Gauss' law-I

Phy 108 course
Zaid Bin Mahbub (ZBM)
DMP, SEPS, NSU

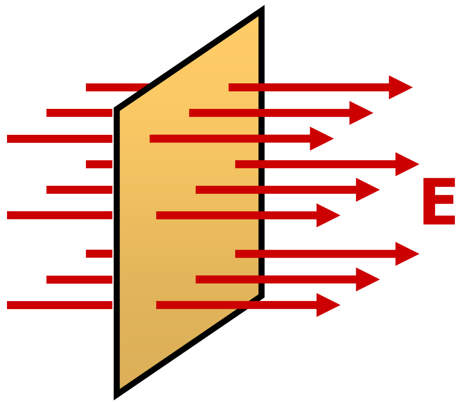
Flux

The concept of **flux** describes how much of something goes through a given area.

More formally, it is the dot product of a vector field (in this chapter, the electric field) with an area.

You may conceptualize the flux of an electric field as a measure of the number of electric field lines passing through an area.

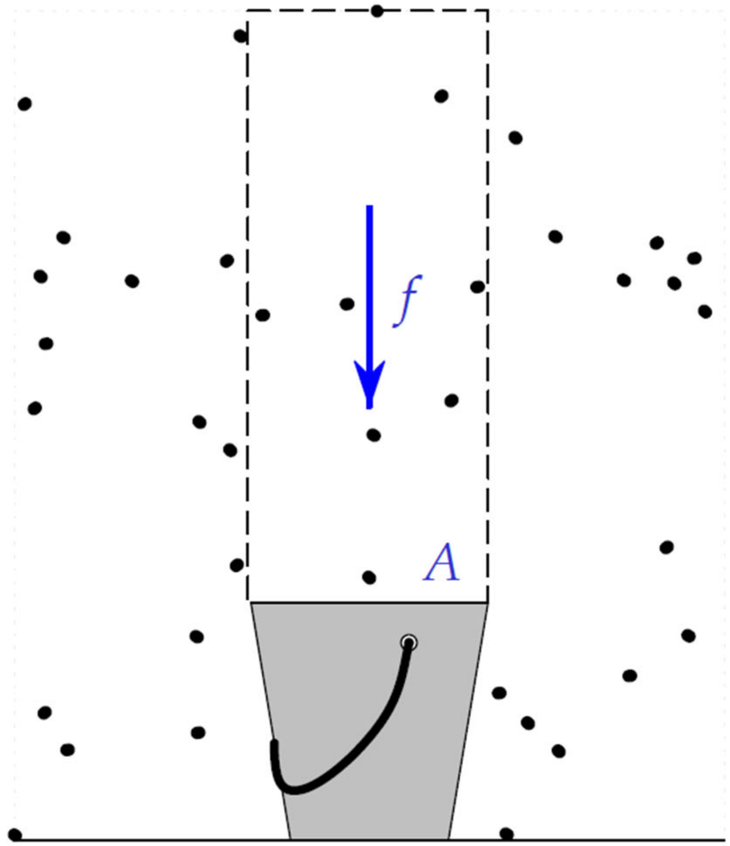
The larger the area, the more field lines go through it and, hence, the greater the flux; similarly, the stronger the electric field is (represented by a greater density of lines), the greater the flux.



Flux

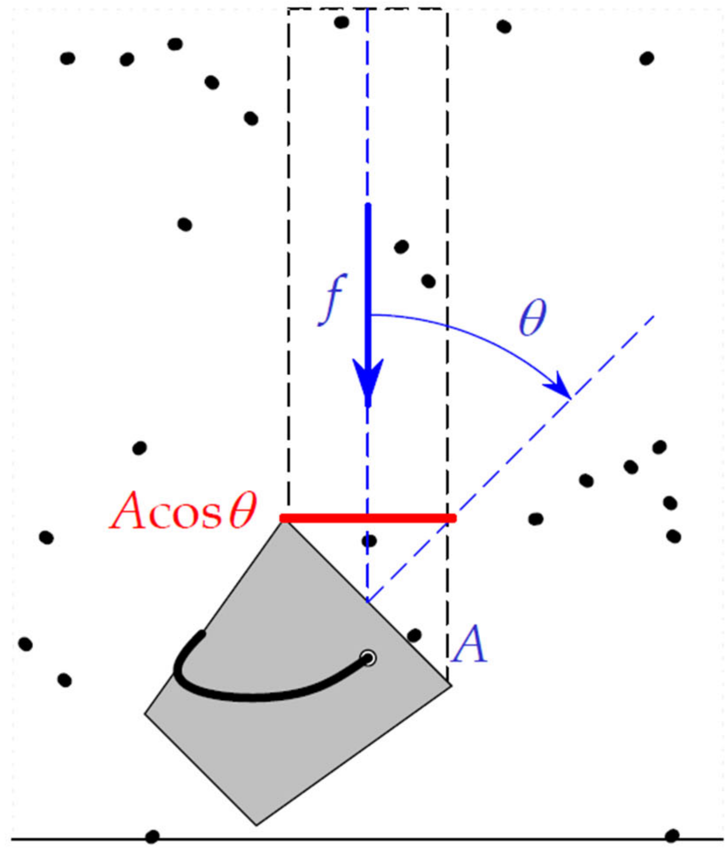
First, the flux of **rain.**

- Suppose rain is falling straight down at a constant rate: call the mass per unit area falling on the ground f.
- You have a bucket sitting flat on the ground, with area A at the top.
- The rate at which the bucket fills up is determined by how big or small the **flux** fA is.



Flux

- Now tip the bucket, so that its axis makes an angle θ with the direction of the raindrops.
- Clearly it will capture water at a smaller rate than before, because its mouth presents a smaller area $A\cos\theta$ to the rain. So the flux, which determines how fast the bucket fills, is more generally $fA\cos\theta$.



Flux Through a Basking Shark's Mouth

Unlike aggressive carnivorous sharks such as great whites, a basking shark feeds passively on plankton in the water that passes through the shark's gills as it swims.

To survive on these tiny organisms requires a huge flux of water through a basking shark's immense mouth, which can be up to a meter across.

The water flux—the product of the shark's speed through the water and the area of its mouth—can be up to 500 liters per Second!!

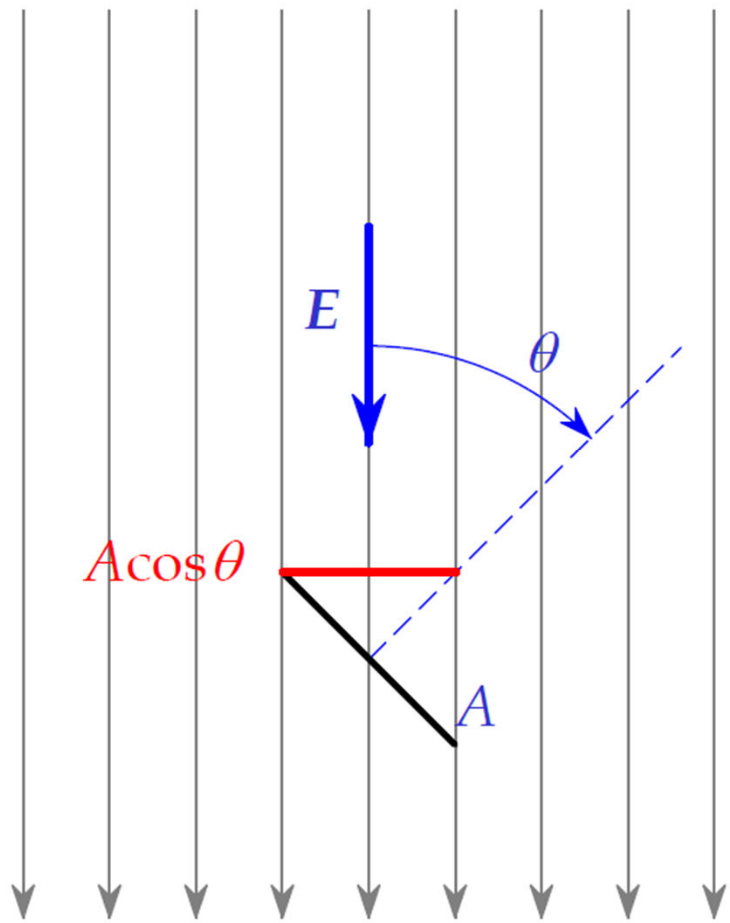
In a similar way, the flux of electric field through a surface depends on the magnitude of the field and the area of the surface (as well as the relative orientation of the field and surface).

