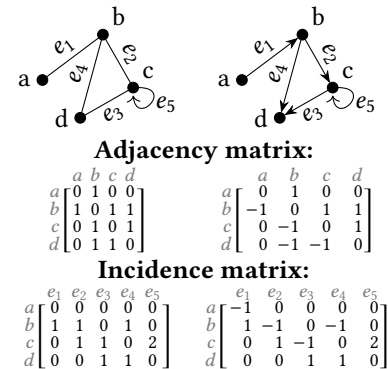
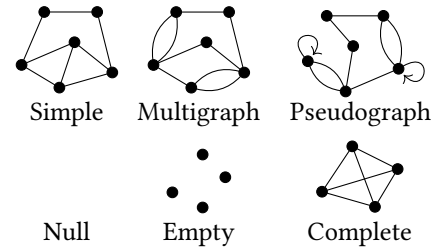


5 Graph Theory Cheatsheet

Glossary

- * **Graph** is an ordered pair $G = \langle V, E \rangle$, where $V = \{v_1, \dots, v_n\}$ is a set of vertices, and $E = \{e_1, \dots, e_m\}$ is a set of edges.
- * Simple **undirected** graphs have $E \subseteq V^{(2)}$, i.e. each edge $e_i \in E$ between vertices u and v is denoted by $\{u, v\} \in V^{(2)}$. Such *undirected edges* are also called *links* or *lines*.
 - $A^{(k)} = \{\{x_1, \dots, x_k\} \mid x_1 \neq \dots \neq x_k \in A\} = \{S \mid S \subseteq A, |S| = k\}$ is the set of k -sized subsets of A .
- * Simple **directed** graphs have $E \subseteq V^2$, i.e. each edge $e_i \in E$ from vertex u to v is denoted by an ordered pair $\langle u, v \rangle \in V^2$. Such *directed edges* are also called *arcs* or *arrows*.
 - $A^k = A \times \dots \times A = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in A\}$ is the set of k -tuples (Cartesian k -power of A).
- * **Multi-edges** are edges that have the same end nodes.
- * **Loop** is an edge that connects a vertex to itself.
- * **Simple graph** is a graph without multi-edges and loops.
- * **Multigraph** is a graph with multi-edges.
- * **Pseudograph** is a multigraph with loops.
- * **Null graph** is a “graph” without vertices.
- * **Trivial graph** is a graph consisting of a single vertex.
- * **Empty (edgeless) graph** is a graph without edges.
- * **Complete graph** K_n is a simple graph in which every pair of distinct vertices is connected by an edge.
- * **Weighted graph** $G = (V, E, w)$ is a graph in which each edge has an associated numerical value (the *weight*) represented by the **weight function** $w : E \rightarrow \text{Num}$.
- * **Subgraph** of a graph $G = \langle V, E \rangle$ is another graph $G' = \langle V', E' \rangle$ such that $V' \subseteq V, E' \subseteq E$. Designated as $G' \subseteq G$.
- * **Spanning (partial) subgraph** is a subgraph that includes all vertices of a graph.
- * **Induces subgraph** of a graph $G = \langle V, E \rangle$ is another graph G' formed from a subset S of the vertices of the graph and *all* the edges (from the original graph) connecting pairs of vertices in that subset. Formally, $G' = G[S] = \langle V', E' \rangle$, where $S \subseteq V, V' = V \cap S, E' = \{e \in E \mid \exists v \in S : e \text{ } I \text{ } v\}$.
- * **Adjacency** is the relation between two vertices connected with an edge.
- * **Adjacency matrix** is a square matrix $A_{V \times V}$ of an adjacency relation.
 - For simple graphs, adjacency matrix is binary, i.e. $A_{ij} \in \{0, 1\}$.
 - For directed graphs, $A_{ij} \in \{0, 1, -1\}$.
 - For multigraphs, adjacency matrix contains edge multiplicities, i.e. $A_{ij} \in \mathbb{N}_0$.
- * **Incidence** is a relation between an edge and its endpoints.
- * **Incidence matrix** is a Boolean matrix $B_{V \times E}$ of an incidence relation.
- * **Degree** $\deg(v)$ the number of edges incident to v (loops are counted twice).
 - $\delta(G) = \min_{v \in V} \deg(v)$ is the **minimum degree**.
 - $\Delta(G) = \max_{v \in V} \deg(v)$ is the **maximum degree**.
 - HANDSHAKING LEMMA. $\sum_{v \in V} \deg(v) = 2|E|$.
- * A graph is called **r -regular** if all its vertices have the same degree: $\forall v \in V : \deg(v) = r$.
- * **Complement graph** of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent iff they are non-adjacent in G .
- * **Intersection graph** of a family of sets $F = \{S_i\}$ is a graph $G = \Omega(F) = \langle V, E \rangle$ such that each vertex $v_i \in V$ denotes the set S_i , i.e. $V = F$, and the two vertices v_i and v_j are adjacent whenever the corresponding sets S_i and S_j have a non-empty intersection, i.e. $E = \{\langle v_i, v_j \rangle \mid i \neq j, S_i \cap S_j \neq \emptyset\}$.
- * **Line graph** of a graph $G = \langle V, E \rangle$ is another graph $L(G) = \Omega(E)$ that represents the adjacencies between edges of G . Each vertex of $L(G)$ represents an edge of G , and two vertices of $L(G)$ are adjacent iff the corresponding edges share a common endpoint in G (i.e. edges are “adjacent”/“incident”).
- * **Walk** is an alternating sequence of vertices and edges in an arbitrary graph traversal.
 - **Trail** is a walk with distinct edges.
 - **Path** is a walk with distinct vertices (and therefore distinct edges).
 - A walk is **closed** if it starts and ends at the same vertex. Otherwise, it is **open**.
 - **Circuit** is a closed trail.
 - **Cycle** is a closed path.
- * **Length** of a path (walk, trail) $l = u \rightsquigarrow v$ is the number of edges in it: $|l| = |E(l)|$.

