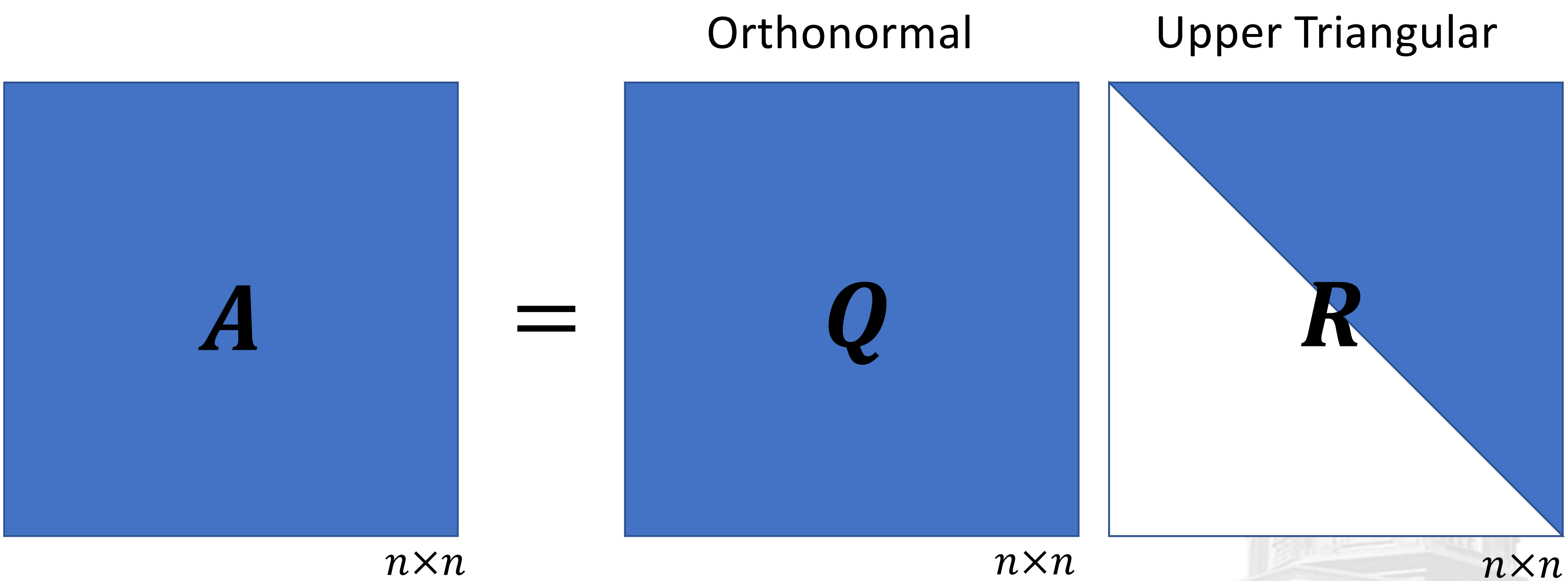
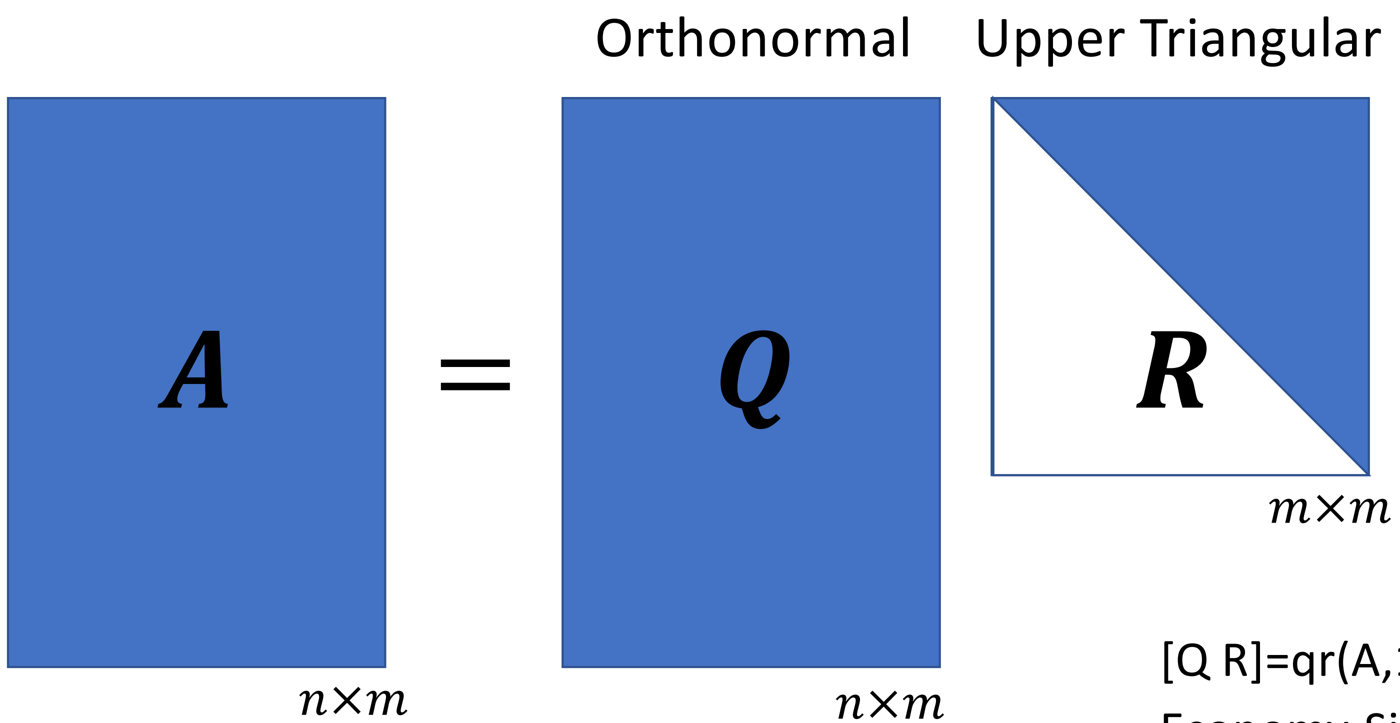
ECSE 343 Numerical Methods in Engineering

