ECSE 343 Numerical Methods in Engineering

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Overdetermined Systems

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

- Assume columns of A are linearly independent
- Columns of A form a 3D subspace Q of $\mathbb{R}^5$

Overdetermined Systems

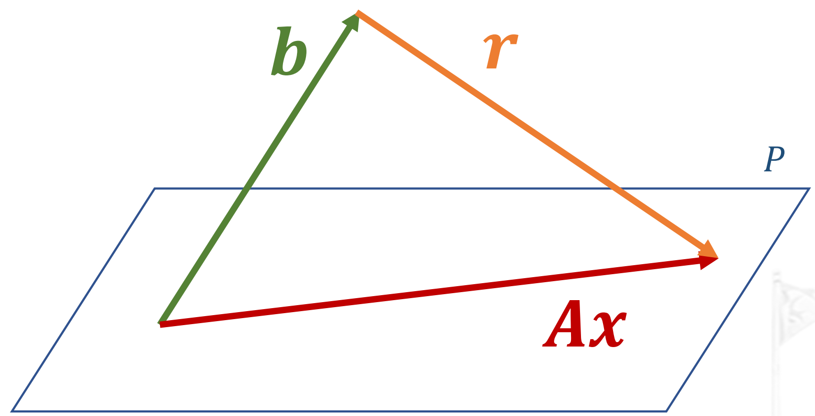
Find $\boldsymbol{x}$ such that: $\boldsymbol{Ax} = \boldsymbol{b}$

Impossible:

$\boldsymbol{Ax} \in P \qquad \boldsymbol{b} \notin P$

Minimize $\boldsymbol{r} = \boldsymbol{Ax} - \boldsymbol{b}$

i.e. minimize $||\boldsymbol{r}|| = ||\boldsymbol{Ax} - \boldsymbol{b}||$



Overdetermined Systems

Minimize $||\boldsymbol{r}|| = ||\boldsymbol{Ax} - \boldsymbol{b}||$

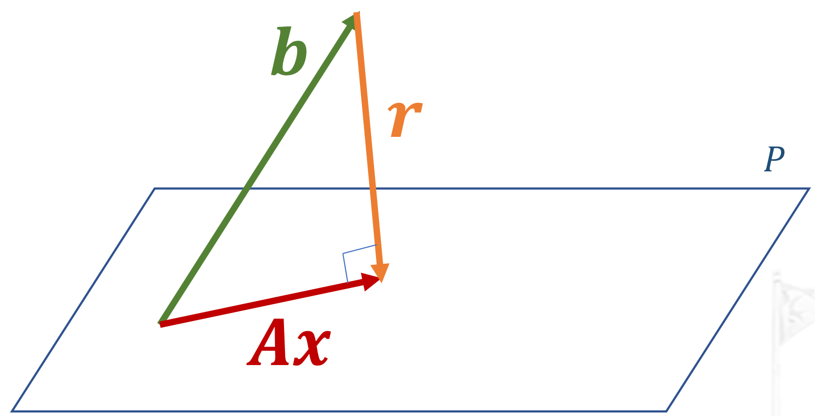
$\Longrightarrow \boldsymbol{r} \perp P$

$\Longrightarrow \boldsymbol{A}^T\boldsymbol{r} = \boldsymbol{0}$

$\Longrightarrow \boldsymbol{A}^T(\boldsymbol{Ax} - \boldsymbol{b}) = \boldsymbol{0}$

$\Longrightarrow \boxed{\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}}$

$\Longrightarrow \boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}$



Avoid $\boldsymbol{A}^T\boldsymbol{A}$ due to possible ill-conditioning

Overdetermined Systems

Minimize $||\boldsymbol{r}|| = ||\boldsymbol{Ax} - \boldsymbol{b}||$

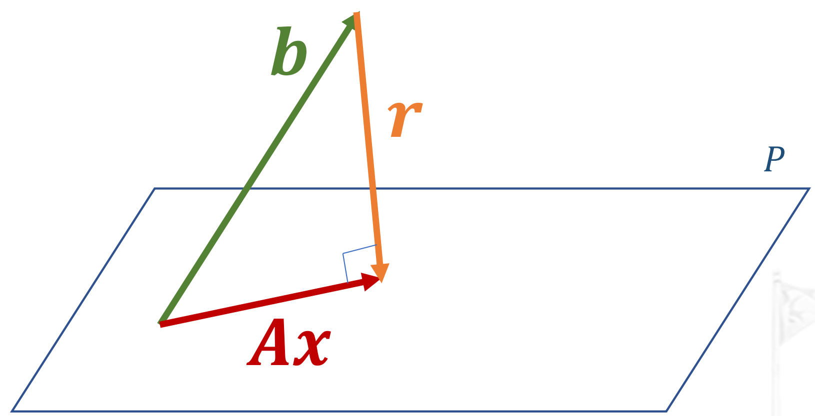
$\Longrightarrow \boldsymbol{r} \perp P$

$\Longrightarrow \boldsymbol{Q}^T\boldsymbol{r} = \mathbf{0}$

$\Longrightarrow \boldsymbol{Q}^T(\boldsymbol{Ax} - \boldsymbol{b}) = \mathbf{0}$

$\Longrightarrow \boxed{\boldsymbol{Q}^T\boldsymbol{Ax} = \boldsymbol{Q}^T\boldsymbol{b}}$

