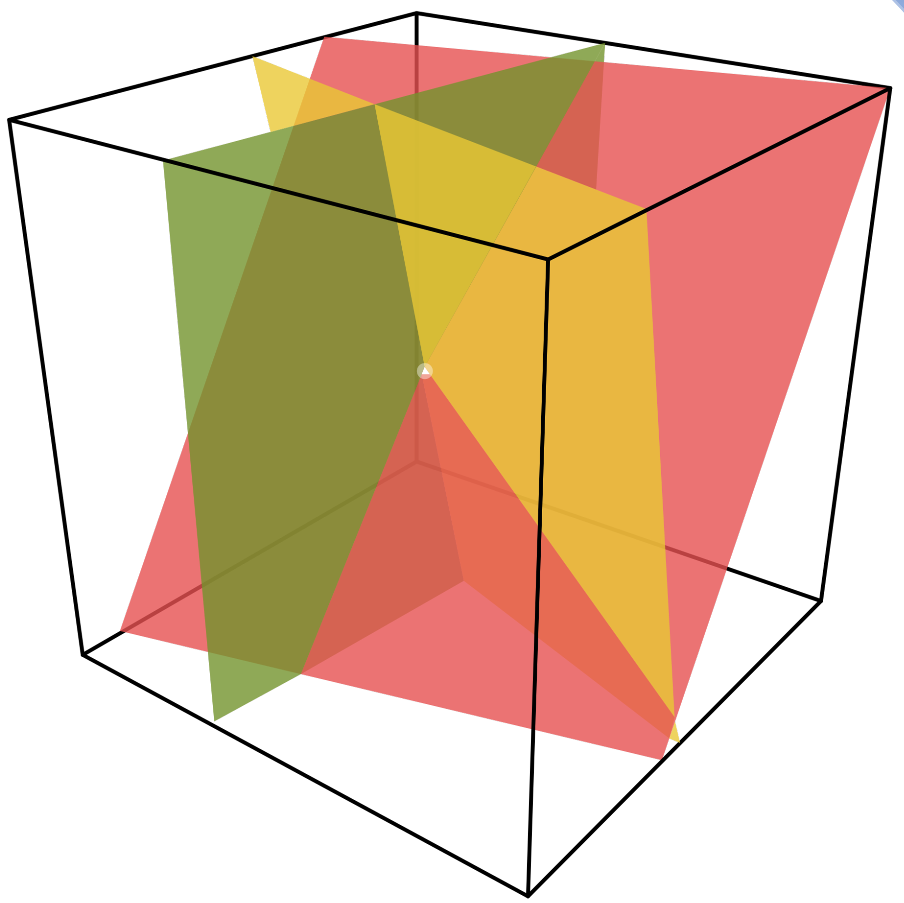
# ECSE 343 Numerical Methods in Engineering

Roni Khazaka

Dept. of Electrical and Computer Engineering

McGill University



# Systems of Linear Equations / Gaussian Elimination


Copyright ©2021 R. Khazaka

# General Case

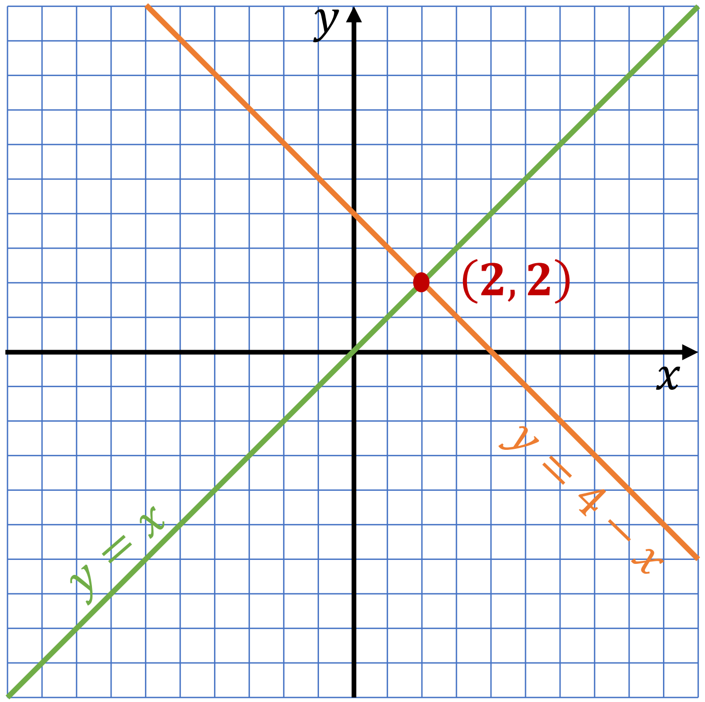
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

$$\boldsymbol{Ax} = \boldsymbol{b} \qquad \boldsymbol{A} \in \mathbb{R}^{n \times n}$$

$$\boldsymbol{b} \in \mathbb{R}^{n}$$

# Geometric Interpretation: Unique solution

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{cases} x + y = 4 \\ x - y = 0 \end{cases}$$



