ECSE 343 Numerical Methods in Engineering

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Vector Space

Definition: A Vector Space P is a set containing vectors $\boldsymbol{v} \in P$ for which we define the two operations: **1) Scalar multiplication** and **2) Addition**.

- Scalar Multiplication: $a\boldsymbol{v} \rightarrow \boldsymbol{u}$

$$\left.\begin{array}{l} a \in \mathbb{R} \\ \boldsymbol{v} \in V \end{array}\right\} \quad \boldsymbol{u} \in V$$

- Addition: $\boldsymbol{u} + \boldsymbol{v} \rightarrow \boldsymbol{w}$

$$\left.\begin{array}{l} \boldsymbol{v} \in V \\ \boldsymbol{u} \in V \end{array}\right\} \quad \boldsymbol{w} \in V$$

Axioms (Addition)

- Axiom #1: Associativity

$$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} \qquad \forall \boldsymbol{u} \in P \quad \boldsymbol{v} \in P \quad \boldsymbol{w} \in P$$

- Axiom #2: Commutativity

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u} \qquad \forall \boldsymbol{u} \in P \quad \boldsymbol{v} \in P$$

- Axiom #3: Existence of “zero” vector $\mathbf{0} \in P$

$$\mathbf{0} + \boldsymbol{v} = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in P$$

- Axiom #4: Existence of “inverse” vector

$$\forall \boldsymbol{v} \in P \quad \exists \boldsymbol{u} = -\boldsymbol{v} \in P \text{ such that } \boldsymbol{v} + \boldsymbol{u} = \boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$$

Axioms (multiplication)

- Axiom #5:

$$1\boldsymbol{v} = \boldsymbol{v} \qquad 1 \in \mathbb{R} \quad \boldsymbol{v} \in P$$

- "Axiom #6":

$$0\boldsymbol{v} = \boldsymbol{0} \qquad 0 \in \mathbb{R} \quad \boldsymbol{v} \in P \quad \boldsymbol{0} \in P$$

Technically not an axiom (can be shown using other axioms)

- Axiom #7: Associativity of Scalar Multiplication

$$a(b\boldsymbol{v}) = (ab)\boldsymbol{v} \qquad a \in \mathbb{R} \quad b \in \mathbb{R} \quad \boldsymbol{v} \in P$$

Distributivity

Consider: $a \in \mathbb{R} \quad b \in \mathbb{R}$ and $\boldsymbol{u} \in P \quad \boldsymbol{v} \in P$

- Axiom 8

$$a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$$

- Axiom 9

$$(a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$$

Exercise

Show "Axiom 6" Using other Axioms:

$0 \in \mathbb{R} \quad \boldsymbol{v} \in P \quad \mathbf{0} \in P$

Show that: $0\boldsymbol{v} = \mathbf{0}$

Lemma 1: Uniqueness of $\mathbf{0}$ vector

If $\boldsymbol{a} + \boldsymbol{v} = \boldsymbol{v} \quad \forall \boldsymbol{v} \in P$ then $\boldsymbol{a} = \mathbf{0}$

$$\boldsymbol{a} + \boldsymbol{v} = \boldsymbol{v} \quad \forall \boldsymbol{v} \in P$$

$$\boldsymbol{a} + \mathbf{0} = \mathbf{0}$$

But: $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$ Axiom #3

$$\boldsymbol{a} = \mathbf{0}$$

Lemma 2 / ”Axiom 6”

Show that $0 \times \boldsymbol{v} = \mathbf{0}$

$\boldsymbol{v} + 0 \times \boldsymbol{v} = 1 \times \boldsymbol{v} + 0 \times \boldsymbol{v}$	Lemma 2
$= (1 + 0) \times \boldsymbol{v}$	Axiom #8
$= 1 \times \boldsymbol{v}$	
$= \boldsymbol{v}$	Axiom #5
$0 \times \boldsymbol{v} = \mathbf{0}$	Lemma 1

