ECSE 343 Numerical Methods in Engineering

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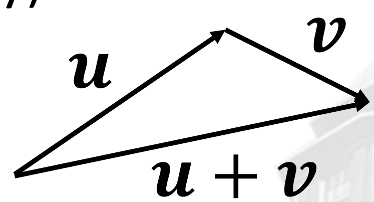
Norm ("Size") of a Vector

Then Norm $\|\boldsymbol{v}\| \in \mathbb{R}$ of a vector $\boldsymbol{v}$ is an indication of its size. It must be defined such that it obeys the following rules:

1. $\|\boldsymbol{v}\| > 0$
2. $\|a\boldsymbol{v}\| = |a|\|\boldsymbol{v}\|$
3. $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ (Triangle Inequality)



L_p Norm

Consider $v \in \mathbb{R}^n$

The p-norm or L_p norm is defined as:

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

Important Special Cases

The 1-norm or L_1 norm is:

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|$$

The Euclidean (L_2) norm is:

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2}$$

Length of the vector

The Infinity (L_∞) norm is:

$$\|v\|_\infty = \max_{1 \le i \le n} |v_i|$$

Norm of a Matrix

Then Norm $\|\boldsymbol{A}\| \in \mathbb{R}$ of a vector $\boldsymbol{A}$ is an indication of its size. It must be defined such that it obeys the following rules:

1. $\|\boldsymbol{A}\| > 0$
2. $\|a\boldsymbol{A}\| = |a|\|\boldsymbol{A}\|$
3. $\|\boldsymbol{A} + \boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$
4. $\|\boldsymbol{AB}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$

Frobenius Norm

$$\|A\|_F = \sqrt{\sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} a_{i,j}^2}$$

Induced Norm

Induced norm $||\boldsymbol{A}||$ of a matrix $\boldsymbol{A}$ based on a vector norm $||\cdot||$

$$||\boldsymbol{A}|| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| \neq 0}} \frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||} \qquad ||\boldsymbol{A}|| = \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| = 1}} ||\boldsymbol{A}\boldsymbol{x}||$$

$$||\boldsymbol{A}\boldsymbol{x}|| \leq ||\boldsymbol{A}||\,||\boldsymbol{x}|| \qquad \forall \boldsymbol{x} \in \mathbb{R}^n$$