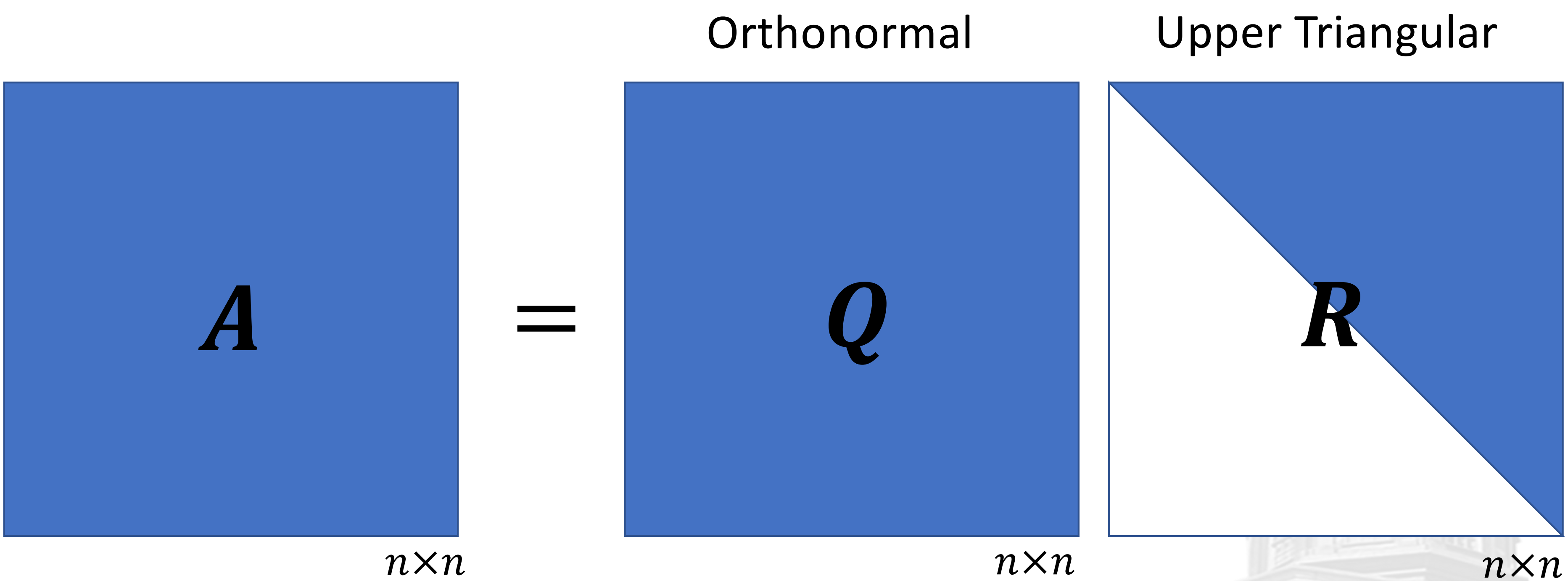
$\Longrightarrow \boldsymbol{Rx} = \boldsymbol{Q}^T\boldsymbol{b}$



$$\boldsymbol{A} = \boldsymbol{QR}$$

$$\text{colsp}\{\boldsymbol{A}\} = \text{colsp}\{\boldsymbol{Q}\}$$

QR Decomposition



Modified Gram-Schmidt

$$\boldsymbol{A}\underbrace{\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3\cdots\boldsymbol{R}_m}_{\boldsymbol{R}^{-1}} = \boldsymbol{Q}$$

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$$

Another Approach

$$\underbrace{\boldsymbol{E}_m \cdots \boldsymbol{E}_4 \boldsymbol{E}_3 \boldsymbol{E}_2 \boldsymbol{E}_1}_{\text{Elimination Matrices}} \boldsymbol{A} = \boldsymbol{U}$$

$$\boldsymbol{L}^{-1}$$

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{Elimination Matrices}} \overbrace{\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix}}^{\boldsymbol{A}} = \overbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}}^{\boldsymbol{U}}$$

Another Approach

$$\underbrace{\boldsymbol{E}_m \cdots \boldsymbol{E}_4 \boldsymbol{E}_3 \boldsymbol{E}_2 \boldsymbol{E}_1}_{\text{Elimination Matrices} \quad \boldsymbol{L}^{-1}} \boldsymbol{A} = \boldsymbol{U}$$

$$\underbrace{\boldsymbol{H}_m \cdots \boldsymbol{H}_4 \boldsymbol{H}_3 \boldsymbol{H}_2 \boldsymbol{H}_1}_{\text{Orthonormal Matrices} \quad \boldsymbol{Q}^{T}} \boldsymbol{A} = \boldsymbol{R}$$

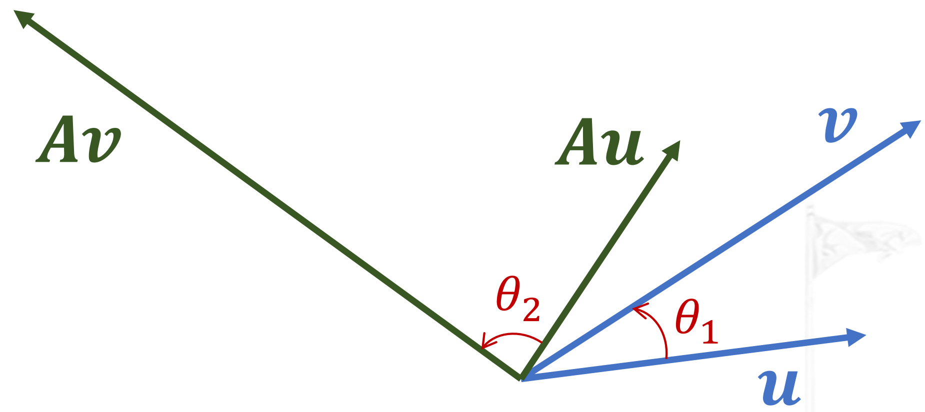
Matrix Vector Multiplication

In General:

$$||Av|| \neq ||v||$$
$$||Au|| \neq ||u||$$

$$\frac{||Au||}{||u||} \neq \frac{||Av||}{||v||}$$

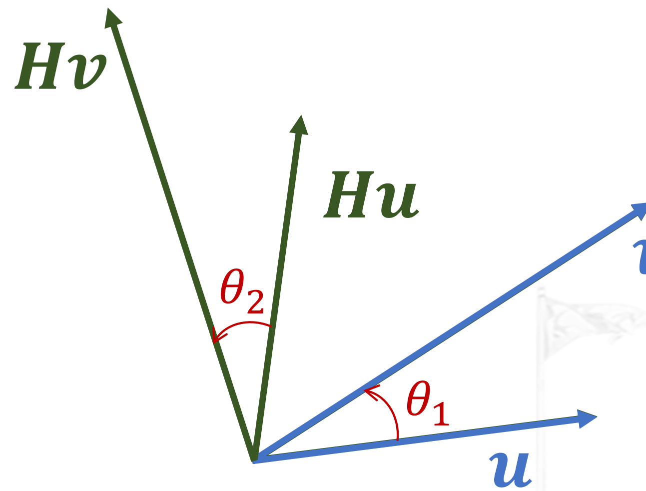
$$\theta_1 \neq \theta_2$$



Multiplication by an Orthonormal Matrix

$$\|Hv\| = \|v\|$$

$$(Hv)^T(Hv) = v^T \overbrace{H^T H}^{U} v = v^T v$$



Similarly: $\|Hu\| = \|u\|$

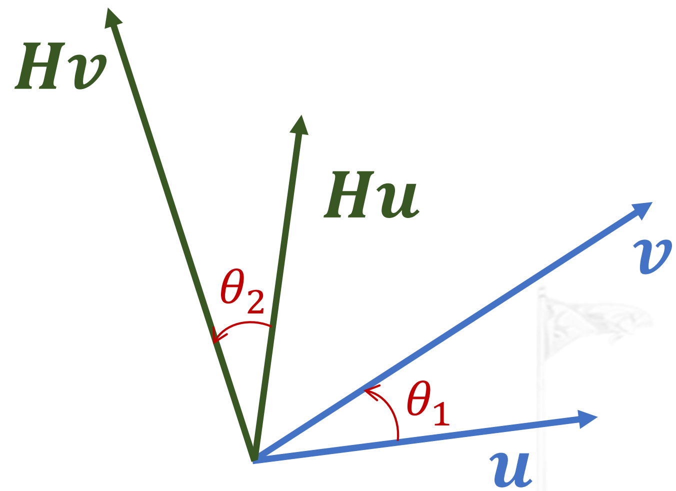
$$\begin{array}{c} \theta_1 = \theta_2 \\ |\theta_1| = |\theta_2| \end{array} \begin{cases} \langle v, u \rangle = \|v\|\|u\|\cos(\theta_1) \\ \langle Hv, Hu \rangle = \|Hv\|\|Hu\|\cos(\theta_2) = \|v\|\|u\|\cos(\theta_2) \\ \langle Hv, Hu \rangle = (Hu)^T(Hv) = u^T H^T H v = u^T v = \langle v, u \rangle \end{cases}$$

Multiplication by an Orthonormal Matrix

$\|\boldsymbol{Hv}\| = \|\boldsymbol{v}\|$ $\|\boldsymbol{Hu}\| = \|\boldsymbol{u}\|$

$\theta_1 = \theta_2$

- Rotation

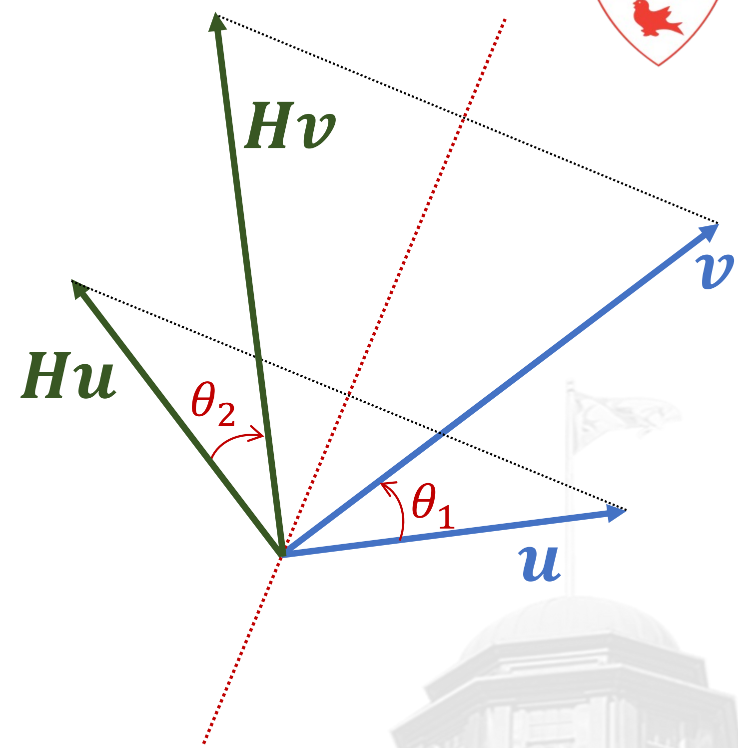


Multiplication by an Orthonormal Matrix

$\|\boldsymbol{Hv}\| = \|\boldsymbol{v}\|$ $\qquad$ $\|\boldsymbol{Hu}\| = \|\boldsymbol{u}\|$

$|\theta_1| = |\theta_2|$ $\qquad$ $\theta_1 = -\theta_2$

- Rotation
- Reflection

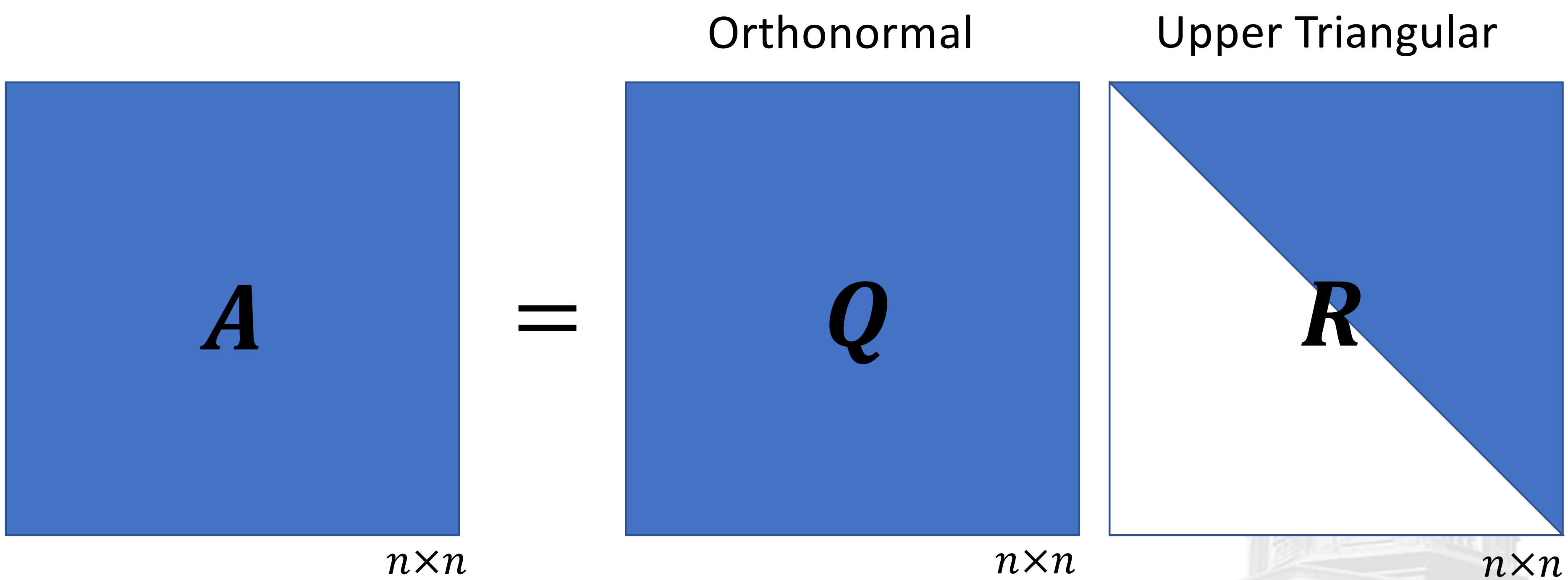


Householder Reflections

$$\underbrace{\boldsymbol{H}_m \cdots \boldsymbol{H}_4\boldsymbol{H}_3\boldsymbol{H}_2\boldsymbol{H}_1}_{\text{Orthonormal Matrices} \quad \boldsymbol{Q}^T}\boldsymbol{A} = \boldsymbol{R}$$

