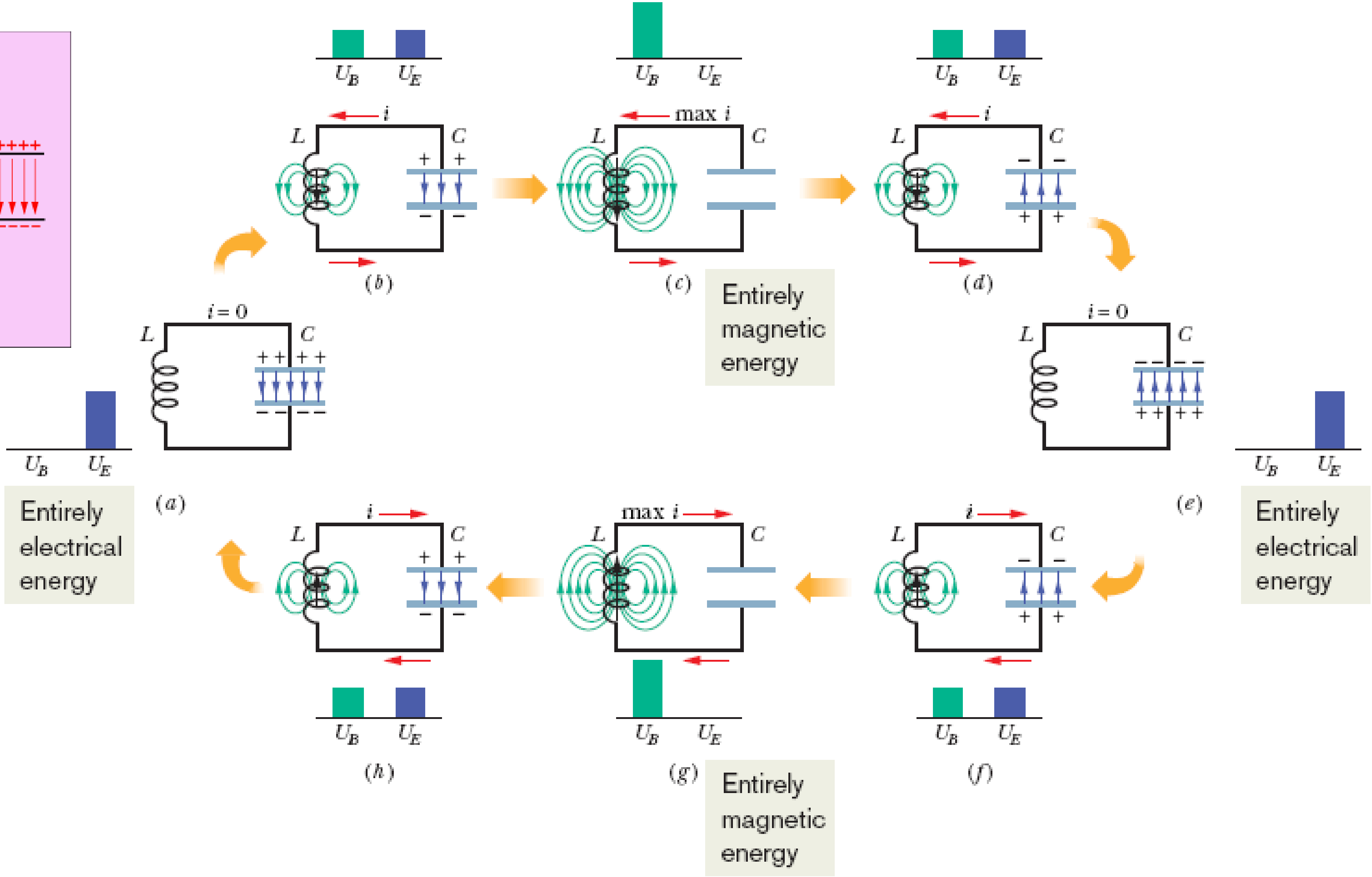
Just as both the charge on the capacitor and the current through the inductor oscillate with time, so does the energy that is contained in the electric field of the capacitor and the magnetic field of the inductor

Even though the energy content of the electric and magnetic fields are varying with time, the sum of the two at any given time is a constant

