ECSE 343 Numerical Methods in Engineering

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Vector Space

Definition: A Vector Space P is a set containing vectors $\boldsymbol{v} \in P$ for which we define the two operations: **1) Scalar multiplication** and **2) Addition**.

- Scalar Multiplication: $a\boldsymbol{v} \to \boldsymbol{u}$

$$\left.\begin{array}{l} a \in \mathbb{R} \\ \boldsymbol{v} \in V \end{array}\right\} \quad \boldsymbol{u} \in V$$

- Addition: $\boldsymbol{u} + \boldsymbol{v} \to \boldsymbol{w}$

$$\left.\begin{array}{l} \boldsymbol{v} \in V \\ \boldsymbol{u} \in V \end{array}\right\} \quad \boldsymbol{w} \in V$$

Axioms (Addition)

- Axiom #1: Associativity

$$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} \qquad \forall \boldsymbol{u} \in P \quad \boldsymbol{v} \in P \quad \boldsymbol{w} \in P$$

- Axiom #2: Commutativity

$$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u} \qquad \forall \boldsymbol{u} \in P \quad \boldsymbol{v} \in P$$

- Axiom #3: Existence of “zero” vector $\mathbf{0} \in P$

$$\mathbf{0} + \boldsymbol{v} = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in P$$

- Axiom #4: Existence of “inverse” vector

$$\forall \boldsymbol{v} \in P \quad \exists \boldsymbol{u} = -\boldsymbol{v} \in P \text{ such that } \boldsymbol{v} + \boldsymbol{u} = \boldsymbol{v} + (-\boldsymbol{v}) = \mathbf{0}$$

Axioms (multiplication)

- Axiom #5:

$$1\boldsymbol{v} = \boldsymbol{v} \qquad 1 \in \mathbb{R} \quad \boldsymbol{v} \in P$$

- "Axiom #6":

$$0\boldsymbol{v} = \mathbf{0} \qquad 0 \in \mathbb{R} \quad \boldsymbol{v} \in P \quad \mathbf{0} \in P$$

  Technically not an axiom (can be shown using other axioms)

- Axiom #7: Associativity of Scalar Multiplication

$$a(b\boldsymbol{v}) = (ab)\boldsymbol{v} \qquad a \in \mathbb{R} \quad b \in \mathbb{R} \quad \boldsymbol{v} \in P$$

Distributivity

Consider: $a \in \mathbb{R} \quad b \in \mathbb{R}$ and $\boldsymbol{u} \in P \quad \boldsymbol{v} \in P$

- Axiom 8

$$a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$$

- Axiom 9

$$(a + b)\boldsymbol{u} = a\boldsymbol{u} + b\boldsymbol{u}$$

Function Space

- A set of functions: $x \mapsto f(x)$ between two sets $\mathcal{D}$ and $\mathcal{F}$
- In General: $x \in \mathcal{D} \mapsto f(x) \in \mathcal{F}$
- Example 1: $x \in \mathbb{R} \mapsto f(x) \in \mathbb{R}$
- Example 2: $x \in [-1,1] \mapsto f(x) \in \mathbb{R}$

Vector Space (Applied to Function Spaces)

Definition: A Vector Space P is a set containing vectors $\boldsymbol{f} \in P$ for which we define the two operations: **1) Scalar multiplication** and **2) Addition**.

- Consider the function space: $x \in [-1,1] \mapsto f(x) \in \mathbb{R}$
- Scalar Multiplication: $a\boldsymbol{f} \to \boldsymbol{g}$

$$\left.\begin{array}{l} a \in \mathbb{R} \\ \boldsymbol{f} \in P \end{array}\right\} \quad (a\boldsymbol{f})(x) = a \times \boldsymbol{f}(x) = \boldsymbol{g}(x) \in P$$

- Addition: $\boldsymbol{f} + \boldsymbol{g} \to \boldsymbol{u}$

$$\left.\begin{array}{l} \boldsymbol{f} \in P \\ \boldsymbol{g} \in P \end{array}\right\} \quad (\boldsymbol{g} + \boldsymbol{f})(x) = \boldsymbol{g}(x) + \boldsymbol{f}(x) = \boldsymbol{u}(x) \in P$$

