Maxwell's Equations

Phy 108 course
Zaid Bin Mahbub (ZBM)
DMP, SEPS, NSU

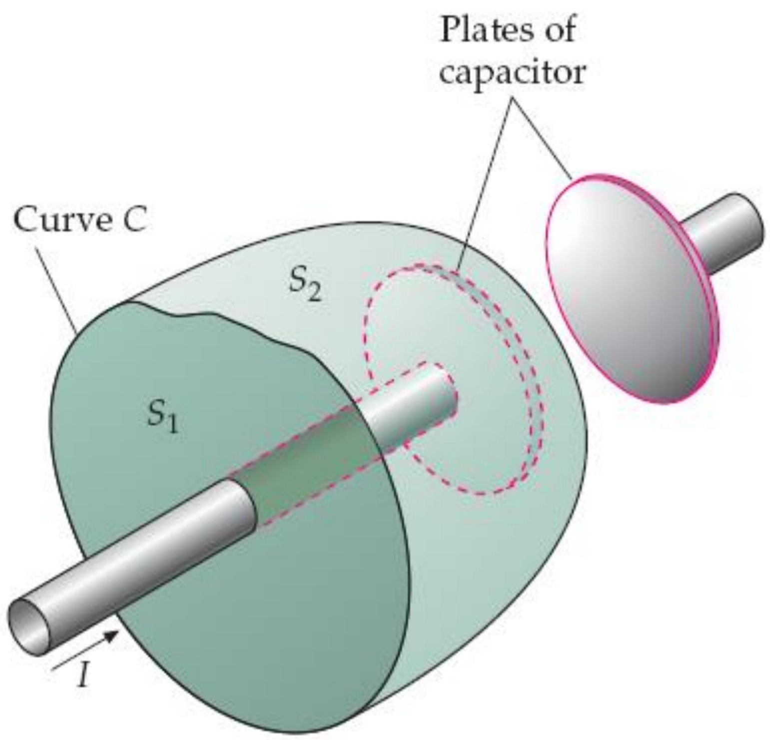
Ampère–Maxwell Law

Ampère's law relates the line integral of the magnetic field around some closed curve to the current that passes through any surface bounded by that curve:

$$\oint \vec{B}.d\vec{l} = \mu_0 I_l$$

Maxwell recognized a flaw in Ampère's law. Figure shows two different surfaces S_1 and S_2, and bounded by the same curve C which encircles a current carrying wire that is connected to a capacitor plate.

The current through surface S_1 is I but no current exists through surface S_2 because the charge stops on the capacitor plate. Thus, ambiguity exists in the phrase "the current through any surface bounded by the curve."



Two surfaces S_1 and S_2 bounded by the same curve C. The current *I* passes through surface S_1 but not through surface S_2

Ampère–Maxwell Law

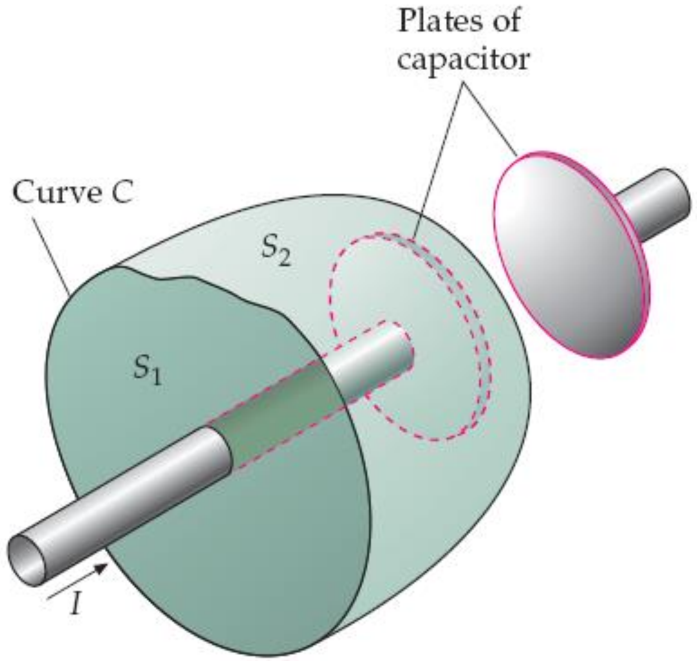
Such a problem arises when the current is not continuous.

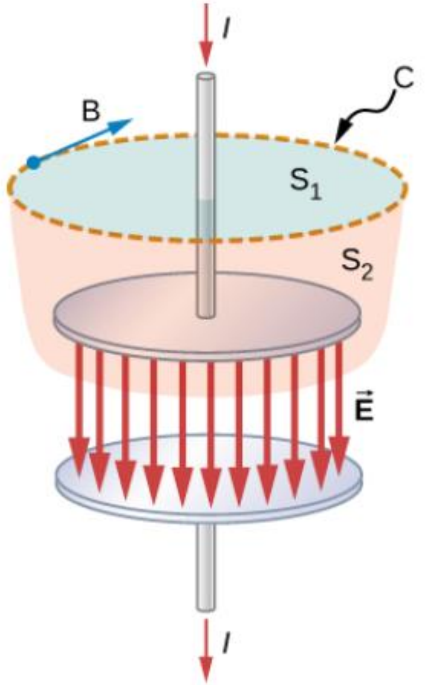
Maxwell showed that the law can be generalized to include all situations if the current in the equation is replaced by the sum of the current and another term I_d called **Maxwell's displacement current,** defined as,

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

where Φ_E is the flux of the electric field through the same surface bounded by the curve *l*.

$$\oint \vec{B}.d\vec{l} = \mu_0(I + I_d)$$





Ampère's law in this form is valid only if any electric fields present are constant in time.

