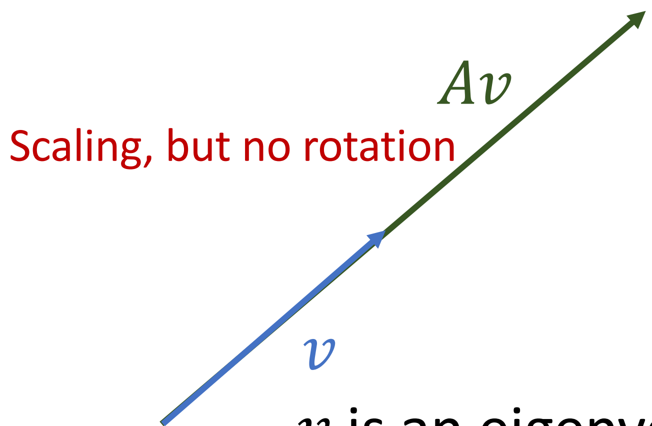
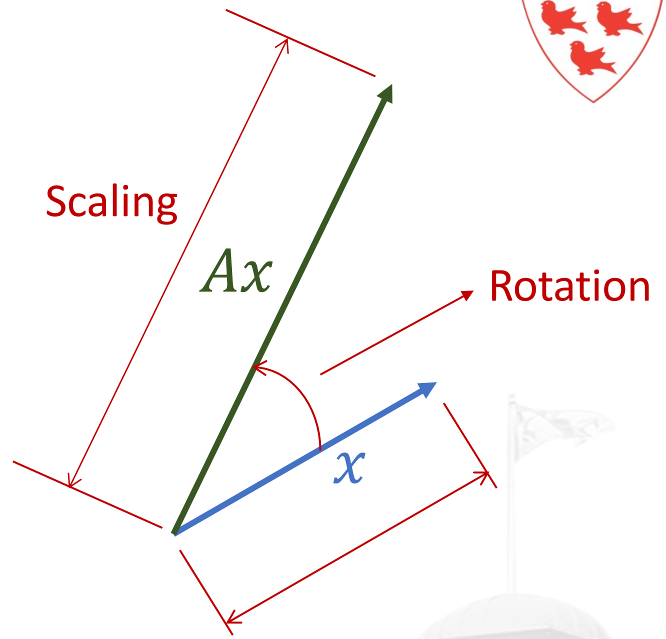
ECSE 343 Numerical Methods in Engineering

Roni Khazaka
Dept. of Electrical and Computer Engineering
McGill University

Eigenvalues / Eigenvectors





v is an eigenvector of A

$$Av = \lambda v$$

eigenvalue (scalar)

Power Method

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be the eigenvectors of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with corresponding eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_n$. Then:

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

Choose a vector $\boldsymbol{y} \in \mathbb{R}^n$, we can express y as:

$$\boldsymbol{y} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$$

Then: $\boldsymbol{A}\boldsymbol{y} = c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + \cdots + c_n\lambda_n\boldsymbol{v}_n$

$$\boldsymbol{A}^2\boldsymbol{y} = c_1\lambda_1^2\boldsymbol{v}_1 + c_2\lambda_2^2\boldsymbol{v}_2 + \cdots + c_n\lambda_n^2\boldsymbol{v}_n$$

Power Method

$$\boldsymbol{A}^k \boldsymbol{y} = c_1 \lambda_1^k \boldsymbol{v}_1 + c_2 \lambda_2^k \boldsymbol{v}_2 + \cdots + c_n \lambda_n^k \boldsymbol{v}_n$$

Assume that that the eigenvectors and eigen values are ordered such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then:

For very large k: $|\lambda_1|^k \gg |\lambda_2|^k \gg \cdots \gg |\lambda_n|^k$

$\boldsymbol{A}^k \boldsymbol{y} \cong c_1 \lambda_1^k \boldsymbol{v}_1$ ⇒ Eigenvector of largest eigenvalue

Power Method (Largest Eigenvalue)

- Choose a vector $\boldsymbol{w}_0$
- $\boldsymbol{w}_1 \leftarrow \boldsymbol{A}\boldsymbol{w}_0$
- $\boldsymbol{w}_1 \leftarrow \boldsymbol{w}_1/\|\boldsymbol{w}_1\|$ (Normalize) Length does not matter (avoid large numbers)
- $\boldsymbol{w}_2 \leftarrow \boldsymbol{w}\boldsymbol{v}_1$
- $\boldsymbol{w}_2 \leftarrow \boldsymbol{w}_2/\|\boldsymbol{w}_2\|$ (Normalize)
- $\boldsymbol{w}_3 \leftarrow \boldsymbol{A}\boldsymbol{w}_2$
- $\boldsymbol{w}_3 \leftarrow \boldsymbol{w}_3/\|\boldsymbol{w}_3\|$ (Normalize)
- Continue until $\boldsymbol{w}_{k+1} \parallel \boldsymbol{w}_k$

