ECSE 343 Numerical Methods in Engineering

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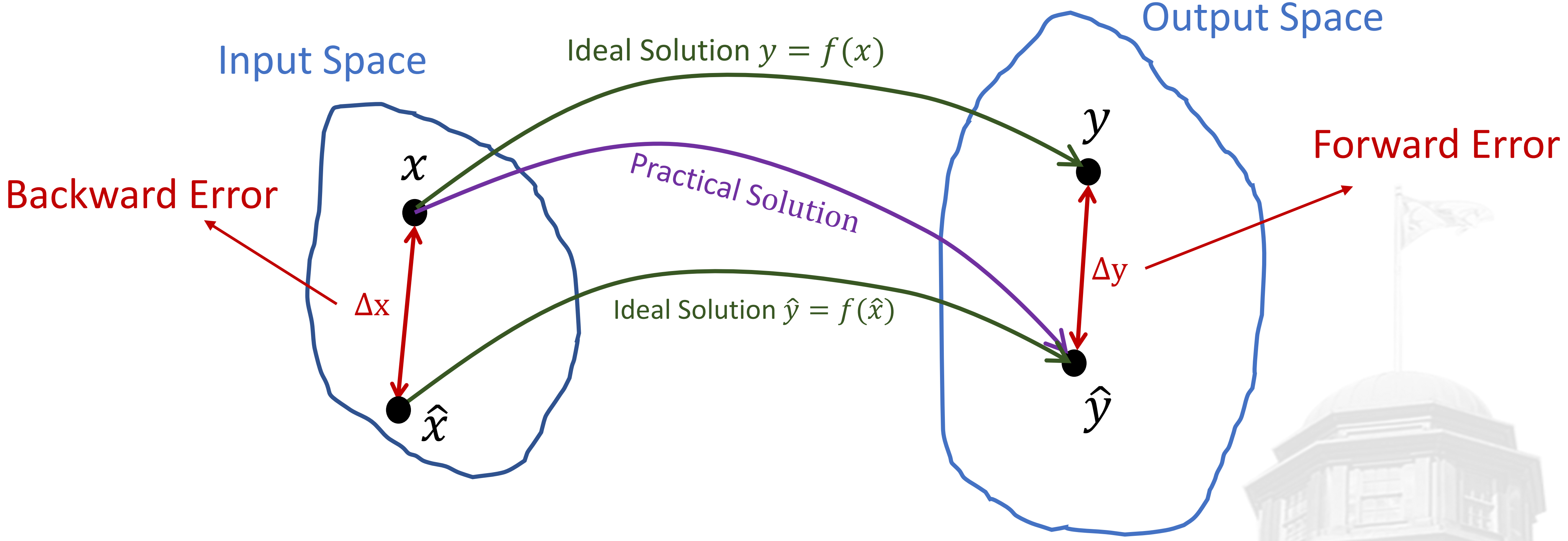
Forward vs Backward Error

- Problem: Find the root of $f(x) = 0$
- Actual solution is $x_o \Longrightarrow f(x_o) \equiv 0$
- Computed (inexact) solution is $\hat{x}$ such that $f(\hat{x}) = \epsilon$
- Forward error: $|x_o - \hat{x}|$