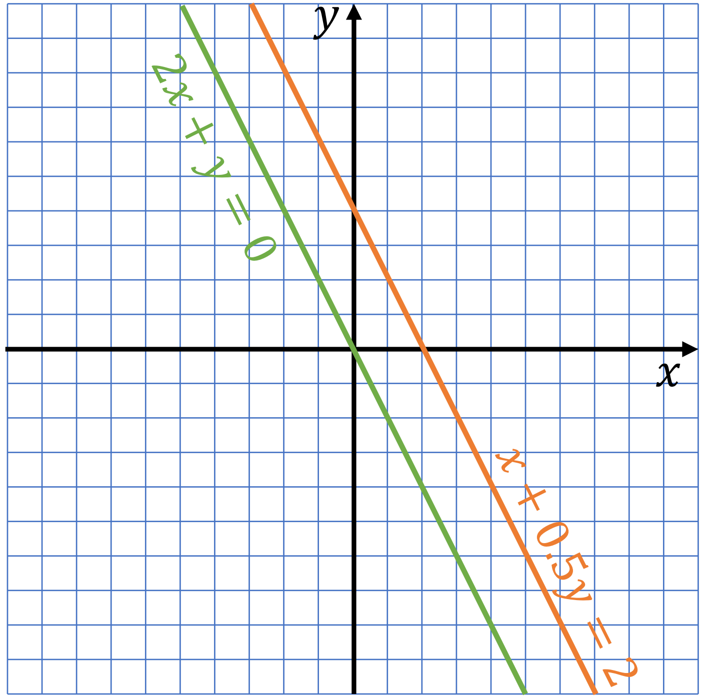

Unique solution:

$$\begin{cases} x = 2 \\ y = 2 \end{cases}$$

Copyright ©2021 R. Khazaka

# Geometric Interpretation: No solution

$$\begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \iff \begin{cases} x + 0.5y = 2 \\ 2x + y = 0 \end{cases}$$

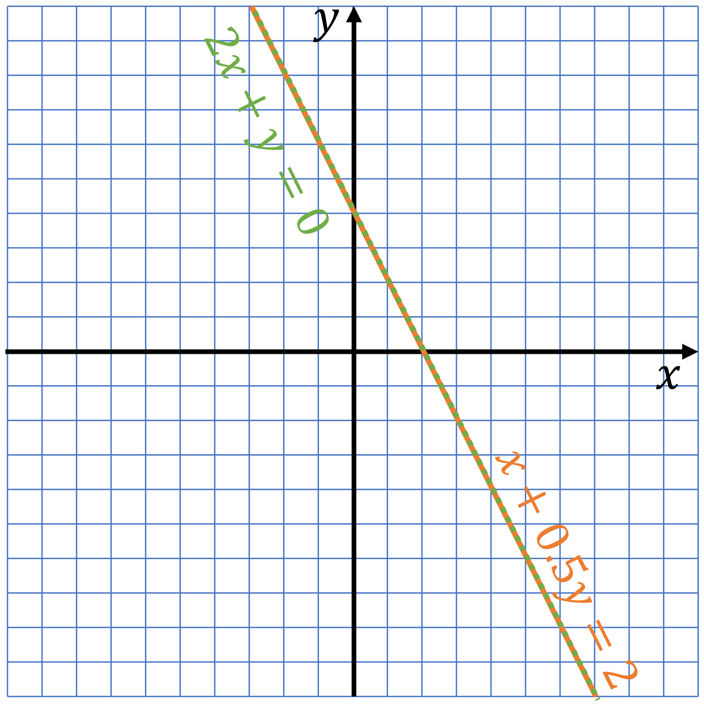



No solution

Copyright ©2021 R. Khazaka

# Geometric Interpretation: Many solutions

$$\begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{cases} x + 0.5y = 2 \\ 2x + y = 4 \end{cases}$$