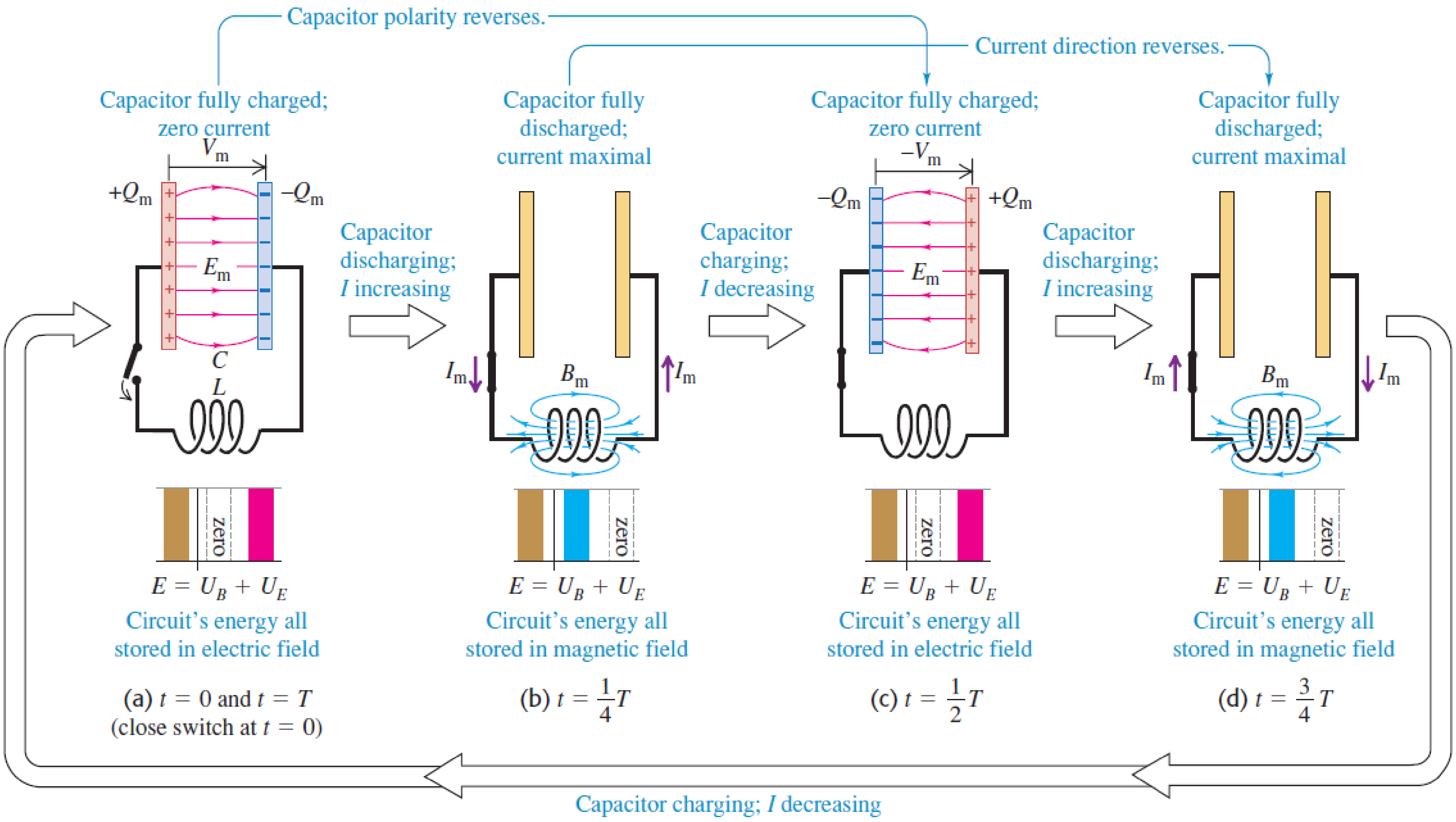








U_B U_E
$i = 0$
L C
(a)
Entirely electrical energy
(b)
i
max i
(c)
Entirely magnetic energy
(d)
(e)
Entirely electrical energy
(f)
(g)
Entirely magnetic energy
(h)

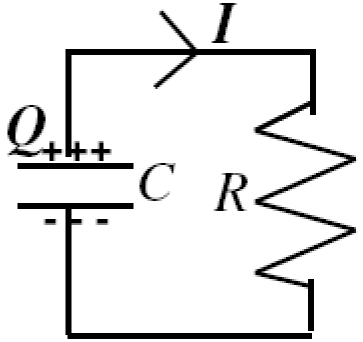


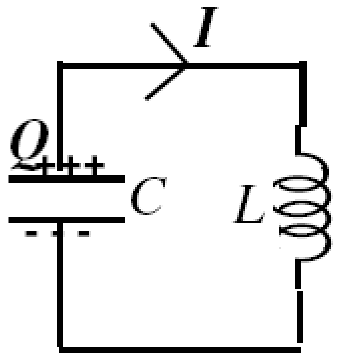
Capacitor polarity reverses.
Current direction reverses.
Capacitor fully charged; zero current
V_m
$+Q_m$
$-Q_m$
E_m
C
L
Capacitor discharging; I increasing
Capacitor fully discharged; current maximal
I_m
B_m
Capacitor charging; I decreasing
Capacitor fully charged; zero current
$-V_m$
Capacitor discharging; I increasing
Capacitor fully discharged; current maximal
zero
$E = U_B + U_E$
Circuit's energy all stored in electric field
Circuit's energy all stored in magnetic field
Circuit's energy all stored in electric field
Circuit's energy all stored in magnetic field
(a) $t = 0$ and $t = T$ (close switch at $t = 0$)
(b) $t = \frac{1}{4}T$
(c) $t = \frac{1}{2}T$
(d) $t = \frac{3}{4}T$
Capacitor charging; I decreasing

q corresponds to x, $1/C$ corresponds to k,
i corresponds to v, and L corresponds to m.