James Clerk Maxwell recognized this limitation and modified Ampère's law to include time-varying electric fields

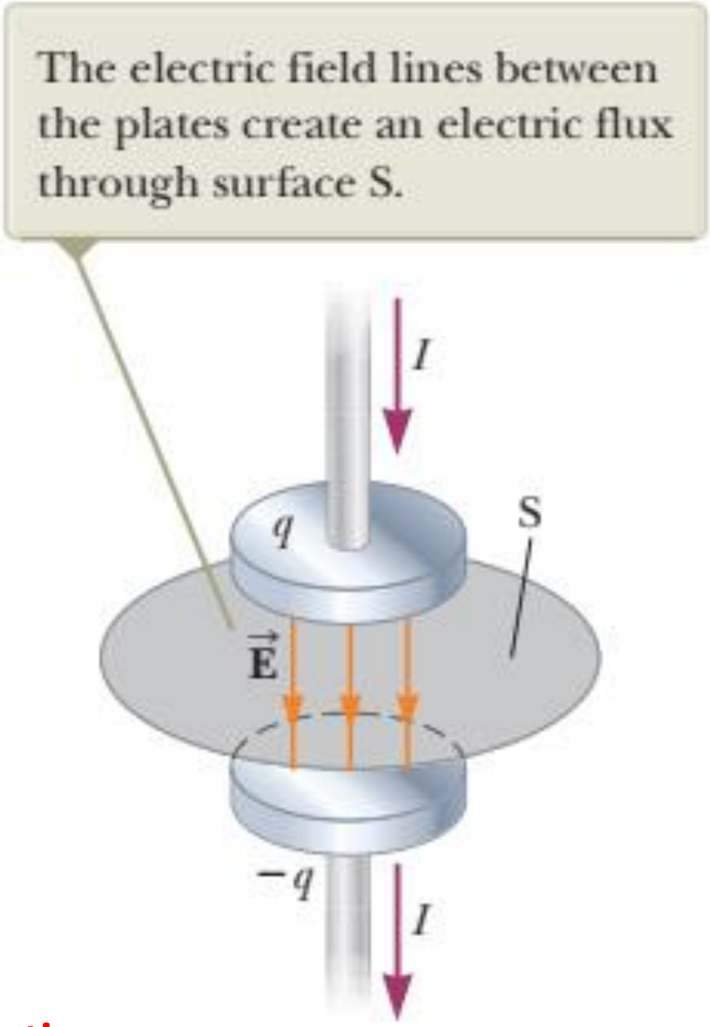
Ampère–Maxwell Law

Maxwell's displacement current, $I_d = \epsilon_0 \frac{d\Phi_E}{dt}$

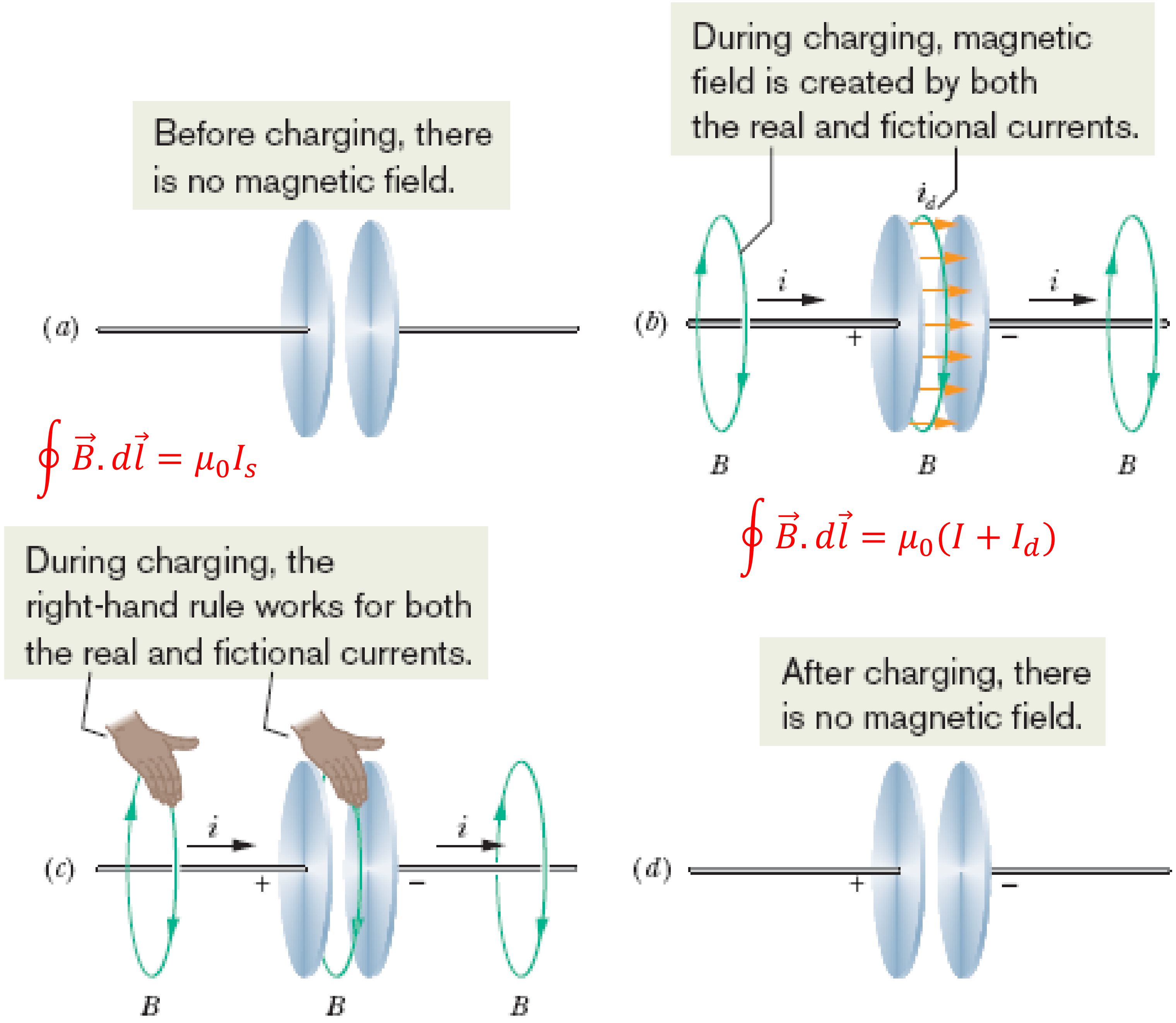
where Φ_E is the flux of the electric field through the same surface bounded by the curve *l*.

$$\oint \vec{B}.d\vec{l} = \mu_0(I + I_d) = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$$

When there is a current but no change in electric flux (such as with a wire carrying a constant current), the second term on the right side of Eq. is zero, and so it reduces to the Ampere's law. When there is a change in electric flux but no current (such as inside or outside the gap of a charging capacitor), the first term on the right side of Eq. is zero, and so it reduces to the Maxwell's law of induction.



