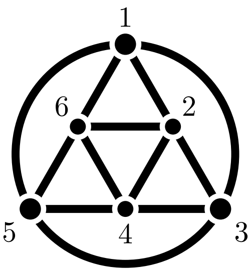
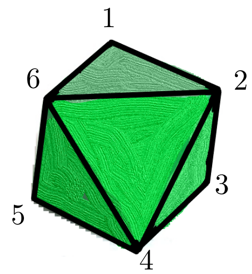


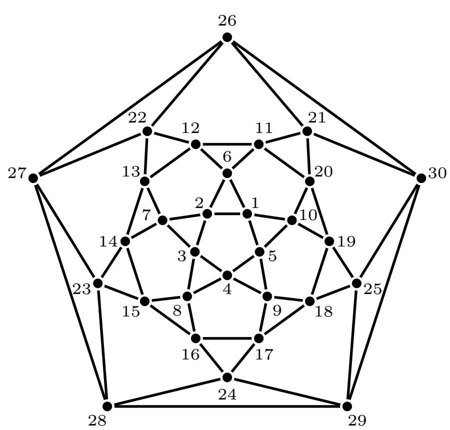


$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Basic Graph Theory: Part Two

June 6, 2023

Shannon Starr
Department of Mathematics
University of Alabama at Birmingham
University Hall, Room 4005
1402 Tenth Avenue South
Birmingham, AL 35294-1241
sℓstarr at *uab* . *edu*



Chapter 1

Euler's closed trail theorem

1 Notation

Let us start with some definitions and notation. Recall that a set is a collection of objects or numbers gathered together, usually between two curly brackets { and }. If we can write a set as

$$\{x_1, \dots, x_n\}\,, \quad \text{such that, whenever } i \neq j \text{, it follows that } x_i \neq x_j, \tag{1.1}$$

then the set has cardinality n, written $|\{x_1, \dots, x_n\}| = n$.

If we have a set $\mathcal{V} = \{v_1, \dots, v_n\}$ with $|\mathcal{V}| = n$ for some $n \in \{1, 2, 3, \dots\}$, then we can call this a finite vertex set. Then the edge set is another set $\mathcal{E}$, written as

$$\mathcal{E} \,=\, \{\{a_1, b_1\}, \dots, \{a_q, b_q\}\}\,, \tag{1.2}$$

for some number $q \in \{0, 1, \dots, n(n-1)/2\}$, and for some vertices $a_1, \dots, a_q, b_1, \dots, b_q \in \mathcal{V}$ such that

- if we have $i \neq j$ then it follows that $\{a_i, b_i\} \neq \{a_j, b_j\}$, meaning that it neither happens that $\big((a_i = a_j)\&(b_i = b_j)\big)$ nor that $\big((a_i = b_j)\&(a_j = b_i)\big)$,
- we have $a_i \neq b_i$ for every i from 1 to q.

Then we may call the pair of the vertex set and the edge set $(\mathcal{V}, \mathcal{E})$ a graph $\mathcal{G}$. A graph is an ordered pair of two sets: the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$ satisfying

$$\mathcal{E} \,\subseteq\, \big\{\{a, b\} \,:\, a, b \in \mathcal{V}\,,\ a \neq b\big\}\,. \tag{1.3}$$

Note that in a *set*, there is no distinction based on order. In other words, $\{a, b\} = \{b, a\}$. Also, repetitions are allowed in sets. That is why we impose the rule that $a \neq b$. In other words, if $\{a, b\}$ is an edge, we do not allow self-edges (sometimes called "loops" by some authors although "loops" frequently refers to more than one type of thing in graph theory). See, for example, Figure 1.1, for a small example of a graph.

For any $k \in \{1, 2, \dots\}$, a *walk* of length k in $\mathcal{G}$ is an ordered $(k+1)$-tuple of vertices

$$\big(x_0\,,\ x_1\,,\ \dots\,,\ x_k\big)\,, \tag{1.4}$$

such that, for each i between 1 and k, we have $\{x_{i-1}, x_i\}$ is in $\mathcal{E}$. (Recall that edges are not oriented because order does not matter.)