Many solutions

Copyright ©2021 R. Khazaka

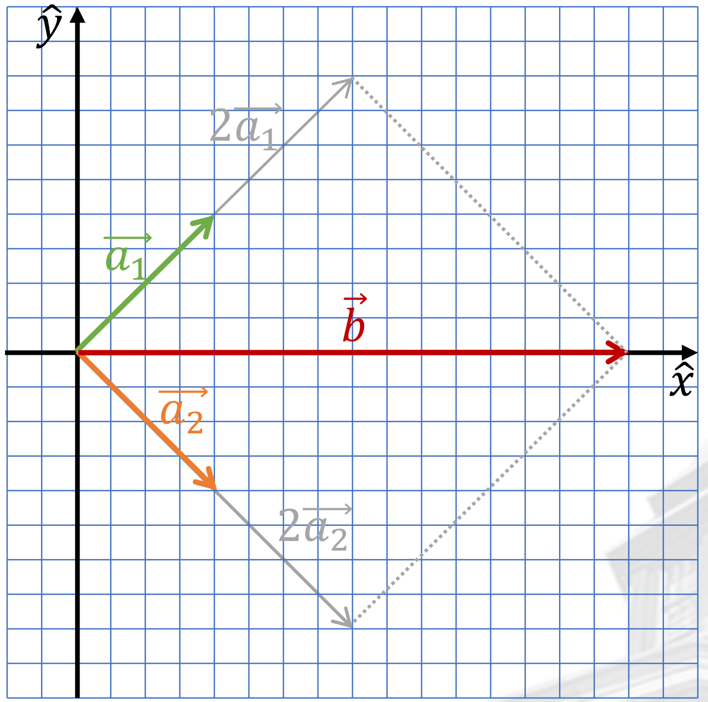
# Interpretation using Vector Space: Unique Solution

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\updownarrow$$

$$\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\overrightarrow{a_1}} x + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\overrightarrow{a_2}} y = \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{\vec{b}}$$

Unique solution: $\begin{cases} x = 2 \\ y = 2 \end{cases}$



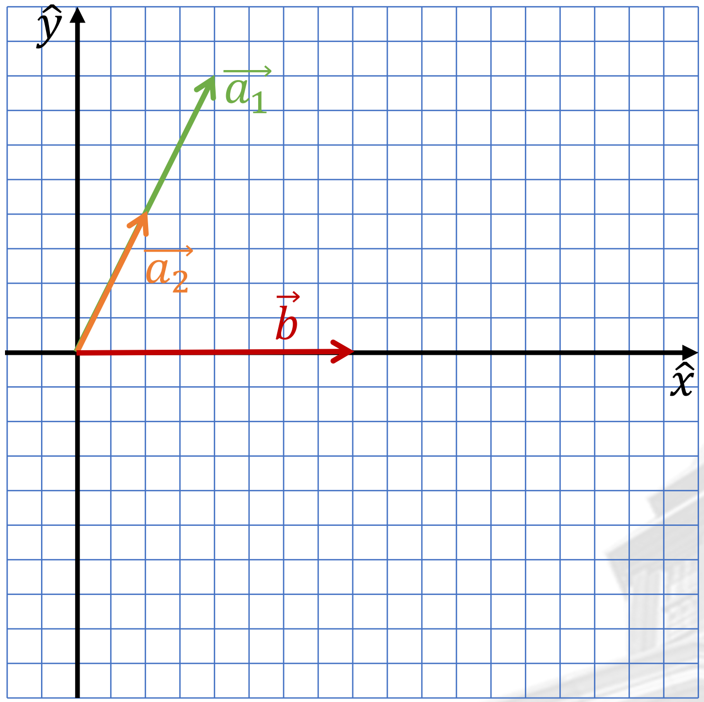

Copyright ©2021 R. Khazaka

# Interpretation using Vector Space: No Solution

$$\begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

No solution

$$\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\overrightarrow{a_1}} x + \underbrace{\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}}_{\overrightarrow{a_2}} y = \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{\vec{b}}$$