Power Method (Smallest Eigenvalue)

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be the eigenvectors of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with corresponding eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then:

$$\boldsymbol{A}^{-1}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

Choose a vector $\boldsymbol{y} \in \mathbb{R}^n$, we can express y as:

$$\boldsymbol{y} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$$

Then: $$\boldsymbol{A}^{-1}\boldsymbol{y} = c_1\lambda_1^{-1}\boldsymbol{v}_1 + c_2\lambda_2^{-1}\boldsymbol{v}_2 + \cdots + c_n\lambda_n^{-1}\boldsymbol{v}_n$$

$$\boldsymbol{A}^{-2}\boldsymbol{y} = c_1\lambda_1^{-2}\boldsymbol{v}_1 + c_2\lambda_2^{-2}\boldsymbol{v}_2 + \cdots + c_n\lambda_n^{-2}\boldsymbol{v}_n$$

Power Method (Smallest Eigenvalue)

$$\boldsymbol{A}^{-k}\boldsymbol{y} = c_1\lambda_1^{-k}\boldsymbol{v}_1 + c_2\lambda_2^{-k}\boldsymbol{v}_2 + \cdots + c_n\lambda_n^{-k}\boldsymbol{v}_n$$

Assume that that the eigenvectors and eigen values are ordered such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then:

For very large k: $|\lambda_1|^{-k} \ll |\lambda_2|^{-k} \ll \cdots, \ll |\lambda_n|^{-k}$

$\boldsymbol{A}^{-k}\boldsymbol{y} \cong c_n\lambda_n^{-k}\boldsymbol{v}_n \Rightarrow$ Eigenvector of smallest eigenvalue

Power Method (Smallest Eigenvalue)

- Choose a vector $\boldsymbol{w}_0$
- Solve: $\boldsymbol{A}\boldsymbol{w}_1 = \boldsymbol{w}_0$
- $\boldsymbol{w}_1 \leftarrow \boldsymbol{w}_1 / \|\boldsymbol{w}_1\|$ (Normalize) Length does not matter (avoid large numbers)
- Solve: $\boldsymbol{A}\boldsymbol{w}_2 = \boldsymbol{w}_1$
- $\boldsymbol{w}_2 \leftarrow \boldsymbol{w}_2 / \|\boldsymbol{w}_2\|$ (Normalize)
- Solve: $\boldsymbol{A}\boldsymbol{w}_3 = \boldsymbol{w}_2$
- $\boldsymbol{w}_3 \leftarrow \boldsymbol{w}_3 / \|\boldsymbol{w}_3\|$ (Normalize)
- Continue until $\boldsymbol{w}_{k+1} \parallel \boldsymbol{w}_k$

Second Eigenvalue

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be the eigenvectors of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with corresponding eigenvalue $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Assume that that the eigenvectors and eigen values are ordered such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$.

Assume that we have already computed $\boldsymbol{v}_1$ using the power method, and we would like to compute $\boldsymbol{v}_2$

Power Method (2nd Eigenvalue)

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be the eigenvectors of $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with corresponding eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_n$. Then:

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$$

Choose a vector $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{y} \perp \boldsymbol{v}_1$ we can express y as:

$$\boldsymbol{y} = 0\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$$

Then: $\boldsymbol{A}\boldsymbol{y} = 0\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + \cdots + c_n\lambda_n\boldsymbol{v}_n$

$$\boldsymbol{A}^2\boldsymbol{y} = 0\lambda_1^2\boldsymbol{v}_1 + c_2\lambda_2^2\boldsymbol{v}_2 + \cdots + c_n\lambda_n^2\boldsymbol{v}_n$$

Power Method

$$\boldsymbol{A}^k\boldsymbol{y} = 0\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2 + \cdots + c_n\lambda_n^k\boldsymbol{v}_n$$

Assume that that the eigenvectors and eigen values are ordered such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Then:

For very large k: $|\lambda_1|^k \gg |\lambda_2|^k \gg \cdots \gg |\lambda_n|^k$

$\boldsymbol{A}^k\boldsymbol{y} \cong c_2\lambda_2^k\boldsymbol{v}_2$ ⇒ Eigenvector of 2nd largest eigenvalue