- Backward error: $|f(x_o) - f(\hat{x})| = |f(\hat{x})| = |\epsilon|$

Forward vs Backward Error

Problem: Compute $y = f(x)$



$Ax = b$

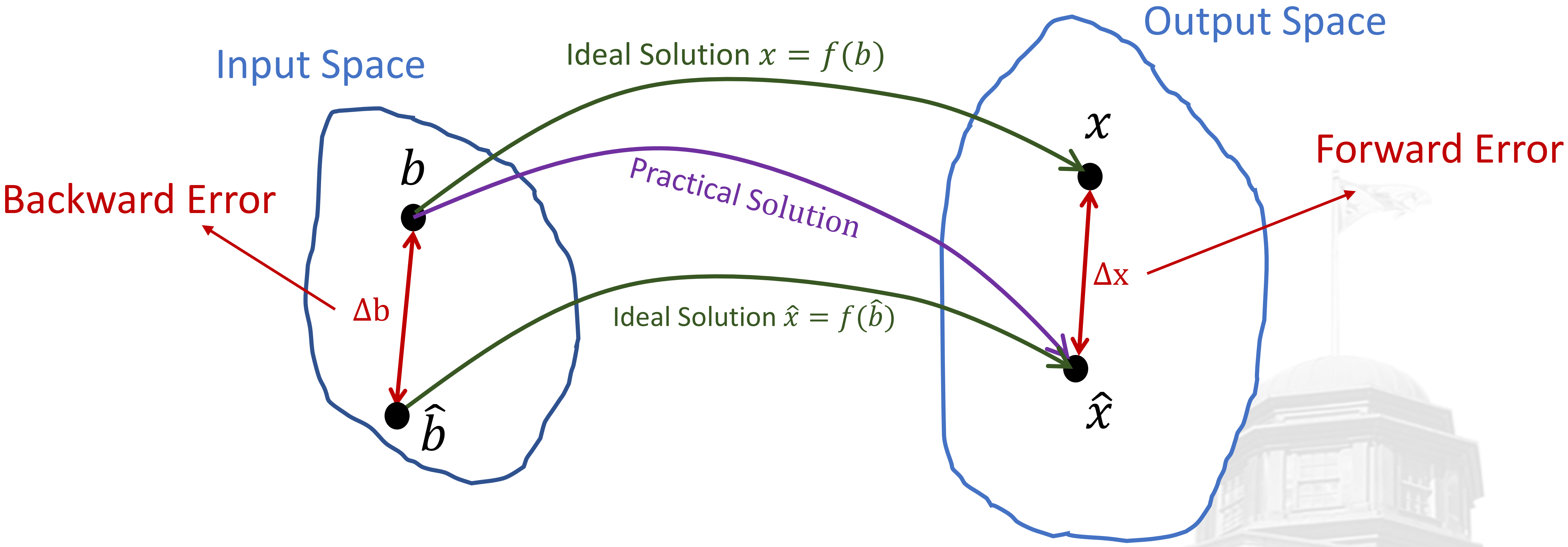
Problem: Find the root of $f(x) = Ax - b$

Problem: Find x such that $Ax = b$

Problem: $x = f(b) = A^{-1}b$

Condition Number for $Ax = b$

Problem: $x = f(b) = A^{-1}b$



Forward vs Backward Error

- Problem: Find the root of $f(x) = Ax - b = 0$
- Actual solution is $x_o \Longrightarrow f(x_o) \equiv Ax_0 - b = 0$
- Computed (inexact) solution is $\hat{x}$ such that $f(\hat{x}) = A\hat{x} - b = \epsilon$
- Forward error: $|x_o - \hat{x}|$
- But: $A\hat{x} - \hat{b} \equiv 0$ and therefore $A\hat{x} \equiv \hat{b}$
- Backward error: $|f(x_o) - f(\hat{x})|$

$$= |f(\hat{x})| = |A\hat{x} - b| = \left|\hat{b} - b\right| = |\epsilon|$$

Condition Number for $Ax = b$

Problem: $x = f(b) = A^{-1}b$

$$J = \frac{df}{db} = A^{-1}$$

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|A^{-1}b\|} = \frac{\|A^{-1}\|\|b\|}{\|A^{-1}b\|} \geq 1$$

Condition Number for $Ax = b$

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|A^{-1}b\|}$$

$$\frac{\|b\|}{\|A^{-1}b\|} = \frac{\|Ax\|}{\|x\|} < \|A\|$$

$$\kappa \le \underbrace{\|A^{-1}\| \|A\|}$$

Condition Number of matrix A $\kappa(A)$

Recall

$$\|\boldsymbol{A}\| \equiv \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \|\boldsymbol{x}\| \neq 0}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

Inner Product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v} \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{\boldsymbol{n}}$$

Generalize for complex vectors $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^* \boldsymbol{v}$ $\qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}^{\boldsymbol{n}}$

$\boldsymbol{u}^*$ is the conjugate transpose of $\boldsymbol{u}$.

