Optimizing Optical Response: Computational Bounds to Quality Factor in Transverse Electric Modes

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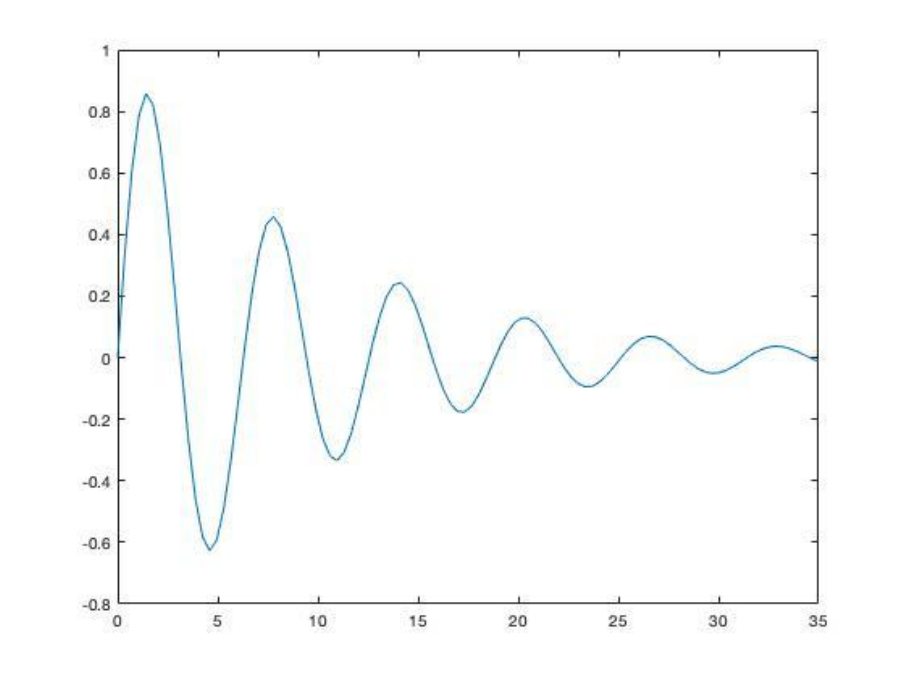
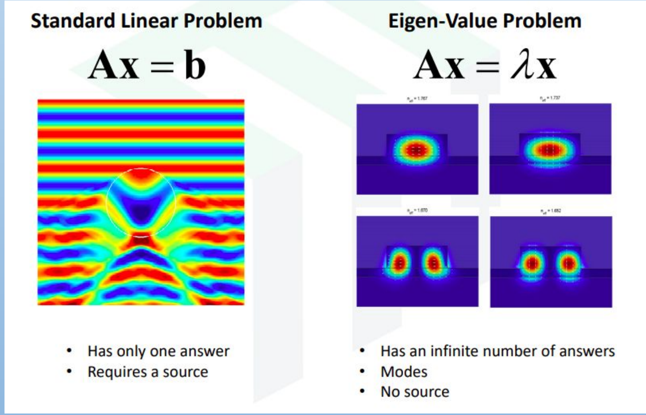
Quality Factor

- Lifetime of resonant energy
- Describes how *underdamped* an oscillator is
- High Q → low damping
- Low Q → high damping
- Can describe: acoustic, electric, or optical resonators