Power Method (Second Eigenvalue)

- Compute $\boldsymbol{v}_1$ (using the power method for example)
- Choose a vector $\boldsymbol{y}_1$
- Use QR to obtain: $[\boldsymbol{v}_1 \quad \boldsymbol{y}_1] = [q_0 \quad \boldsymbol{w}_1]\boldsymbol{R}$
  - $\|\boldsymbol{w}_1\| = 1$ (Normalized)
  - $\boldsymbol{w}_1 \perp \boldsymbol{v}_1$
- $\boldsymbol{y}_2 \leftarrow \boldsymbol{A}\boldsymbol{w}_1$
- Use QR to obtain: $[\boldsymbol{v}_1 \quad \boldsymbol{y}_2] = [q_0 \quad \boldsymbol{w}_2]\boldsymbol{R}$
  - $\|\boldsymbol{w}_2\| = 1$ (Normalized)
  - $\boldsymbol{w}_2 \perp \boldsymbol{v}_1$
- $\boldsymbol{y}_3 \leftarrow \boldsymbol{A}\boldsymbol{w}_2$
- Continue until we converge to an eigenvector.

Some applications of Eigenvalues and Eigenvectors

Least Squares Approximation

Consider n data points (t_i, y_i)

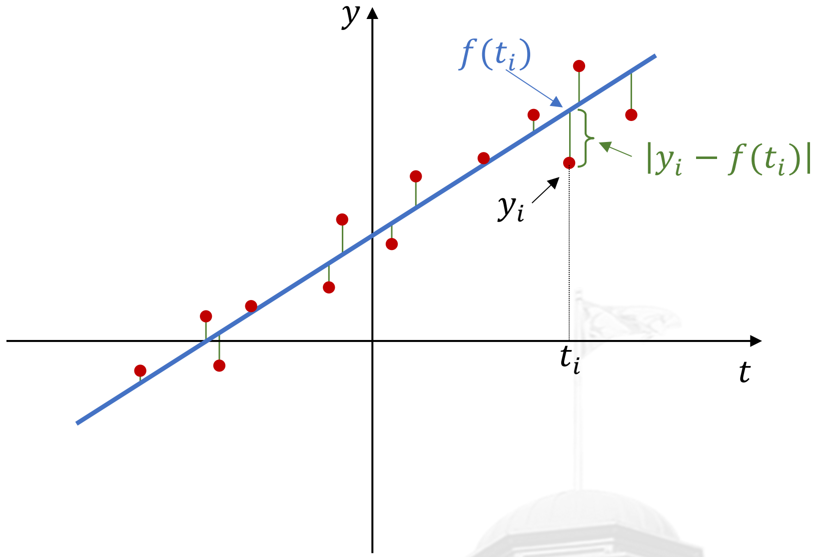
Approximate data with a model:

$$y = f(t) = a_o + a_1 t$$

a_o and a_1 are the model parameters.

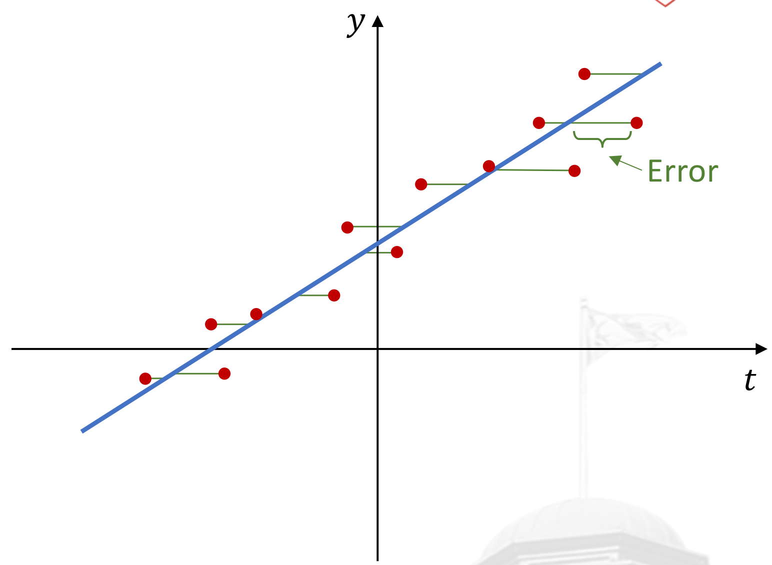
Choose the parameters to minimize:

$$e = \sum_{i=1}^{n} (f(t_i) - y_i)^2$$



Least Squares Approximation

The problem could have been formulated to minimize the error along the t axis.



