On the Accuracy of Coupled Mode Theory With Applications in Integrated Photonic Devices

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Abstract

This paper outlines the development and accuracy of coupled mode theory (CMT). We examine a few examples within the conventional framework, in both time and space, examine its accuracy. We then examine a scattering problem which exposes some shortcomings of conventional CMT and from there develop an alternative framework by treating the system as non-Hermitian. This alternative framework yields much more accurate results in the case of a scatterer.

1 Introduction

Integrated photonic systems offer a slew of applications in the world of electronics, in particular because of their potential to replace electronics altogether. While electronic systems operate in dimensions that are much smaller than their constituent wavelength, photonic systems, such as waveguides and cavities, have physical components with dimensions on the wavelength scale. In order to analyze such systems then, we need a tool that describes and relates the system's constituent devices. The most common formalism we use is known as coupled mode theory (CMT). Coupled mode theory is a perturbational approach that provides accurate physical insights and allows engineers and scientists to design devices "a priori" (from first principles). There are however, some glaring issues regarding the accuracy of conventional CMT. Firstly, the conventional approach assumes the system to be closed and Hermitian, rendering inaccurate results in problems with significant radiation loss. Another issue is that it lacks any modal phase information, rendering it essentially useless in studying interference based phenomena. To account for these inaccuracies, we can formulate some alternative approaches by addressing non-orthogonality or by treating the system as non-Hermitian. We will compare the validity of these approaches with that of the conventional formalism as well as some numerical results.

2 Background and History

Historically, early versions of coupled mode theory began to appear in the 1950s within the context of microwave transmission lines. The approaches that were taken were quite heuristic: Coupled modes were extrapolated through the known modes of the uncoupled system. An early example of this can be seen in the telegrapher's equations, which describe the coupling between current and voltage on an electrical transmission line with space and time [1]. Eventually, by the late 1950s, Hermann A. Haus demonstrated that a set of spatial coupled mode equations were derivable through a variational principle along along with approximating the fields of the coupled system as a linear superposition of the uncoupled fields:

$$P^{\omega}(z) = \sum_{m}^{n} P_m^{\omega}(z) \backsim \sum_{m}^{n} |a_m^{\omega}(z)|^2$$

where $P_m^{\omega}(z)$ and a_m^{ω} describe the modal power and modal amplitude, respectively, of mode m and frequency ω [2]. By the 1990s, due to the success of fiber optic technologies, integrated photonics started to become an area of interest, and CMT naturally became one of the most reliable mathematical formalism for describing certain devices. These include optical waveguides, directional couplers, laser arrays, and distributed Bragg reflectors, among many others.

3 Conventional CMT

To engage in our study of coupled mode theory, it is prudent to first solidify our interpretation of electromagnetic modes their coupling. We consider modes in this context to be packets of electromagnetic power that propagate independent of anything else. In other words, they experience no interference in a linear medium. While the properties of some oscillating electromagnetic fields depend highly on their source charge, we consider electromagnetic modes that radiate continually through space without the effect of their source charge.

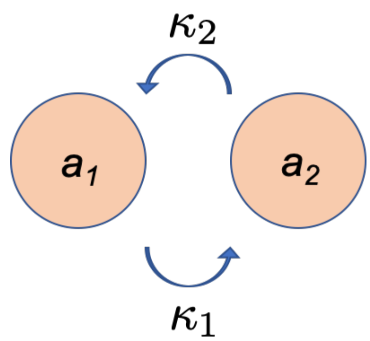

Figure 1: A pair of lossless, weakly coupled resonators.

