CHAPTER 29

# Magnetic Fields Due to Currents

## 29-1 MAGNETIC FIELD DUE TO A CURRENT

### Learning Objectives

*After reading this module, you should be able to . . .*

**29.01** Sketch a current-length element in a wire and indicate the direction of the magnetic field that it sets up at a given point near the wire.

**29.02** For a given point near a wire and a given current-element in the wire, determine the magnitude and direction of the magnetic field due to that element.

**29.03** Identify the magnitude of the magnetic field set up by a current-length element at a point in line with the direction of that element.

**29.04** For a point to one side of a long straight wire carrying current, apply the relationship between the magnetic field magnitude, the current, and the distance to the point.

**29.05** For a point to one side of a long straight wire carrying current, use a right-hand rule to determine the direction of the field vector.

**29.06** Identify that around a long straight wire carrying current, the magnetic field lines form circles.

**29.07** For a point to one side of the end of a semi-infinite wire carrying current, apply the relationship between the magnetic field magnitude, the current, and the distance to the point.

**29.08** For the center of curvature of a circular arc of wire carrying current, apply the relationship between the magnetic field magnitude, the current, the radius of curvature, and the angle subtended by the arc (in radians).

**29.09** For a point to one side of a short straight wire carrying current, integrate the Biot–Savart law to find the magnetic field set up at the point by the current.

### Key Ideas

- The magnetic field set up by a current-carrying conductor can be found from the Biot–Savart law. This law asserts that the contribution $d\vec{B}$ to the field produced by a current-length element $i\,d\vec{s}$ at a point $P$ located a distance $r$ from the current element is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \hat{r}}{r^2} \quad \text{(Biot–Savart law).}$$

Here $\hat{r}$ is a unit vector that points from the element toward $P$. The quantity $\mu_0$, called the permeability constant, has the value

$$4\pi \times 10^{-7}\ \mathrm{T\cdot m/A} \approx 1.26 \times 10^{-6}\ \mathrm{T\cdot m/A}.$$

- For a long straight wire carrying a current $i$, the Biot–Savart law gives, for the magnitude of the magnetic field at a perpendicular distance $R$ from the wire,

$$B = \frac{\mu_0 i}{2\pi R} \quad \text{(long straight wire).}$$

- The magnitude of the magnetic field at the center of a circular arc, of radius $R$ and central angle $\phi$ (in radians), carrying current $i$, is

$$B = \frac{\mu_0 i \phi}{4\pi R} \quad \text{(at center of circular arc).}$$

### What Is Physics?

One basic observation of physics is that a moving charged particle produces a magnetic field around itself. Thus a current of moving charged particles produces a magnetic field around the current. This feature of *electromagnetism,* which is the combined study of electric and magnetic effects, came as a surprise to the people who discovered it. Surprise or not, this feature has become enormously important in everyday life because it is the basis of countless electromagnetic devices. For example, a magnetic field is produced in maglev trains and other devices used to lift heavy loads.

Our first step in this chapter is to find the magnetic field due to the current in a very small section of current-carrying wire. Then we shall find the magnetic field due to the entire wire for several different arrangements of the wire.

## Calculating the Magnetic Field Due to a Current

Figure 29-1 shows a wire of arbitrary shape carrying a current $i$. We want to find the magnetic field $\vec{B}$ at a nearby point $P$. We first mentally divide the wire into differential elements $ds$ and then define for each element a length vector $d\vec{s}$ that has length $ds$ and whose direction is the direction of the current in $ds$. We can then define a differential *current-length element* to be $i\,d\vec{s}$; we wish to calculate the field $d\vec{B}$ produced at $P$ by a typical current-length element. From experiment we find that magnetic fields, like electric fields, can be superimposed to find a net field. Thus, we can calculate the net field $\vec{B}$ at $P$ by summing, via integration, the contributions $d\vec{B}$ from all the current-length elements. However, this summation is more challenging than the process associated with electric fields because of a complexity; whereas a charge element $dq$ producing an electric field is a scalar, a current-length element $i\,d\vec{s}$ producing a magnetic field is a vector, being the product of a scalar and a vector.

***Magnitude.*** The magnitude of the field $d\vec{B}$ produced at point $P$ at distance $r$ by a current-length element $i\,d\vec{s}$ turns out to be

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}, \tag{29-1}$$

where $\theta$ is the angle between the directions of $d\vec{s}$ and $\hat{r}$, a unit vector that points from $ds$ toward $P$. Symbol $\mu_0$ is a constant, called the *permeability constant,* whose value is defined to be exactly

$$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A} \approx 1.26 \times 10^{-6}\,\mathrm{T\cdot m/A}. \tag{29-2}$$

***Direction.*** The direction of $d\vec{B}$, shown as being into the page in Fig. 29-1, is that of the cross product $d\vec{s} \times \hat{r}$. We can therefore write Eq. 29-1 in vector form as

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s}\times\hat{r}}{r^2} \quad \text{(Biot–Savart law).} \tag{29-3}$$

This vector equation and its scalar form, Eq. 29-1, are known as the **law of Biot and Savart** (rhymes with "Leo and bazaar"). The law, which is experimentally deduced, is an inverse-square law. We shall use this law to calculate the net magnetic field $\vec{B}$ produced at a point by various distributions of current.

Here is one easy distribution: If current in a wire is either directly toward or directly away from a point $P$ of measurement, can you see from Eq. 29-1 that the magnetic field at $P$ from the current is simply zero (the angle $\theta$ is either 0° for *toward* or 180° for *away*, and both result in $\sin\theta = 0$)?

This element of current creates a magnetic field at $P$, into the page.

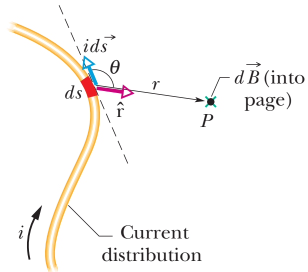


**Figure 29-1** A current-length element $i\,d\vec{s}$ produces a differential magnetic field $d\vec{B}$ at point $P$. The green × (the tail of an arrow) at the dot for point $P$ indicates that $d\vec{B}$ is directed *into* the page there.

### Magnetic Field Due to a Current in a Long Straight Wire

Shortly we shall use the law of Biot and Savart to prove that the magnitude of the magnetic field at a perpendicular distance $R$ from a long (infinite) straight wire carrying a current $i$ is given by

$$B = \frac{\mu_0 i}{2\pi R} \quad \text{(long straight wire).} \tag{29-4}$$

The field magnitude $B$ in Eq. 29-4 depends only on the current and the perpendicular distance $R$ of the point from the wire. We shall show in our derivation that the field lines of $\vec{B}$ form concentric circles around the wire, as Fig. 29-2 shows

The magnetic field vector at any point is tangent to a circle.

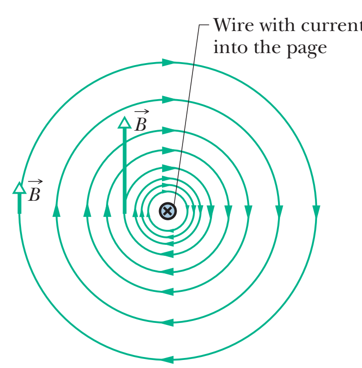


**Figure 29-2** The magnetic field lines produced by a current in a long straight wire form concentric circles around the wire. Here the current is into the page, as indicated by the ×.

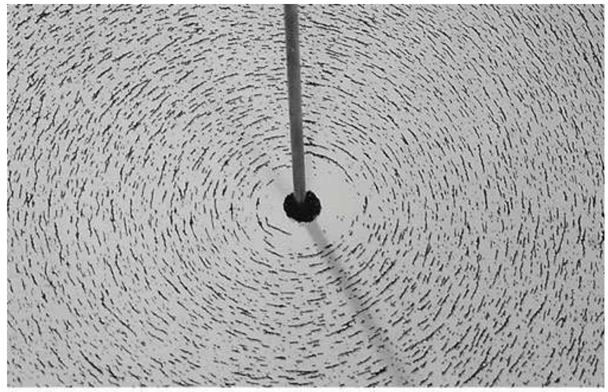

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**Figure 29-3** Iron filings that have been sprinkled onto cardboard collect in concentric circles when current is sent through the central wire. The alignment, which is along magnetic field lines, is caused by the magnetic field produced by the current.

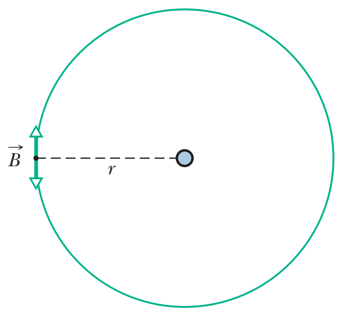


**Figure 29-4** The magnetic field vector $\vec{B}$ is perpendicular to the radial line extending from a long straight wire with current, but which of the two perpendicular vectors is it?

and as the iron filings in Fig. 29-3 suggest. The increase in the spacing of the lines in Fig. 29-2 with increasing distance from the wire represents the $1/R$ decrease in the magnitude of $\vec{B}$ predicted by Eq. 29-4. The lengths of the two vectors $\vec{B}$ in the figure also show the $1/R$ decrease.

***Directions.*** Plugging values into Eq. 29-4 to find the field magnitude $B$ at a given radius is easy. What is difficult for many students is finding the direction of a field vector $\vec{B}$ at a given point. The field lines form circles around a long straight wire, and the field vector at any point on a circle must be tangent to the circle. That means it must be perpendicular to a radial line extending to the point from the wire. But there are two possible directions for that perpendicular vector, as shown in Fig. 29-4. One is correct for current into the figure, and the other is correct for current out of the figure. How can you tell which is which? Here is a simple right-hand rule for telling which vector is correct:

> *Curled–straight right-hand rule:* Grasp the element in your right hand with your extended thumb pointing in the direction of the current. Your fingers will then naturally curl around in the direction of the magnetic field lines due to that element.

