Mock Quiz

1. $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ is a unitary (or orthonormal) matrix if and only if $\boldsymbol{V}^t\boldsymbol{V} = \boldsymbol{U}$ where $\boldsymbol{U}$ is the identity Matrix. Show that $\|\boldsymbol{Vx}\|_2 = \|\boldsymbol{x}\|_2$.

Hint. $\|x\|_2^2 = x^T x \quad , x \in \mathbb{R}^n$

$V \in \mathbb{R}^{n\times n}$ (V is a $n\times n$ real valued matrix)

$x \in \mathbb{R}^n$ (x " " $n\times 1$ " " vector)

$\Rightarrow Vx \in \mathbb{R}^n$

Square of 2-norm of Vx can be written as

$$\|Vx\|_2^2 = (Vx)^T(Vx)$$
$$= x^T V^T V x$$

Since V is orthonormal matrix, $V^T V = U_{n\times n}$ (identity)

$$\Rightarrow \|Vx\|_2^2 = x^T U x$$
$$= x^T x$$
$$\Rightarrow \|Vx\|_2^2 = \|x\|_2^2$$

Thus $\|Vx\|_2 = \|x\|_2$.

2. Using singular value decomposition, a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ can be decomposed into matrix $\boldsymbol{A} = \boldsymbol{USV}^t$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary and $\boldsymbol{S}$ is a diagonal matrix with the positive singular values on the diagonal. In class we showed that $\|\boldsymbol{A}\|_2 = \sigma_{max}$, where σ_{max} is the largest singular value of matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Based on this result show that $\|\boldsymbol{A}^{-1}\|_2 = \frac{1}{\sigma_{min}}$, where σ_{min} is the smallest singular value of matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$. Find an expression for the condition of matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.

$$A = USV^T \leftarrow$$

$$\|A\|_2 = \sigma_{max} \quad . \quad \boxed{\|A^{-1}\|_2 = \frac{1}{\sigma_{min}}.}$$

Definition of matrix norm,

$$\|A\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

Thus assuming A is invertible, we can write 2-norm of A^{-1} as,

$$\|A^{-1}\|_2 = \max_{\|x\|_2 \neq 0} \frac{\|A^{-1}x\|_2}{\|x\|_2}$$

Suppose, $y = A^{-1}x \Rightarrow Ay = x.$

$$\Rightarrow \|A^{-1}\|_2 = \max_{\|y\|_2 \neq 0} \frac{\|y\|_2}{\|Ay\|_2}$$

The above can be written as,

$$\|A^{-1}\| = \max_{\|y\|_2 \neq 0} \left(\frac{1}{\|Ay\|_2 / \|y\|_2} \right)$$

$$\|A^{-1}\|_2 = \frac{1}{\min\limits_{\|y\|_2 \neq 0} \left(\frac{\|Ay\|_2}{\|y\|_2} \right)} = \frac{1}{\min\limits_{\|y\|_2 = 1} \|Ay\|_2}$$

The matrix A can be decomposed using SVD.
as, $A = USV^T$

$$\Rightarrow \|A^{-1}\|_2 = \frac{1}{\min\limits_{\|y\|_2=1} \|USV^Ty\|_2}$$

Since U is orthonormal matrix therefore $\|Uw\|_2 = \|w\|_2$ for any vector w.

$$\|A^{-1}\|_2 = \frac{1}{\min\limits_{\|y\|_2=1} \|U\underbrace{SV^Ty}_{w}\|_2} = \frac{1}{\min\limits_{\|y\|_2=1} \|SV^Ty\|_2}$$

Similarly V is also a orthonormal matrix

$$\Rightarrow \|A^{-1}\| = \frac{1}{\min\limits_{\|y\|_2=1} \|S\underbrace{V^Ty}_{w}\|_2} = \frac{1}{\min\limits_{\|y\|_2=1} \|Sy\|_2}$$

$$S = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}, \quad \sigma_1 > \sigma_2 > \dots > \sigma_n$$

the minimum of $\|Sy\|_2$ happens when $y = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{n \times 1}$

$\Rightarrow \min\limits_{\|y\|_2=1} \|Sy\|_2 = \sigma_n$ ← minimimum singular value.

$$\Rightarrow \|A^{-1}\|_2 = \frac{1}{\sigma_{min}}$$

!!! $\kappa(A) = \|A\|_2 \, \|A^{-1}\|_2 = \frac{\sigma_{max}}{\sigma_{min}}$ ← ratio of maximum & minimum singular values.

What can be said about singular values of A^TA ?

Hint: $A^TA = (USV^T)^T(USV)$

$= \dots\dots$

3. Given a square ***lower triangular*** matrix $\boldsymbol{L}, \boldsymbol{L} \in \mathbb{R}^{n \times n}$. We need to solve for the solve for the following system of equations.

$$\boldsymbol{L}\,\boldsymbol{y} = \boldsymbol{b}$$

$$\begin{bmatrix} l_{11} & & & & \\ l_{21} & l_{22} & & & \\ l_{31} & l_{32} & l_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Do we need to invert the matrix L? How can we solve for y?