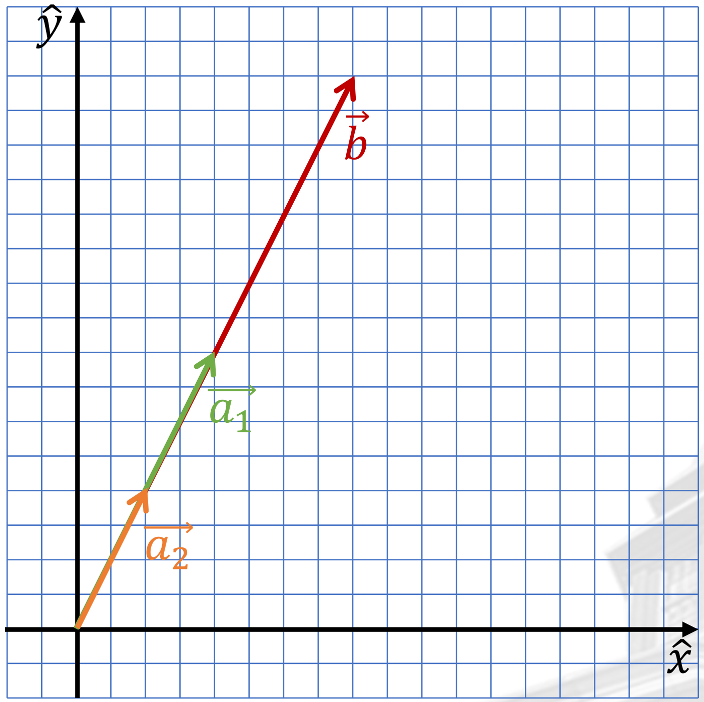
# Interpretation using Vector Space: Infinitely many Solutions

$$\begin{bmatrix} 1 & 0.5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

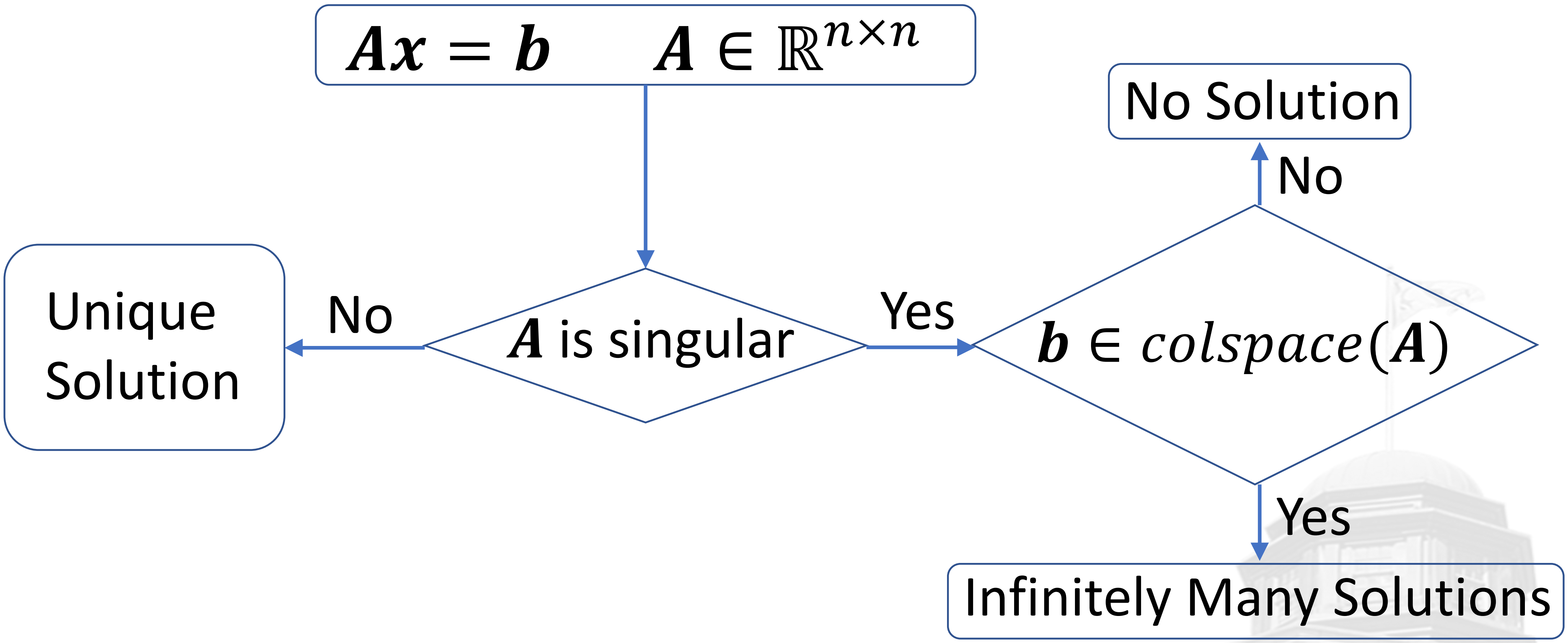
➡ <u>Infinitely many solution</u>

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\overrightarrow{a_1}} x + \underbrace{\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}}_{\overrightarrow{a_2}} y = \underbrace{\begin{bmatrix} 2 \\ 4 \end{bmatrix}}_{\vec{b}}$$