The result of applying this right-hand rule to the current in the straight wire of Fig. 29-2 is shown in a side view in Fig. 29-5*a*. To determine the direction of the magnetic field $\vec{B}$ set up at any particular point by this current, mentally wrap your right hand around the wire with your thumb in the direction of the current. Let your fingertips pass through the point; their direction is then the direction of the magnetic field at that point. In the view of Fig. 29-2, $\vec{B}$ at any point is *tangent to a magnetic field line;* in the view of Fig. 29-5, it is *perpendicular to a dashed radial line connecting the point and the current.*

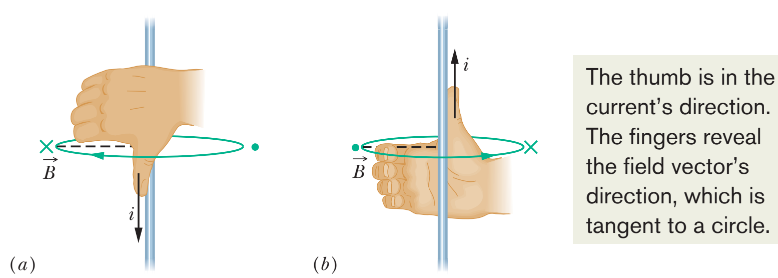


**Figure 29-5** A right-hand rule gives the direction of the magnetic field due to a current in a wire. (*a*) The situation of Fig. 29-2, seen from the side. The magnetic field $\vec{B}$ at any point to the left of the wire is perpendicular to the dashed radial line and directed into the page, in the direction of the fingertips, as indicated by the ×. (*b*) If the current is reversed, $\vec{B}$ at any point to the left is still perpendicular to the dashed radial line but now is directed out of the page, as indicated by the dot.

## Proof of Equation 29-4

Figure 29-6, which is just like Fig. 29-1 except that now the wire is straight and of infinite length, illustrates the task at hand. We seek the field $\vec{B}$ at point $P$, a perpendicular distance $R$ from the wire. The magnitude of the differential magnetic field produced at $P$ by the current-length element $i\,d\vec{s}$ located a distance $r$ from $P$ is given by Eq. 29-1:

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds\sin\theta}{r^2}.$$

The direction of $d\vec{B}$ in Fig. 29-6 is that of the vector $d\vec{s} \times \hat{r}$—namely, directly into the page.

Note that $d\vec{B}$ at point $P$ has this same direction for all the current-length elements into which the wire can be divided. Thus, we can find the magnitude of the magnetic field produced at $P$ by the current-length elements in the upper half of the infinitely long wire by integrating $dB$ in Eq. 29-1 from 0 to $\infty$.

Now consider a current-length element in the lower half of the wire, one that is as far below $P$ as $d\vec{s}$ is above $P$. By Eq. 29-3, the magnetic field produced at $P$ by this current-length element has the same magnitude and direction as that from element $i\,d\vec{s}$ in Fig. 29-6. Further, the magnetic field produced by the lower half of the wire is exactly the same as that produced by the upper half. To find the magnitude of the *total* magnetic field $\vec{B}$ at $P$, we need only multiply the result of our integration by 2. We get

$$B = 2\int_0^\infty dB = \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{\sin\theta\,ds}{r^2}. \qquad (29\text{-}5)$$

The variables $\theta$, $s$, and $r$ in this equation are not independent; Fig. 29-6 shows that they are related by

$$r = \sqrt{s^2 + R^2}$$

and

$$\sin\theta = \sin(\pi - \theta) = \frac{R}{\sqrt{s^2 + R^2}}.$$

With these substitutions and integral 19 in Appendix E, Eq. 29-5 becomes

$$\begin{aligned} B &= \frac{\mu_0 i}{2\pi}\int_0^\infty \frac{R\,ds}{(s^2 + R^2)^{3/2}} \\ &= \frac{\mu_0 i}{2\pi R}\left[\frac{s}{(s^2 + R^2)^{1/2}}\right]_0^\infty = \frac{\mu_0 i}{2\pi R}, \end{aligned} \qquad (29\text{-}6)$$

as we wanted. Note that the magnetic field at $P$ due to either the lower half or the upper half of the infinite wire in Fig. 29-6 is half this value; that is,

$$B = \frac{\mu_0 i}{4\pi R} \quad \text{(semi-infinite straight wire)}. \qquad (29\text{-}7)$$

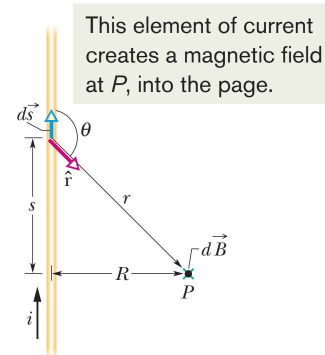


**Figure 29-6** Calculating the magnetic field produced by a current $i$ in a long straight wire. The field $d\vec{B}$ at $P$ associated with the current-length element $i\,d\vec{s}$ is directed into the page, as shown.

## Magnetic Field Due to a Current in a Circular Arc of Wire

To find the magnetic field produced at a point by a current in a curved wire, we would again use Eq. 29-1 to write the magnitude of the field produced by a single current-length element, and we would again integrate to find the net field produced by all the current-length elements. That integration can be difficult, depending on the shape of the wire; it is fairly straightforward, however, when the wire is a circular arc and the point is the center of curvature.

Figure 29-7*a* shows such an arc-shaped wire with central angle $\phi$, radius $R$, and center $C$, carrying current $i$. At $C$, each current-length element $i\,d\vec{s}$ of the wire produces a magnetic field of magnitude $dB$ given by Eq. 29-1. Moreover, as Fig. 29-7*b* shows, no matter where the element is located on the wire, the angle $\theta$

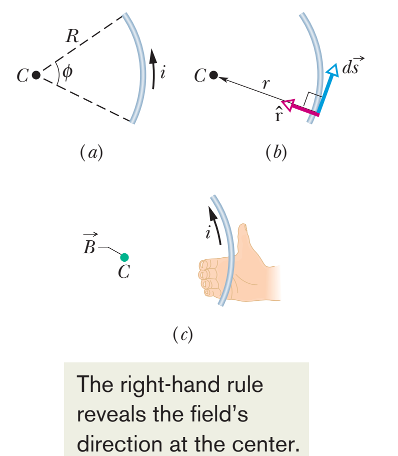


**Figure 29-7** (*a*) A wire in the shape of a circular arc with center $C$ carries current $i$. (*b*) For any element of wire along the arc, the angle between the directions of $d\vec{s}$ and $\hat{r}$ is 90°. (*c*) Determining the direction of the magnetic field at the center $C$ due to the current in the wire; the field is out of the page, in the direction of the fingertips, as indicated by the colored dot at $C$.

between the vectors $d\vec{s}$ and $\hat{r}$ is 90°; also, $r = R$. Thus, by substituting $R$ for $r$ and 90° for $\theta$ in Eq. 29-1, we obtain

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds \sin 90°}{R^2} = \frac{\mu_0}{4\pi}\frac{i\,ds}{R^2}. \tag{29-8}$$

The field at $C$ due to each current-length element in the arc has this magnitude.

***Directions.*** How about the direction of the differential field $d\vec{B}$ set up by an element? From above we know that the vector must be perpendicular to a radial line extending through point $C$ from the element, either into the plane of Fig. 29-7*a* or out of it. To tell which direction is correct, we use the right-hand rule for any of the elements, as shown in Fig. 29-7*c*. Grasping the wire with the thumb in the direction of the current and bringing the fingers into the region near $C$, we see that the vector $d\vec{B}$ due to any of the differential elements is out of the plane of the figure, not into it.

***Total Field.*** To find the total field at $C$ due to all the elements on the arc, we need to add all the differential field vectors $d\vec{B}$. However, because the vectors are all in the same direction, we do not need to find components. We just sum the magnitudes $dB$ as given by Eq. 29-8. Since we have a vast number of those magnitudes, we sum via integration. We want the result to indicate how the total field depends on the angle $\phi$ of the arc (rather than the arc length). So, in Eq. 29-8 we switch from $ds$ to $d\phi$ by using the identity $ds = R\,d\phi$. The summation by integration then becomes

$$B = \int dB = \int_0^{\phi} \frac{\mu_0}{4\pi}\frac{iR\,d\phi}{R^2} = \frac{\mu_0 i}{4\pi R}\int_0^{\phi} d\phi.$$

Integrating, we find that

$$B = \frac{\mu_0 i \phi}{4\pi R} \quad \text{(at center of circular arc)}. \tag{29-9}$$

***Heads Up.*** Note that this equation gives us the magnetic field *only* at the center of curvature of a circular arc of current. When you insert data into the equation, you must be careful to express $\phi$ in radians rather than degrees. For example, to find the magnitude of the magnetic field at the center of a full circle of current, you would substitute $2\pi$ rad for $\phi$ in Eq. 29-9, finding

$$B = \frac{\mu_0 i(2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \quad \text{(at center of full circle)}. \tag{29-10}$$

## Sample Problem 29.01 Magnetic field at the center of a circular arc of current

The wire in Fig. 29-8*a* carries a current $i$ and consists of a circular arc of radius $R$ and central angle $\pi/2$ rad, and two straight sections whose extensions intersect the center $C$ of the arc. What magnetic field $\vec{B}$ (magnitude and direction) does the current produce at $C$?

### KEY IDEAS

We can find the magnetic field $\vec{B}$ at point $C$ by applying the Biot–Savart law of Eq. 29-3 to the wire, point by point along the full length of the wire. However, the application of Eq. 29-3 can be simplified by evaluating $\vec{B}$ separately for the three distinguishable sections of the wire—namely, (1) the straight section at the left, (2) the straight section at the right, and (3) the circular arc.