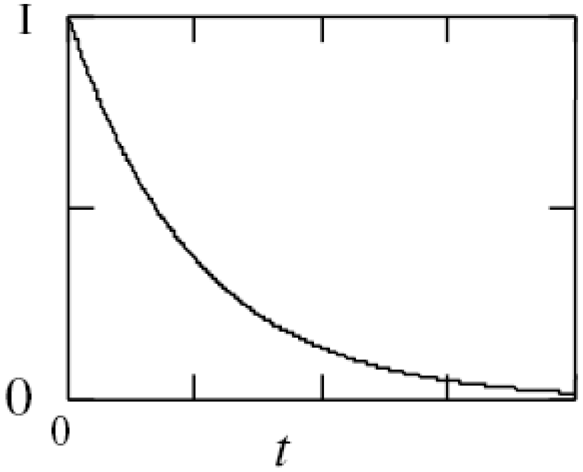
Mass-Spring System	Inductor-Capacitor Circuit
Kinetic energy $= \frac{1}{2}m{v_x}^2$	Magnetic energy $= \frac{1}{2}Li^2$
Potential energy $= \frac{1}{2}kx^2$	Electric energy $= q^2/2C$
$\frac{1}{2}m{v_x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$	$\frac{1}{2}Li^2 + q^2/2C = Q^2/2C$
$v_x = \pm\sqrt{k/m}\sqrt{A^2 - x^2}$	$i = \pm\sqrt{1/LC}\sqrt{Q^2 - q^2}$
$v_x = dx/dt$	$i = dq/dt$
$\omega = \sqrt{\frac{k}{m}}$	$\omega = \sqrt{\frac{1}{LC}}$
$x = A\cos(\omega t + \phi)$	$q = Q\cos(\omega t + \phi)$

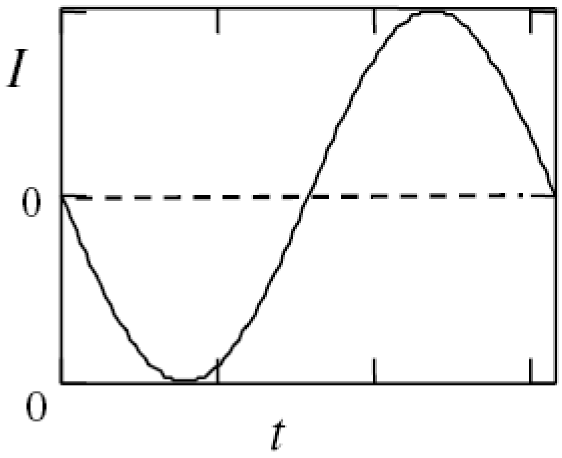
RC/LC Circuits





RC:	LC:
current decays exponentially	current oscillates





Energy Oscillations in LC Circuits

➔ Total energy in circuit is conserved. Let's see why

$$L\frac{di}{dt} + \frac{q}{C} = 0$$ Equation of LC circuit

$$L\frac{di}{dt}i + \frac{q}{C}\frac{dq}{dt} = 0$$ Multiply by i = dq/dt

$$\frac{L}{2}\frac{d}{dt}\left(i^2\right) + \frac{1}{2C}\frac{d}{dt}\left(q^2\right) = 0$$ Use $\frac{dx^2}{dt} = 2x\frac{dx}{dt}$

$$\frac{d}{dt}\left(\tfrac{1}{2}Li^2 + \tfrac{1}{2}\frac{q^2}{C}\right) = 0$$

$$\boxed{\tfrac{1}{2}Li^2 + \tfrac{1}{2}\frac{q^2}{C} = \text{const}}$$

$U_L + U_C = \text{const}$

Oscillation of Energies

➔ Energies can be written as (using $\omega^2 = 1/LC$)

$$U_C = \frac{q^2}{2C} = \frac{q_{\max}^2}{2C}\cos^2(\omega t + \theta)$$

$$U_L = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L\omega^2 q_{\max}^2 \sin^2(\omega t + \theta) = \frac{q_{\max}^2}{2C}\sin^2(\omega t + \theta)$$

➔ Conservation of energy: $U_C + U_L = \frac{q_{\max}^2}{2C} = \text{const}$

➔ Energy oscillates between capacitor and inductor

- Endless oscillation between electrical and magnetic energy
- Just like oscillation between potential energy and kinetic energy for mass on spring

Energy Check for LC circuits

Energy in Capacitor

$$\boxed{U_E(t) = \frac{1}{2C} Q_0^2 \cos^2(\omega t + \phi)}$$

Energy in Inductor

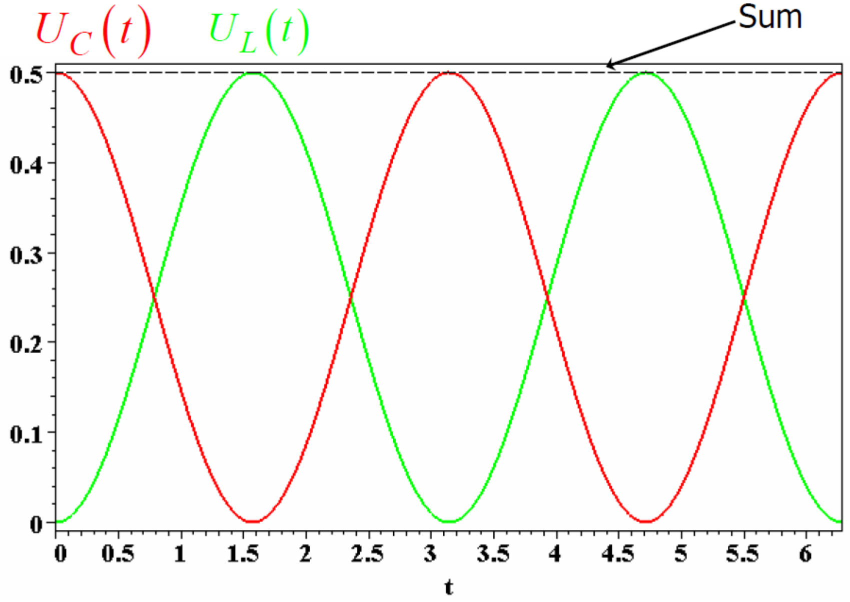
$$U_B(t) = \frac{1}{2} L\omega^2 Q_0^2 \sin^2(\omega t + \phi)$$