With this in mind, we can now discuss the concept of coupling. Coupling refers simply to the concept of two systems exchanging energy with one another. In our case, this would be travelling electromagnetic modes exchanging power periodically. Let us first consider the example of two waveguides in close proximity. If we pump electromagnetic energy into one of the waveguides, the mode will propagate freely through the entirety of the waveguide's length. However, as it radiates, it will slowly begin to spill power into the other guide. At some point there is even zero power in the original guide, and this periodic exchange of power will continue as far as the length of the guide. What CMT does is it ultimately treats this problem perturbatively, by assuming that the coupled modes can be represented as a weighted sum of the in individually guided modes. This can be written as an approximation of the form:

$$\mathbf{E} = a_1(z)\mathbf{E}_1 + a_2(z)\mathbf{E}_2 \tag{1}$$

and

$$\mathbf{H} = a_1(z)\mathbf{H}_1 + a_2(z)\mathbf{H}_2 \tag{2}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field components of modes 1 and 2, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the first and second mode amplitudes, respectively. Here, the modes exchange power as a function of space, z, but modes are also capable of coupling in time.

3.1 Coupling of Modes in Time

To investigate the coupling of modes in time, we start by examining the case of two weakly coupled, lossless resonators, each with an exponential time dependence $e^{j\omega t}$. Figure 1 shows a schematic of such a system. If we let our modes be described by complex field amplitudes and assume the system to be closed and Hermitian, we arrive at the following set of coupled mode [3]:

$$\frac{da_1}{dt} = j\omega_1 a_1 + j\kappa_1 2 a_2 \tag{3}$$

and

$$\frac{da_2}{dt} = j\omega_2 a_2 + j\kappa_1 2 a_1 \tag{4}$$

It is not readily obvious here as to why coupling is proportional to the amplitude of the other resonator, rather than the time derivative or integral of the modal amplitude. Upon looking closely

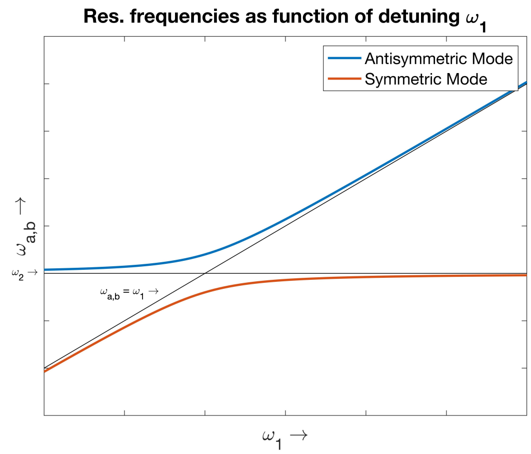

Figure 2: Resonant frequencies of symmetric (lower cuve) and anti-symmetric (upper curve) modes of a pair of coupled resonator as a function of detuning ω_1. Reproduced from [4] using MATLAB.

however, we might realize that, since we have assumed weak coupling, the time dependence is really only perturbed weakly. Therefore, we only have significant coupling when the resonances are approximately equal. When κ is small then compared to ω, using an approximation instead of the time derivative or integral causes an error of high order that can ultimately be ignored. We therefore have this set of approximated temporal coupled mode equations for a system of two weakly coupled resonators.

An interesting consequence of these equations becomes readily apparent when we address the orthogonality of the mode energies. We can first normalize our mode amplitudes such that, in the presence of coupling,

$$E_{tot} = |a_1|^2 + |a_2|^2 \tag{5}$$

If we differentiate the total energy as defined above with respect the time, and we assume that $\kappa_{12} = \kappa_{21} = \kappa$, we find that an interesting phenomenon occurs, namely frequency splitting. Say we wish to keep one of the frequencies ω_2 constant. When we begin to tune ω_1 to a value far from ω_2, we begin to notice that the set of possible frequencies splits into two, one for symmetric modes and one for anti-symmetric modes (see figure 2). The solutions are given as

$$\omega = \frac{\omega_1 + \omega_2}{2} \pm \sqrt{(\frac{\omega_1 - \omega_2}{2})^2 + |\kappa|^2} \tag{6}$$

At the crossover point, the solutions are symmetric and anti-symmetric combination of modes a_1 and a_2. As we move away from this crossover point, the solutions start to become more characteristic of the mode with the closest eigenfrequency [5].

3.2 Coupling of Modes in Space

Now, we move on to the coupling of modes in space. First, consider two modes in a linear system, such as that of the parallel waveguides in close proximity, with propagation constants β_1 and β_2.