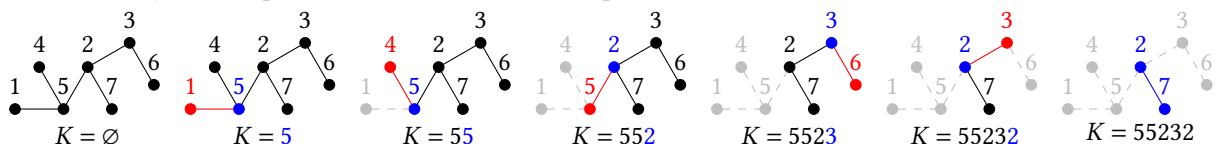
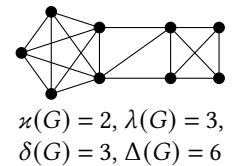
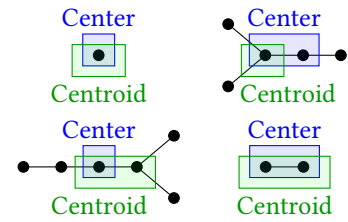


Term	V ¹	E ²	Closed
Walk	+	+	Closed walk
Trail	+	-	Circuit
Path	-	-	Cycle
	-	+	(impossible)

¹ Can vertices be repeated?

² Can edges be repeated?

