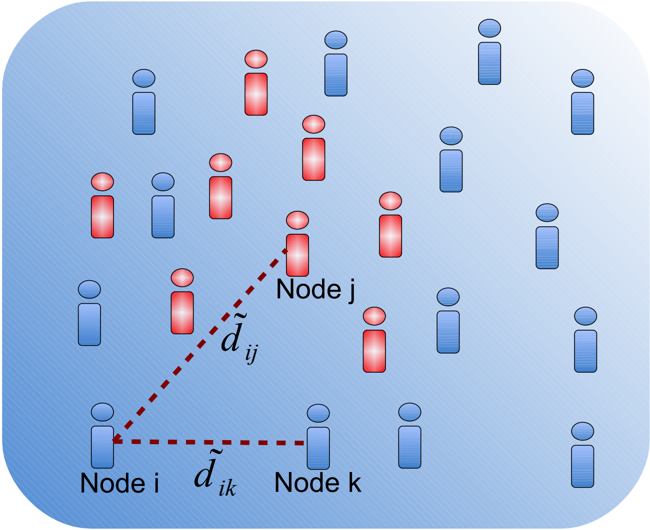
Identifying latent space geometry of network formation models via analysis of curvature

Shane Lubold, Arun Chandrasekhar, Tyler McCormick

May 17th, 2021

Latent Space Model



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- Estimating the geometry of a network is critical!
- **Our contribution: propose the first data-driven, HT framework for estimating geometry!**

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Contribution: Provide a HT framework to answer this!

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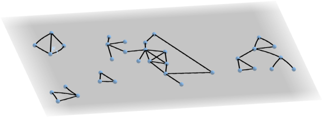

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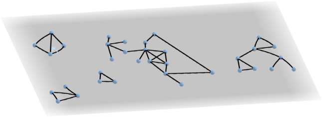

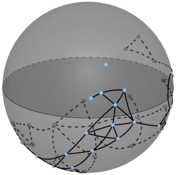

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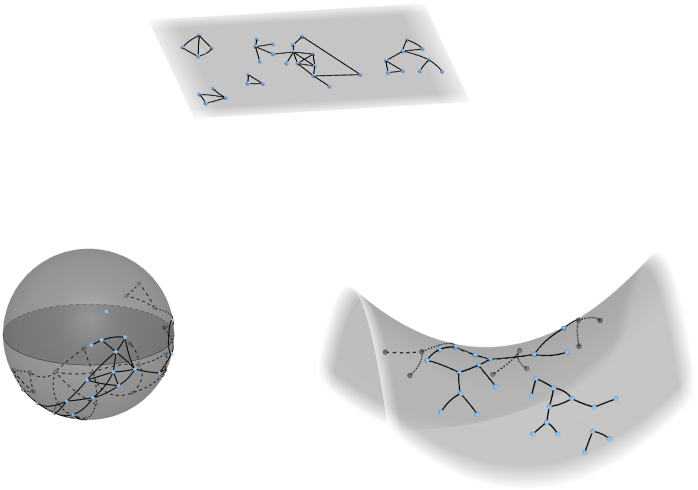

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Assume that $\kappa \in [-b, -a] \cup \{0\} \cup [a, b]$**, with** $b > a > 0$.

Convert hypotheses into easily tested conditions!

HT Framework

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Embedding Conditions

Theorem ([Schoenberg, 1935])

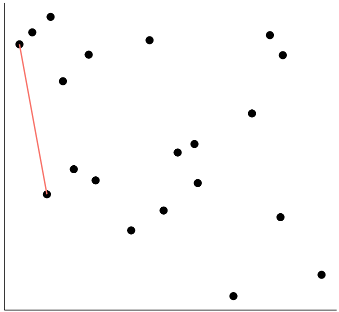
Let D contains distances between $z_1, \ldots, z_K$. Then, $\{z_1, \ldots, z_K\} \overset{isom}{\to} \mathbb{R}^p$ for some p if and only if $F(D)$ is psd, where

$$F(D) = -1/2 JD \circ DJ \ .$$

and J is the $K \times K$ centering matrix.