$$\omega = \frac{1}{\sqrt{LC}} \quad \Downarrow$$

$$\boxed{U_B(t) = \frac{1}{2C} Q_0^2 \sin^2(\omega t + \phi)}$$

Therefore, $\boxed{U_E(t) + U_B(t) = \frac{Q_0^2}{2C}}$

Plot Energies vs t



LC Circuit Example

- Parameters
  - C = 20μF
  - L = 200 mH
  - Capacitor initially charged to 40V, no current initially
- Calculate ω, f and T
  - ω = 500 rad/s
  - f = ω/2π = 79.6 Hz
  - T = 1/f = 0.0126 sec

$$\omega = 1/\sqrt{LC} = 1/\sqrt{\left(2\times10^{-5}\right)(0.2)} = 500$$

- Calculate q_{max} and i_{max}
  - $q_{max} = CV = 800\ \mu C = 8 \times 10^{-4}$ C
  - $i_{max} = \omega q_{max} = 500 \times 8 \times 10^{-4} = 0.4$ A
- Calculate maximum energies
  - $U_C = q^2_{max}/2C = 0.016$J $U_L = Li^2_{max}/2 = 0.016$J
- Charge and current

$$q = 0.0008\cos(500t) \qquad i = \frac{dq}{dt} = -0.4\sin(500t)$$

- Energies

$$U_C = 0.016\cos^2(500t) \quad U_L = 0.016\sin^2(500t)$$

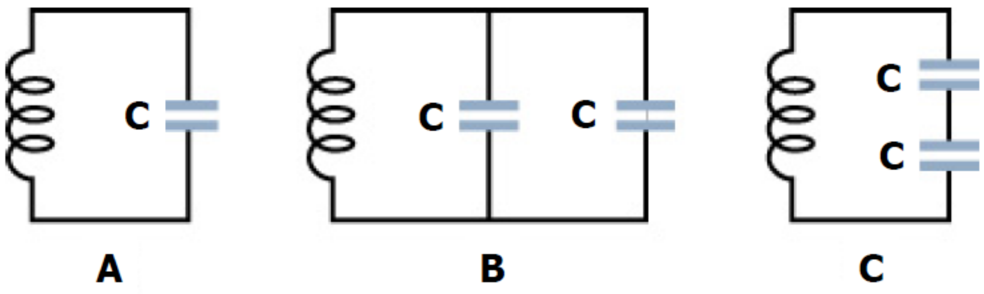
- Voltages

$$V_C = q/C = 40\cos(500t)$$

$$V_L = Ldi/dt = -L\omega i_{max}\cos(500t) = -40\cos(500t)$$

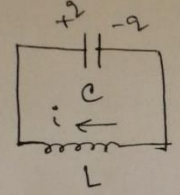
- Note how voltages sum to zero, as they must!

→Below are shown 3 LC circuits. Which one takes the least time to fully discharge the capacitors during the oscillations?



$$\omega = 1/\sqrt{LC}$$

C has smallest capacitance, therefore highest frequency, therefore shortest period



Kirchoff's law of circuit,

as ~~one~~ current direction clockwise,

$$-V_L - V_c = 0$$

$$-L\frac{di}{dt} - \frac{q}{c} = 0$$

Since, $i = \frac{dq}{dt}$ definition of current

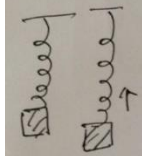
$$\frac{di}{dt} = \frac{d^2q}{dt^2}$$

Then, $L\frac{d^2q}{dt^2} + \frac{q}{c} = 0$

$$\Rightarrow \boxed{\frac{d^2q}{dt^2} + \frac{1}{Lc}q = 0}$$ LC circuit differential equation

For a spring-mass system, $F = -Kx$ and $F = ma$

combing the expressions



$ma = -Kx \Rightarrow m\frac{d^2x}{dt^2} = -Kx$ $\left[a = \frac{dv}{dt} = \frac{d^2x}{dt^2}\right.$

$a \to$ acceleration

$x \to$ displacement

$v \to$ velocity$]$

$$\Rightarrow m\frac{d^2x}{dt^2} + Kx = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{K}{m}x = 0}$$ simple harmonic motion differential equation.

solution. $x = A\cos(\omega t + \phi)$

where, $\omega = \sqrt{\frac{K}{m}}$ angular frequency.