Roni Khazaka
Dept. of Electrical and Computer Engineering
McGill University

QR Decomposition



QR Decomposition

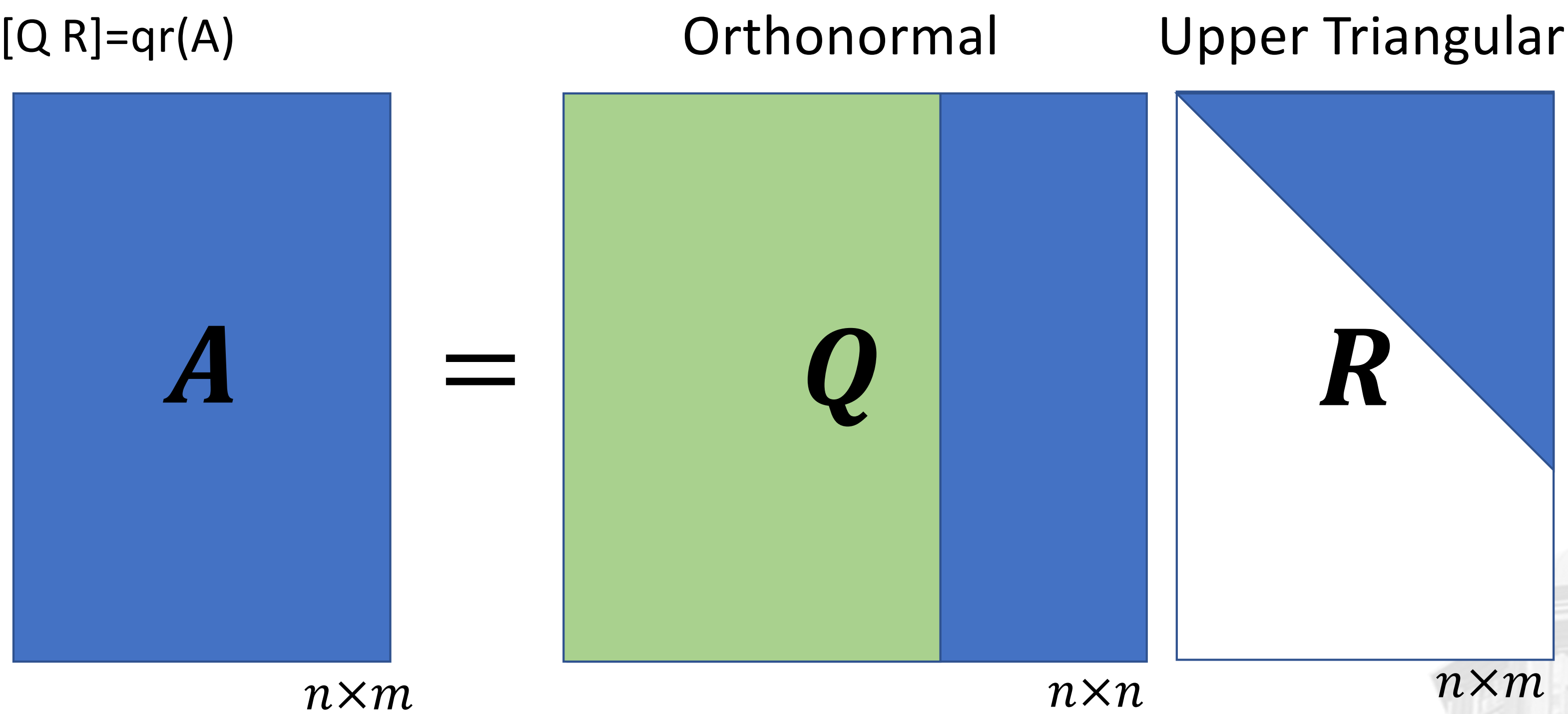


[Q R]=qr(A,1)

Economy-Size

QR Decomposition

[Q R]=qr(A)



QR Decompostion

Orthonormal

$$
\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}
$$

$$
\boldsymbol{q}_i^T \boldsymbol{q}_j = \begin{cases} 1\,; i = j \\ 0\,; i \neq j \end{cases}
$$

Gram-Schmidt Algorithm

Orthonormal

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}$$

$$\boldsymbol{a}_1 = r_{11}\boldsymbol{q}_1$$

$$\boldsymbol{a}_2 = r_{12}\boldsymbol{q}_1 + r_{22}\boldsymbol{q}_2$$