These waves will have the following spatial dependencies: $e^{-j\beta_1 z}$ and $e^{-j\beta_2 z}$. If we assume the periodicity of the structures to be Λ, then the coupling coefficient is of the form: $2\kappa \cos \frac{2\pi z}{\Lambda}$ and the spatial coupled mode equations are given by

$$\frac{da_1}{dz} = -j\beta_1 a_1 - j2\kappa_{12} \cos \frac{2\pi z}{\Lambda} a_2 \tag{7}$$

and

$$\frac{da_2}{dz} = -j\beta_2 a_2 - j2\kappa_{21} \cos \frac{2\pi z}{\Lambda} a_1 \tag{8}$$

If we assuming the propagation constants of each respective mode to be closely synchronized [4], then we can assume solutions of the form

$$a_1(z) = A_1 \exp(-j\frac{\beta_1 + \beta_2}{2} + \frac{\pi}{\Lambda})z \tag{9}$$

and

$$a_2(z) = A_2 \exp(-j\frac{\beta_1 + \beta_2}{2} + \frac{\pi}{\Lambda})z \tag{10}$$

We now assume the modes to be matched in phase, and consider the waveguides to be infinite in length. Since $\beta_1 = \beta_2$, we know $\beta_1 - \beta_2 = 2\pi/\Lambda$. Using our trial solutions and the assumption of periodicity, we can get a simplified version of our coupled mode equations with and a solution of the form:

$$A_1 = A_+ e^{-j|\kappa|z} + A_- e^{j|\kappa|z} \tag{11}$$

and

$$A_2 = \frac{|\kappa}{\kappa}(A_+ e^{-j|\kappa|z} + A_- e^{j|\kappa|z}) \tag{12}$$

This is an elegant description of what is called "co-directional coupling", that is, the individual group velocities of the modes are identical. But what if the group velocities are opposite to one another? Such is the case of a device known as a backward wave oscillator (figure 3), created by the coupling of a backward moving electromagnetic mode to a forward moving circuit wave. The transfer function of this system will be given by:

$$|\frac{A_1(l)}{A_1(0)}|^2 = \frac{1}{\cos^2 \kappa l} \tag{13}$$

This formalism offers excellent accuracy when dealing with lossless coupled waveguide modes, but to what degree is it accurate when we extend it to some other systems? To address this, we compare the results of CMT to those of numerical solutions in the case of photonic crystal coupled resonator devices.

3.3 Accuracy of Spatial CMT

In this section, we consider the case of a side coupled integrated sequence of resonators (SCISSORs) and evaluate its transfer function using both spatial CMT and the finite-difference time-domain method, the latter of which is a common computational method used photonic design. The case of the SCISSOR is similar to that of two waveguides in close proximity, however in this scenario we a singular photonic-crystal waveguide coupled to a sequence of photonic crystal defect cavities.

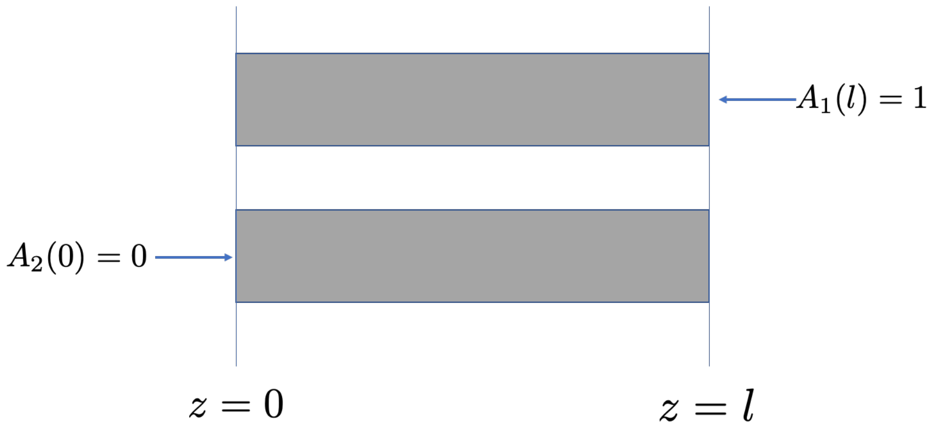

Figure 3: Schematic of a backward wave oscillator. Electromagnetic energy is pumped into A_1 at $z = l$ and an unexcited circuit wave is injected into A_0 at $z = 0$.

