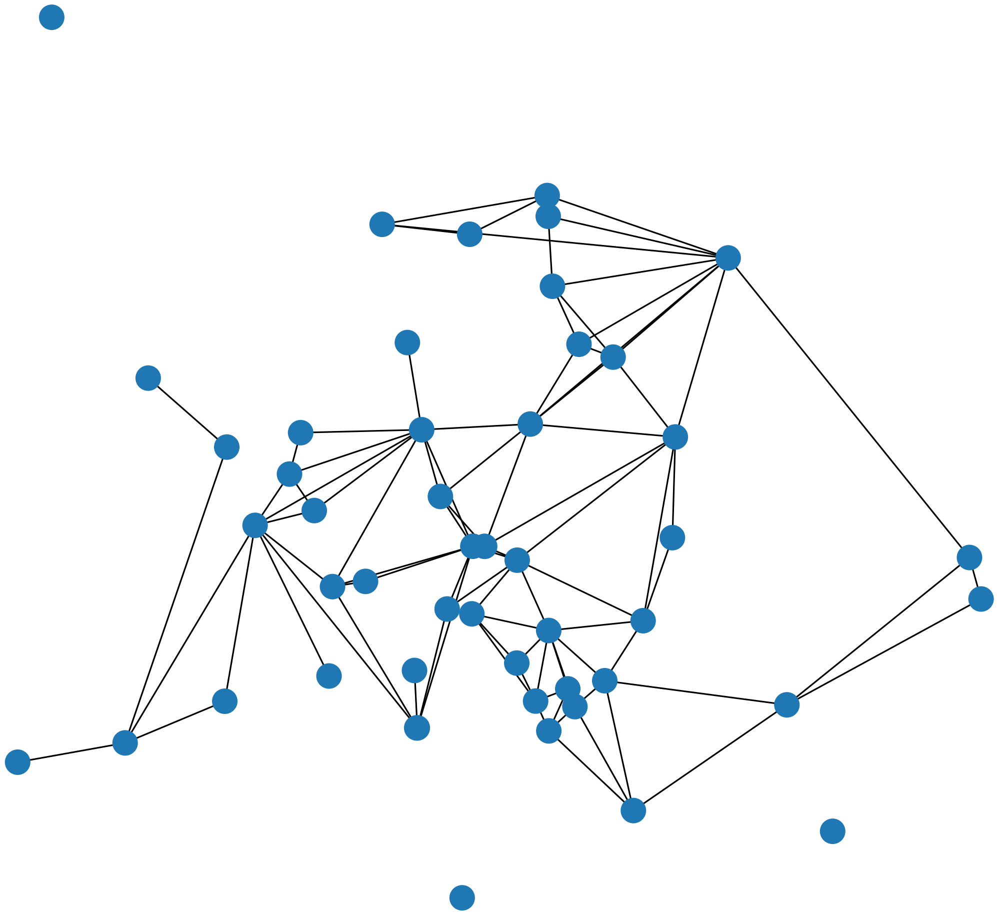
1. The graph[1] of Europe $\mathcal{G}^* = \langle V, E \rangle$ is defined as follows: each vertex $v \in V$ is a Europe country[2]; two vertices are adjacent ($\{u, v\} \in E$) if the corresponding countries share a land border. Let $\mathcal{G}$ be the largest connected component of $\mathcal{G}^*$.

   (a) Draw $\mathcal{G}^*$ with the minimum number of intersecting edges.



   (b) Find $|V|$, $|E|$, $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$, $\mathrm{rad}(\mathcal{G})$, $\mathrm{diam}(\mathcal{G})$, $\mathrm{girth}(\mathcal{G})$, $\mathrm{center}(\mathcal{G})$, $\varkappa(\mathcal{G})$, $\lambda(\mathcal{G})$.

   (c) Find the minimum vertex coloring $Z : V \to \mathbb{N}$ of $\mathcal{G}$.

   (d) Find the minimum edge coloring $X : E \to \mathbb{N}$ of $\mathcal{G}$.

   (e) Find the maximum clique $Q \subseteq V$ of $\mathcal{G}$.

   (f) Find the maximum stable set $S \subseteq V$ of $\mathcal{G}$.

   (g) Find the maximum matching $M \subseteq E$ of $\mathcal{G}$.

   (h) Find the minimum vertex cover $R \subseteq V$ of $\mathcal{G}$.

   (i) Find the minimum edge cover $F \subseteq E$ of $\mathcal{G}$.

   (j) Find the shortest closed walk W that visits every vertex of $\mathcal{G}$.

   (k) Find the shortest closed walk U that visits every edge of $\mathcal{G}$.

   (l) Find all biconnected components (blocks) and draw the block-cut tree of $\mathcal{G}^*$.

   (m) Find all 2-edge-connected components of $\mathcal{G}^*$.

   (n) Construct an SPQR tree of the largest biconected component of $\mathcal{G}^*$.

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[1] Hereinafter, "graphs" are "simple, finite, undirected and unweighted", unless stated otherwise.

[2] `https://simple.wikipedia.org/wiki/List_of_European_countries` used as reference.

(o) Add the weight function $w : E \longrightarrow \mathbb{R}$ denoting the distance between capitals. Find the minimum (*w.r.t.* the total weight of edges) spanning tree T for the largest connected component of the weighted Europe graph $\mathcal{G}_w^* = (V, E, w)$.

(p) Find centroid(T) (*w.r.t.* the ege weight function w).

(q) Construct the Prüfer code for T.

2. Prove *rigorously* the following theorems:

Theorem 1 (Triangle Inequality). For any connected graph $G = \langle V, E\rangle$:

$$\forall x, y, z \in V : \operatorname{dist}(x, y) + \operatorname{dist}(y, z) \geq \operatorname{dist}(x, z)$$

Theorem 2 (Tree). A connected graph $G = \langle V, E\rangle$ is a tree (*i.e.* acyclic graph) *iff* $|E| = |V| - 1$.

Theorem 3 (Whitney). For any graph $G : \varkappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 4 (Chartrand). For a connected graph $G = \langle V, E\rangle$: if $\delta(G) \geq \lfloor \frac{|V|}{2} \rfloor$, then $\lambda(G) = \delta(G)$.

Theorem 5 (Menger). For any pair of non-adjacent vertices u and v in an undirected graph, the size of the minimum *vertex cut* is equal to the maximum number of pairwise *internally vertex-disjoint paths* from u to v.

Theorem 6 (Harary). Every block of a block graph[3] is a clique.

[3] A block graph $H = \mathrm{B}(G)$ is an intersection graph of all blocks (biconnected components) of G, *i.e.* each vertex $v \in V(H)$ corresponds to a block of G, and there is an edge $\{v, u\} \in E(H)$ iff "blocks" v and u share a cut vertex.