$$\boldsymbol{a}_3 = r_{13}\boldsymbol{q}_1 + r_{23}\boldsymbol{q}_2 + r_{33}\boldsymbol{q}_3$$

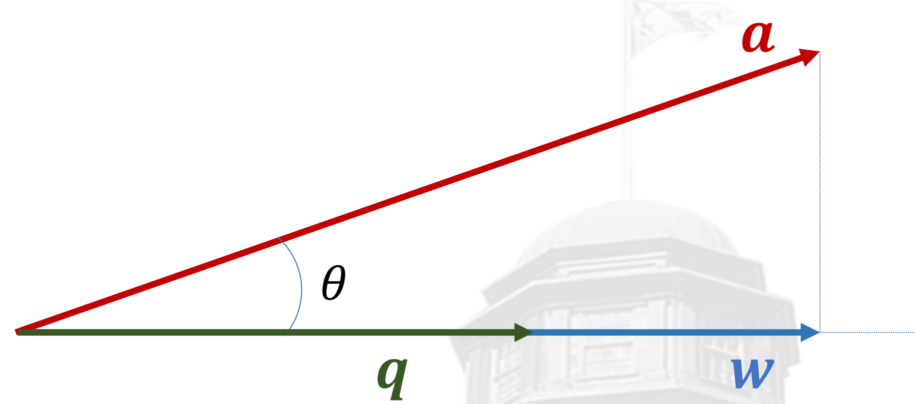
$$colspan\{\boldsymbol{A}\} = colspan\{\boldsymbol{Q}\}$$

Projectors Review

$$\|\boldsymbol{q}\|_2 = \mathbf{1}$$

$$\langle \boldsymbol{a}, \boldsymbol{q} \rangle = \boldsymbol{q}^T \boldsymbol{a} = \|\boldsymbol{q}\|_2 \|\boldsymbol{a}\|_2 \cos(\theta) = \|\boldsymbol{a}\|_2 \cos(\theta)$$

$$\boldsymbol{w} = \boldsymbol{q} \langle \boldsymbol{a}, \boldsymbol{q} \rangle = \boldsymbol{q}\boldsymbol{q}^T \boldsymbol{a}$$