Carefully choose reflection matrices

QR Decomposition



Householder Reflections

Orthonormal

Upper Triangular

$$\underbrace{\boldsymbol{Q}^T}_{\boldsymbol{H}_m \cdots \boldsymbol{H}_4 \boldsymbol{H}_3 \boldsymbol{H}_2 \boldsymbol{H}_1} \boldsymbol{A} = \boldsymbol{R}$$

Elimination Matrices

$$\underbrace{E_m \cdots E_4 E_3 E_2 E_1}_{\text{Elimination Matrices} \quad L^{-1}} A = U$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Elimination Matrices}} \overbrace{\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix}}^{A} = \overbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}}^{U}$$

QR Decomposition

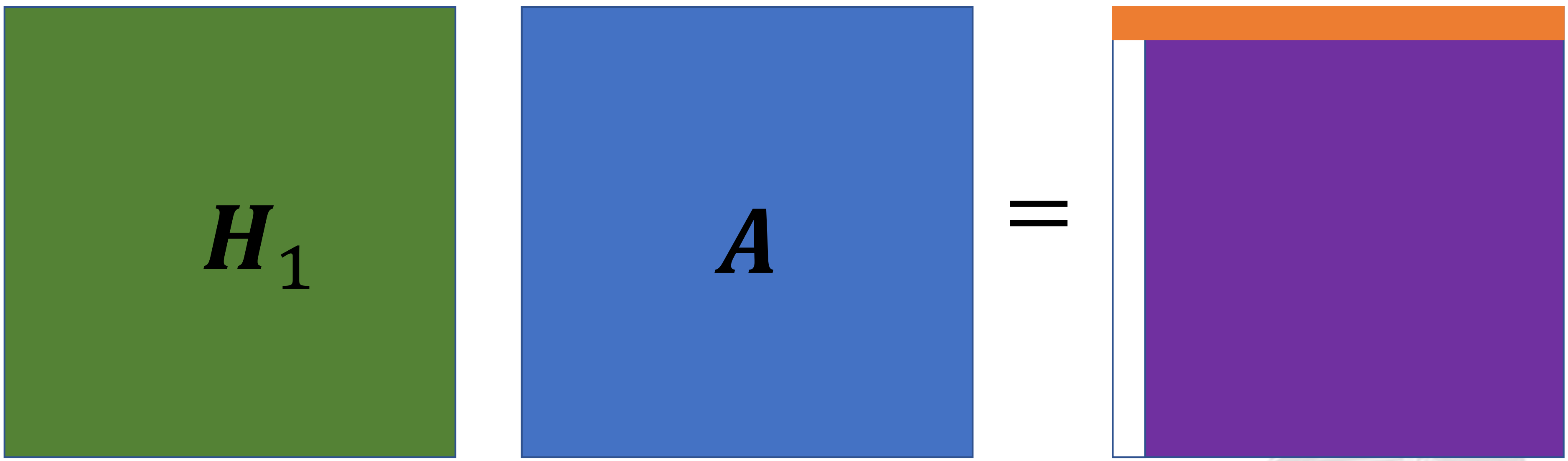
Orthonormal

Upper Triangular

$$\underbrace{\boldsymbol{Q}^T}_{\boldsymbol{H}_m \cdots \boldsymbol{H}_4 \boldsymbol{H}_3 \boldsymbol{H}_2 \boldsymbol{H}_1} \boldsymbol{A} = \boldsymbol{R}$$

QR Decomposition

Orthonormal



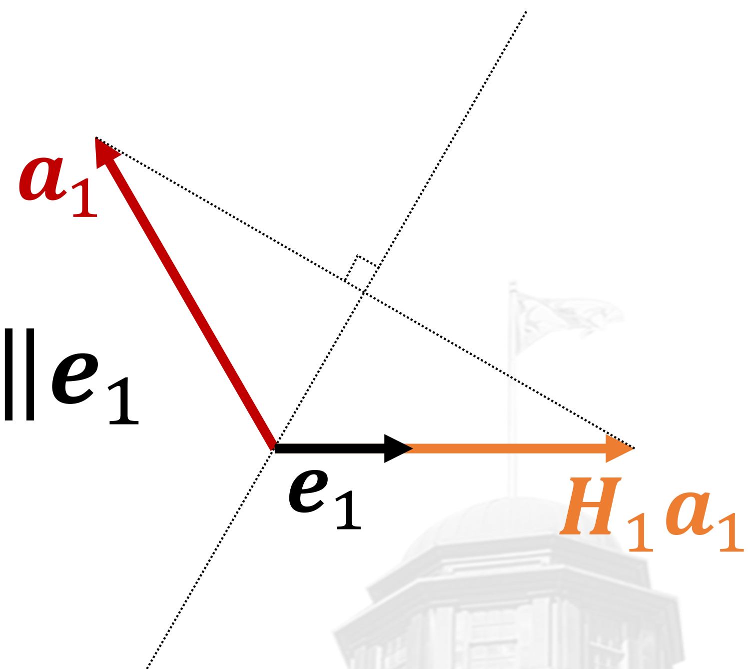
Householder Reflections

$$\begin{bmatrix} & & \\ & \boldsymbol{H}_1 & \\ & & \end{bmatrix}\begin{bmatrix}\begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix}\end{bmatrix} = \begin{bmatrix}\begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{12} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix}\end{bmatrix}$$

Householder Reflections

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = r_{11}\boldsymbol{e}_1$$

$$= \pm\|\boldsymbol{a}_1\|\boldsymbol{e}_1$$



We know what $\boldsymbol{H}_1\boldsymbol{a}_1$ is (or must be) even before we learn how to find the reflection matrix $\boldsymbol{H}_1$