$A \to$ amplitude, $\phi \to$ phase angle

depends on the initial conditions.

$$\frac{d^2q}{dt^2} + \frac{1}{Lc}q = 0 \qquad q \Leftrightarrow x$$

Solution similar form, $\boxed{q = Q\cos(\omega t + \phi)}$

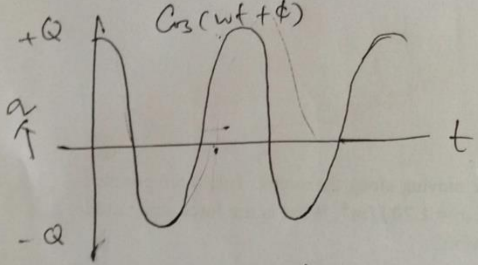
where, $\omega = \sqrt{\frac{1}{Lc}}$ ~~angle~~ angular frequency of LC circuit oscillation

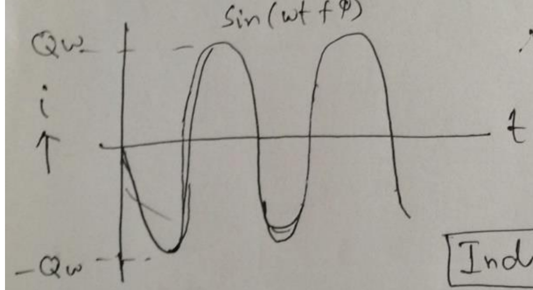
$Q \to$ amplitude, $\phi \to$ phase angle, $\boxed{i \Leftrightarrow v}$

Then corresponding current, $i = \frac{dq}{dt}$

$$\boxed{i = -Q\omega\sin(\omega t + \phi)} = -i_0\sin(\omega t + \phi)$$

$$i_0 = Q\omega$$





Now, the corresponding energy,

Capacitor $\Leftrightarrow$ elastic potential energy

electric field energy $\frac{1}{2}\frac{q^2}{c}$ $\qquad \frac{1}{2}kx^2$

Inductor $\qquad$ Kinetic energy $\frac{1}{2}mv^2$

magnetic field energy $\frac{1}{2}Li^2$

Thus, total energy

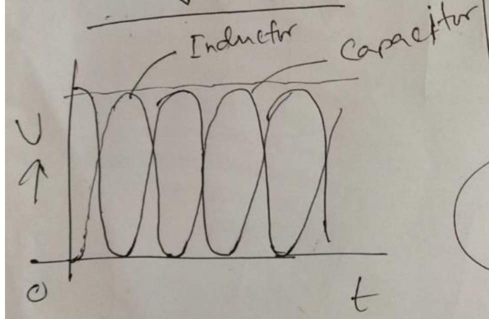
Spring-mass system

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t+\phi) + \frac{1}{2} K A^2 \cos^2(\omega t+\phi)$$

$$\boxed{E = \frac{1}{2} K A^2} \qquad \omega = \sqrt{\frac{K}{m}}$$

$$v_x = \pm\sqrt{\frac{K}{m}(A^2 - x^2)}$$



LC circuit

$$U = U_B + U_E$$

magnetic field (U_B), electric field (U_E)

$$= \frac{1}{2} L i^2 + \frac{1}{2} \frac{q^2}{C}$$

$$= \frac{1}{2} L Q^2 \omega^2 \sin^2(\omega t+\phi) + \frac{1}{2C} Q^2 \cos^2(\omega t+\phi)$$

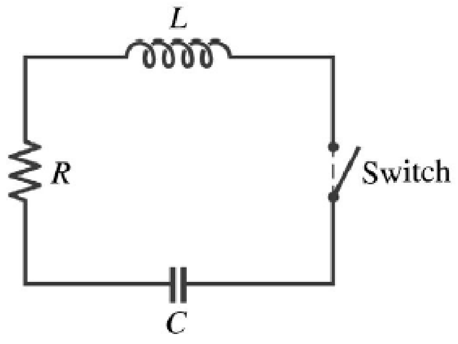
$$\boxed{U = \frac{Q^2}{2C}} \qquad \omega = \sqrt{\frac{1}{LC}}$$

$$i = \pm\sqrt{\frac{1}{LC}(Q^2 - q^2)} \qquad \omega^2 L = \frac{1}{C}$$

$$\frac{1}{2}\frac{Q^2}{C} \sin^2(\omega t+\phi) + \frac{1}{2}\frac{Q^2}{C} \cos^2(\omega t+\phi)$$

L-R-C Circuit

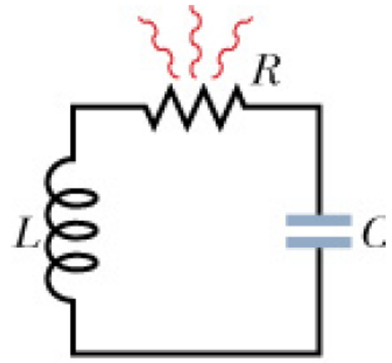
Instead of just having an L-C circuit with no resistance, what happens when there is a resistance R in the circuit