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \boldsymbol{u}^* = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \end{bmatrix}$$

$\bar{u}_n$ is the complex conjugate of u_n

Properties of Inner Product

- $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$
- $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
- $\langle c\boldsymbol{u}, \boldsymbol{v} \rangle = \bar{c}\langle \boldsymbol{u}, \boldsymbol{v} \rangle$
- $\langle \boldsymbol{u}, c\boldsymbol{v} \rangle = c\langle \boldsymbol{u}, \boldsymbol{v} \rangle$
- $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = \mathbf{0} \Leftrightarrow \boldsymbol{u} = \mathbf{0}$

c is a scalar $c \in \mathbb{C}$

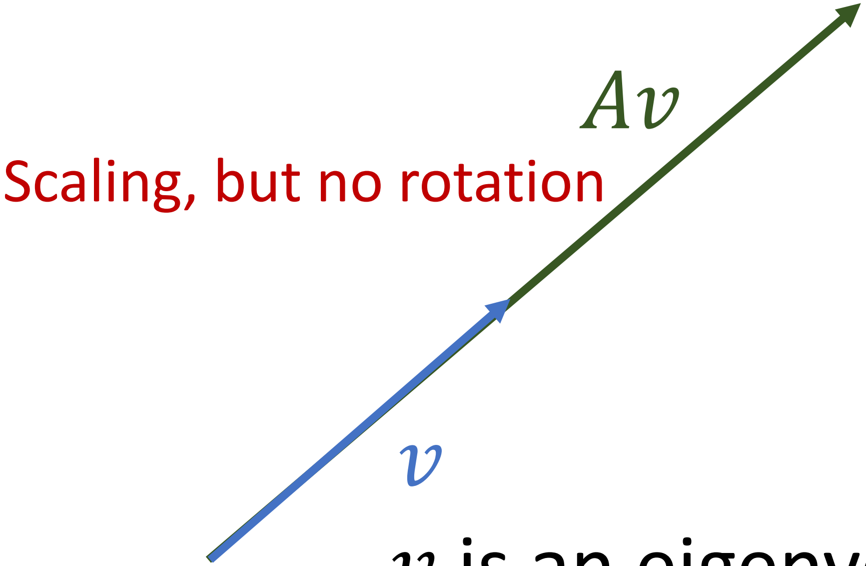
Inner Product and Hermitian Matrices

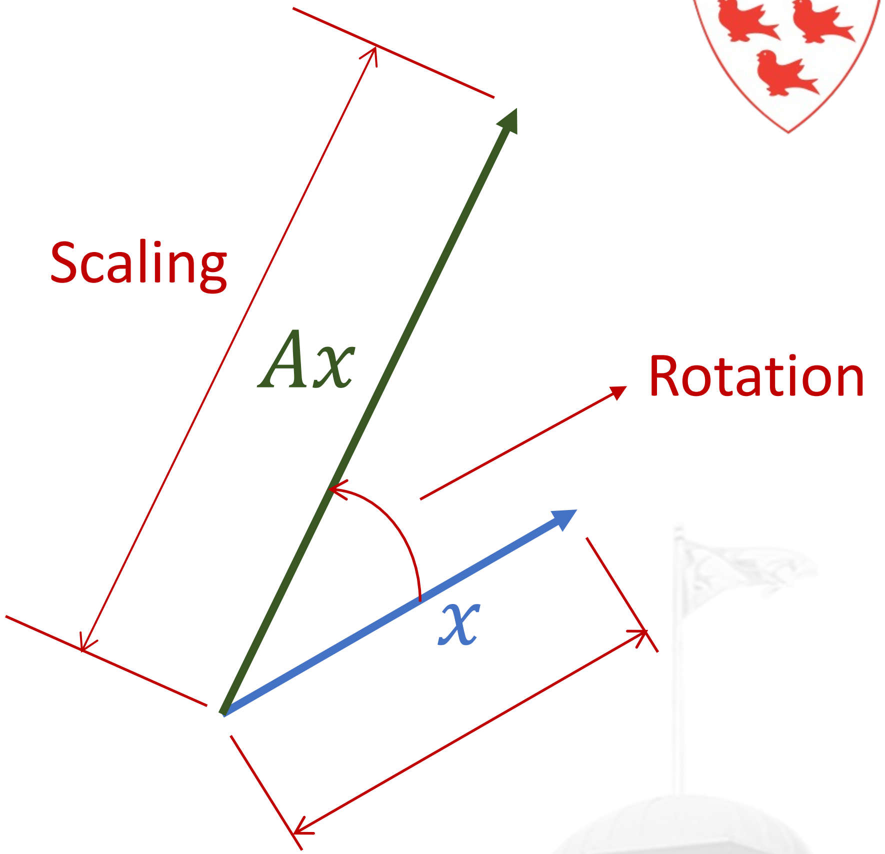
If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is Hermitian (i.e $\boldsymbol{A}^* = \boldsymbol{A}$) then $\langle \boldsymbol{Au}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \boldsymbol{Av} \rangle$.