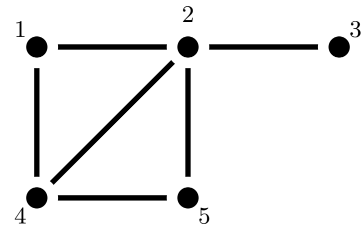


Figure 1.1: This is the plot of an example of a graph with $n = 5$ vertices and $q = 6$ edges. If we label the vertices as $\mathcal{V} = \{1, 2, 3, 4, 5\}$, then the edges are: $\{1, 2\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$ and $\{4, 5\}$. So, to restate: $\mathcal{E} = \{\ \{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\ \}$.

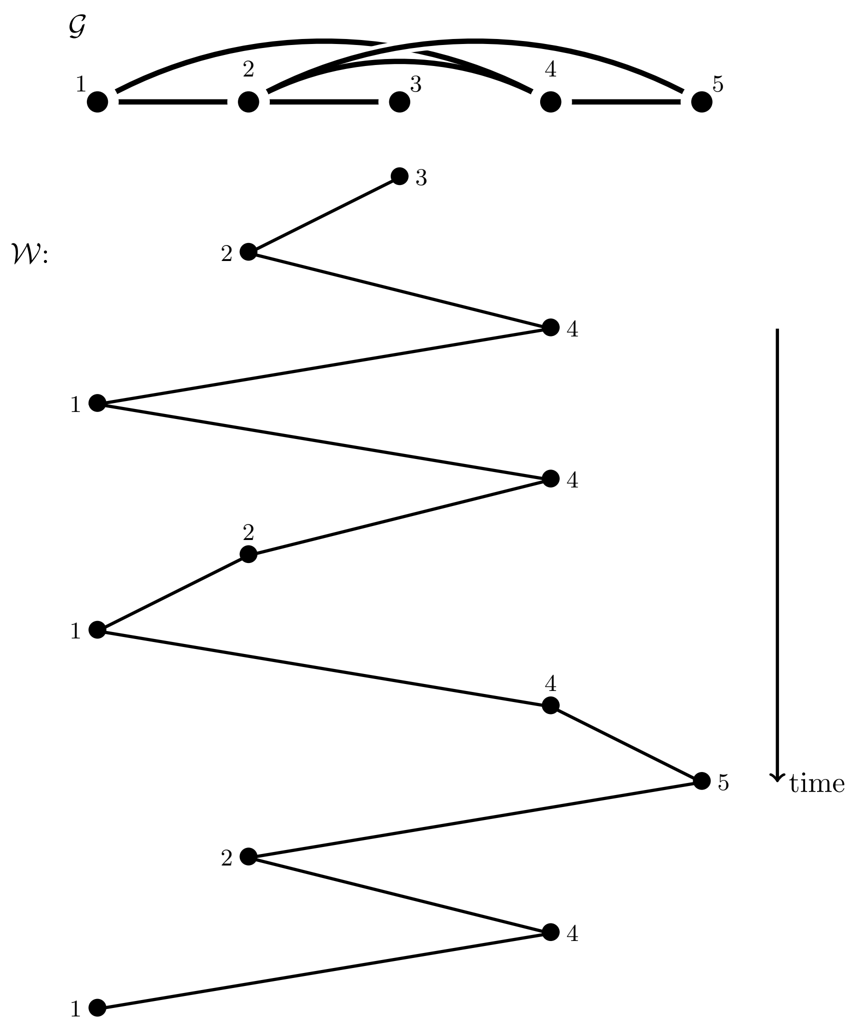


Figure 1.2: This is one way of drawing the walk $\mathcal{W} = (3, 2, 4, 1, 4, 2, 1, 4, 5, 2, 4, 1)$ in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the graph from Figure 1.1. We draw it such that time is going down, from top to bottom.

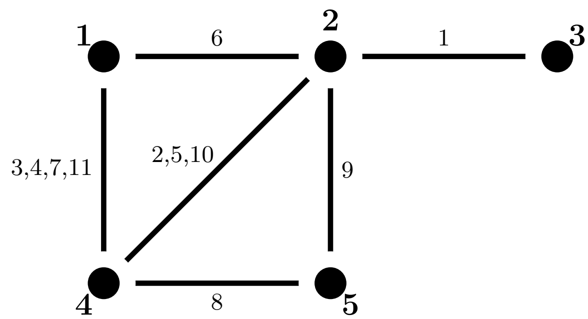


Figure 1.3: This is another way of indicating the walk $\mathcal{W} = (3, 2, 4, 1, 4, 2, 1, 4, 5, 2, 4, 1)$ in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the graph from Figure 1.1. We now list which order each edge is traversed in the walk.

