ECSE 343 Numerical Methods in Engineering

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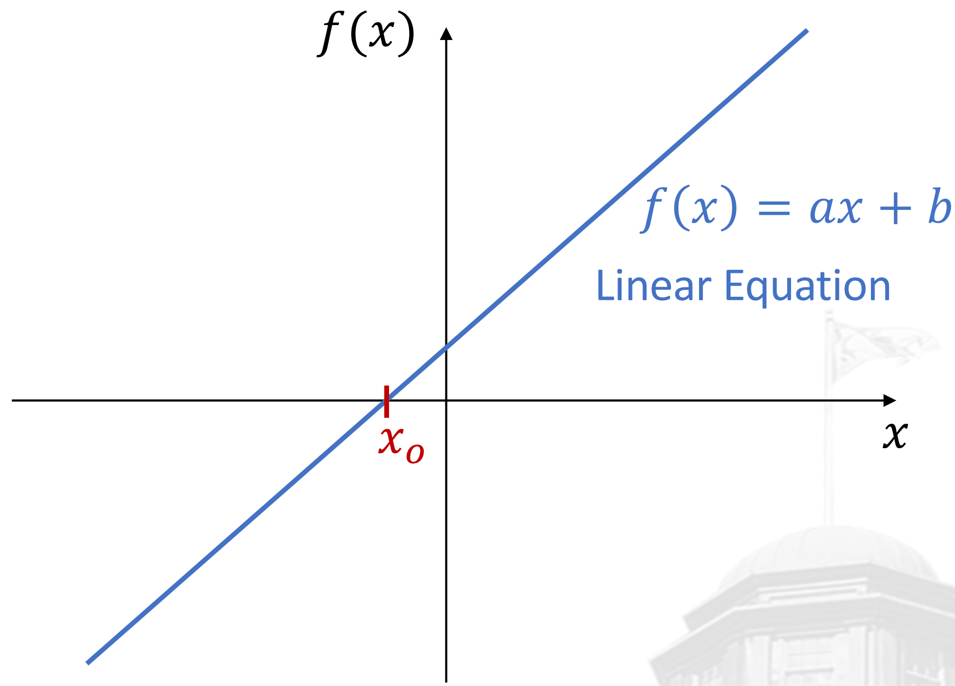
Roots of Linear Equations

Find x such that $f(x) = 0$

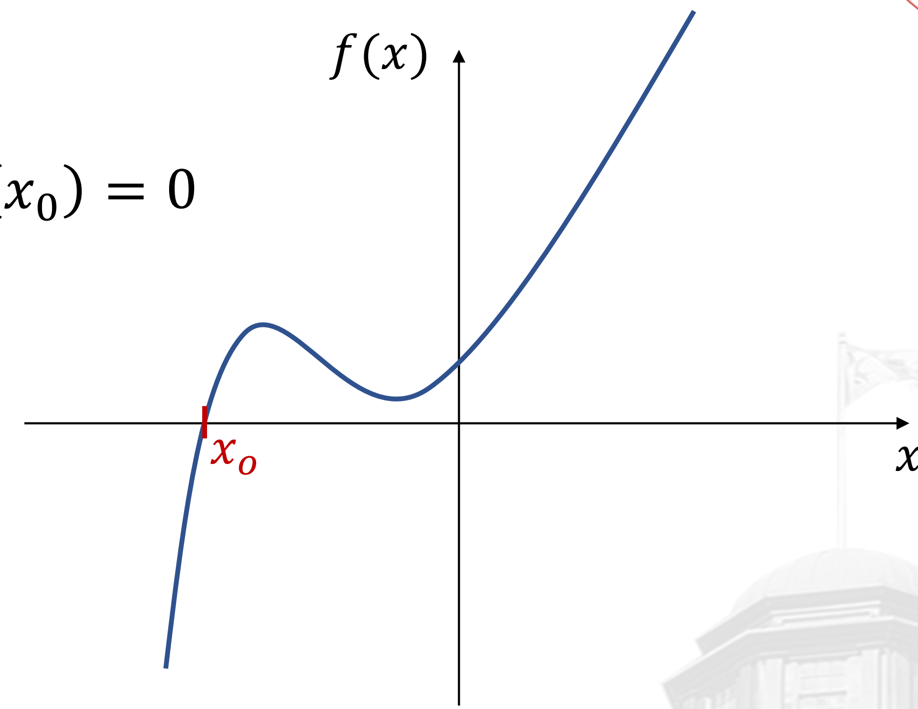
One root $x = x_o \Rightarrow f(x_0) = 0$



Roots of Nonlinear Equations

Find x_o such that $f(x_o) = 0$

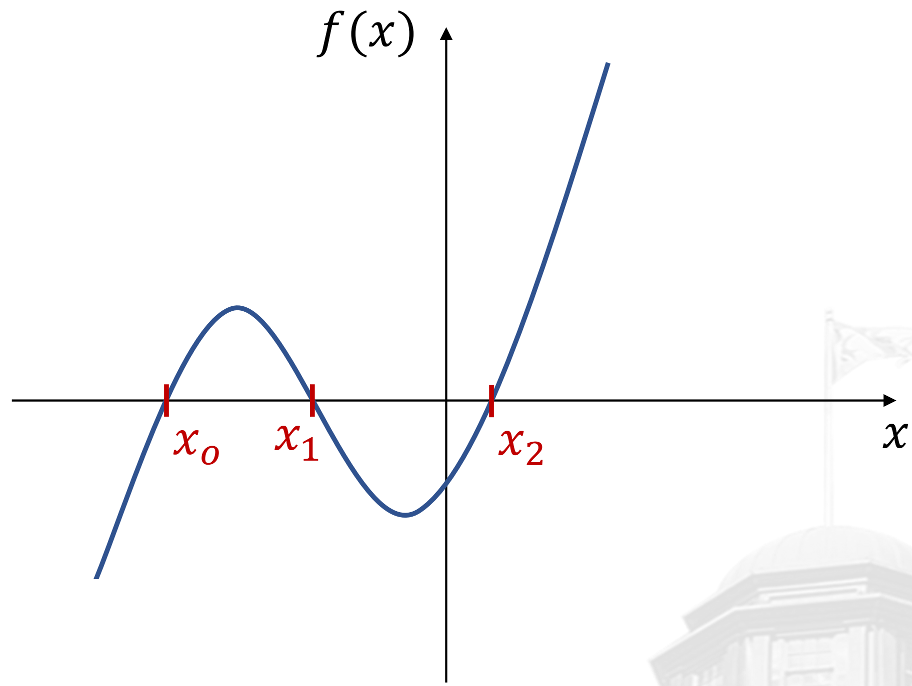
Possibly one root $x = x_o \Rightarrow f(x_0) = 0$



Roots of Nonlinear Equations

Find x_o such that $f(x_o) = 0$

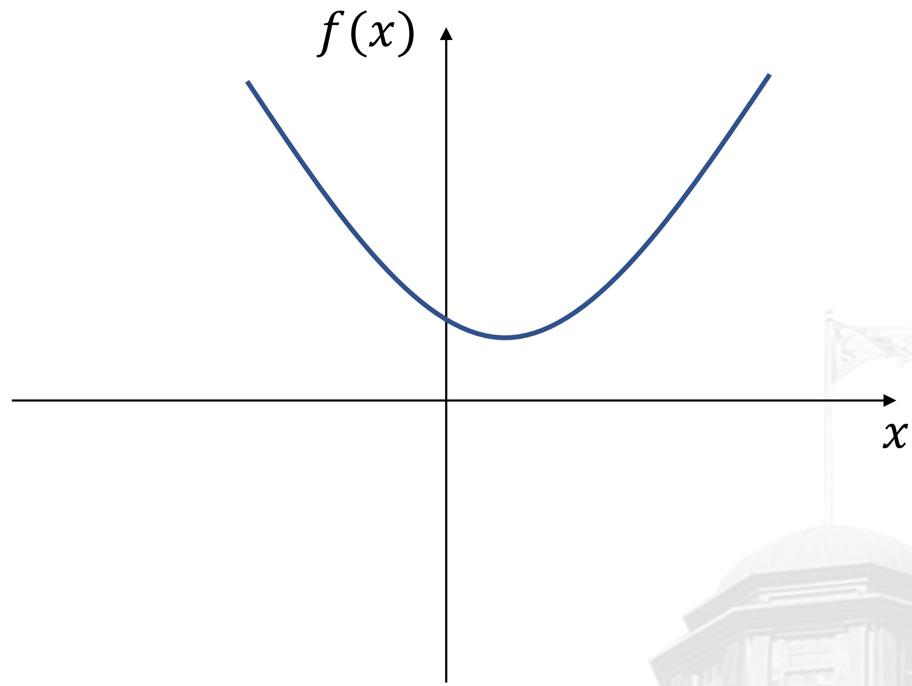
Three roots: x_0, x_1, and x_2



Roots of Nonlinear Equations

Find x_o such that $f(x_o) = 0$

No roots.

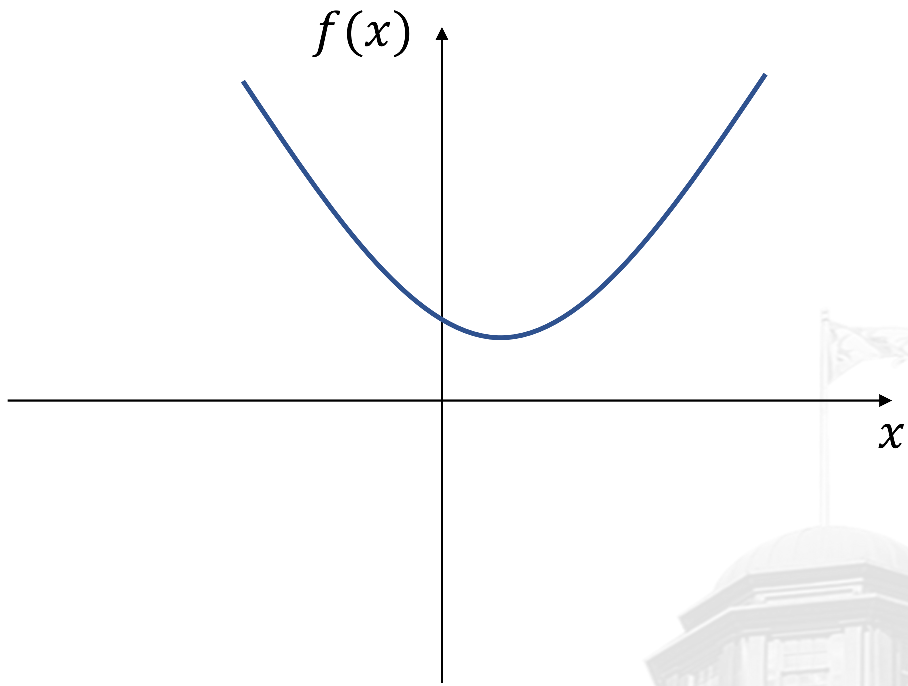


Continuous Function

$$x \to y$$

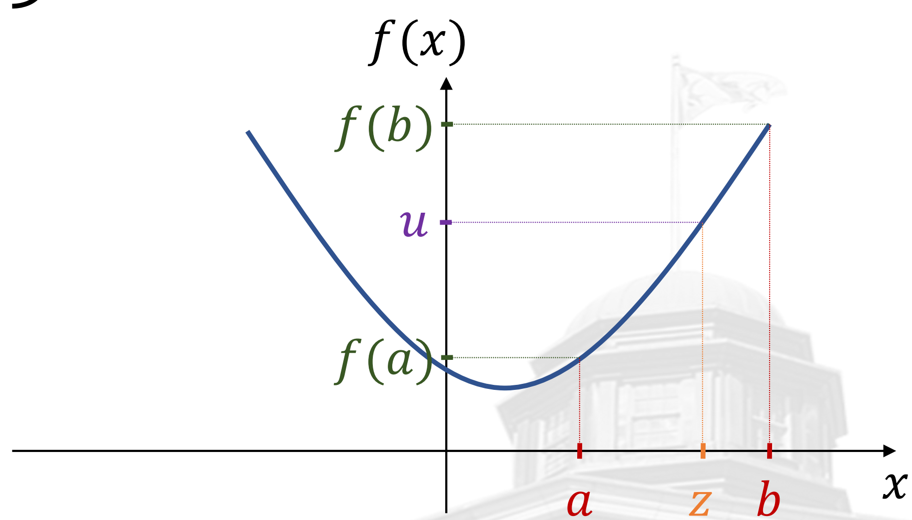
$$\Downarrow$$

$$f(x) \to f(y)$$



Intermediate Value Theorem

$$\left.\begin{array}{l} f(x) \text{ is continuous over interval } [a,b] \\ f(a) < u < f(b) \text{ or } f(b) < u < f(a) \end{array}\right\} \Rightarrow \begin{array}{l} \exists z \in (a,b) \text{ such that} \\ f(z) = u \end{array}$$

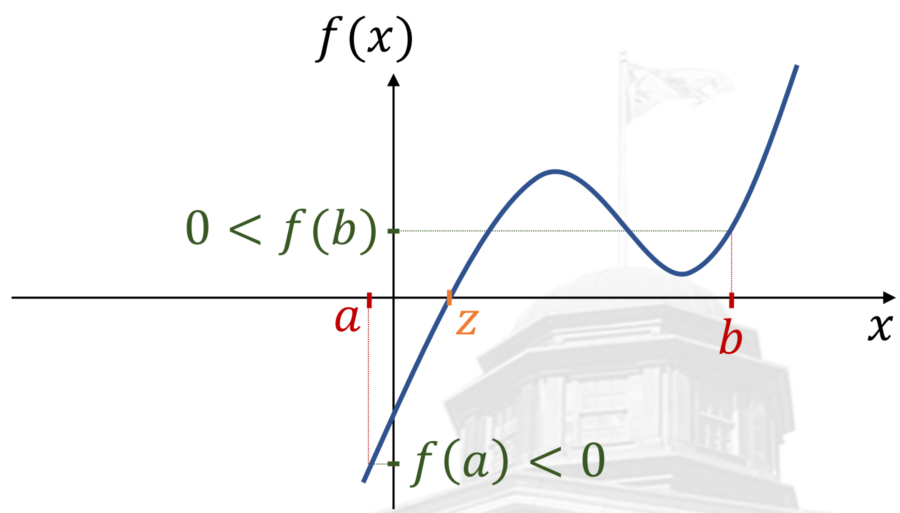


Application to Root finding

$f(x)$ is continuous over interval $[a, b]$
$f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$

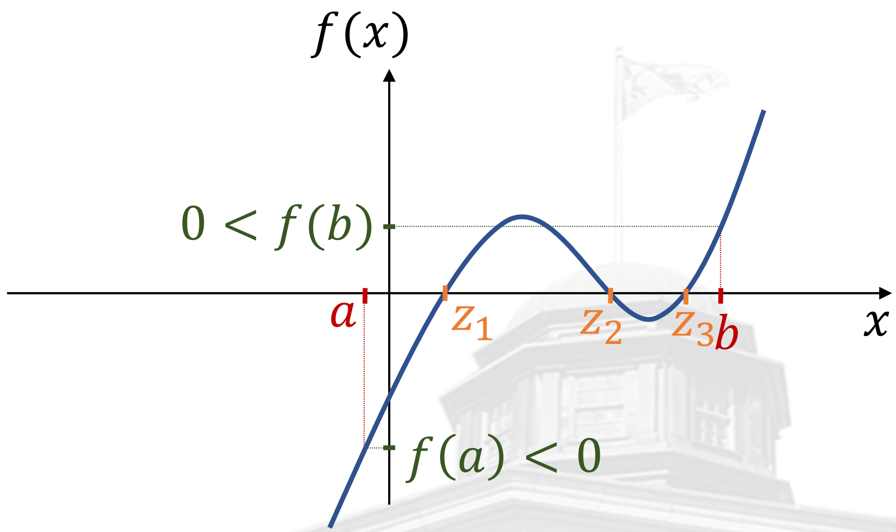
$f(a)$ and $f(b)$ have opposite signs

$\exists z \in (a, b)$ such that
$f(z) = 0$



Application to Root finding

$$\underbrace{\left.\begin{array}{l} f(x) \text{ is continuous over interval } [a,b] \\ f(a) < 0 < f(b) \text{ or } f(b) < 0 < f(a) \end{array}\right\}}_{f(a) \text{ and } f(b) \text{ have opposite signs}} \Rightarrow \begin{array}{l} \exists z \in (a,b) \text{ such that} \\ f(z) = 0 \end{array}$$