Despite the differences between this and the previous case, it is functionally similar: we inject a forward propagating mode into the waveguide, which in turn excites the cavity modes to near their resonant frequency, ω_0. By evaluating the transfer function in such a scenario both numerically and through spatial CMT, we can acquire some insight to the accuracy of the theory.

Assuming spatial coupling of the form of equations 7 and 8, we can numerically evaluate the modes and frequencies of such a system, and compare to the discretized solutions of the FDTD method. The results of such a comparison were completed by one 2011 study and are show in figure 4. The results here demonstrate that spatial CMT is really only accurate when the bandwidth of the cavity resonance is wide. In the bottom figure, one can readily see that CMT predicts a far different bandwidth (roughly 50% smaller) than the FDTD predicts [6].

4 Alternative Formalism

In reality, the majority of coupled systems are, to some degree, lossy. This means that our Hermitian approach will ultimately be inaccurate, especially when we are dealing with systems of low Q-factor. To correct for this, we will investigate an alternative formalism to conventional CMT

4.1 Scattering Approach

To begin, let us attempt to analyze a simple scattering problem using conventional CMT. Consider the case of an incident field an a simple scatterer (shown in figure 5). Using the conventional approach, we find a pair of coupled equations of the form

$$i(\Omega - \omega)\mathbf{a} = D^T \mathbf{c}_{in} \tag{14}$$

$$\mathbf{c}_{out} = S_{bg}\mathbf{c}_{in} + K\mathbf{a} \tag{15}$$

where Ω is a matrix that encodes the phase and amplitude evolution, D^T couples the input waves to the modal amplitudes, K couples the modal amplitudes to the output waves, and S_{bg} denotes

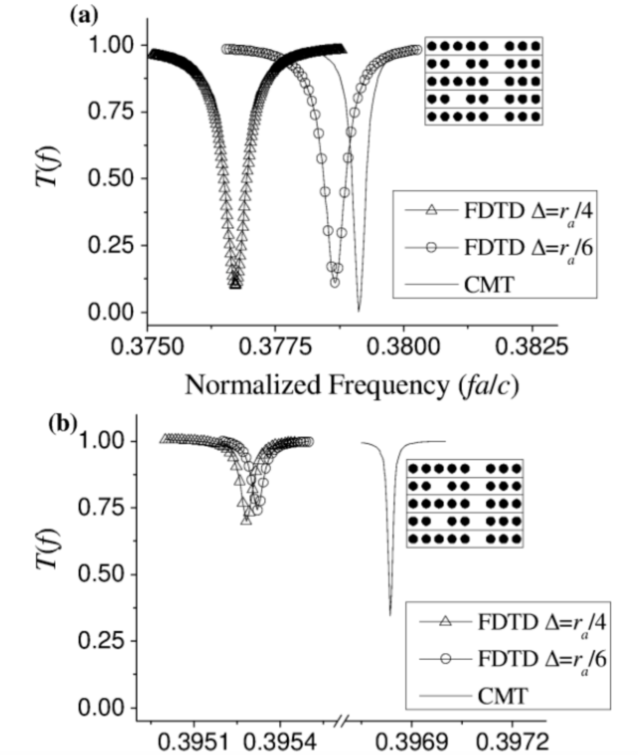

Figure 4: Transfer functions of SCISSOR evaluated by CMT and FDTD (with two different resolutions). Top: broad resonance. Bottom: narrow resonance [6]

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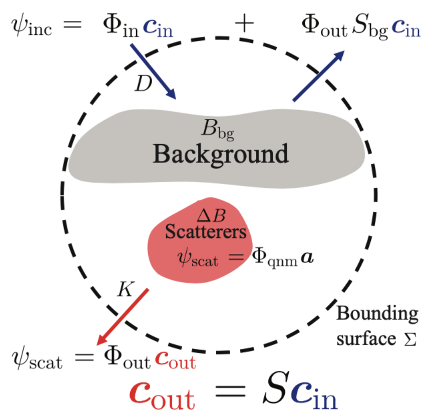