Inner Product of Two Vectors

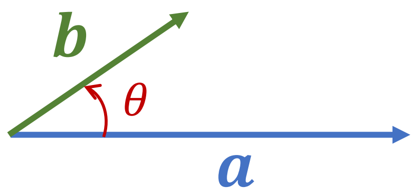
- $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$
- $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
- $\langle c\boldsymbol{u}, \boldsymbol{v} \rangle = \bar{c}\langle \boldsymbol{u}, \boldsymbol{v} \rangle$
- $\langle \boldsymbol{u}, c\boldsymbol{v} \rangle = c\langle \boldsymbol{u}, \boldsymbol{v} \rangle$

c is a scalar $c \in \mathbb{C}$

- $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = \boldsymbol{0} \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$
- $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > \boldsymbol{0}$ if $\boldsymbol{u} \neq \boldsymbol{0}$

Euclidian Inner Product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v} \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{\boldsymbol{n}}$$



$$\langle a, b \rangle = |a||b|\cos(\theta)$$

Functional Inner Product

- $x \mapsto f(x)$ between two sets $\mathcal{D}$ and $\mathcal{F}$; $\boldsymbol{f} \in P$
- $x \mapsto g(x)$ between two sets $\mathcal{D}$ and $\mathcal{F}$; $\boldsymbol{g} \in P$

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_{\mathcal{D}} f(x) g(x) dx$$

Functional Norm

- $x \mapsto f(x)$ between two sets $\mathcal{D}$ and $\mathcal{F}$; $\boldsymbol{f} \in P$

$$\|\boldsymbol{f}\| = \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle}$$

Example Basis: Taylor Expansion

Let P be the set of functions: $x \mapsto f(x)$ between two sets $[-1,1]$ and $\mathbb{R}$ whose Taylor expansion converges in $[-1,1]$

$$f(x) = a_0 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$$

Monomials
Basis Functions

Example Basis: Taylor Expansion

Let P be the set of functions: $x \mapsto f(x)$ between two sets $[-1,1]$ and $\mathbb{R}$ whose Taylor expansion converges in $[-1,1]$

$$f(x) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots$$

$$g(x) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + \cdots$$

$$(f+g)(x) = (a_0 + b_0) + (a_1 + b_1)t +$$

$$+(a_2 + b_2)t^2 + (a_3 + b_3)t^3 + \cdots$$

Example Hermite Polynomials

Let P be the set of functions: $x \mapsto f(x)$ between $\mathbb{R}$ and $\mathbb{R}$

$$f(x) = a_0 H_o(x) + a_1 H_1(x) + a_2 H_2(x) + a_3 H_3(x) + \cdots$$

Hermite Polynomials as
Basis Functions

Fourier Series

Let P be the set of functions, with period T, $x \mapsto f(x)$ between $\mathbb{R}$ and $\mathbb{R}$

Let $\omega_o = \frac{2\pi}{T}$

$$f(t) = a_0 + a_1 \cos \omega_o t + b_1 \sin \omega_o t +$$

$$+ a_2 \cos 2\omega_o t + b_2 \sin 2\omega_o t + a_3 \cos 3\omega_o t + b_3 \sin 3\omega_o t +$$

$$f(x) = a_0 + \sum_k (a_k \cos k\omega_o t + b_{\mathrm{k}} \sin k\omega_o t)$$

Fourier Series

Let P be the set of functions, with period T, $x \mapsto f(x)$ between $\mathbb{R}$ and $\mathbb{R}$

Let $\omega_o = \frac{2\pi}{T}$

$$f(t) = a_0 1 + \sum_k (a_k \cos k\omega_o t + b_\mathrm{k} \sin k\omega_o t)$$

Orthogonal Basis Functions

Functional Inner Product

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \int_0^T f(t)g(t)dx$$

Orthogonality

$$\int_0^T 1\times\cos(k\omega_0 t)dt = 0$$

$$T = \frac{2\pi}{\omega_o}$$

$$\int_0^T 1\times\sin(k\omega_0 t)dt = 0$$

Orthogonality

$$\int_0^T \cos(k\omega_0 t)\cos(l\omega_0 t)dt = 0 \qquad\qquad k \neq l$$

$$\int_0^T \sin(k\omega_0 t)\sin(l\omega_0 t)dt = 0 \qquad\qquad k \neq l$$

$$\int_0^T \sin(k\omega_0 t)\cos(l\omega_0 t)dt = 0$$

Orthogonality

$$\int_0^T \cos(k\omega_0 t)\cos(k\omega_0 t)dt = \pi$$

$$\int_0^T \sin(k\omega_0 t)\sin(kt)dt = \pi$$

Fourier Series

$$f(t) = a_0 1 + \sum_k (a_k \cos k\omega_o t + b_k \sin k\omega_o t)$$

$$\int_0^T f(t)dt = \int_0^T \left[a_0 1 + \sum_k (a_k \cos k\omega_o t + b_k \sin k\omega_o t) \right] dt$$

$$= T a_0$$

$$\boxed{a_0 = \frac{1}{T}\int_0^T f(t)dt}$$

Fourier Series

$$f(t) = a_0 1 + \sum_k (a_k \cos k\omega_o t + b_\mathrm{k} \sin k\omega_o t)$$

$$\int_0^T f(t) \cos(k\omega_0 t)\, dt =$$

$$= \int_0^T \cos(k\omega_0 t) \left[a_0 + \sum_k (a_k \cos k\omega_o t + b_\mathrm{k} \sin k\omega_o t) \right] dt$$