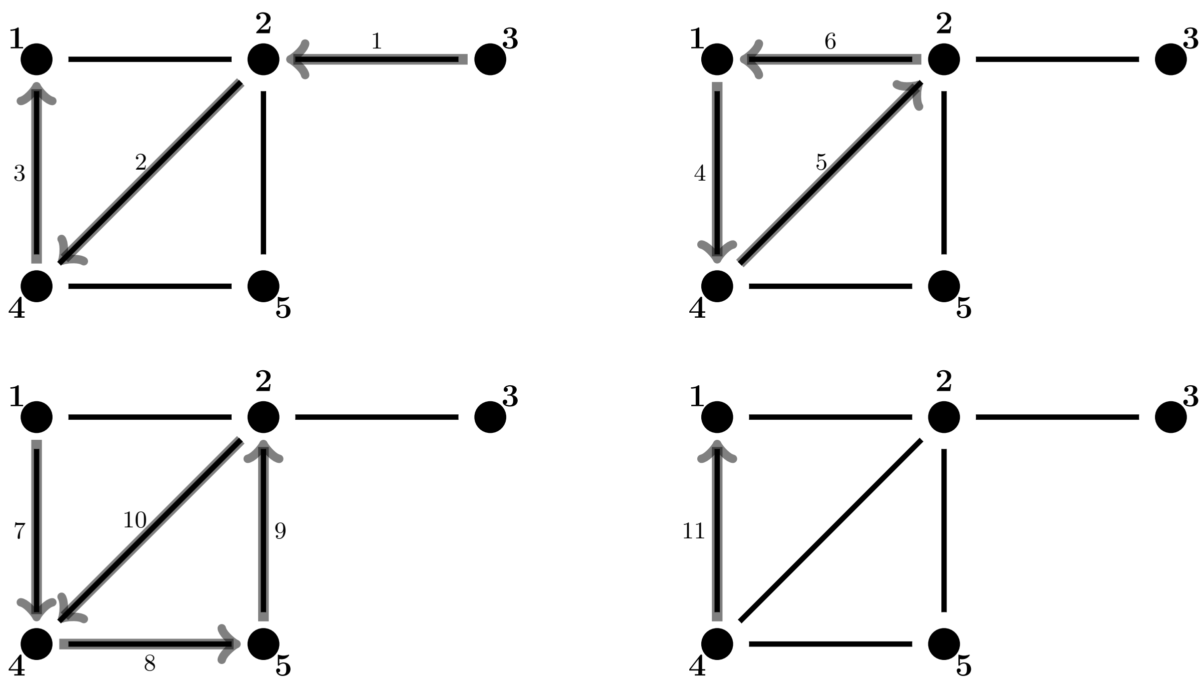


Figure 1.4: This is an amplification of the second way of indicating the walk $\mathcal{W} = (3, 2, 4, 1, 4, 2, 1, 4, 5, 2, 4, 1)$ in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the graph from Figure 1.1. We now list which order each edge is traversed in the walk, and we decompose the walk into successive trails, so that it is easier to visualize each trail using arrows overlaid on the graph. The trails are: $(3, 2, 4, 1)$; then $(1, 4, 2, 1)$; then $(1, 4, 5, 2, 4)$; and finally $(4, 1)$.

A walk on $\mathcal{G}$ may be called a trail if no edge is traversed more than once: for any i and j from 1 to k, if $i \neq j$ then $\{x_{i-1}, x_i\}$ is a distinct set from $\{x_{j-1}, x_j\}$ which means it is neither true that $\big((x_{i-1} = x_{j-1})\&(x_i = x_j)\big)$ nor that $\big((x_{i-1} = x_j)\&(x_{j-1} = x_i)\big)$. Using trails we may visualize a walk in a second, less cumbersome way. We may simply list the order of each edge in the walk, listing which steps of the walk $\{x_{i-1}, x_i\}$ traverse each edge. So, to each edge is assigned a subset of $\{1, \dots, k\}$. This is shown in Figure 1.3 for our earlier example. Or, if we want a better pictorial indication, then we may decompose our walk into successive trails, each of which is easier to indicate as an overlay to the original graph by putting arrows on each edge as well as enumerating the order each edge is traversed as in the second way. See Figure 1.4.

If a walk of length k satisfies that the 0th vertex x_0 is the same as the kth vertex x_k, then the walk is called a *closed* walk.