Norm ("Size") of a Vector

Then Norm $\|\boldsymbol{v}\| \in \mathbb{R}$ of a vector $\boldsymbol{v}$ is an indication of its size. It must be defined such that it obeys the following rules:

1. $\|\boldsymbol{v}\| > 0$
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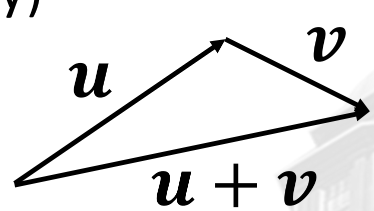
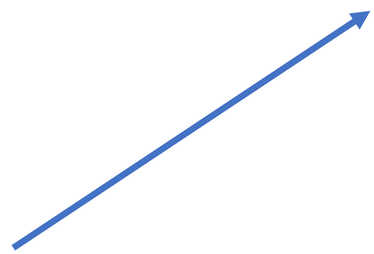
3. $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (Triangle Inequality)



Example of a Vector Space

- Magnitude
- Direction



- Difference in location between two cities.
- Magnitude direction of a force, speed, electric field…

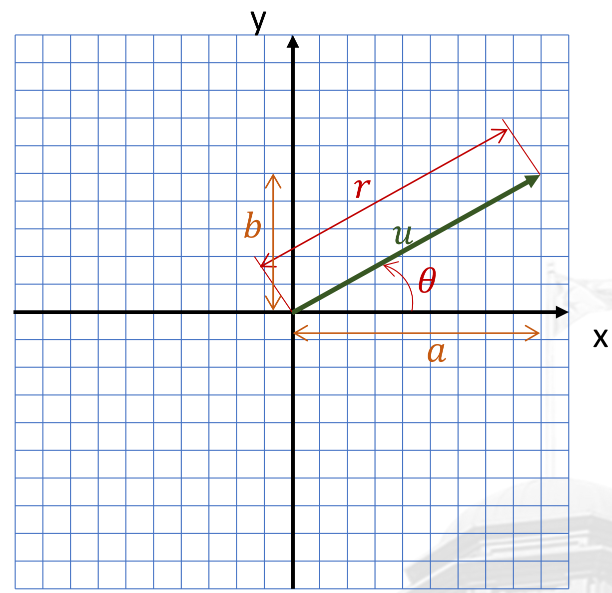
Cartesian vs Polar Coordinates 2D

- Polar Coordinates:

$$u = r\angle\theta = (r, \theta)$$

- Cartesian Coordinates:

$$u = \begin{bmatrix} a \\ b \end{bmatrix}$$



$$a = r\cos(\theta)$$
$$b = r\sin(\theta)$$
$$r^2 = a^2 + b^2$$

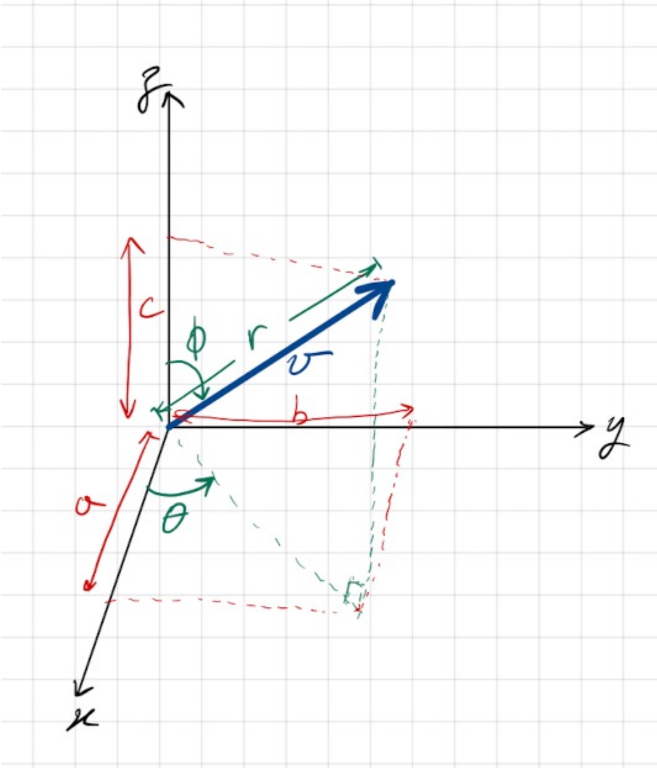
Cartesian vs Polar Coordinates 3D

- Polar Coordinates:

$$u = (r, \theta, \phi)$$

- Cartesian Coordinates:

$$u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



n Dimensional Space: $\mathbb{R}^n$

- Cartesian Coordinates:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

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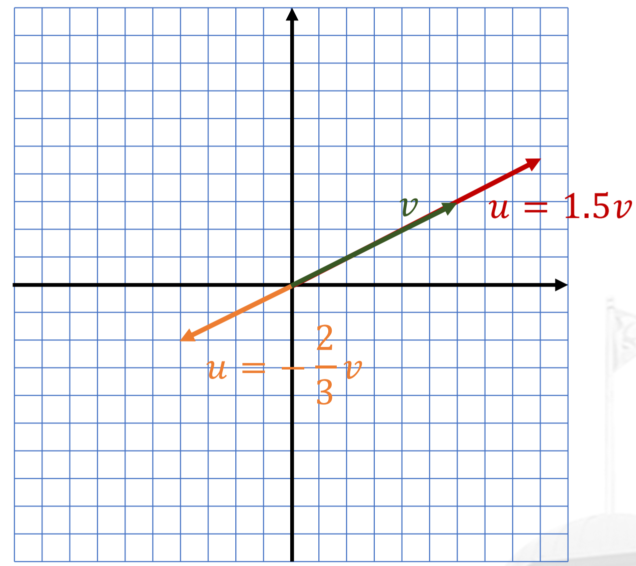
Scalar Multiplication

- $v \in \mathbb{R}^2 \quad ; \quad \alpha \in \mathbb{R}$
- $v \rightarrow u = \alpha v$
- u is in the same direction as v
- Length of u is
  $|\alpha|$ times length of v
- If α is negative u and v would have opposite orientations.

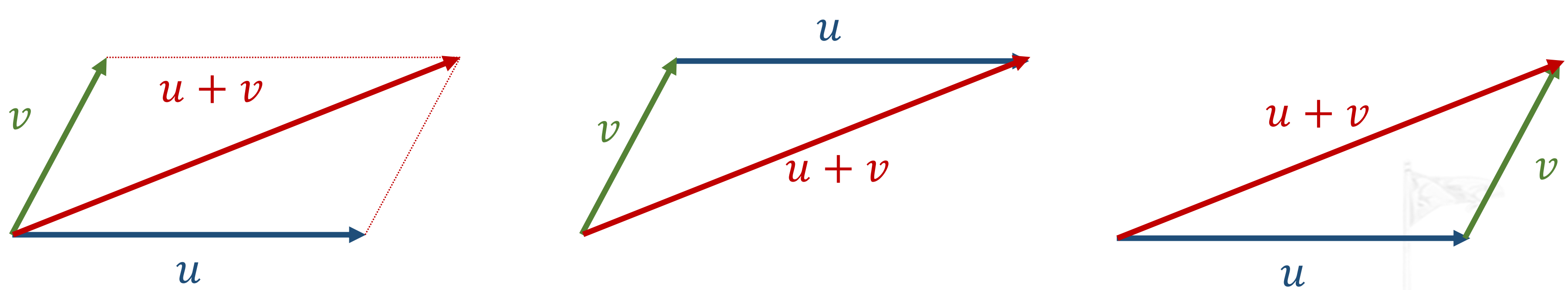


Scalar Multiplication

$$u \in \mathbb{R}^n \qquad \alpha \in \mathbb{R}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \alpha u = \begin{bmatrix} \alpha u_1 \\ \alpha u_2 \\ \vdots \\ \alpha u_n \end{bmatrix}$$

