1. The graph[1] of Europe $\mathcal{G}^* = \langle V, E\rangle$ is defined as follows: each vertex $v \in V$ is a Europe country[2]; two vertices are adjacent ($\{u, v\} \in E$) if the corresponding countries share a land border. Let $\mathcal{G}$ be the largest connected component of $\mathcal{G}^*$.
   (a) Draw[3] $\mathcal{G}^*$ with the minimum number of intersecting edges[4].
   (b) Find $|V|$, $|E|$, $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$, $\mathrm{rad}(\mathcal{G})$, $\mathrm{diam}(\mathcal{G})$, $\mathrm{girth}(\mathcal{G})$, $\mathrm{center}(\mathcal{G})$, $\varkappa(\mathcal{G})$, $\lambda(\mathcal{G})$.
   (c) Find the minimum vertex coloring $Z : V \to \mathbb{N}$ of $\mathcal{G}$.
   (d) Find the minimum edge coloring $X : E \to \mathbb{N}$ of $\mathcal{G}$.
   (e) Find the maximum clique $Q \subseteq V$ of $\mathcal{G}$.
   (f) Find the maximum stable set $S \subseteq V$ of $\mathcal{G}$.
   (g) Find the maximum matching $M \subseteq E$ of $\mathcal{G}$.
   (h) Find the minimum vertex cover $R \subseteq V$ of $\mathcal{G}$.
   (i) Find the minimum edge cover $F \subseteq E$ of $\mathcal{G}$.
   (j) Find the shortest closed walk $W$ that visits every vertex of $\mathcal{G}$.
   (k) Find the shortest closed walk $U$ that visits every edge of $\mathcal{G}$.
   (l) Find all biconnected components (blocks) and draw the block-cut tree of $\mathcal{G}^*$.
   (m) Find all 2-edge-connected components of $\mathcal{G}^*$.
   (n) Construct an SPQR tree of the largest biconnected component of $\mathcal{G}$.
   (o) Add the weight function $w : E \to \mathbb{R}$ denoting the distance[5] between capitals. Find the minimum (*w.r.t.* the total weight of edges) spanning tree $T$ for the largest connected component of the weighted Europe graph $\mathcal{G}^*_w = (V, E, w)$.
   (p) Find $\mathrm{centroid}(T)$ (*w.r.t.* the edge weight function $w$).
   (q) Construct the Prüfer code for $T$.

2. Prove *rigorously* the following theorems:

   **Theorem 1** (Triangle Inequality)**.** For any connected graph $G = \langle V, E\rangle$:
   $$\forall x, y, z \in V : \mathrm{dist}(x, y) + \mathrm{dist}(y, z) \geq \mathrm{dist}(x, z)$$

   **Theorem 2** (Tree)**.** A connected graph $G = \langle V, E\rangle$ is a tree (*i.e.* acyclic graph) *iff* $|E| = |V| - 1$.

   **Theorem 3** (Whitney)**.** For any graph $G$: $\varkappa(G) \leq \lambda(G) \leq \delta(G)$.

   **Theorem 4** (Chartrand)**.** For a connected graph $G$: if $\delta(G) \geq \lfloor |V|/2 \rfloor$, then $\lambda(G) = \delta(G)$.

   **Theorem 5** (Menger)**.** For any pair of non-adjacent vertices $u$ and $v$ in an undirected graph, the size of the minimum *vertex cut* is equal to the maximum number of pairwise *internally vertex-disjoint paths* from $u$ to $v$.

   **Theorem 6** (Harary)**.** Every block of a block graph[6] is a clique.

[1] Hereinafter, "graphs" are "simple, finite, undirected, and unweighted", unless stated otherwise.
[2] Since the absolute geopolitical correctness is not necessary to accomplish this task, you can simply use `https://simple.wikipedia.org/wiki/List_of_European_countries` or any other similar source as a reference.
[3] You may either preserve the original spatial relationships between vertices, or use an automatic layout software. Take a look at Graphviz, Gephi, Cytoscape.
[4] Since $\mathcal{G}^*$ naturally corresponds to a "map", it *has* to be planar. However, due to *some broken countries*, it might be impossible to draw $\mathcal{G}^*$ completely without edge intersections, so you just have to minimize them.
[5] You may choose geodesic or road distance, to your preference. You have to use *more-or-less real* data.
[6] A block graph $H = \mathrm{B}(G)$ is an intersection graph of all blocks (biconnected components) of $G$, *i.e.* each vertex $v \in V(H)$ corresponds to a block of $G$, and there is an edge $\{v, u\} \in E(H)$ iff "blocks" $v$ and $u$ share a cut vertex.

### Minor Cheatsheet

- **Graph** $G$ is a pair $\langle V, E \rangle$ of a set of vertices $V = \{v_1, \dots, v_n\}$ and a set of edges $E = \{e_1, \dots, e_m\}$.
- Simple **directed** graphs have $E \subseteq V^2$. Simple **undirected** graphs have $E \subseteq V^{(2)}$.
  - $A^k = A \times \cdots \times A = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in A\}$ is the set of $k$-tuples (Cartesian $k$-power of $A$).
  - $A^{(k)} = \{\{x_1, \dots, x_k\} \mid x_1 \neq \cdots \neq x_k \in A\} = \{S \mid S \subseteq A, |S| = k\}$ is the set of $k$-sized subsets of $A$.
- **Degree** $\deg(v)$ of a vertex $v$ is the number of edges indident to $v$ (loops are counted twice).
  - $\delta(G) = \min\limits_{v \in V} \deg(v)$ is the **minimum degree**.
  - $\Delta(G) = \max\limits_{v \in V} \deg(v)$ is the **maximum degree**.
- **Walk** is an alternating sequence of vertices and edges in an arbitrary traversal of a graph.
  - **Trail** is a walk with distinct edges.
  - **Path** is a walk with distinct vertices (and therefore distinct edges).
  - The walk is **closed** if it starts and ends at the same vertex. Otherwise, it is **open**.
  - **Circuit** is a closed trail.
  - **Cycle** is a closed path.
- **Distance** $\operatorname{dist}(u, v)$ between two vertices is the length of the shortest path $u \rightsquigarrow v$.
  - $\varepsilon(v) = \max\limits_{u \in V} \operatorname{dist}(v, u)$ is the **eccentricity** of the vertex $v$.
  - $\operatorname{rad}(G) = \min\limits_{v \in V} \varepsilon(v)$ is the **radius** of the graph $G$.
  - $\operatorname{diam}(G) = \max\limits_{v \in V} \varepsilon(v)$ is the **diameter** of the graph $G$.
- **Girth** is the length of the shortest cycle in the graph.
- **Clique** $Q \subseteq V$ is a set of vertices inducing a complete subgraph.
- **Stable set** $S \subseteq V$ is a set of independent (pairwise non-adjacent) vertices.
- **Matching** $M \subseteq E$ is a set of independent (pairwise non-adjacent) edges.
- **Vertex cover** $R \subseteq V$ is a set of vertices “covering” all edges.
- **Edge cover** $F \subseteq E$ is a set of edges “covering” all vertices.
- Note the distinction between the terms **maximum** (“globally”) and **maximal** (“locally”):
  - Some set $A^*$ is **maximum (by cardinality)** if there is no other set $A$ such that $|A| > |A^*|$.
  - Some set $A'$ is **maximal (by inclusion)** if there is no other set $A$ such that $A \supset A'$.
  - Similarly for **minimum** / **minimal**.