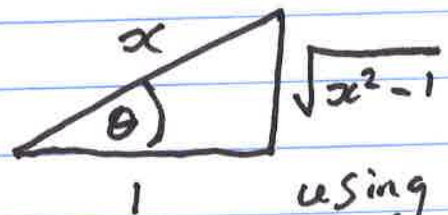
Math 146-003, Test II: Solutions, March 9, 2017

① Let $x = \sec\theta$ so $dx = \sec\theta\tan\theta\, d\theta$ and I (always the integral in the question) becomes

$$\int \frac{\sqrt{\sec^2\theta - 1}}{\sec^3\theta} \sec\theta\tan\theta\, d\theta = \int \frac{\tan\theta}{\sec^3\theta} \sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\tan^2\theta}{\sec^2\theta} = \int \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta\, d\theta = \int \sin^2\theta\, d\theta$$

$$= \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$



so $I = \frac{1}{2}\left(\sec^{-1}x - \frac{\sqrt{x^2-1}}{x^2}\right) + C$

using $\sin 2\theta = 2\sin\theta\cos\theta$.

② $$\frac{3x^2+4x+14}{(x-1)(x^2+2x+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+4}$$

so $$3x^2+4x+14 = A(x^2+2x+4) + (Bx+C)(x-1)$$
$$= (A+B)x^2 + (2A-B+C)x + (4A-C)$$

$$\left.\begin{aligned} A + B \quad\quad &= 3 \\ 2A - B + C &= 4 \\ 4A \quad\quad - C &= 14. \end{aligned}\right\}$$ solve.. For instance

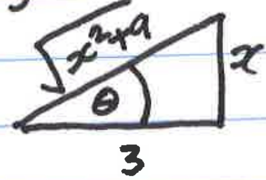
add all 3 equations to get $7A = 21$, so $A = 3$.

then $B = 0$, $C = -2$.

The answer is $\dfrac{3}{x-1} - \dfrac{2}{x^2+2x+4}$

(3) a) Let $x = 3\tan\theta$ so $I = \displaystyle\int \frac{1}{3\sec\theta} \cdot 3\sec^2\theta \, d\theta$
so $dx = 3\sec^2\theta \, d\theta$

$= \displaystyle\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$



$= \ln\left| \dfrac{\sqrt{x^2+9}}{3} + \dfrac{x}{3} \right| + C$

b) Note that $\dfrac{x^2}{x^5+x^3+x^2+x+11} \leq \dfrac{x^2}{x^5} = \dfrac{1}{x^3}$.

Now $\displaystyle\int_1^\infty \frac{1}{x^3} dx$ converges by the p-test with $p = 3 > 1$ so

I also converges by the comparison test.

(4) $I = \dfrac{1}{2}\displaystyle\int \frac{2x+10}{x^2+6x+10} dx = \frac{1}{2}\left(\int \frac{2x+6}{x^2+6x+10} dx + 4\int \frac{1}{x^2+6x+10} dx\right)$

$= \dfrac{1}{2}\ln|x^2+6x+10| + 2\displaystyle\int \frac{1}{(x+3)^2+1} dx$

$= \dfrac{1}{2}\ln|x^2+6x+10| + 2\tan^{-1}(x+3) + C$.

(5) a) $I = \lim_{t \to 3^-} \int_2^t (3-x)^{-1/2}\, dx$

$= \lim_{t \to 3^-} -(3-x)^{1/2} \Big|_2^t = \lim_{t\to 3^-} -\sqrt{3-t} + 1 = 1.$

Therefore I converges

b) $\lim_{n\to\infty} \frac{(3n+1)!}{(3n+2)!} = \lim_{n\to\infty} \frac{(3n+1)!}{(3n+2)\cdot(3n+1)!} = \lim_{n\to\infty} \frac{1}{3n+2} = 0.$

so Yes, it converges (to zero).

(6) a) Diverges, by the dangerous theorem since $\lim \frac{n^2+1}{3n^2+1} = \frac{1}{3} \neq 0$

b) Use limit comparison with $\sum \frac{1}{\sqrt{n}}$ which diverges, by p-test with $p = \frac{1}{2} \leq 1$. Now $\lim \frac{1}{\sqrt{n+8}} \Big/ \frac{1}{\sqrt{n}} = \lim \sqrt{\frac{n}{n+8}} = 1 \neq 0$ so both series have the same behaviour. The given series DIVERGES

c) ~~[illegible]~~ Since $\frac{1+7^n}{5^n} \geq \frac{7^n}{5^n} = \left(\frac{7}{5}\right)^n$

and $\sum \left(\frac{7}{5}\right)^n$ DIVERGES (it's a G.S. with $r = \frac{7}{5} > 1$) the given series also DIVERGES.

(7) a) Since $0 \le \frac{\sin^2 n}{n^7+3} \le \frac{1}{n^7+3} \le \frac{1}{n^7}$ and $\sum \frac{1}{n^7}$ converges by the p-test with $p = 7 > 1$ the comparison test shows that the given series converges

b). Since $\frac{\ln n}{n} > \frac{1}{n}$ for $n \geq 5$.

and $\sum \frac{1}{n}$ diverges (it's the harmonic series) the given series also diverges by comparison.

(8) a) Since $\lim \frac{1}{n^2-1} / \frac{1}{n^2} = 1 \neq 0$ and $\sum \frac{1}{n^2}$

converges by the p-test with $p = 2 > 1$ the

given series converges by the limit comparison. test

b) The integral converges by the integral test (which says that $\sum_2^\infty \frac{1}{n^2-1}$ and $\int_2^\infty \frac{1}{x^2-1}\,dx$

have the same behavioar.).