Vector Addition



Vector addition in $\mathbb{R}^n$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Axioms (Addition)

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Vector Subspaces

Consider the vector space $V = \mathbb{R}^n$. W is a subspace of V ($W \subset V$) iff:

- $u \in W \implies u \in V$
- $u \in W \implies \alpha u \in W \quad (\forall\, \alpha \in \mathbb{R})$
- $(u \in W \,\&\, v \in W) \implies u + v \in W$

Note: The above implies that the vector $\mathbf{0}$ must be in the subspace W.

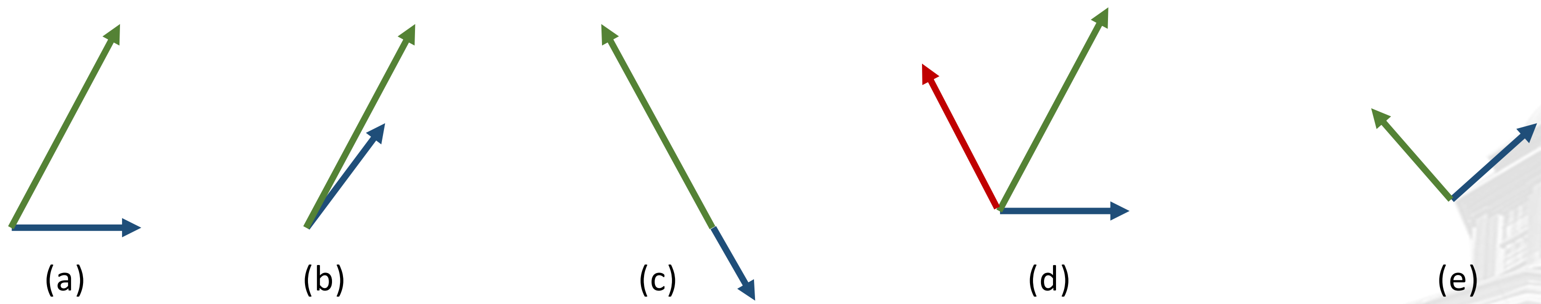
Basis of a Subspace

The vectors $u_1, u_2, \cdots, u_p$ form a basis for a vector space or a subspace P if and only if:

- All vectors in P can be written as a linear combination of $u_1, u_2, \cdots, u_p$

  $$v \in P \iff v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_p.$$

- The Vectors $u_1, u_2, \cdots, u_p$ are linearly independent.



Basis of a Subspace

$$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_p$$

$$v = \begin{bmatrix} \begin{bmatrix} u_1 \end{bmatrix} & \begin{bmatrix} u_2 \end{bmatrix} & \cdots & \begin{bmatrix} u_p \end{bmatrix} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix}$$

Example $\mathbb{R}^5$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \quad b = b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Canonical Basis for $\mathbb{R}^5$

Example $\mathbb{R}^5$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \qquad b = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{The identity matrix } U \in \mathbb{R}^{n\times n} \text{ is the canonical basis for } \mathbb{R}^n} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Example $\mathbb{R}^5$

Consider the matrix $A \in \mathbb{R}^{5\times 5}$, under what condition do A (or the columns of A) forma basis for $\mathbb{R}^5$?

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \in \mathbb{R}^5 \Rightarrow b = Ub = b_1 e_1 + b_2 e_2 + \cdots + b_5 e_5$$

Identity Matrix

Columns of Identity Matrix

$$x = A^{-1}\mathrm{b}$$

If A the columns of A are linearly independent (i.e. A is invertible):

Example $\mathbb{R}^5$

Consider the matrix $A \in \mathbb{R}^{5\times5}$, under what condition do A (or the columns of A) forma basis for $\mathbb{R}^5$?

If A the columns of A are linearly independent (i.e. A is invertible):

$$x = A^{-1}\mathrm{b} \quad \Longleftrightarrow \quad b = Ax$$

b is a linear combination of the columns of A.