Electric Flux

Now, $\boldsymbol{E}$:

- Electric field doesn't flow, but the direction and density of lines of $\boldsymbol{E}$ function like the velocity and density of raindrops.
- For uniform $\boldsymbol{E}$ and planar A, the flux of $\boldsymbol{E}$ is therefore

$$\Phi = EA\cos\theta$$

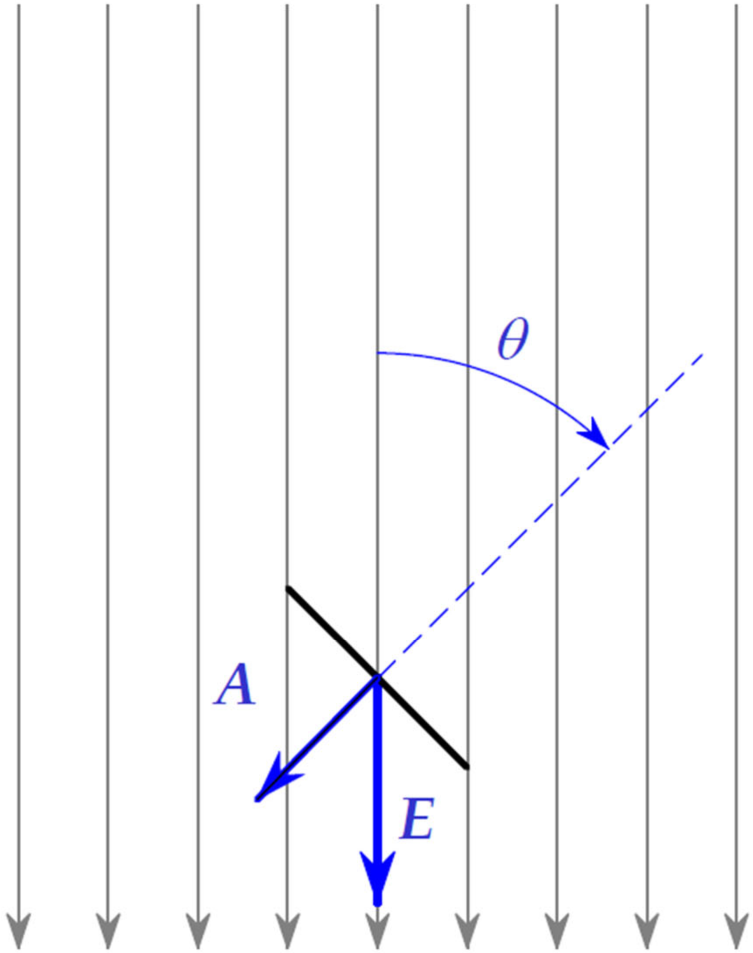


Electric Flux

- Or, inventing a vector $\boldsymbol{A}$ which has magnitude equal to the area A and direction perpendicular to the area, we can write the flux in a vector shorthand that will turn out to be convenient:

  Flux is the scalar ("dot") product of the vectors $\boldsymbol{E}$ and $\boldsymbol{A}$:

$$\Phi = \boldsymbol{E} \bullet \boldsymbol{A} \quad .$$



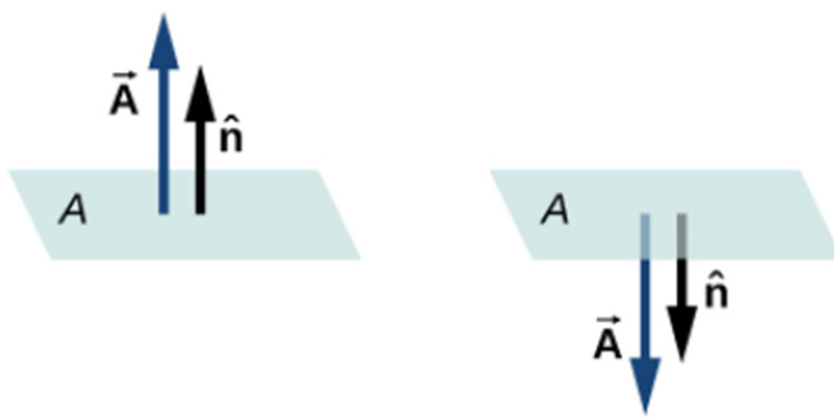
Electric flux is a scalar quantity and SI unit of newton-meters squared per coulomb ($N \cdot m^2 / C$).

The **area vector** of a flat surface of area *A* has the following magnitude and direction:

Magnitude is equal to area (*A*)
Direction is along the normal to the surface, that is, perpendicular to the surface.

Since the normal to a flat surface can point in either direction from the surface, the direction of the area vector of an open surface needs to be chosen.

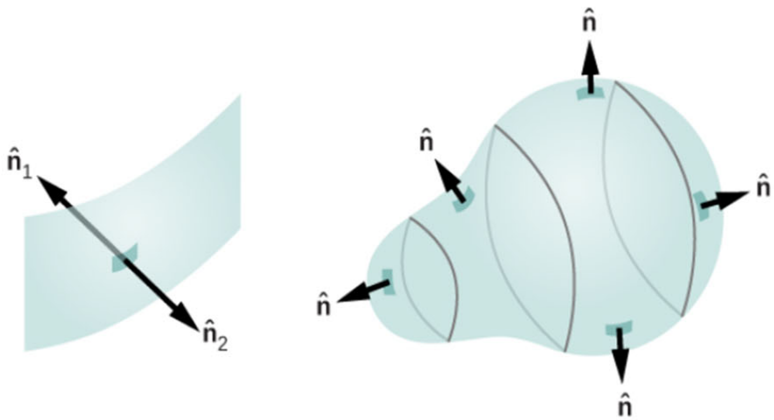
The direction of the area vector of an open surface needs to be chosen; it could be either of the two cases displayed here.



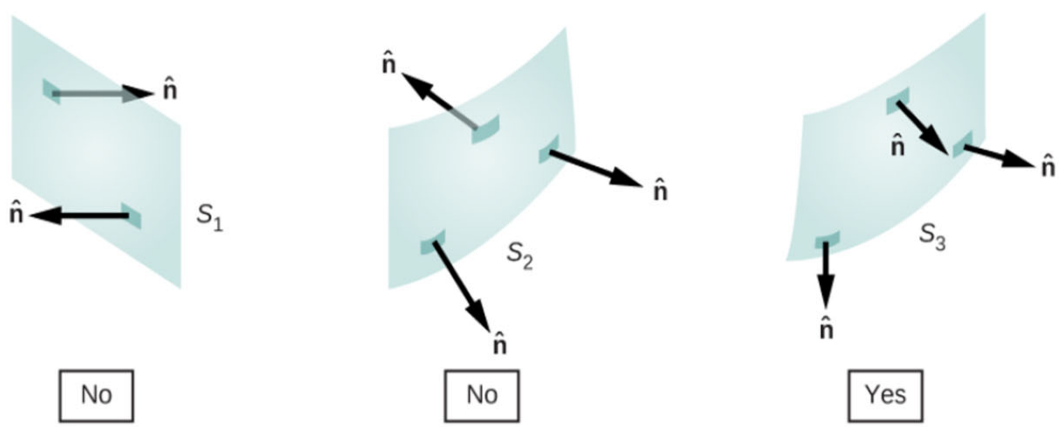
If the surface is closed, means the surface encloses a volume.

In that case, the direction of the **normal vector** at any point on the surface points from the inside to the outside.

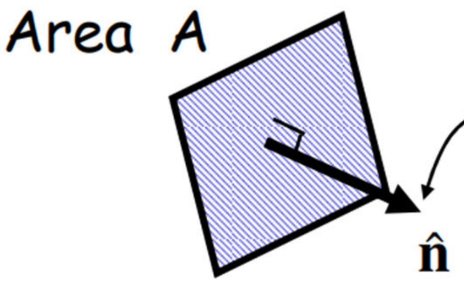
The area vector of a part of a closed surface is defined to point from the inside of the closed space to the outside. This rule gives a unique direction.



(a) (b)

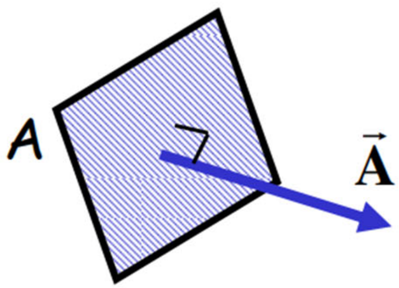


On a *closed surface* such as that of Figure, is chosen to be the *outward normal* at every point, to be consistent with the sign convention for electric charge.



