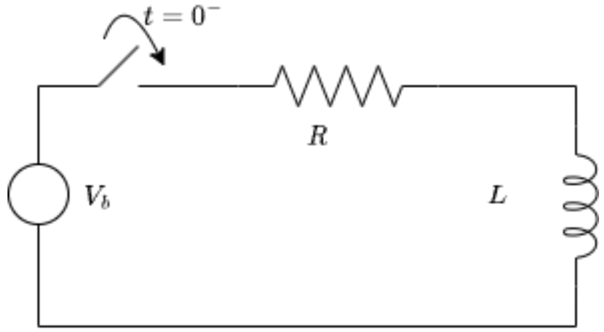
Mock Quiz # 8

1. You are given the following circuit,



The KVL equation for the above circuit can be written as,

$$V_b - I(t)\,R - L\,\frac{\mathrm{d}}{\mathrm{d}t} I(t)$$

where, R and L are the resistance and the inductance, respectively. $\boldsymbol{V_b}$ is the DC source voltage and $\boldsymbol{I(t)}$ is the loop current at time t. Suppose the initial value of the current at time $\boldsymbol{t = t_0}$ is $\boldsymbol{I_0}$.

a. Use Forward- Euler method to find the expression for current I(t) for time values $\boldsymbol{t = t_1}$ and $\boldsymbol{t = t_2}$.

See the MATLAB file

b. Use Backward- Euler method to find the expression for current I(t) for time values $\boldsymbol{t = t_1}$ and $\boldsymbol{t = t_2}$.

See the MATLAB file

2. In this question, your task is to find the solution of the following differential equation,

$$\frac{d\,\boldsymbol{X}(\boldsymbol{t})}{\boldsymbol{dt}} = \boldsymbol{A}\,\boldsymbol{X}(\boldsymbol{t})$$

Where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{X}(\boldsymbol{t}) \in \mathbb{R}^{n}$. You are also provided with the initial condition.

$$\boldsymbol{X}(\boldsymbol{t}=\boldsymbol{0}) = \boldsymbol{X_0}$$

a. Use Forward- Euler method to find the expression **X**(t) for time values $\boldsymbol{t} = \boldsymbol{t_1}$ and $\boldsymbol{t} = \boldsymbol{t_2}$.

We start by writing the given differential eqn. at $t = t_n$.

$$\frac{dX(t_n)}{dt} = A\,X(t_n)$$

We can use X_n to denote $X(t_n)$. Write above equation using above notation we get,

$$\frac{dX_n}{dt} = A\,X_n$$

We use forward Euler approximation for $\frac{dX_n}{dt}$

$$\frac{dX_n}{dt} \simeq \frac{X_{n+1} - X_n}{\Delta t}$$

Δt ← time step.

Substitute this in differential equation we get the DIFFERENCE equation.

$$\Rightarrow \frac{X_{n+1} - X_n}{\Delta t} = A\,X_n$$

We are given the initial condition i.e value of $X(t)$ at time $t = t_0$. Therefore, we can rearrange our DIFFERENCE equation to solve for X_{n+1}.

$$\Rightarrow \quad \frac{X_{n+1}}{\Delta t} = A X_n + \frac{X_n}{\Delta t}$$

$$\Rightarrow \boxed{X_{n+1} = \Delta t \left(A + \mathbb{I} \right) X_n}$$

identity matrix of size of A.

The above equation can be used to compute X_{n+1} (i.e., the solution at time $t = t_{n+1}$) using the solution at previous time step, (i.e. X_n).

We are given $\mathbf{X_0}$, solution at initial value at $t = t_0$.

The solution at $t = t_1$ can be computed as,

$$X_1 = \Delta t \left(A + \mathbb{I} \right) X_0$$

we were given this.

Now we can compute sol. at $t = t_2$ as,

$$X_2 = \Delta t \left(A + \mathbb{I} \right) X_1$$

we already computed this before.

b. Use Backward- Euler method to find the expression for **X**(t) for time values $\boldsymbol{t} = \boldsymbol{t_1}$ and $\boldsymbol{t} = \boldsymbol{t_2}$.

The given differential equation at time $t=t_n$ is

$$\frac{dX_n}{dt} = A X_n$$

Use Backward Euler approximation for $\frac{dX_n}{dt}$,

$$\frac{dX_n}{dt} \simeq \frac{X_n - X_{n-1}}{\Delta t}$$

Substitute this in the given differential equation

$$\frac{X_n - X_{n-1}}{\Delta t} = A X_n$$

DIFFERENCE EQUATION.

$$\Rightarrow \frac{X_n}{\Delta t} - A X_n = \frac{X_{n-1}}{\Delta t}$$

$$\Rightarrow \left(\frac{\mathbb{I}}{\Delta t} - A\right) X_n = \frac{X_{n-1}}{\Delta t}$$

identity matrix of same size as A.

$$\underbrace{(\mathbb{I} - \Delta t\, A)}_{M} X_n = X_{n-1}$$

M is a square matrix. To solve for X_n we decompose M into lower and upper triangular matrices as,

$$L\, U\, X_n = X_{n-1}$$

After this we can use the backward and forward substitution methods. to solve for X_n.

__Note__: In __most__ applications you will need to use row pivoting in LU decomposition to avoid numerical errors. However, pivoting is omitted in the above explanation.

At $t = t_1$,

$$(\mathbb{I} - \Delta t\, A)\, X_1 = X_0$$ ← we are given this.

$$L\, U\, X_1 = X_0$$

After forward & backward substitution we can solve for X_1,

At $t = t_2$

$$(\mathbb{I} - \Delta A)\, X_2 = X_1$$

$$L U\, X_2 = X_1$$

solve for X_2.

3. Suppose you are given the following second order ODE,

$$\ddot{y} + 5\dot{y} + 6y = 10sin(t)$$

You are also provided with initial conditions, $\boldsymbol{y(t = t_0) = 0}$ and $\boldsymbol{\dot{y}(t = t_0) = 5}$. Use Backward- Euler method to find the expression for **X**(t) for time values $\boldsymbol{t = t_1}$ and $\boldsymbol{t = t_2}$.

See the MATLAB file.