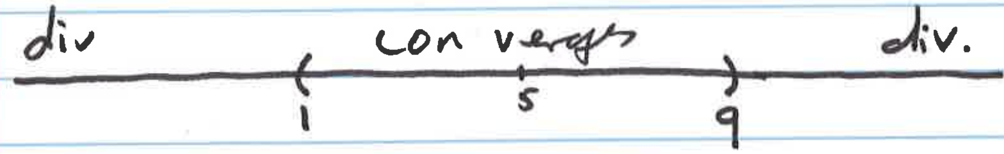
Math 146-003, Test III: Solutions, April 20, 2017

① Let $\rho = \lim \left| \dfrac{(-1)^{n+1}(x-5)^{n+1}}{4^{n+1}(n+1)} \Big/ \dfrac{(-1)^n (x-5)^n}{4^n\, n} \right|$

$$= \lim \left| \frac{(x-5)}{4} \frac{n}{n+1} \right| = \left| \frac{x-5}{4} \right| .$$

Converges when $\left| \frac{x-5}{4} \right| < 1$, i.e when

$|x-5| < 4$, or $-4 < x-5 < 4$ or.
$1 < x < 9$.



Endpoints

When $x=1$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{4^n\, n} = \sum_{n=1}^{\infty} \frac{1}{n}$
which diverges (Harmonic series).

When $x=9$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges
by the alternating series test. The interval of convergence is therefore $(1, 9]$.

(2) a) $\rho = \lim \left| \frac{x^{n+1}}{(2(n+1))!} \Big/ \frac{x^n}{(2n)!} \right| = \lim \left| \frac{x}{(2n+2)(2n+1)} \right| = 0$

< 1 for all x so the interval is

$(-\infty, \infty)$.

b) $\rho = \lim \left| (2(n+1))!\, x^{n+1} / (2n)!\, x^n \right|$

$= \lim \left| (2n+2)(2n+1)\, x \right| = \begin{cases} \infty & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

so the series converges <u>only</u> when $x = 0$.
The "interval" is $[0,0]$ (strange notation though).

(3) a) $f(x) = \frac{1}{2} \frac{1}{1 + \frac{1}{2}x^4}$ so replace x with

$-\frac{1}{2}x^4$ in our "main theorem" to get

$f(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \left(-\frac{1}{2}x^4\right)^n \right)$ for $\left|-\frac{1}{2}x^4\right| < 1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{n+1}}$ for $\left|\frac{1}{2}x^4\right| < 1$

or $|x^4| < 2$

or $-\sqrt[4]{2} < x < \sqrt[4]{2}$.

Radius of convergence is $\sqrt[4]{2}$.

b) $g(x) = x^3 f(x)$

$= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2^{n+1}}$.

(4) Now $\frac{1}{5-x} = \frac{1}{5}\frac{1}{1-x/5} = \frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n$

$= \sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$ so $\ln(5-x) = -\int\frac{1}{5-x}\,dx$

$= -\int\left(\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}\right)dx = -\sum_{n=0}^{\infty}\left(\int\frac{x^n}{5^{n+1}}\,dx\right)$

$= C - \sum_{n=0}^{\infty}\frac{1}{5^{n+1}}\frac{x^{n+1}}{(n+1)}$. Setting $x=0$ gives $C = \ln 5$ so

$\ln(5-x) = \ln 5 - \sum_{n=0}^{\infty}\frac{1}{5^{n+1}(n+1)}x^{n+1}$.