Again let us start with the capacitor fully charged with a charge Q_0 on it

The switch is now closed

RLC Circuit



- ➔The loop rule tells us

$$L\frac{di}{dt}+Ri+\frac{q}{C}=0$$

- ➔Use i = dq/dt, divide by L

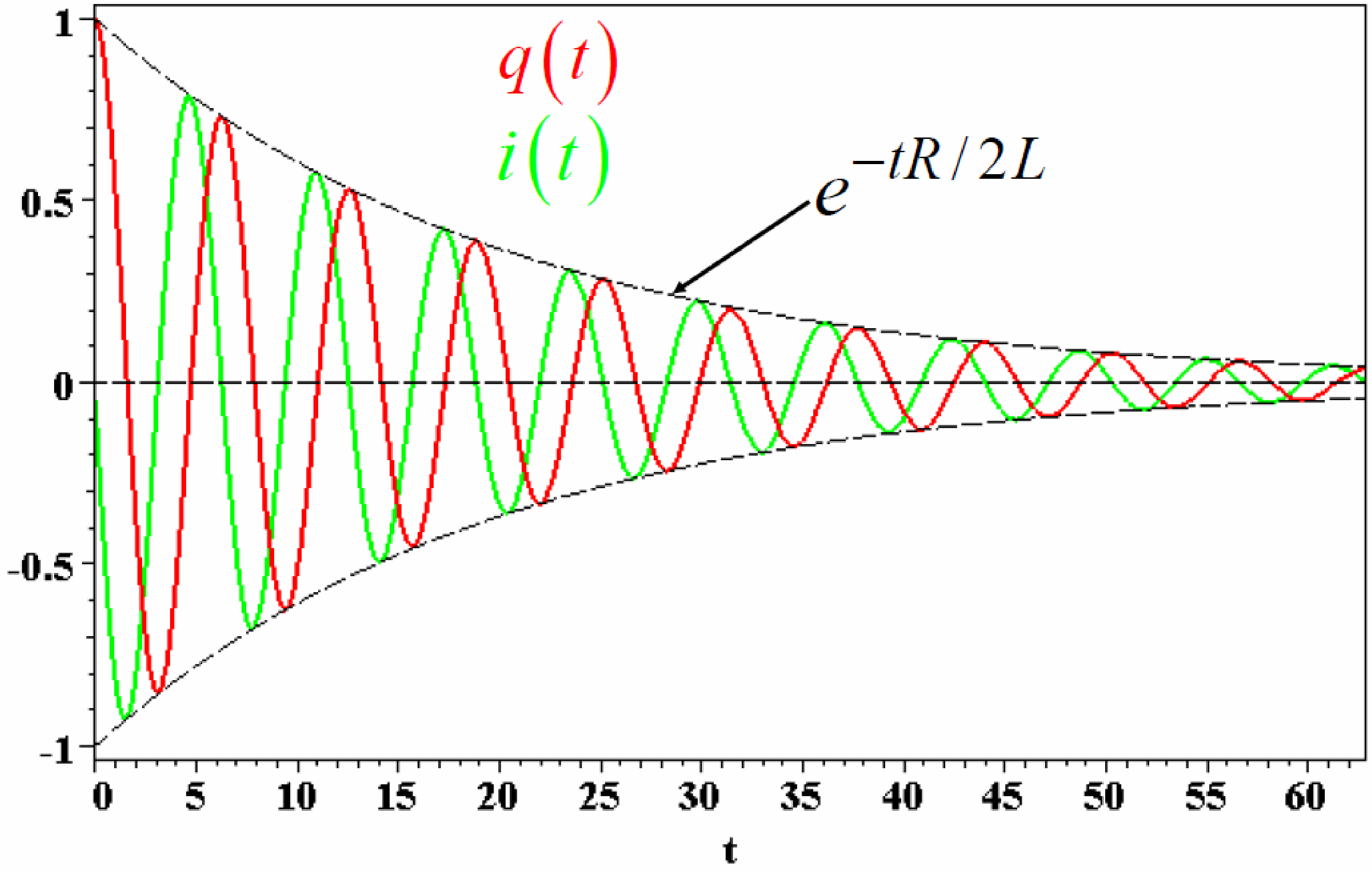
$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=0$$

- ➔Solution slightly more complicated than LC case

$$\boxed{q=q_{\max}e^{-tR/2L}\cos(\omega' t+\theta)} \quad \omega'=\sqrt{1/LC-(R/2L)^2}$$

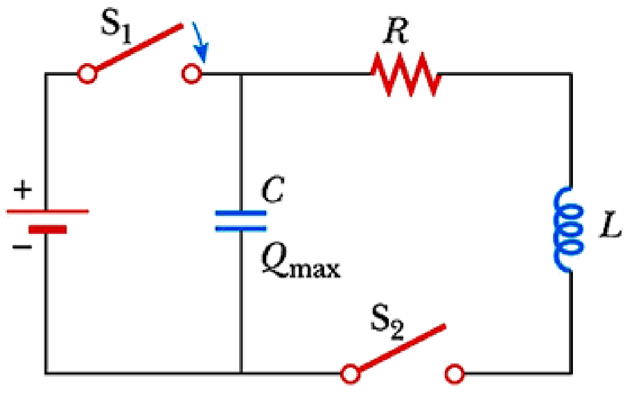
- ➔This is a damped oscillator (similar to mechanical case)
  - ◆ Amplitude of oscillations falls exponentially

Charge and Current vs t in RLC Circuit



The RLC Circuit

A real circuit with resistance.
The resistor transforms electrical energy to internal energy (heat).



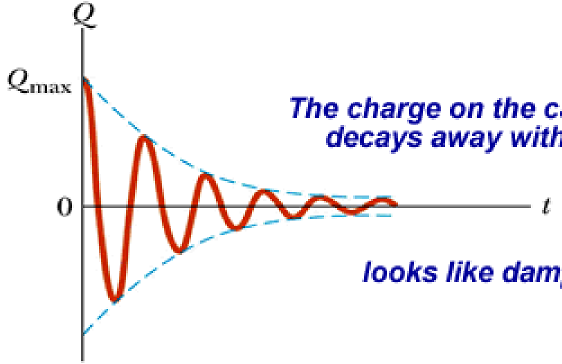
S_1 closed and S_2 open, capacitor is charged.
S_1 open and S_2 closed capacitor discharges thru the resistor.