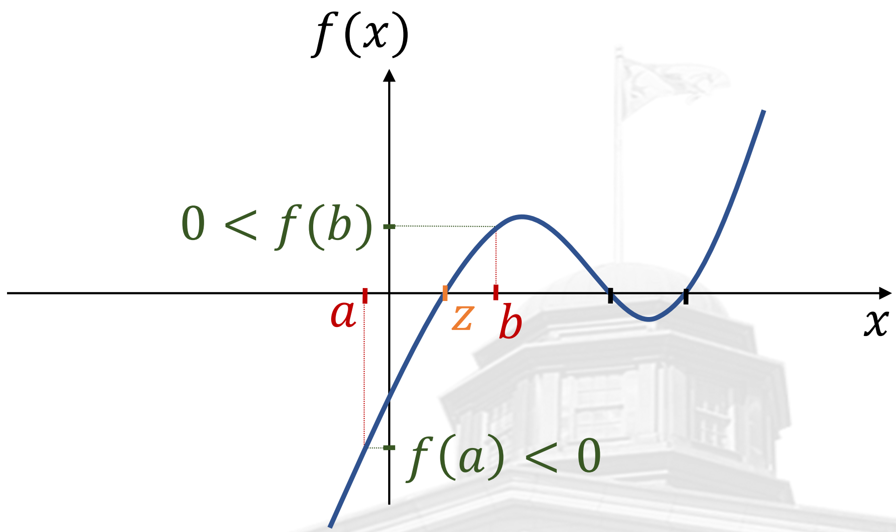


Application to Root finding

$f(x)$ is continuous over interval $[a, b]$
$f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$

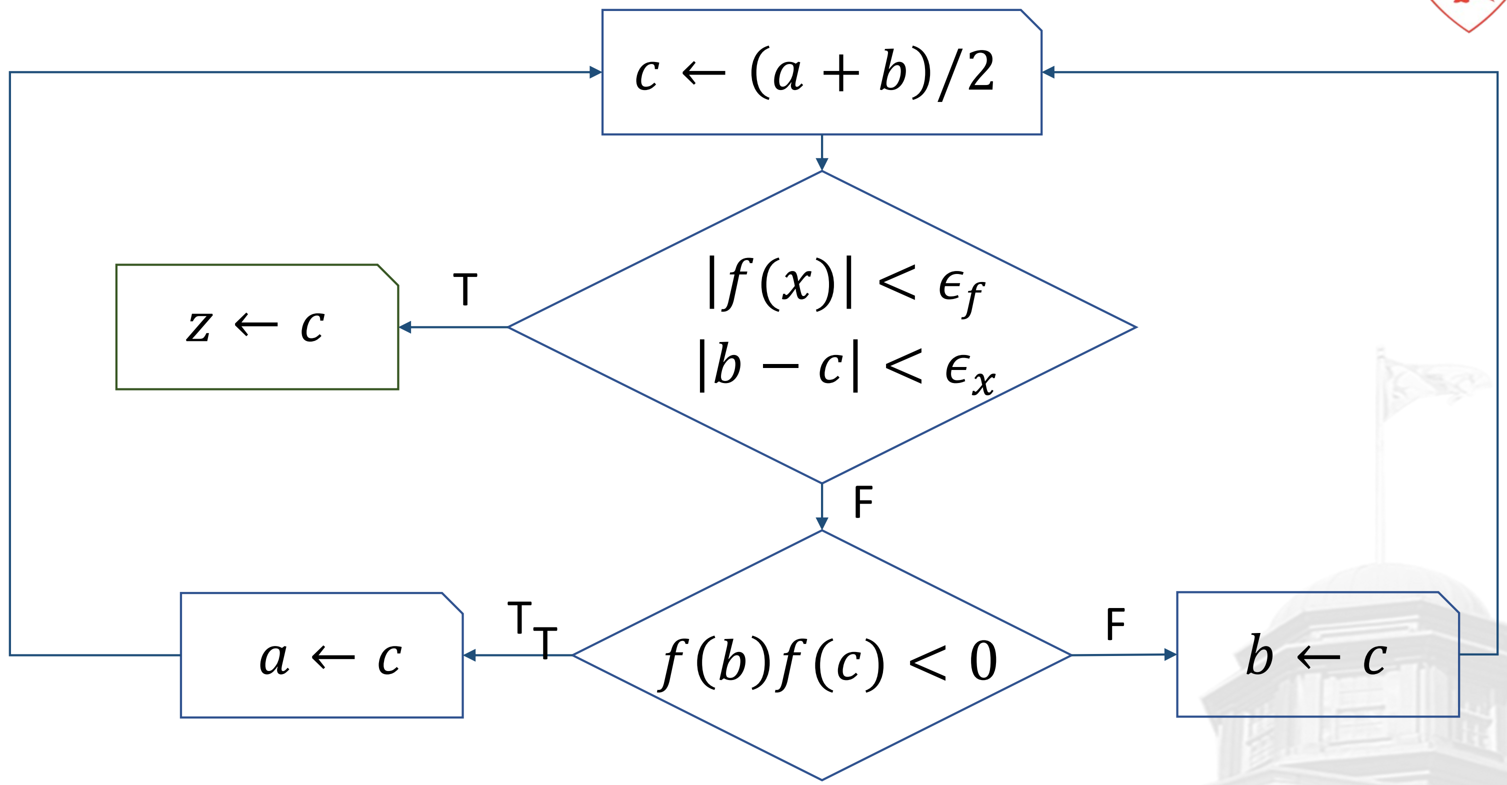
$\underbrace{\qquad\qquad}_{}$ $f(a)$ and $f(b)$ have opposite signs

$\Rightarrow$ $\exists z \in (a, b)$ such that $f(z) = 0$

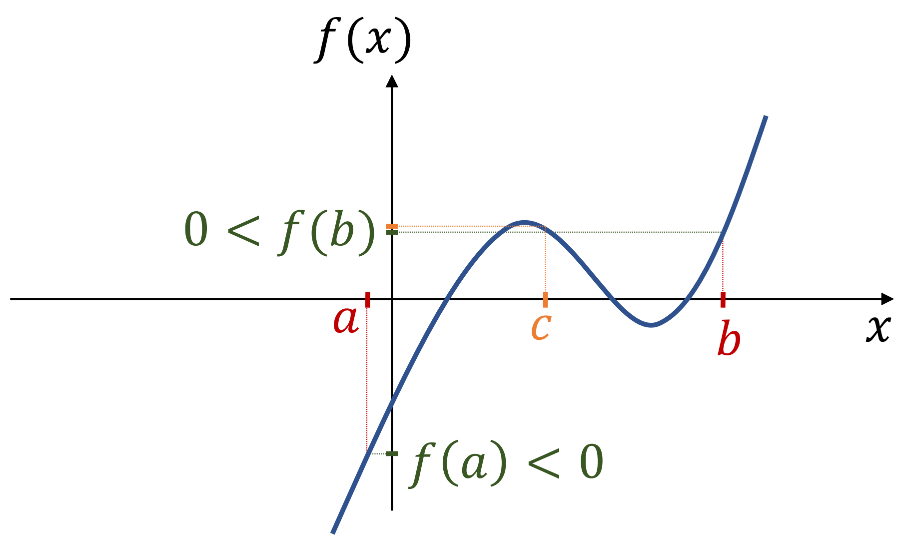


As a starting point, we need a, and b ($b > a$)such that: $f(a)$ and $f(b)$ have opposite signs.

Bisection Method



Bisection Method Example

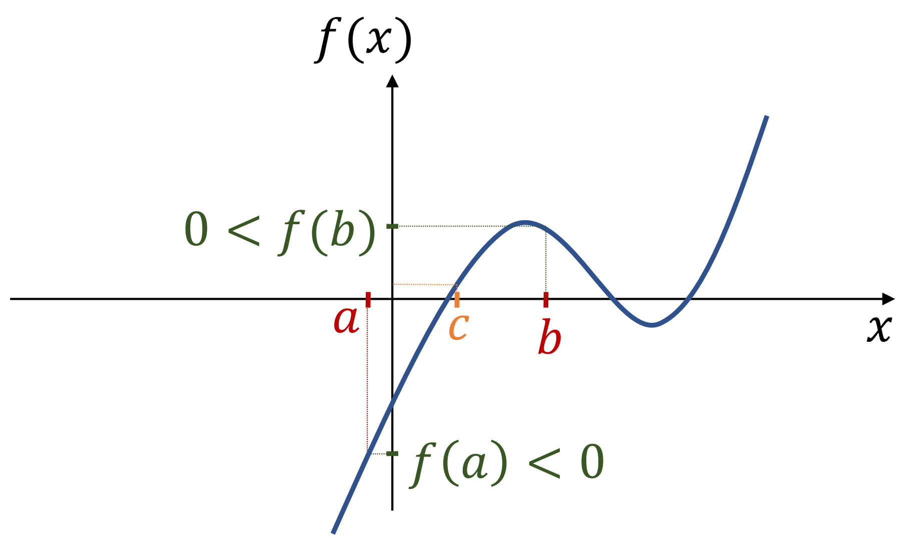


$$c \leftarrow (a+b)/2$$

$$\cancel{f(b)f(c) < 0}$$

$$b \leftarrow c$$

Bisection Method Example



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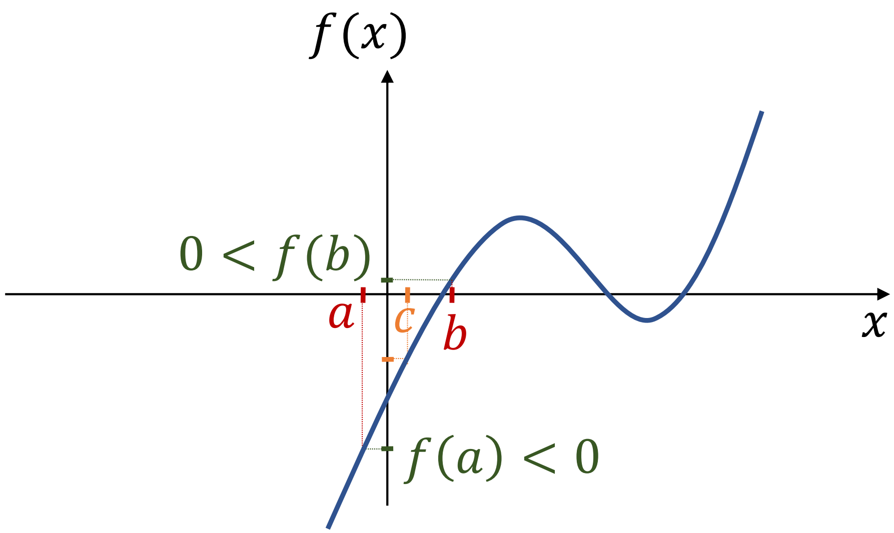
Bisection Method Example

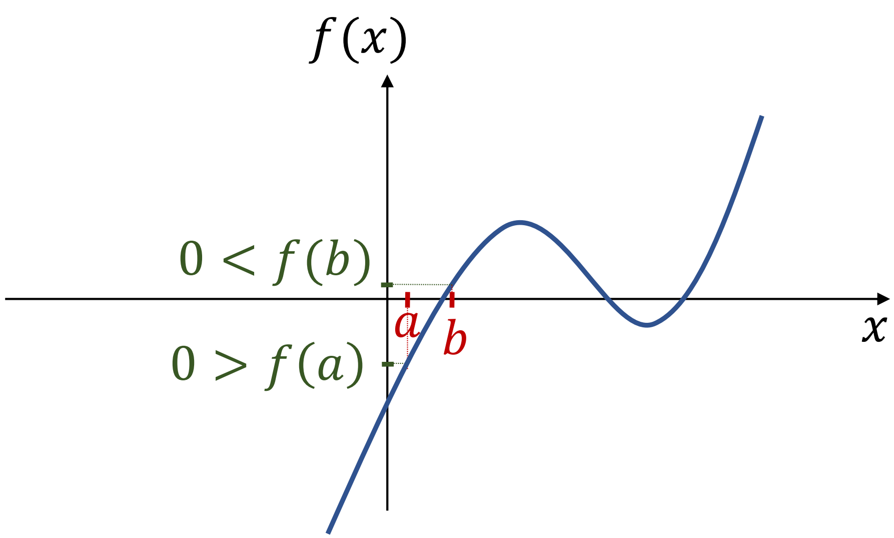


$c \leftarrow (a+b)/2$

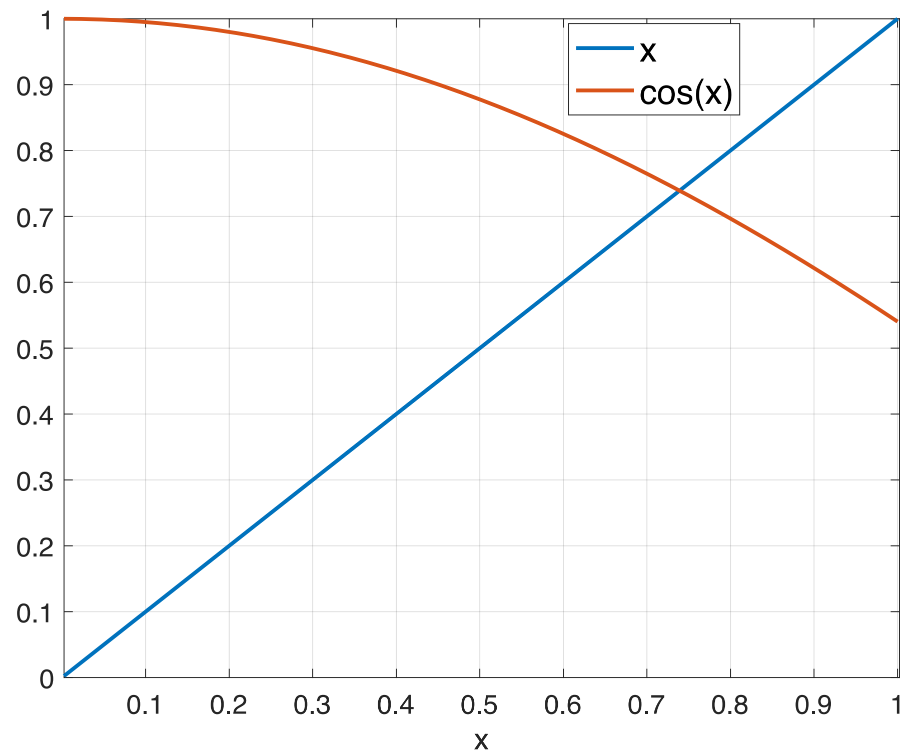
$f(b)f(c) < 0$ ✓

$a \leftarrow c$

Bisection Method Example



Bisection Example $f(x) = x - \cos(x)$



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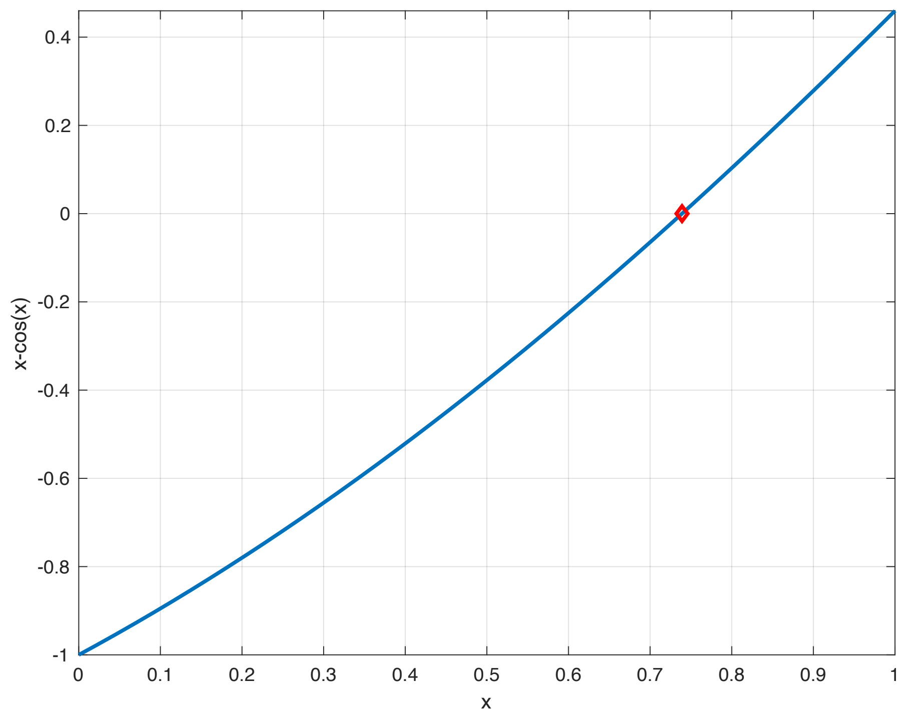