Absolute vs Relative Error

Problem: Find the root of $f(x) = 0$

Actual solution is x

Computed solution is $\hat{x}$

Absolute error: $|x - \hat{x}|$

Relative error: $\frac{|x-\hat{x}|}{|x|}$

Example

Absolute Error

$R_1 = 1 \pm 0.01$ → 1% Relative Error

$R_2 = 10^5 \pm 0.01$ → 0.00001% Relative Error

Catastrophic Cancellation

Absolute Error

$x = 1.00 \pm 0.004$ ⟹ 0.4% Relative Error

$y = 0.99 \pm 0.004$ ⟹ 0.4% Relative Error

$d = x - y = 0.01 \pm 0.008$ ⟹ 80% Relative Error

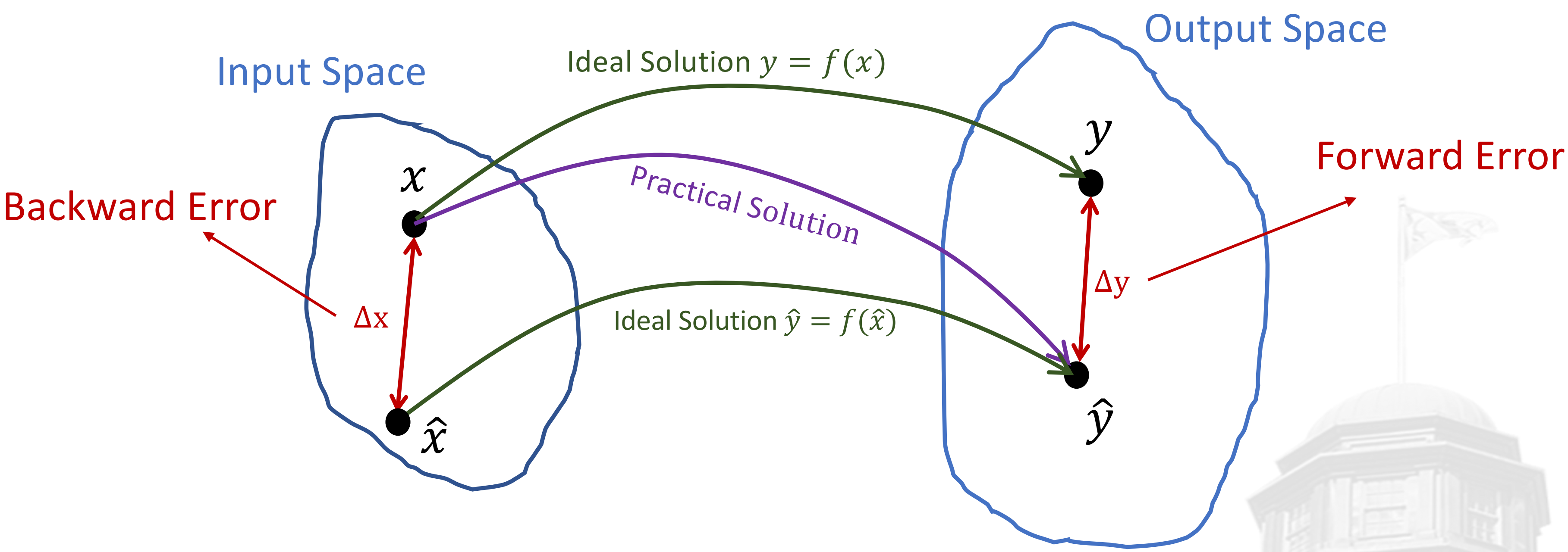
Forward vs Backward Error

- Problem: Find the root of $f(x) = 0$
- Actual solution is $x_o \Longrightarrow f(x_o) \equiv 0$
- Computed (inexact) solution is $\hat{x}$ such that $f(\hat{x}) = \epsilon$
- Forward error: $|x_o - \hat{x}|$

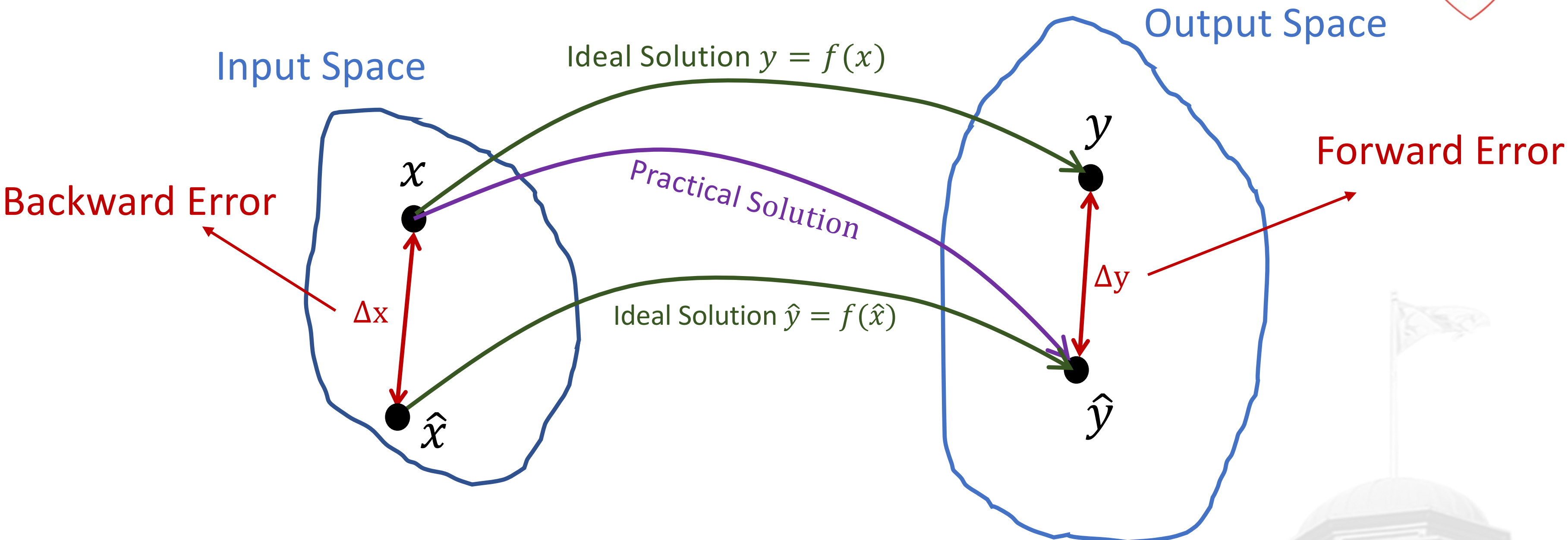
- Backward error: $|f(x_o) - f(\hat{x})| = |f(\hat{x})| = |\epsilon|$