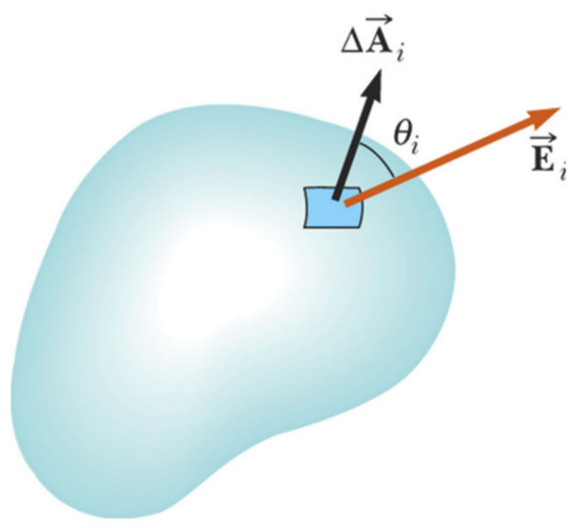
Unit vector $\perp$ surface
("unit normal")

Define "area vector": $\vec{\mathbf{A}} = (area) \cdot \hat{\mathbf{n}}$

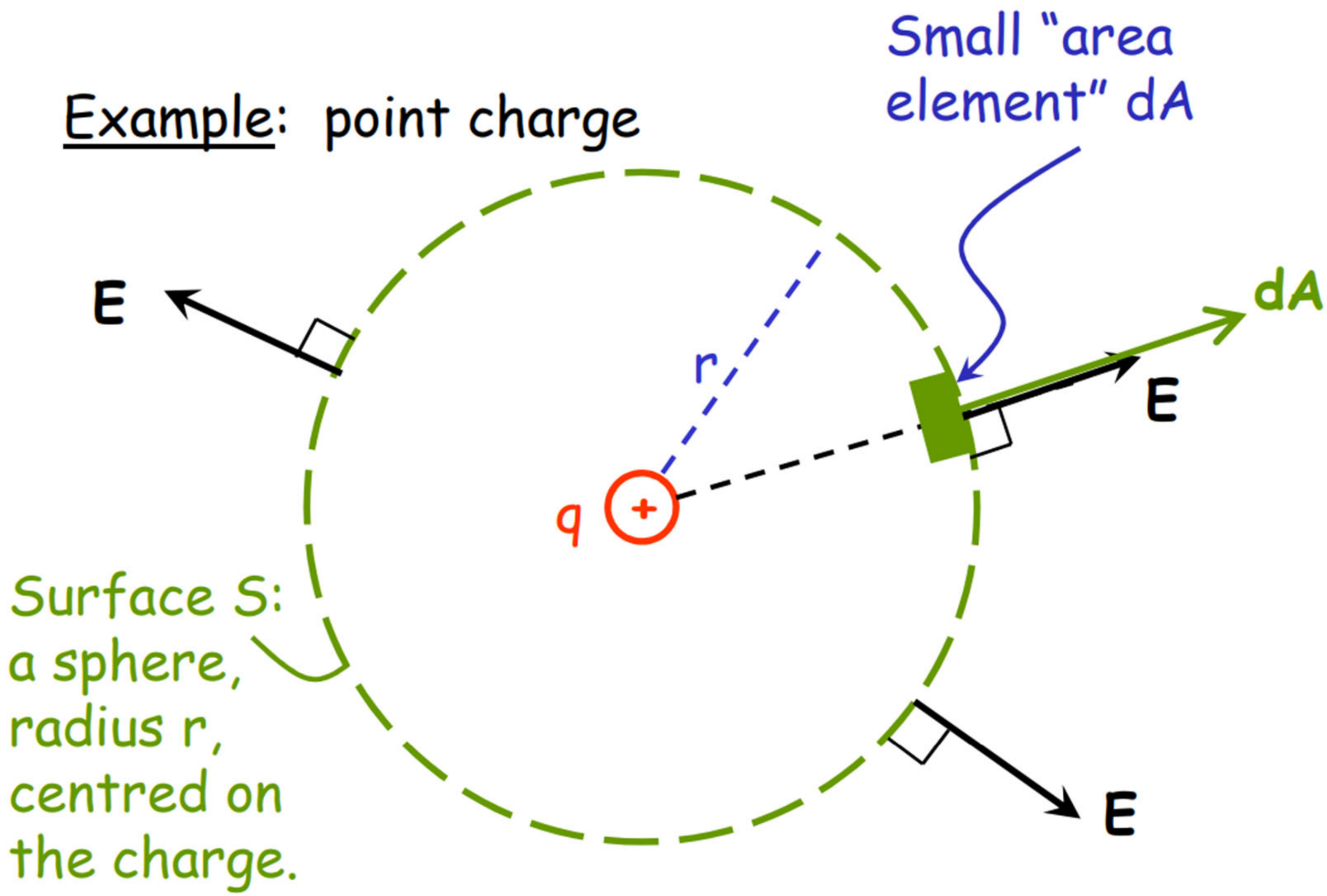


$|\,\vec{\mathbf{A}}\,| = \text{area}$

$\vec{\mathbf{A}} \parallel \hat{\mathbf{n}}$ *(perpendicular to surface)*



Example: point charge

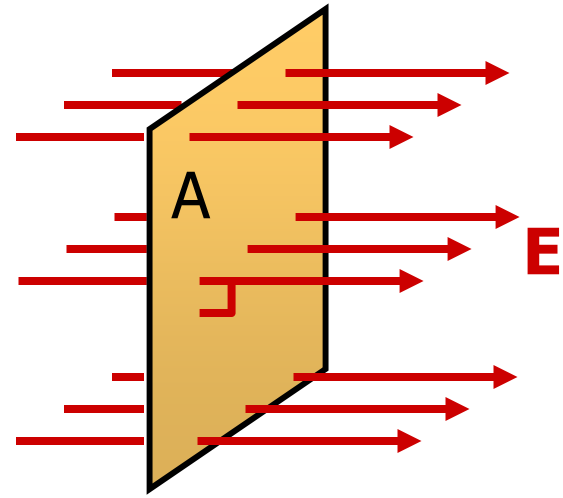


Electric Flux

The **electric flux** passing through a surface is the number of electric field lines that pass through it.

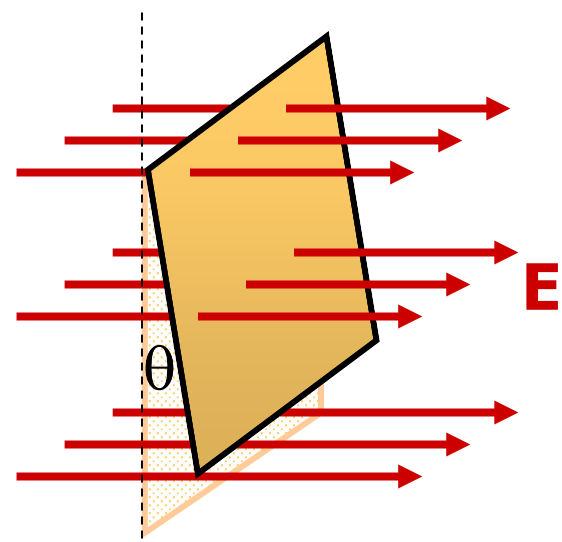
Because electric field lines are drawn arbitrarily, we **quantify** electric flux like this:

$$\Phi_E = EA.$$



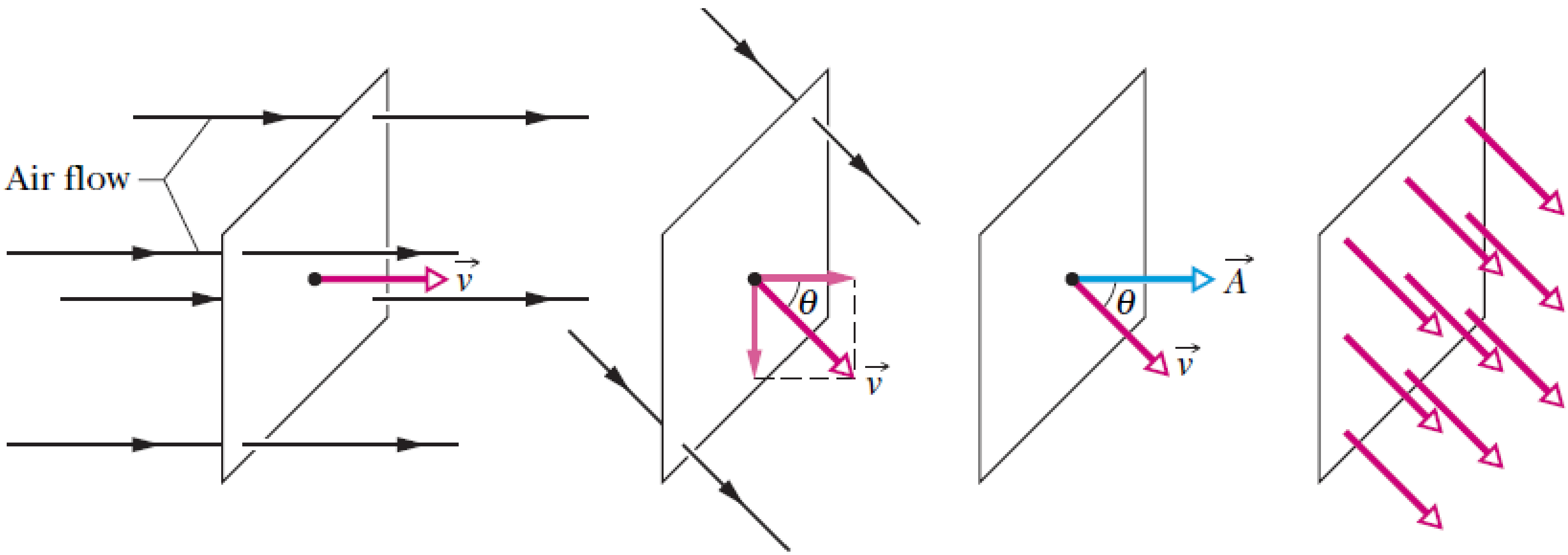
If the surface is tilted, fewer lines cut the surface.