Embedding Conditions - Example

$z_i \in \mathbb{R}^2$



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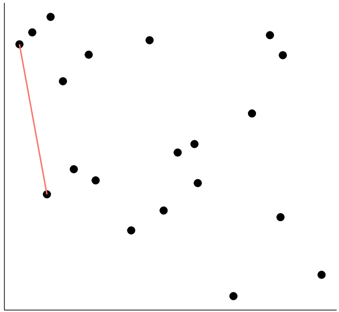

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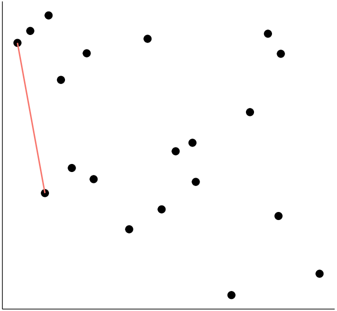

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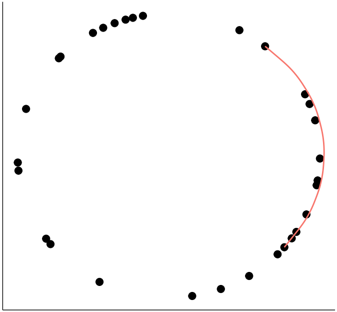

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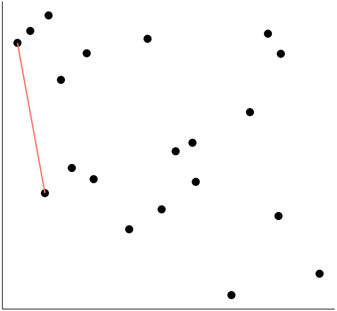

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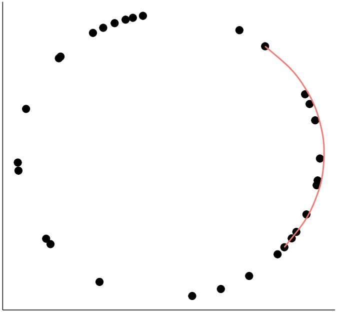

$$\lambda_1(F(D)) < 0$$

$F(D)$ is not psd

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- **Main point: Eigenvalues of distance matrix tell us the geometry! So how do we find points on $\mathcal{M}$?**

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But how do we estimate $P(G_{ij} = 1|z)$?

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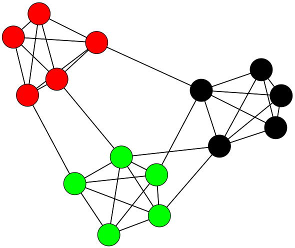

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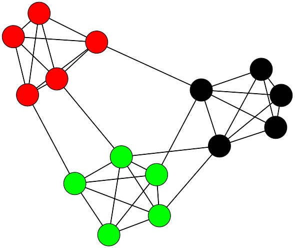

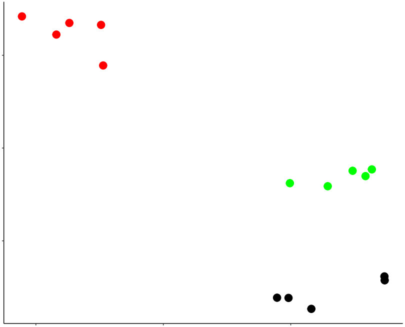

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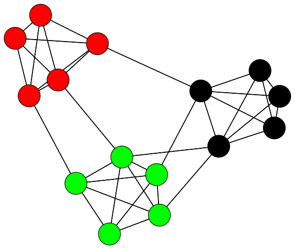

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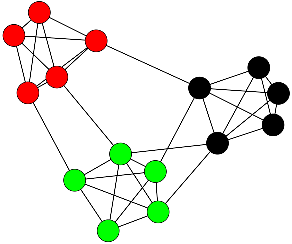

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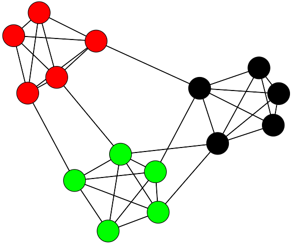

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Find K cliques of size ℓ and compute $\hat{D}$ using these cliques.

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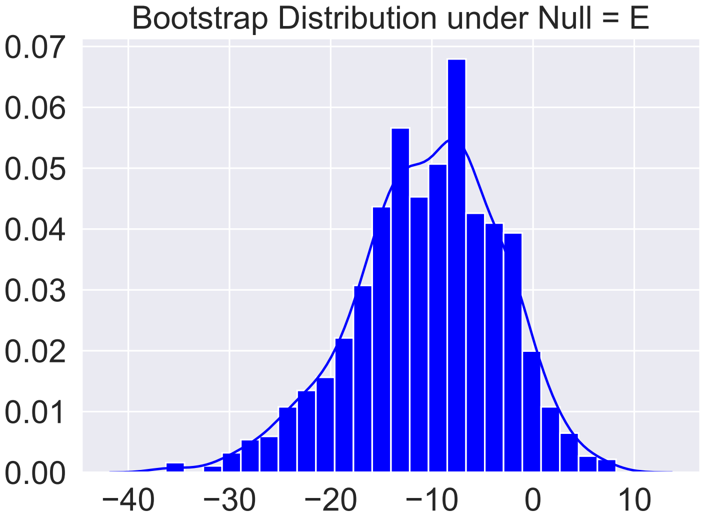
Bootstrapping $\lambda_{\min}(F(D))$