Householder Reflections

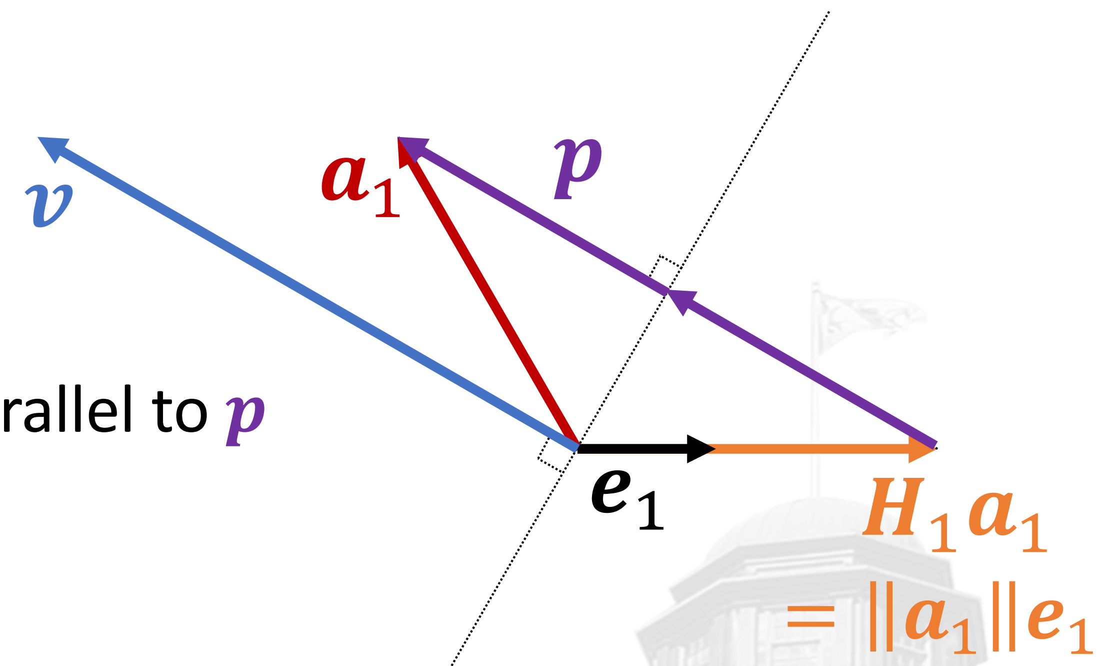
$$\boldsymbol{H}_1\boldsymbol{a}_1 = \pm\|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \boldsymbol{a}_1 - 2\boldsymbol{p}$$

Assume we have a vector $\boldsymbol{v}$ parallel to $\boldsymbol{p}$

The length of $\boldsymbol{v}$ is arbitrary



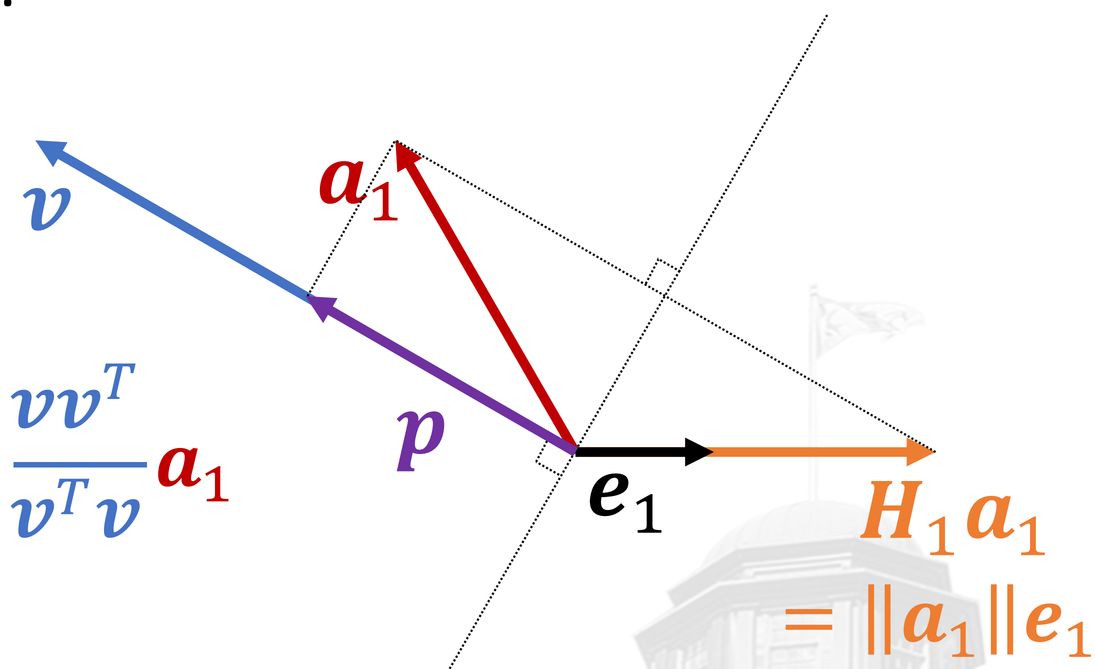
Householder Reflections

$\boldsymbol{p}$ is the projection of $\boldsymbol{a}_1$ on $\boldsymbol{v}$.

$$\boldsymbol{p} = \underbrace{\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}}_{\text{Projection matrix onto } \boldsymbol{v}} \boldsymbol{a}_1$$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \boldsymbol{a}_1 - 2\boldsymbol{p} = \boldsymbol{a}_1 - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}\boldsymbol{a}_1$$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \left(\boldsymbol{U} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}\right)\boldsymbol{a}_1$$