***Straight sections:*** For any current-length element in section 1, the angle $\theta$ between $d\vec{s}$ and $\hat{r}$ is zero (Fig. 29-8*b*); so Eq. 29-1 gives us

$$dB_1 = \frac{\mu_0}{4\pi}\frac{i\,ds \sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{i\,ds \sin 0}{r^2} = 0.$$

Thus, the current along the entire length of straight section 1 contributes no magnetic field at $C$:

$$B_1 = 0.$$

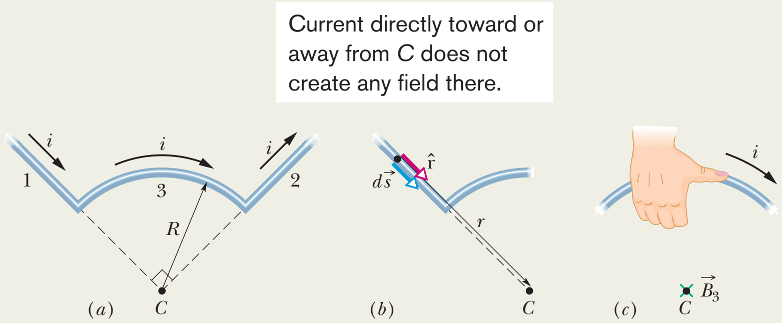


**Figure 29-8** (*a*) A wire consists of two straight sections (1 and 2) and a circular arc (3), and carries current $i$. (*b*) For a current-length element in section 1, the angle between $d\vec{s}$ and $\hat{r}$ is zero. (*c*) Determining the direction of magnetic field $\vec{B}_3$ at $C$ due to the current in the circular arc; the field is into the page there.

The same situation prevails in straight section 2, where the angle $\theta$ between $d\vec{s}$ and $\hat{r}$ for any current-length element is 180°. Thus,

$$B_2 = 0.$$

***Circular arc:*** Application of the Biot–Savart law to evaluate the magnetic field at the center of a circular arc leads to Eq. 29-9 ($B = \mu_0 i\phi/4\pi R$). Here the central angle $\phi$ of the arc is $\pi/2$ rad. Thus from Eq. 29-9, the magnitude of the magnetic field $\vec{B}_3$ at the arc's center $C$ is

$$B_3 = \frac{\mu_0 i(\pi/2)}{4\pi R} = \frac{\mu_0 i}{8R}.$$

To find the direction of $\vec{B}_3$, we apply the right-hand rule displayed in Fig. 29-5. Mentally grasp the circular arc with your right hand as in Fig. 29-8*c*, with your thumb in the direction of the current. The direction in which your fingers curl around the wire indicates the direction of the magnetic field lines around the wire. They form circles around the wire, coming out of the page above the arc and going into the page inside the arc. In the region of point $C$ (inside the arc), your fingertips point *into the plane* of the page. Thus, $\vec{B}_3$ is directed into that plane.

***Net field:*** Generally, we combine multiple magnetic fields as vectors. Here, however, only the circular arc produces a magnetic field at point $C$. Thus, we can write the magnitude of the net field $\vec{B}$ as

$$B = B_1 + B_2 + B_3 = 0 + 0 + \frac{\mu_0 i}{8R} = \frac{\mu_0 i}{8R}. \qquad \text{(Answer)}$$

The direction of $\vec{B}$ is the direction of $\vec{B}_3$—namely, into the plane of Fig. 29-8.

## Sample Problem 29.02 Magnetic field off to the side of two long straight currents

Figure 29-9*a* shows two long parallel wires carrying currents $i_1$ and $i_2$ in opposite directions. What are the magnitude and direction of the net magnetic field at point $P$? Assume the following values: $i_1 = 15$ A, $i_2 = 32$ A, and $d = 5.3$ cm.

### KEY IDEAS

(1) The net magnetic field $\vec{B}$ at point $P$ is the vector sum of the magnetic fields due to the currents in the two wires. (2) We can find the magnetic field due to any current by applying the Biot–Savart law to the current. For points near the current in a long straight wire, that law leads to Eq. 29-4.

***Finding the vectors:*** In Fig. 29-9*a*, point $P$ is distance $R$ from both currents $i_1$ and $i_2$. Thus, Eq. 29-4 tells us that at point $P$ those currents produce magnetic fields $\vec{B}_1$ and $\vec{B}_2$ with magnitudes

$$B_1 = \frac{\mu_0 i_1}{2\pi R} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi R}.$$

In the right triangle of Fig. 29-9*a*, note that the base angles (between sides $R$ and $d$) are both 45°. This allows us to write

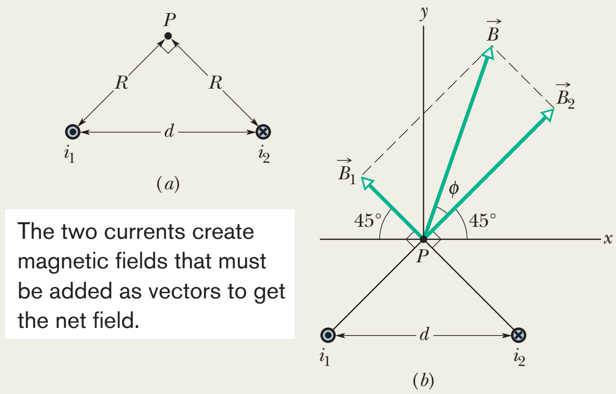


**Figure 29-9** (*a*) Two wires carry currents $i_1$ and $i_2$ in opposite directions (out of and into the page). Note the right angle at $P$. (*b*) The separate fields $\vec{B}_1$ and $\vec{B}_2$ are combined vectorially to yield the net field $\vec{B}$.

$\cos 45° = R/d$ and replace $R$ with $d \cos 45°$. Then the field magnitudes $B_1$ and $B_2$ become

$$B_1 = \frac{\mu_0 i_1}{2\pi d \cos 45°} \quad \text{and} \quad B_2 = \frac{\mu_0 i_2}{2\pi d \cos 45°}.$$

We want to combine $\vec{B}_1$ and $\vec{B}_2$ to find their vector sum, which is the net field $\vec{B}$ at $P$. To find the directions of $\vec{B}_1$ and $\vec{B}_2$, we apply the right-hand rule of Fig. 29-5 to each current in Fig. 29-9*a*. For wire 1, with current out of the page, we mentally grasp the wire with the right hand, with the thumb pointing out of the page. Then the curled fingers indicate that the field lines run counterclockwise. In particular, in the region of point $P$, they are directed upward to the left. Recall that the magnetic field at a point near a long, straight current-carrying wire must be directed perpendicular to a radial line between the point and the current. Thus, $\vec{B}_1$ must be directed upward to the left as drawn in Fig. 29-9*b*. (Note carefully the perpendicular symbol between vector $\vec{B}_1$ and the line connecting point $P$ and wire 1.)

Repeating this analysis for the current in wire 2, we find that $\vec{B}_2$ is directed upward to the right as drawn in Fig. 29-9*b*.

***Adding the vectors:*** We can now vectorially add $\vec{B}_1$ and $\vec{B}_2$ to find the net magnetic field $\vec{B}$ at point $P$, either by using a vector-capable calculator or by resolving the vectors into components and then combining the components of $\vec{B}$. However, in Fig. 29-9*b*, there is a third method: Because $\vec{B}_1$ and $\vec{B}_2$ are perpendicular to each other, they form the legs of a right triangle, with $\vec{B}$ as the hypotenuse. So,

$$B = \sqrt{B_1^2 + B_2^2} = \frac{\mu_0}{2\pi d(\cos 45°)}\sqrt{i_1^2 + i_2^2}$$
$$= \frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})\sqrt{(15\ \text{A})^2 + (32\ \text{A})^2}}{(2\pi)(5.3 \times 10^{-2}\ \text{m})(\cos 45°)}$$
$$= 1.89 \times 10^{-4}\ \text{T} \approx 190\ \mu\text{T}. \qquad \text{(Answer)}$$

The angle $\phi$ between the directions of $\vec{B}$ and $\vec{B}_2$ in Fig. 29-9*b* follows from

$$\phi = \tan^{-1}\frac{B_1}{B_2},$$

which, with $B_1$ and $B_2$ as given above, yields

$$\phi = \tan^{-1}\frac{i_1}{i_2} = \tan^{-1}\frac{15\ \text{A}}{32\ \text{A}} = 25°.$$

The angle between $\vec{B}$ and the $x$ axis shown in Fig. 29-9*b* is then

$$\phi + 45° = 25° + 45° = 70°. \qquad \text{(Answer)}$$

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# 29-2 FORCE BETWEEN TWO PARALLEL CURRENTS

## Learning Objectives

*After reading this module, you should be able to . . .*

**29.10** Given two parallel or antiparallel currents, find the magnetic field of the first current at the location of the second current and then find the force acting on that second current.

**29.11** Identify that parallel currents attract each other, and antiparallel currents repel each other.

**29.12** Describe how a rail gun works.

## Key Ideas

- Parallel wires carrying currents in the same direction attract each other, whereas parallel wires carrying currents in opposite directions repel each other. The magnitude of the force on a length $L$ of either wire is

$$F_{ba} = i_b L B_a \sin 90° = \frac{\mu_0 L i_a i_b}{2\pi d},$$

where $d$ is the wire separation, and $i_a$ and $i_b$ are the currents in the wires.

## Force Between Two Parallel Currents

Two long parallel wires carrying currents exert forces on each other. Figure 29-10 shows two such wires, separated by a distance $d$ and carrying currents $i_a$ and $i_b$. Let us analyze the forces on these wires due to each other.

We seek first the force on wire $b$ in Fig. 29-10 due to the current in wire $a$. That current produces a magnetic field $\vec{B}_a$, and it is this magnetic field that actually causes the force we seek. To find the force, then, we need the magnitude and direction of the field $\vec{B}_a$ *at the site of wire b*. The magnitude of $\vec{B}_a$ at every point of wire $b$ is, from Eq. 29-4,

$$B_a = \frac{\mu_0 i_a}{2\pi d}. \qquad (29\text{-}11)$$

The curled–straight right-hand rule tells us that the direction of $\vec{B}_a$ at wire $b$ is down, as Fig. 29-10 shows. Now that we have the field, we can find the force it produces on wire $b$. Equation 28-26 tells us that the force $\vec{F}_{ba}$ on a length $L$ of wire $b$ due to the external magnetic field $\vec{B}_a$ is

$$\vec{F}_{ba} = i_b\vec{L} \times \vec{B}_a, \tag{29-12}$$

where $\vec{L}$ is the length vector of the wire. In Fig. 29-10, vectors $\vec{L}$ and $\vec{B}_a$ are perpendicular to each other, and so with Eq. 29-11, we can write

$$F_{ba} = i_bLB_a \sin 90° = \frac{\mu_0 L i_a i_b}{2\pi d}. \tag{29-13}$$

The direction of $\vec{F}_{ba}$ is the direction of the cross product $\vec{L} \times \vec{B}_a$. Applying the right-hand rule for cross products to $\vec{L}$ and $\vec{B}_a$ in Fig. 29-10, we see that $\vec{F}_{ba}$ is directly toward wire $a$, as shown.

