- 1. The graph¹ of Europe² $\mathcal{G}^* = \langle V, E \rangle$ is defined as follows: each vertex $v \in V$ is a Europe country; two vertices are adjacent $(\{u, v\} \in E)$ if the corresponding countries share a land border. Let \mathcal{G} be the largest connected component of \mathcal{G}^* .
 - (a) Draw G^* with the minimum number of intersecting edges.
 - (b) Find |V|, |E|, $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$, rad (\mathcal{G}) , diam (\mathcal{G}) , girth (\mathcal{G}) , center (\mathcal{G}) , $\varkappa(\mathcal{G})$, $\lambda(\mathcal{G})$.
 - (c) Find the minimum vertex coloring $Z: V \to \mathbb{N}$ of \mathcal{G} .
 - (d) Find the minimum edge coloring $X : E \to \mathbb{N}$ of \mathcal{G} .
 - (e) Find the maximum clique $Q \subseteq V$ of G.
 - (f) Find the maximum stable set $S \subseteq V$ of G.
 - (g) Find the maximum matching $M \subseteq E$ of G.
 - (h) Find the minimum vertex cover $R \subseteq V$ of G.
 - (i) Find the minimum edge cover $F \subseteq E$ of G.
 - (j) Find the shortest closed walk W that visits every vertex of G.
 - (k) Find the shortest closed walk U that visits every edge of \mathcal{G} .
 - (l) Find all biconnected components (blocks) and draw the block-cut tree of \mathcal{G}^* .
 - (m) Find all 2-edge-connected components of \mathcal{G}^* .
 - (n) Construct an SPQR tree of the largest biconected component of \mathcal{G}^* .
 - (o) Add the weight function $w: E \to \mathbb{R}$ denoting the distance between capitals. Find the minimum (*w.r.t.* the total weight of edges) spanning tree T for the largest connected component of the weighted Europe graph $\mathcal{G}_w^* = (V, E, w)$.
 - (p) Find centroid(T) (w.r.t. the ege weight function w).
 - (q) Construct the Prüfer code for *T*.
- 2. Prove *rigorously* the following theorems:

Theorem 1 (Triangle Inequality). For any connected graph $G = \langle V, E \rangle$:

$$\forall x, y, z \in V : dist(x, y) + dist(y, z) \ge dist(x, z)$$

Theorem 2 (Tree). A connected graph $G = \langle V, E \rangle$ is a tree (i.e. acyclic graph) iff |E| = |V| - 1.

Theorem 3 (Whitney). For any graph $G : \varkappa(G) \le \lambda(G) \le \delta(G)$.

Theorem 4 (Chartrand). For a connected graph $G = \langle V, E \rangle$: if $\delta(G) \ge \lfloor \frac{|V|}{2} \rfloor$, then $\lambda(G) = \delta(G)$.

Theorem 5 (Menger). For any pair of non-adjacent vertices u and v in an undirected graph, the size of the minimum *vertex cut* is equal to the maximum number of pairwise *internally vertex-disjoint paths* from u to v.

Theorem 6 (HARARY). Every block of a block graph³ is a clique.

¹ Hereinafter, "graphs" are "simple, finite, undirected and unweighted", unless stated otherwise.

² https://simple.wikipedia.org/wiki/List_of_European_countries used as reference.

³ A block graph H = B(G) is an intersection graph of all blocks (biconnected components) of G, *i.e.* each vertex $v \in V(H)$ corresponds to a block of G, and there is an edge $\{v, u\} \in E(H)$ iff "blocks" v and u share a cut vertex.