Given any vertex $x \in \mathcal{V}$, we say that its *degree* is $\deg(x)$, defined as

$$\deg(x) = \big|\{y \in \mathcal{V} : \{x, y\} \in \mathcal{E}\}\big| = \sum_{y \in \mathcal{V}} \mathbf{1}_{\mathcal{E}}(\{x, y\}). \tag{1.5}$$

Then we may state the two main theorems of this chapter.

2 Statement of two main theorems for (undirected) graphs

Theorem 2.1 (Veblen's theorem) *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph with $n = |\mathcal{V}|$ in $\{1, 2, \dots\}$, and if we have the following property*

- *for every $x \in \mathcal{V}$, we have that $\deg(x)$ is an even nonnegative integer, meaning it is in $\{0, 2, 4, 6, \dots, 2k-2, 2k, 2k+2, \dots\}$,*

then the edge set $\mathcal{E}$ may be decomposed into (edge) disjoint cycles.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ we say that $\mathcal{G}$ is *connected* if, for each $x, y \in \mathcal{V}$ we have that there is a finite walk $\mathcal{W} = (z_0, z_1, \dots, z_k)$ for some $k \in \{1, 2, \dots\}$, such that $z_0 = x$ and $z_k = y$.

Theorem 2.2 *If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a <u>connected</u> graph with $n = |\mathcal{V}|$ in $\{1, 2, \dots\}$, and if we have the following property*

- *for every $x \in \mathcal{V}$, we have that $\deg(x)$ is an even nonnegative integer, meaning it is in $\{0, 2, 4, 6, \dots, 2k-2, 2k, 2k+2, \dots\}$,*

then there is a single closed trail in $\mathcal{G}$ such that every edge of $\mathcal{E}$ is an edge of the closed trail, and the length of the trail is the same as $|\mathcal{E}|$.

By the pigeon hole principle, this means that the closed trail in question possesses each edge of $\mathcal{E}$ as an edge of the trail, once and only once. We call such a closed trail a *closed Eulerian trail.*

3 Directed graphs as a tool for proving Veblen's theorem

Suppose that we have a finite vertex set $\mathcal{V}$, with $n = |\mathcal{V}|$. A directed edge set is a set of ordered pairs $\vec{\mathcal{E}} = \{(a_1, b_1), \dots, (a_q, b_q)\}$ for some q between 0 and $n(n-1)$ such that each (a_i, b_i) satisfies

$a_i, b_i \in \mathcal{V}$ and $a_i \neq b_i$. But now if x and y are two distinct vertices in $\mathcal{V}$, then we do distinguish between the directed edge (x, y) and the directed edge (y, x).

For the directed edge (x, y) we would say it is the directed edge from x to y. We can say that x is the initial endpoint and y is the terminal endpoint of the directed edge. Now we may define two *degrees* for each vertex, the *in-degree*

$$\deg^-(x) = \sum_{y \in \mathscr{V}} \mathbf{1}_{\vec{\mathcal{E}}}\big((y, x)\big), \tag{1.6}$$

and the *out-degree*

$$\deg^+(x) = \sum_{y \in \mathscr{V}} \mathbf{1}_{\vec{\mathcal{E}}}\big((x, y)\big). \tag{1.7}$$

Let us say that the directed graph $(\mathcal{V}, \vec{\mathcal{E}})$ is *balanced* if the following property is true:

- for every vertex x in $\mathcal{V}$ we have $\deg^-(x) = \deg^+(x)$.

Let us define a walk of length k in the directed graph $(\mathcal{V}, \vec{\mathcal{E}})$ to be an ordered $(k+1)$-tuple $(x_0, x_1, \ldots, x_k)$ such that each (x_{i-1}, x_i) is a directed edge in $\vec{\mathcal{E}}$. Let us say that a walk of length k in the directed graph $(\mathcal{V}, \vec{\mathcal{E}})$ is a cycle if

$$|\{x_1, \ldots, x_k\}| = k \quad \text{and} \quad x_0 = x_k\,. \tag{1.8}$$

Theorem 3.1 (Veblen's theorem for directed graphs) *If $\mathcal{V}$ is a set with $n = |\mathcal{V}|$ in $\{1, 2, \ldots\}$ and if $(\mathcal{V}, \vec{\mathcal{E}})$ is a directed graph, which is also balanced, then $\vec{\mathcal{E}}$ can be decomposed into (edge) disjoint cycles.*