If the area rotated so that the plane is aligned with the field lines, none will pass through and there will be no flux.



Remember the dot product from Physics 107?

Velocity Flux



(a) (b) (c) (d)

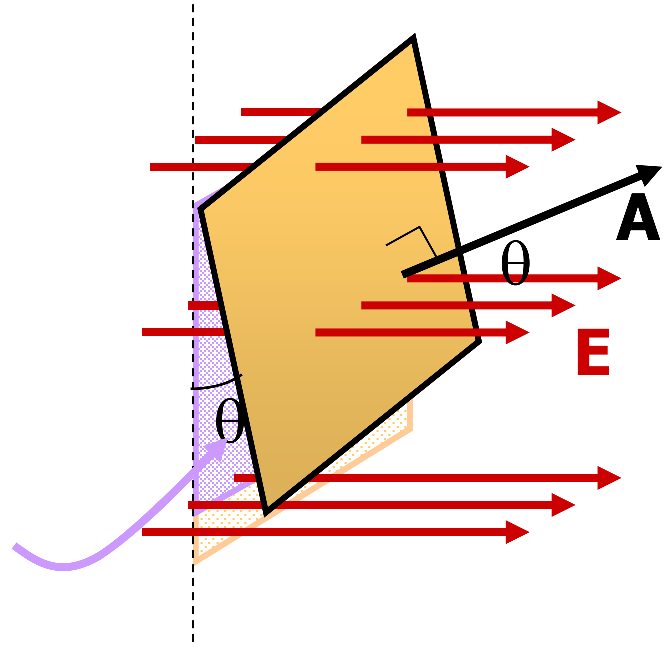
(a) A uniform airstream of velocity is perpendicular to the plane of a square loop of area ***A***.
(b) The component of perpendicular to the plane of the loop is $v\cos\theta$, where θ is the angle between ***v*** and a normal to the plane.
(c) The area vector ***A*** is perpendicular to the plane of the loop and makes an angle θ with ***v***. (d) The velocity field intercepted by the area of the loop.

The **rate of volume flow** through the loop is $F = (v\cos\theta)A$.

$$\Phi = vA\cos\theta = \vec{v}\cdot\vec{A},$$

Electric Flux

We define $\vec{A}$ to be a vector having a magnitude equal to the area of the surface, in a direction normal to the surface.



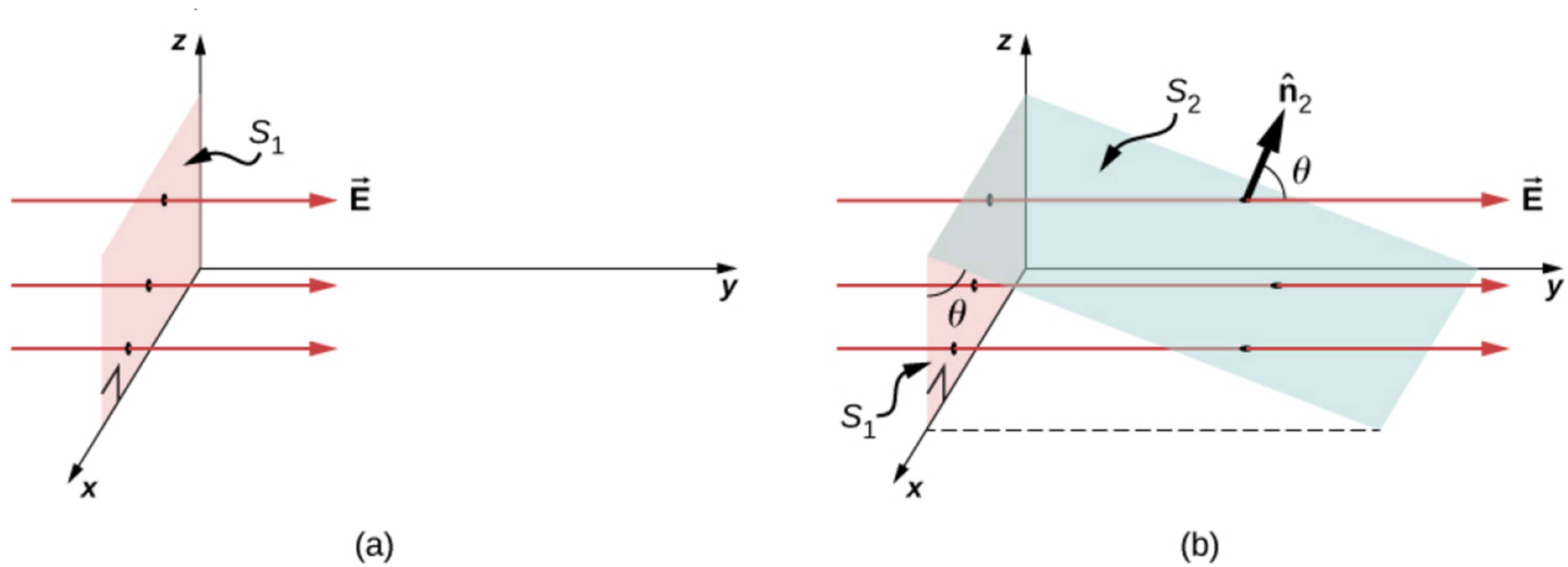
The "amount of surface" perpendicular to the electric field is A cos θ.

Because $\vec{A}$ is perpendicular to the surface, the amount of $\vec{A}$ parallel to the electric field is A cos θ.

$$A_{\|} = A\cos\theta \quad \text{so} \quad \Phi_E = EA_{\|} = EA\cos\theta.$$

$$\Phi_E = \vec{E}\cdot\vec{A}$$

Electric flux is a scalar quantity and SI unit of newton-meters squared per coulomb ($N\cdot m^2/C$).



Consider the field near a positive point charge q: imagine a surface (radius r) surrounding q.

A planar surface $S1$ of area $A1$ that is perpendicular to the uniform electric field $\vec{E}$

$$\vec{E} = E\hat{y}$$

If *N* field lines pass through $S1$, then we know from the definition of electric field lines that $N/S1 \propto E$, or $N \propto ES1$.

The quantity $ES1$ is the **electric flux** through $S1$

(a) Surface is face-on to electric field:

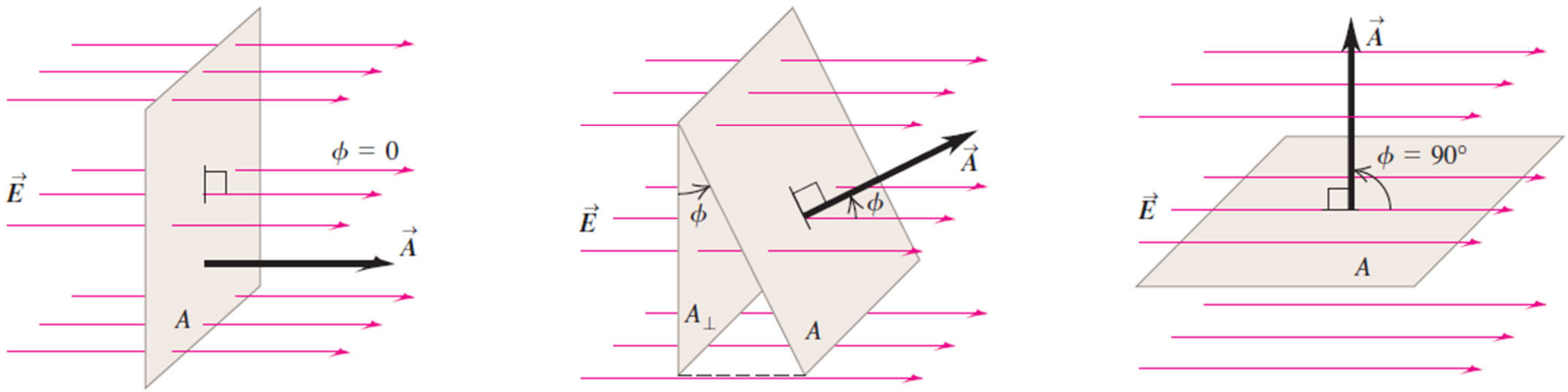
- $\vec{E}$ and $\vec{A}$ are parallel (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 0$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$.

