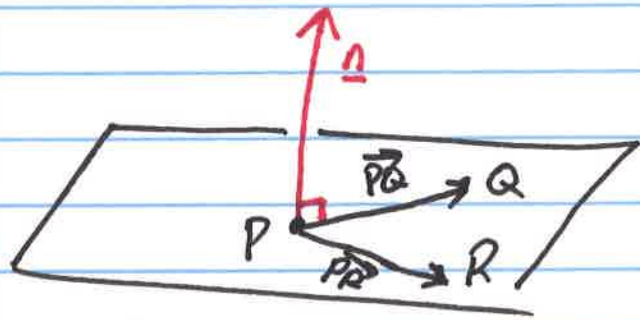
MATH 146-003 Test I: Solutions, Feb. 9, 2017

(1)



We need a normal vector to the plane, $\underline{n}$ say. We can take

$$\underline{n} = \vec{PQ} \times \vec{PR} = \langle 0, -3, -6 \rangle \times \langle -1, 3, 3 \rangle$$

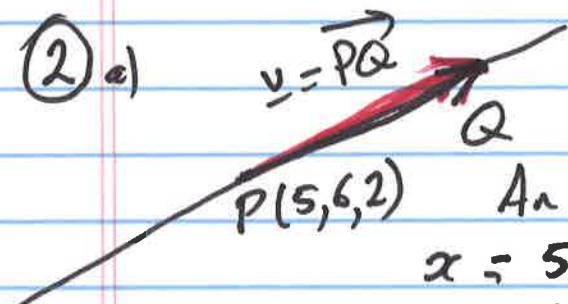
$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & -3 & -6 \\ -1 & 3 & 3 \end{vmatrix} = 9\underline{i} + 6\underline{j} - 3\underline{k}.$$

An equation of the plane has the form $9x + 6y - 3z = D$. Plug in the coordinates of P (say) to find D.

$9 + 6 - 3 = D = 12$. An answer is:

$9x + 6y - 3z = 12$, as we can easily check by plugging in the coordinates of Q and R.

(2) a)



Let $\underline{v} = \vec{PQ} = \langle -6, -12, 6 \rangle$.

An equation of L is

$$\left.\begin{aligned} x &= 5 - 6t \\ y &= 6 - 12t \\ z &= 2 + 6t. \end{aligned}\right\} (*)$$

b) Plug in the expression at (*)

$2(5-6t) - 5(6-12t) - 2(2+6t) = 0$, so

$10 - 12t - 30 + 60t - 4 - 12t = 0$,

so $-24+36t=0$ and $t=\frac{2}{3}$. Plug that into $(*)$ to get the point of intersection: $(1,-2,6)$

③ a) Take $\frac{1}{|\underline{v}|}\underline{v} = \frac{1}{\sqrt{1+9+25}}\underline{v} = \frac{1}{\sqrt{35}}\underline{v}$, which has length 1, and multiply it by 6 to get $\frac{6}{\sqrt{35}}\langle 1,3,-5\rangle$ $= \left\langle \frac{6}{\sqrt{35}}, \frac{18}{\sqrt{35}}, \frac{-30}{\sqrt{35}} \right\rangle$.

b) Use $\underline{v}\cdot\underline{w} = |\underline{v}|\,|\underline{w}|\cos\theta$ where θ is the angle in question. We compute. $\underline{v}\cdot\underline{w} = 2+18+5=25$

$|\underline{v}|=\sqrt{35}$ $\quad |\underline{w}| = \sqrt{4+36+1} = \sqrt{41}$ so

$25 = \sqrt{35}\sqrt{41}\cos\theta$ and $\theta = \cos^{-1}\frac{25}{\sqrt{35}\sqrt{41}}$.

④ Change the parameter in the equation of one of the lines: L_2 is $x=1-s$, $y=1+s$, $z=12+2s$, say. Solve

$$2+t = 1-s \quad ①$$
$$4-3t = 1+s \quad ②$$
$$10-2t = 12+2s \quad ③.$$

I'll add ① and ② to get $6-2t=2$ so if there's a solution it has $t=2$. If so then ① gives $s=-3$. Yes $t=2$ gives $(4,-2,6)$ on L_1 and $s=-3$ gives $(4,-2,6)$ on L_2.

(5) a) $= \int \sin^4 x \cos^4 x \, \boxed{\cos x \, dx} = \int \sin^4 x (\cos^2 x)^2 \boxed{\cos x \, dx}$

$= \int \sin^4 x \, (1-\sin^2 x)^2 \boxed{\cos x \, dx}$, Let $u = \sin x$
so $du = \cos x \, dx$

$= \int u^4 (1-u^2)^2 du = \int u^4 (1-2u^2+u^4) du$

$= \int (u^4 - 2u^6 + u^8) du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$

$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

b) $= \int \frac{1-\cos 6x}{2} dx = \frac{1}{2}\left(x - \frac{1}{6}\sin 6x\right) + C$

(6) a) $= \int \tan^2 x \sec^2 x \; \boxed{\tan x \sec x \, dx}$

$= \int (\sec^2 x - 1)\sec^2 x \; \boxed{\tan x \sec x \, dx}$ Let $u = \sec x$ so
$du = \tan x \sec x \, dx$

$= \int (u^2 - 1) u^2 du = \int (u^4 - u^2) du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

b) $= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du$ (with $u = \cos x$)

$= -\ln|u| + c = -\ln|\cos x| + C.$

(7) a) $= x^2 e^x - \int e^x 2x\,dx = x^2 e^x - 2\int x e^x\,dx$

$= x^2 e^x - 2\left(x e^x - \int e^x \cdot 1\,dx\right)$

$= x^2 e^x - 2x e^x + 2e^x + c$

b) $= \int 1 \cdot \tan^{-1} x = (\tan^{-1} x)\,x - \int x \frac{1}{1+x^2}\,dx$

$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C$

(8) a) Let $L = \lim\limits_{x\to 0^+} (x)^{2x^2}$ so $\ln L = \ln \lim\limits_{x\to 0^+} (x)^{2x^2}$

$= \lim\limits_{x\to 0^+} \ln\left[(x)^{2x^2}\right] = \lim\limits_{x\to 0^+} 2x^2 \ln x$

$= \lim\limits_{x\to 0^+} 2\,\frac{\ln x}{x^{-2}} \underset{L'H}{=} \lim\limits_{x\to 0^+} 2\,\frac{1/x}{-2x^{-3}} = \lim\limits_{x\to 0^+} -\frac{x^3}{x}$

$= \lim\limits_{x\to 0^+} -x^2 = 0$. Therefore $\ln L = 0$ and so

$L = e^0 = 1.$

b) Done in class.

(9)

Let $I = \int \sin^n x \, dx = \int \sin^{n-1} x \cdot \sin x \, dx$

$$= \sin^{n-1} x \, (-\cos x) - \int (-\cos x) \, (n-1) \sin^{n-2} x \cdot \cos x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

So $I = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) I$

and $nI = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx.$

~~so~~ which implies

$$I = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

as required.