The general procedure for finding the force on a current-carrying wire is this:

> To find the force on a current-carrying wire due to a second current-carrying wire, first find the field due to the second wire at the site of the first wire. Then find the force on the first wire due to that field.

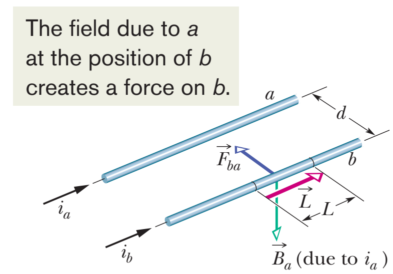



**Figure 29-10** Two parallel wires carrying currents in the same direction attract each other. $\vec{B}_a$ is the magnetic field at wire $b$ produced by the current in wire $a$. $\vec{F}_{ba}$ is the resulting force acting on wire $b$ because it carries current in $\vec{B}_a$.

We could now use this procedure to compute the force on wire $a$ due to the current in wire $b$. We would find that the force is directly toward wire $b$; hence, the two wires with parallel currents attract each other. Similarly, if the two currents were antiparallel, we could show that the two wires repel each other. Thus,

> Parallel currents attract each other, and antiparallel currents repel each other.

The force acting between currents in parallel wires is the basis for the definition of the ampere, which is one of the seven SI base units. The definition, adopted in 1946, is this: The ampere is that constant current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross section, and placed 1 m apart in vacuum, would produce on each of these conductors a force of magnitude $2 \times 10^{-7}$ newton per meter of wire length.

## Rail Gun

The basics of a rail gun are shown in Fig. 29-11$a$. A large current is sent out along one of two parallel conducting rails, across a conducting "fuse" (such as a narrow piece of copper) between the rails, and then back to the current source along the second rail. The projectile to be fired lies on the far side of the fuse and fits loosely between the rails. Immediately after the current begins, the fuse element melts and vaporizes, creating a conducting gas between the rails where the fuse had been.

The curled–straight right-hand rule of Fig. 29-5 reveals that the currents in the rails of Fig. 29-11$a$ produce magnetic fields that are directed downward between the rails. The net magnetic field $\vec{B}$ exerts a force $\vec{F}$ on the gas due to the current $i$ through the gas (Fig. 29-11$b$). With Eq. 29-12 and the right-hand rule for cross products, we find that $\vec{F}$ points outward along the rails. As the gas is forced outward along the rails, it pushes the projectile, accelerating it by as much as $5 \times 10^6g$, and then launches it with a speed of 10 km/s, all within 1 ms. Someday rail guns may be used to launch materials into space from mining operations on the Moon or an asteroid.

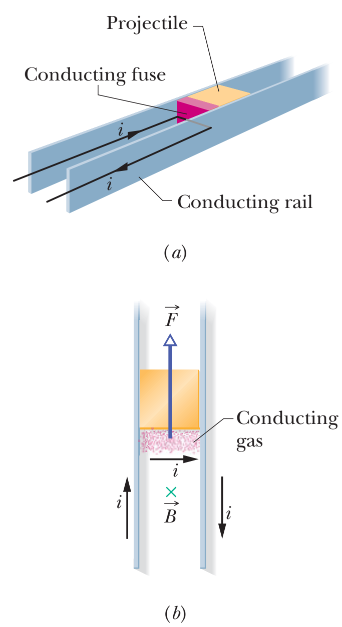
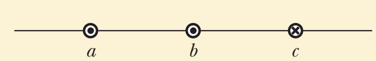



**Figure 29-11** ($a$) A rail gun, as a current $i$ is set up in it. The current rapidly causes the conducting fuse to vaporize. ($b$) The current produces a magnetic field $\vec{B}$ between the rails, and the field causes a force $\vec{F}$ to act on the conducting gas, which is part of the current path. The gas propels the projectile along the rails, launching it.

### Checkpoint 1

The figure here shows three long, straight, parallel, equally spaced wires with identical currents either into or out of the page. Rank the wires according to the magnitude of the force on each due to the currents in the other two wires, greatest first.



# 29-3 AMPERE'S LAW

## Learning Objectives

*After reading this module, you should be able to . . .*

**29.13** Apply Ampere's law to a loop that encircles current.

**29.14** With Ampere's law, use a right-hand rule for determining the algebraic sign of an encircled current.

**29.15** For more than one current within an Amperian loop, determine the net current to be used in Ampere's law.

**29.16** Apply Ampere's law to a long straight wire with current, to find the magnetic field magnitude inside and outside the wire, identifying that only the current encircled by the loop matters.

## Key Idea

- Ampere's law states that

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} \quad \text{(Ampere's law).}$$

The line integral in this equation is evaluated around a closed loop called an Amperian loop. The current $i$ on the right side is the *net* current encircled by the loop.

## Ampere's Law

We can find the net electric field due to *any* distribution of charges by first writing the differential electric field $d\vec{E}$ due to a charge element and then summing the contributions of $d\vec{E}$ from all the elements. However, if the distribution is complicated, we may have to use a computer. Recall, however, that if the distribution has planar, cylindrical, or spherical symmetry, we can apply Gauss' law to find the net electric field with considerably less effort.

Similarly, we can find the net magnetic field due to *any* distribution of currents by first writing the differential magnetic field $d\vec{B}$ (Eq. 29-3) due to a current-length element and then summing the contributions of $d\vec{B}$ from all the elements. Again we may have to use a computer for a complicated distribution. However, if the distribution has some symmetry, we may be able to apply **Ampere's law** to find the magnetic field with considerably less effort. This law, which can be derived from the Biot–Savart law, has traditionally been credited to André-Marie Ampère (1775–1836), for whom the SI unit of current is named. However, the law actually was advanced by English physicist James Clerk Maxwell. Ampere's law is

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} \quad \text{(Ampere's law).} \tag{29-14}$$

The loop on the integral sign means that the scalar (dot) product $\vec{B} \cdot d\vec{s}$ is to be integrated around a *closed* loop, called an *Amperian loop.* The current $i_{enc}$ is the *net* current encircled by that closed loop.

To see the meaning of the scalar product $\vec{B} \cdot d\vec{s}$ and its integral, let us first apply Ampere's law to the general situation of Fig. 29-12. The figure shows cross sections of three long straight wires that carry currents $i_1$, $i_2$, and $i_3$ either directly into or directly out of the page. An arbitrary Amperian loop lying in the plane of the page encircles two of the currents but not the third. The counterclockwise direction marked on the loop indicates the arbitrarily chosen direction of integration for Eq. 29-14.

To apply Ampere's law, we mentally divide the loop into differential vector elements $d\vec{s}$ that are everywhere directed along the tangent to the loop in the direction of integration. Assume that at the location of the element $d\vec{s}$ shown in Fig. 29-12, the net magnetic field due to the three currents is $\vec{B}$. Because the wires are perpendicular to the page, we know that the magnetic field at $d\vec{s}$ due to each current is in the plane of Fig. 29-12; thus, their net magnetic field $\vec{B}$ at $d\vec{s}$ must also be in that plane. However, we do not know the orientation of $\vec{B}$ within the plane. In Fig. 29-12, $\vec{B}$ is arbitrarily drawn at an angle $\theta$ to the direction of $d\vec{s}$. The scalar product $\vec{B}\cdot d\vec{s}$ on the left side of Eq. 29-14 is equal to $B \cos\theta\, ds$. Thus, Ampere's law can be written as

$$\oint \vec{B}\cdot d\vec{s} = \oint B\cos\theta\, ds = \mu_0 i_{\text{enc}}. \tag{29-15}$$

We can now interpret the scalar product $\vec{B}\cdot d\vec{s}$ as being the product of a length $ds$ of the Amperian loop and the field component $B\cos\theta$ tangent to the loop. Then we can interpret the integration as being the summation of all such products around the entire loop.

***Signs.*** When we can actually perform this integration, we do not need to know the direction of $\vec{B}$ before integrating. Instead, we arbitrarily assume $\vec{B}$ to be generally in the direction of integration (as in Fig. 29-12). Then we use the following curled–straight right-hand rule to assign a plus sign or a minus sign to each of the currents that make up the net encircled current $i_{\text{enc}}$:

> Curl your right hand around the Amperian loop, with the fingers pointing in the direction of integration. A current through the loop in the general direction of your outstretched thumb is assigned a plus sign, and a current generally in the opposite direction is assigned a minus sign.

Finally, we solve Eq. 29-15 for the magnitude of $\vec{B}$. If $B$ turns out positive, then the direction we assumed for $\vec{B}$ is correct. If it turns out negative, we neglect the minus sign and redraw $\vec{B}$ in the opposite direction.

***Net Current.*** In Fig. 29-13 we apply the curled–straight right-hand rule for Ampere's law to the situation of Fig. 29-12. With the indicated counterclockwise direction of integration, the net current encircled by the loop is

$$i_{\text{enc}} = i_1 - i_2.$$

(Current $i_3$ is not encircled by the loop.) We can then rewrite Eq. 29-15 as

$$\oint B\cos\theta\, ds = \mu_0(i_1 - i_2). \tag{29-16}$$

You might wonder why, since current $i_3$ contributes to the magnetic-field magnitude $B$ on the left side of Eq. 29-16, it is not needed on the right side. The answer is that the contributions of current $i_3$ to the magnetic field cancel out because the integration in Eq. 29-16 is made around the full loop. In contrast, the contributions of an encircled current to the magnetic field do not cancel out.

We cannot solve Eq. 29-16 for the magnitude $B$ of the magnetic field because for the situation of Fig. 29-12 we do not have enough information to simplify and solve the integral. However, we do know the outcome of the integration; it must be equal to $\mu_0(i_1 - i_2)$, the value of which is set by the net current passing through the loop.

We shall now apply Ampere's law to two situations in which symmetry does allow us to simplify and solve the integral, hence to find the magnetic field.