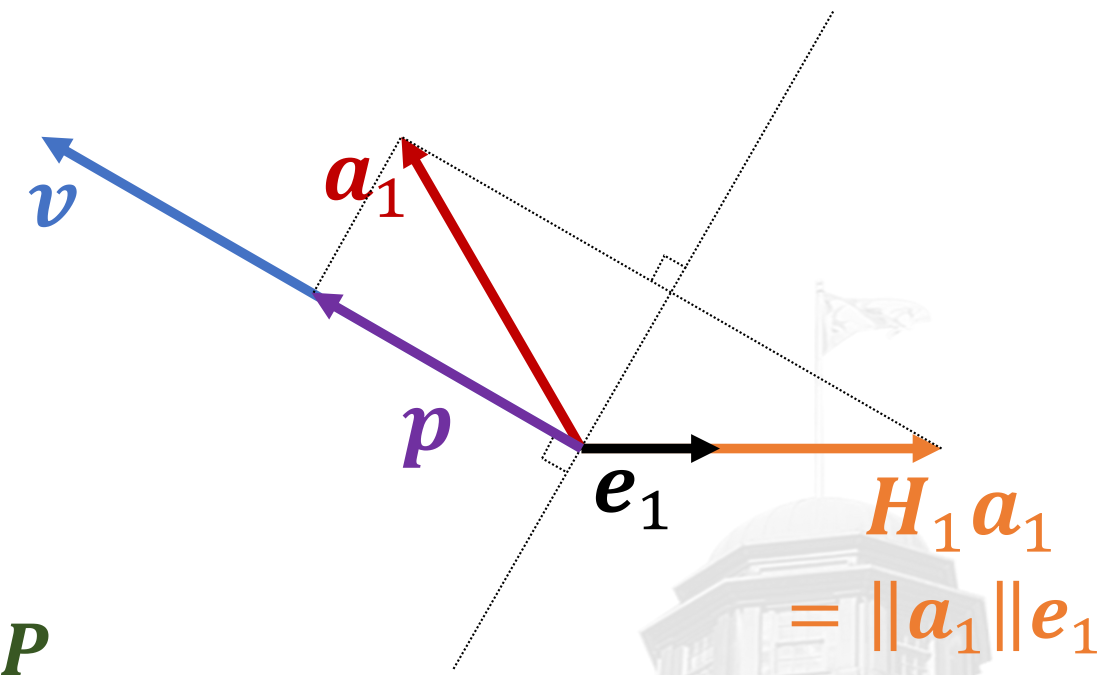
Householder Reflections

$$\boldsymbol{p} = \underbrace{\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}}_{} \boldsymbol{a}_1 = \boldsymbol{P}\boldsymbol{a}_1$$

Projection matrix onto $\boldsymbol{v}$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \left(\boldsymbol{U} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}\right)\boldsymbol{a}_1$$

$$\boldsymbol{H}_1 = \boldsymbol{U} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} = \boldsymbol{U} - 2\boldsymbol{P}$$

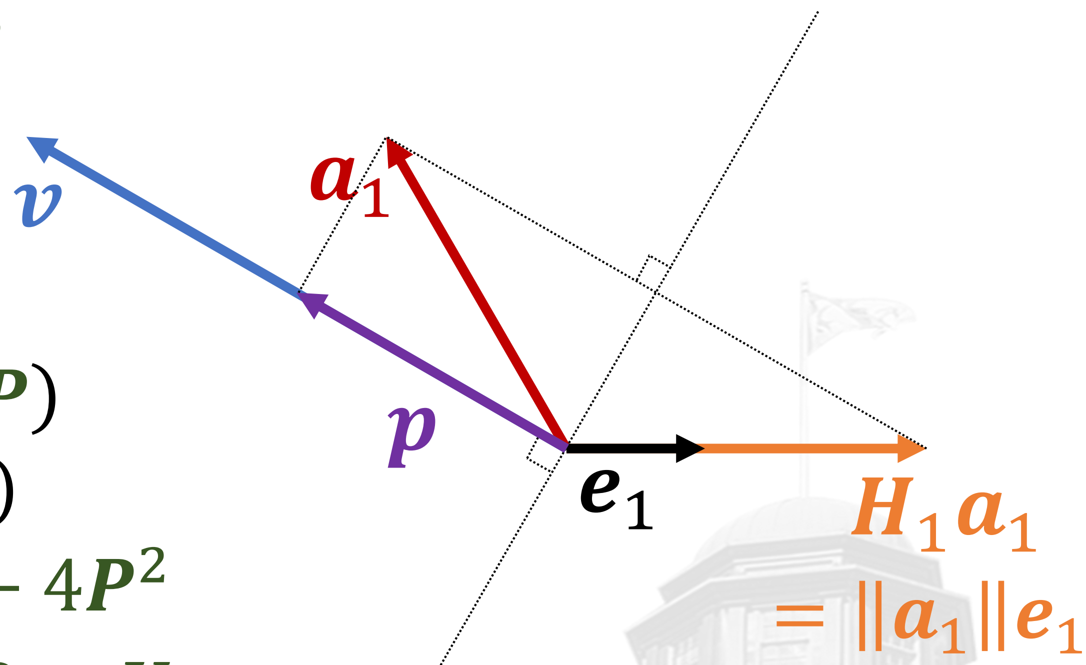


Is $\boldsymbol{H}_1$ Orthonormal?

$$\boldsymbol{H}_1 = \boldsymbol{U} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} = \boldsymbol{U} - 2\boldsymbol{P}$$

$$\boldsymbol{P} = \boldsymbol{P}^T \qquad \boldsymbol{P}^2 = \boldsymbol{P}$$

$$\begin{aligned}\boldsymbol{H}_1^T\boldsymbol{H}_1 &= (\boldsymbol{U} - 2\boldsymbol{P})^T(\boldsymbol{U} - 2\boldsymbol{P}) \\ &= (\boldsymbol{U} - 2\boldsymbol{P})(\boldsymbol{U} - 2\boldsymbol{P}) \\ &= \boldsymbol{U} - 2\boldsymbol{U}\boldsymbol{P} - 2\boldsymbol{P}\boldsymbol{U} + 4\boldsymbol{P}^2 \\ &= \boldsymbol{U} - 2\boldsymbol{P} - 2\boldsymbol{P} + 4\boldsymbol{P} = \boldsymbol{U}\end{aligned}$$



Choice of $\boldsymbol{v}$

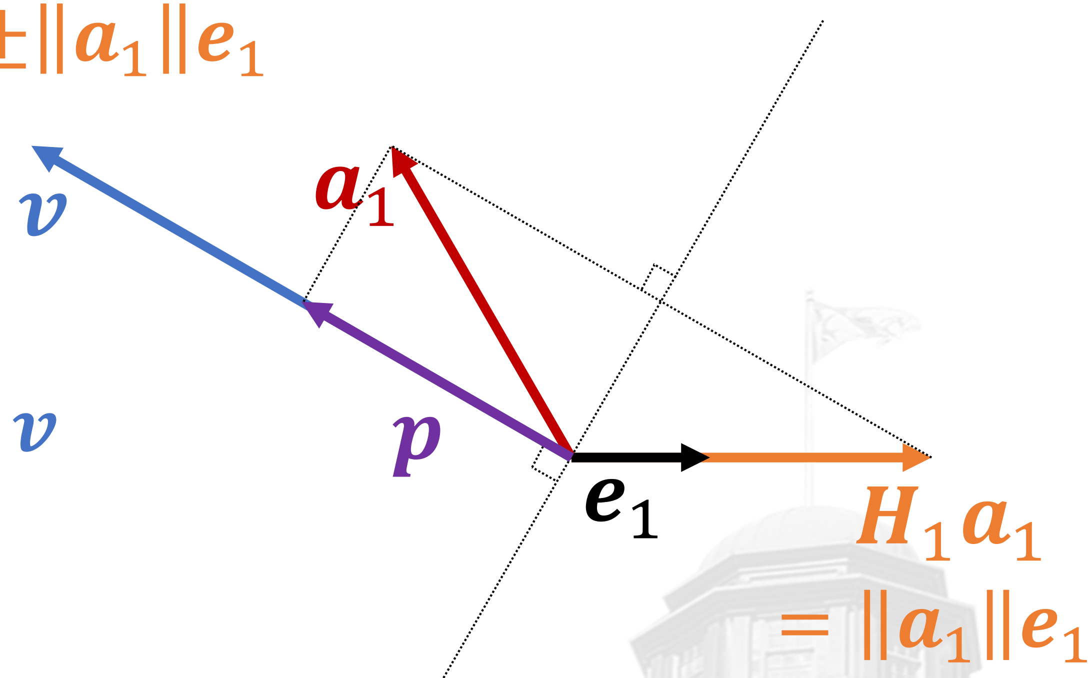
$$\boldsymbol{H}_1\boldsymbol{a}_1 = \boldsymbol{a}_1 - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}\boldsymbol{a}_1 = \pm\|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{a}_1 \pm \|\boldsymbol{a}_1\|\boldsymbol{e}_1 = 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}}\boldsymbol{a}_1$$