The central point of this formalism is that magnetic fields are produced *both* by conduction currents *and* by time-varying electric fields.



$$\oint \vec{B}.d\vec{l} = \mu_0 I_s$$

$$\oint \vec{B}.d\vec{l} = \mu_0 (I + I_d)$$

(*a*) Before and (*d*) after the plates are charged, there is no magnetic field. (*b*) During the charging, magnetic field is created by both the real current and the (fictional) displacement current. (*c*) The same right hand rule works for both currents to give the direction of the magnetic field.

Maxwell's law of induction

Induced Electric fields and **Induced Magnetic Fields**

A changing magnetic flux induces an electric field, and we ended up with Faraday's law of induction ($\mathcal{E} = \oint \vec{E}.d\vec{l}$) in the form,

$$\oint \vec{E}.d\vec{l} = -\frac{d\Phi_B}{dt}$$

Here $\vec{E}$ is the electric field induced along a closed loop by the changing magnetic flux Φ_B encircled by that loop.

Can a changing electric flux induce a magnetic field?

$$\oint \vec{\boldsymbol{B}}.\boldsymbol{d}\vec{\boldsymbol{l}} = \mu_0\epsilon_0\frac{\boldsymbol{d\Phi_E}}{\boldsymbol{dt}}$$

(Maxwell's law of induction).

Here $\vec{B}$ is the magnetic field induced along a closed loop by the changing electric flux Φ_E encircled by that loop.

Maxwell’s law of induction

According to Faraday’s law, a changing magnetic flux produces an electric field whose line integral around a closed curve is proportional to the rate of change of magnetic flux through any surface bounded by the curve.

Maxwell’s modification of Ampère’s law shows that a changing electric flux produces a magnetic field whose line integral around a curve is proportional to the rate of change of the electric flux.

We thus have the interesting reciprocal result that a changing magnetic field produces an electric field (Faraday’s law) and a changing electric field produces a magnetic field (generalized form of Ampère’s law).

Maxwell's Equations: integral form

Gauss' law for electricity: $$\boldsymbol{\Phi_E = \oint_s \vec{E}.d\vec{A} = q/\epsilon_0}$$

Relates net electric flux to net enclosed electric charge

Gauss' law for magnetism: $$\boldsymbol{\Phi_B = \oint_s \vec{B}.d\vec{A} = 0}$$

Relates net magnetic flux to net enclosed magnetic charge

Faraday's law: $$\boldsymbol{\oint_c \vec{E}.d\vec{l} = -\frac{d\Phi_B}{dt}}$$

Relates induced electric field to changing magnetic flux

Ampère-Maxwell law: $$\boldsymbol{\oint_c \vec{B}.d\vec{l} = \mu_0(I + I_d) = \mu_0 I + \mu_0\epsilon_0 \frac{d\Phi_E}{dt}}$$

Relates induced magnetic field to changing electric flux and to current

* Written on the assumption that no dielectric or magnetic materials are present.

** In all four of Maxwell's equations, the integration paths *C* and the integration surfaces A are at rest and the integrations take place at an instant in time.

1.Gauss's law for static electric fields

2.Gauss's law for static magnetic fields

3.Faraday's law which says a changing magnetic field (changing with time) produces an electric field

4.Ampere-Maxwell's law which says a changing electric field (changing with time) produces a magnetic field

Maxwell's Equations

Gauss's law of electricity states that the flux of the electric field through any closed surface equals multiplied by the net charge inside the surface.

This law implies that the electric field due to a point charge varies inversely as the square of the distance from the charge.
This law describes how electric field lines diverge from a positive charge and converge on a negative charge.
Its experimental basis is Coulomb's law.

Gauss's law for magnetism states that the flux of the magnetic field through *any* closed surface is zero.

This equation describes the experimental observation that magnetic field lines do not diverge from any point in space or converge to any point in space; that is, it implies that isolated magnetic poles do not exist.

Maxwell’s Equations

Faraday’s law states that the line integral of the electric field around any closed curve equals the negative of the rate of change of the flux of the magnetic field through any surface bounded by curve (is not a closed surface, so the magnetic flux through is not necessarily zero.)

Faraday’s law describes how electric field lines encircle any area through which the magnetic flux is changing, and it relates the electric field vector to the rate of change of the magnetic field vector

Ampère’s law modified to include Maxwell’s displacement current states that the line integral of the magnetic field around any closed curve equals multiplied by the sum of the current through any surface bounded by the curve and the displacement current through the same surface.

This law describes how the magnetic field lines encircle an area through which a current or a displacement current is passing.

Maxwell's Equations: Differential form

Gauss' law (for electricity): Electrical charges are the source of the electric field

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon}$$

The electric field produced by electric charge diverges from positive charge and converges upon negative charge.

Gauss' law (for magnetism): There are no magnetic monopoles and the magnetic field lines can only circulate

$$\vec{\nabla}.\vec{B} = 0$$

- The divergence of the magnetic field at any point is zero.
- The assumption that there are no magnetic monopoles.

Faraday's law: The Curl of the electric field is caused by changing magnetic fields. A changing magnetic field can produce electric fields with field lines that close on themselves

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

A circulating electric field is produced by a magnetic field that changes with time.

Ampère-Maxwell law: The Curl of the magnetic field is caused by current of charged particles (J) or of the field they produce (dE/dt). The strength of the field depends on the material

$$\vec{\nabla} \times \vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

A circulating magnetic field is produced by an electric current and by an electric field that changes with time.

Two other vector fields are required when describing propagation of electromagnetism through matter (rather than through vacuum): the electric displacement $\vec{\boldsymbol{D}}$ and the magnetic field intensity $\vec{\boldsymbol{H}}$ (also called the "magnetizing force" or the "auxiliary field").

We assume that any material is linear, isotropic, and homogeneous.

$$\vec{\boldsymbol{D}} = \boldsymbol{\epsilon}\vec{\boldsymbol{E}} \text{ and } \vec{\boldsymbol{H}} = \frac{\vec{\boldsymbol{B}}}{\boldsymbol{\mu}}$$

where ε and μ are the electric permittivity and magnetic permeability of the material, respectively.