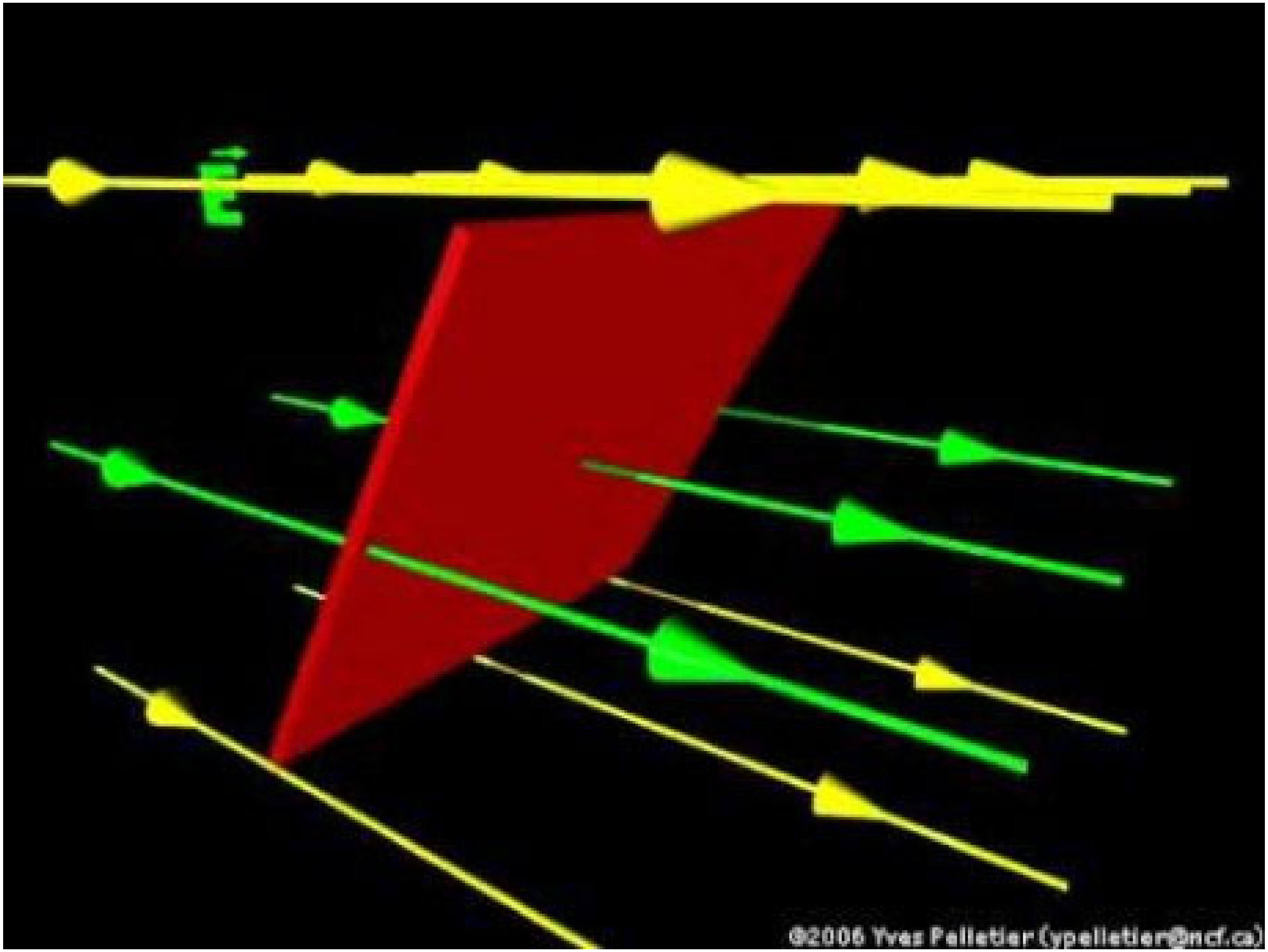
(b) Surface is tilted from a face-on orientation by an angle ϕ:

- The angle between $\vec{E}$ and $\vec{A}$ is ϕ.
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$.

(c) Surface is edge-on to electric field:

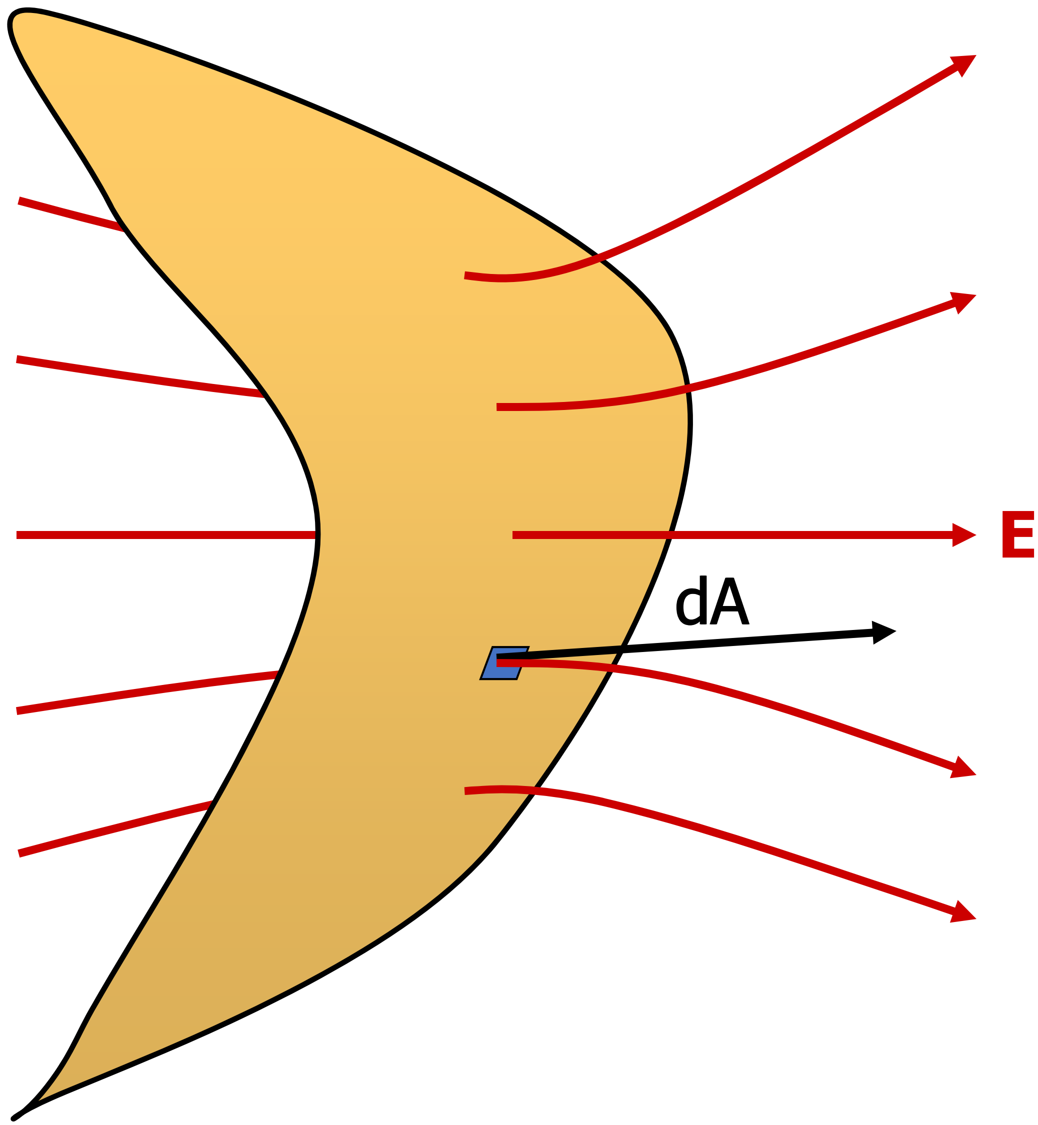
- $\vec{E}$ and $\vec{A}$ are perpendicular (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 90°$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos 90° = 0$.





https://www.youtube.com/watch?v=_xsN9zDHRcA

If the electric field is not uniform, or the surface is not flat…

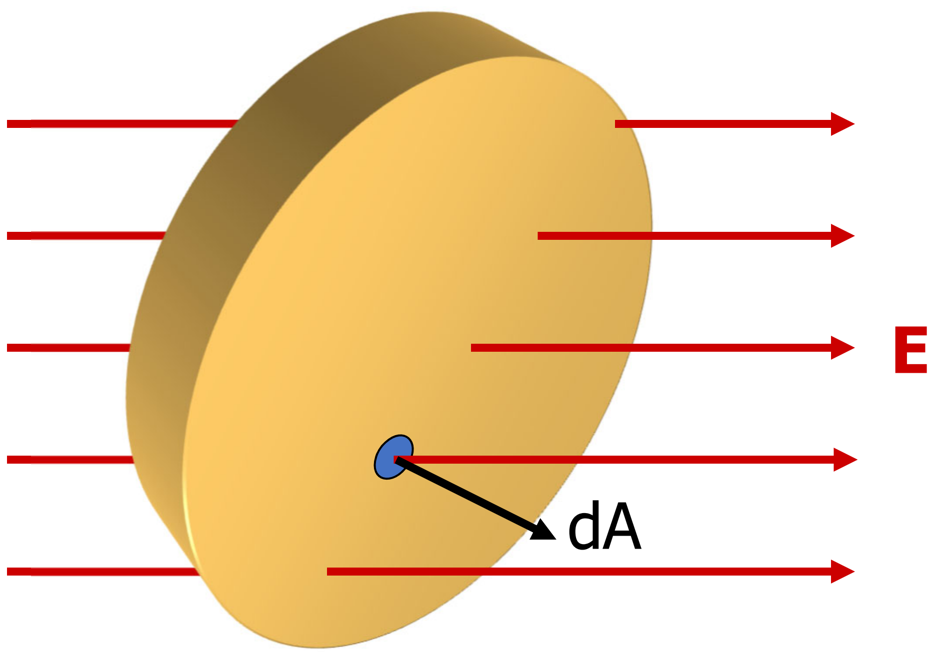


divide the surface into infinitesimal surface elements and add the flux through each…

$$\Phi_E = \lim_{\Delta A_i \to 0} \sum_i \vec{E}_i \cdot \Delta\vec{A}_i$$

$$\boxed{\Phi_E = \int \vec{E} \cdot d\vec{A}}$$

If the surface is closed (completely encloses a volume)…



lines outward as positive

lines inward as negative

$$\Phi_E = \oint \vec{E} \cdot d\vec{A}$$

a surface integral, therefore a double integral $\iint$

Different equations for electric flux

$\Phi_E = EA$ Flat surface, $\vec{E} \parallel \vec{A}$, $\vec{E}$ constant over surface. Easy!