Recall: $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^* \boldsymbol{v}$

$$\langle \boldsymbol{Au}, \boldsymbol{v} \rangle = (\boldsymbol{Au})^* \boldsymbol{v} = \boldsymbol{u}^* \boldsymbol{A}^* \boldsymbol{v} = \boldsymbol{u}^* \boldsymbol{A} \boldsymbol{v} = \boldsymbol{u}^* (\boldsymbol{Av}) = \langle \boldsymbol{u}, \boldsymbol{Av} \rangle$$

Eigenvalues / Eigenvectors





v is an eigenvector of A

$$Av = \lambda v$$

eigenvalue (scalar)

Characteristic Polynomial

Identity Matrix

λ is an eigenvalue of $\boldsymbol{A}$ ⟹ $\boldsymbol{Av} = \lambda\boldsymbol{v} = \lambda\boldsymbol{Uv}$

$$(\boldsymbol{A} - \lambda\boldsymbol{U})\boldsymbol{v} = \boldsymbol{0} \qquad \boldsymbol{v} \neq \boldsymbol{0}$$

λ is an eigenvalue of $\boldsymbol{A}$ iff $(\boldsymbol{A} - \lambda\boldsymbol{U})$ is singular.
λ is an eigenvalue of $\boldsymbol{A}$ iff $\det(\boldsymbol{A} - \lambda\boldsymbol{U}) = \boldsymbol{0}$ is singular.
Characteristic polynomial of of $\boldsymbol{A}$ is: $p_A(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{U})$.

Eigenvalues and Eigenvectors

- Eigenvalues of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ are the roots of the its characteristic polynomial.
- The order of the characteristic polynomial is n.
- We cannot have closed form expressions for roots of polynomials of order ≥ 5 (Abel-Ruffini theorem).
- We cannot have closed form expressions for eigenvalues of matrices of size $n \geq 5$.
- The determinant method is useful from a theoretical standpoint but is not used in numerical algorithms (CPU cost, numerical conditioning).

Diagonalization

Consider a full rank matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with distinct eigen values λ_i and corresponding eigenvectors v_i, $1 \le i \le n$.

$$\boldsymbol{A}\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1 \qquad \boldsymbol{A}\boldsymbol{v}_2 = \lambda_1\boldsymbol{v}_2 \qquad \cdots \qquad \boldsymbol{A}\boldsymbol{v}_n = \lambda_n\boldsymbol{v}_n$$

$$\boldsymbol{A}[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n] = [\lambda_1\boldsymbol{v}_1 \quad \lambda_2\boldsymbol{v}_2 \quad \cdots \quad \lambda_n\boldsymbol{v}_n] = \mathbf{V\Gamma}$$

$$\boldsymbol{A}\mathbf{V} = \mathbf{V\Gamma} \qquad \mathbf{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n] \qquad \mathbf{\Gamma} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\boldsymbol{V}^{-1}\boldsymbol{A}\mathbf{V} = \mathbf{\Gamma} \qquad \boldsymbol{A} = \mathbf{V\Gamma}\boldsymbol{V}^{-1}$$

