ECSE 343 Numerical Methods in Engineering

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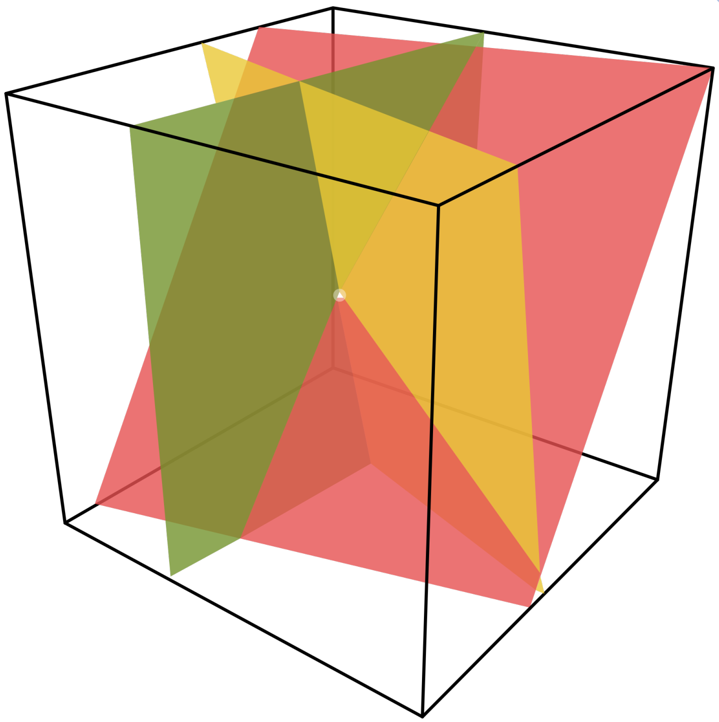


Systems of Linear Equations / Gaussian Elimination

Systems of Linear Equations

$$\left.\begin{aligned} 2x_1 + 1x_2 - 2x_3 &= 2 \\ 4x_1 + 5x_2 - 3x_3 &= 11 \\ 6x_1 + 9x_2 - 2x_3 &= 22 \end{aligned}\right\}$$

3 Linear Equations
3 unknowns

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

Matrix Format

$$\begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2x_1 + 1x_2 - 2x_3 = 2$$

System of Linear Equations

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}}_{\boldsymbol{b}}$$

$$\boxed{\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}} \qquad \boldsymbol{A} \in \mathbb{R}^{n\times n} \qquad \boldsymbol{b} \in \mathbb{R}^{n} \qquad \boldsymbol{x} \in \mathbb{R}^{n}$$

Solution using Matrix Inverse

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\Downarrow$$

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

- ✓ Once $\boldsymbol{A}^{-1}$ is computed, the system can be easily solved for different right-hand side (RHS) vectors $\boldsymbol{b}$.
  - Numerical complexity is $O(n^3)$ even when $\boldsymbol{A}$ is sparse.
  - The Matrix $\boldsymbol{A}^{-1}$ is in general **dense** even when $\boldsymbol{A}$ is sparse.
  - Solving using matrix inversion is not as numerically stable as other methods.
- ✓ This approach is useful for theoretical considerations.

Solution using Gaussian Elimination

Gaussian Elimination: Overview

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

Difficult to solve

Row operations

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

- Upper-triangular
- Can be easily solved using back-substitution

Gaussian Elimination

$$\begin{cases} 2x_1 + 1x_2 - 2x_3 = 2 & (1) \\ 4x_1 + 5x_2 - 3x_3 = 11 & (2) \\ 6x_1 + 9x_2 - 2x_3 = 22 & (3) \end{cases}$$

$$\begin{cases} 2x_1 + 1x_2 - 2x_3 = 2 & (1) \\ 0x_1 + 3x_2 + 1x_3 = 7 & (2) \; - 2 \times (1) \\ 0x_1 + 6x_2 + 4x_3 = 16 & (3) \; - 3 \times (1) \end{cases}$$

Gaussian Elimination

$$\begin{cases} 2x_1 + 1x_2 - 2x_3 = 2 & \text{①} \\ 0x_1 + 3x_2 + 1x_3 = 7 & \text{②} \\ 0x_1 + 6x_2 + 4x_3 = 16 & \text{③} \end{cases}$$

$$\begin{cases} 2x_1 + 1x_2 - 2x_3 = 2 & \text{①} \\ 0x_1 + 3x_2 + 1x_3 = 7 & \text{②} \\ 0x_1 + 0x_2 + 2x_3 = 2 & \text{③} - 2 \times \text{②} \end{cases}$$