Forward vs Backward Error

Problem: Compute $y = f(x)$



Forward vs Backward Error



$$\boxed{|\Delta y| \le \hat{\kappa}|\Delta y|}$$

$$\frac{|\Delta y|}{|\Delta x|} \le \hat{\kappa}$$

Forward vs Backward Error

$$\boxed{|\Delta y| \leq \hat{\kappa}|\Delta y|} \qquad \frac{|\Delta y|}{|\Delta x|} \leq \hat{\kappa}$$

- How sensitive is the output to small changes in the input.
- If $\hat{\kappa}$ is small, then the practical solution $\hat{y}$ is also the ideal solution of a *nearby problem*.
- If $\hat{\kappa}$ is small the problem is said to be well conditioned.

Absolute Condition Number

$$\hat{\kappa} = \lim_{\epsilon \to 0} \max_{|\Delta x| < \epsilon} \frac{|\Delta y|}{|\Delta x|}$$

Note: $y = f(x)$

For scalar x and y: $\hat{\kappa} = \left|\frac{df}{dx}\right|$

Scaling / Normalization

$x = f(t) = vt$ $\qquad\qquad$ $\hat{\kappa} = \left|\frac{df}{dt}\right| = |v|$

t in hours

$v = 3600m/h \qquad \Longrightarrow x = 3600t \qquad \Longrightarrow \hat{\kappa} = 3600m/h$

$v = 1m/s \qquad \Longrightarrow x = 1t \qquad \Longrightarrow \hat{\kappa} = 1m/s$

t in seconds

Relative Condition Number

$$\frac{|\Delta y|}{|\Delta x|} \leq \hat{\kappa} \qquad\qquad \frac{|\Delta y|/|y|}{|\Delta x|/|x|} \leq \kappa$$

$$\hat{\kappa} = \lim_{\epsilon \to 0} \max_{|\Delta x| < \epsilon} \frac{|\Delta y|/|y|}{|\Delta x|/|x|}$$

$$\kappa = \hat{\kappa} \frac{|x|}{|y|} = \hat{\kappa} \frac{|x|}{|f(x)|}$$

Relative Condition Number

$$x = f(t) = vt$$

$$\hat{\kappa} = \left|\frac{df}{dt}\right| = |v|$$

$$\kappa = \hat{\kappa}\frac{|t|}{|f(t)|} = |v|\frac{|t|}{|vt|} = 1$$

Example 2

$$y = f(x) = x^2$$

$$\hat{\kappa} = \left|\frac{df}{dx}\right| = |2x|$$

$$\kappa = \left|\frac{df}{dx}\right| \frac{|x|}{|f(x)|} = |2x| \frac{|x|}{|x^2|} = 2$$

Absolute Condition Number

x and y are vectors

$$\hat{\kappa} = \lim_{\epsilon \to 0} \sup_{||\Delta x|| < \epsilon} \frac{||\Delta y||}{||\Delta x||}$$