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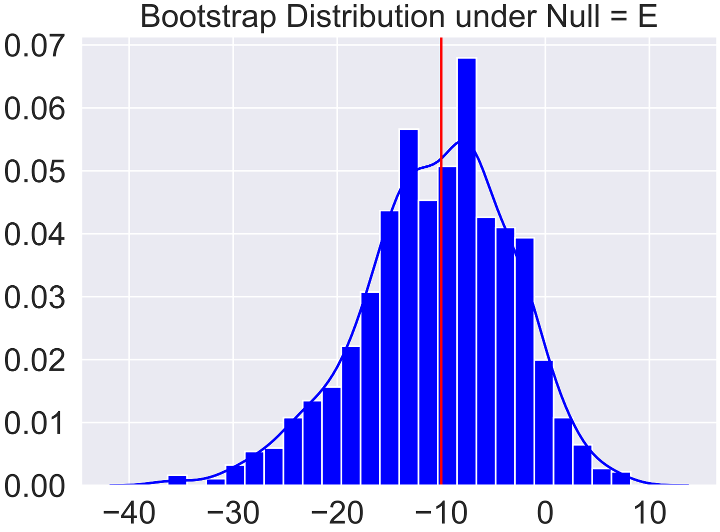

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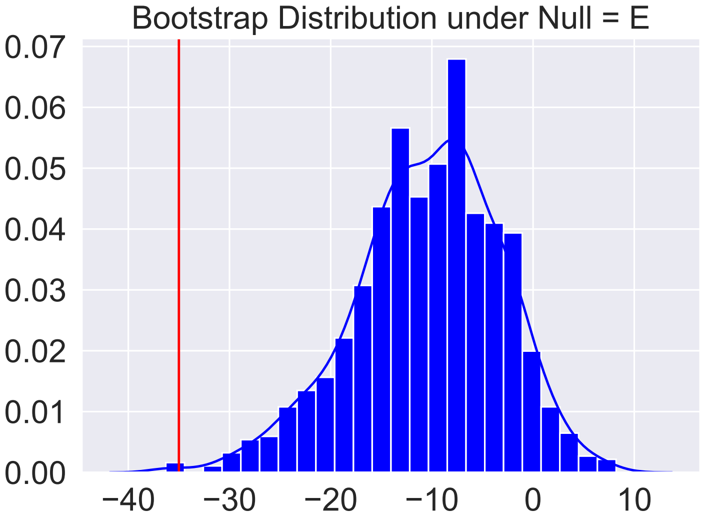

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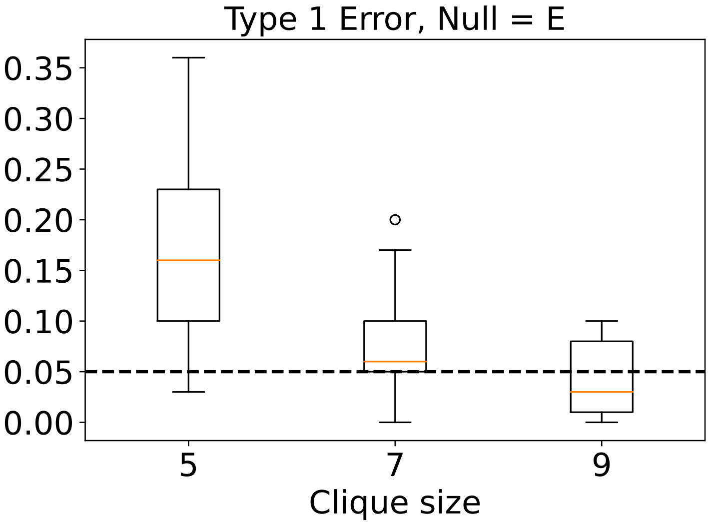
  - Distribute LS locations around these group centers.
- We draw $\nu_i \overset{\text{i.i.d.}}{\sim} \text{Unif}(-0.2, 0)$.