- * **Girth** is the length of the shortest cycle in the graph.
- * **Distance** $\text{dist}(u, v)$ between two vertices is the length of the shortest path $u \rightsquigarrow v$.
 - $\varepsilon(v) = \max_{u \in V} \text{dist}(v, u)$ is the **eccentricity** of the vertex v .
 - $\text{rad}(G) = \min_{v \in V} \varepsilon(v)$ is the **radius** of the graph G .
 - $\text{diam}(G) = \max_{v \in V} \varepsilon(v)$ is the **diameter** of the graph G .
 - $\text{center}(G) = \{v \mid \varepsilon(v) = \text{rad}(G)\}$ is the **center** of the graph G .
- * **Clique** $Q \subseteq V$ is a set of vertices inducing a complete subgraph.
- * **Stable set** $S \subseteq V$ is a set of independent (pairwise non-adjacent) vertices.
- * **Matching** $M \subseteq E$ is a set of independent (pairwise non-adjacent) edges.
- * **Vertex cover** $R \subseteq V$ is a set of vertices “covering” all edges.
- * **Edge cover** $F \subseteq E$ is a set of edges “covering” all vertices.
- * **Vertex connectivity** $\kappa(G)$ is the minimum number of vertices that has to be removed in order to make the graph disconnected or trivial. Equivalently, it is the largest k for which the graph G is k -vertex-connected.
- * **k -vertex-connected graph** is a graph that remains connected after less than k vertices are removed, i.e. $\kappa(G) \geq k$.
 - Corollary of Menger’s theorem: graph $G = \langle V, E \rangle$ is k -vertex-connected if, for every pair of vertices $u, v \in V$, it is possible to find k *vertex-independent* (*internally vertex-disjoint*) paths between u and v .
 - k -vertex-connected graphs are also called simply *k -connected*.
 - 1-connected graph is called *connected*, 2-connected is *biconnected*, 3-connected is *triconnected*.
- * **Edge connectivity** $\lambda(G)$ is the minimum number of edges that has to be removed in order to make the graph disconnected. Equivalently, it is the largest k for which the graph G is k -edge-connected.
- * **k -edge-connected graph** is a graph that remains connected after less than k edges are removed, i.e. $\lambda(G) \geq k$.
 - Corollary of Menger’s theorem: graph $G = \langle V, E \rangle$ is k -edge-connected if, for every pair of vertices $u, v \in V$, it is possible to find k *edge-disjoint* paths between u and v .
- * WHITNEY’S THEOREM. $\kappa(G) \leq \lambda(G) \leq \delta(G)$.
- * **Tree** is a connected undirected acyclic graph.
- * **Forest** is an undirected acyclic graph, i.e. a disjoint union of trees.
- * An **unrooted tree (free tree)** is a tree without any designated *root*.
- * A **rooted tree** is a tree in which one vertex has been designated the *root*.
 - In a rooted tree, the **parent** of a vertex v is the vertex connected to v on the path to the root.
 - A **child** of a vertex v is a vertex of which v is the parent.
 - A **sibling** to a vertex v is any other vertex on the tree which has the same parent as v .
 - A **leaf** is a vertex with no children. Equivalently, **leaf** is a *pendant vertex*, i.e. $\deg(v) = 1$.
 - An **internal vertex** is a vertex that is not a leaf.
 - A **k -ary tree** is a rooted tree in which each vertex has at most k children. 2-ary trees are called **binary trees**.
- * A **labeled tree** is a tree in which each vertex is given a unique *label*, e.g., $1, 2, \dots, n$.
- * CAYLEY’S FORMULA. Number of labeled trees on n vertices is n^{n-2} .
- * **Prüfer code** is a unique sequence of labels $\{1, \dots, n\}$ of length $(n-2)$ associated with the labeled tree on n vertices.
 - **ENCODING** (iterative algorithm for converting tree T labeled with $\{1, \dots, n\}$ into a Prüfer sequence K):
 - On each iteration, remove the leaf with *the smallest label*, and extend K with *a single neighbour* of this leaf.
 - After $(n-2)$ iterations, the tree will be left with two adjacent vertices — there is no need to encode them, because there is only one unique tree on 2 vertices, which requires 0 bits of information to encode.



- **DECODING** (iterative algorithm for converting a Prüfer sequence K into a tree T):
 - Given a Prüfer code K of length $(n-2)$, construct a set of “leaves” $W = \{1, \dots, n\} \setminus K$.
 - On each iteration:
 - (1) Pop the *first* element of K (denote it as k) and the *minimum* label in W (denote it as w).
 - (2) Connect k and w with an edge $\langle k, w \rangle$ in the tree T .
 - (3) If $k \notin K$, then extend the set of “leaves” $W := W \cup \{k\}$.
 - After $(n-2)$ iterations, the sequence K will be empty, and the set W will contain exactly two vertices — connect them with an edge.