### Magnetic Field Outside a Long Straight Wire with Current

Figure 29-14 shows a long straight wire that carries current $i$ directly out of the page. Equation 29-4 tells us that the magnetic field $\vec{B}$ produced by the current has the same magnitude at all points that are the same distance $r$ from the wire; that is, the field $\vec{B}$ has cylindrical symmetry about the wire. We can take advantage of that symmetry to simplify the integral in Ampere's law (Eqs. 29-14 and 29-15) if we encircle the wire with a concentric circular Amperian loop of radius $r$, as in Fig. 29-14. The magnetic field then has the same magnitude $B$ at every point on the loop. We shall integrate counterclockwise, so that $d\vec{s}$ has the direction shown in Fig. 29-14.

Only the currents encircled by the loop are used in Ampere's law.

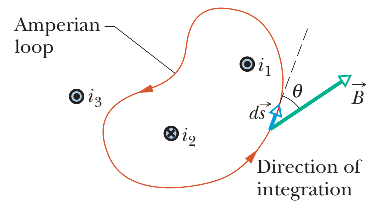



**Figure 29-12** Ampere's law applied to an arbitrary Amperian loop that encircles two long straight wires but excludes a third wire. Note the directions of the currents.

This is how to assign a sign to a current used in Ampere's law.

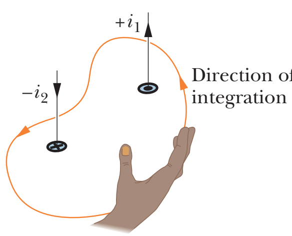



**Figure 29-13** A right-hand rule for Ampere's law, to determine the signs for currents encircled by an Amperian loop. The situation is that of Fig. 29-12.

All of the current is encircled and thus all is used in Ampere's law.

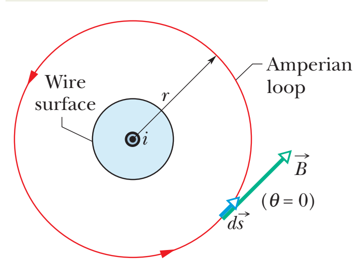



**Figure 29-14** Using Ampere's law to find the magnetic field that a current $i$ produces outside a long straight wire of circular cross section. The Amperian loop is a concentric circle that lies outside the wire.

We can further simplify the quantity $B \cos \theta$ in Eq. 29-15 by noting that $\vec{B}$ is tangent to the loop at every point along the loop, as is $d\vec{s}$. Thus, $\vec{B}$ and $d\vec{s}$ are either parallel or antiparallel at each point of the loop, and we shall arbitrarily assume the former. Then at every point the angle $\theta$ between $d\vec{s}$ and $\vec{B}$ is $0°$, so $\cos \theta = \cos 0° = 1$. The integral in Eq. 29-15 then becomes

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta \, ds = B \oint ds = B(2\pi r).$$

Note that $\oint ds$ is the summation of all the line segment lengths $ds$ around the circular loop; that is, it simply gives the circumference $2\pi r$ of the loop.

Our right-hand rule gives us a plus sign for the current of Fig. 29-14. The right side of Ampere's law becomes $+\mu_0 i$, and we then have

$$B(2\pi r) = \mu_0 i$$

or

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{(outside straight wire).} \tag{29-17}$$

With a slight change in notation, this is Eq. 29-4, which we derived earlier—with considerably more effort—using the law of Biot and Savart. In addition, because the magnitude $B$ turned out positive, we know that the correct direction of $\vec{B}$ must be the one shown in Fig. 29-14.

### Magnetic Field Inside a Long Straight Wire with Current

Figure 29-15 shows the cross section of a long straight wire of radius $R$ that carries a uniformly distributed current $i$ directly out of the page. Because the current is uniformly distributed over a cross section of the wire, the magnetic field $\vec{B}$ produced by the current must be cylindrically symmetrical. Thus, to find the magnetic field at points inside the wire, we can again use an Amperian loop of radius $r$, as shown in Fig. 29-15, where now $r < R$. Symmetry again suggests that $\vec{B}$ is tangent to the loop, as shown; so the left side of Ampere's law again yields

Only the current encircled by the loop is used in Ampere's law.

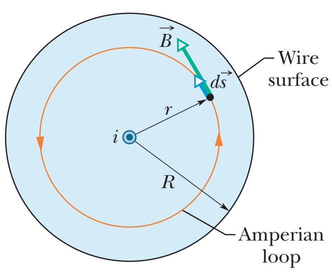



**Figure 29-15** Using Ampere's law to find the magnetic field that a current $i$ produces inside a long straight wire of circular cross section. The current is uniformly distributed over the cross section of the wire and emerges from the page. An Amperian loop is drawn inside the wire.

$$\oint \vec{B} \cdot d\vec{s} = B \oint ds = B(2\pi r). \tag{29-18}$$

Because the current is uniformly distributed, the current $i_{enc}$ encircled by the loop is proportional to the area encircled by the loop; that is,

$$i_{enc} = i \frac{\pi r^2}{\pi R^2}. \tag{29-19}$$

Our right-hand rule tells us that $i_{enc}$ gets a plus sign. Then Ampere's law gives us

$$B(2\pi r) = \mu_0 i \frac{\pi r^2}{\pi R^2}$$

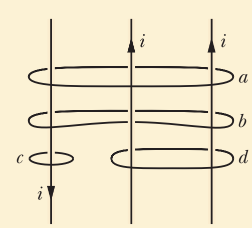
or

$$B = \left(\frac{\mu_0 i}{2\pi R^2}\right) r \quad \text{(inside straight wire).} \tag{29-20}$$

Thus, inside the wire, the magnitude $B$ of the magnetic field is proportional to $r$, is zero at the center, and is maximum at $r = R$ (the surface). Note that Eqs. 29-17 and 29-20 give the same value for $B$ at the surface.

### Checkpoint 2

The figure here shows three equal currents $i$ (two parallel and one antiparallel) and four Amperian loops. Rank the loops according to the magnitude of $\oint \vec{B} \cdot d\vec{s}$ along each, greatest first.



### Sample Problem 29.03 Ampere's law to find the field inside a long cylinder of current

Figure 29-16*a* shows the cross section of a long conducting cylinder with inner radius $a = 2.0$ cm and outer radius $b = 4.0$ cm. The cylinder carries a current out of the page, and the magnitude of the current density in the cross section is given by $J = cr^2$, with $c = 3.0 \times 10^6$ A/m$^4$ and $r$ in meters. What is the magnetic field $\vec{B}$ at the dot in Fig. 29-16*a*, which is at radius $r = 3.0$ cm from the central axis of the cylinder?

#### KEY IDEAS

The point at which we want to evaluate $\vec{B}$ is inside the material of the conducting cylinder, between its inner and outer radii. We note that the current distribution has cylindrical symmetry (it is the same all around the cross section for any given radius). Thus, the symmetry allows us to use Ampere's law to find $\vec{B}$ at the point. We first draw the Amperian loop shown in Fig. 29-16*b*. The loop is concentric with the cylinder and has radius $r = 3.0$ cm because we want to evaluate $\vec{B}$ at that distance from the cylinder's central axis.

Next, we must compute the current $i_{enc}$ that is encircled by the Amperian loop. However, we *cannot* set up a proportionality as in Eq. 29-19, because here the current is not uniformly distributed. Instead, we must integrate the current density magnitude from the cylinder's inner radius $a$ to the loop radius $r$, using the steps shown in Figs. 29-16*c* through *h*.

***Calculations:*** We write the integral as

$$i_{enc} = \int J\,dA = \int_a^r cr^2(2\pi r\,dr)$$

$$= 2\pi c \int_a^r r^3\,dr = 2\pi c\left[\frac{r^4}{4}\right]_a^r$$

$$= \frac{\pi c(r^4 - a^4)}{2}.$$

Note that in these steps we took the differential area $dA$ to be the area of the thin ring in Figs. 29-16*d–f* and then

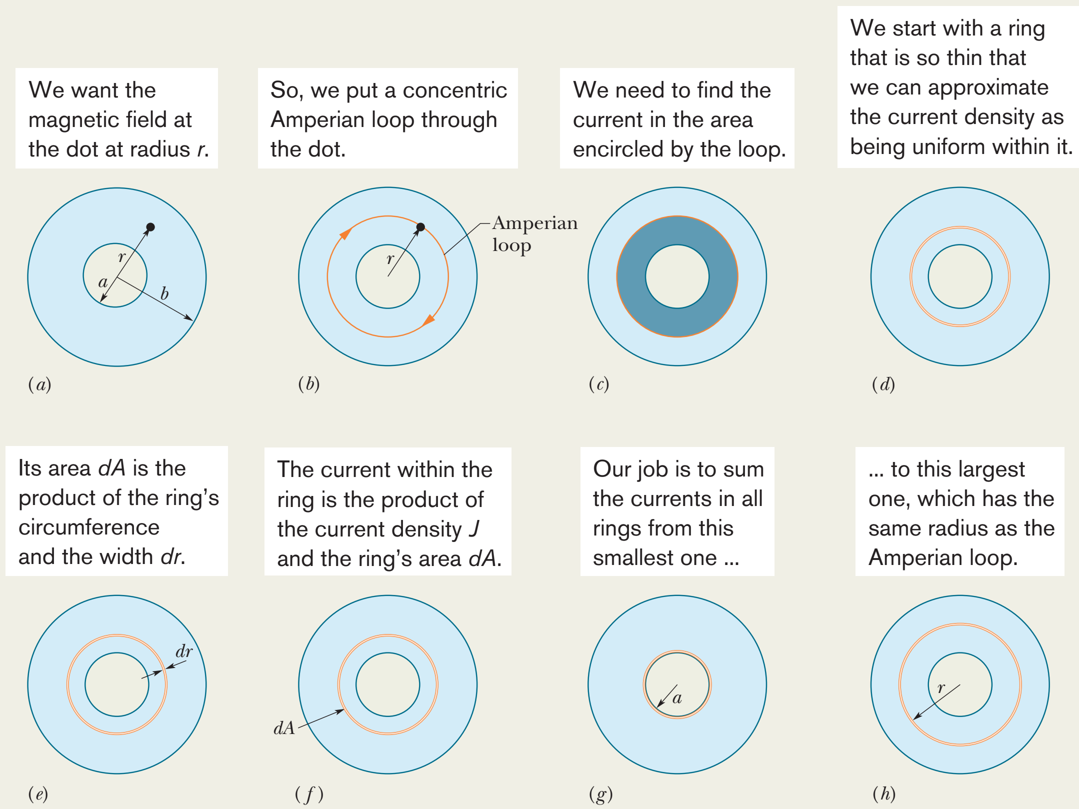



**Figure 29-16** (*a*)–(*b*) To find the magnetic field at a point within this conducting cylinder, we use a concentric Amperian loop through the point. We then need the current encircled by the loop. (*c*)–(*h*) Because the current density is nonuniform, we start with a thin ring and then sum (via integration) the currents in all such rings in the encircled area.

replaced it with its equivalent, the product of the ring's circumference $2\pi r$ and its thickness $dr$.