Norm ("Size") of a Vector

Then Norm $\|\boldsymbol{v}\| \in \mathbb{R}$ of a vector $\boldsymbol{v}$ is an indication of its size. It must be defined such that it obeys the following rules:

1. $\|\boldsymbol{v}\| > 0$
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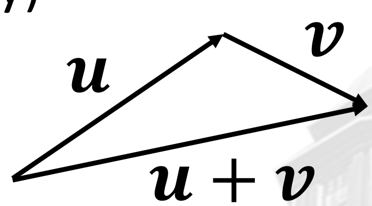


L_p Norm

Consider $v \in \mathbb{R}^n$

The p-norm or L_p norm is defined as:

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

Important Special Cases

The 1-norm or L_1 norm is:

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

The Euclidean (L_2) norm is:

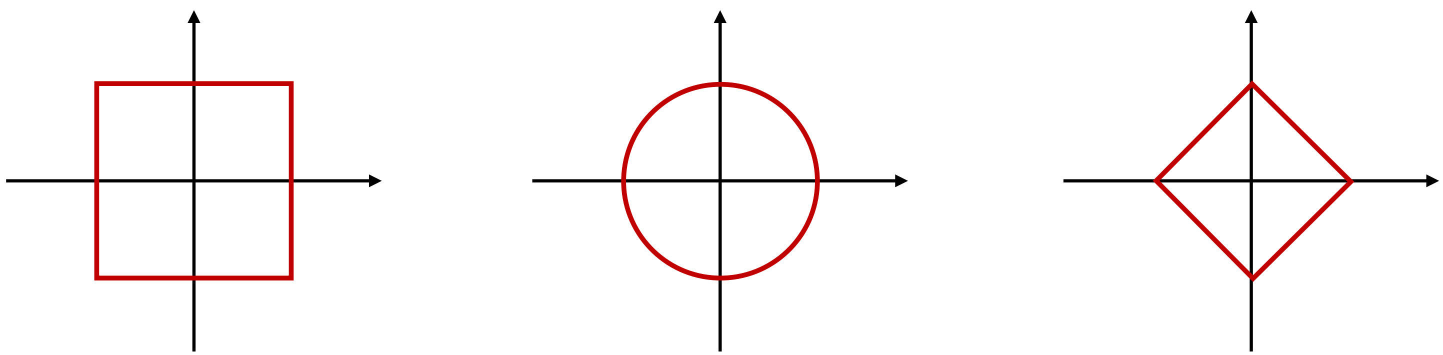
$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

Length of the vector

The Infinity (L_∞) norm is:

$$\|v\|_\infty = \max_{1 \le i \le n} |v_i|$$

Geometric Interpretation



$$\|v\|_\infty = \max|v_i|$$

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2} = 1$$

$$\|v\|_1 = \sum_{i=1}^{n} |v_i| = 1$$

Norm of a Matrix

Then Norm $\|\boldsymbol{A}\| \in \mathbb{R}$ of a vector $\boldsymbol{A}$ is an indication of its size. It must be defined such that it obeys the following rules:

1. $\|\boldsymbol{A}\| > 0$
2. $\|a\boldsymbol{A}\| = |a|\|\boldsymbol{A}\|$
3. $\|\boldsymbol{A} + \boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$
4. $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$

Frobenius Norm

$$\|A\|_F = \sqrt{\sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} a_{i,j}^2}$$

Induced Norm

Induced norm $||\boldsymbol{A}||$ of a matrix $\boldsymbol{A}$ based on a vector norm $||\cdot||$

$$||\boldsymbol{A}|| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| \neq 0}} \frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||}$$

$$||\boldsymbol{A}|| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| = 1}} ||\boldsymbol{A}\boldsymbol{x}||$$

$$||\boldsymbol{A}\boldsymbol{x}|| < ||\boldsymbol{A}||\,||\boldsymbol{x}|| \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$