In order to prove Veblen's theorem, let us note the following lemma that we call pre-Veblen's lemma

Lemma 3.2 (pre-Veblen's lemma) *Suppose $\mathcal{V}$ is a set with $n = |\mathcal{V}|$ in $\{1, 2, \ldots\}$ and suppose that $(\mathcal{V}, \vec{\mathcal{E}})$ is a directed graph satisfying the following properties for a pair of distinct vertices x and y in $\mathscr{V}$:*

- *we have* $\deg^+(x) - \deg^-(x) = 1$*;*
- *we have* $\deg^+(y) - \deg^-(y) = -1$*;*
- *for every $z \in \mathcal{V} \setminus \{x, y\}$, we have* $\deg^+(z) = \deg^-(z)$.

Then there is a walk of some length k, $(z_0, z_1, \ldots, z_k)$ in the directed graph $(\mathcal{V}, \vec{\mathcal{E}})$ such that $z_0 = z$ and $z_k = y$.

Proof: This is proved by induction on $q = |\vec{\mathcal{E}}|$. If $q = 1$ then $\vec{\mathcal{E}}$ must be equal to (x, y) because

$$\deg^+(x) = \deg^-(x) + 1 \geq 1 \quad \text{and} \quad \deg^-(y) = 1 + \deg^+(y) \geq 1\,. \tag{1.9}$$

In that case there is a walk of length 1 in $(\mathcal{V}, \vec{\mathcal{E}})$ which attains the desired conclusion, namely (z_0, z_1) with $z_0 = x$ and $z_1 = y$.

Now for the induction hypothesis, suppose we have an integer $q > 1$ and let us suppose that whenever we have a directed graph with fewer than q directed edges we may prove the statement in the lemma. But for the induction step, let us assume that $|\vec{\mathcal{E}}| = q$, the first integer not covered already by our induction hypothesis. We will prove the lemma for $(\mathcal{V}, \vec{\mathcal{E}})$ by reducing to a case amenable to the induction hypothesis.

Let us write $\deg^{\pm}(z)$ as $\deg^{\pm}(z; \vec{\mathcal{E}})$ in order to emphasize its dependence on the directed edge set. By (1.9), we know that $\deg^+(x) \geq 1$. So there is an element $z \in \mathcal{V}$ such that $(x, z) \in \vec{\mathcal{E}}$. If $z = y$ then we may take our walk to be z_0, z_1 with $z_0 = x$ and $z_1 = z = y$, as in the base case.

If $z \neq y$, then consider the new directed edge-set $\vec{\mathcal{E}}' = \vec{\mathcal{E}} \setminus \{(x, z)\}$. Then it is easy to see that

$$\deg^{\pm}(w; \vec{\mathcal{E}}') \;=\; \deg^{\pm}(w; \vec{\mathcal{E}}) - \frac{1}{2} \cdot \mathbf{1}_{\{x,z\}}(w) \pm \frac{1}{2} \left(\mathbf{1}_{\{z\}}(w) - \mathbf{1}_{\{x\}}(w)\right) \,, \tag{1.10}$$

because we only change the degrees of x and z in going from $\vec{\mathcal{E}}$ to $\vec{\mathcal{E}}'$, by decreasing the out-degree of the former by 1, and decreasing the in-degree of the latter by 1. But then we see that

$$\deg^+(x; \vec{\mathcal{E}}') \;=\; \deg^-(x; \vec{\mathcal{E}}') \quad \text{and} \quad \deg^+(z; \vec{\mathcal{E}}') - \deg^-(x; \vec{\mathcal{E}}') \;=\; 1\,. \tag{1.11}$$

Therefore, since $|\vec{\mathcal{E}}'| = q - 1$, the induction hypothesis applies to guarantee that there is a walk in $(\mathcal{V}, \vec{\mathcal{E}}')$, of some length k', written

$$w_0\,,\; w_1\,,\; \dots\,,\; w_{k'}\,, \tag{1.12}$$

such that $w_0 = z$ and $w_{k'} = y$. But then to complete the induction step, let $k = k' + 1$ and define the wall $(z_0, z_1, \dots, z_k)$ by taking $z_0 = x$ and for $i \geq 1$ let $z_i = w_{i-1}$. Since (x, z) is a directed edge of $\vec{\mathcal{E}}$, we do have that $(z_0, z_1, \dots, z_k)$ is a walk in $(\mathcal{V}, \vec{\mathcal{E}})$. So the induction step has been proved. □

Corollary 3.3 *We may assume that the walk in the last lemma is a trail, meaning that no directed edge of $\vec{\mathcal{E}}$ is used more than once.*

Proof: This follows by consideration of the proof from before. For the base case, where $q = 1$, or more generally for the case of the walk having $k = 1$ so that $z_0 = x$ and $z_1 = y$, the walk only has 1 directed edge. So it is a trail, trivially.