For the Amperian loop, the direction of integration indicated in Fig. 29-16*b* is (arbitrarily) clockwise. Applying the right-hand rule for Ampere's law to that loop, we find that we should take $i_{enc}$ as negative because the current is directed out of the page but our thumb is directed into the page.

We next evaluate the left side of Ampere's law as we did in Fig. 29-15, and we again obtain Eq. 29-18. Then Ampere's law,

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{enc},$$

gives us

$$B(2\pi r) = -\frac{\mu_0 \pi c}{2}(r^4 - a^4).$$

Solving for $B$ and substituting known data yield

$$B = -\frac{\mu_0 c}{4r}(r^4 - a^4)$$

$$= -\frac{(4\pi\times 10^{-7}\,\text{T}\cdot\text{m/A})(3.0\times 10^6\,\text{A/m}^4)}{4(0.030\,\text{m})} \times [(0.030\,\text{m})^4 - (0.020\,\text{m})^4]$$

$$= -2.0\times 10^{-5}\,\text{T}.$$

Thus, the magnetic field $\vec{B}$ at a point 3.0 cm from the central axis has magnitude

$$B = 2.0\times 10^{-5}\,\text{T} \qquad \text{(Answer)}$$

and forms magnetic field lines that are directed opposite our direction of integration, hence counterclockwise in Fig. 29-16*b*.

WILEY PLUS Additional examples, video, and practice available at *WileyPLUS*

# 29-4 SOLENOIDS AND TOROIDS

## Learning Objectives

*After reading this module, you should be able to . . .*

**29.17** Describe a solenoid and a toroid and sketch their magnetic field lines.

**29.18** Explain how Ampere's law is used to find the magnetic field inside a solenoid.

**29.19** Apply the relationship between a solenoid's internal magnetic field $B$, the current $i$, and the number of turns per unit length $n$ of the solenoid.

**29.20** Explain how Ampere's law is used to find the magnetic field inside a toroid.

**29.21** Apply the relationship between a toroid's internal magnetic field $B$, the current $i$, the radius $r$, and the total number of turns $N$.

## Key Ideas

- Inside a long solenoid carrying current $i$, at points not near its ends, the magnitude $B$ of the magnetic field is

$$B = \mu_0 i n \quad \text{(ideal solenoid)},$$

where $n$ is the number of turns per unit length.

- At a point inside a toroid, the magnitude $B$ of the magnetic field is

$$B = \frac{\mu_0 i N}{2\pi}\frac{1}{r} \quad \text{(toroid)},$$

where $r$ is the distance from the center of the toroid to the point.

## Solenoids and Toroids

### Magnetic Field of a Solenoid

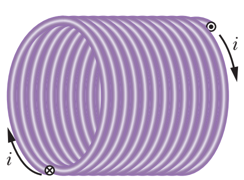



**Figure 29-17** A solenoid carrying current $i$.

We now turn our attention to another situation in which Ampere's law proves useful. It concerns the magnetic field produced by the current in a long, tightly wound helical coil of wire. Such a coil is called a **solenoid** (Fig. 29-17). We assume that the length of the solenoid is much greater than the diameter.

Figure 29-18 shows a section through a portion of a "stretched-out" solenoid. The solenoid's magnetic field is the vector sum of the fields produced by the individual turns (*windings*) that make up the solenoid. For points very

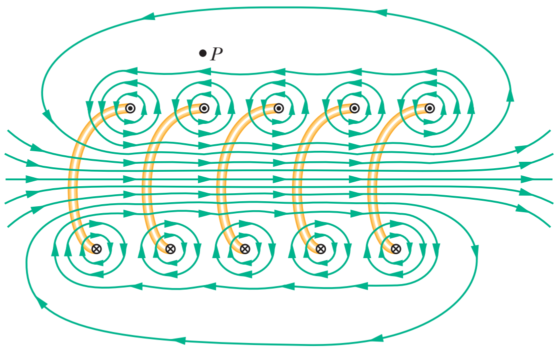



**Figure 29-18** A vertical cross section through the central axis of a "stretched-out" solenoid. The back portions of five turns are shown, as are the magnetic field lines due to a current through the solenoid. Each turn produces circular magnetic field lines near itself. Near the solenoid's axis, the field lines combine into a net magnetic field that is directed along the axis. The closely spaced field lines there indicate a strong magnetic field. Outside the solenoid the field lines are widely spaced; the field there is very weak.

close to a turn, the wire behaves magnetically almost like a long straight wire, and the lines of $\vec{B}$ there are almost concentric circles. Figure 29-18 suggests that the field tends to cancel between adjacent turns. It also suggests that, at points inside the solenoid and reasonably far from the wire, $\vec{B}$ is approximately parallel to the (central) solenoid axis. In the limiting case of an *ideal solenoid,* which is infinitely long and consists of tightly packed (*close-packed*) turns of square wire, the field inside the coil is uniform and parallel to the solenoid axis.

At points above the solenoid, such as $P$ in Fig. 29-18, the magnetic field set up by the upper parts of the solenoid turns (these upper turns are marked $\odot$) is directed to the left (as drawn near $P$) and tends to cancel the field set up at $P$ by the lower parts of the turns (these lower turns are marked $\otimes$), which is directed to the right (not drawn). In the limiting case of an ideal solenoid, the magnetic field outside the solenoid is zero. Taking the external field to be zero is an excellent assumption for a real solenoid if its length is much greater than its diameter and if we consider external points such as point $P$ that are not at either end of the solenoid. The direction of the magnetic field along the solenoid axis is given by a curled–straight right-hand rule: Grasp the solenoid with your right hand so that your fingers follow the direction of the current in the windings; your extended right thumb then points in the direction of the axial magnetic field.

Figure 29-19 shows the lines of $\vec{B}$ for a real solenoid. The spacing of these lines in the central region shows that the field inside the coil is fairly strong and uniform over the cross section of the coil. The external field, however, is relatively weak.

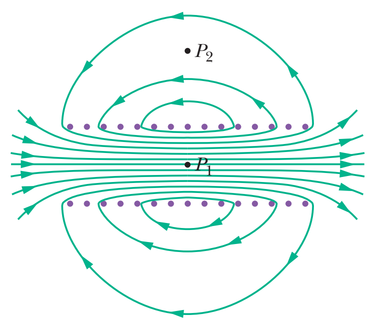



**Figure 29-19** Magnetic field lines for a real solenoid of finite length. The field is strong and uniform at interior points such as $P_1$ but relatively weak at external points such as $P_2$.

***Ampere's Law.*** Let us now apply Ampere's law,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}, \tag{29-21}$$

to the ideal solenoid of Fig. 29-20, where $\vec{B}$ is uniform within the solenoid and zero outside it, using the rectangular Amperian loop *abcda*. We write $\oint \vec{B} \cdot d\vec{s}$ as the sum of four integrals, one for each loop segment:

$$\oint \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^c \vec{B} \cdot d\vec{s} + \int_c^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s}. \tag{29-22}$$

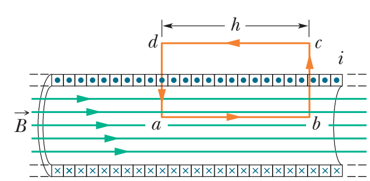



**Figure 29-20** Application of Ampere's law to a section of a long ideal solenoid carrying a current $i$. The Amperian loop is the rectangle *abcda*.

The first integral on the right of Eq. 29-22 is $Bh$, where $B$ is the magnitude of the uniform field $\vec{B}$ inside the solenoid and $h$ is the (arbitrary) length of the segment from $a$ to $b$. The second and fourth integrals are zero because for every element $ds$ of these segments, $\vec{B}$ either is perpendicular to $ds$ or is zero, and thus the product $\vec{B} \cdot d\vec{s}$ is zero. The third integral, which is taken along a segment that lies outside the solenoid, is zero because $B = 0$ at all external points. Thus, $\oint \vec{B} \cdot d\vec{s}$ for the entire rectangular loop has the value $Bh$.

***Net Current.*** The net current $i_{enc}$ encircled by the rectangular Amperian loop in Fig. 29-20 is not the same as the current $i$ in the solenoid windings because the windings pass more than once through this loop. Let $n$ be the number of turns per unit length of the solenoid; then the loop encloses $nh$ turns and

$$i_{enc} = i(nh).$$

Ampere's law then gives us

$$Bh = \mu_0 inh$$

or

$$B = \mu_0 in \quad \text{(ideal solenoid).} \tag{29-23}$$

Although we derived Eq. 29-23 for an infinitely long ideal solenoid, it holds quite well for actual solenoids if we apply it only at interior points and well away from the solenoid ends. Equation 29-23 is consistent with the experimental fact that the magnetic field magnitude $B$ within a solenoid does not depend on the diameter or the length of the solenoid and that $B$ is uniform over the solenoidal cross section. A solenoid thus provides a practical way to set up a known uniform magnetic field for experimentation, just as a parallel-plate capacitor provides a practical way to set up a known uniform electric field.

## Magnetic Field of a Toroid

Figure 29-21$a$ shows a **toroid,** which we may describe as a (hollow) solenoid that has been curved until its two ends meet, forming a sort of hollow bracelet. What magnetic field $\vec{B}$ is set up inside the toroid (inside the hollow of the bracelet)? We can find out from Ampere's law and the symmetry of the bracelet.