Back Substitution

$$\begin{cases} 2x_1 + 1x_2 - 2x_3 = 2 & (1) \\ 0x_1 + 3x_2 + 1x_3 = 7 & (2) \\ 0x_1 + 0x_2 + 2x_3 = 2 & (3) \end{cases}$$

$$\begin{cases} (3) \Rightarrow 2x_3 = 2 \Rightarrow x_3 = \dfrac{2}{2} = 1 \\ (2) \Rightarrow 3x_2 = 7 - x_3 \Rightarrow x_2 = \dfrac{7-1}{3} = 2 \\ (1) \Rightarrow 2x_1 = 2 - x_2 + 2x_3 \Rightarrow x_1 = \dfrac{2-2+2}{2} = 1 \end{cases}$$

Gaussian Elimination: Matrix Format

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 22 \end{bmatrix} \quad row\ 2 - 2 \times row\ 1$$

The same row operations are done on the matrix A and the RHS vector b.

Gaussian Elimination: Augmented Matrix

$$\begin{cases} 2x_1 + 1x_2 - 2x_3 = 2 & (1) \\ 4x_1 + 5x_2 - 3x_3 = 11 & (2) \\ 6x_1 + 9x_2 - 2x_3 = 22 & (3) \end{cases}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix} \Rightarrow \overbrace{\begin{bmatrix} 2 & 1 & -2 & 2 \\ 4 & 5 & -3 & 11 \\ 6 & 9 & -2 & 22 \end{bmatrix}}^{\text{Augmented Matrix}}$$

Gaussian Elimination: Augmented Matrix Format

Pivot

$$\begin{bmatrix} 2 & 1 & -2 & 2 \\ 4 & 5 & -3 & 11 \\ 6 & 9 & -2 & 22 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -2 & 2 \\ 0 & 3 & +1 & 7 \\ 6 & 9 & -2 & 22 \end{bmatrix} \quad (row\ 2) - 2\times(row\ 1)$$

$\frac{4}{2}$ ← Pivot

Pivot

$$\begin{bmatrix} 2 & 1 & -2 & 2 \\ 0 & 3 & +1 & 7 \\ 6 & 9 & -2 & 22 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -2 & 2 \\ 0 & 3 & +1 & 7 \\ 0 & 6 & +4 & 16 \end{bmatrix} \quad (row\ 3) - 3\times(row\ 1)$$

$\frac{6}{2}$ ← Pivot

Gaussian Elimination: Augmented Matrix Format

Pivot

$$\begin{bmatrix} 2 & 1 & -2 & 2 \\ 0 & 3 & +1 & 7 \\ 0 & 6 & +4 & 16 \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} 2 & 1 & -2 & 2 \\ 0 & 3 & +1 & 7 \\ 0 & 0 & +2 & 2 \end{bmatrix}}_{\text{Augmented matrix in row echelon form.}}$$

$\frac{6}{3}$ ⟵ Pivot

$(row\ 3) - 2 \times (row\ 2)$

$$\begin{bmatrix} 2 & 1 & -2 & 2 \\ 0 & 3 & +1 & 7 \\ 0 & 0 & +2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$$

Back Substitution

$$\underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}}_{\text{Upper triangular}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$$

$$2x_3 = 2 \Rightarrow x_3 = \frac{2}{2} = 1$$

$$3x_2 = 7 - x_3 \Rightarrow x_2 = \frac{7-1}{3} = 2$$

$$2x_1 = 2 - x_2 + 2x_3 \Rightarrow x_1 = \frac{2-2+2}{2} = 1$$

Gaussian Elimination

- ✓ Systematic Algorithm: Can be easily implemented as a computer program.
- ✓ Accurate.
- ✓ Can be much faster that matrix inversion for sparse systems.

Row operations are preformed on the augmented matrix (matrix A and RHS vector b simultaneously)

→ Must restart the process from the beginning if we need to solve the system for a different RHS.

Solution using LU Decomposition

LU Factorization

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Solution Using LU Decomposition

- Consider the system: $\boldsymbol{Ax = b}$
- Decompose $\boldsymbol{A}$ such that: $\boldsymbol{A = LU}$
  - $\boldsymbol{L}$ is a lower triangular matrix.
  - $\boldsymbol{U}$ is an upper triangular matrix.

$$\boldsymbol{L}\overbrace{(\boldsymbol{Ux})}^{\boldsymbol{y}} = \boldsymbol{b}$$

- We can now express the equation as: $\boldsymbol{LUx = b}$
  - Solve $\boldsymbol{Ly = b}$ for $\boldsymbol{y}$ using forward substitution.
  - Solve $\boldsymbol{Ux = y}$ for $\boldsymbol{x}$ using back substitution.