Application: Powers of $\boldsymbol{A}$

$$\boldsymbol{A} = \mathbf{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}$$

$$\boldsymbol{A}^2 = (\mathbf{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1})\,(\mathbf{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}) \;= \mathbf{V}\boldsymbol{\Gamma}(\boldsymbol{V}^{-1}\boldsymbol{V})\,\boldsymbol{\Gamma}\boldsymbol{V}^{-1} = \mathbf{V}\boldsymbol{\Gamma}^2\boldsymbol{V}^{-1}$$

$$\boldsymbol{A}^m = \mathbf{V}\boldsymbol{\Gamma}^{\mathrm{m}}\boldsymbol{V}^{-1}$$

$$\boldsymbol{\Gamma} = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_1^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

Eigenvalues/Eigenvector of a Hermitian Matrix

If $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is Hermitian (i.e $\boldsymbol{A}^* = \boldsymbol{A}$) then:

- $\boldsymbol{A}$ has real eigenvalues.
- The eigenvectors of $\boldsymbol{A}$ are orthogonal to each other.
- (As a consequence, the matrix of eigenvectors can be chosen to be unitary)

Eigenvalues of a Hermitian Matrix

λ is an eigenvalue of $\boldsymbol{A}$ with corresponding eigenvector $\boldsymbol{v}$ then $\boldsymbol{Av} = \lambda \boldsymbol{v}$

$\boldsymbol{A}$ is Hermitian then $\langle \boldsymbol{Av}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{Av} \rangle$.

$$\langle \lambda \boldsymbol{v}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \lambda \boldsymbol{v} \rangle$$

$$\bar{\lambda} \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \lambda \langle \boldsymbol{v}, \lambda \boldsymbol{v} \rangle$$

$$\bar{\lambda} = \lambda$$

λ is real.

$\boldsymbol{A}$ has real eigenvalues

Eigenvectors of a Hermitian Matrix

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be eigenvectors of a Hermitian matrix $\boldsymbol{A}$ corresponding to eigenvalues λ and μ ($\lambda \neq \mu$). Note that λ and μ are real.

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}\,, \boldsymbol{A}\boldsymbol{y} = \mu\boldsymbol{y}$$

$\boldsymbol{A}$ is Hermitian then $\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} \rangle$.

$$\langle \lambda\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \mu\boldsymbol{y} \rangle$$

$$\lambda\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \mu\langle \boldsymbol{x}, \boldsymbol{y} \rangle$$

Recall that λ and μ are real.

$$(\lambda - \mu)\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$$

Recall that $\lambda \neq \mu$

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$$

$\boldsymbol{A}$ has orthogonal eigenvectors

Diagonalization of a Hermitian Matrix

Consider a full rank Hermitian matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ with distinct real eigenvalues λ_i and corresponding eigenvectors v_i, $1 \le i \le n$, $\|v_i\| = 1$

$$\boldsymbol{A}\mathbf{V} = \mathbf{V\Gamma} \qquad \underbrace{\mathbf{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n]}_{\text{Unitary matrix } \mathbf{V}^*\mathbf{V} = \boldsymbol{U}} \qquad \mathbf{\Gamma} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\boldsymbol{A} = \mathbf{V\Gamma V}^* \qquad \mathbf{V}^*\boldsymbol{A}\mathbf{V} = \mathbf{\Gamma}$$

Similar Matrices

- $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar matrices iff: $\boldsymbol{A} = \boldsymbol{M}\boldsymbol{B}\boldsymbol{M}^{-1}$
- Similar Matrices share the same eigenvalues

$$\boldsymbol{B} = \mathbf{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}$$

$$\boldsymbol{A} = \boldsymbol{M}\boldsymbol{B}\boldsymbol{M}^{-1} = \mathbf{M}\mathbf{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}\boldsymbol{M}^{-1}$$

$\boldsymbol{AB}$ and $\boldsymbol{BA}$

- $\boldsymbol{A}$ and $\boldsymbol{B}$ are two invertible matrices.
- In general, $\boldsymbol{AB} \neq \boldsymbol{BA}$
- $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are similar matrices: $\boldsymbol{AB} = \boldsymbol{B}^{-1}(\boldsymbol{BA})\boldsymbol{B}$
- $\boldsymbol{AB}$ and $\boldsymbol{BA}$ Share the same eigen values.