Type 1 error approaches α as clique size increases.



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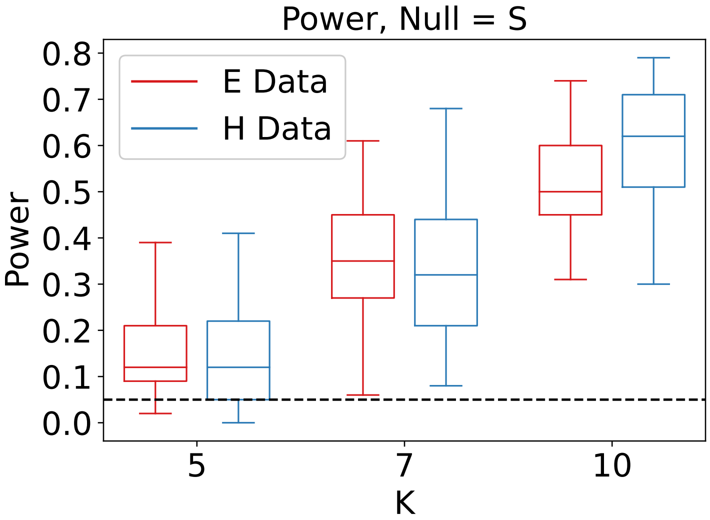
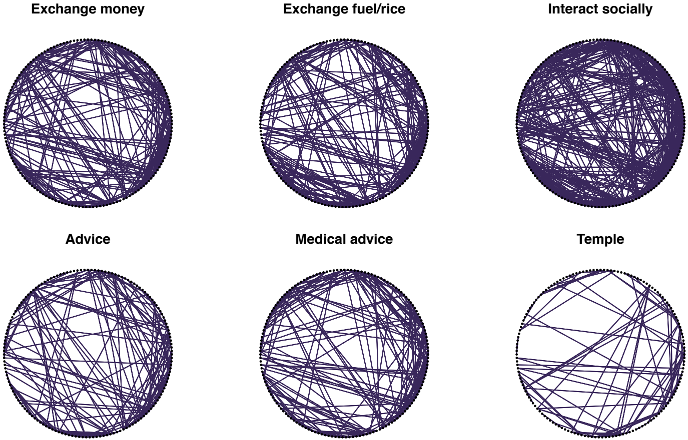


Banerjee, Duflo, Chandrasekhar and Jackson Data

- 75 villages from Karnakata, India.
- We explored, for example, how predicted geometry affects likelihood of loans between villagers.



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Future Work

- How to combine three HTs into one estimate of geometry?
- Can we test if the constant curvature assumption is satisfied?
- What if there are few cliques? Different geometry characterizations might lead to better tests [Gu et al., 2019]?
- The sub-sampling method requires several parameters. Can we use a simpler resampling method, like [Levin and Levina, 2019]?

More Information

At https://slubold.github.io/

- Arxiv pre-print
- Code
- Blog post on major points of paper
- These slides!

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Clique sizes

- We do not need all cliques!
- We

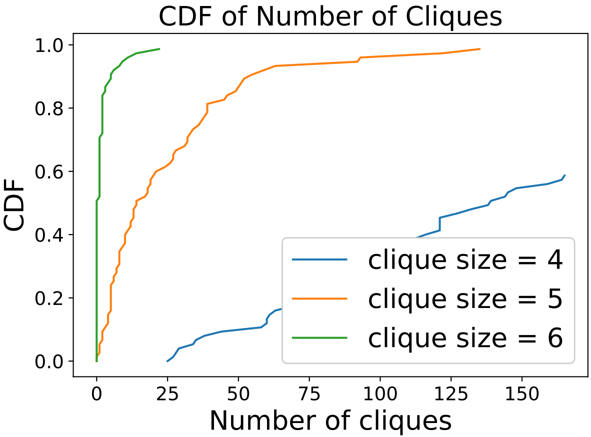

Figure 1: Number of cliques of size $\ell \in \{4, 5, 6\}$ in the Indian villages data

Picking Cliques

$$\begin{aligned}\hat{C}_1, \ldots, \hat{C}_K \in \underset{C_1,\ldots,C_k}{\text{argmin}} \sum_{i,j}^{K} |C_i \cap C_j| \\ \text{such that} \quad |C_i| = \ell \text{ for each } i \text{ and} \\ \hat{P}(C_1, \ldots, C_K) \text{ does not contains a 0.}\end{aligned} \tag{2}$$