Example: Solution using LU

$$\underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

Same upper triangular matrix we obtained using Gaussian Elimination

Step #1: Decompose $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{LU}$

$$\underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix}}_{\boldsymbol{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}}_{\boldsymbol{U}}$$

Example: Solution using LU

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 2 & 1\end{bmatrix}\underbrace{\begin{bmatrix}2 & 1 & -2\\0 & 3 & +1\\0 & 0 & +2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\boldsymbol{U}\boldsymbol{x}=\boldsymbol{y}}=\begin{bmatrix}2\\11\\22\end{bmatrix}$$

Step #2a: Solve $\boldsymbol{L}\boldsymbol{y}=\boldsymbol{b}$ using forward substitution.

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 2 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\11\\22\end{bmatrix}$$

Forward Substitution

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

$$\left.\begin{aligned} y_1 &= 2 \\ y_2 &= 11 - 2y_1 = 7 \\ y_3 &= 22 - 3y_1 - 2y_2 = 2 \end{aligned}\right\} \Rightarrow \boldsymbol{y} = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$$

Same RHS vector we obtained after row operations on the $\boldsymbol{b}$ vector during Gaussian Elimination.

Back Substitution

<u>Step #2b</u>: Solve $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ using back substitution.

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$$

$$2x_3 = 2 \quad \Rightarrow \quad x_3 = \frac{2}{2} = 1$$

$$3x_2 = 7 - x_3 \quad \Rightarrow \quad x_2 = \frac{7-1}{3} = 2$$

$$2x_1 = 2 - x_2 + 2x_3 \quad \Rightarrow \quad x_1 = \frac{2-2+2}{2} = 1$$

Same $\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$ equation we solved as the last step of Gaussian Elimination

Forward Substitution

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Compute sequentially using forward substitution.

$$y_1 = \frac{1}{l_{11}} b_1 \; ; \; y_2 = \frac{1}{l_{22}} (b_2 - l_{21} y_1) \; ; y_3 = \frac{1}{l_{33}} (b_3 - l_{31} y_1 - l_{32} y_2)$$

$$y_i = \frac{1}{l_{ii}} \left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right)$$

Back-Substitution

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Compute sequentially using backward substitution.

$$x_3 = \frac{1}{u_{33}} y_3 \; ; \; x_2 = \frac{1}{u_{22}} (y_2 - u_{23} x_3) \; ; x_1 = \frac{1}{u_{11}} (y_1 - u_{12} x_2 - u_{13} x_3)$$

$$x_i = \frac{1}{u_{ii}} \left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j \right)$$

For a matrix of size n.

CPU Cost: Solution Using LU Decomposition

LU Decomposition

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

- Similar CPU cost compared to Gaussian Elimination.
- Can be reused for different values of the RHS vector $\boldsymbol{b}$.

Forward and Back Substitution

$$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$$

$$\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$$

- Relatively low CPU cost (compared to LU).
- Repeated for each new RHS vector $\boldsymbol{b}$.

LU Decomposition Using Gaussian Steps

Matrix Vector Multiplication

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$

➡ Multiplying a matrix $\boldsymbol{A}$ by a vector, results in a vector $\boldsymbol{Ax}$ which is a linear combination of the columns of $\boldsymbol{A}$.

Matrix Vector Multiplication

- Left-multiplying a matrix $\boldsymbol{A}$ by a row vector vector, results in a row vector $\boldsymbol{x}^T\boldsymbol{A}$ which is a linear combination of the rows of $\boldsymbol{A}$.

$$[x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = x_1[a_{11} \quad a_{12} \quad a_{13}] + \\ +x_2[a_{21} \quad a_{22} \quad a_{23}] + x_3[a_{31} \quad a_{32} \quad a_{33}]$$

$$[-2 \quad 1 \quad 0]\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} = -2[2 \quad 1 \quad -2] + 1[4 \quad 5 \quad -3] \\ = [0 \quad 3 \quad 1] \quad \mathit{row}\ 2 - 2\times \mathit{row}\ 1$$

Elimination Matrices

$$\left.\begin{array}{l}
\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}2 & 1 & -2\\4 & 5 & -3\\6 & 9 & -2\end{bmatrix} = \begin{bmatrix}2 & 1 & -2\end{bmatrix}\\[2ex]
\begin{bmatrix}-2 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 1 & -2\\4 & 5 & -3\\6 & 9 & -2\end{bmatrix} = \underset{row\ 2 - 2\times row\ 1}{\begin{bmatrix}0 & 3 & 1\end{bmatrix}}\\[2ex]
\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & -2\\4 & 5 & -3\\6 & 9 & -2\end{bmatrix} = \begin{bmatrix}6 & 9 & -2\end{bmatrix}
\end{array}\right\}\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & -2\\4 & 5 & -3\\6 & 9 & -2\end{bmatrix} = \begin{bmatrix}2 & 1 & -2\\0 & 3 & 1\\6 & 9 & -2\end{bmatrix}$$