Figure 5: Schematic of scattering problem. We treat the input, output, and scattered waves as ports.

the background scattering, and couples incoming waves directly to outgoing waves. If we assume that the system is reciprocal, follows energy conservation, and allows time-reversal symmetry, we can readily find the scattering response that relates the incoming and outgoing modes [7].

$$S = S_{bg} - iK(\Omega - \omega)^{-1}D^T \tag{16}$$

While this is certainly a valid solution, these equations are really only valid for systems of single or small, isolated resonances. Upon increasing the complexity of the resonant response, or by considering highly symmetric scatterers, these equations begin to break down. This is in large part due to the assumption from the conventional approach, namely, that the system is closed, with little to no energy loss. When applying this to scattering problems however, it is clear that this will fail, as scattering inherently involves radiation loss.

The approach we will then take is to treat the system as non-Hermitian. Beginning with a Maxwellian scattering framework, we split the total field into its incident and scattered components. We define the incident field to be the sum of its input and output components:

$$\psi_{inc} = \Phi_{in}\mathbf{c}_{in} + \Phi_{out}S_{bg}\mathbf{c}_{in} \tag{17}$$

where $\mathbf{c}_{in}$ is the incoming wave amplitude and Φ represents the input/output vector field basis. All that is left now is to solve for the resonances and the outgoing mode amplitudes. Firstly, to solve for the resonances, we use our assumption of non-Hermiticity, as the system is open due to radiation loss. In this context, we call the modes "Quasinormal" [7]. These "quasinormal modes" (QNMs) will give us complex eigenfrequency, which we find by solving the non-Hermitian eigenproblems

$$\Theta\psi_{R,m} = j\tilde{\omega}_m B(\tilde{\omega}_m)\psi_{R,m} \tag{18}$$

and

$$\Theta\psi_{L,n} = j\tilde{\omega}_n B^T(\tilde{\omega}_n)\psi_{L,n} \tag{19}$$

where ψ_m denotes the left and right moving fields, Θ has is standard off-diagonal Maxwell curl operator, and B is the material tensor. From here the goal is to relate the QNM amplitudes to the incoming wave amplitudes as we did with the conventional case. To obtain this result, we require the QNM response for a given incident field. First, we separate the material tensor into its background and scattered components (as seen in figure 5), and assuming the scatterer to be non-dispersive, we can manipulate Maxwell's equations carefully to arrive at the following coupled mode equations:

$$i(\Omega - \omega)\mathbf{a} = D^T(\omega)\mathbf{c}_{in} \tag{20}$$

$$\mathbf{c}_{out} = [S_{bg} + \frac{1}{4\alpha\beta^*}i\omega(\Phi_{inc}^{TR}, \Delta B\Phi_{inc})]\mathbf{c}_{inc} + K(\omega)\mathbf{a} \tag{21}$$

where we again have K and Ω as our coupling matrices. While this set of equations is similar to equations 14-15, they have a few key differences. First, our coupling matrices are frequency dependent, despite the fact that we have assumed a non-dispersive medium. Second, we have a time reverse of the incident field, Φ_{inc}^{TR}, which we can find with the free space Green's Function. The impications of these changes can be readily seen when we find the scattering response, which is given by

$$S = S_{bg} + H(\omega) + i\tilde{K}\Omega^{-1}\tilde{D}^T - i\tilde{K}(\Omega - \omega)^{-1}\tilde{D^T} \tag{22}$$

This is again similar to equation 16, but now we have a few extra terms in addition to our resonance terms. These are the first three terms, which correspond to background scattering. In particular, $H(\omega)$ represents a frequency dependent background term known as "born scattering".