Gram-Schmidt Algorithm

$$\left[\begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix}\right] = \left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix}\right] \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}$$

$$\left.\begin{aligned} \boldsymbol{a}_1 &= \boldsymbol{q}_1 r_{11} \\ \|\boldsymbol{q}_1\| &= 1 \end{aligned}\right\} \quad \begin{aligned} r_{11} &= \|\boldsymbol{a}_1\| \\ \boldsymbol{q}_1 &= \frac{\boldsymbol{a}_1}{r_{11}} \end{aligned}$$

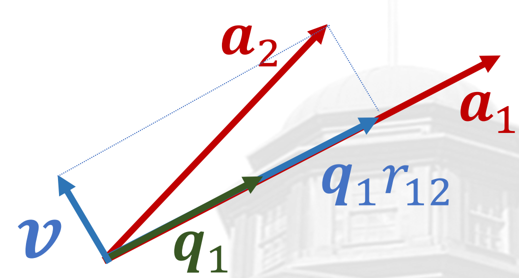
Gram-Schmidt Algorithm

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}$$

$$\boldsymbol{a}_2 = \boldsymbol{q}_1 r_{12} + \boldsymbol{q}_2 r_{22}$$

$$\boldsymbol{q}_1^T \boldsymbol{a}_2 = \boldsymbol{q}_1^T \boldsymbol{q}_1 r_{12} + \boldsymbol{q}_1^T \boldsymbol{q}_2 r_{22} = r_{12}$$

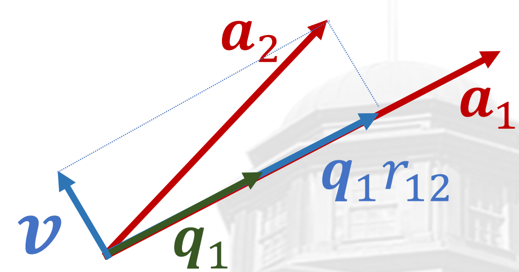
$$\boldsymbol{v} = \boldsymbol{a}_2 - \boldsymbol{q}_1 r_{12} = \boldsymbol{q}_2 r_{22}$$



Gram-Schmidt Algorithm

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}$$

$$\left.\begin{aligned} \boldsymbol{v} &= \boldsymbol{q}_2 r_{22} \\ \|\boldsymbol{q}_2\| &= 1 \end{aligned}\right\} \begin{aligned} r_{22} &= \|\boldsymbol{v}\| \\ \boldsymbol{q}_2 &= \frac{\boldsymbol{v}}{r_{22}} \end{aligned}$$



Gram-Schmidt Algorithm

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}$$

$$\boldsymbol{a}_3 = \boldsymbol{q}_1 r_{13} + \boldsymbol{q}_2 r_{23} + \boldsymbol{q}_3 r_{33}$$

$$\boldsymbol{q}_1^T \boldsymbol{a}_3 = r_{13}$$

$$\boldsymbol{q}_2^T \boldsymbol{a}_3 = r_{23}$$

$$\boldsymbol{v} = \boldsymbol{a}_3 - \boldsymbol{q}_1 r_{13} - \boldsymbol{q}_2 r_{23} = \boldsymbol{q}_3 r_{33}$$

Gram-Schmidt Algorithm

$$\left[\begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix}\right] = \left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \boldsymbol{q}_m \\ \\ \end{bmatrix}\right] \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ & r_{22} & r_{23} & \cdots & r_{2m} \\ & & r_{33} & \cdots & r_{3m} \\ & & & \ddots & \vdots \\ & & & & r_{mm} \end{bmatrix}$$

$$\left.\begin{aligned} \boldsymbol{v} &= \boldsymbol{q}_3 r_{34} \\ \|\boldsymbol{q}_3\| &= 1 \end{aligned}\right\} \begin{aligned} r_{33} &= \|\boldsymbol{v}\| \\ \boldsymbol{q}_3 &= \frac{\boldsymbol{v}}{r_{33}} \end{aligned}$$

