Homework Assignment 4 – MA 637-OV

Graph Theory and Combinatorics

Due: Round 1, July 3, **Final due date:** Round 2, July 14

1. Let q be an integer in $\{2, 3, \dots\}$ and let $\mathcal{C}_n = \{0, \dots, q-1\}^n$ for each $n \in \{1, 2, \dots\}$. Define $\mathcal{C}_0$ to be the set $\{\emptyset\}$: it is the set containing only one element, and that element is itself the empty set. For $n \in \{1, 2, \dots\}$, let us take $A_n \subset \mathcal{C}_n$ to be the set

$$A_n = \mathcal{C}_n \setminus \{(0, \dots, 0)\}. \tag{1}$$

Explain why all of the $q^n - 1$ elements $(x_1, \dots, x_n)$ remaining in A_n necessarily have a largest element of the set $\{k \in \{1, \dots, n\} : x_k \neq 0\}$.

2. Continuing with problem 1, let $(x_1, \dots, x_n)$ be an element of A_n and let r be the largest element of the set $\{k \in \{1, \dots, n\} : x_k \neq 0\}$. Let $\rho(x_1, \dots, x_n)$ be equal to this r. Define $\phi : A_n \to \{1, \dots, q-1\}$ by the formula

$$\phi(x_1, \dots, x_n) = x_{\rho(x_1, \dots, _n)}. \tag{2}$$

Now, for each $r \in \{1, \dots, n\}$, let $B_n(r)$ denote the set

$$B_{n,r} = \{(x_1, \dots, x_n) \in A_n : \rho(x_1, \dots, x_n) = r\}. \tag{3}$$

For $r \in \{2, \dots, n\}$, define $\Psi_r : B_{n,r} \to \mathcal{C}_{r-1}$ to be the mapping such that

$$\Psi_r(x_1, \dots, x_n) = (x_1, \dots, x_{\rho(x_1, \dots, x_n)-1}). \tag{4}$$

For the special case of $r = 1$, define $\Psi_1 : B_{n,1} \to \mathcal{C}_0$ to just be $\Psi(x_1, \dots, x_n) = \emptyset$.

Explain why $|B_n(r)| = (q-1)q^{n-1}$.

3. Why does problem 2 imply that $q^n - 1$ is equal to $(q-1)$-times-$\sum_{k=0}^{n-1} q^k$? *(Show your work.)*

4. Adapt the computer program from class to find a (random) closed Eulerian trail from 1 to 7 in the following graph