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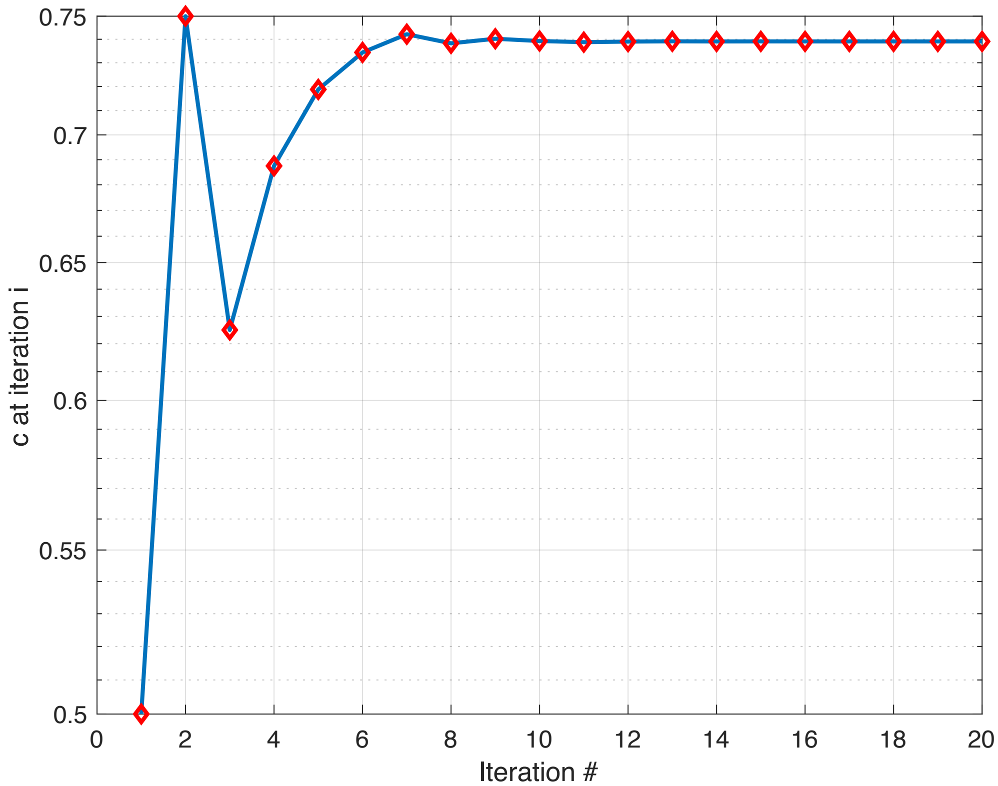
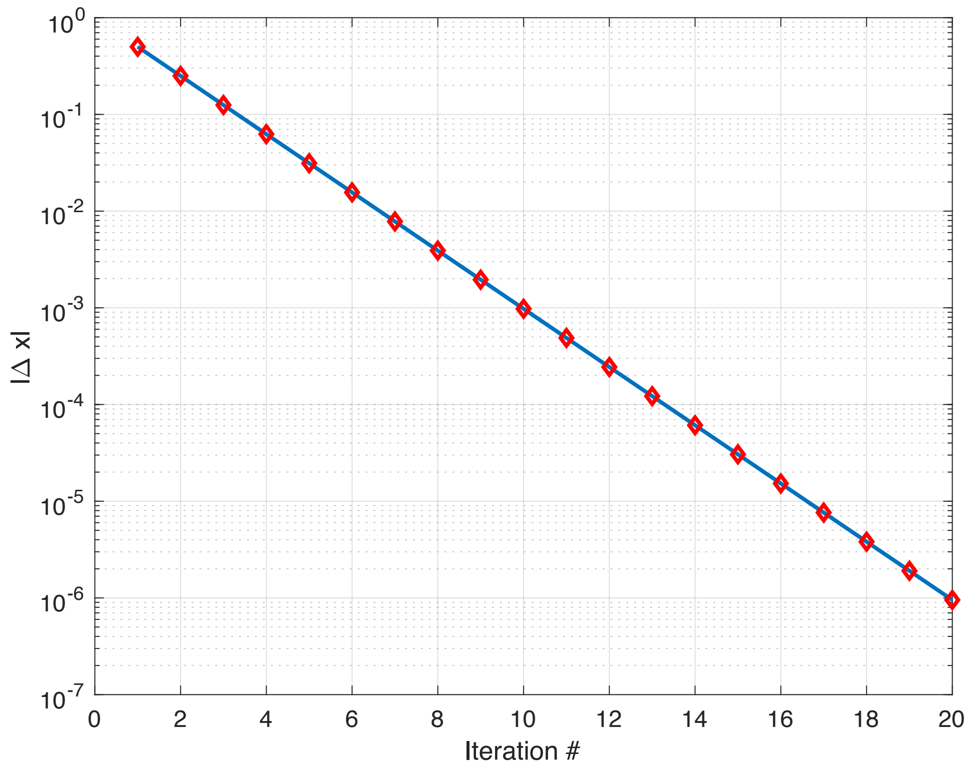


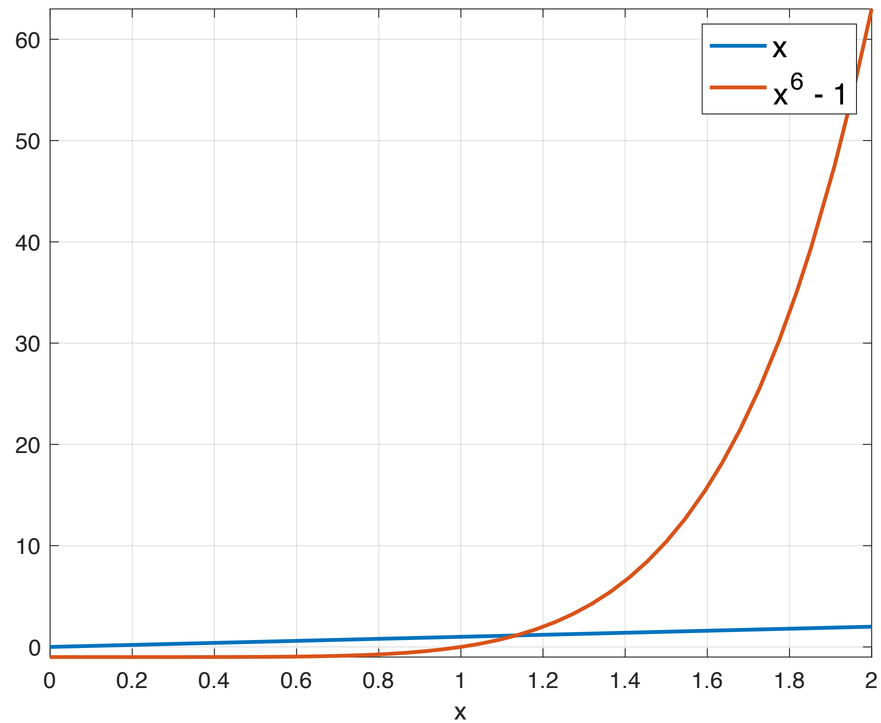
Solution converges to:

$$z = 0.7391$$

Bisection Example $f(x) = x - \cos(x)$



Bisection Example $f(x) = x^6 - x - 1$



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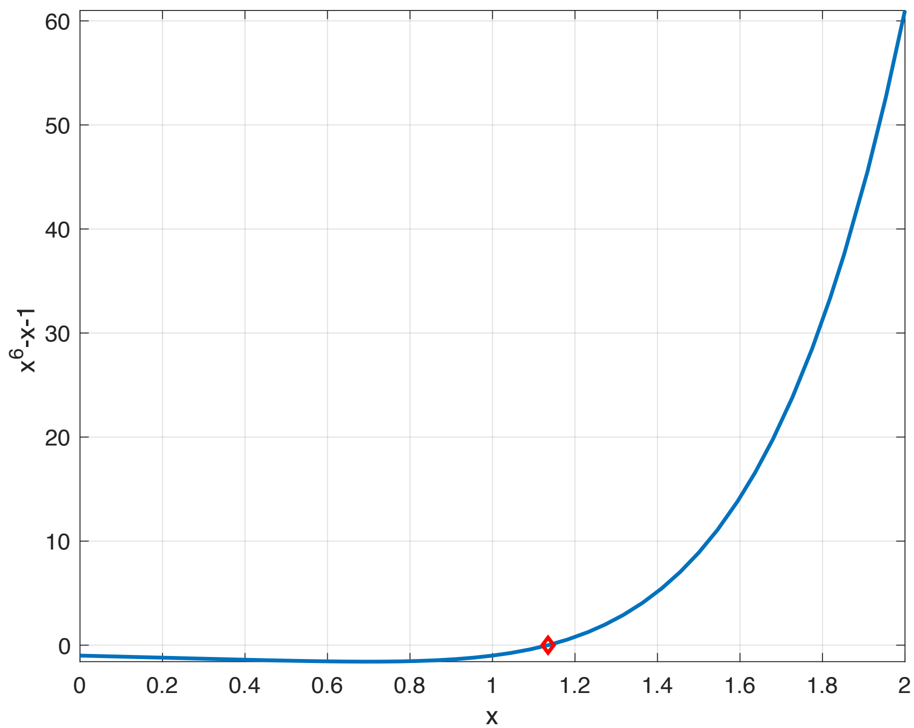

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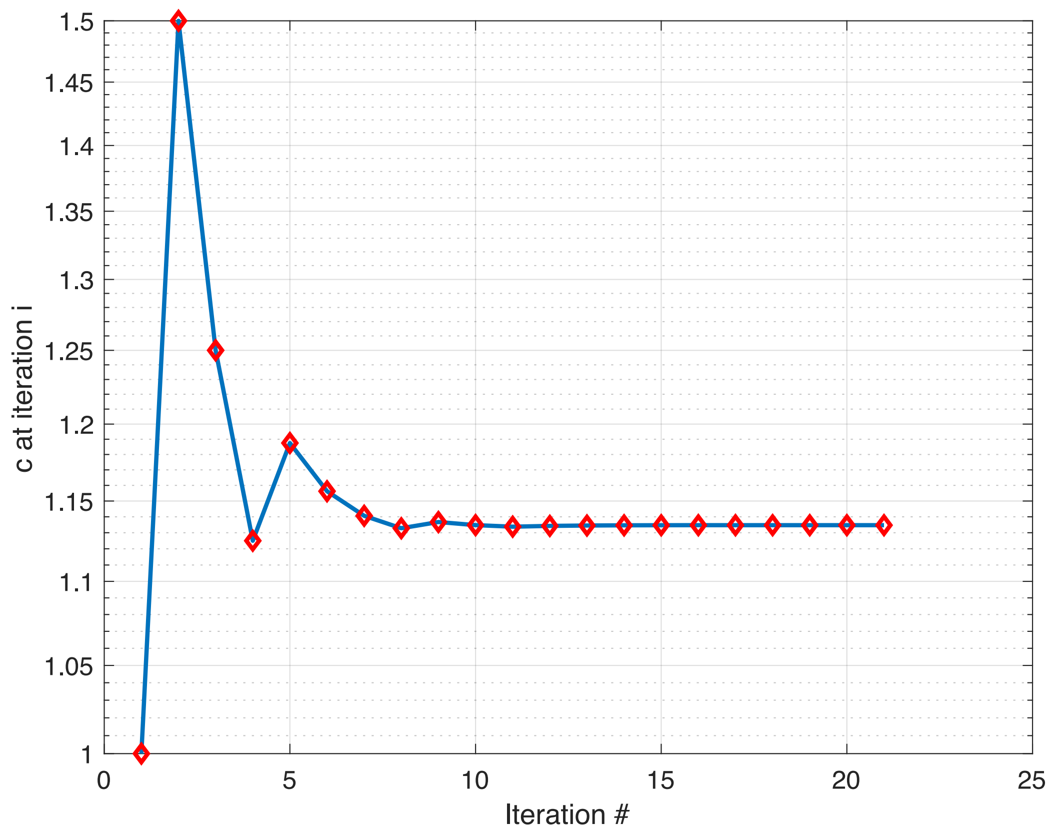
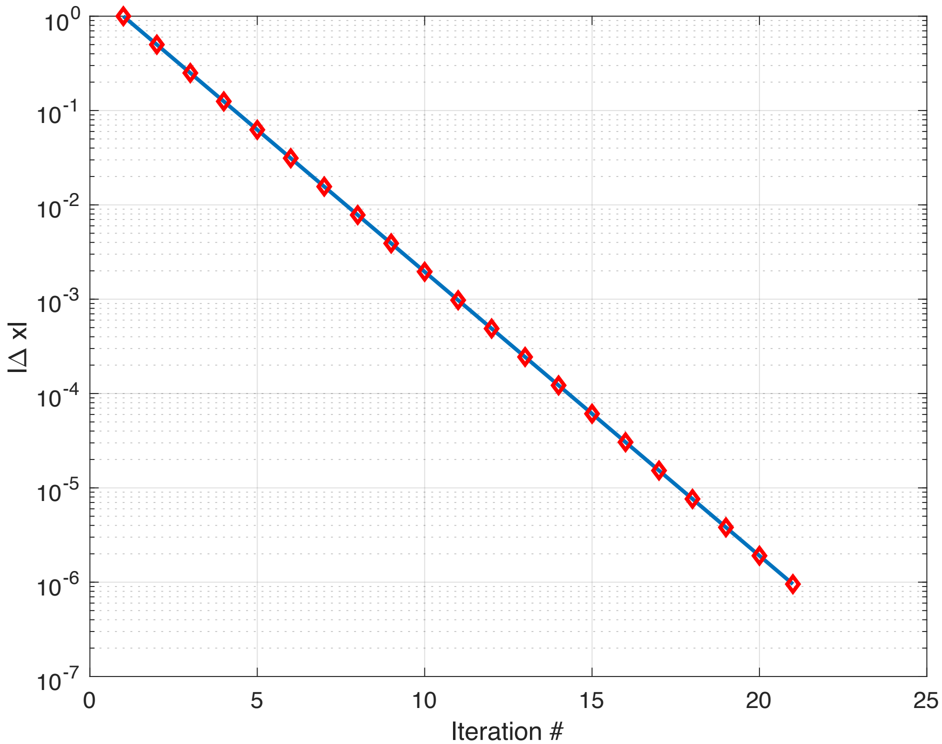