Simple Damped Oscillation of low Q

Resonant Modes

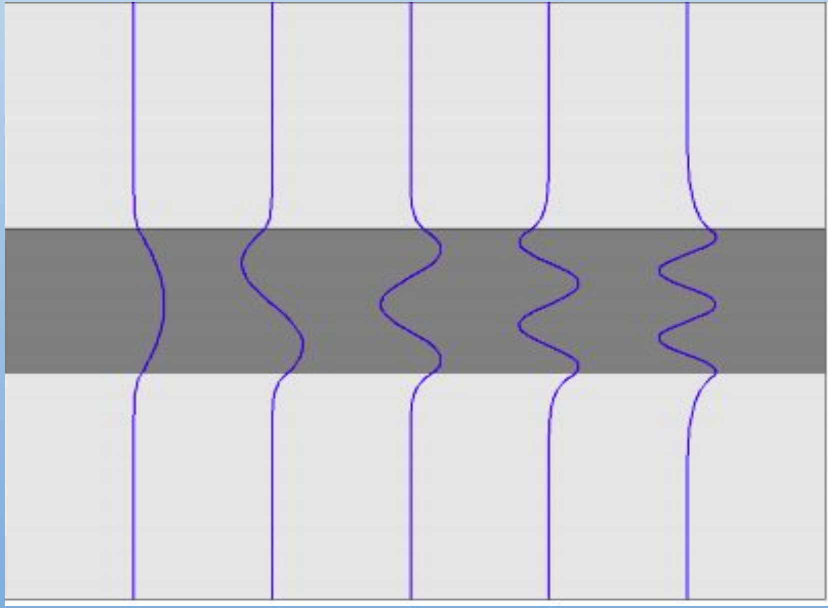


Linear Problem vs Eigenvalue Problem

- Resonant modes of oscillators with losses
- Example: the vibrations of a tuning fork
- Electromagnetic Context: modes of non-Hermitian systems (i.e. Quasi-Normal Modes QNMs) with complex frequency
- Yields a solvable generalized eigenproblem

Solving and Normalizing QNMs

- Electromagnetic Mode problems can be solved computationally
- We consider non-magnetic material (μ = 1) and Transverse Electric Modes
- Harmonic, time-varying electric field: $\nabla \times \nabla \times E(x) = \omega^2 \varepsilon(x) E(x)$
    - Generalized eigenproblem: $\tilde{A}v = \lambda Bv$
- Include "Perfectly Matched Layer" (PML)



Modes of 1D Waveguide Slab

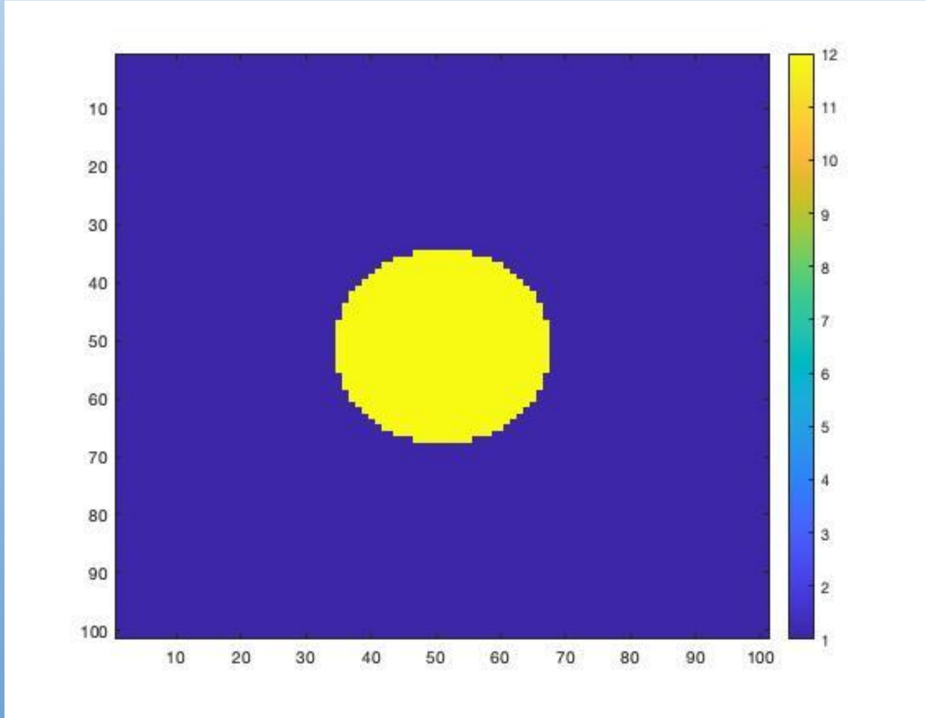
MATLAB Implementation

- Generalized eigenproblem: $\tilde{A}v = \lambda Bv$

  [V,D] = eigs(X,n, omega^2)