Total Least Squares Approximation

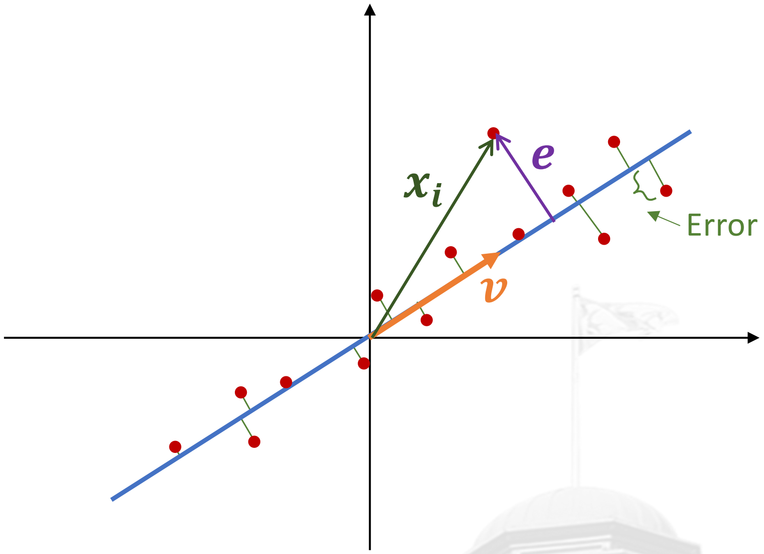
Minimize the error in the orthogonal direction.
Finding the vector $\boldsymbol{v}$ defines the line.

Assume $\boldsymbol{v}$ is normalized: $\|\boldsymbol{v}\| = 1$

$$\boldsymbol{e}_i = \boldsymbol{x}_i - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}$$

Choose $\boldsymbol{v}$ to minimize:

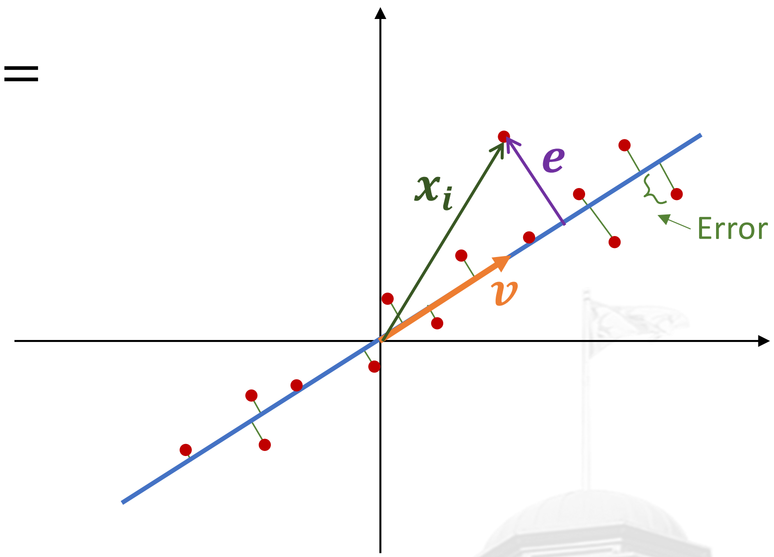
$$\sum_i \|\boldsymbol{e}_i\| = \sum_i \|\boldsymbol{x}_i - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}\|$$



Total Least Squares Approximation

Choose $\boldsymbol{v}$ to minimize: $\sum_i \left\| \boldsymbol{x}_i - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v} \right\|^2 =$

$$= \sum_i \left(\boldsymbol{x}_i - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}\right)^T \left(\boldsymbol{x}_i - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}\right)$$



$$= \sum_i \left(\boldsymbol{x}_i^T \boldsymbol{x}_i - \boldsymbol{x}_i^T (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v} - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}^T \boldsymbol{x}_i + (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}^T (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v}\right)$$