These are measures of the ability of the electric and magnetic fields to "permeate" the medium; if ε is increased, then a larger electric field exists within the material, if μ is larger, then the magnetic field intensity H does not penetrate as far into the medium.

Maxwell's Equations: Differential form

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon}$$

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}.\vec{D} = \rho$$

$$\vec{\nabla}.\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

Maxwell's Equations: Integral form

$$\oint_s \vec{E}.d\vec{A} = q/\epsilon_0$$

$$\oint_s \vec{B}.d\vec{A} = 0$$

$$\oint_c \vec{E}.d\vec{A} = -\frac{d\Phi_B}{dt}$$

$$\oint_c \vec{B}.d\vec{A} = \mu_0(I + I_d) = \mu_0 I + \mu_0\epsilon_0\frac{d\Phi_E}{dt}$$

$$\oint_s \vec{D}.d\vec{A} = \iiint \rho\, dV$$

$$\oint_s \vec{B}.d\vec{A} = 0$$

$$\oint_c \vec{E}.d\vec{A} = -\iint \frac{d\vec{B}}{dt}.d\vec{A}$$

$$\oint_c \vec{H}.d\vec{A} = \iint \vec{J}.d\vec{A} + \iint \frac{\partial\vec{D}}{\partial t}.d\vec{A}$$

Maxwell's equations in empty space

$$\oint \vec{E} \cdot d\vec{A} = q_{enc}/\epsilon_0$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} + \frac{1}{c^2}\frac{d\Phi_E}{dt}$$

$$\oint \vec{E} \cdot d\vec{A} = 0$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \frac{1}{c^2}\frac{d\Phi_E}{dt}$$

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} = \left(\frac{1}{4\pi \times 10^{-7}} 4\pi \times 9 \times 10^9\right)^{1/2} = \left(9 \times 10^{16}\right)^{1/2} = 3 \times 10^8 \, \frac{m}{s}$$

Speed of light

Electromagnetic wave equation

Assume a charge free, homogeneous, linear and isotropic medium, charge density, $\rho = 0$, current density, $\vec{J} = 0$

Now Maxwell's 3rd equation: Faraday's law,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (1)}$$

Now taking curl both sides,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) \quad \text{--- (2)}$$

From Ampere's-Maxwell equation,

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{--- (3)}$$

From vector calculus relation,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \cdot \vec{\nabla} \vec{E}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} \quad \text{--- (4)}$$

From equations (2), (3) and (4)

Thus, $\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu \vec{J} + \mu \epsilon \frac{\partial \vec{E}}{\partial t}\right)$ --- (5)

Since, $\rho = 0$ and $J = 0$

$$-\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu \epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \boxed{\nabla^2 \vec{E} = \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}} \quad \text{wave equation.}$$

$$\nabla^2 E_x = \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2} \qquad \text{[Scalar Wave Equation]}$$

$$\frac{\partial^2 E_x}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

Wave Equation

$$\frac{\partial^2 y}{\partial^2 x} = \frac{1}{v^2} \frac{\partial^2 y}{\partial^2 t}$$

A Solution

$$y(x,t) = y_m \sin(kx - \omega t)$$

$$v = \omega / k$$

Electromagnetic wave equation

From Ampere-Maxwell equation,

$$\vec{\nabla} \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \quad \text{—— (1)}$$

Taking curl both sides,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}\right) \quad \text{—— (2)}$$

With, $\rho = 0$ and $J = 0$

and vector rule, $\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}$

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \vec{\nabla} \times \left(\mu\epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

From Gauss' law, $\vec{\nabla} \cdot \vec{B} = 0$ —— (3)

$$-\nabla^2 \vec{B} = \mu\epsilon \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) \quad \text{—— (4)}$$

Now, from faraday's law, $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ —— (5)

Equation (4) and (5), $-\nabla^2 \vec{B} = -\mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$

$$\boxed{\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}}$$ wave equation.

If we compare with wave equation in mechanics,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$ where, v is the phase velocity.

Thus, $v = \frac{1}{\sqrt{\mu\epsilon}}$ is the speed of the EM wave.

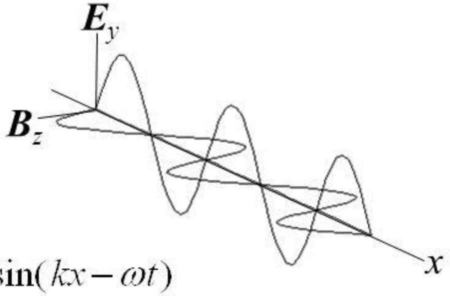
Ultimately, $c = \frac{1}{\sqrt{\mu\epsilon}}$

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$ Wave equation for ***E***

Wave equation for ***B*** $$\frac{\partial^2 \vec{B}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

These equations describe electric and magnetic waves traveling in the *x* direction

Sinusoidal Electromagnetic Waves



$$\vec{E} = \hat{y} E_{max} \sin(kx - \omega t)$$

$$\vec{B} = \hat{z} B_{max} \sin(kx - \omega t)$$

$$k = \frac{2\pi}{\lambda} \qquad \omega = 2\pi f$$

$$\lambda f = \frac{\omega}{k} = v$$

$$E_{max} = v B_{max}$$

$$\frac{1}{v^2} = \mu_0 \varepsilon_0$$

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\mu_0 \varepsilon_0 = (0.4\pi \ \mu\text{H/m})(8.85 \text{ pF/m}) = 1.11 \times 10^{-17} \text{ s}^2/\text{m}^2$$

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3.0 \times 10^8 \text{ m/s}$$

the speed of light!