In the case of the induction step, note that $(w_0, w_1, \dots, w_k')$ does not have the edge (x, z) even once because it is a walk in $(\mathcal{V}, \vec{\mathcal{E}}')$ and $\vec{\mathcal{E}}'$ has (x, z) removed. So if $(w_0, w_1, \dots, w_k')$ is a trail in $(\mathcal{V}, \vec{\mathcal{E}}')$, then $(x, w_0, w_1, \dots, w_{k'})$ is a trail in $(\mathcal{V}, \vec{\mathcal{E}})$. □

Before finishing the proof of Veblen's theorem for the directed graph, which will be accomplished using the pre-Veblen lemma and induction, let us consider the connection between directed graphs and undirected graphs in another section.

4 The pre-Veblen lemma for undirected graphs and "directing"

Lemma 4.1 (pre-Veblen's lemma for undirected graphs) *Suppose $\mathcal{V}$ is a set with $n = |\mathcal{V}|$ in $\{1, 2, \dots\}$ and if $(\mathcal{V}, \mathcal{E})$ is a graph satisfying the following properties for a pair of distinct vertices x and y in $\mathscr{V}$:*

- *we have* $\deg(x)$ *and deg(y) are each odd positive integers;*
- *for every $z \in \mathcal{V} \setminus \{x, y\}$, we have* $\deg(z)$ *is an even nonnegative integer.*

Then there is a walk of some length k, $(z_0, z_1, \ldots, z_k)$ in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ such that $z_0 = z$ and $z_k = y$.

Proof: The proof is so similar to the proof of the corresponding lemma for directed graphs that we leave it as an exercise for the reader. □

If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph, and if $\vec{\mathcal{E}}$ is a directed edge set on $\mathcal{V}$, then let us say that $(\mathcal{V}, \vec{\mathcal{E}})$ is a *directing* of $\mathcal{G}$ if

$$\mathcal{E} \;=\; \{\{x, y\} \;:\; (x, y) \in \mathcal{V}\}\,. \tag{1.13}$$

Of particular interest will be how to go from a graph $\mathcal{G}$ where every vertex has even degree to a directing of $\mathcal{G}$ with $(\mathcal{V}, \vec{\mathcal{E}})$ such that every vertex is balanced. But before we become too concerned with the complexity of this task, note that if we succeed in proving Veblen's lemma then we may simply orient each of the cycles, and thereby obtain such a balanced directing. The reason for why we would want to do this may elude the reader, but let us hold off until we mention the combinatorial question related to this.

Bibliography

[1] Subho Bannerjee and Blake Wilkerson. Asymptotic expansions of Lambert series and related q-series. *Int. J. Number Theory* **13**, no. 8, 2097–2113 (2017).

[2] Kai-Lai Chung and William Feller. On fluctuations in coin-tossing. *Proc. Nat. Acad. Sci.* **35**, 605–608 (1949).

[3] Herbert B.Enderton. *Elements of Set Theory, 1st Edition.* Academic Press, San Diego, CA 1977.

[4] William Feller. *An Introduction to Probability Theory and Its Applications, Volume 1, 3rd Edition.* John Wiley & Sons, USA 1968.

[5] Paul Lévy. Sur certains processus stochastiques homogènes. Compositio Mathematica **7**, 283–339 (1940).

[6] Alberto Grünbaum. A Feynman-Kac approach to a paper of Chung and Feller on fluctuations in the coin-tossing game. *Proc. Amer. Math. Soc. to appear.* `https://doi.org/10.1090/proc/14758`

[7] Henry McKean. *Probability: The Classical Limit Theorems.* Cambridge University Press, Cambridge, UK 2014.

[8] G. N. Watson. The Final Problem: An Account of the Mock Theta functions. *J. London Math. Soc.* **11**, 55–80 (1936).

[9] Herbert S. Wilf. *Generatingfunctionology.* Academic Press, Boston, MA 1990.