# Interpretation using Vector Space



# Solution using Matrix Inverse

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ $\quad$ $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ $\quad$ $\boldsymbol{A}$ is full rank

$$\Rightarrow \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Copyright ©2021 R. Khazaka

# Matrix Inverse

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\Downarrow$$

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

- Numerical complexity is $O(n^3)$.
- The Matrix $\boldsymbol{A}^{-1}$ is **dense** even when $\boldsymbol{A}$ is sparse.
- This approach is useful for theoretical considerations.
- Matrix inversion is not typically used in practical applications.

# Overdetermined System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left.\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}\right\}$$

- 3 degrees of freedom (unknowns)
- 5 constraints (equations)
- Solution may or may not exist.

Least squares method: Can find approximate solution by minimizing the residual: $||\boldsymbol{Ax} - \boldsymbol{b}||$

$$\boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\,\boldsymbol{A}^T\boldsymbol{b}$$

# Underdetermined System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- 5 degrees of freedom
- 3 constraints
- Infinitely many solutions.

Restate the problem as that of finding the smallest $\boldsymbol{x}$ (minimum $||\boldsymbol{x}||$) that satisfies the system of equations. This is obtained using the Pseudo-inverse method:

$$\boldsymbol{x} = \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\,\boldsymbol{b}$$

# Stability of the LU based approach

$$\begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Use infinite precision arithmetics

$$\begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10^{-4} & 1 \\ 10^{4} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10^{-4} & 1 \\ 10^{4} & -9999 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 10^{4} & 1 \end{bmatrix} \begin{bmatrix} 10^{-4} & 1 \\ 0 & -9999 \end{bmatrix}$$

Copyright ©2017 R. Khazaka

# Infinite Precision Arithmetics

$$\begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix} \begin{bmatrix} 10^{-4} & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} y_1 = 1 \\ y_2 = 2 - 10^4 = -9998 \end{cases}$$

$$\begin{bmatrix} 10^{-4} & 1 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9998 \end{bmatrix} \Rightarrow \begin{cases} x_2 = 0.9999 \\ x_1 = 1.0001 \end{cases}$$

# Results using Finite Precision Arithmetics

$$\begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

➔ Use computer that can only store three significant digits:

$$\begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10^{-4} & 1 \\ 10^{4} & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 10^{-4} & 1 \\ 10^{4} & -9999 \end{bmatrix} \Rightarrow \begin{bmatrix} 10^{-4} & 1 \\ 10^{4} & -10^{4} \end{bmatrix}$$

Rounding error of 1

$$\underbrace{\begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix}}_{\boldsymbol{A}} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 10^{4} & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 10^{-4} & 1 \\ 0 & -10^{4} \end{bmatrix}}_{\boldsymbol{U}} = \begin{bmatrix} 10^{-4} & 1 \\ 1 & 0 \end{bmatrix}$$

# Forward Backward Substitution

Finite precision LU:
$$\begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix}\begin{bmatrix} 10^{-4} & 1 \\ 0 & -10^4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 10^4 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \begin{aligned} y_1 &= 1 \\ y_2 &= 2 - 10^4 = -9998 = -10^4 \end{aligned}$$

$$\begin{bmatrix} 10^{-4} & 1 \\ 0 & -10^4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -10^4 \end{bmatrix} \qquad \left\{ \begin{aligned} x_2 &= 1 \\ \boxed{x_1 = 0} \end{aligned} \right.$$

→ Very far from accurate answer.

# Reorder Equations

$$\begin{bmatrix} 10^{-4} & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 10^{-4} \end{bmatrix}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Type equation here.

LU Decomposition:

$$\begin{bmatrix} 1 & 1 \\ 1 & 10^{-4} \end{bmatrix} \xrightarrow[\text{Divide by pivot}]{} \begin{bmatrix} 1 & 1 \\ 1 & 10^{-4} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 10^{-4} - 1\times 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -0.9999 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Rounding error of $10^{-4}$

# LU results

$$\begin{bmatrix} 1 & 1 \\ 1 & 10^{-4} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}_{\boldsymbol{U}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 10^{-4} \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

# Forward Backward Substitution

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$$

→ Only a small error compared to exact results