Total Least Squares Approximation

Choose v to minimize: $\sum_i \left\| x_i - (x_i^T v) v \right\|^2 = Const. - \sum_i (x_i^T v)^2$

$$X = \begin{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} & \begin{bmatrix} x_2 \end{bmatrix} & \cdots & \begin{bmatrix} x_n \end{bmatrix} \end{bmatrix}$$

$$X^T v = \begin{bmatrix} [& x_1^T &] \\ [& x_2^T &] \\ & \vdots & \\ [& x_n^T &] \end{bmatrix} v = \begin{bmatrix} x_1^T v \\ x_2^T v \\ \vdots \\ x_n^T v \end{bmatrix}$$

$$\sum_i (x_i^T v)^2 = \left\| X^T v \right\|_2^2$$

Total Least Squares Approximation

Choose $\boldsymbol{v}$ to minimize: $\sum_i \left\| \boldsymbol{x}_i - (\boldsymbol{x}_i^T \boldsymbol{v})\boldsymbol{v} \right\|^2 = Const. - \sum_i (\boldsymbol{x}_i^T \boldsymbol{v})^2$

$$= Const. - \left\| \boldsymbol{X}^T \boldsymbol{v} \right\|_2^2$$

Choose $\boldsymbol{v}$ to maximize: $\left\| \boldsymbol{X}^T \boldsymbol{v} \right\|_2^2$

Subject to: $\|\boldsymbol{v}\|_2 = 1$

Choose $\boldsymbol{v}$ to maximize: $\boldsymbol{v}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v}$

Subject to: $\|\boldsymbol{v}\|_2 = 1$

Total Least Squares Approximation

Choose $\boldsymbol{v}$ to maximize: $\boldsymbol{v}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v}$

Subject to: $\|\boldsymbol{v}\|_2 = 1$

$\boldsymbol{v}$ is the eigenvector corresponding to the largest eigenvalue of $\boldsymbol{X}\boldsymbol{X}^T$

Proof using the Lagrange Multiplier:

$$\mathcal{L}(\boldsymbol{v}, \lambda) = \boldsymbol{v}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v} - \lambda(\boldsymbol{v}^T \boldsymbol{v} - 1)$$

$$\frac{d}{d\boldsymbol{v}} \mathcal{L}(\boldsymbol{v}, \lambda) = 2\boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v} - 2\lambda \boldsymbol{v} = \boldsymbol{0}$$

Total Least Squares Approximation

Proof using the Lagrange Multiplier:

$$\mathcal{L}(\boldsymbol{v}, \lambda) = \boldsymbol{v}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v} - \lambda(\boldsymbol{v}^T \boldsymbol{v} - 1)$$

$$\frac{d}{d\boldsymbol{v}} \mathcal{L}(\boldsymbol{v}, \lambda) = 2\boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v} - 2\lambda \boldsymbol{v} = \boldsymbol{0}$$

Critical points when $\boldsymbol{v}$ is an eigenvector: $\boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v} = \lambda \boldsymbol{v}$

At the critical points: $\boldsymbol{v}^T \boldsymbol{X}\boldsymbol{X}^T \boldsymbol{v} = \lambda \boldsymbol{v}^T \boldsymbol{v} = \lambda$

The maximum occurs when $\boldsymbol{v}$ is the eigenvector corresponding to the largest eigenvalue of $\boldsymbol{X}\boldsymbol{X}^T$

Diagonalization

Consider a full rank matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ with distinct eigen values λ_i and corresponding eigenvectors v_i, $1 \leq i \leq n$.

$$\boldsymbol{A}\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1 \qquad \boldsymbol{A}\boldsymbol{v}_2 = \lambda_1\boldsymbol{v}_2 \qquad \cdots \qquad \boldsymbol{A}\boldsymbol{v}_n = \lambda_n\boldsymbol{v}_n$$

$$\boldsymbol{A}[\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n] = [\lambda_1\boldsymbol{v}_1 \quad \lambda_2\boldsymbol{v}_2 \quad \cdots \quad \lambda_n\boldsymbol{v}_n] = \mathbf{V\Gamma}$$

$$\boldsymbol{A}\mathbf{V} = \mathbf{V\Gamma} \qquad \mathbf{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n] \qquad \mathbf{\Gamma} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\boldsymbol{V}^{-1}\boldsymbol{A}\mathbf{V} = \mathbf{\Gamma} \qquad \boldsymbol{A} = \mathbf{V\Gamma}\boldsymbol{V}^{-1}$$

Application: Powers of a matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \mathbf{V\Gamma}\boldsymbol{V}^{-1}$$

$$\boldsymbol{A}^2 = (\mathbf{V\Gamma}\boldsymbol{V}^{-1})\,(\mathbf{V\Gamma}\boldsymbol{V}^{-1}) \;\; = \mathbf{V\Gamma}(\boldsymbol{V}^{-1}\boldsymbol{V})\,\mathbf{\Gamma}\boldsymbol{V}^{-1} = \mathbf{V\Gamma}^2\boldsymbol{V}^{-1}$$

$$\boldsymbol{A}^m = \mathbf{V\Gamma}^{\mathrm{m}}\boldsymbol{V}^{-1}$$

$$\mathbf{\Gamma}^{\mathrm{m}} = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_1^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

Matlab exp vs expm

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\exp(\boldsymbol{A}) = \begin{bmatrix} e^{a_{11}} & e^{a_{12}} \\ e^{a_{21}} & e^{a_{22}} \end{bmatrix} \neq e^{\boldsymbol{A}t}$$

Useful function but not the matrix exponential.