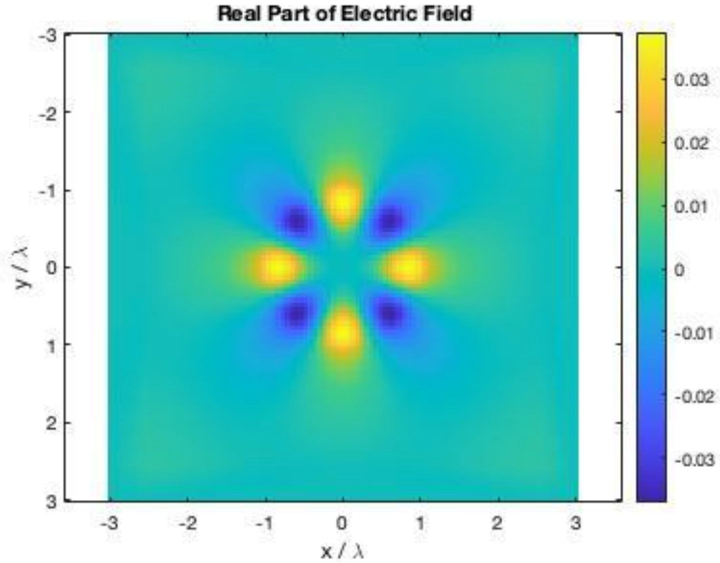
- Geometry: 2D cylinder

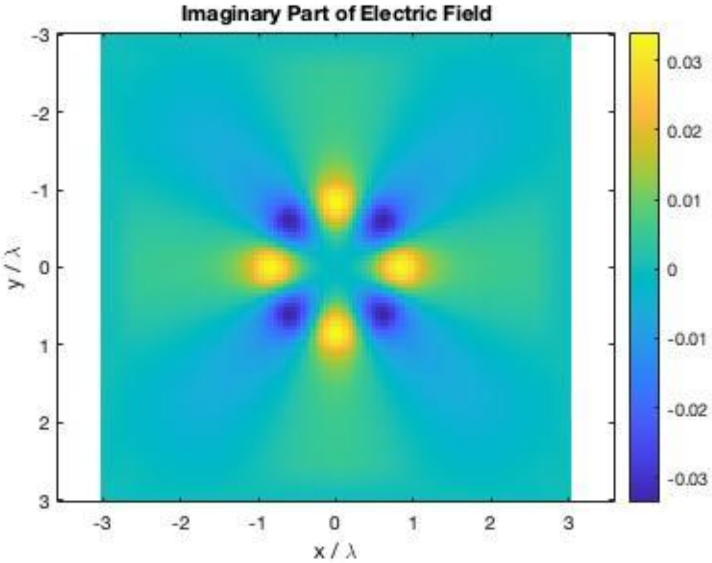
Dielectric Cylinder



Eigenmode Example

- $\omega = 1.7997 - 0.0020865i$





Quality Factor: Analytical and VIE Definition

- $Q = 2\pi \frac{Stored\ Energy}{Power\ Loss}$
- Complex frequency → analytical Q: $Q = -\frac{Re(\omega)}{2Im(\omega)}$
- Assuming high Q → $\omega \approx \omega_r$

$$Q \approx \omega_r \frac{U_{store}}{P_{loss}}$$

Both U and P can be computed by frequency dependent volume integrals

Dyadic Green's Function Implementation

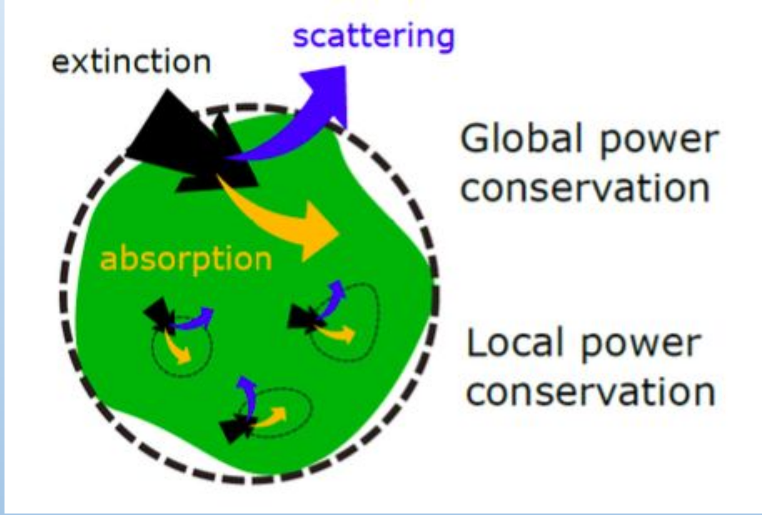
- Dyadic Green's Function: impulse response to linear operator between all points
- Design region inside cylinder: $N_x N_y = N$