$(\boldsymbol{a}_1 \pm \|\boldsymbol{a}_1\|\boldsymbol{e}_1)$ is parallel to $\boldsymbol{v}$

Note: Length of $\boldsymbol{v}$ is arbitrary

$$\boldsymbol{v} \equiv \boldsymbol{a}_1 \pm \|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

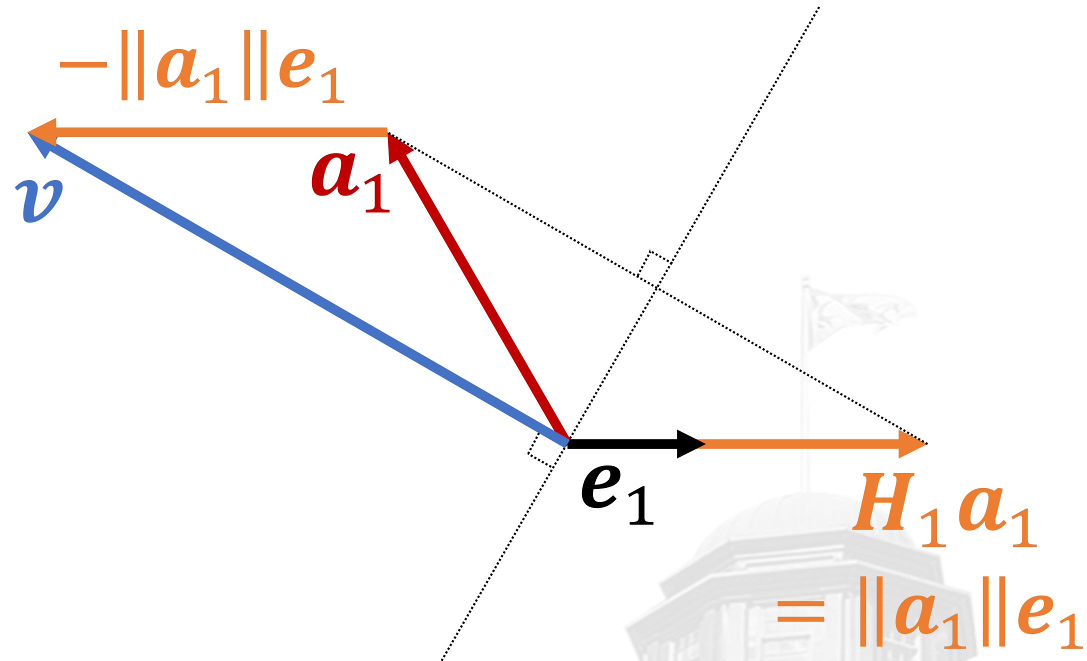


Choice of $\boldsymbol{v}$ Option #1

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \pm\|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{v} \equiv \boldsymbol{a}_1 - \|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

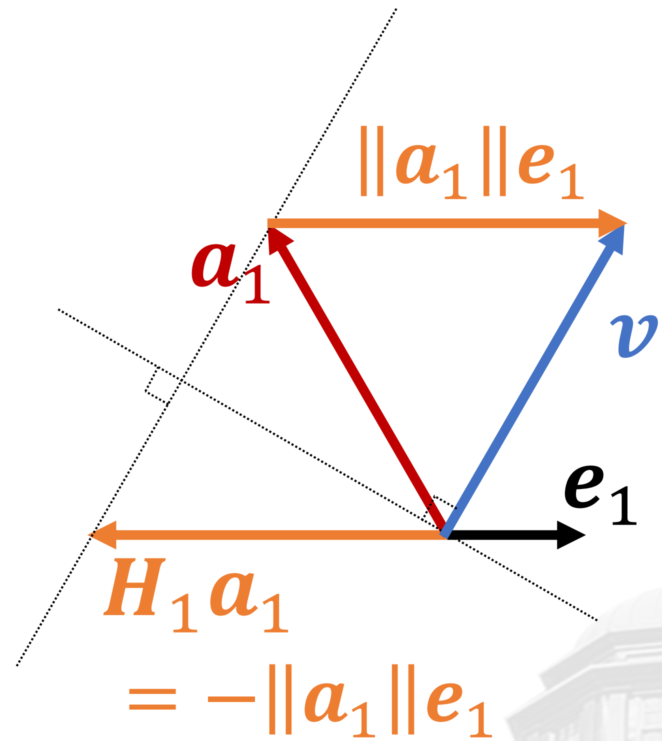


Choice of $\boldsymbol{v}$ Option #2

$$\boldsymbol{H}_1\boldsymbol{a}_1 = \pm\|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{H}_1\boldsymbol{a}_1 = -\|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

$$\boldsymbol{v} \equiv \boldsymbol{a}_1 + \|\boldsymbol{a}_1\|\boldsymbol{e}_1$$



Householder Reflections

$$\begin{bmatrix} & & \\ & \boldsymbol{H}_1 & \\ & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{12} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$\boldsymbol{H}_1 = \boldsymbol{U} - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}} \qquad \boldsymbol{v} \equiv \boldsymbol{a}_1 \pm \|\boldsymbol{a}_1\|\boldsymbol{e}_1$$

Chose sign that makes $\boldsymbol{v}^T\boldsymbol{v}$ larger and therefore avoid dividing by a small value

Next Steps

$$\begin{bmatrix} \\ \boldsymbol{H}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{12} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$\tilde{\boldsymbol{A}}$$