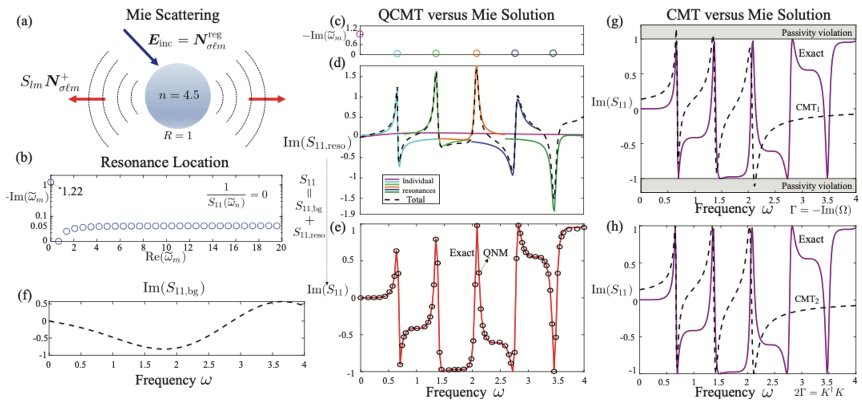

Figure 6: Contributions to the S_{11} element of the scattering matrix. c,d, and e show QMCT versus the Mie Solution. e): solid red line shows Mie solution and black circles show QMCT solution. g and h show conventional CMT (dotted black) compared to the Mie solution (solid colored). Conventional CMT begins to fail at high frequency [7].

4.2 QCMT vs Conventional CMT

Now, let us test the accuracy of these newly derived coupled equations on a particular scattering example: Mie scattering, which describes the scattering of a plane wave by a homogeneous sphere. As can be seen in figure 6, QCMT accurately models the scattering of a Mie sphere, while CMT begins to break down, especially at high frequency. Also worth noting is that background scattering (figure 6f) provides a significant shift in getting an accurate answer for a total scattering response.

The question then arises: when is conventional CMT really accurate, especially in scattering situations? We can answer this by identifying the conditions that allow our QCMT equations to simplify to conventional CMT equations. Firstly, this will occur when our background scattering term is small. Second, the coupling strengths must be roughly frequency independent within the bandwidth of interest, which is a sensible conclusion considering that conventional CMT was only ever intended for modes of high Q factor.

To better understand the differences between these formalisms, we can switch to the time domain with an inverse Fourier transform. In our operator notation, the conventional, closed system has the following coupled mode equations:

$$\frac{d}{dt}\mathbf{a}(t) = -i\Omega\mathbf{a}(t) + D^T\mathbf{c}_{in}(t) \tag{23}$$

$$\mathbf{c}_{out}(t) = S_{bg}\mathbf{c}_{in}(t) + K\mathbf{a}(t) \tag{24}$$

while, our new, non-Hermitian system will be given as

$$\frac{d}{dt}\mathbf{a}(t) = -i\Omega\mathbf{a}(t) + \int D^T(t-t^{'})\mathbf{c}_{in}(t-t^{'})dt^{'} \tag{25}$$

$$\mathbf{c}_{out}(t) = \int [S_{bg}(t-t^{'}) + E(t-t^{'})]\mathbf{c}_{in}(t^{'})dt^{'} + \int K(t-t^{'}\mathbf{a}(t^{'})dt^{'} \tag{26}$$

From equations 25-26, it becomes apparent that the relationships between the scattering mode amplitudes and the incoming and outgoing modes are convolutions in time, which is an understandable result considering that the coupling of our QNMs is frequency-dependent. We can also note that the conventional formalism, equations 23-24, is valid when the time dependent coupling matrices have sharp peaks (i.e. small bandwidth).

5 Conclusions

This paper has outlined conventional coupled mode theory in both the space and time domain. We have noted some interesting phenomena associated with coupling as well the accuracy of the theory and its applications. We have then shown an example of the limits of this conventional formalism, that being the scattering problem. It is clear that, when radiation loss cannot be ignored, a new formalism is required where we treat the system as non-Hermitian, giving us QNMs and an adjusted framework. This new Quasinormal Coupled Mode theory has been shown to be far more accurate than the conventional approach in scattering problems.

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