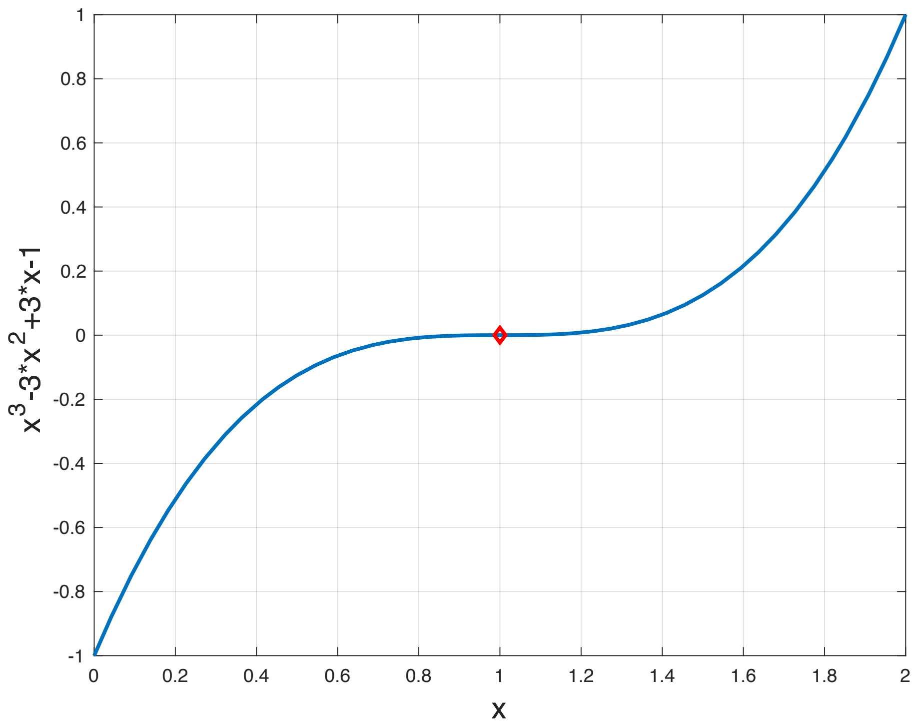
Solution converges to:

$$z = 0.1.1347$$

Bisection Example $f(x) = x^6 - x - 1$



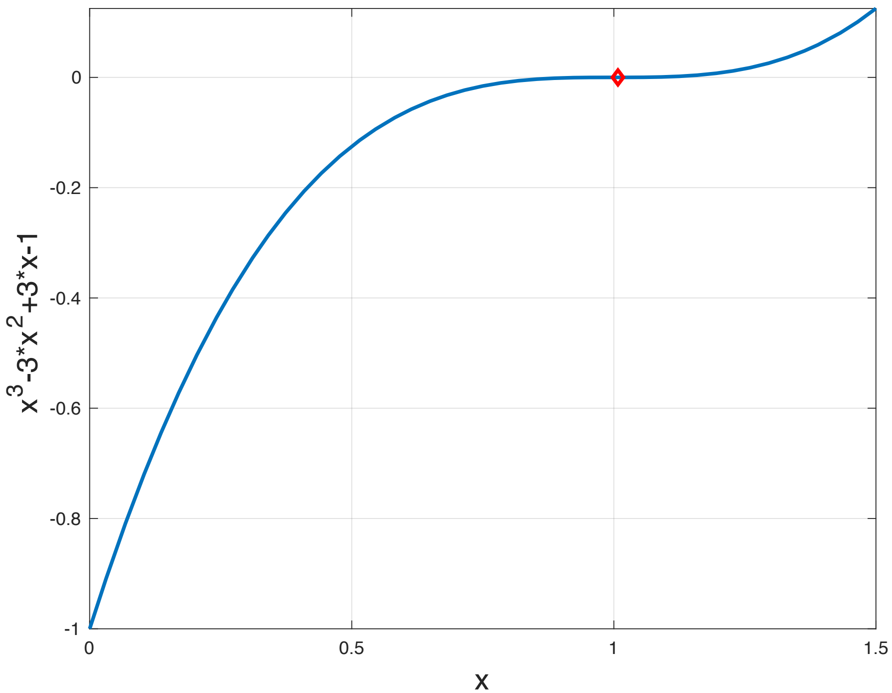
Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



Solution converges to:

$$z = 1$$

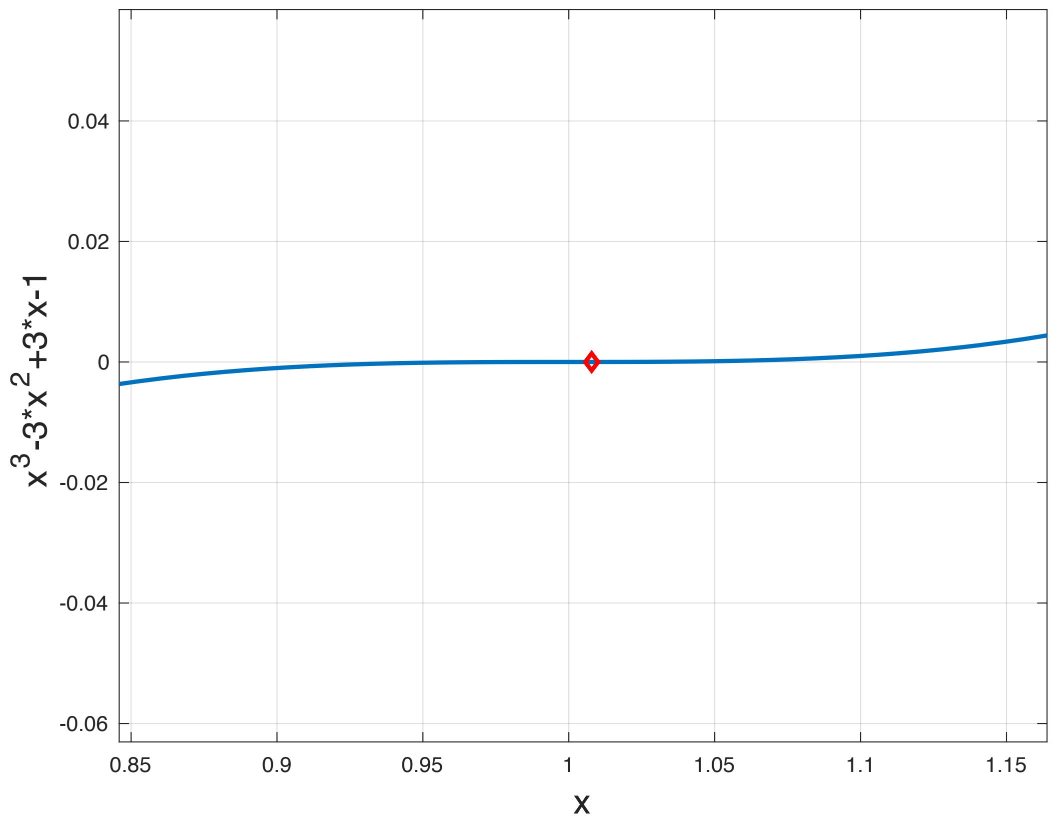
Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



Solution converges to:

$$z = 1.0078$$

Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$

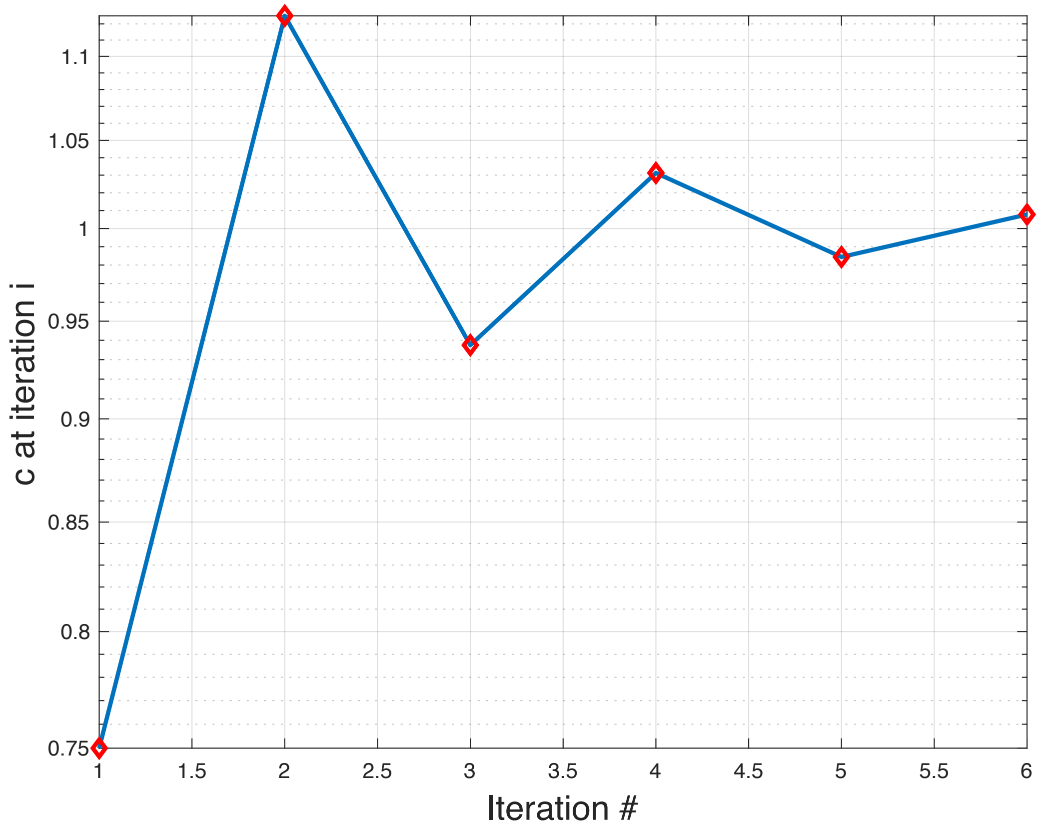


Solution converges to:

$$z = 1.0078$$

$$f(z) = 4.7684 \times 10^{-7}$$

Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



Solution converges to:

$$z = 1.0078$$

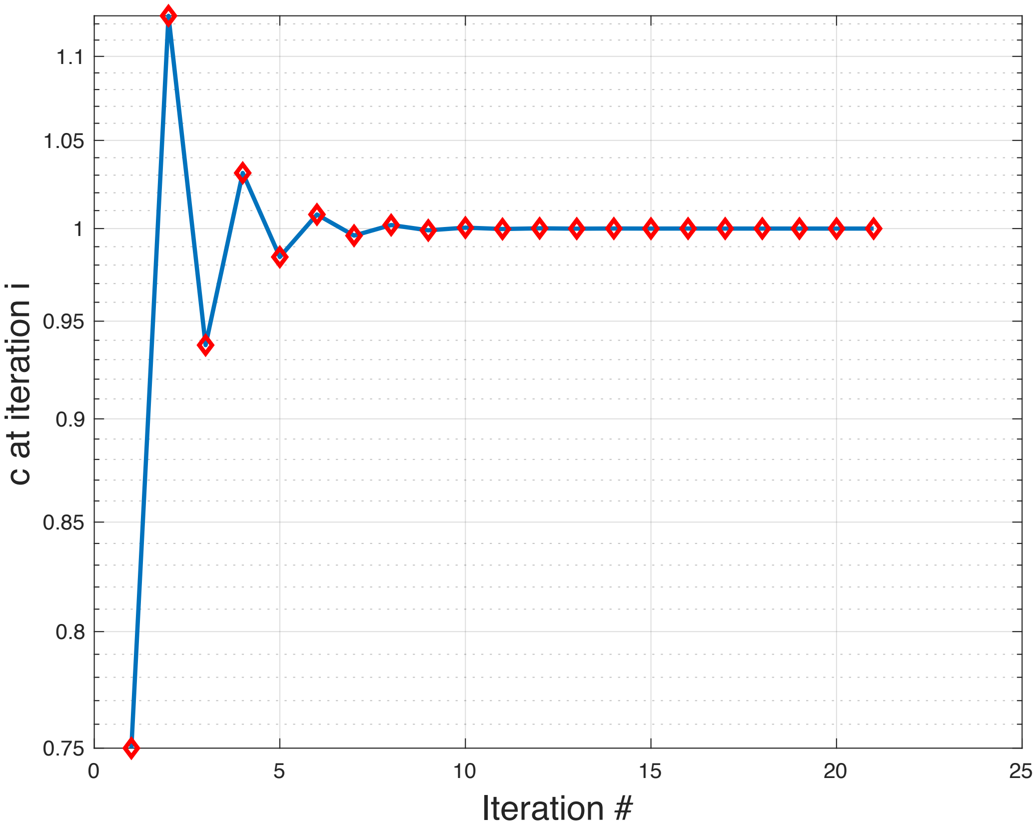
$$f(z) = 4.7684\times10^{-7}$$

Stop if:

$$\epsilon_f < 10^{-6}$$

OR $\epsilon_x < 10^{-6}$

Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



Solution converges to:

$$z = 0.999985$$

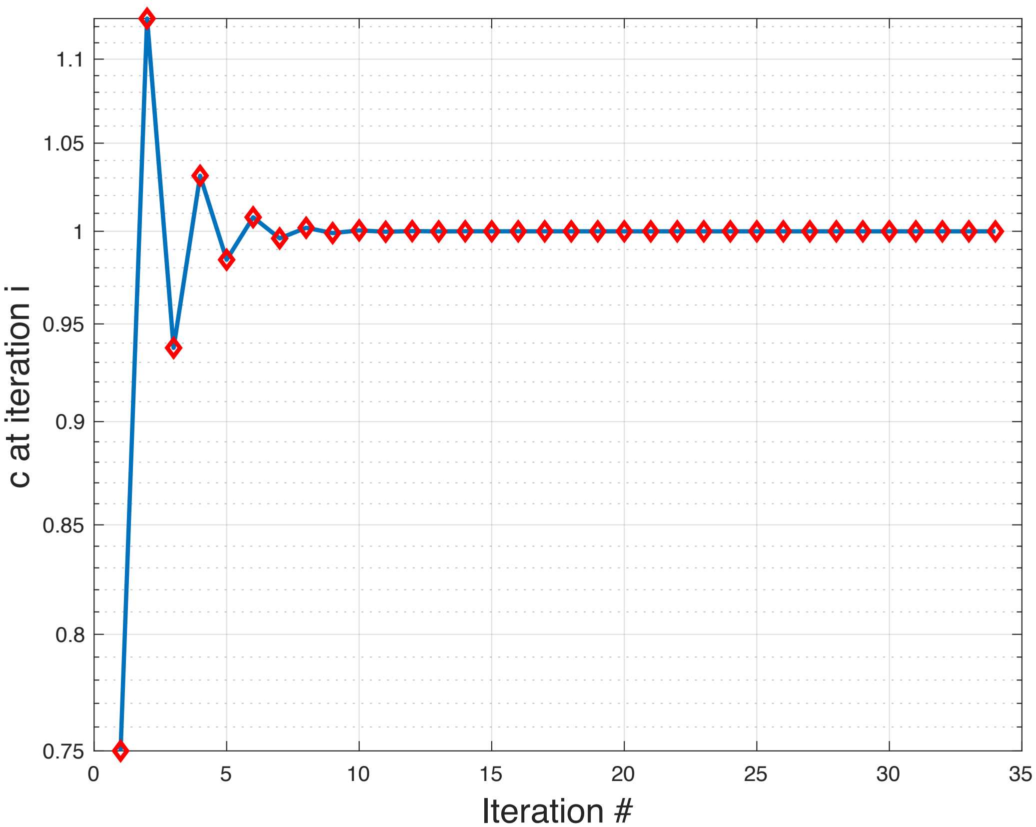
$$f(z) = 3.1 \times 10^{-15}$$

Stop if:

$$\epsilon_f < 10^{-6}$$

AND $\epsilon_x < 10^{-6}$

Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



Solution converges to:

$$z = 0.999985$$

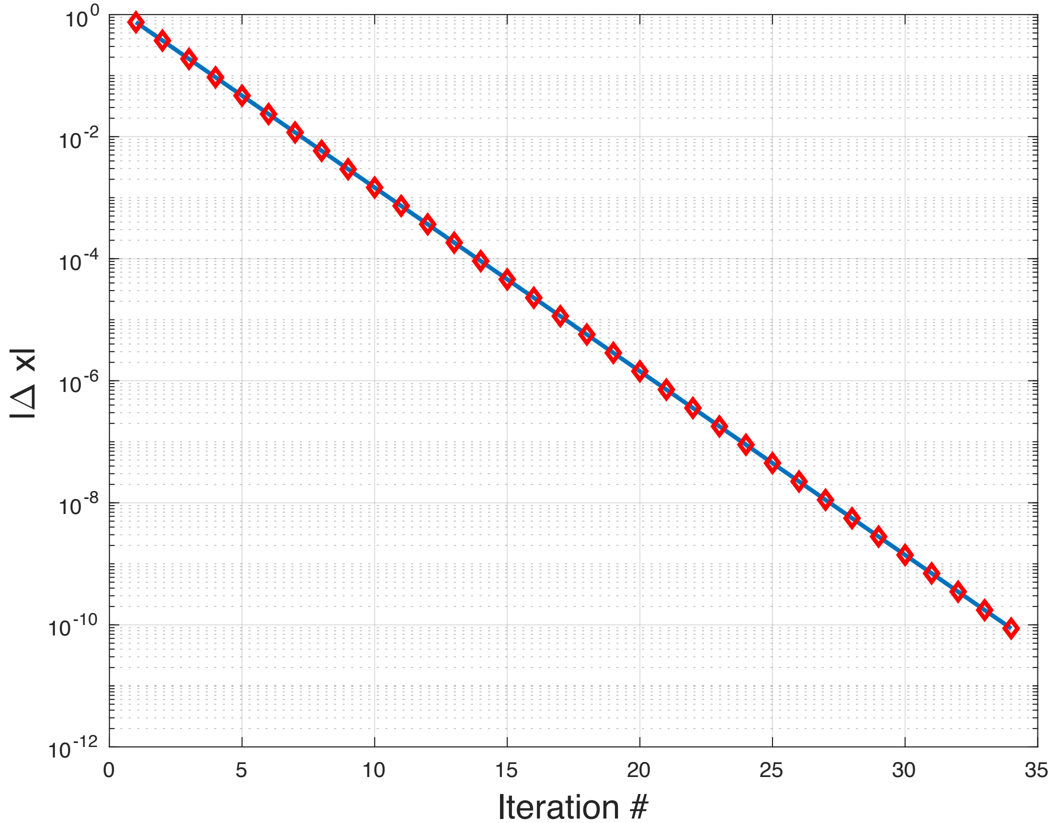
$$f(z) = 3.1\times10^{-15}$$

Stop if:

$$\epsilon_f < 10^{-10}$$

AND $\epsilon_x < 10^{-10}$

Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



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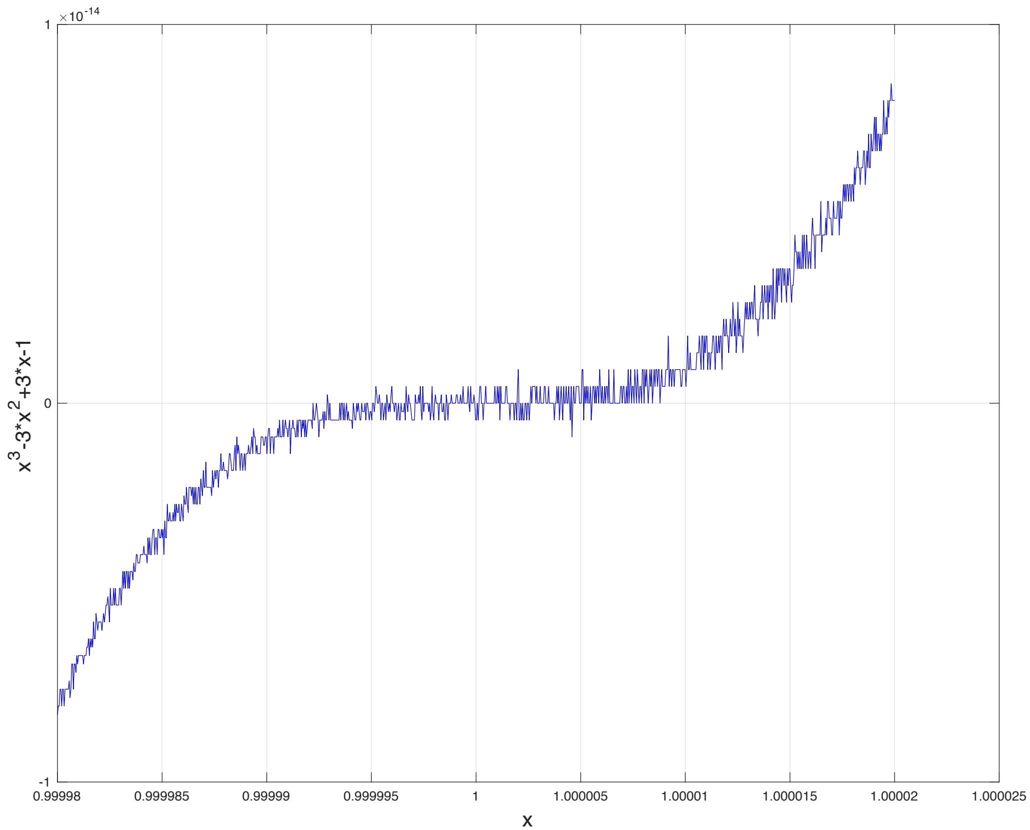
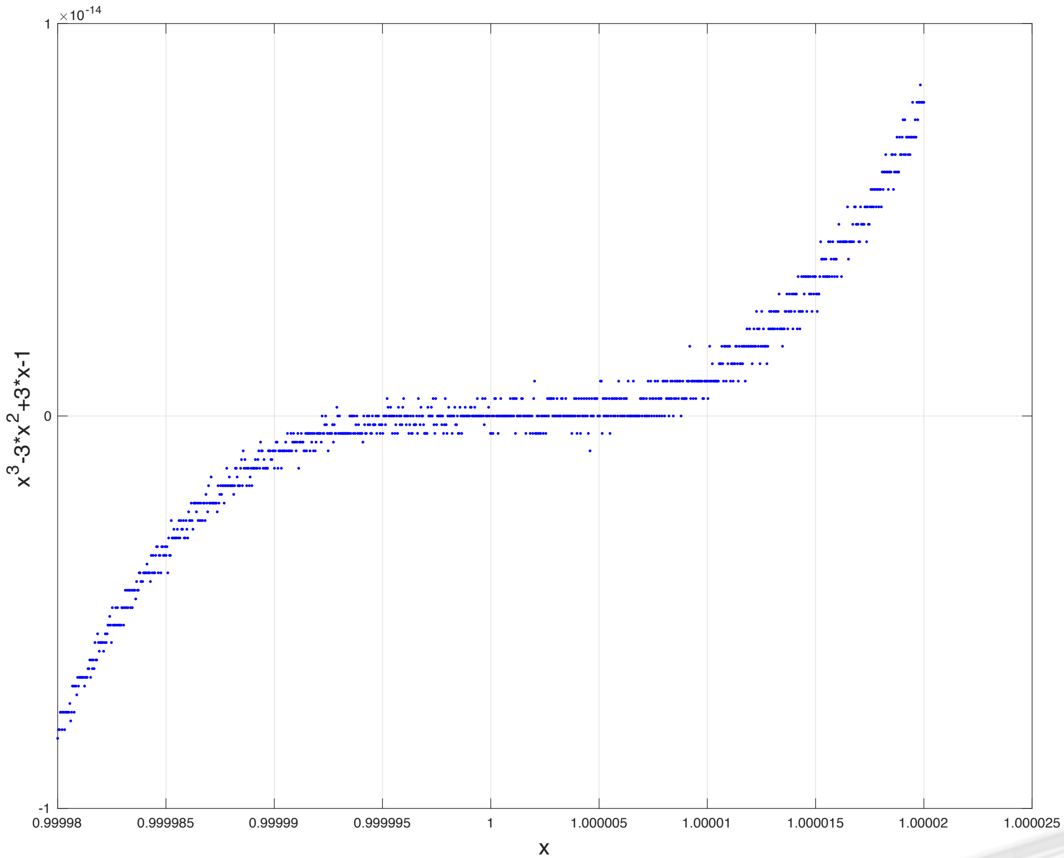


Bisection Example $f(x) = x^3 - 3x^2 + 3x - 1$



Lipschitz Continuous Functions

$$\exists c; \quad |g(x) - g(y)| \leq c|x - y| \qquad\qquad \forall x, y$$

$g(x)$ must be continuous, but not necessarily differentiable.
Lipschitz functions are limited in their rate of change.

Contractions

$$\exists\ 0 < c < 1; \quad |g(x) - g(y)| \le c|x - y| \qquad \forall x, y$$

The distance between x and y contacts when operated on by $g(\ )$.

Hence the name contraction.

Fixed Point Iteration

Find the fixed point of $g(.)$:

$$g(x) = x$$

Equivalent to finding a root of f$(.)$:

$$f(x) = g(x) - x$$

Fixed Point Iteration

Choose initial guess: $x^{(0)}$

Update guess $x^{(1)} = g(x^{(0)})$

Update guess $x^{(2)} = g(x^{(1)})$

Update guess $x^{(k)} = g(x^{(k-1)})$

Until : $x^{(n+1)} = x^{(n)}$

Converges if $g(x)$ is a contraction.

Fixed Point Iteration

Assume that x^* is the fixed-point solution.

Then: $g(x^*) = x^*$

Note, as per the algorithm: $x^{(k+1)} = g(x^{(k)})$

Distance between guess and x^*: $E_k = \left|x^{(k)} - x^*\right|$

$g(x)$ is a contraction:

$$\left|g(x^{(k)}) - g(x^*)\right| < \left|x^{(k)} - x^*\right|$$

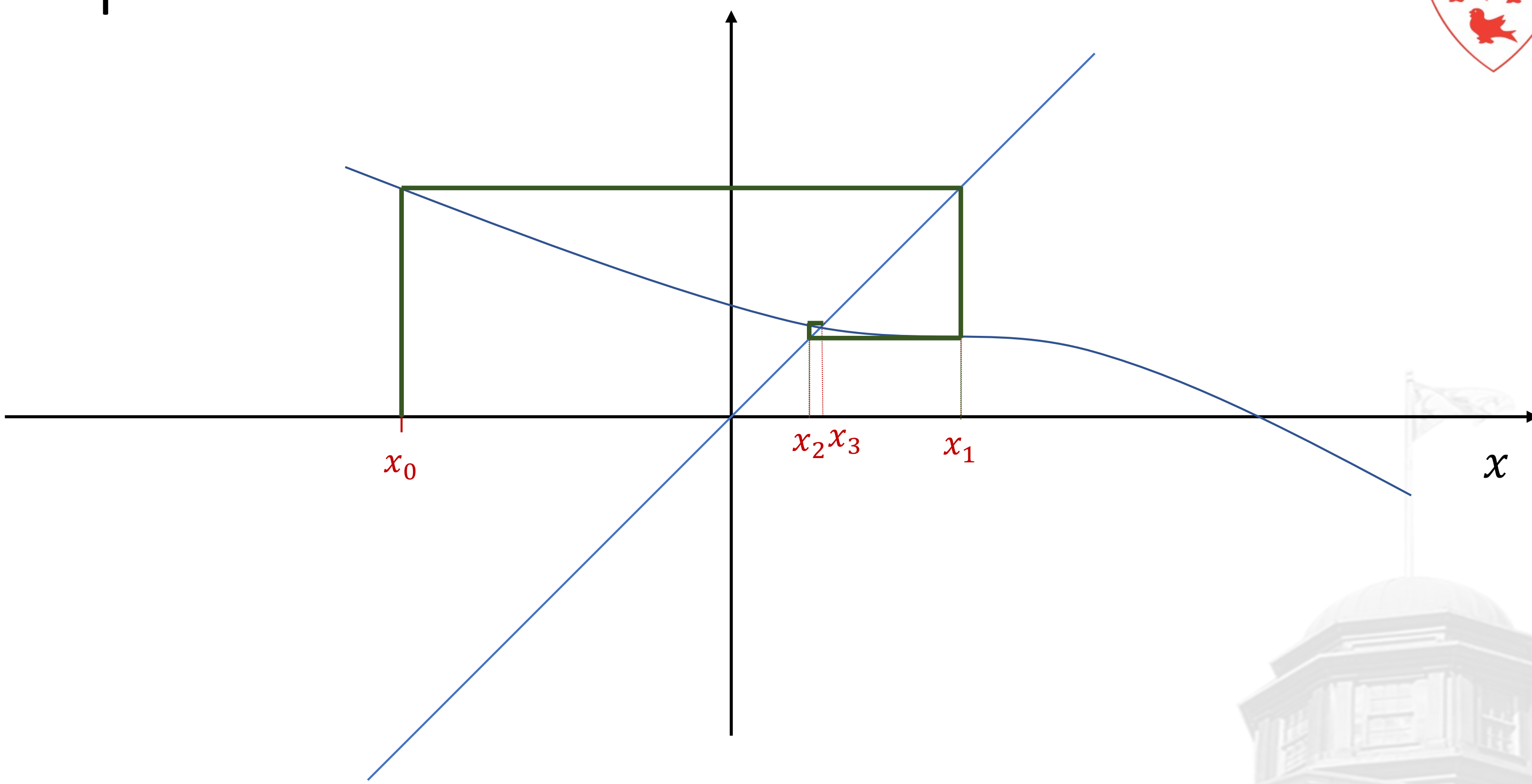
Fixed Point Iteration

$$\left|g(x^{(k)}) - g(x^*)\right| < \left|x^{(k)} - x^*\right|$$

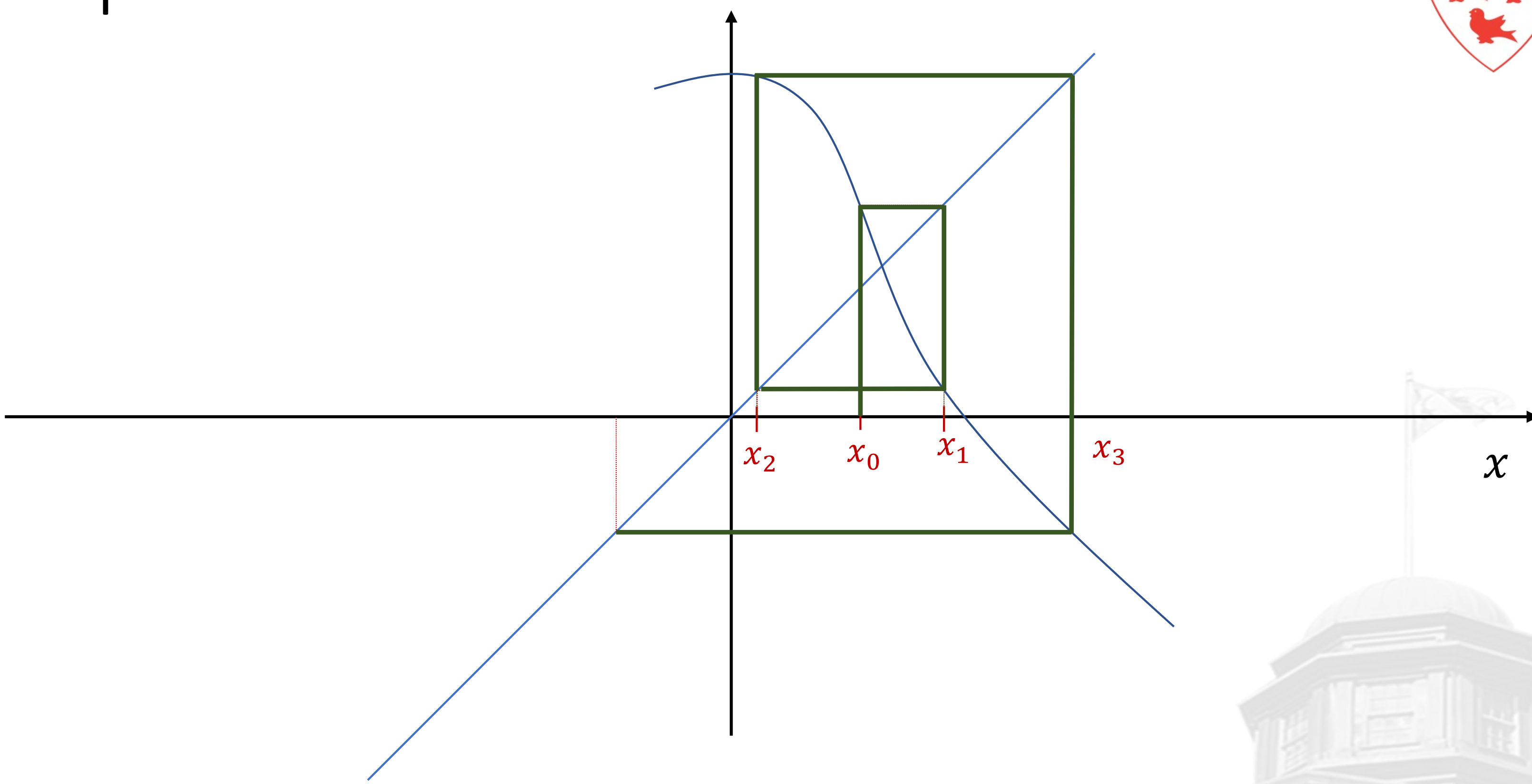
$$\left|x^{(k+1)} - x^*\right| < \left|x^{(k)} - x^*\right|$$

$$E_{k+1} < E_k$$

Example



Example



Key Advantage Compared to Bisection

Fixed point iteration can be easily applied to systems of nolinear equations

Find the fixed point of $g(.)$: $\quad g(x) = x$

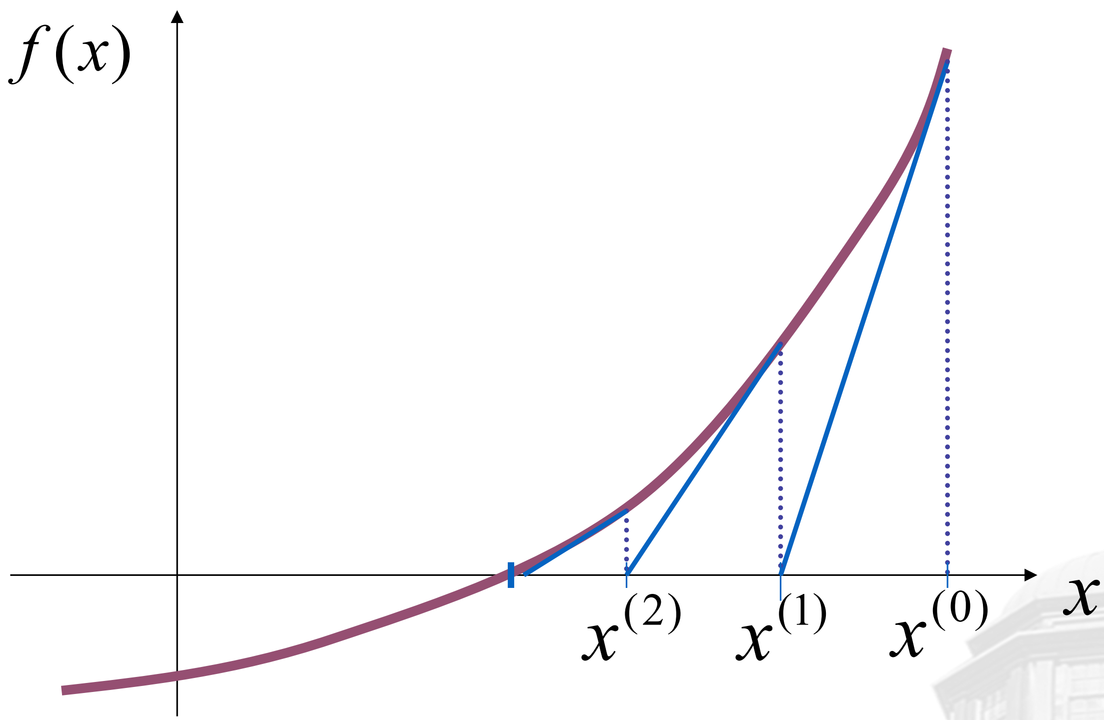
$$g(x) \in \mathbb{R}^n \qquad x \in \mathbb{R}^m$$