Induced Norm

Induced norm $||\boldsymbol{A}||$ of a matrix $\boldsymbol{A}$ based on a vector norm $||\cdot||$

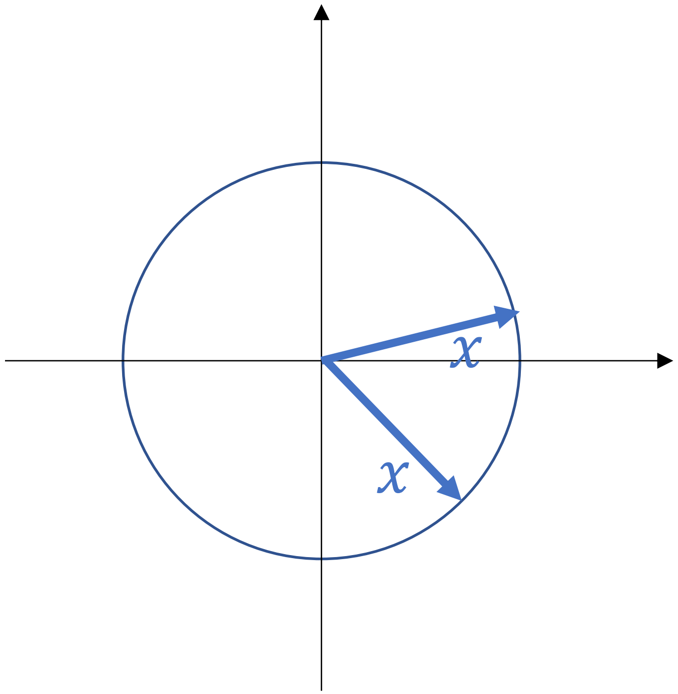
$$||\boldsymbol{A}|| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| \neq 0}} \frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||}$$

$$||\boldsymbol{A}|| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| = 1}} ||\boldsymbol{A}\boldsymbol{x}||$$

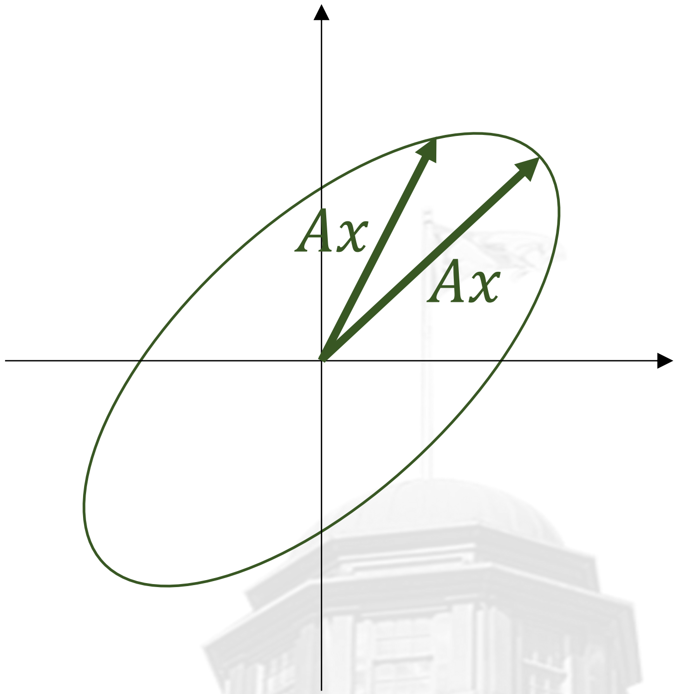
Induced Norm

$$\|\boldsymbol{A}\| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \|\boldsymbol{x}\| = 1}} \|\boldsymbol{A}\boldsymbol{x}\|$$