From the symmetry, we see that the lines of $\vec{B}$ form concentric circles inside the toroid, directed as shown in Fig. 29-21$b$. Let us choose a concentric circle of radius $r$ as an Amperian loop and traverse it in the clockwise direction. Ampere's law (Eq. 29-14) yields

$$(B)(2\pi r) = \mu_0 iN,$$

where $i$ is the current in the toroid windings (and is positive for those windings enclosed by the Amperian loop) and $N$ is the total number of turns. This gives

$$B = \frac{\mu_0 iN}{2\pi} \frac{1}{r} \quad \text{(toroid).} \tag{29-24}$$

In contrast to the situation for a solenoid, $B$ is not constant over the cross section of a toroid.

It is easy to show, with Ampere's law, that $B = 0$ for points outside an ideal toroid (as if the toroid were made from an ideal solenoid). The direction of the magnetic field within a toroid follows from our curled–straight right-hand rule: Grasp the toroid with the fingers of your right hand curled in the direction of the current in the windings; your extended right thumb points in the direction of the magnetic field.

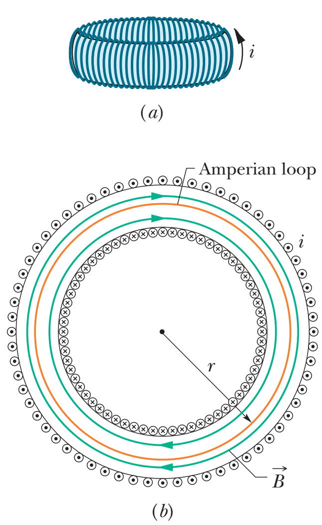



**Figure 29-21** ($a$) A toroid carrying a current $i$. ($b$) A horizontal cross section of the toroid. The interior magnetic field (inside the bracelet-shaped tube) can be found by applying Ampere's law with the Amperian loop shown.

### Sample Problem 29.04 The field inside a solenoid (a long coil of current)

A solenoid has length $L = 1.23$ m and inner diameter $d = 3.55$ cm, and it carries a current $i = 5.57$ A. It consists of five close-packed layers, each with 850 turns along length $L$. What is $B$ at its center?

**KEY IDEA**

The magnitude $B$ of the magnetic field along the solenoid's central axis is related to the solenoid's current $i$ and number of turns per unit length $n$ by Eq. 29-23 ($B = \mu_0 in$).

***Calculation:*** Because $B$ does not depend on the diameter of the windings, the value of $n$ for five identical layers is simply five times the value for each layer. Equation 29-23 then tells us

$$B = \mu_0 in = (4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(5.57\ \text{A})\frac{5 \times 850\ \text{turns}}{1.23\ \text{m}}$$

$$= 2.42 \times 10^{-2}\ \text{T} = 24.2\ \text{mT}. \qquad \text{(Answer)}$$

To a good approximation, this is the field magnitude throughout most of the solenoid.

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# 29-5 A CURRENT-CARRYING COIL AS A MAGNETIC DIPOLE

### Learning Objectives

*After reading this module, you should be able to . . .*

**29.22** Sketch the magnetic field lines of a flat coil that is carrying current.

**29.23** For a current-carrying coil, apply the relationship between the dipole moment magnitude $\mu$ and the coil's current $i$, number of turns $N$, and area per turn $A$.

**29.24** For a point along the central axis, apply the relationship between the magnetic field magnitude $B$, the magnetic moment $\mu$, and the distance $z$ from the center of the coil.

### Key Idea

- The magnetic field produced by a current-carrying coil, which is a magnetic dipole, at a point $P$ located a distance $z$ along the coil's perpendicular central axis is parallel to the axis and is given by

$$\vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3},$$

where $\vec{\mu}$ is the dipole moment of the coil. This equation applies only when $z$ is much greater than the dimensions of the coil.

## A Current-Carrying Coil as a Magnetic Dipole

So far we have examined the magnetic fields produced by current in a long straight wire, a solenoid, and a toroid. We turn our attention here to the field produced by a coil carrying a current. You saw in Module 28-8 that such a coil behaves as a magnetic dipole in that, if we place it in an external magnetic field $\vec{B}$, a torque $\vec{\tau}$ given by

$$\vec{\tau} = \vec{\mu} \times \vec{B} \qquad (29\text{-}25)$$

acts on it. Here $\vec{\mu}$ is the magnetic dipole moment of the coil and has the magnitude $NiA$, where $N$ is the number of turns, $i$ is the current in each turn, and $A$ is the area enclosed by each turn. (*Caution:* Don't confuse the magnetic dipole moment $\vec{\mu}$ with the permeability constant $\mu_0$.)

Recall that the direction of $\vec{\mu}$ is given by a curled–straight right-hand rule: Grasp the coil so that the fingers of your right hand curl around it in the direction of the current; your extended thumb then points in the direction of the dipole moment $\vec{\mu}$.

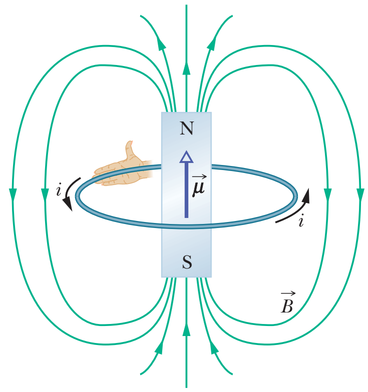



**Figure 29-22** A current loop produces a magnetic field like that of a bar magnet and thus has associated north and south poles. The magnetic dipole moment $\vec{\mu}$ of the loop, its direction given by a curled–straight right-hand rule, points from the south pole to the north pole, in the direction of the field $\vec{B}$ within the loop.

## Magnetic Field of a Coil

We turn now to the other aspect of a current-carrying coil as a magnetic dipole. What magnetic field does *it* produce at a point in the surrounding space? The problem does not have enough symmetry to make Ampere's law useful; so we must turn to the law of Biot and Savart. For simplicity, we first consider only a coil with a single circular loop and only points on its perpendicular central axis, which we take to be a $z$ axis. We shall show that the magnitude of the magnetic field at such points is

$$B(z) = \frac{\mu_0 i R^2}{2(R^2 + z^2)^{3/2}}, \tag{29-26}$$

in which $R$ is the radius of the circular loop and $z$ is the distance of the point in question from the center of the loop. Furthermore, the direction of the magnetic field $\vec{B}$ is the same as the direction of the magnetic dipole moment $\vec{\mu}$ of the loop.

***Large z.*** For axial points far from the loop, we have $z \gg R$ in Eq. 29-26. With that approximation, the equation reduces to

$$B(z) \approx \frac{\mu_0 i R^2}{2z^3}.$$

Recalling that $\pi R^2$ is the area $A$ of the loop and extending our result to include a coil of $N$ turns, we can write this equation as

$$B(z) = \frac{\mu_0}{2\pi}\frac{NiA}{z^3}.$$

Further, because $\vec{B}$ and $\vec{\mu}$ have the same direction, we can write the equation in vector form, substituting from the identity $\mu = NiA$:

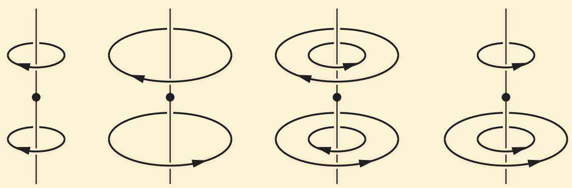
$$\vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3} \quad \text{(current-carrying coil).} \tag{29-27}$$

Thus, we have two ways in which we can regard a current-carrying coil as a magnetic dipole: (1) it experiences a torque when we place it in an external magnetic field; (2) it generates its own intrinsic magnetic field, given, for distant points along its axis, by Eq. 29-27. Figure 29-22 shows the magnetic field of a current loop; one side of the loop acts as a north pole (in the direction of $\vec{\mu}$)

and the other side as a south pole, as suggested by the lightly drawn magnet in the figure. If we were to place a current-carrying coil in an external magnetic field, it would tend to rotate just like a bar magnet would.

### Checkpoint 3

The figure here shows four arrangements of circular loops of radius $r$ or $2r$, centered on vertical axes (perpendicular to the loops) and carrying identical currents in the directions indicated. Rank the arrangements according to the magnitude of the net magnetic field at the dot, midway between the loops on the central axis, greatest first.




### Proof of Equation 29-26

Figure 29-23 shows the back half of a circular loop of radius $R$ carrying a current $i$. Consider a point $P$ on the central axis of the loop, a distance $z$ from its plane. Let us apply the law of Biot and Savart to a differential element $ds$ of the loop, located at the left side of the loop. The length vector $d\vec{s}$ for this element points perpendicularly out of the page. The angle $\theta$ between $d\vec{s}$ and $\hat{r}$ in Fig. 29-23 is 90°; the plane formed by these two vectors is perpendicular to the plane of the page and contains both $\hat{r}$ and $d\vec{s}$. From the law of Biot and Savart (and the right-hand rule), the differential field $d\vec{B}$ produced at point $P$ by the current in this element is perpendicular to this plane and thus is directed in the plane of the figure, perpendicular to $\hat{r}$, as indicated in Fig. 29-23.

Let us resolve $d\vec{B}$ into two components: $dB_\parallel$ along the axis of the loop and $dB_\perp$ perpendicular to this axis. From the symmetry, the vector sum of all the perpendicular components $dB_\perp$ due to all the loop elements $ds$ is zero. This leaves only the axial (parallel) components $dB_\parallel$ and we have

$$B = \int dB_\parallel .$$

For the element $d\vec{s}$ in Fig. 29-23, the law of Biot and Savart (Eq. 29-1) tells us that the magnetic field at distance $r$ is

$$dB = \frac{\mu_0}{4\pi}\frac{i\,ds \sin 90°}{r^2}.$$

We also have

$$dB_\parallel = dB \cos \alpha.$$

Combining these two relations, we obtain

$$dB_\parallel = \frac{\mu_0 i \cos \alpha\, ds}{4\pi r^2}. \tag{29-28}$$

Figure 29-23 shows that $r$ and $\alpha$ are related to each other. Let us express each in terms of the variable $z$, the distance between point $P$ and the center of the loop. The relations are

$$r = \sqrt{R^2 + z^2} \tag{29-29}$$

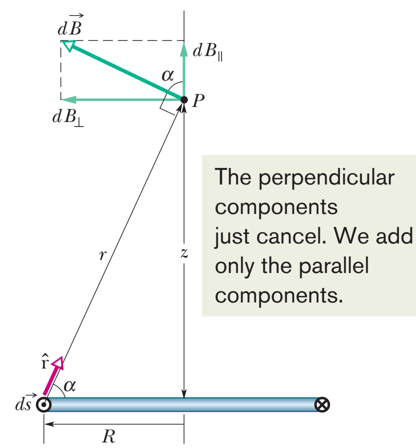



**Figure 29-23** Cross section through a current loop of radius $R$. The plane of the loop is perpendicular to the page, and only the back half of the loop is shown. We use the law of Biot and Savart to find the magnetic field at point $P$ on the central perpendicular axis of the loop.

and
$$\cos\alpha = \frac{R}{r} = \frac{R}{\sqrt{R^2 + z^2}}. \tag{29-30}$$

Substituting Eqs. 29-29 and 29-30 into Eq. 29-28, we find

$$dB_{\parallel} = \frac{\mu_0 iR}{4\pi(R^2 + z^2)^{3/2}}\,ds.$$

Note that $i$, $R$, and $z$ have the same values for all elements $ds$ around the loop; so when we integrate this equation, we find that

$$B = \int dB_{\parallel} = \frac{\mu_0 iR}{4\pi(R^2 + z^2)^{3/2}}\int ds$$

or, because $\int ds$ is simply the circumference $2\pi R$ of the loop,

$$B(z) = \frac{\mu_0 iR^2}{2(R^2 + z^2)^{3/2}}.$$

This is Eq. 29-26, the relation we sought to prove.