$$G_0 p = E_{rad}$$

$$\begin{pmatrix} G_0\big((x_1,y_1),(x_1,y_1)\big) & G_0\big((x_1,y_1),(x_2,y_1)\big) & \cdots \\ G_0\big((x_2,y_1),(x_1,y_1)\big) & \ddots & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} p(x_1,y_1) \\ p(x_2,y_1) \\ \vdots \\ p(N_x,N_y) \end{pmatrix} = \begin{pmatrix} E_{rad}(x_1,y_1) \\ E_{rad}(x_2,y_1) \\ \vdots \\ E_{rad}(x_{N_x},y_{N_y}) \end{pmatrix}$$

Local Power Constraints

- Energy Conservation Method:
  - Maximize *f*
  - Subject to "energy conservation constraint
- Quasi-Convex Optimization Problem
- D matrix constraints chosen by those maximally violated by polarization current constraint

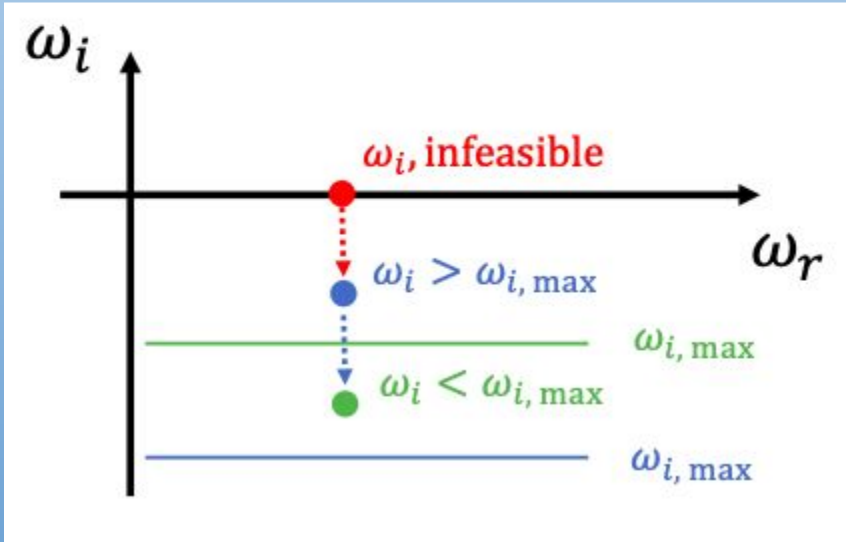


Constraints for Feasibility Problem

$$
\begin{aligned}
\underset{p,\mathrm{Im}\{\omega\}}{\text{maximize}} \quad & \mathrm{Im}\{\omega\} \\
\text{s.t.} \quad & p^\dagger \,\mathrm{Im}\{\omega^* \mathbb{G}_0(\omega)\}\, p + p^\dagger \,\mathrm{Im}\{\omega^* \xi(\omega)\}\, p = 0 \\
& p^\dagger \,\mathrm{Re}\{\mathbb{D}_i \left[\omega^* \mathbb{G}_0(\omega) + \omega^* \xi(\omega)\right]\}\, p = 0, \quad i = 1, \ldots, N \\
& p^\dagger p = 1.
\end{aligned}
$$