Gram-Schmidt Algorithm

$$r_{11} = \|\boldsymbol{a}_1\| \qquad \boldsymbol{q}_1 = \frac{\boldsymbol{a}_1}{r_{11}}$$

$$r_{12} = \boldsymbol{q}_1^T \boldsymbol{a}_2 \qquad \boldsymbol{v} = \boldsymbol{a}_2 - \underbrace{\boldsymbol{q}_1 r_{12}}_{\boldsymbol{p}} \qquad r_{22} = \|\boldsymbol{v}\| \qquad \boldsymbol{q}_2 = \frac{\boldsymbol{v}}{r_{22}}$$

$$r_{13} = \boldsymbol{q}_1^T \boldsymbol{a}_3 \qquad \boldsymbol{v} = \boldsymbol{a}_3 - \underbrace{\boldsymbol{q}_1 r_{13} - \boldsymbol{q}_2 r_{23}}_{\boldsymbol{p}} \qquad r_{33} = \|\boldsymbol{v}\|$$

$$r_{23} = \boldsymbol{q}_2^T \boldsymbol{a}_3 \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{v}}{r_{33}}$$

Gram-Schmidt Pseudocode

```
r_11 ← ||a_1||
q_1 ← a_1/r_11
for j ← 2,3,⋯,m
        p ← 0
        for i ← 1,2,⋯,j − 1
                r_ij ← q_i^T a_j
                p ← p + r_ij q_i
        endfor
        v ← a_j − p
        r_jj ← ||v||
        q_j ← v/r_jj
endfor
```

Gram-Schmidt

- For each vector in A starting from the left, remove its projection on all previously computed vectors in Q, then normalize to length 1.
- The entries of R are computed column by column.

Modified Gram-Schmidt Pseudocode

for $\mathrm{i} \leftarrow 1,2,\cdots,\mathrm{m}$

  $\mathrm{r_{ii}} \leftarrow \|\mathrm{a_i}\|$

  $\mathrm{q_i} \leftarrow \mathrm{a_i}/\mathrm{r_{ii}}$

  for $\mathrm{j} \leftarrow \mathrm{i}+1,\cdots,\mathrm{m}$

    $\mathrm{r_{ij}} \leftarrow \mathrm{q_i^{\mathit{T}}} \mathrm{a_j}$

    $\mathrm{a_j} \leftarrow \mathrm{a_j} - \mathrm{r_{ij}} \mathrm{q_i}$

  endfor

endfor

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_3 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix}\right]$$

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_3 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix}\right]$$

$$r_{11} \leftarrow \|a_1\|$$
$$q_1 \leftarrow a_1/r_{11}$$

Modified Gram-Schmidt

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_3 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} \Rightarrow \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix}$$

for $j \leftarrow 2, \cdots, m$

$\quad r_{1j} \leftarrow q_1^T a_j$

$\quad a_j \leftarrow a_j - r_{1j} q_1$

endfor

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \widetilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \widetilde{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \widetilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix}\right]$$

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \widetilde{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \widetilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix}\right]$$

$$r_{22} \leftarrow \|a_2\|$$

$$q_2 \leftarrow a_2/r_{22}$$

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} \quad \cdots \quad \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix}\right] \Rightarrow \left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \hat{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} \quad \cdots \quad \begin{bmatrix} \\ \hat{\boldsymbol{a}}_m \\ \\ \end{bmatrix}\right]$$

for $\mathrm{j} \leftarrow 3, \cdots, \mathrm{m}$

$\quad \mathrm{r}_{2\mathrm{j}} \leftarrow \mathrm{q}_2^T \mathrm{a}_\mathrm{j}$

$\quad \mathrm{a}_\mathrm{j} \leftarrow \mathrm{a}_\mathrm{j} - \mathrm{r}_{2\mathrm{j}} \mathrm{q}_2$

endfor

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \hat{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \hat{\boldsymbol{a}}_m \\ \\ \end{bmatrix}\right]$$

Modified Gram-Schmidt

$$\left[\begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_3 \\ \\ \end{bmatrix} \cdots \begin{bmatrix} \\ \widehat{\boldsymbol{a}}_m \\ \\ \end{bmatrix}\right]$$

$$r_{33} \leftarrow \|a_3\|$$
$$q_3 \leftarrow a_3 / r_{33}$$

Modified Gram-Schmidt

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_3 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \widehat{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix} \Rightarrow \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_2 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{q}_3 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \breve{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix}$$

for j $\leftarrow 4, \cdots, \mathrm{m}$

$\quad \mathrm{r}_{3\mathrm{j}} \leftarrow \mathrm{q}_3^T \mathrm{a}_\mathrm{j}$

$\quad \mathrm{a}_\mathrm{j} \leftarrow \mathrm{a}_\mathrm{j} - \mathrm{r}_{3\mathrm{j}} \mathrm{q}_3$