$\Phi_E = EA\cos\theta$ Flat surface, $\vec{E}$ not $\parallel \vec{A}$, $\vec{E}$ constant over surface.

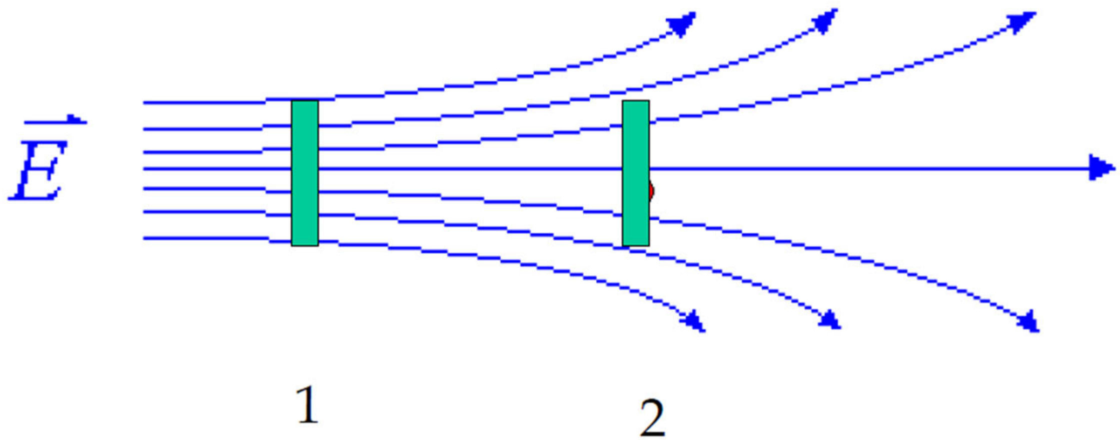
$\Phi_E = \vec{E}\cdot\vec{A}$ Flat surface, $\vec{E}$ not $\parallel \vec{A}$, $\vec{E}$ constant over surface.

$\Phi_E = \int \vec{E}\cdot d\vec{A}$ Surface not flat, $\vec{E}$ not uniform.

$\Phi_E = \oint \vec{E}\cdot d\vec{A}$ Closed surface. Most general. Most complex.

If the surface is closed, you may be able to "break it up" into simple segments and still use $\Phi_E=\vec{E}\cdot\vec{A}$ for each segment.

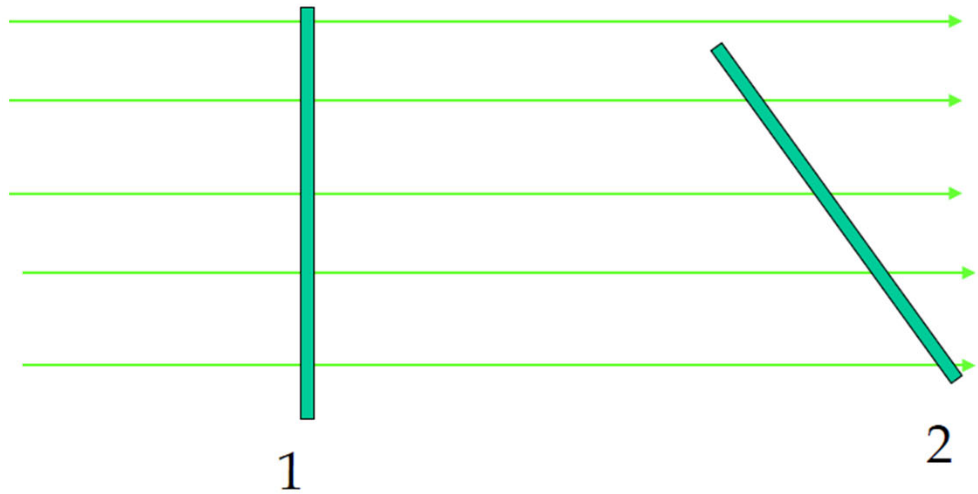
Compare fluxes



Which flux is larger?

$\boxed{1)\ \Phi_1 > \Phi_2}$ 2) $\Phi_1 < \Phi_2$ 3) $\Phi_1 = \Phi_2$

Compare fluxes



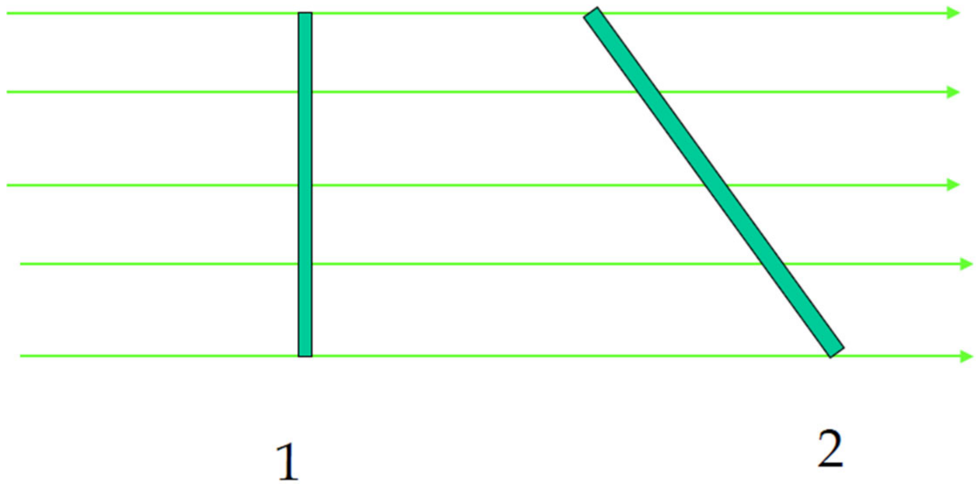
Which flux is larger?

1) $\Phi_1 > \Phi_2$

2) $\Phi_1 < \Phi_2$

3) $\Phi_1 = \Phi_2$

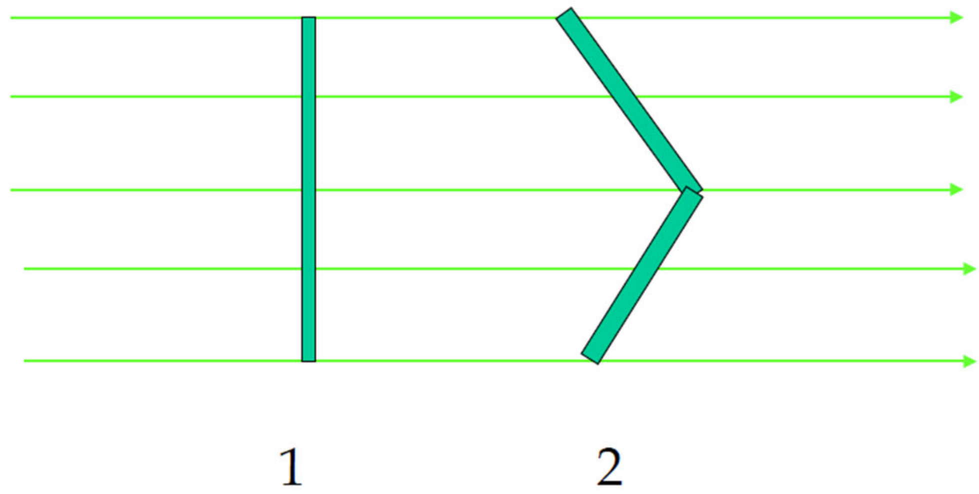
Compare fluxes



Which flux is larger?