Fourier Series

$$\int_0^T f(t)\cos(k\omega_0 t)\,dt =$$

$$= \int_0^T \cos(k\omega_0 t)\left[a_0 + \sum_k (a_k \cos k\omega_o t + b_\mathrm{k} \sin k\omega_o t)\right]dt$$

$$= \int_0^T a_k \cos(k\omega_0 t)\cos(k\omega_0 t)\,dt = \pi a_k$$

Fourier Series

$$a_k = \frac{1}{\pi}\int_0^T f(t)\cos(k\omega_0 t)\,dt = \frac{1}{\pi}\langle f(t), \cos(k\omega_0 t)\rangle$$

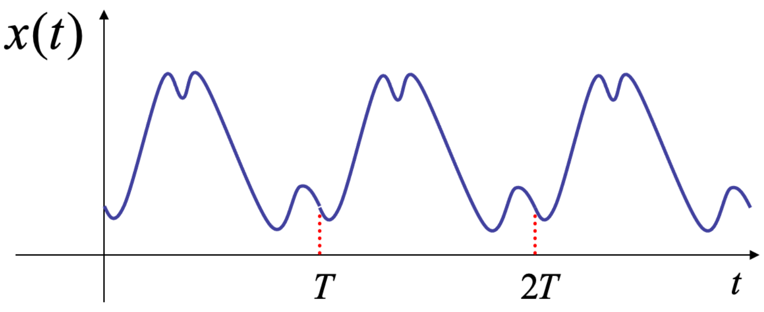
$$b_k = \frac{1}{\pi}\int_0^T f(t)\sin(k\omega_0 t)\,dt = \frac{1}{\pi}\langle f(t), \sin(k\omega_0 t)\rangle$$

Fourier Series

$$a_k = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(k\omega_0 t)\,dt = \frac{1}{\pi}\langle f(t), \cos(k\omega_0 t)\rangle$$

$$b_k = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(k\omega_0 t)\,dt = \frac{1}{\pi}\langle f(t), \sin(k\omega_0 t)\rangle$$

Discrete Fourier Transform (DFT)

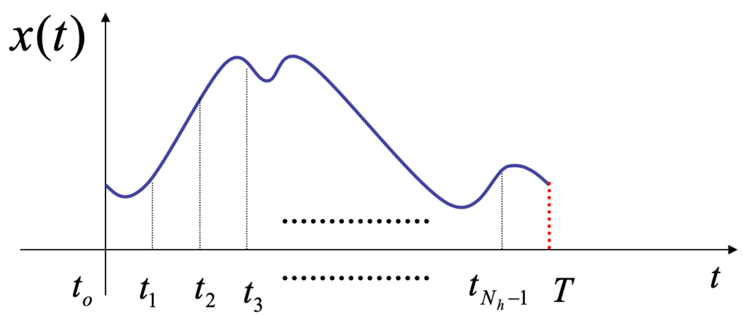


Periodic signal with a period T

$$x(t) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t) + b_k \sin(k\omega t) \right)$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Discrete Fourier Transform



Divide the period [0,T) into N_h equally spaced time points:

$$[t_o, t_1, t_2, \ldots, t_{N_h-1}] \qquad N_h = 2H+1$$

$$t_n = n\frac{T}{N_h} \quad ; \quad n = 0,1,2,\ldots,N_h-1$$

Discrete Fourier Transform

$$x(t) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t) + b_k \sin(k\omega t) \right)$$

Evaluate the Fourier series at the N_h equally spaced time points in the interval [0,T):

$$[t_o, t_1, t_2, \ldots, t_{N_h - 1}]$$

$$x(t_o) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_o) + b_k \sin(k\omega t_o) \right)$$

$$x(t_1) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_1) + b_k \sin(k\omega t_1) \right)$$