Matlab exp vs expm

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{expm}(\boldsymbol{A}) = \sum_{m=0}^{\infty} \frac{\boldsymbol{A}^m}{m!} \quad = e^{\boldsymbol{A}}$$

Matlab exp vs expm

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \boldsymbol{V\Gamma V^{-1}}$$

$$e^{\boldsymbol{A}} = \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^m}{m!} = \sum_{m=0}^{\infty} \frac{\boldsymbol{V\Gamma}^m \boldsymbol{V}^{-1}}{m!} = \boldsymbol{V}\left(\sum_{m=0}^{\infty} \frac{\boldsymbol{\Gamma}^m}{m!}\right)\boldsymbol{V}^{-1}$$

$$e^{\boldsymbol{A}} = \boldsymbol{V}\left(\boldsymbol{e}^{\boldsymbol{\Gamma}}\right)\boldsymbol{V}^{-1}$$

Matlab exp vs expm

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}$$

$$e^{\boldsymbol{A}} = \boldsymbol{V}\left(\sum_{m=0}^{\infty}\frac{\boldsymbol{\Gamma}^m}{m!}\right)\boldsymbol{V}^{-1}$$

$$\boldsymbol{\Gamma}^m = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_1^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

$$\sum_{m=0}^{\infty}\frac{\boldsymbol{\Gamma}^m}{m!} = \begin{bmatrix} \sum_{m=0}^{\infty}\frac{\lambda_1^m}{n!} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \sum_{m=0}^{\infty}\frac{\lambda_n^m}{n!} \end{bmatrix}$$

Matlab exp vs expm

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}$$

$$e^{\boldsymbol{A}} = \boldsymbol{V}\left(\sum_{m=0}^{\infty}\frac{\boldsymbol{\Gamma}^{m}}{m!}\right)\boldsymbol{V}^{-1}$$

$$\boldsymbol{\Gamma}^{m} = \begin{bmatrix} \lambda_1^m & & & \\ & \lambda_1^m & & \\ & & \ddots & \\ & & & \lambda_n^m \end{bmatrix}$$

$$\sum_{m=0}^{\infty}\frac{\boldsymbol{\Gamma}^{m}}{m!} = \begin{bmatrix} e^{\lambda_1^m} & & & \\ & e^{\lambda_2^m} & & \\ & & \ddots & \\ & & & e^{\lambda_n^m} \end{bmatrix} = \boldsymbol{e}^{\boldsymbol{\Gamma}}$$

Decoupled Diff Equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{B}$$

Constant vector

Matrix Exponential

Initial Value Problem:
We know the boundary condition $\boldsymbol{x}(0)$

Decoupled Diff Equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{B}$$

Constant vector

Matrix Exponential

$$\boldsymbol{x}(0) = e^{0t}\ \boldsymbol{B} = \boldsymbol{B}$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0)$$

Initial Value

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1}$$

$$\boldsymbol{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \cdots \quad \boldsymbol{v}_n]$$

$$\boldsymbol{x}(0) = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{V}\boldsymbol{c}$$

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$\boldsymbol{V}\boldsymbol{c} = \boldsymbol{x}(0)$$

$$\boldsymbol{c} = \boldsymbol{V}^{-1}\boldsymbol{x}(0)$$

Decoupled Diff Equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$$

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(0) = \left(\boldsymbol{V}\left(e^{\boldsymbol{\Gamma}\mathrm{t}}\right)\boldsymbol{V}^{-1}\right)(\boldsymbol{V}\boldsymbol{c}) = \boldsymbol{V}e^{\boldsymbol{\Gamma}t}\mathbf{c}$$

$$\boldsymbol{x}(t) = \boldsymbol{V}\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_3 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_1 e^{\lambda_2 t}\boldsymbol{v}_2 + \cdots + c_1 e^{\lambda_n t}\boldsymbol{v}_n$$