1) $\Phi_1 > \Phi_2$ 2) $\Phi_1 < \Phi_2$ 3) $\Phi_1 = \Phi_2$

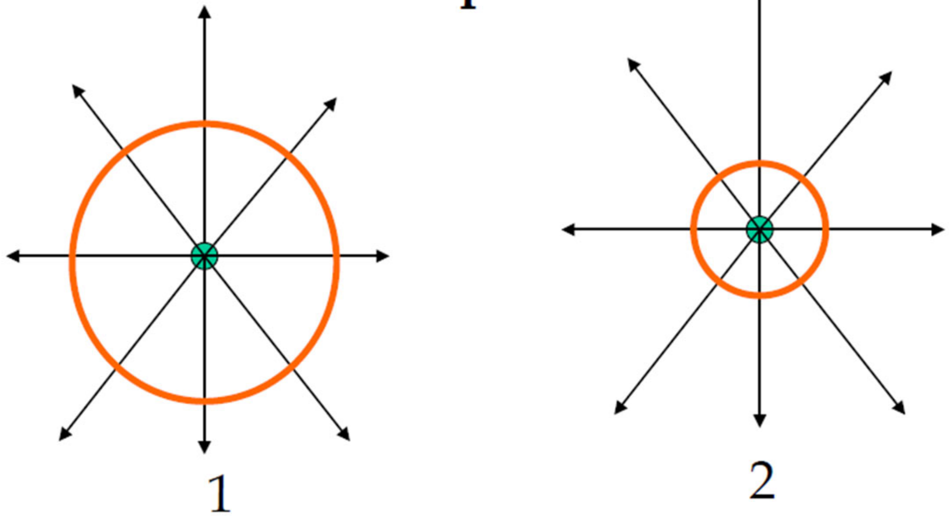
Compare fluxes



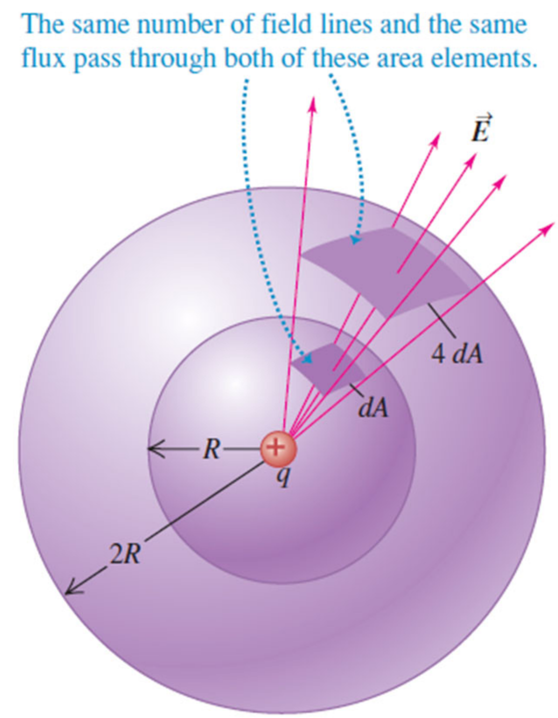
Which flux is larger?

1) $\Phi_1 > \Phi_2$ 2) $\Phi_1 < \Phi_2$ 3) $\boxed{\Phi_1 = \Phi_2}$

Compare fluxes



22.11 Projection of an element of area dA of a sphere of radius R onto a concentric sphere of radius $2R$. The projection multiplies each linear dimension by 2, so the area element on the larger sphere is $4\,dA$.



Which flux is larger?

1) $\Phi_1 > \Phi_2$

2) $\Phi_1 < \Phi_2$

3) $\boxed{\Phi_1 = \Phi_2}$

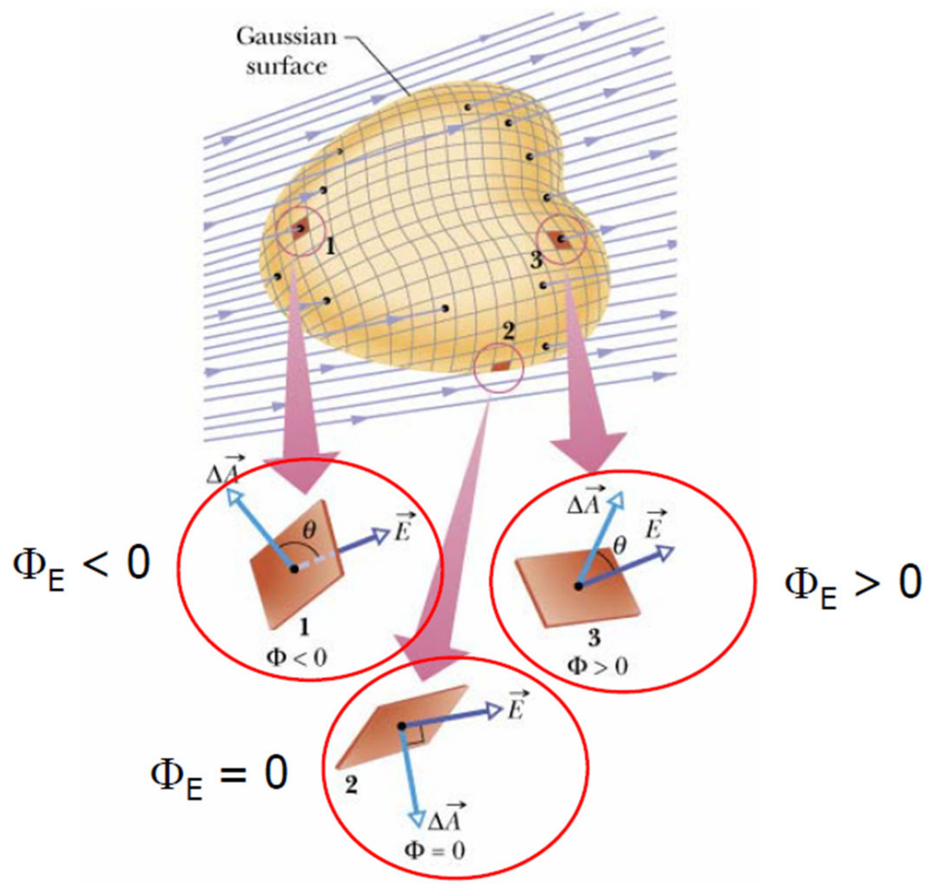
Electric Flux

The electric flux through a surface is defined to be the inner product of the electric field and the surface vector:

$$\Phi_E = \vec{E} \bullet \Delta\vec{A}$$

For a closed surface, it is $\Phi = \sum \vec{E} \cdot \Delta\vec{A}.$

$$\Phi = \oint \vec{E} \cdot d\vec{A} \qquad \text{(electric flux through a Gaussian surface).}$$



Electric flux is a scalar quantity and SI unit of newton-meters squared per coulomb ($N \cdot m^2 / C$).

Electric Flux Φ_E Through a Cube

Surface1: $\vec{E}$ antiparallel to $\vec{A}$

Φ_E = E A cos(180°) = $-EL^2$

Surface 2: $\vec{E} \parallel \vec{A}$

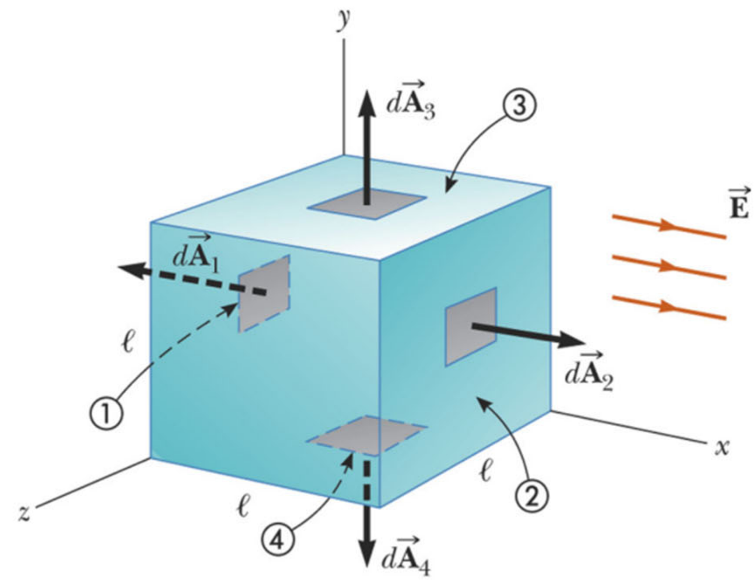
Φ_E = E A cos(0°) = $+EL^2$

Top & Bottom: $\vec{E} \perp \vec{A}$

Φ_E = E A cos(90°) = 0

Each side:

Φ_E = E A cos(90°) = 0



Net Φ_E =
$0+0+0+0+ EL^2 - EL^2 =$ **0**

Example, Flux through a closed cylinder, uniform field:

Figure 23-4 shows a Gaussian surface in the form of a cylinder of radius R immersed in a uniform electric field $\vec{E}$, with the cylinder axis parallel to the field. What is the flux Φ of the electric field through this closed surface?

KEY IDEA

We can find the flux Φ through the Gaussian surface by integrating the scalar product $\vec{E}\cdot d\vec{A}$ over that surface.