Elimination Matrices

$$\begin{array}{cc} & \text{column } j \\ \text{row } i & \underbrace{\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & -l_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix}}_{\boldsymbol{E}_{ij}} \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

➔ Perform the row operation $(row\ i) \leftarrow (row\ i) - l_{ij}(row\ j)$

Inverse of Elimination Matrices

$$\underbrace{\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & l_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix}}_{M = E_{ij}^{-1}\ ?} \quad \underbrace{\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & -l_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix}}_{E_{ij}}$$

➔ E_{ij} Perform the row operation $(row\ i) \leftarrow (row\ i) - l_{ij}(row\ j)$

➔ M Perform the row operation $(row\ i) \leftarrow (row\ i) + l_{ij}(row\ j)$

Inverse of Elimination Matrices

$$\underbrace{\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & l_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix}}_{E_{ij}^{-1}\ \checkmark} \quad \underbrace{\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & -l_{ij} & & \ddots & \\ & & & & 1 \end{bmatrix}}_{E_{ij}}$$

$$E_{ij}^{-1} E_{ij} A = A \qquad (row\ i) \leftarrow (row\ i) - l_{ij}(row\ j) + l_{ij}(row\ j)$$

Gaussian Elimination

Pivot

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

Pivot $\dfrac{4}{2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

$(row\ 2) \leftarrow row\ 2 - 2 \times row\ 1$

Elimination Matrix $\boldsymbol{E}_{21}$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 6 & 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 22 \end{bmatrix}$$

Gaussian Elimination

Pivot

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 6 & 9 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 22 \end{bmatrix}$$

Pivot $\frac{6}{2}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 6 & 9 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}\begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

$(row\ 3) \leftarrow row\ 3 - 3 \times row\ 1$

Elimination Matrix $\boldsymbol{E}_{31}$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 6 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 16 \end{bmatrix}$$

Gaussian Elimination

Pivot

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 16 \end{bmatrix}$$

Pivot $\frac{6}{3}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 6 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

$(row\ 3) \leftarrow row\ 3 - 2 \times row\ 2$

Elimination Matrix $\boldsymbol{E}_{32}$

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$$

Gaussian Elimination vs LU Factorization

$$\underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix}$$

Gaussian Elimination ←→ LU Factorization

$$\underbrace{\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix}}_{\boldsymbol{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\overbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}}}_{\boldsymbol{U}}$$

LU Factorization

$$\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}$$

$$\overbrace{\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{\boldsymbol{E}_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{\boldsymbol{E}_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{E}_{21}} \begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 11 \\ 22 \end{bmatrix}$$

LU Factorization

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}\overbrace{\begin{bmatrix}2&1&-2\\4&5&-3\\6&9&-2\end{bmatrix}}^{\boldsymbol{A}}=\overbrace{\begin{bmatrix}2&1&-2\\0&3&+1\\0&0&+2\end{bmatrix}}^{\boldsymbol{U}}$$

$$\begin{bmatrix}2&1&-2\\4&5&-3\\6&9&-2\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}^{-1}\begin{bmatrix}2&1&-2\\0&3&+1\\0&0&+2\end{bmatrix}$$

$$\begin{bmatrix}2&1&-2\\4&5&-3\\6&9&-2\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}2&1&-2\\0&3&+1\\0&0&+2\end{bmatrix}$$

LU Factorization

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}$$

$$= \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}$$

LU Factorization

$$\begin{bmatrix} 2 & 1 & -2 \\ 4 & 5 & -3 \\ 6 & 9 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & +1 \\ 0 & 0 & +2 \end{bmatrix}}^{\text{Row Echelon Form}}$$

Multipliers used in row operations

LU Algorithm using Gaussian Steps

Example

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1.5 & 3 & 2.5 \\ -2 & 0 & 2 & 6 \\ 0.5 & 2.5 & 9 & 13.5 \end{bmatrix}$$

First row of $\boldsymbol{U}$

Divide Column by the pivot

Multipliers (First column of $\boldsymbol{L}$)

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1.5 & 3 & 2.5 \\ -2 & 0 & 2 & 6 \\ 0.5 & 2.5 & 9 & 13.5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 6 & 8 \\ 0.5 & 2 & 8 & 13 \end{bmatrix} \begin{array}{l} \\ row\ 2 - 0.5 \times row\ 1 \\ row\ 3 + 2 \times row\ 1 \\ row\ 4 - 0.5 \times row\ 1 \end{array}$$