The Laplacian operator

The vector electric field

The electric permittivity of free space

The vector electric field

$$\underbrace{\nabla^2 \vec{E}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

The second derivative of the vector electric field over space

The magnetic permeability of free space

The second derivative of the vector electric field with time

$$E = E_m \sin(kx - \omega t)$$

$$B = B_m \sin(kx - \omega t)$$

The Laplacian operator

The vector magnetic field

The electric permittivity of free space

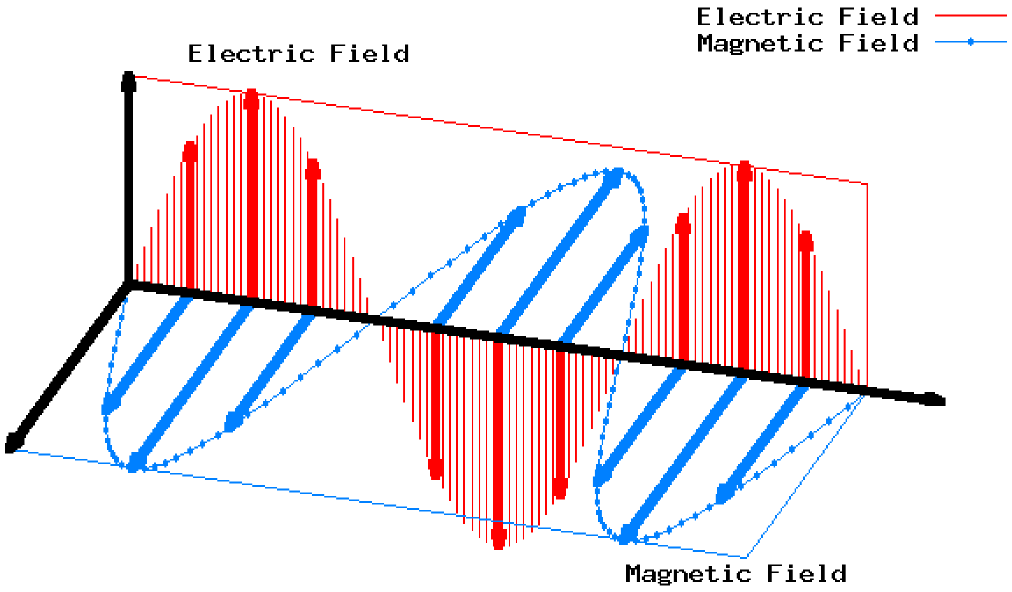
The vector magnetic field

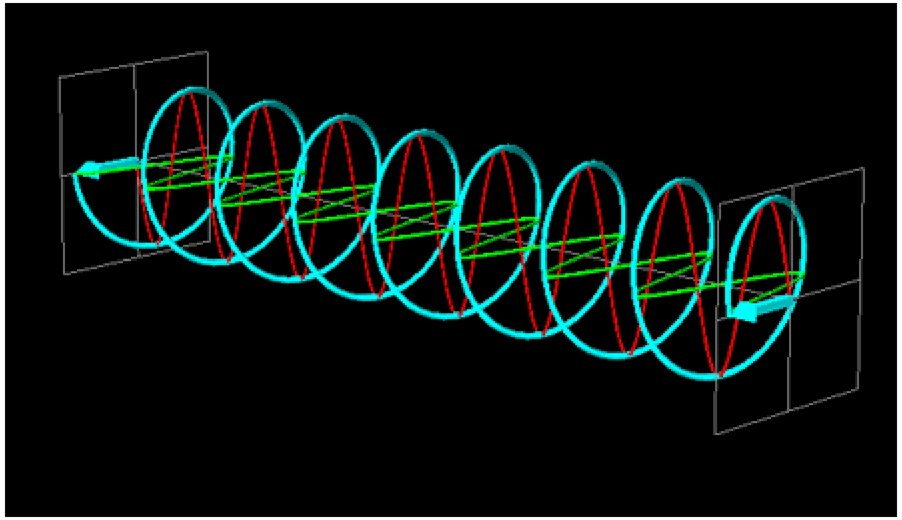
$$\underbrace{\nabla^2 \vec{B}} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