$\|\boldsymbol{x}\| = 1$



Ax



Induced Norm

$$\|\boldsymbol{A}\| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \|\boldsymbol{x}\| = 1}} \|\boldsymbol{A}\boldsymbol{x}\|$$

$\boldsymbol{B} = \boldsymbol{A}^*\boldsymbol{A}$ is Hermitian $\quad \boldsymbol{B} = \mathbf{V\Gamma V}^*$

$$\|\boldsymbol{A}\boldsymbol{x}\|^2 = (\boldsymbol{A}\boldsymbol{x})^*(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{x}^*\overbrace{\boldsymbol{A}^*\boldsymbol{A}}\boldsymbol{x} = \underbrace{\boldsymbol{x}^*\mathbf{V}}_{\boldsymbol{z}^*}\mathbf{\Gamma}\underbrace{\mathbf{V}^*\boldsymbol{x}}_{\boldsymbol{z} = \mathbf{V}^*\boldsymbol{x}} = \boldsymbol{z}^*\mathbf{\Gamma}\boldsymbol{z}$$

$$\|\boldsymbol{z}\| = 1$$

$$\|\boldsymbol{A}\boldsymbol{x}\|^2 = \langle \boldsymbol{x}, \boldsymbol{B}\boldsymbol{x} \rangle$$

Induced Norm

$$\|\boldsymbol{Ax}\|^2 = \mathbf{z}^*\boldsymbol{\Gamma}\mathbf{z} = [\bar{z}_1 \quad \bar{z}_2 \quad \cdots \quad \bar{z}_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Note: $\bar{z}_i z_i = |z_i|^2$

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$$

$$\|\boldsymbol{Ax}\|^2 = \mathbf{z}^*\boldsymbol{\Gamma}\mathbf{z} = \sum_{i=1}^{n} \lambda_i |z_i|^2$$

Recall: $\|\mathbf{z}\|^2 = \sum_{i=1}^{n} |z_i|^2 = 1$

Singular Values

$$\|\boldsymbol{Ax}\|^2 = \boldsymbol{z}^*\boldsymbol{\Gamma z} = [\bar{z}_1 \quad \bar{z}_2 \quad \cdots \quad \bar{z}_n]\begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Note: $\bar{z}_i z_i = |z_i|^2$

$$\sigma_1^2 > \sigma_2^2 > \cdots > \sigma_n^2 > 0$$

Singular Values $\sigma_i = \sqrt{\lambda_i}$

$$\|\boldsymbol{Ax}\|^2 = \boldsymbol{z}^*\boldsymbol{\Gamma z} = \sum_{i=1}^{n} \sigma_i^2 |z_i|^2$$

Recall: $\|\boldsymbol{z}\|^2 = \sum_{i=1}^{n} |z_i|^2 = 1$

Induced Norm

$$\|\boldsymbol{Ax}\|^2 = \boldsymbol{z}^*\boldsymbol{\Gamma}\boldsymbol{z} = \sum_{i=1}^{n} \sigma_i^2 |z_i|^2$$

$$\max_{\|\boldsymbol{z}\|=1} \|\boldsymbol{Ax}\|^2 = \sigma_1^2$$

Occurs when $\boldsymbol{z} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Spectral Norm $\|\boldsymbol{A}\| = \sigma_1$

σ_1 is the largest singular value of $\boldsymbol{A}$

Condition Number for $Ax = b$

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|A^{-1}b\|}$$

$$\frac{\|b\|}{\|A^{-1}b\|} = \frac{\|Ax\|}{\|x\|} < \|A\|$$

$$\kappa \le \underbrace{\|A^{-1}\|\|A\|}_{\text{Condition Number of matrix A } \kappa(A)}$$

Recall

$$\|\boldsymbol{A}\| \equiv \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ \|\boldsymbol{x}\| \neq 0}} \frac{\|\boldsymbol{Ax}\|}{\|\boldsymbol{x}\|}$$