For each entry a_{ij} replace with $a_{ij} - a_{i1} \times a_{1j}$

$$1.5 \longrightarrow 1.5 - 0.5 \times 1 = 1$$
$$3 \longrightarrow 3 - 0.5 \times 2 = 2$$
$$2.5 \longrightarrow 2.5 - 0.5 \times 1 = 2$$
$$0 \longrightarrow 0 - (-2) \times 1 = 2$$
$$2 \longrightarrow 2 - (-2) \times 2 = 6$$

Repeat the same process for the submatrix

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 6 & 8 \\ 0.5 & 2 & 8 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & \boxed{1 \quad 2 \quad 2} \\ -2 & 2 & 6 & 8 \\ 0.5 & 2 & 8 & 13 \end{bmatrix}$$

2^{nd} row of $\boldsymbol{U}$

Divide Column by the pivot

Multipliers (2^{nd} column of $\boldsymbol{L}$)

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 6 & 8 \\ 0.5 & 2 & 8 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 2 & 4 \\ 0.5 & 2 & 4 & 9 \end{bmatrix} \begin{array}{l} \\ \\ row\ 3 - 2 \times row\ 2 \\ row\ 4 - 2 \times row\ 2 \end{array}$$

For each entry a_{ij} replace with $a_{ij} - a_{i2} \times a_{2j}$

$$6 \longrightarrow 6 - 2 \times 2 = 2$$
$$8 \longrightarrow 8 - 2 \times 2 = 4$$
$$8 \longrightarrow 8 - 2 \times 2 = 4$$
$$13 \longrightarrow 13 - 2 \times 2 = 9$$

Repeat the same process for the submatrix

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 2 & 4 \\ 0.5 & 2 & \boxed{4} & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & \boxed{2 \quad 4} \\ 0.5 & 2 & \boxed{2} & 9 \end{bmatrix}$$

Divide Column by the pivot

3rd row of $\boldsymbol{U}$

Multipliers (3rd column of $\boldsymbol{L}$)

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 2 & 4 \\ 0.5 & 2 & 2 & \boxed{9} \end{bmatrix} \Rightarrow \begin{array}{cl} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 2 & 4 \\ 0.5 & 2 & 2 & 1 \end{bmatrix} & \\ & row\ 4 - 2 \times row\ 3 \end{array}$$

For each entry a_{ij} replace with $a_{ij} - a_{i3} \times a_{3j}$

$$9 \longrightarrow 9 - 2 \times 4 = 1$$

LU Algorithm using Gaussian Steps

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 1 & 2 & 2 \\ -2 & 2 & 2 & 4 \\ 0.5 & 2 & 2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0.5 & 2 & 2 & 1 \end{bmatrix}$$

LU Algorithm using Gaussian Steps

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0.5 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Doolittle Algorithm

Doolittle Algorithm for LU

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$u_{11} = 2$$
$$u_{12} = 1$$
$$u_{13} = 2$$
$$u_{14} = 1$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$2l_{21} = 1 \Rightarrow l_{21} = 0.5$$

$$2l_{31} = -4 \Rightarrow l_{31} = -2$$

$$2l_{41} = 1 \Rightarrow l_{41} = 0.5$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \boxed{0.5 \quad 1 \quad 0 \quad 0} \\ -2 & l_{32} & 1 & 0 \\ 0.5 & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$0.5 \times 1 + u_{22} = 1.5 \Rightarrow u_{22} = 1$$

$$0.5 \times 2 + u_{23} = 3 \Rightarrow u_{23} = 2$$

$$0.5 \times 1 + u_{24} = 2.5 \Rightarrow u_{24} = 2$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ -2 & l_{32} & 1 & 0 \\ 0.5 & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$-2 \times 1 + l_{32} \times 1 = 0 \quad \Rightarrow \quad l_{32} = (0 + 2 \times 1)/1 = 2$$

$$0.5 \times 1 + l_{42} \times 1 = 2.5 \quad \Rightarrow \quad l_{42} = (2.5 - 0.5 \times 1)/1 = 2$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ \boxed{-2 \quad 2 \quad 1 \quad 0} \\ 0.5 & 2 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$-2\times 2 + 2\times 2 + u_{33} = 2 \Rightarrow u_{33} = 2 + 2\times 2 - 2\times 2 = 2$$

$$-2\times 1 + 2\times 2 + u_{34} = 6 \Rightarrow u_{34} = 6 + 2\times 1 - 2\times 2 = 4$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0.5 & 2 & l_{43} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$0.5 \times 2 + 2 \times 2 + l_{43} \times 2 = 9 \quad \Longrightarrow \quad l_{43} = (9 - 0.5 \times 2 - 2 \times 2)/2 = 2$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ \boxed{0.5 \quad 2 \quad 2} & & & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$0.5 \times 1 + 2 \times 2 + 2 \times 4 + u_{44} = 13.5$$

$$\Rightarrow u_{44} = 13.5 - 0.5 \times 1 - 2 \times 2 - 2 \times 4 = 1$$

Example: Doolittle Algorithm

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 1.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0.5 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