The second derivative of the vector magnetic field over space

The magnetic permeability of free space

The second derivative of the vector magnetic field with time



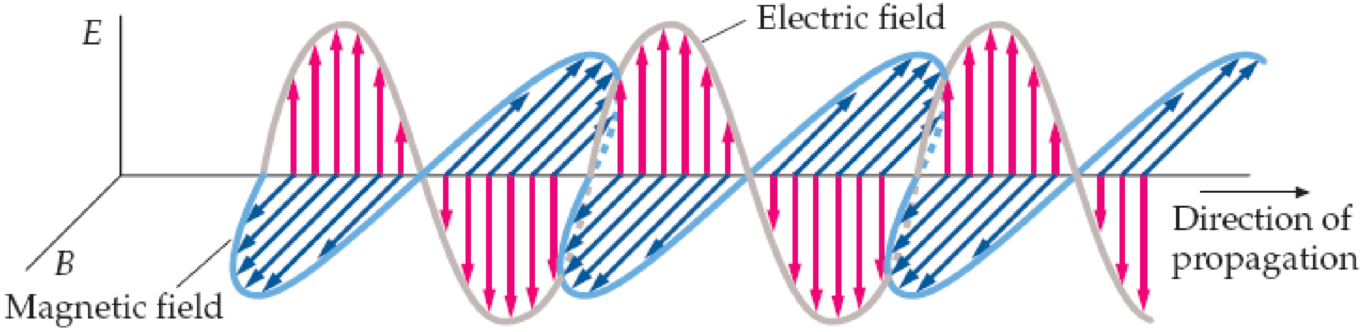


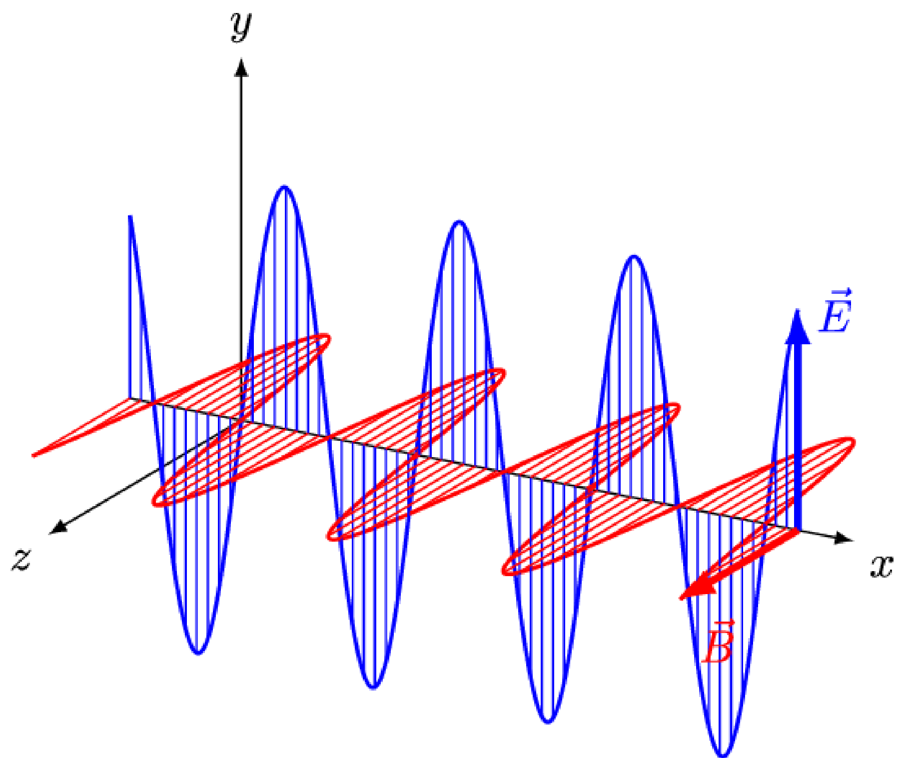
As is true for the refractive index *n*, the permittivity and permeability in matter are larger than in vacuum, $\varepsilon > \varepsilon_0$, and $\mu > \mu_0$.

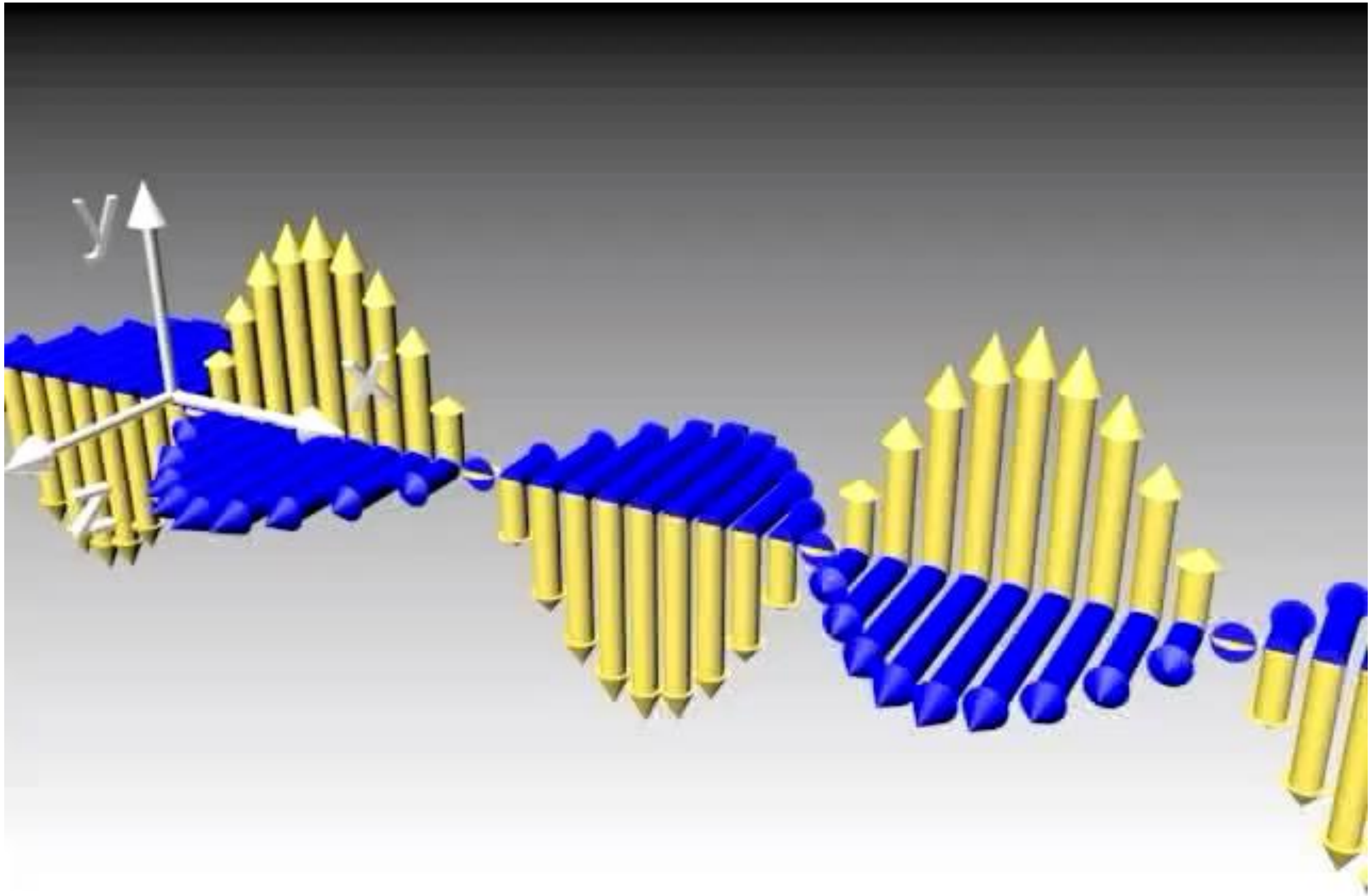
In fact (though we won't discuss it in detail), ε and μ determine the phase velocity *v* and the refractive index n via: $n = \frac{c}{v}$ and $v = \frac{1}{\sqrt{\mu\epsilon}}$

For free space, speed of light, $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$

All electromagnetic waves, including visible light, have the same speed *c* in vacuum. $\boldsymbol{c = \frac{E}{B}}$