Can easily apply: $x^{(k)} = g(x^{(k-1)})$

Newton-Raphson Method

Find x such that:

$$f(x) = 0$$



Newton-Raphson Method

Current guess: $x^{(0)}$

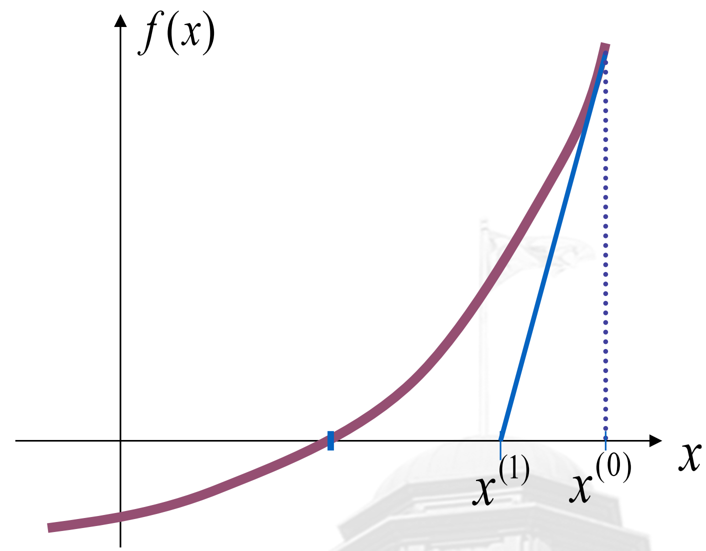
Taylor expansion at $x^{(0)}$:

$$f(x) = f\left(x^{(0)}\right) + f'\left(x^{(0)}\right)\left(x - x^{(0)}\right) + \frac{1}{2} f''\left(x^{(0)}\right)\left(x - x^{(0)}\right)^2 + \cdots$$

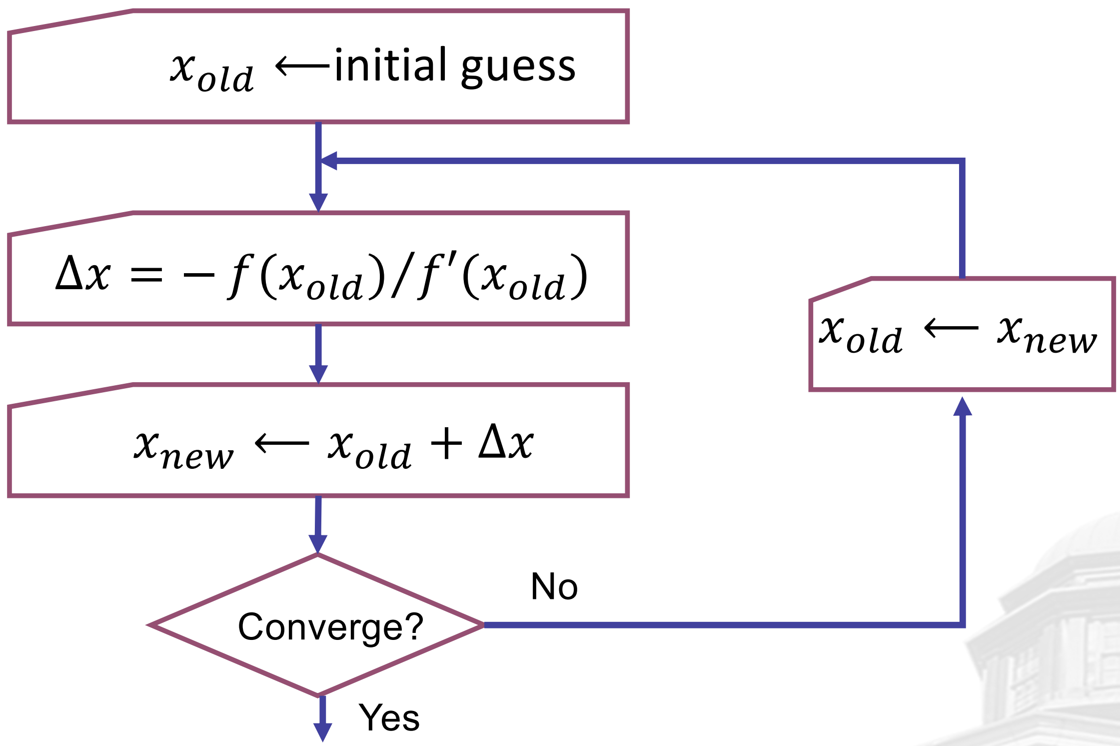
Linear Approximation:

$$f(x) \cong f\left(x^{(0)}\right) + f'\left(x^{(0)}\right)\left(x - x^{(0)}\right)$$

Find the root: $x^{(1)} = x^{(0)} - \dfrac{f\left(x^{(0)}\right)}{f'\left(x^{(0)}\right)}$



Newton-Raphson Method



N-R Method Example

Solve: $\frac{2}{3}x + e^{40x} - \frac{5}{3} = 0$

Find x such that: $f(x) = \frac{2}{3}x + e^{40x} - \frac{5}{3} = 0$

Note: $f'(x) = \frac{2}{3} + 40e^{40x}$

N-R Method Example

Start with initial guess: $x^{(0)} = 0.1$

$$f\left(x^{(0)}\right) = \frac{2}{3} \times 0.1 + e^{40\times 0.1} - \frac{5}{3} = 52.998$$

$$f'\left(x^{(0)}\right) = \frac{2}{3} + 40e^{40\times 0.1} = 2184.6$$

$$\Delta x^{(0)} = -\frac{f\left(x^{(0)}\right)}{f'\left(x^{(0)}\right)} = -2.426 \times 10^{-2}$$

N-R Method Example

$$x^{(1)} = x^{(0)} + \Delta x^{(0)} = 0.1 - 2.426\times10^{-2} = 7.574\times10^{-2}$$

$$f\left(x^{(1)}\right) = \frac{2}{3}\times7.574\times10^{-2} + e^{40\times7.574\times10^{-2}} - \frac{5}{3} = 1.9073\times10^{1}$$

$$f'\left(x^{(1)}\right) = \frac{2}{3} + 40e^{40\times7.574\times10^{-2}} = 8.2823\times10^{2}$$

$$\Delta x^{(1)} = -\frac{f\left(x^{(1)}\right)}{f'\left(x^{(1)}\right)} = -2.30285\times10^{-2}$$

N-R Method Example

$$x^{(2)} = x^{(1)} + \Delta x^{(1)} = 7.574\times10^{-2} - 2.30285\times10^{-2} = 5.271\times10^{-2}$$

$$f\left(x^{(2)}\right) = \frac{2}{3}\times x^{(2)} + e^{40\times x^{(2)}} - \frac{5}{3} = 6.60403$$

$$f'\left(x^{(2)}\right) = \frac{2}{3} + 40e^{40\times x^{(2)}} = 3.3009\times10^{2}$$

$$\Delta x^{(2)} = -\frac{f\left(x^{(2)}\right)}{f'\left(x^{(2)}\right)} = -2.00068\times10^{-2}$$

N-R Method Example

$$x^{(3)} = x^{(2)} + \Delta x^{(2)} = 3.2705\times 10^{-2}$$

$$f\left(x^{(3)}\right) = \frac{2}{3}\times x^{(3)} + e^{40\times x^{(3)}} - \frac{5}{3} = 2.0546$$

$$f'\left(x^{(3)}\right) = \frac{2}{3} + 40e^{40\times x^{(3)}} = 1.4865\times 10^{2}$$

$$\Delta x^{(3)} = -\frac{f\left(x^{(3)}\right)}{f'\left(x^{(3)}\right)} = -1.3822\times 10^{-2}$$

N-R Method Example

$$x^{(4)} = x^{(3)} + \Delta x^{(3)} = 1.8883 \times 10^{-2}$$

$$f\left(x^{(4)}\right) = \frac{2}{3} \times x^{(4)} + e^{40 \times x^{(4)}} - \frac{5}{3} = 4.7417 \times 10^{-1}$$

$$f'\left(x^{(4)}\right) = \frac{2}{3} + 40 e^{40 \times x^{(4)}} = 8.5797 \times 10^{1}$$

$$\Delta x^{(4)} = -\frac{f\left(x^{(4)}\right)}{f'\left(x^{(4)}\right)} = -5.5267 \times 10^{-3}$$

N-R Method Example

$$x^{(5)} = x^{(4)} + \Delta x^{(4)} = 1.3356\times 10^{-2}$$

$$f\left(x^{(5)}\right) = \frac{2}{3}\times x^{(5)} + e^{40\times x^{(5)}} - \frac{5}{3} = 4.8376\times 10^{-2}$$

$$f'\left(x^{(5)}\right) = \frac{2}{3} + 40e^{40\times x^{(5)}} = 6.8912\times 10^{1}$$

$$\Delta x^{(5)} = -\frac{f\left(x^{(5)}\right)}{f'\left(x^{(5)}\right)} = -7.0199\times 10^{-4}$$

N-R Method Example

$$x^{(6)} = x^{(5)} + \Delta x^{(5)} = 1.2654 \times 10^{-2}$$

$$f\left(x^{(6)}\right) = \frac{2}{3} \times x^{(6)} + e^{40 \times x^{(6)}} - \frac{5}{3} = 6.6636 \times 10^{-4}$$

$$f'\left(x^{(6)}\right) = \frac{2}{3} + 40e^{40 \times x^{(6)}} = 6.7022 \times 10^{1}$$

$$\Delta x^{(6)} = -\frac{f\left(x^{(6)}\right)}{f'\left(x^{(6)}\right)} = -9.9424 \times 10^{-6}$$

N-R Method Example

$$x^{(7)} = x^{(6)} + \Delta x^{(6)} = 1.2644 \times 10^{-2}$$

$$f\left(x^{(7)}\right) = \frac{2}{3} \times x^{(7)} + e^{40 \times x^{(7)}} - \frac{5}{3} = 1.31169 \times 10^{-7}$$

$$f'\left(x^{(7)}\right) = \frac{2}{3} + 40 e^{40 \times x^{(7)}} = 6.6996 \times 10^{1}$$

$$\Delta x^{(7)} = -\frac{f\left(x^{(7)}\right)}{f'\left(x^{(7)}\right)} = -1.9579 \times 10^{-9}$$

N-R Method Example

$$x^{(8)} = x^{(7)} + \Delta x^{(7)} = 1.2644 \times 10^{-2}$$

Exit Iteration

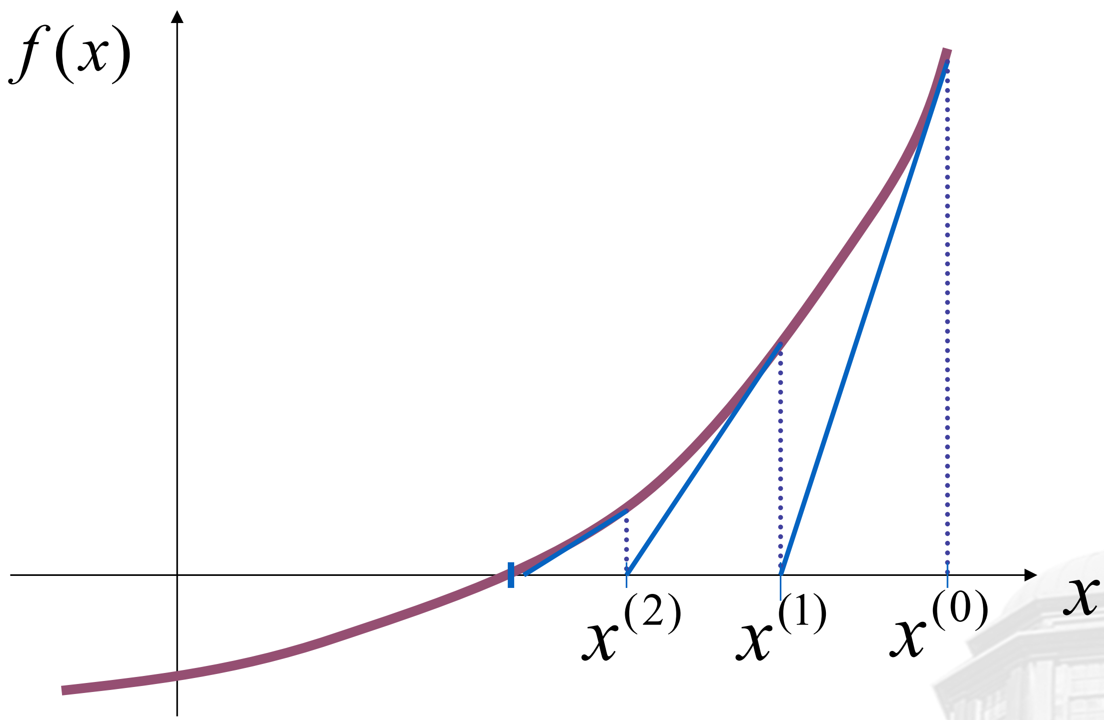
N-R Method Example

k	$x^{(k)}$	$\Delta x^{(k-1)}$
1	7.574×10^{-2}	-2.426×10^{-2}
2	5.2712×10^{-2}	-2.3029×10^{-2}
3	3.2705×10^{-2}	-2.0007×10^{-2}
4	1.8883×10^{-2}	-1.3822×10^{-2}
5	1.3356×10^{-2}	-5.5267×10^{-3}
6	1.2654×10^{-2}	-7.0199×10^{-4}
7	1.2644×10^{-2}	-9.9424×10^{-6}
8	1.2644×10^{-2}	-1.9579×10^{-9}

Newton-Raphson Method

Find x such that:

$$f(x) = 0$$



Newton-Raphson Method

Current guess: $x^{(0)}$

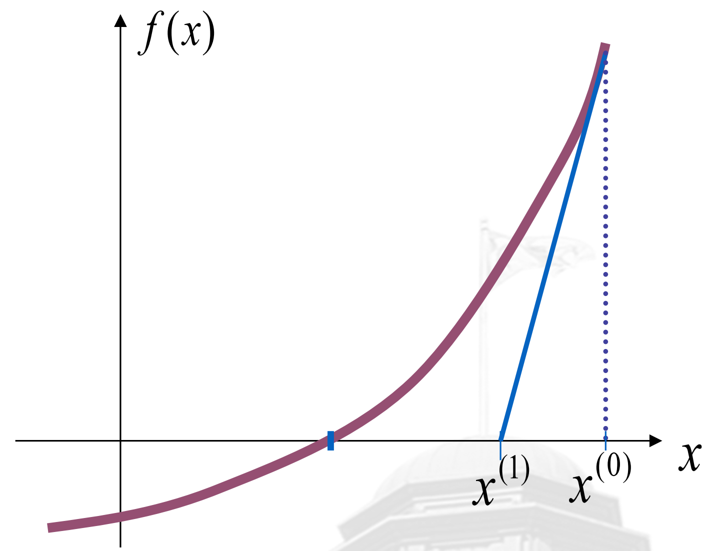
Taylor expansion at $x^{(0)}$:

$$f(x) = f\left(x^{(0)}\right) + f'\left(x^{(0)}\right)\left(x - x^{(0)}\right) + \frac{1}{2} f''\left(x^{(0)}\right)\left(x - x^{(0)}\right)^2 + \cdots$$