Next Steps

Orthonormal

Next Steps

$$\begin{bmatrix} \quad \boldsymbol{F}_2 \quad \end{bmatrix} \overset{\tilde{\boldsymbol{A}}}{\begin{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{a}}_2 \end{bmatrix} & \begin{bmatrix} \tilde{\boldsymbol{a}}_3 \end{bmatrix} & \cdots & \begin{bmatrix} \tilde{\boldsymbol{a}}_m \end{bmatrix} \end{bmatrix}} = \begin{bmatrix} \begin{bmatrix} r_{22} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{23} \\ \begin{bmatrix} \hat{\boldsymbol{a}}_3 \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{2m} \\ \begin{bmatrix} \hat{\boldsymbol{a}}_m \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

We can find $\boldsymbol{F}_2$ using the same process we used to find $\boldsymbol{H}_1$

Second Householder Reflection

$$\boldsymbol{H}_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & & & \\ \vdots & & & \boldsymbol{F}_2 & \\ 0 & & & & \end{bmatrix}$$

Next Steps

$$\begin{bmatrix} & & \\ & \boldsymbol{H}_1 & \\ & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{12} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & \boldsymbol{H}_2\boldsymbol{H}_1 & \\ & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = ?$$

Next Steps

$$\begin{bmatrix} & & \\ & \boldsymbol{H}_2 & \\ & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{12} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \end{bmatrix} = \textbf{?}$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & \boldsymbol{F}_2 & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \begin{bmatrix} \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{12} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \end{bmatrix} = \textbf{?}$$

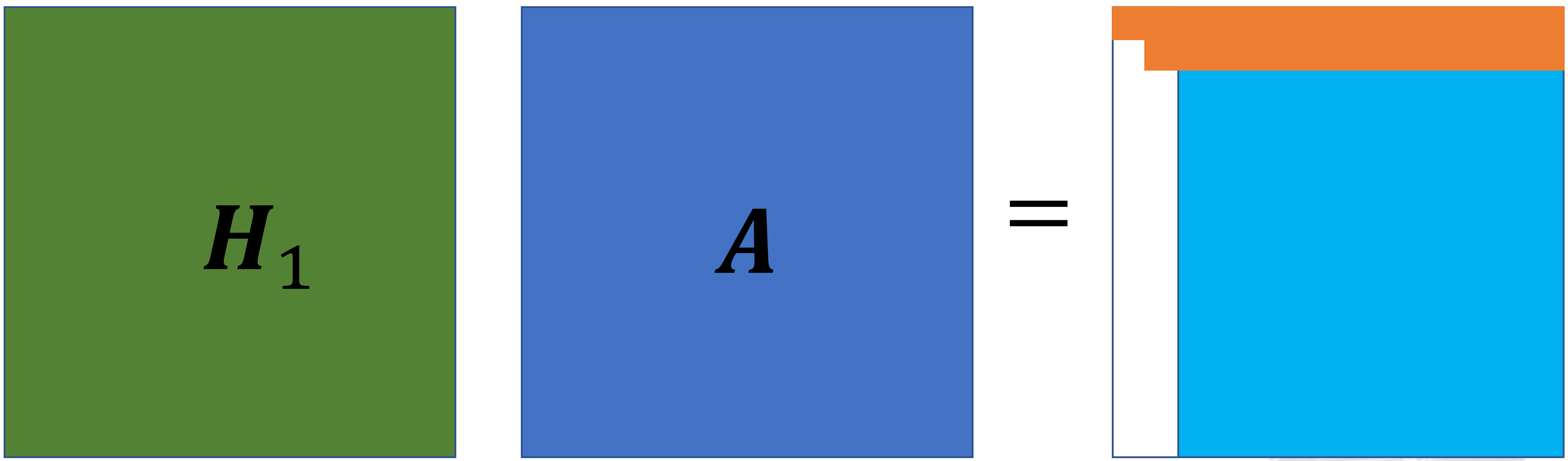
Next Steps

$$\begin{bmatrix} \boldsymbol{H}_2\boldsymbol{H}_1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} =$$

$$= \begin{bmatrix} \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{11} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} r_{11} \\ r_{22} \\ \begin{bmatrix} \\ \hat{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} \end{bmatrix} & \cdots & \begin{bmatrix} r_{1m} \\ r_{2m} \\ \begin{bmatrix} \\ \hat{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \end{bmatrix}$$

QR Decomposition

Orthonormal



Third Householder Reflection

$$
\boldsymbol{H}_3 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \boldsymbol{F}_3 & \\ 0 & 0 & & & \end{bmatrix}
$$

QR Using Givens Rotations

$$
\boldsymbol{H}_i = \begin{bmatrix}
1 & 0 & & & \cdots & & & 0 \\
0 & 1 & \cdots & & \cdots & & & 0 \\
 & & \ddots & & & & & \\
 & & & \alpha & & \lambda & & \\
\vdots & \vdots & & & \ddots & & & \\
 & & & -\lambda & & \alpha & & \\
 & & & & & & \ddots & 0 \\
0 & 0 & & & & & 0 & 1
\end{bmatrix}
$$

Example

$$
\boldsymbol{H}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

Example

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Orthonormal iff $\alpha^2 + \lambda^2 = 1$

$$\boldsymbol{H}^T\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & -\lambda & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{U}$$

QR using Givens Rotations

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

QR Using Givens Rotations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \lambda \\ 0 & 0 & 0 & -\lambda & \alpha \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & \tilde{a}_{44} & \tilde{a}_{45} \\ 0 & \tilde{a}_{52} & \tilde{a}_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{bmatrix}$$

$$\begin{cases} \alpha^2 + \lambda^2 = 1 \\ -\lambda a_{41} + \alpha a_{51} = 0 \end{cases}$$

$$\alpha = \frac{a_{41}}{\sqrt{a_{41}^2 + a_{51}^2}} \qquad \lambda = \frac{a_{51}}{\sqrt{a_{41}^2 + a_{51}^2}}$$

Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \lambda & 0 \\ 0 & 0 & -\lambda & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & \tilde{a}_{44} & \tilde{a}_{45} \\ 0 & \tilde{a}_{52} & \tilde{a}_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & \hat{a}_{34} & \hat{a}_{35} \\ 0 & \hat{a}_{42} & \hat{a}_{43} & \hat{a}_{44} & \hat{a}_{45} \\ 0 & \tilde{a}_{52} & \tilde{a}_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{bmatrix}$$

$$\begin{cases} \alpha^2 + \lambda^2 = 1 \\ -\lambda a_{31} + \alpha \tilde{a}_{41} = 0 \end{cases}$$

$$\alpha = \frac{a_{31}}{\sqrt{a_{41}^2 + a_{51}^2}} \quad \lambda = \frac{\tilde{a}_{41}}{\sqrt{a_{41}^2 + a_{51}^2}}$$