LU Decomposition Algorithm

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{11} = l_{11}u_{11} \;\; ; \;\; a_{23} = l_{21}u_{13} + l_{22}u_{23} \;\; ; \;\; a_{21} = l_{21}u_{11}$$

For the general case: $a_{ij} = \sum_{k=1}^{\min(i,j)} l_{ik}u_{kj}$

For a Matrix of size n we have n^2 constraints and $n^2 + n$ degrees of freedom.
→ When a solution exists, it is not unique.

Doolittle's Algorithm

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ l_{21} & 1 & 0 & \\ l_{31} & l_{32} & 1 & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots \\ 0 & u_{22} & u_{23} & \\ 0 & 0 & u_{33} & \\ \vdots & & & \ddots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \\ a_{21} & a_{22} & a_{23} & \\ a_{31} & a_{32} & a_{33} & \\ & & & \ddots \end{bmatrix}$$

First row of $\boldsymbol{U}$: $u_{1k} = a_{1k} \quad ; \ \mathrm{k} = 1 \dots n$

First Column of $\boldsymbol{L}$: $l_{k1} = a_{k1}/u_{11} \ ; \ \mathrm{k} = 2 \dots n$

i^{th} row of $\boldsymbol{U}$: $u_{ik} = a_{ik} - \sum_{m=1}^{i-1} l_{im} u_{mk} \quad ; \quad \mathrm{k} = i \dots n$

j^{th} column of $\boldsymbol{L}$: $l_{kj} = \left(a_{kj} - \sum_{m=1}^{j-1} l_{km} u_{mj} \right) / u_{jj} \ ; \quad \mathrm{k} = (j+1) \dots n$

Pivoting

LU Algorithm using Gaussian Steps

Example

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 0.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 0.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 0.5 & 3 & 2.5 \\ -2 & 0 & 2 & 6 \\ 0.5 & 2.5 & 9 & 13.5 \end{bmatrix}$$

First row of $\boldsymbol{U}$

Divide Column by the pivot

Multipliers (First column of $\boldsymbol{L}$)

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 0.5 & 3 & 2.5 \\ -2 & 0 & 2 & 6 \\ 0.5 & 2.5 & 9 & 13.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 0 & 2 & 2 \\ -2 & 2 & 6 & 8 \\ 0.5 & 2 & 8 & 13 \end{bmatrix} \quad \begin{array}{l} row\ 2 - 0.5 \times row\ 1 \\ row\ 3 + 2 \times row\ 1 \\ row\ 4 - 0.5 \times row\ 1 \end{array}$$

For each entry a_{ij} replace with $a_{ij} - a_{i1} \times a_{1j}$

$$1.5 \longrightarrow 1.5 - 0.5 \times 1 = 1$$
$$3 \longrightarrow 3 - 0.5 \times 2 = 2$$
$$2.5 \longrightarrow 2.5 - 0.5 \times 1 = 2$$
$$0 \longrightarrow 0 - (-2) \times 1 = 2$$
$$2 \longrightarrow 2 - (-2) \times 2 = 6$$

Repeat the same process for the submatrix

LU Algorithm using Gaussian Steps

Cannot be a pivot

pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 0.5 & 0 & 2 & 2 \\ -2 & 2 & 6 & 8 \\ 0.5 & 2 & 8 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 2 & 8 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 1 & 8 & 13 \end{bmatrix}$$

Divide Column by the pivot

LU Algorithm using Gaussian Steps

Pivot

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 1 & 8 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 1 & 2 & 5 \end{bmatrix}$$

Pivot

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 1 & 1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 1 & 1 & 3 \end{bmatrix}$$

LU Algorithm using Gaussian Steps

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ -2 & 2 & 6 & 8 \\ 0.5 & 0 & 2 & 2 \\ 0.5 & 1 & 1 & 3 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 2 & 6 & 8 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \\ 0.5 & 1 & 1 & 1 \end{bmatrix}$$

Pivoting Example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0.5 & 0 & 1 & 0 \\ 0.5 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 2 & 6 & 8 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 1 \\ -4 & 0 & 2 & 6 \\ 1 & 0.5 & 3 & 2.5 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 0.5 & 3 & 2.5 \\ -4 & 0 & 2 & 6 \\ 1 & 2.5 & 9 & 13.5 \end{bmatrix}$$

Row Pivoting

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_4 \\ b_3 \end{bmatrix}$$

- Exchanging rows 3 and 4 is the same as switching the order of the equation.
- The order of the variables in the vector of unknowns remains the same.
- Note the impact on the right hand side (RHS) vector.