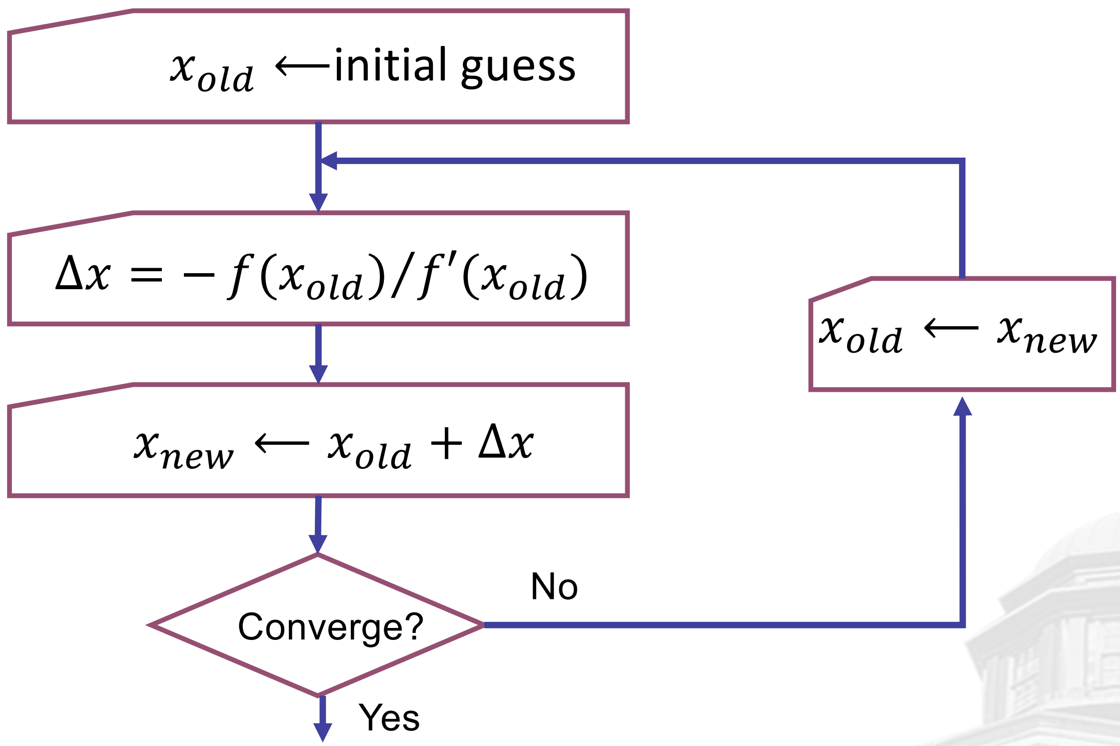
Linear Approximation:

$$f(x) \cong f\left(x^{(0)}\right) + f'\left(x^{(0)}\right)\left(x - x^{(0)}\right)$$

Find the root: $x^{(1)} = x^{(0)} - \dfrac{f\left(x^{(0)}\right)}{f'\left(x^{(0)}\right)}$



Newton-Raphson Method



N-R Method Example

Solve: $\frac{2}{3}x + e^{40x} - \frac{5}{3} = 0$

Find x such that: $f(x) = \frac{2}{3}x + e^{40x} - \frac{5}{3} = 0$

Note: $f'(x) = \frac{2}{3} + 40e^{40x}$

N-R Method Example

Start with initial guess: $x^{(0)} = 0.1$

$$f\left(x^{(0)}\right) = \frac{2}{3} \times 0.1 + e^{40\times 0.1} - \frac{5}{3} = 52.998$$

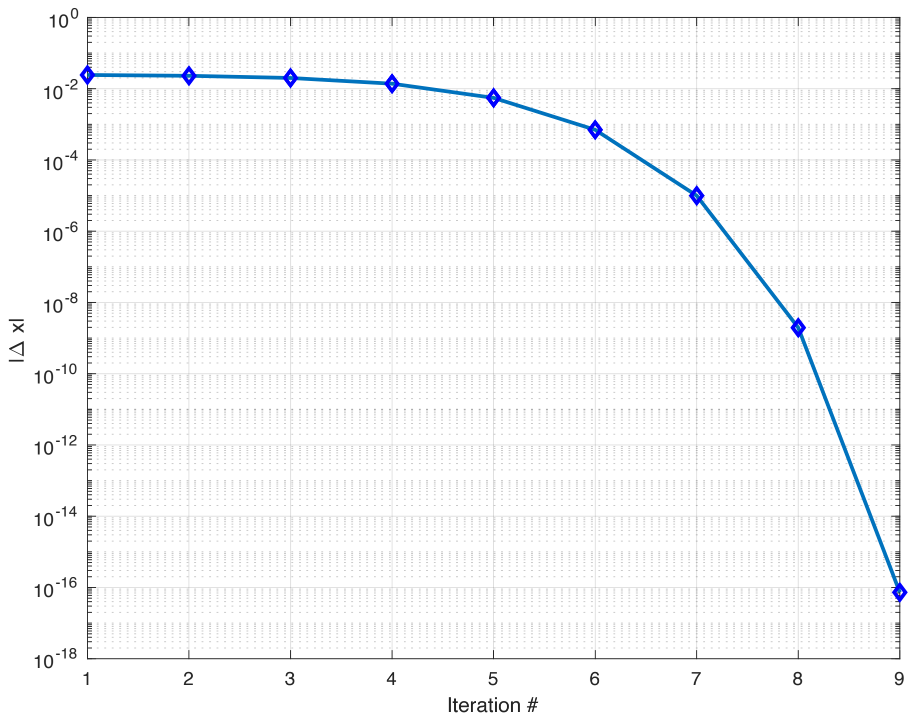
$$f'\left(x^{(0)}\right) = \frac{2}{3} + 40e^{40\times 0.1} = 2184.6$$

$$\Delta x^{(0)} = -\frac{f\left(x^{(0)}\right)}{f'\left(x^{(0)}\right)} = -2.426\times 10^{-2}$$

N-R Method Example

k	$x^{(k)}$	$\Delta x^{(k-1)}$
1	7.574×10^{-2}	-2.426×10^{-2}
2	5.2712×10^{-2}	-2.3029×10^{-2}
3	3.2705×10^{-2}	-2.0007×10^{-2}
4	1.8883×10^{-2}	-1.3822×10^{-2}
5	1.3356×10^{-2}	-5.5267×10^{-3}
6	1.2654×10^{-2}	-7.0199×10^{-4}
7	1.2644×10^{-2}	-9.9424×10^{-6}
8	1.2644×10^{-2}	-1.9579×10^{-9}

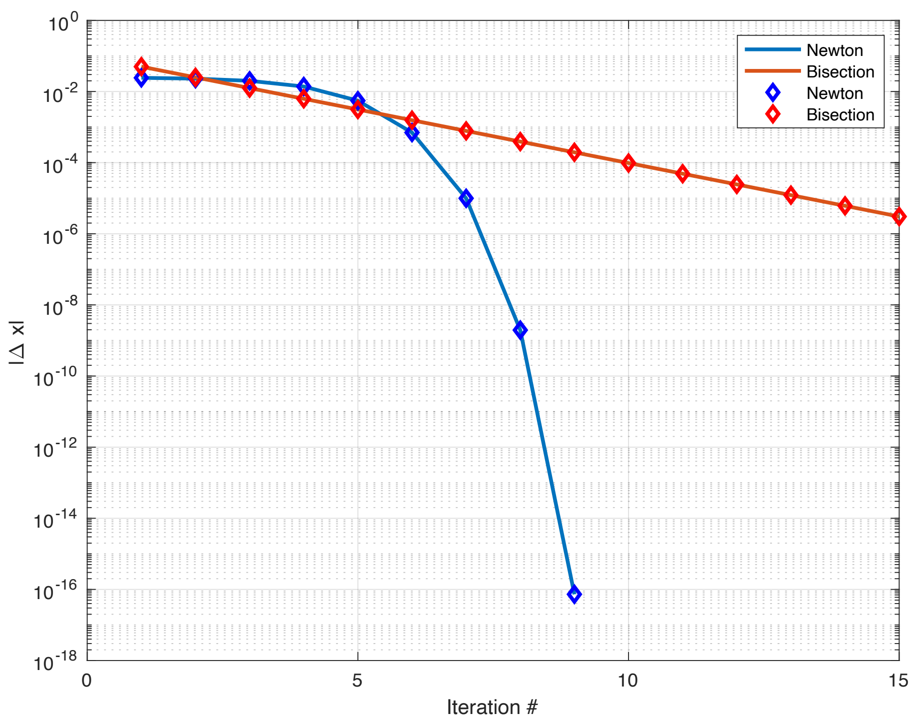
N-R Convergence Rate



$$f(x) = \frac{2}{3}x + e^{40x} - \frac{5}{3}$$

Initial Guess $x_0 = 0.1$

Bisection vs N-R

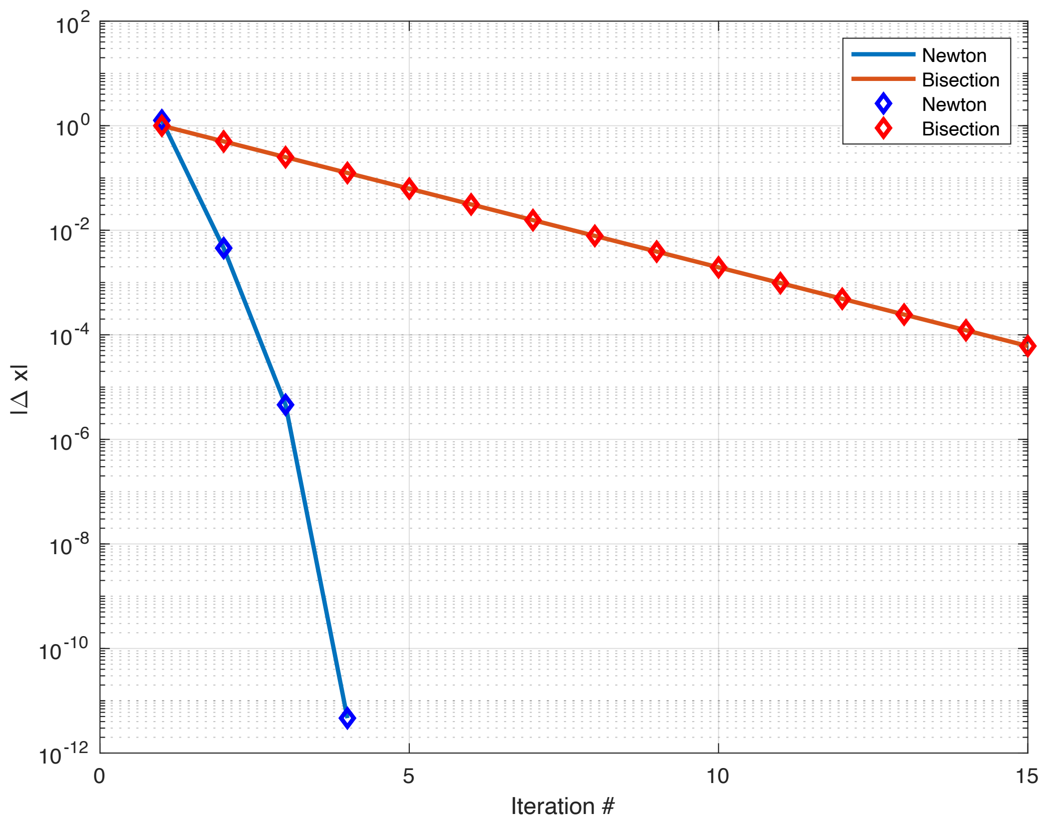


$$f(x) = \frac{2}{3}x + e^{40x} - \frac{5}{3}$$

Initial Guess $\quad x_0 = 0.1$

Initial Interval $[0, 0.1]$

Bisection vs N-R



$$f(x) = x - \cos(x)$$

Initial Guess $x_0 = 0.1$

Initial Interval $[0, 0.1]$

Secant Method

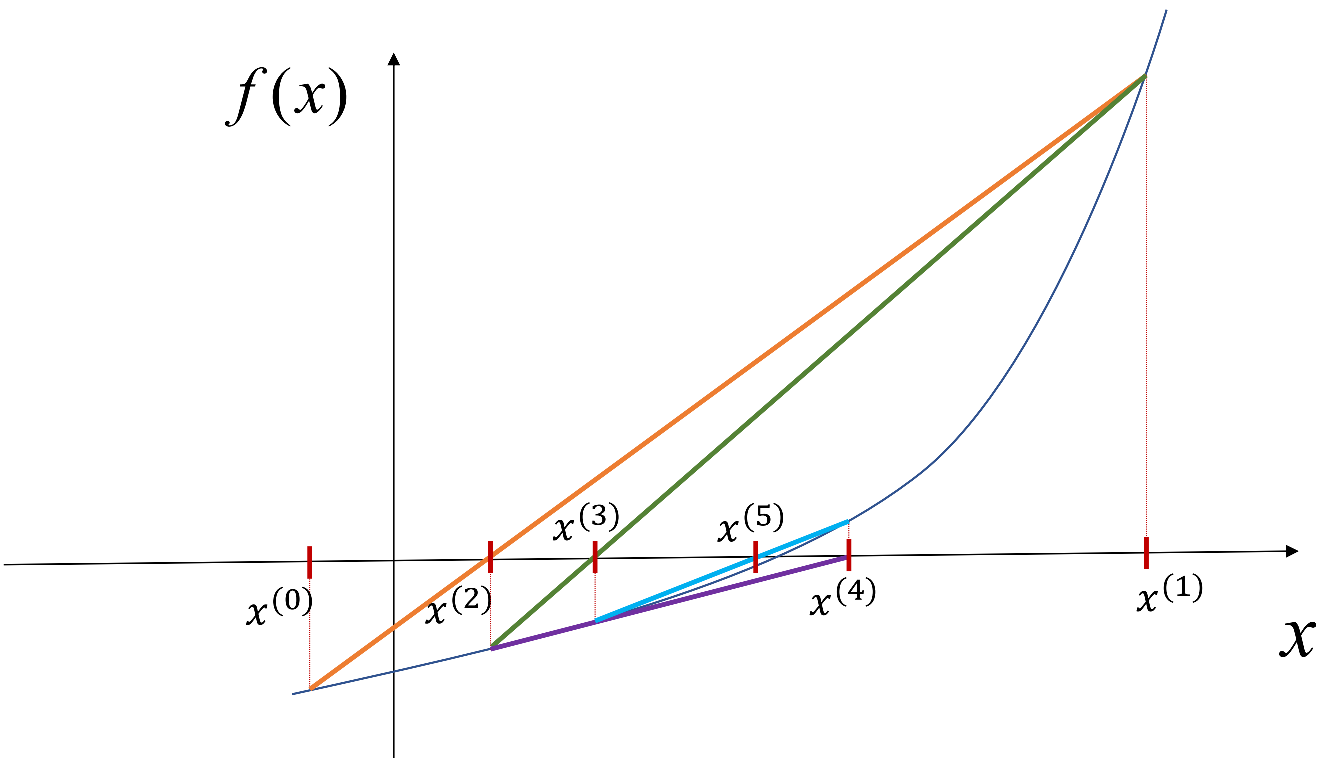
Newton Update: $x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{f'\left(x^{(k)}\right)}$

Approximate: $f'\left(x^{(k)}\right) \approx \frac{f\left(x^{(k)}\right)-f\left(x^{(k-1)}\right)}{x^{(k)}-x^{(k-1)}}$

Secant Method: $x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)\left(x^{(k)}-x^{(k-1)}\right)}{f\left(x^{(k)}\right)-f\left(x^{(k-1)}\right)}$

Need two starting points.