endfor

Modified Gram-Schmidt

$$\begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{a}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \boldsymbol{a}_2 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \boldsymbol{a}_m \\ \\ \end{bmatrix} \end{bmatrix} \overbrace{\begin{bmatrix} \frac{1}{r_{11}} & -\frac{r_{12}}{r_{11}} & -\frac{r_{13}}{r_{11}} & \cdots & -\frac{r_{1m}}{r_{11}} \\ & 1 & 0 & \cdots & 0 \\ & & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}}^{\boldsymbol{R}_1} =$$

$$\mathrm{q}_1 \leftarrow \mathrm{a}_1/\mathrm{r}_{11}$$

$$\mathrm{a}_2 \leftarrow \mathrm{a}_2 - \frac{r_{12}}{r_{11}}\mathrm{a}_1 = \mathrm{a}_2 - \mathrm{r}_{12}\mathrm{q}_1$$

$$\mathrm{a}_3 \leftarrow \mathrm{a}_3 - \frac{r_{13}}{r_{11}}\mathrm{a}_1 = \mathrm{a}_3 - \mathrm{r}_{13}\mathrm{q}_1$$

$$= \begin{bmatrix} \begin{bmatrix} \\ \boldsymbol{q}_1 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_2 \\ \\ \end{bmatrix} & \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_3 \\ \\ \end{bmatrix} & \cdots & \begin{bmatrix} \\ \tilde{\boldsymbol{a}}_m \\ \\ \end{bmatrix} \end{bmatrix}$$

Modified Gram-Schmidt

$$\begin{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \end{bmatrix} & \begin{bmatrix} \tilde{\boldsymbol{a}}_2 \end{bmatrix} & \begin{bmatrix} \tilde{\boldsymbol{a}}_3 \end{bmatrix} & \cdots & \begin{bmatrix} \tilde{\boldsymbol{a}}_m \end{bmatrix} \end{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ & \dfrac{1}{r_{22}} & -\dfrac{r_{23}}{r_{22}} & \cdots & -\dfrac{r_{2m}}{r_{22}} \\ & & 1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix}}^{\boldsymbol{R}_2} =$$

$$q_2 \leftarrow a_2/r_{22}$$

$$a_3 \leftarrow a_3 - \frac{r_{23}}{r_{22}} a_2 = a_3 - r_{23} q_2$$

$$a_4 \leftarrow a_4 - \frac{r_{24}}{r_{22}} a_2 = a_4 - r_{24} q_2$$

$$= \begin{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 \end{bmatrix} & \begin{bmatrix} \boldsymbol{q}_2 \end{bmatrix} & \begin{bmatrix} \hat{\boldsymbol{a}}_3 \end{bmatrix} & \cdots & \begin{bmatrix} \hat{\boldsymbol{a}}_m \end{bmatrix} \end{bmatrix}$$

Modified Gram-Schmidt

$$\boldsymbol{A}\underbrace{\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3\cdots\boldsymbol{R}_m}_{\boldsymbol{R}^{-1}} = \boldsymbol{Q}$$