## Review & Summary

**The Biot–Savart Law** The magnetic field set up by a current-carrying conductor can be found from the *Biot–Savart law.* This law asserts that the contribution $d\vec{B}$ to the field produced by a current-length element $i\,d\vec{s}$ at a point $P$ located a distance $r$ from the current element is

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{i\,d\vec{s} \times \hat{r}}{r^2} \quad \text{(Biot–Savart law)}. \tag{29-3}$$

Here $\hat{r}$ is a unit vector that points from the element toward $P$. The quantity $\mu_0$, called the permeability constant, has the value

$$4\pi \times 10^{-7}\ \mathrm{T\cdot m/A} \approx 1.26 \times 10^{-6}\ \mathrm{T\cdot m/A}.$$

**Magnetic Field of a Long Straight Wire** For a long straight wire carrying a current $i$, the Biot–Savart law gives, for the magnitude of the magnetic field at a perpendicular distance $R$ from the wire,

$$B = \frac{\mu_0 i}{2\pi R} \quad \text{(long straight wire)}. \tag{29-4}$$

**Magnetic Field of a Circular Arc** The magnitude of the magnetic field at the center of a circular arc, of radius $R$ and central angle $\phi$ (in radians), carrying current $i$, is

$$B = \frac{\mu_0 i\phi}{4\pi R} \quad \text{(at center of circular arc)}. \tag{29-9}$$

**Force Between Parallel Currents** Parallel wires carrying currents in the same direction attract each other, whereas parallel wires carrying currents in opposite directions repel each other. The magnitude of the force on a length $L$ of either wire is

$$F_{ba} = i_b L B_a \sin 90^\circ = \frac{\mu_0 L i_a i_b}{2\pi d}, \tag{29-13}$$

where $d$ is the wire separation, and $i_a$ and $i_b$ are the currents in the wires.

**Ampere's Law** **Ampere's law** states that

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 i_{\text{enc}} \quad \text{(Ampere's law)}. \tag{29-14}$$

The line integral in this equation is evaluated around a closed loop called an *Amperian loop.* The current $i$ on the right side is the *net* current encircled by the loop. For some current distributions, Eq. 29-14 is easier to use than Eq. 29-3 to calculate the magnetic field due to the currents.

**Fields of a Solenoid and a Toroid** Inside a *long solenoid* carrying current $i$, at points not near its ends, the magnitude $B$ of the magnetic field is

$$B = \mu_0 in \quad \text{(ideal solenoid)}, \tag{29-23}$$

where $n$ is the number of turns per unit length. Thus the internal magnetic field is uniform. Outside the solenoid, the magnetic field is approximately zero.

At a point inside a *toroid*, the magnitude $B$ of the magnetic field is

$$B = \frac{\mu_0 iN}{2\pi}\frac{1}{r} \quad \text{(toroid)}, \tag{29-24}$$

where $r$ is the distance from the center of the toroid to the point.

**Field of a Magnetic Dipole** The magnetic field produced by a current-carrying coil, which is a *magnetic dipole,* at a point $P$ located a distance $z$ along the coil's perpendicular central axis is parallel to the axis and is given by

$$\vec{B}(z) = \frac{\mu_0}{2\pi}\frac{\vec{\mu}}{z^3}, \tag{29-27}$$

where $\vec{\mu}$ is the dipole moment of the coil. This equation applies only when $z$ is much greater than the dimensions of the coil.

# Questions

**1** Figure 29-24 shows three circuits, each consisting of two radial lengths and two concentric circular arcs, one of radius $r$ and the other of radius $R > r$. The circuits have the same current through them and the same angle between the two radial lengths. Rank the circuits according to the magnitude of the net magnetic field at the center, greatest first.

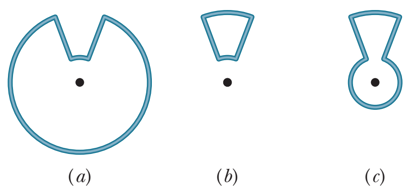



**Figure 29-24** Question 1.

**2** Figure 29-25 represents a snapshot of the velocity vectors of four electrons near a wire carrying current $i$. The four velocities have the same magnitude; velocity $\vec{v}_2$ is directed into the page. Electrons 1 and 2 are at the same distance from the wire, as are electrons 3 and 4. Rank the electrons according to the magnitudes of the magnetic forces on them due to current $i$, greatest first.

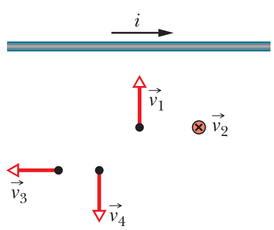



**Figure 29-25** Question 2.

**3** Figure 29-26 shows four arrangements in which long parallel wires carry equal currents directly into or out of the page at the corners of identical squares. Rank the arrangements according to the magnitude of the net magnetic field at the center of the square, greatest first.

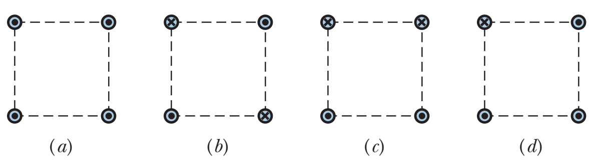



**Figure 29-26** Question 3.

**4** Figure 29-27 shows cross sections of two long straight wires; the left-hand wire carries current $i_1$ directly out of the page. If the net magnetic field due to the two currents is to be zero at point $P$, (a) should the direction of current $i_2$ in the right-hand wire be directly into or out of the page and (b) should $i_2$ be greater than, less than, or equal to $i_1$?

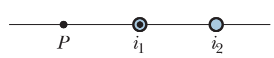



**Figure 29-27** Question 4.

**5** Figure 29-28 shows three circuits consisting of straight radial lengths and concentric circular arcs (either half- or quarter-circles of radii $r$, $2r$, and $3r$). The circuits carry the same current. Rank them according to the magnitude of the magnetic field produced at the center of curvature (the dot), greatest first.

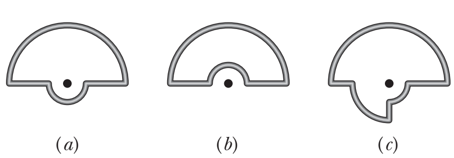



**Figure 29-28** Question 5.

**6** Figure 29-29 gives, as a function of radial distance $r$, the magnitude $B$ of the magnetic field inside and outside four wires ($a$, $b$, $c$, and $d$), each of which carries a current that is uniformly distributed across the wire's cross section. Overlapping portions of the plots (drawn slightly separated) are indicated by double labels. Rank the wires according to (a) radius, (b) the magnitude of the magnetic field on the surface, and (c) the value of the current, greatest first. (d) Is the magnitude of the current density in wire $a$ greater than, less than, or equal to that in wire $c$?

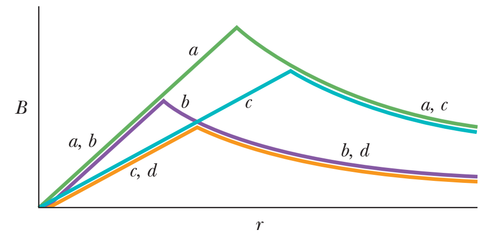



**Figure 29-29** Question 6.

**7** Figure 29-30 shows four circular Amperian loops ($a$, $b$, $c$, $d$) concentric with a wire whose current is directed out of the page. The current is uniform across the wire's circular cross section (the shaded region). Rank the loops according to the magnitude of $\oint \vec{B} \cdot d\vec{s}$ around each, greatest first.

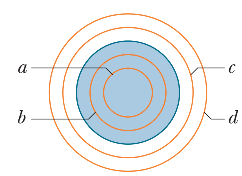



**Figure 29-30** Question 7.

**8** Figure 29-31 shows four arrangements in which long, parallel, equally spaced wires carry equal currents directly into or out of the page. Rank the arrangements according to the magnitude of the net force on the central wire due to the currents in the other wires, greatest first.

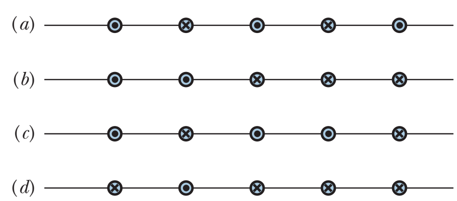



**Figure 29-31** Question 8.

**9** Figure 29-32 shows four circular Amperian loops ($a$, $b$, $c$, $d$) and, in cross section, four long circular conductors (the shaded regions), all of which are concentric. Three of the conductors are hollow cylinders; the central conductor is a solid cylinder. The currents in the conductors are, from smallest radius to largest radius, 4 A out of

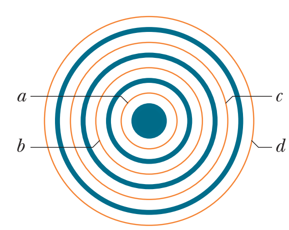



**Figure 29-32** Question 9.