(5) The required series is $\sum_{n=0}^{\infty} a_n x^n$ where

$a_n = \frac{1}{n!}f^{(n)}(0)$. Now

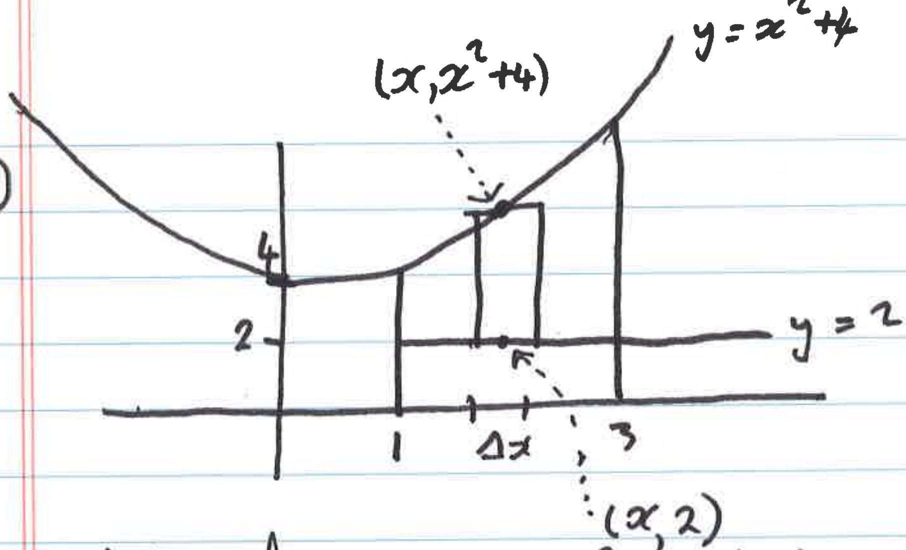
$f^{(0)}(x) = (1-x)^{-3}$	$f^{(0)}(0) = 1$
$f'(x) = -3(1-x)^{-4}(-1)$	$f'(0) = 3$
$f''(x) = (-4)3(1-x)^{-5}(-1)$	$f''(0) = 4\cdot 3$.
$f'''(x) = (-5)\cdot 4\cdot 3(1-x)^{-6}(-1)$	$f'''(0) = 5\cdot 4\cdot 3$.
$f^{(4)}(x) = (-6)\cdot 5\cdot 4\cdot 3(1-x)^{-7}(-1)$	$f^{(4)}(0) = 6\cdot 5\cdot 4\cdot 3$.

We guess $f^{(n)}(0) = \frac{(n+2)!}{2}$ so

$a_n = \frac{1}{2}(n+2)(n+1)$. The series is

$$\sum_{n=0}^{\infty}\frac{1}{2}(n+2)(n+1)\,x^n$$

⑥



Vol of Washer is $\pi(R^2 - r^2)\Delta x$

$$= \pi\left((x^2+4)^2 - 2^2\right)\Delta x = \pi(x^4+8x^2+12)\Delta x.$$

Vol of Solid is $\pi\int_1^3 (x^4+8x^2+12)\,dx$

$$= \pi\left(\frac{1}{5}x^5 + 8/3\,x^3 + 12x\right)\Big|_1^3 = \pi\left[\frac{1}{5}3^5 + 8\cdot 9 + 36 - \frac{1}{5} - \frac{8}{3} - 12\right]$$

⑦ Let $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$. Then

$f'(x) = \frac{1}{4}x^2 - \frac{1}{x^2}$ and the arc length is

$$\int_1^3 \sqrt{1+[f'(x)]^2}\,dx = \int_1^3 \sqrt{1+\left(\frac{1}{4}x^2 - \frac{1}{x^2}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{1+\frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4}}\,dx$$

$$= \int_1^3 \sqrt{\frac{1}{16}x^4 + \frac{1}{2} + \frac{1}{x^4}}\,dx$$

$$= \int_1^3 \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2}\, dx = \int_1^3 \left(\frac{1}{4}x^2 + \frac{1}{x^2}\right) dx$$

$$= \left(\frac{1}{12}x^3 - \frac{1}{x}\right)\Bigg|_1^3 = \frac{1}{12}\left(3^3 - 1\right) - \left(\frac{1}{3} - 1\right).$$

⑧ a) We are given that $\frac{dy}{dx} = 15x^2 y$ and $y(0) = 2$

so $\quad \frac{1}{y}\, dy = 15x^2\, dx \quad$ so

$\int \frac{1}{y}\, dy = \int 15x^2\, dx$ and $\ln|y| = 5x^3 + C$.

Plug in $x = 0$ and $y = 2$ to get $\ln 2 = C$.

so

$\ln|y| = 5x^3 + \ln 2$ and

$$y = e^{5x^3 + \ln 2} = e^{5x^3} \cdot e^{\ln 2}$$

$$= 2e^{5x^3}$$

b) $x = r\cos\theta = -2\cos\left(-\frac{\pi}{3}\right) = -2\left(\frac{1}{2}\right) = -1$

$y = r\sin\theta = -2\sin\left(-\frac{\pi}{3}\right) = -2\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$