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$$

Another Approach

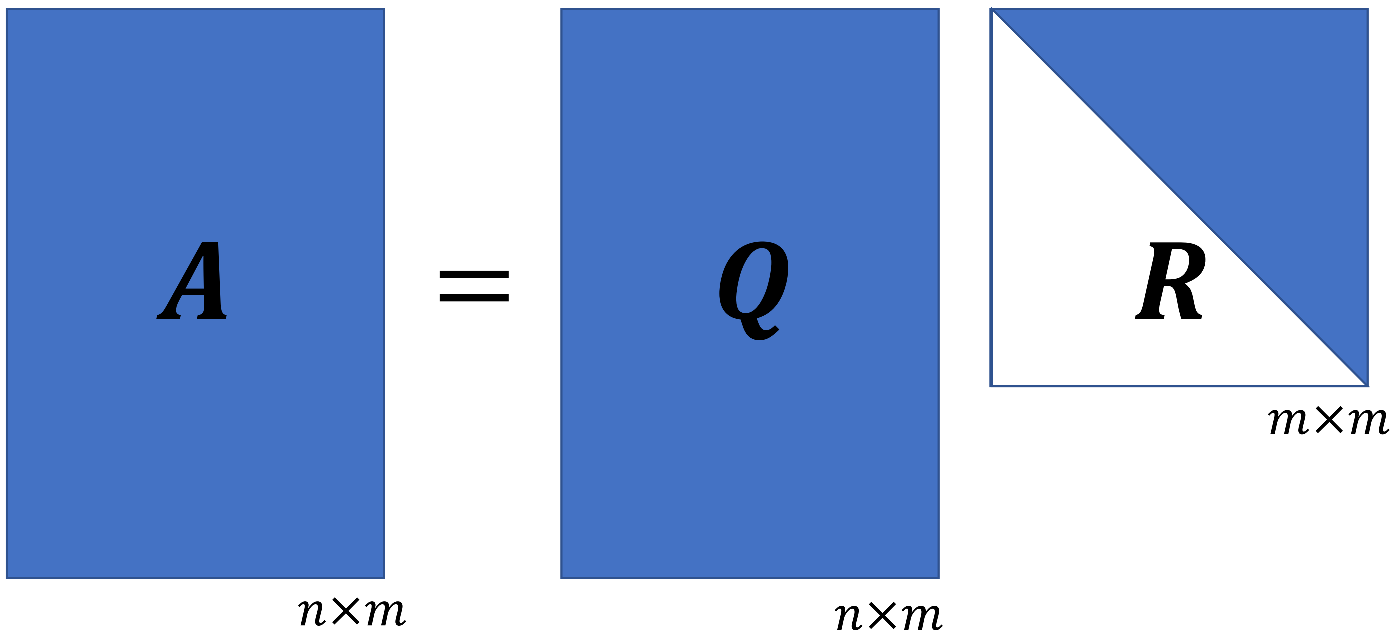
$$\underbrace{\boldsymbol{E}_m \cdots \boldsymbol{E}_4 \boldsymbol{E}_3 \boldsymbol{E}_2 \boldsymbol{E}_1}_{\text{Elimination Matrices} \quad \boldsymbol{L}^{-1}} \boldsymbol{A} = \boldsymbol{U}$$

$$\underbrace{\boldsymbol{H}_m \cdots \boldsymbol{H}_4 \boldsymbol{H}_3 \boldsymbol{H}_2 \boldsymbol{H}_1}_{\text{Orthonormal Matrices} \quad \boldsymbol{Q}^T} \boldsymbol{A} = \boldsymbol{R}$$

LU Factorization

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}}_{\text{Elimination Matrices}}\underbrace{\begin{bmatrix}2 & 1 & -2\\4 & 5 & -3\\6 & 9 & -2\end{bmatrix}}_{\boldsymbol{A}}=\underbrace{\begin{bmatrix}2 & 1 & -2\\0 & 3 & +1\\0 & 0 & +2\end{bmatrix}}_{\boldsymbol{U}}$$

QR Decomposition: Key Property



$$\boxed{\text{colsp}\{\boldsymbol{A}\} = \text{colsp}\{\boldsymbol{Q}\}}$$

Overdetermined Systems

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

- Assume columns of A are linearly independent
- Columns of A form a 3D subspace Q of $\mathbb{R}^5$

Overdetermined Systems

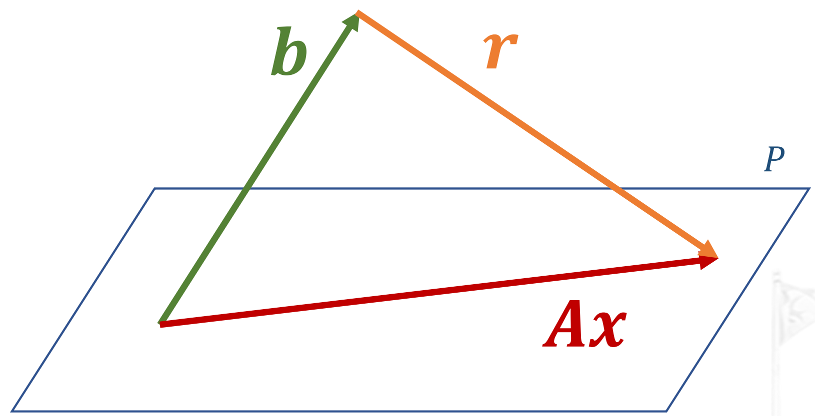
Find $\boldsymbol{x}$ such that: $\boldsymbol{Ax} = \boldsymbol{b}$

