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Avaliação 4 - Questão 9

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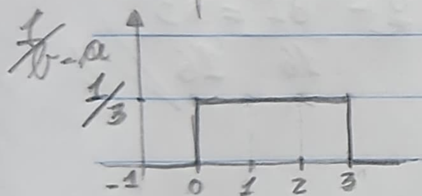
(9) Sejam $X_1, X_2 \sim Unif([0,3])$ variáveis aleatórias sorteadas independentemente.

a) Sejam: $Y_1 = X_1^2$, $Y_2 = X_2^2$, $Y_3 = X_1 \cdot X_2$. determinar o vetor média e a matriz covariância do vetor aleatório $\vec{Y} = [Y_1\ Y_2\ Y_3]^T$.

Uniforme([0,3]):



vetor média $\vec{\mu}_X = E[\vec{Y}]$:

$$\begin{bmatrix} E[Y_1] \\ E[Y_2] \\ E[Y_3] \end{bmatrix} \begin{matrix} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{matrix} \begin{bmatrix} E[X_1^2] \\ E[X_2^2] \\ E[X_1] . E[X_2] \end{bmatrix}$$

$$E[X_1^2] = \int_0^3 \frac{1}{3} x^2 dx$$

$$= \frac{1}{3} \cdot \left[\frac{x^3}{3}\right]_0^3 = 3$$

$$E[X_1^2] = E[X_2^2] = 3$$

$$E[X_1] = E[X_2] = \int_0^3 \frac{1}{3} x\, dx = \frac{x^2}{2}\Big|_0^3 = \frac{9}{6} = \frac{3}{2}$$

$$E[Y_3] = E[X_1] . E[X_2] = 9/4$$

$$E[\vec{Y}] = [3 \quad 3 \quad 9/4] \Rightarrow E[Y]^T = \begin{bmatrix} 3 \\ 3 \\ 9/4 \end{bmatrix}$$

Matriz Covariancia:

$$E[X_1^4] = 1/3 \int_0^3 x^4\,dx = \frac{1}{3}\left[\frac{x^5}{5}\right]_0^3 = \frac{243}{15} = \boxed{\frac{81}{5}}$$

$$\begin{pmatrix} var[Y_1] & cov[Y_1,Y_2] & cov[Y_1,Y_3] \\ cov[Y_2,Y_1] & var[Y_2] & cov[Y_2,Y_3] \\ cov[Y_3,Y_1] & cov[Y_3,Y_2] & var[Y_3] \end{pmatrix}$$

$$var[Y_1] = E[Y_1^2] - E[Y_1]^2 \Rightarrow E[(X_1^2)^2] - E[X_1^2] \Longrightarrow$$

$$\Longrightarrow \frac{81}{5} - 3 = \frac{81-15}{5} = \frac{66}{5}$$

$(X_1)^2 (X_2)^2$

$$var[Y_1] = var[Y_2] = 66/5$$

$$var[Y_3] = E[Y_3^2] - E[Y_3]^2 \Rightarrow E[(X_1X_2)^2] - E[Y_3]^2$$

$$\Rightarrow E[X_1^2].E[X_2^2] - E[Y_3]^2 \Rightarrow 3.3 - \left(\frac{9}{4}\right)^2 = 9 - \frac{81}{16} = \frac{63}{16}$$

$$cov[Y_1,Y_2] = cov[Y_2,Y_1] = 0$$

$$cov[Y_1 Y_3] = E[Y_1Y_3] - E[Y_1].E[Y_3] \Rightarrow E[X_1^2 . X_1X_2] - E[Y_3].E[Y_1]$$

$$= E[X_1^3].E[X_2] - E[Y_3].E[Y_1]$$

$$= \frac{27}{4}.\frac{3}{2} - \frac{9}{4}.3 \Rightarrow 27/8$$

$$E[X_1^3] = \frac{1}{3}\int_0^3 x^3\,dx$$

$$= \frac{1}{3}\left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{81}{12} = \frac{27}{4}$$

$$cov[Y_1,Y_3] = cov[Y_2,Y_3] = 27/8$$

$$\begin{pmatrix} 66/5 & 0 & 27/8 \\ 0 & 66/5 & 27/8 \\ 27/8 & 27/8 & 63/16 \end{pmatrix} = C$$

tilibra

b) Sejam $z_1 = Y_1$, $z_2 = Y_1 + Y_2$, $z_3 = Y_1 + Y_2 + Y_3$, determine o vetor médio e a matriz covariância do vetor aleatório $\vec{z} = [z_1\ z_2\ z_3]^T$. Utilize a formulação matricial.

$6 + \frac{9}{4} = \frac{24+9}{4} = \frac{33}{4}$

$$E[\vec{z}] = A \cdot E[\vec{Y}] + \vec{b}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \\ 9/4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 6 \\ 33/4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$E[\vec{z}] = \begin{pmatrix} 3 \\ 6 \\ 33/4 \end{pmatrix}$$

Matriz Covariância de z

$$C_z = A \cdot C_{\vec{y}} \cdot A^T$$

$$C_z = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 66/5 & 0 & 27/8 \\ 0 & 66/5 & 27/8 \\ 27/8 & 27/8 & 63/16 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Wolfram $\Rightarrow$ $C_{\vec{z}} = \begin{pmatrix} 66/5 & 66/5 & 66/5 \\ 66/5 & 132/5 & 132/5 \\ 663/40 & 663/20 & 663/20 \end{pmatrix}$