Kirchoff's Power Sum Rule

$$LI\frac{dI}{dt} + I^2R + \frac{Q}{C}\frac{dQ}{dt} = 0$$

dividing by I $\quad I = \frac{dQ}{dt}$

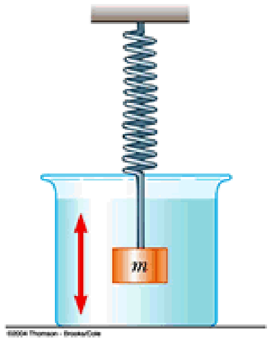
$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$



The charge on the capacitor decays away with time

looks like damping

mechanical analog
spring and block system

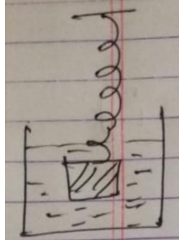


$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Inductance (L) acts like mass of the block
Resistance acts like the damping coefficient (b)
Capacitance acts like the inverse spring constant (k)

Damped Harmonic oscillation

Spring - mass system



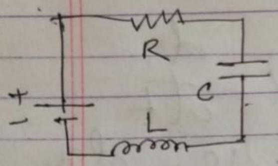
$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

damping term

$$x(t) = x_0 \exp\left(-\frac{t}{2\tau}\right)\cos(\omega' t + \phi)$$

$$\tau = \frac{m}{b} \; ; \; \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

LRC circuit



$$V_R + V_c + V_L = 0$$

$$iR + \frac{q}{c} + L\frac{di}{dt} = 0$$

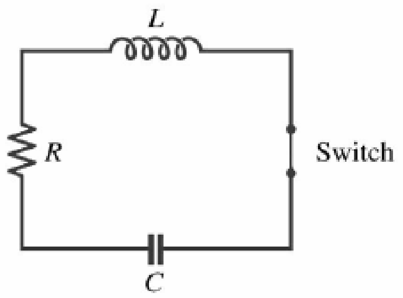
$i = \frac{dq}{dt}$ $\quad \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{Lc}q = 0$

Then, $q = q_0 e^{-t/2\tau_L}\cos(\omega' t + \phi)$

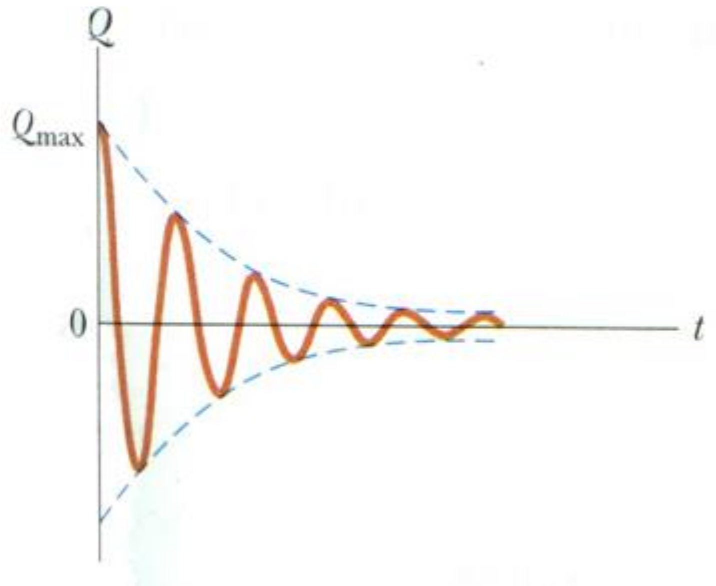
$$\omega' = \sqrt{\frac{1}{Lc} - \frac{R^2}{4L^2}}$$

L-R-C Circuit

The circuit now looks like



The capacitor will start to discharge and a current will start to flow



We apply Kirchoff's Law to this circuit and get

$$-iR - L\frac{di}{dt} - \frac{q}{C} = 0$$

And remembering that $i = {dq}/{dt}$ we get

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

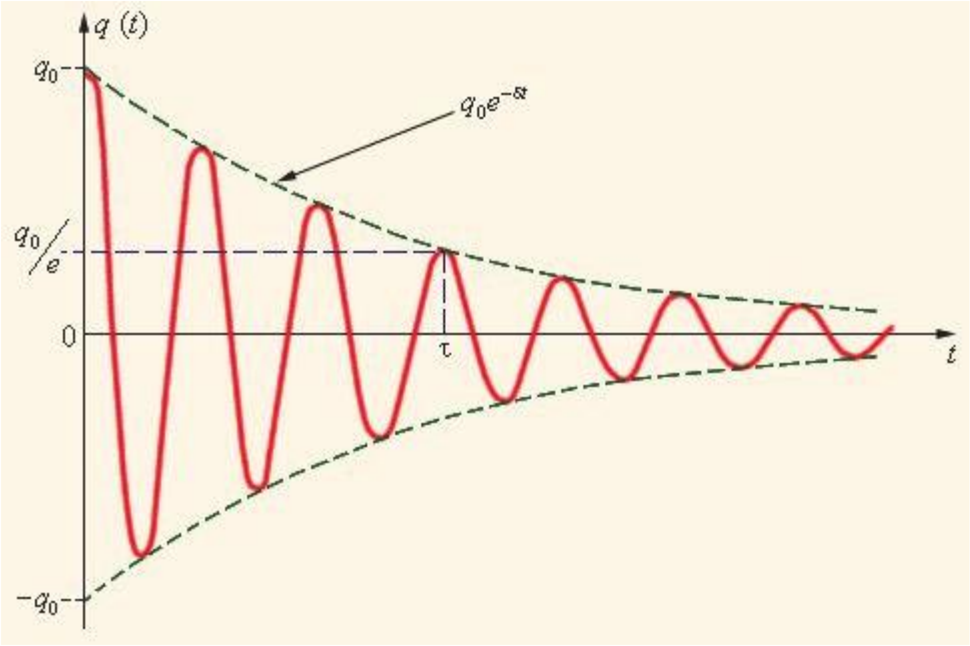
$$q = q_0 e^{-\frac{t}{\tau}} \cos(\omega' t + \varphi) \text{ where } \tau = \frac{2L}{R}$$