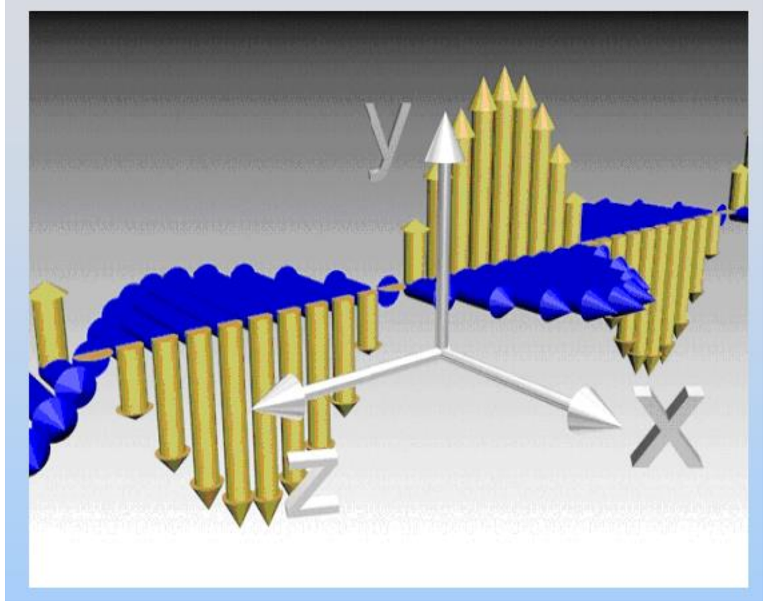

y
x
z

Direction of Propagation

$$\vec{\mathbf{E}} = \hat{\mathbf{E}} E_0 \sin(k(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}}) - \omega t); \quad \vec{\mathbf{B}} = \hat{\mathbf{B}} B_0 \sin(k(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}}) - \omega t)$$

$$\boxed{\hat{\mathbf{E}} \times \hat{\mathbf{B}} = \hat{\mathbf{p}}}$$

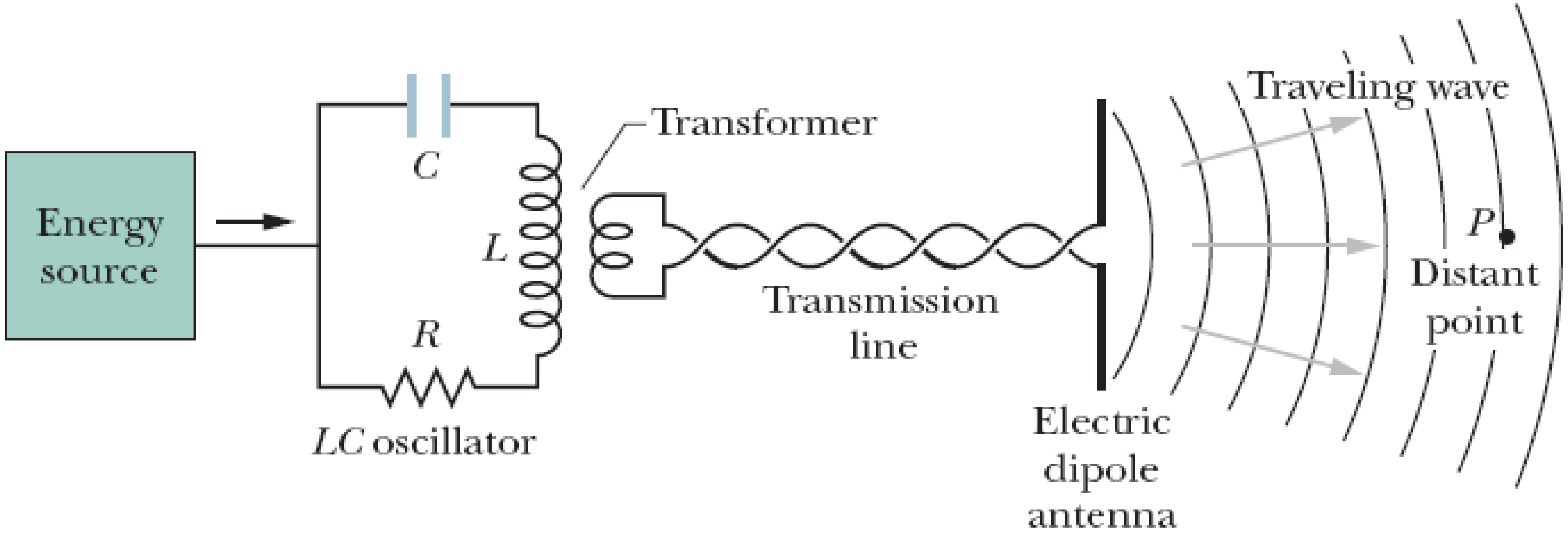
$\hat{\mathbf{E}}$	$\hat{\mathbf{B}}$	$\hat{\mathbf{p}}$	$(\hat{\mathbf{p}} \cdot \vec{\mathbf{r}})$
$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	z
$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	x
$\hat{\mathbf{k}}$	$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	y
$\hat{\mathbf{j}}$	$\hat{\mathbf{i}}$	$-\hat{\mathbf{k}}$	$-z$
$\hat{\mathbf{k}}$	$\hat{\mathbf{j}}$	$-\hat{\mathbf{i}}$	$-x$
$\hat{\mathbf{i}}$	$\hat{\mathbf{k}}$	$-\hat{\mathbf{j}}$	$-y$



Here, E_y and B_z are "the same," traveling along x axis

$$\frac{\partial B_z}{\partial x} = - \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}$$

The Traveling Electromagnetic Wave



An arrangement for generating a traveling electromagnetic wave in the shortwave radio region of the spectrum: an *LC* oscillator produces a sinusoidal current in the antenna, which generates the wave. *P* is a distant point at which a detector can monitor the wave traveling past it.