Row Pivoting Matrix $\boldsymbol{P}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Permutation Matrix $\boldsymbol{P}$

$\boldsymbol{e}_i$ is the i^{th} row of the identity matrix

Note: $\boldsymbol{P}^t = \boldsymbol{P}^{-1}$

$$\boldsymbol{P} = \begin{bmatrix} [\quad \boldsymbol{e}_1 \quad] \\ [\quad \boldsymbol{e}_2 \quad] \\ [\quad \boldsymbol{e}_4 \quad] \\ [\quad \boldsymbol{e}_3 \quad] \end{bmatrix}$$

Column Pivoting

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{14} & a_{13} \\ a_{21} & a_{22} & a_{24} & a_{23} \\ a_{31} & a_{32} & a_{34} & a_{33} \\ a_{41} & a_{42} & a_{44} & a_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- Exchanging columns 3 and 4 is the same as switching the order of the unknowns in $\boldsymbol{x}$.
- The order of the equations remains the same.
- The RHS vector remains the same.

Column Pivoting Matrix $\boldsymbol{Q}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} a_{11} & a_{12} & a_{14} & a_{13} \\ a_{21} & a_{22} & a_{24} & a_{23} \\ a_{31} & a_{32} & a_{34} & a_{33} \\ a_{41} & a_{42} & a_{44} & a_{43} \end{bmatrix}$$

Permutation Matrix $\mathbf{Q} = [\boldsymbol{e}_1 \quad \boldsymbol{e}_2 \quad \boldsymbol{e}_4 \quad \boldsymbol{e}_3]$ where $\boldsymbol{e}_i$ is the i^{th} column of the identity matrix

Note:$\boldsymbol{Q}^t = \boldsymbol{Q}^{-1}$

Full Pivoting

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{12} & a_{11} & a_{13} & a_{14} \\ a_{22} & a_{21} & a_{23} & a_{24} \\ a_{42} & a_{41} & a_{43} & a_{44} \\ a_{32} & a_{31} & a_{33} & a_{34} \end{bmatrix}$$

Solving Linear Systems using LU Decomposition

$\boldsymbol{A}, \boldsymbol{b}$ $\qquad$ $\boldsymbol{Ax} = \boldsymbol{b}$

Factor: $\boldsymbol{A} = \boldsymbol{LU}$

$\boldsymbol{L}, \boldsymbol{U}$ $\qquad$ $\boldsymbol{LUx} = \boldsymbol{b}$

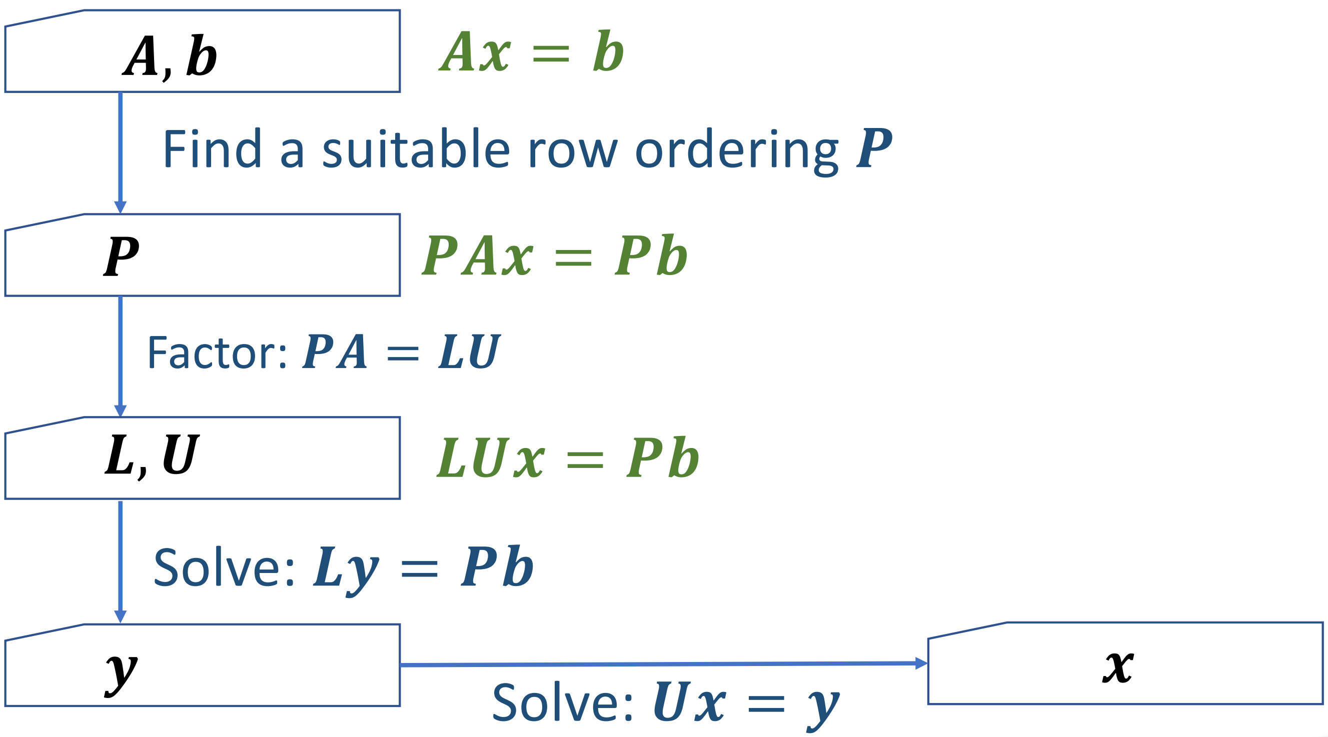
Solve: $\boldsymbol{Ly} = \boldsymbol{b}$ using Forward Substitution