Proposed Algorithm for Quality Factor Bound

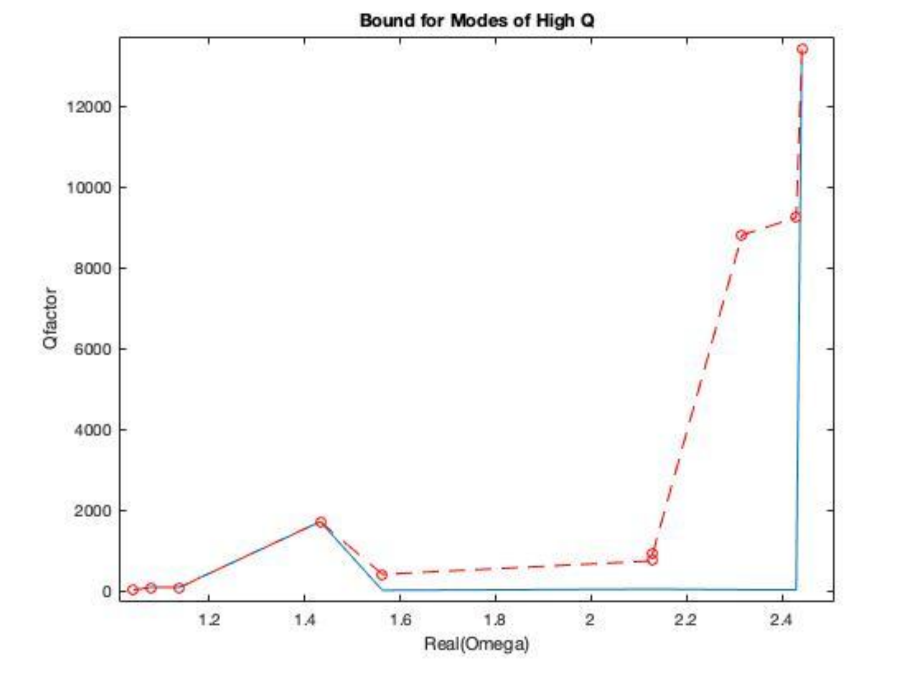
1. Start with $\omega_i = 0$ (mathematically infeasible)
2. If infeasible, reduce ω_i
3. If feasible, deduce $Q_{max} = -\frac{\omega_r}{2\omega_{i,max}}$
4. Repeat for new ω_r



Approximate Q Bound

- Choose mode with known high Q (ω = 1.813 - 0.002i)
- Computed 20 Modes with ND = 5

```
w = 1.040376 0.000000i
Testing wi = 0.000000
Warning: 3 of the 6 requested eigenvalues converged. Eigenvalues that did not converge are NaN.
> In eigs (line 176)
  In extract_p_opt (line 8)
  In bound_Ds (line 29)
  In cal_DmatrixBound (line 32)
  In Q_factor_Bound_test3 (line 82)
add a random D matrix as Dopt ...
 D matrix: 3 / 5: 10.07 seconds, + status: Infeasible
```



Discussion & Further Work

- Approximate bound is possible in calculating Q factors for QNMs
- Requires more rigorous testing
- Applications in optics, nanophotonics, and plasmonics
- Opens possibility of minimum mode volume problem

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- Qingqing Zhao

References

EMPossible. (2020, November 12). *Computational Electromagnetics*. https://empossible.net/academics/emp5337/

Faryad, M., & Lakhtakia, A. (2018). *Infinite-Space Dyadic Green Functions in Electromagnetism (Iop Concise Physics)*. IOP Concise Physics.

Kuang, Z., & Miller, O. D. (in press). Computational bounds to light–matter interactions via local conservation laws. *Department of Applied Physics and Energy Sciences Institute, Yale University*.

Lalanne, P., Yan, W., Vynck, K., Sauvan, C., & Hugonin, J.-P. (2018). Light Interaction with Photonic and Plasmonic Resonances. *Laser & Photonics Reviews*, *12*(5), 1–38. https://doi.org/10.1002/lpor.201700113

Luo, Z.-, Ma, W.-, So, A., Ye, Y., & Zhang, S. (2010). Semidefinite Relaxation of Quadratic Optimization Problems. *IEEE Signal Processing Magazine*, *27*(3), 20–34. https://doi.org/10.1109/msp.2010.936019