Homework Assignment 2 – MA 637-OV

Graph Theory and Combinatorics

Due: Round 1, June 19, **Final due date:** Round 2, June 29

1. The capital-pi product notation is

$$\prod_{i=1}^{n} f(i) \;=\; g(n)\,,$$

where we define $g(0)$ to be 1 by convention, and for all $n \in \mathbb{N} = \{1, 2, \dots\}$, we define $g(n)$ inductively as

$$g(n) \;=\; \left(\prod_{i=1}^{n-1} f(i)\right) \cdot f(n)\,, \quad \text{so} \quad g(n) \;=\; g(n-1) \cdot f(n)\,.$$

For any number $n \in \{0, 1, \dots\}$ and any $x \in \{0, 1, 2, \dots\}$ such that $x \geq n$, we define the binomial coefficient x-choose-n as

$$\binom{x}{n} \;=\; \frac{x(x-1)(x-2)\cdots\big(x-(n-1)\big)}{1 \cdot 2 \cdot 3 \cdots n}\,.$$

Suppose that x is fixed. Find a function $g(1), g(2), \dots, g(x)$ such that

$$\binom{x}{n} \;=\; \prod_{i=1}^{n} g(i)\,.$$

Note: The function $g(i)$ does depend on x, but we leave that dependence implicit in this problem.

2. Continuing with the last problem, let us now make the dependence on x explicit by writing $G_i(x)$ for the function from Problem 1 that we called $g(i)$, there. Prove that $\prod_{i=1}^{n} G_i(x)$ is equal to 0 if x is any number $0, 1, 2, \dots, n-1$.

3. Use proof-by-induction to prove that

$$\sum_{i=1}^{n} \binom{i}{k} \;=\; \binom{n+1}{k+1}\,. \tag{1}$$

(Show your work.)

4. Find a bijective proof of the same result from Problem 3.