$\boldsymbol{y}$

Solve: $\boldsymbol{Ux} = \boldsymbol{y}$ using Back Substitution

$\boldsymbol{x}$

Solving Linear Systems using LU Decomposition with Row Pivoting



Solving Linear Systems using LU Decomposition with Column Pivoting

$\boldsymbol{A, b}$ — $\boldsymbol{Ax = b}$

Find a suitable column ordering $\boldsymbol{Q}$

$\boldsymbol{Q}$ — $\boldsymbol{AQQ^t x = b}$

Factor: $\boldsymbol{AQ = LU}$

$\boldsymbol{L, U}$ — $\boldsymbol{LUQ^t x = b}$

Solve: $\boldsymbol{Ly = b}$

$\boldsymbol{y}$ $\xrightarrow{\text{Solve: } \boldsymbol{Uz = y}}$ $\boldsymbol{z}$ $\xrightarrow[\text{Solve: } \boldsymbol{Q^t x = z}]{\text{Reorder}}$ $\boldsymbol{x}$

Note: $\boldsymbol{z = Q^t x}$ is a reordered version of $\boldsymbol{x}$.
$\boldsymbol{Q^t = Q^{-1}}$

Solving Linear Systems using LU Decomposition with Full Pivoting

$\boldsymbol{A}, \boldsymbol{b}$ — $\boldsymbol{Ax} = \boldsymbol{b}$

Find a suitable ordering $\boldsymbol{P}, \boldsymbol{Q}$

$\boldsymbol{P}, \boldsymbol{Q}$ — $\boldsymbol{PAQQ^t x} = \boldsymbol{Pb}$

Factor: $\boldsymbol{PAQ} = \boldsymbol{LU}$

$\boldsymbol{L}, \boldsymbol{U}$ — $\boldsymbol{LUQ^t x} = \boldsymbol{Pb}$

Solve: $\boldsymbol{Ly} = \boldsymbol{Pb}$

$\boldsymbol{y}$ $\xrightarrow{\text{Solve: } \boldsymbol{Uz} = \boldsymbol{y}}$ $\boldsymbol{z}$ $\xrightarrow[\text{Solve: } \boldsymbol{Q^t x} = \boldsymbol{z}]{\text{Reorder}}$ $\boldsymbol{x}$

Note: $\boldsymbol{z} = \boldsymbol{Q^t x}$ is a reordered version of $\boldsymbol{x}$.
$\boldsymbol{Q}^t = \boldsymbol{Q}^{-1}$

Pivoting

- LU decomposition algorithms (e.g. Doolittle's algorithm) assume the **pivot on the diagonal is never zero**. In general, this cannot be guaranteed.
- A non-zero but **small pivot** can also lead to numerical inaccuracies in finite precision computing (more about this later).
- Matrix ordering (pivoting) can be used to address these issues.
- For Sparse matrices, pivoting has an impact on sparsity.
- It is possible to do row pivoting, column pivoting, or both (full pivoting).

Summary

- Systems of linear equations.
- Gaussian elimination.
- LU decomposition.
- Doolittle's algorithm.
- Introduction to pivoting.