Note: $y = f(x)$

Taylor Expansion

$$f(x) = f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o} (x - x_o) + \cdots$$

$$f(x) \cong f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o} (x - x_o)$$

Taylor Expansion: Scalar function of multiple variables

$$f(x_1, x_2, x_3) \cong f\left(x_{1,o}, x_{2,o}, x_{3,o}\right) +$$

$$\left.\frac{\partial f}{\partial x_1}\right|_{\substack{x_1 = x_{1,o} \\ x_2 = x_{2,o} \\ x_3 = x_{3,o}}} \left(x_1 - x_{1,o}\right) + \left.\frac{\partial f}{\partial x_2}\right|_{\substack{x_1 = x_{1,o} \\ x_2 = x_{2,o} \\ x_3 = x_{3,o}}} \left(x_2 - x_{2,o}\right) +$$

$$+ \left.\frac{\partial f}{\partial x_3}\right|_{\substack{x_1 = x_{1,o} \\ x_2 = x_{2,o} \\ x_3 = x_{3,o}}} \left(x_3 - x_{3,o}\right)$$

Taylor Expansion: Scalar function of multiple variables

$$f(x) \cong f(x_o) +$$

$$+\left.\frac{\partial f}{\partial x_1}\right|_{x=x_o}(x_1 - x_{1,o}) + \left.\frac{\partial f}{\partial x_2}\right|_{x=x_o}(x_2 - x_{2,o}) +$$

$$+\left.\frac{\partial f}{\partial x_3}\right|_{x=x_o}(x_3 - x_{3,o})$$

Taylor Expansion: Scalar function of multiple variables

$$f(x) \cong f(x_o) + \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3}\right]\Bigg|_{x=x_o} \begin{bmatrix} (x_1 - x_{1,o}) \\ (x_2 - x_{2,o}) \\ (x_3 - x_{3,o}) \end{bmatrix}$$

$$f(x) \cong f(x_o) + \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3}\right]\Bigg|_{x=x_o} \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} x_{1,o} \\ x_{2,o} \\ x_{3,o} \end{bmatrix} \right)$$

Taylor Expansion: Scalar function of multiple variables

$$f(x) \cong f(x_o) + \underbrace{\left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2} \quad \frac{\partial f}{\partial x_3}\right]\Bigg|_{x=x_o}}_{\text{Jacobian } \frac{df}{dx}} (x - x_o)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Taylor Expansion: Vector function of multiple variables

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Taylor Expansion: Vector function of multiple variables

$$f(x) \cong f(x_o) + \underbrace{\left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & & & \frac{\partial f_n}{\partial x_m} \end{bmatrix}\right|_{x=x_o}}_{\text{Jacobian } \frac{df}{dx}} (x - x_o)$$

Jacobian

$$J = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & & & \frac{\partial f_n}{\partial x_m} \end{bmatrix}$$

Absolute Condition Number

$$\hat{\kappa} = \lim_{\epsilon \to 0} \sup_{||\Delta x|| < \epsilon} \frac{||\Delta y||}{||\Delta x||}$$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\Delta y \cong J \Delta x \qquad J = \frac{df}{dx} \qquad \hat{\kappa} = \sup_{||\Delta x||} \frac{||J \Delta x||}{||\Delta x||} = ||J||$$

Relative Condition Number

$$\kappa = \lim_{\epsilon \to 0} \max_{|\Delta x| < \epsilon} \frac{||\Delta y|| / ||y||}{||\Delta x|| / ||x||}$$

$$\kappa = \hat{\kappa} \frac{||x||}{||y||} = ||J|| \frac{||x||}{||f(x)||}$$

Example: Subtraction

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad f(x) = x_1 - x_2$$

$$J = \frac{df}{dx} = \begin{bmatrix} \frac{\partial f}{x_1} & \frac{\partial f}{x_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\|J\|_2 = 1.41 \qquad \kappa = \|J\| \frac{\|x\|}{\|x_1 - x_2\|}$$