Calculations: We can do the integration by writing the flux as the sum of three terms: integrals over the left cylinder cap a, the cylindrical surface b, and the right cap c. Thus, from Eq. 23-4,

$$\Phi = \oint \vec{E}\cdot d\vec{A}$$

$$= \int_a \vec{E}\cdot d\vec{A} + \int_b \vec{E}\cdot d\vec{A} + \int_c \vec{E}\cdot d\vec{A}. \qquad (23\text{-}5)$$

For all points on the left cap, the angle θ between $\vec{E}$ and $d\vec{A}$ is 180° and the magnitude E of the field is uniform. Thus,

$$\int_a \vec{E}\cdot d\vec{A} = \int E(\cos 180°)\, dA = -E\int dA = -EA,$$

where $\int dA$ gives the cap's area A $(= \pi R^2)$. Similarly, for the

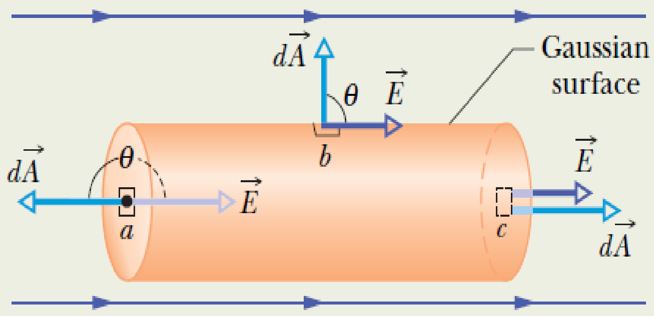


Fig. 23-4 A cylindrical Gaussian surface, closed by end caps, is immersed in a uniform electric field. The cylinder axis is parallel to the field direction.

right cap, where $\theta = 0$ for all points,

$$\int_c \vec{E}\cdot d\vec{A} = \int E(\cos 0)\, dA = EA.$$

Finally, for the cylindrical surface, where the angle θ is 90° at all points,

$$\int_b \vec{E}\cdot d\vec{A} = \int E(\cos 90°)\, dA = 0.$$

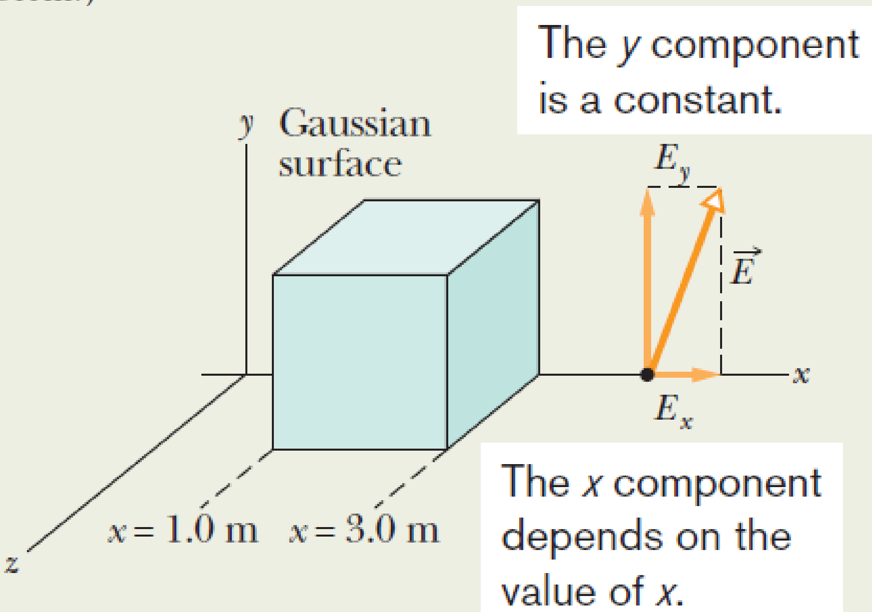
Substituting these results into Eq. 23-5 leads us to

$$\Phi = -EA + 0 + EA = 0. \qquad \text{(Answer)}$$

The net flux is zero because the field lines that represent the electric field all pass entirely through the Gaussian surface, from the left to the right.

Example, Flux through a closed cube, Non-uniform field:

A *nonuniform* electric field given by $\vec{E} = 3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}$ pierces the Gaussian cube shown in Fig. 23-5*a*. (E is in newtons per coulomb and x is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in another sample problem.)



(*a*)

Right face: An area vector **A** is always perpendicular to its surface and always points away from the interior of a Gaussian surface. Thus, the vector for any area element **dA** (small section) on the right face of the cube must point in the positive direction of the x axis. The most convenient way to express the vector is in unit-vector notation,

$$d\vec{A} = dA\hat{\mathrm{i}}.$$

$$\Phi_r = \int \vec{E} \cdot d\vec{A} = \int (3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}) \cdot (dA\hat{\mathrm{i}})$$

$$= \int [(3.0x)(dA)\hat{\mathrm{i}} \cdot \hat{\mathrm{i}} + (4.0)(dA)\hat{\mathrm{j}} \cdot \hat{\mathrm{i}}]$$

$$= \int (3.0x\, dA + 0) = 3.0 \int x\, dA.$$

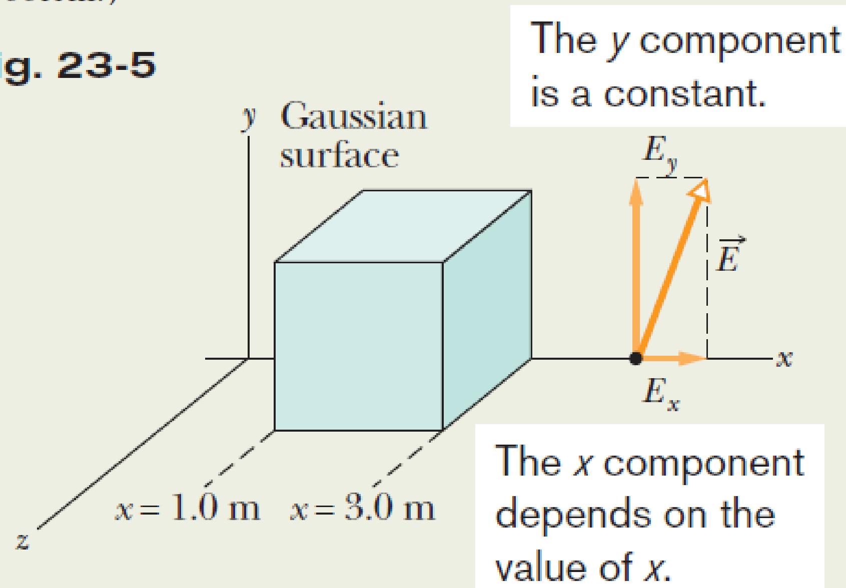
Although x is certainly a variable as we move left to right across the figure, because the right face is perpendicular to the x *axis,* every point on the face has the same x coordinate. (The y and z coordinates do not matter in our integral.) Thus, we have

$$\Phi_r = 3.0 \int (3.0)\, dA = 9.0 \int dA. = (9.0\ \mathrm{N/C})(4.0\ \mathrm{m}^2) = 36\ \mathrm{N \cdot m^2/C}. \qquad \text{(Answer)}$$

Example, Flux through a closed cube, Non-uniform field:

A *nonuniform* electric field given by $\vec{E} = 3.0x\hat{i} + 4.0\hat{j}$ pierces the Gaussian cube shown in Fig. 23-5*a*. (E is in newtons per coulomb and x is in meters.) What is the electric flux through the right face, the left face, and the top face? (We consider the other faces in another sample problem.)

Fig. 23-5



Left face: The procedure for finding the flux through the left face is the same as that for the right face. However, two factors change. (1) The differential area vector $d\vec{A}$ points in the negative direction of the x axis, and thus $d\vec{A} = -dA\hat{i}$ (Fig. 23-5*d*). (2) The term x again appears in our integration, and it is again constant over the face being considered. However, on the left face, $x = 1.0$ m. With these two changes, we find that the flux Φ_l through the left face is

$$\Phi_l = -12\ \mathrm{N \cdot m^2/C}. \qquad \text{(Answer)}$$

Top face: The differential area vector $d\vec{A}$ points in the positive direction of the y axis, and thus $d\vec{A} = dA\hat{j}$ (Fig. 23-5*e*). The flux Φ_t through the top face is then

$$\begin{aligned}\Phi_t &= \int (3.0x\hat{i} + 4.0\hat{j}) \cdot (dA\hat{j}) \\ &= \int [(3.0x)(dA)\hat{i} \cdot \hat{j} + (4.0)(dA)\hat{j} \cdot \hat{j}] \\ &= \int (0 + 4.0\,dA) = 4.0 \int dA \\ &= 16\ \mathrm{N \cdot m^2/C}. \qquad \text{(Answer)}\end{aligned}$$

Summary: Topics

- Identify that Gauss' law relates the electric field at points on a closed surface (real or imaginary, said to be a Gaussian surface) to the net charge enclosed by that surface.

- Identify that an area vector for a flat surface is a vector that is perpendicular to the surface and that has a magnitude equal to the area of the surface.

- For a closed surface, explain the algebraic signs associated with inward flux and outward flux.