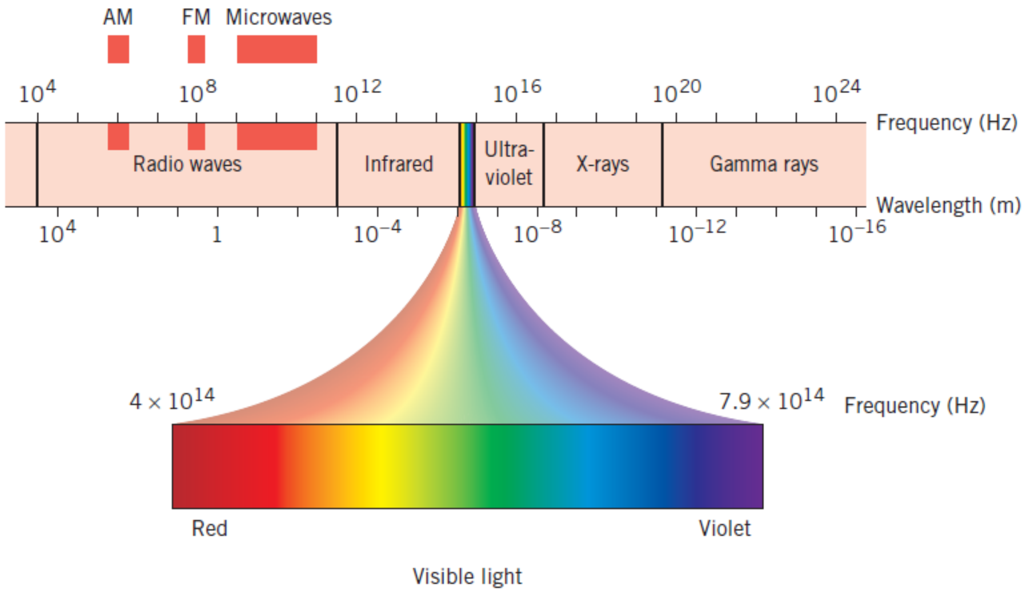
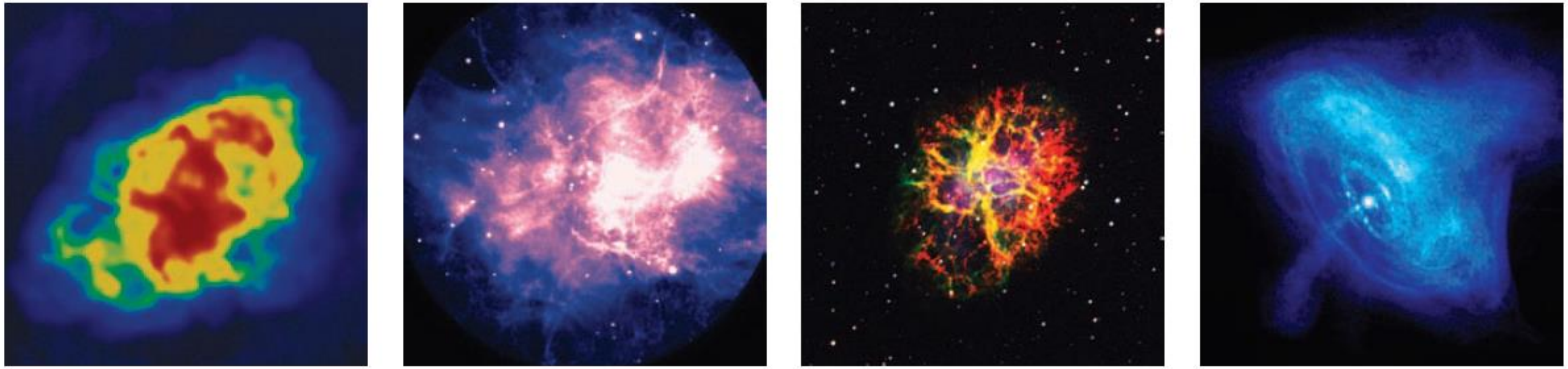


Figure 24.9 The electromagnetic spectrum.



(*a*) Radio wave   (*b*) Infrared   (*c*) Visible   (*d*) X-ray

The human body, like any object, radiates infrared radiation, and the amount emitted depends on the temperature of the body. Although infrared radiation cannot be seen by the human eye, it can be detected by sensors. An ear thermometer, like the pyroelectric thermometer shown in Figure 24.11, measures the body's temperature by determining the amount of infrared radiation that emanates from the eardrum and surrounding tissue

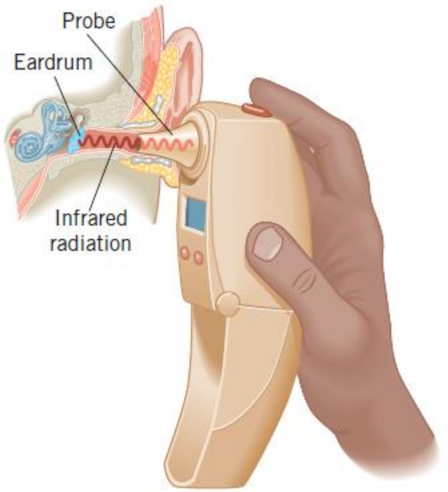
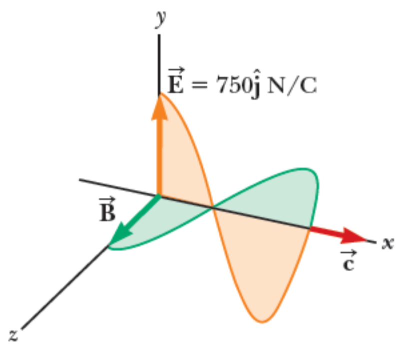


Figure 24.11 A pyroelectric thermometer measures body temperature by determining the amount of infrared radiation emitted by the eardrum and surrounding tissue.

A sinusoidal electromagnetic wave of frequency 40.0 MHz travels in free space in the *x* direction as in Figure

(A) Determine the wavelength and period of the wave.

(B) At some point and at some instant, the electric field has its maximum value of 750 N/C and is directed along the *y* axis. Calculate the magnitude and direction of the magnetic field at this position and time.



The electric field component of an electromagnetic wave traveling in a vacuum is given by $\boldsymbol{E_y} = \boldsymbol{E_0}\ \mathbf{sin}\ (\boldsymbol{kx} - \boldsymbol{\omega t})$, where **$E_0$ = 300 V/m** and **k=10^7/m**. What are the frequency of the oscillations (**in Hz**) and the direction of propagation?

The electric field component of an electromagnetic plane wave traveling in a vacuum is given by $\vec{\boldsymbol{E}} = \boldsymbol{E_0}\ \mathbf{sin}\ (\boldsymbol{kx} + \boldsymbol{\omega t})\hat{\boldsymbol{y}}$, where **$E_0$ = 300 V/m** and **k=10^7/m**. What is the magnetic field component of the electromagnetic wave? What is the frequency of the electromagnetic wave (**in Hz**)?