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

https://iwant2study.org/lookangejss/05electricitynmagnetism_17AC/ejss_model_springRLC/springRLC_Simulation.xhtml

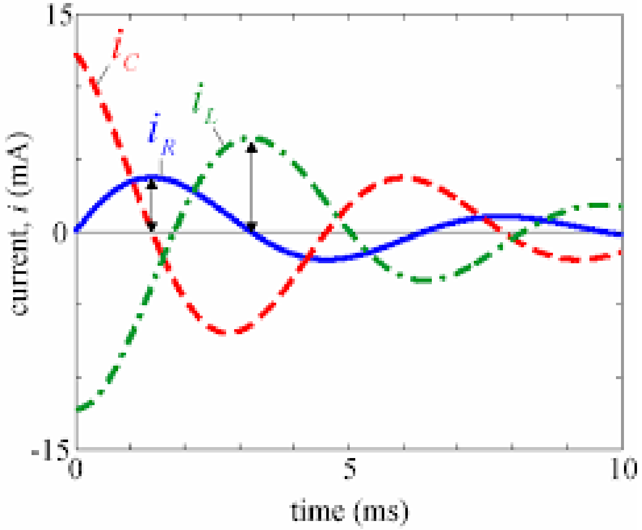
https://iwant2study.org/lookangejss/02_newtonianmechanics_8oscillations/ejss_model_SHM21SLS_dampingcar/SHM21SLS_dampingcar_Simulation.xhtml

This solution corresponds to the *underdamped* behavior with frequency, $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$



the function represents a sinusoidal oscillation with an exponentially decaying amplitude, $q_0 e^{-\frac{t}{2L/R}}$.

Note that the exponential factor $e^{-\frac{t}{2L/R}}$ is *not* the same as the factor $e^{-\frac{t}{L/R}}$ that we encountered in describing the *R*-*L* circuit



When R =0 reduces to the oscillations in an *L*-*C* circuit.

If is not zero, the angular frequency of the oscillation is *less* because of the term containing R

L-R-C Circuit

The solution to this second order differential equation is similar to that of the damped harmonic oscillator

The are three different solutions

Underdamped

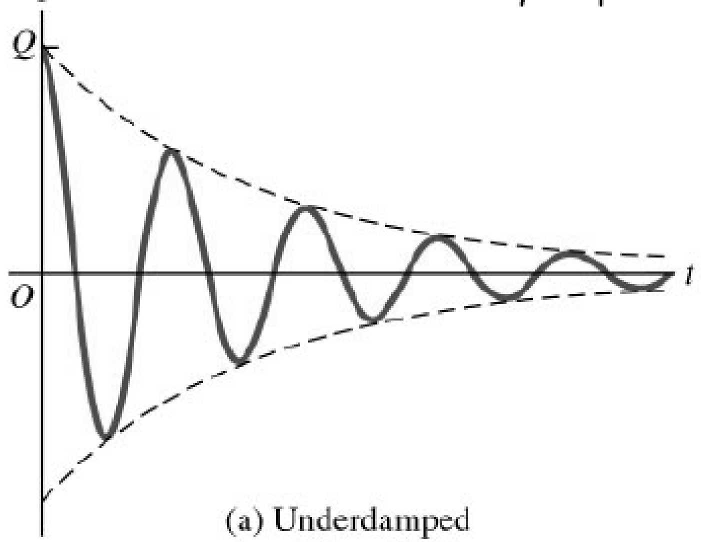
$$q = q_0 e^{-\frac{t}{\tau}} \cos(\omega' t + \varphi)$$

Critically Damped

Overdamped

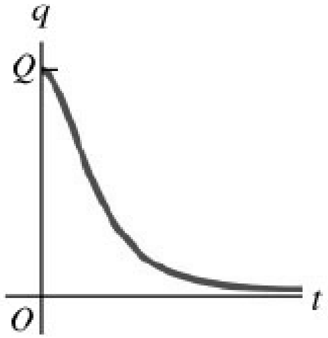
Which solution we have is dependent upon the relative values of R^2 and $4L/C$

Very important !!



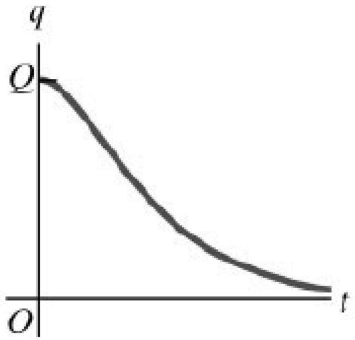
(a) Underdamped circuit (small R)

$$\frac{1}{LC} > \frac{R^2}{4L^2}$$



(b) Critically damped circuit (larger R)

$$\frac{1}{LC} = \frac{R^2}{4L^2}$$



(c) Overdamped circuit (very large R)

$$\frac{1}{LC} < \frac{R^2}{4L^2}$$