Illconditioned when $x_1 \cong x_2$

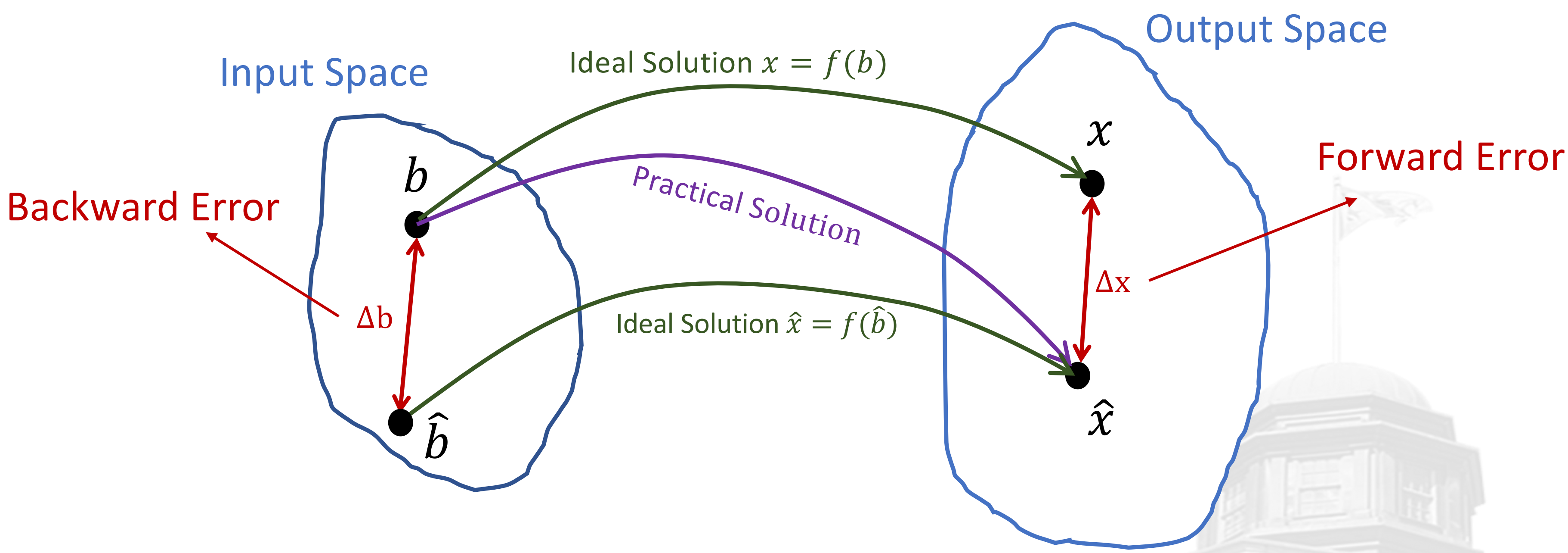
Condition Number for $Ax = b$

Problem: Find x such that $Ax = b$

Problem: $x = f(b) = A^{-1}b$

Condition Number for $Ax = b$

Problem: $x = f(b) = A^{-1}b$



Condition Number for $Ax = b$

Problem: $x = f(b) = A^{-1}b$

$$J = \frac{df}{db} = A^{-1}$$

$$\kappa = \|A^{-1}\| \frac{\|b\|}{\|A^{-1}b\|} = \frac{\|A^{-1}\|\|b\|}{\|A^{-1}b\|} \geq 1$$

Condition Number for $Ax = b$

$$\kappa = ||A^{-1}|| \frac{||b||}{||A^{-1}b||}$$

$$\frac{||b||}{||A^{-1}b||} = \frac{||Ax||}{||x||} < ||A||$$

$$\kappa \leq \underbrace{||A^{-1}||\,||A||}_{\text{Condition Number of matrix A } \kappa(A)}$$

Recall

$$||\boldsymbol{A}|| \equiv \max_{\substack{\boldsymbol{x} \in \mathbb{R}^n \\ ||\boldsymbol{x}|| \neq 0}} \frac{||\boldsymbol{A}\boldsymbol{x}||}{||\boldsymbol{x}||}$$