Secant Method



Multi-variate Case

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix}$$

Find the roots of $\boldsymbol{f}(\boldsymbol{x})$

Newton-Raphson

Update method, starting with guess $\boldsymbol{x}^{(k)}$

Approximate $\boldsymbol{f}(\boldsymbol{x})$ with:

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) + \left(\frac{d}{d\boldsymbol{x}}\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)\right)\left(\boldsymbol{x} - \boldsymbol{x}^{(k)}\right)$$

Solve:

$$\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) + \left(\frac{d}{d\boldsymbol{x}}\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)\right)\left(\boldsymbol{x} - \boldsymbol{x}^{(k)}\right) = 0$$

Newton-Raphson

Solve: $\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) + \boldsymbol{J}_k\left(\boldsymbol{x} - \boldsymbol{x}^{(k)}\right) = 0$

Solution is $\boldsymbol{x}^{(k+1)}$

$$\boldsymbol{J}_k = \frac{d}{d\boldsymbol{x}}\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}^{(k)}}$$

Newton-Raphson

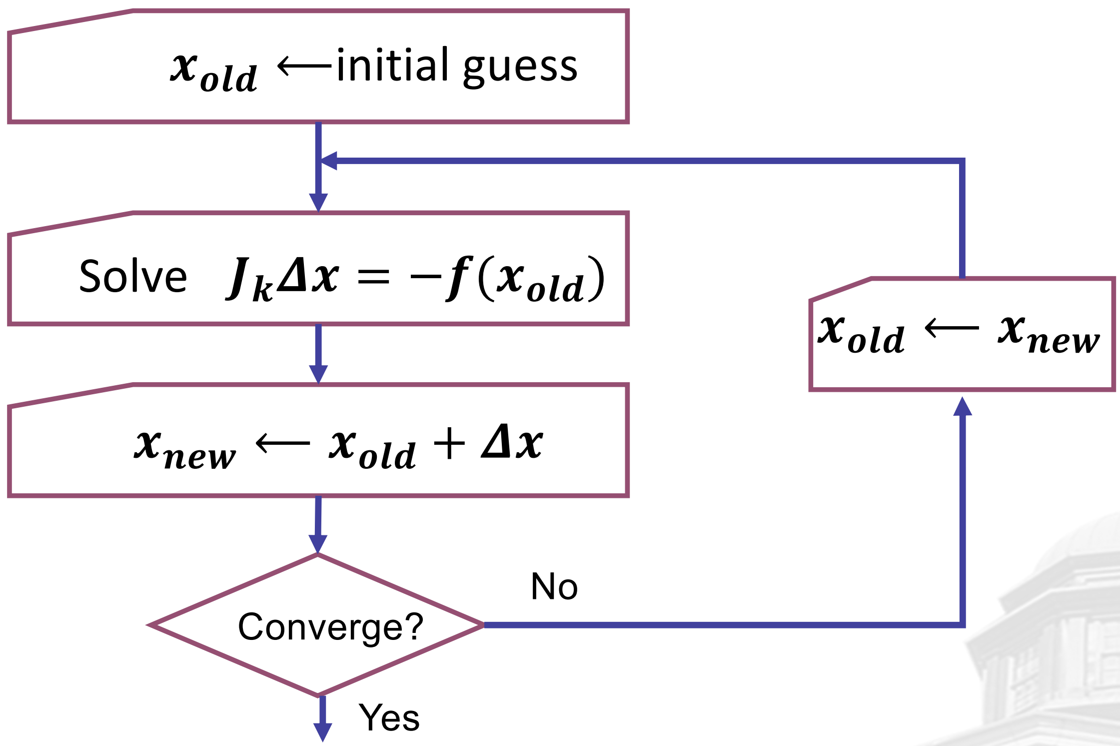
Solve: $$\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) + \boldsymbol{J}\left(\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right) = 0$$

$$\boldsymbol{J}\boldsymbol{x}^{(k+1)} = \boldsymbol{J}\boldsymbol{x}^{(k)} - \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$$

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} \underbrace{- \boldsymbol{J}_{\boldsymbol{k}}^{-1}\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)}_{\boldsymbol{\Delta x}}$$

$$\boldsymbol{J}_{\boldsymbol{k}}\boldsymbol{\Delta x} = -\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$$ $\Rightarrow$ System of Linear Equations

Newton-Raphson Method



Broyden's Method

Just like the secant method avoided computing the derivative, Broyden's method allow us to avoid computing the Jacobian.

Newton's Update: $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \boldsymbol{J}_{\boldsymbol{k}}^{-1} \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$

$$\boldsymbol{J}_{\boldsymbol{k}} = \frac{d}{d\boldsymbol{x}} \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$$

How do we approximate $\boldsymbol{J}_{\boldsymbol{k}}$? Not easy for multidimensional problems

Broyden's Method

$$\boldsymbol{J_k} = \frac{d}{d\boldsymbol{x}}\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$$

$\boldsymbol{J_k}$ must satisfy: $\boldsymbol{J_k}\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right) = \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) - \boldsymbol{f}\left(\boldsymbol{x}^{(k-1)}\right)$

The above formula is not enough to compute $\boldsymbol{J_k}$

It constrains the action of $\boldsymbol{J_k}$ only in one direction: $\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right)$

Broyden's Method

$$\boldsymbol{J_k} = \frac{d}{d\boldsymbol{x}}\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$$

$\boldsymbol{J_k}$ must satisfy: $\boldsymbol{J_k}\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right) = \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) - \boldsymbol{f}\left(\boldsymbol{x}^{(k-1)}\right)$

Assume that we know $\boldsymbol{J_{k-1}}$

Find $\boldsymbol{J_k}$ such that: $||\boldsymbol{J_k} - \boldsymbol{J_{k-1}}||_F$ is minimum

Subject to $\boldsymbol{J_k}\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right) = \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) - \boldsymbol{f}\left(\boldsymbol{x}^{(k-1)}\right)$

Broyden's Method

$$\boldsymbol{J}_k = \boldsymbol{J}_{k-1} + \underbrace{\frac{\left(\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) - \boldsymbol{f}\left(\boldsymbol{x}^{(k-1)}\right) - \boldsymbol{J}_{k-1}\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right)\right)}{\left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right\|_2^2}\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right)^T}_{\text{Rank 1 Update}}$$

$$\Delta \boldsymbol{J} = \boldsymbol{J}_k - \boldsymbol{J}_{k-1} = \boldsymbol{u}\boldsymbol{v}^T$$

In the absence of better information, we use: $\boldsymbol{J}_0 = \boldsymbol{U}$

Broyden's Method

Just like the secant method avoided computing the derivative, Broyden's method allow us to avoid computing the Jacobian.

Broyden's Update: $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \boldsymbol{J}_{\boldsymbol{k}}^{-1} \boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right)$

Broyden's method allow us to approximate $\boldsymbol{J}_{\boldsymbol{k}}$

We still need to solve the system $\boldsymbol{J}_{\boldsymbol{k}} \boldsymbol{\Delta x} = -\boldsymbol{f}(\boldsymbol{x}^{(k)})$

Sherman-Morisson Formula

$$(\boldsymbol{A} + \boldsymbol{u}\boldsymbol{v}^T)^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{u}\boldsymbol{v}^T\boldsymbol{A}^{-1}}{1 + \boldsymbol{v}^T\boldsymbol{A}^{-1}\boldsymbol{u}}$$

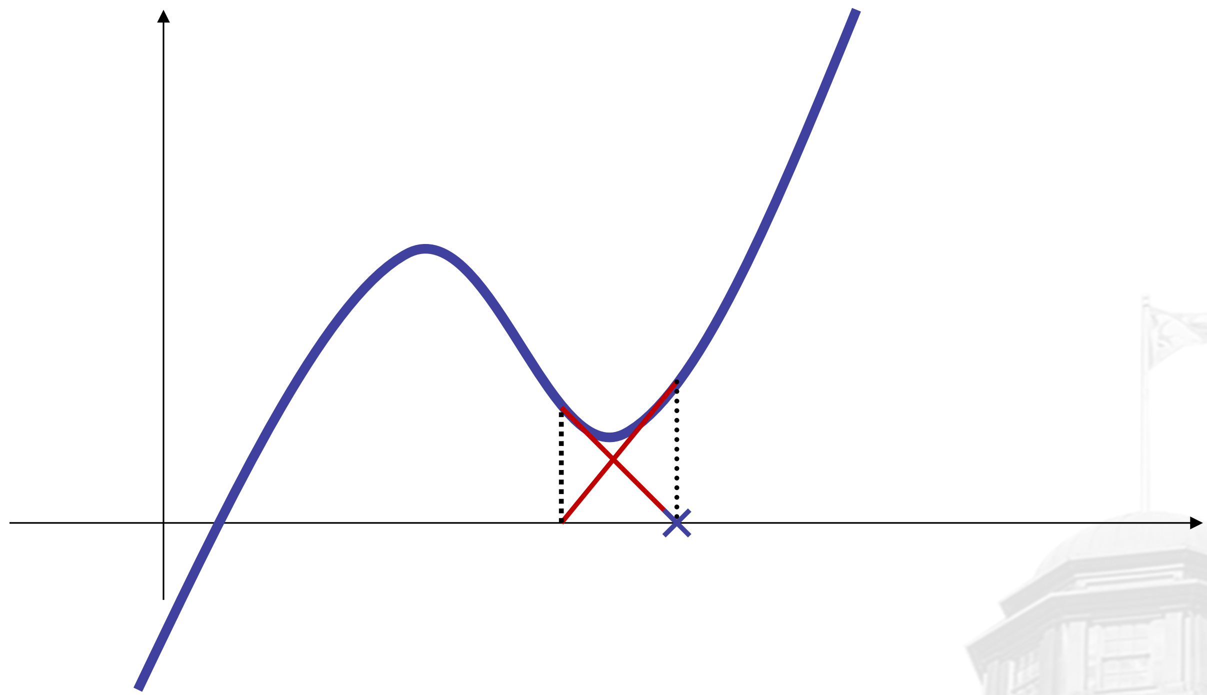
Update Inverse Instead

$$\boldsymbol{J}_k^{-1} = \boldsymbol{J}_{k-1}^{-1} + \frac{\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)} - \boldsymbol{J}_{k-1}^{-1}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) - \boldsymbol{f}\left(\boldsymbol{x}^{(k-1)}\right)\right)}{(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)})^T \boldsymbol{J}_{k-1}^{-1}\left(\boldsymbol{f}\left(\boldsymbol{x}^{(k)}\right) - \boldsymbol{f}\left(\boldsymbol{x}^{(k-1)}\right)\right)}\left(\left(\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k-1)}\right)^T \boldsymbol{J}_{k-1}^{-1}\right)$$

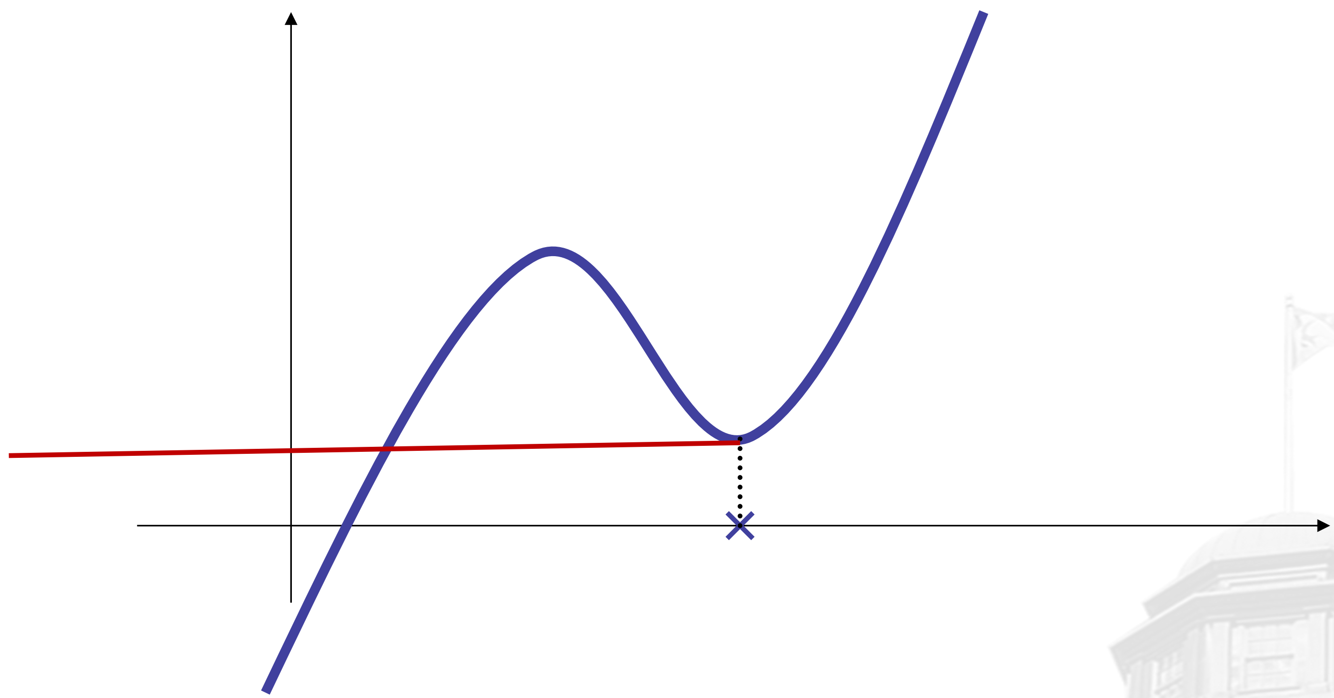
If we use $\boldsymbol{J}_0 = \boldsymbol{U}$

The inverse of $\boldsymbol{J}_0$ is trivial

Convergence Problems



Convergence problems



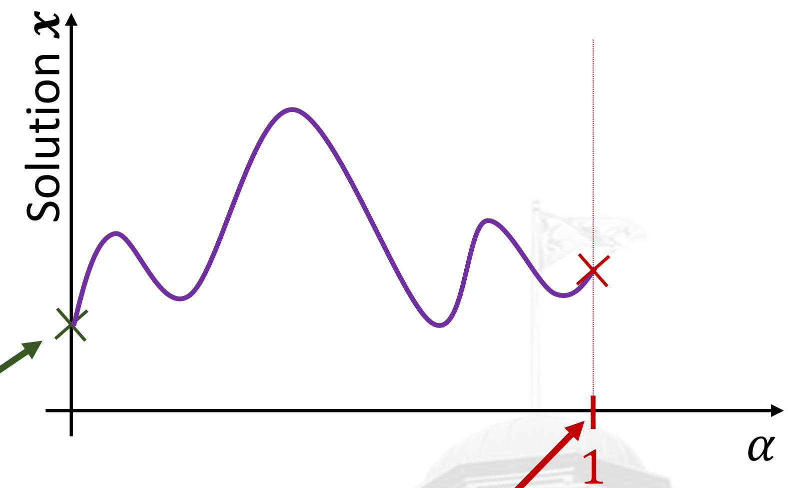
Continuation Methods (Homotopy)

Initial Problem: $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$

Embed parameter α

Modified Problem:

$$\boldsymbol{g}(\boldsymbol{x}, \alpha) = \boldsymbol{0}$$



Solution of $\boldsymbol{g}(\boldsymbol{x}, 0) = \boldsymbol{0}$ is trivial

$\boldsymbol{g}(\boldsymbol{x}, 1) = \boldsymbol{f}(\boldsymbol{x})$

Example

$$f(x) = \frac{2}{3}x + e^{40x} - \frac{5}{3}$$

$$g(x, \alpha) = \frac{2}{3}x + \alpha e^{40x} - \frac{5}{3}$$

$$g(x, 0) = \frac{2}{3}x - \frac{5}{3}$$

$$g(x, 1) = \frac{2}{3}x + e^{40x} - \frac{5}{3} = f(x)$$

Example

$$f(x) = x^3 - 6x^2 + 1$$

$$g(x,\alpha) = x^3 - 6\alpha x^2 + 1$$

$$g(x,0) = x^3 + 1 \qquad g(x,1) = x^3 - 6x^2 + 1$$

$$g(x,\alpha) = x^3 - 6x^2 + \alpha$$

$$g(x,0) = x^3 - 6x^2 \qquad g(x,1) = x^3 - 6x^2 + 1$$

Homotopy Transformation

Initial Problem: $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$

Embed parameter α

Modified Problem:

$$\boldsymbol{\Psi}(\boldsymbol{x}, \alpha) = \alpha \boldsymbol{f}(\boldsymbol{x}) + (1 - \alpha)\boldsymbol{g}(\boldsymbol{x})$$

$$\boldsymbol{\Psi}(\boldsymbol{x}, 0) = \boldsymbol{g}(\boldsymbol{x}) \qquad \boldsymbol{\Psi}(\boldsymbol{x}, 1) = \boldsymbol{f}(\boldsymbol{x})$$