Discrete Fourier Transform

$$x(t_o) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_o) + b_k \sin(k\omega t_o) \right)$$

$$x(t_1) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_1) + b_k \sin(k\omega t_1) \right)$$

$$\vdots$$

$$x(t_{N_h-1}) = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_{N_h-1}) + b_k \sin(k\omega t_{N_h-1}) \right)$$

Discrete Fourier Transform

$$x_o = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_o) + b_k \sin(k\omega t_o) \right)$$

$$x_1 = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_2) + b_k \sin(k\omega t_2) \right)$$

$$\vdots$$

$$x_{N_h-1} = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_{N_h-1}) + b_k \sin(k\omega t_{N_h-1}) \right)$$

Discrete Fourier Transform

$$x_o = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_o) + b_k \sin(k\omega t_o) \right)$$

$$x_1 = a_o + \sum_{k=1}^{H} \left(a_k \cos(k\omega t_1) + b_k \sin(k\omega t_1) \right)$$

$$\begin{bmatrix} x_o \\ x_1 \\ \vdots \\ x_{N_h-1} \end{bmatrix} = \begin{bmatrix} 1 & \cos(\omega t_o) & \sin(\omega t_o) & \cdots & \cos(H\omega t_o) & \sin(H\omega t_o) \\ 1 & \cos(\omega t_1) & \sin(\omega t_1) & \cdots & \cos(H\omega t_1) & \sin(H\omega t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\omega t_{N_h-1}) & \sin(\omega t_{N_h-1}) & \cdots & \cos(H\omega t_{N_h-1}) & \sin(H\omega t_{N_h-1}) \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ b_1 \\ \vdots \\ a_H \\ b_H \end{bmatrix}$$

Discrete Fourier Transform

Discrete Fourier Transform

$$
\begin{bmatrix} x_o \\ x_1 \\ \vdots \\ x_{N_h-1} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} & \overbrace{\hphantom{\cos(\omega t_{N_h-1}) \quad \sin(\omega t_{N_h-1})}}^{k=1} & & \overbrace{\hphantom{\cos(H\omega t_{N_h-1}) \quad \sin(H\omega t_{N_h-1})}}^{k=H} \end{array} \\
\begin{bmatrix}
1 & \cos(\omega t_o) & \sin(\omega t_o) & \cdots & \cos(H\omega t_o) & \sin(H\omega t_o) \\
1 & \cos(\omega t_1) & \sin(\omega t_1) & \cdots & \cos(H\omega t_1) & \sin(H\omega t_1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cos(\omega t_{N_h-1}) & \sin(\omega t_{N_h-1}) & \cdots & \cos(H\omega t_{N_h-1}) & \sin(H\omega t_{N_h-1})
\end{bmatrix}
\end{array}
\begin{bmatrix} a_o \\ a_1 \\ b_1 \\ \vdots \\ a_H \\ b_H \end{bmatrix}
$$

$$
k\omega t_n = k\left(\frac{2\pi}{T}\right)\left(n\frac{T}{N_h}\right) = kn\left(\frac{2\pi}{N_h}\right)
$$

Discrete Fourier Transform

$$\overset{X_s}{\begin{bmatrix} x_o \\ x_1 \\ \vdots \\ x_{N_h-1} \end{bmatrix}} = \overset{\Gamma}{\begin{bmatrix} 1 & \cos(\Theta_{0,1}) & \sin(\Theta_{0,1}) & \cdots & \cos(\Theta_{0,H}) & \sin(\Theta_{0,H}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\Theta_{n,1}) & \sin(\Theta_{n,1}) & \cdots & \cos(\Theta_{n,H}) & \sin(\Theta_{n,H}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(\Theta_{N_h-1,1}) & \sin(\Theta_{N_h-1,1}) & \cdots & \cos(\Theta_{N_h-1,H}) & \sin(\Theta_{N_h-1,H}) \end{bmatrix}} \overset{X_h}{\begin{bmatrix} a_o \\ a_1 \\ b_1 \\ \vdots \\ a_H \\ b_H \end{bmatrix}}$$

$$\Theta_{n,k} = kn\left(\frac{2\pi}{N_h}\right)$$

Discrete Fourier Transform

$$X_h = \Gamma^{-1} X_s$$ ➡ Discrete Fourier Transform (DFT)

$$X_s = \Gamma X_h$$ ➡ Inverse Discrete Fourier Transform (IDFT)

- ➔ Γ is only a function of the number of samples
- ➔ Cost of DFT is $O(n^2)$.
- ➔ FFT is numerically equivalent to DFT (for n power of 2).
- ➔ Cost of FFT is O(n*ln(n))