Singular Values

The singular values σ_n of $\boldsymbol{A}$ are the square roots of the eigenvalues λ_n of $\boldsymbol{A}^*\boldsymbol{A}$

$(\boldsymbol{A}^{-1})^*$vs $(\boldsymbol{A}^*)^{-1} \rightarrow \boldsymbol{A}^{-*}$

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{U} \qquad\qquad \boldsymbol{A}^*(\boldsymbol{A}^*)^{-1} = \boldsymbol{U}$$

$$\boldsymbol{A}^*(\boldsymbol{A}^{-1})^* = \boldsymbol{U}$$

$$\boldsymbol{A}^*(\boldsymbol{A}^*)^{-1} = \boldsymbol{U}$$

$$\boldsymbol{A}^*(\boldsymbol{A}^{-1})^* = \boldsymbol{A}^*(\boldsymbol{A}^*)^{-1} = \boldsymbol{U}$$

$$(\boldsymbol{A}^{-1})^* = (\boldsymbol{A}^*)^{-1}$$

$$(\boldsymbol{A}^{-1})^* = (\boldsymbol{A}^*)^{-1} = \boldsymbol{A}^{-*}$$

Norm of $\boldsymbol{A}^{-1}$

- The singular values of $\boldsymbol{A}^{-1}$ are the square roots of the eigenvalues λ_n of $\boldsymbol{A}^{-*}\,\boldsymbol{A}^{-1}$
- Note: $(\boldsymbol{A}^{-1})^* = (\boldsymbol{A}^*)^{-1} = \boldsymbol{A}^{-*}$
- $\boldsymbol{A}^{-*}\,\boldsymbol{A}^{-1}$ and $\boldsymbol{A}^{-1}\boldsymbol{A}^{-*}$ are similar and thus share the same eigenvalues.
- $\boldsymbol{A}^{-1}\boldsymbol{A}^{-*} = (\boldsymbol{A}^*\boldsymbol{A})^{-1}$
- The singular values of $\boldsymbol{A}^{-1}$ are the inverse of the singular values of $\boldsymbol{A}$

Singular Values

- The singular values σ_i of $\boldsymbol{A}$ are the square roots of the eigenvalues λ_i of $\boldsymbol{A}^*\boldsymbol{A}$
- $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$
- The singular values of $\boldsymbol{A}^{-1}$ are $1/\sigma_i$
- $\|\boldsymbol{A}\| = \sigma_1$ (largest singular value of $\boldsymbol{A}$)
- $\|\boldsymbol{A}^{-1}\| = 1/\sigma_n$ (largest singular value of $\boldsymbol{A}^{-1}$)

Condition Number for matrix A

$$\text{cond}(A) = \|A^{-1}\|\|A\| = \underbrace{\frac{\sigma_1}{\sigma_n}}_{\text{Ratio of largest and smallest singular values of } A}$$

Condition Number – Singular Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 4 & -4 \end{bmatrix} \longrightarrow \text{cond}(A) = 4$$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 3.9 \end{bmatrix} \longrightarrow \text{cond}(A) = 332.1$$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 3.99 \end{bmatrix} \longrightarrow \text{cond}(A) = 3{,}392$$

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 3.999 \end{bmatrix} \longrightarrow \text{cond}(A) = 33{,}992$$

Large condition number means the matrix is close to singular

Determinant – Singular Matrix

- If $\boldsymbol{A}$ is singular, then $\det(\boldsymbol{A}) = 0$.
- Is $\boldsymbol{A}$ close to singular when $\det(\boldsymbol{A}) \rightarrow 0$?

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 3.999 \end{bmatrix} \Longrightarrow \det(A) = 0.001$$

$$A = \begin{bmatrix} 0.1 & & & \\ & 0.1 & & \\ & & 0.1 & \\ & & & 0.1 \end{bmatrix} \Longrightarrow \begin{array}{l} \det(A) = 0.0001 \\ \mathrm{cond}(A) = 1 \end{array}$$