Row 1: $l_{11} y_1 = b_1 \Rightarrow y_1 = b_1 / l_{11}$

Row 2: $l_{21} y_1 + l_{22} y_2 = b_2 \Rightarrow y_2 = \frac{1}{l_{22}} (b_2 - y_1 l_{21})$

Row 3: $l_{31} y_1 + l_{32} y_2 + l_{33} y_3 = b_3$

$$y_3 = \frac{1}{l_{33}} \left(b_3 - l_{31} y_1 - l_{32} y_2 \right)$$

$$y_n = \frac{1}{l_{nn}} \left(b_n \underbrace{- l_{n1} y_1 - l_{n2} y_2 - l_{n,n-1} y_{n-1}} \right)$$

$$y_n = \frac{1}{l_{nn}} \left(b_n - \sum_{j=1}^{n-1} l_{nj} y_j \right) \leftarrow$$

$y = [0; 0; \cdots 0]_{n\times 1}$ (initiate a zero valued $n\times 1$ column vector).

$$y_1 = \frac{b_1}{\ell_{11}}.$$

for $I = 2$ to n

$$y_I = \frac{1}{\ell_{II}}\left(b_I - \sum_{J=1}^{I-1} \ell_{IJ}\, y_J\right)$$

end

Since y is initiated as a column matrix of zeros and L is a lower triangular matrix. The summation, $\sum_{j=1}^{N-1} \ell_{Ij}\, y_j$ can be written as. $\ell_{I,:}\, y$

y → column (vector) y.

$\ell_{I,:}$ ← I^{th} row of L matrix

4. Given a square ***upper triangular*** matrix $\boldsymbol{U}$, $U \in \mathbb{R}^{n\times n}$. We need to solve for the solve for the following system of equations.

$$\boldsymbol{U\,x = y}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

Do we need to invert the matrix $\boldsymbol{U}$? How can we solve for x?

$$u_{nn}\, x_n = y_n \iff x_n = y_n / x_n.$$

$$u_{33}\, x_3 + u_{34}\, x_4 + \cdots + u_{3n}\, x_n = y_3$$

$$u_{33}\, x_3 = y_3 - (u_{34} x_4 + \cdots + u_{3n} x_n)$$

$$x_3 = \frac{1}{u_{33}} \left(y_3 - (u_{34}\, x_4 + \cdots + u_{3n} x_n) \right)$$

$$x_3 = \frac{1}{u_{33}} \left(y_3 - \underbrace{U(3,4{:}n)}_{3^{rd}\text{ Row columns 4 to } n} * X(4:n) \right)$$

Generalise,

$$X(I) = \frac{1}{U(I,I)} \left(y(I) - \underbrace{U(I, I{+}1{:}n)}_{\substack{I^{th}\text{ Row} \\ \text{of } U}} * X(I{+}1:n) \right)$$

$X = [0 ; 0 ; \dots ; 0]_{n \times 1}$ % initiate x as a column vector. of zeros.

$x_n = y_n / U_{nn}$

for $I = n-1$ to 1 % $n > 1$

$$x_I = \frac{1}{u_{II}} \left(y_I - \sum_{j=I+1}^{n} u_{Ij} x_j \right)$$

end.

$U_{I,:} * X$

5. Given a system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ ($A \in \mathbb{R}^{n\times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^n$), where matrix $\boldsymbol{A}$ is nonsingular. How will you solve for $\boldsymbol{x}$ using LU decomposition? Show your steps.

$$Ax = b.$$

$$L\underbrace{Ux}_{y} = b.$$

① $Ly = b$

Solve this by. Forward Sub.

$Ux = y$

Solve this by backward Sub.

6. Given an over determined system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ (i.e., $A \in \mathbb{R}^{m\times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$, with $m > n$). How will you best $\boldsymbol{x}$ using method of least squares? Show your steps. What are possible algorithms for you could use?

$$A_{m\times n} \; x_{n\times 1} = b_{m\times 1}$$

Residual, $r = Ax - b$.

$$\|r\|_2 = \|Ax - b\|_2$$

We minimize the norm of residual, in lecture we saw that this happens when residual, $\boldsymbol{r}$ is orthogonal (i.e. perpendicular) to column space of A.

$$\Rightarrow \quad A^T r = 0$$

$$A^T(Ax - b) = 0$$

$$A^T A x = A^T b$$

The above set of equations are called normal equations.

$$(A_{m\times n})^T A_{m\times n} = \underbrace{A^T_{n\times m} A_{m\times n}}_{n\times n \text{ (square)}}.$$

* $A^T A x = A^T b$ can be solved using cholesky then use forward and backward substitutions.

$$\underbrace{A^T A}_{LL^T} x = A^T b.$$

, L is a lower triangular matrix.

$$\Rightarrow \quad L \underbrace{L^T x}_{y} = A^T b$$

$Ly = A^T b \longrightarrow$ solve using Forward substitution.

$L^T x = y \rightarrow$ solve using Backward substitution.