Impossible:

$\boldsymbol{Ax} \in P \qquad \boldsymbol{b} \notin P$

Minimize $\boldsymbol{r} = \boldsymbol{Ax} - \boldsymbol{b}$

i.e. minimize $||\boldsymbol{r}|| = ||\boldsymbol{Ax} - \boldsymbol{b}||$



Overdetermined Systems

Minimize $||\boldsymbol{r}|| = ||\boldsymbol{Ax} - \boldsymbol{b}||$

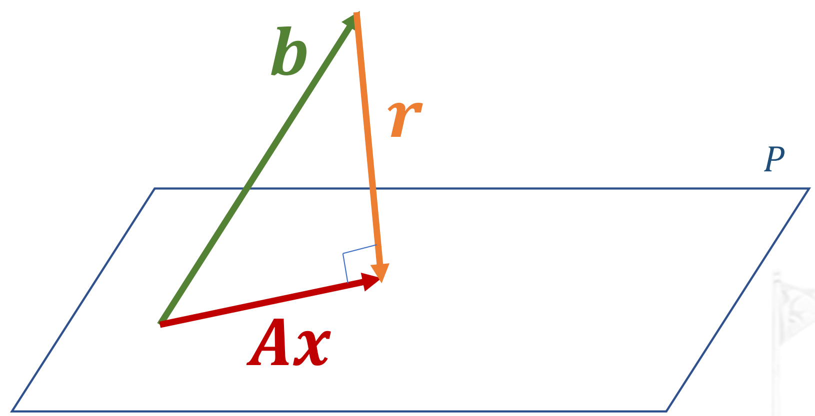
$\Longrightarrow \boldsymbol{r} \perp P$

$\Longrightarrow \boldsymbol{A}^T\boldsymbol{r} = \boldsymbol{0}$

$\Longrightarrow \boldsymbol{A}^T(\boldsymbol{Ax} - \boldsymbol{b}) = \boldsymbol{0}$

$\Longrightarrow \boxed{\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}}$

$\Longrightarrow \boldsymbol{x} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}$



Avoid $\boldsymbol{A}^T\boldsymbol{A}$ due to possible ill-conditioning

Overdetermined Systems

Minimize $||\boldsymbol{r}|| = ||\boldsymbol{Ax} - \boldsymbol{b}||$

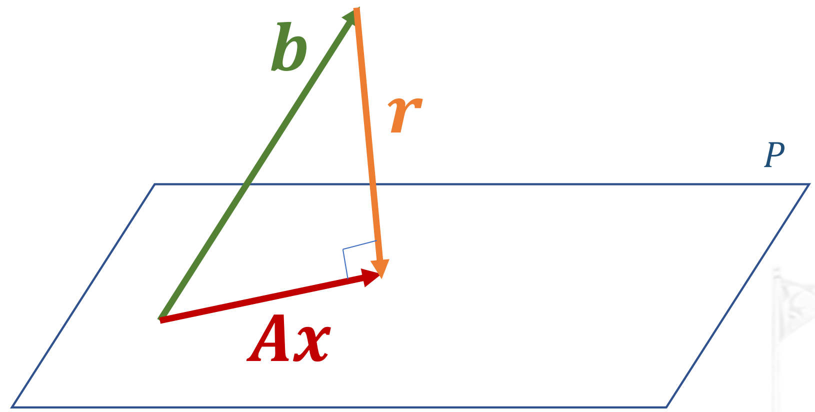
$\Longrightarrow \boldsymbol{r} \perp P$

$\Longrightarrow \boldsymbol{Q}^T \boldsymbol{r} = \mathbf{0}$

$\Longrightarrow \boldsymbol{Q}^T(\boldsymbol{Ax} - \boldsymbol{b}) = \mathbf{0}$

$\Longrightarrow \boxed{\boldsymbol{Q}^T \boldsymbol{Ax} = \boldsymbol{Q}^T \boldsymbol{b}}$

$\Longrightarrow \boldsymbol{Rx} = \boldsymbol{Q}^T \boldsymbol{b}$



$$\boldsymbol{A} = \boldsymbol{QR}$$

$$\text{colsp}\{\boldsymbol{A}\} = \text{colsp}\{\boldsymbol{Q}\}$$