THE LAPLACE TRANSFORM [p 654]

- GENERALIZATION OF FOURIER TRANSFORM
- USES e^{st} BASIS FUNCTIONS

  ($s = j\omega$ => F.T.)

- CAN IN_CLUDE UNSTABLE SIGNALS

(BILATERAL) LAPLACE TRANSFORM

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$$

IF $h(t)$ IS THE IMPULSE OF A LTI SYSTEM, THEN

$H(s)$ IS CALLED THE <u>SYSTEM FUNCTION</u>

SET $s = \sigma + j\omega$

$$\therefore \quad X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} \left(x(t)\, e^{-\sigma t} \right) e^{-j\omega t}\, dt$$

$$= \text{F.T. OF } \; x(t)\, e^{-\sigma t}$$

EXAMPLE: (p 656) $\quad x(t) = e^{-at}\, u(t)$, a REAL

FIRST RECALL $\quad X(j\omega) = \dfrac{1}{j\omega + a}$, $a > 0$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(a+s)t}\, dt$$

$$= \frac{1}{s+a} \quad \text{if} \quad Re\{a+s\} > 0$$

$$\text{ie} \quad Re\{s\} > -a$$

EXAMPLE: $\quad x(t) = -e^{-at}\, u(-t)$

$$X(s) = -\int_{-\infty}^{0} e^{-(a+s)t}\, dt$$

$$= \frac{1}{s+a} \quad \text{if} \quad Re\{s+a\} < 0$$

$$\text{ie} \quad Re\{s\} < -a$$

* **ROC** : REGION OF CONVERGENCE

  : VALUES OF s FOR WHICH THE $\int$ CONVERGES

EXAMPLE: (p658)

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

WHAT IS THE R.O.C. ? WHAT IS THE L.T.?

ANSWER:

$$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1} \qquad Re\{s\} > -1$$

$$e^{-2t}u(t) \longleftrightarrow \frac{1}{s+2} \qquad Re\{s\} > -2$$

$$\therefore x(t) \longleftrightarrow X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

$$= \frac{s-1}{s^2+3s+2}, \quad \underbrace{Re\{s\} > -1}_{\text{R.O.C.}}$$

* **POLES & ZEROS**

A POLE IS LOCATED WHERE THE T/F $= \infty$

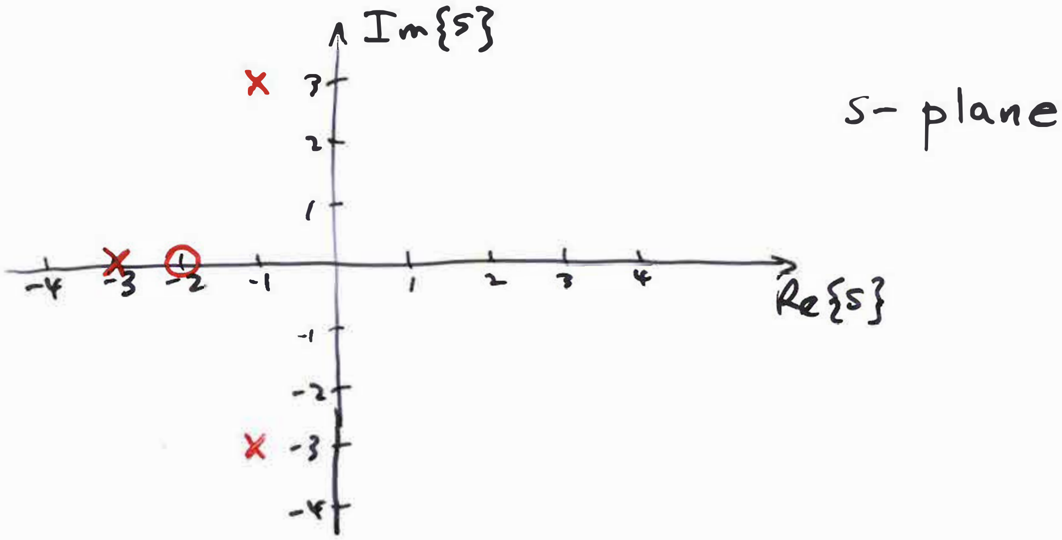
A ZERO IS LOCATED WHERE THE T/F $= 0$

EXAMPLE:
$$X(s) = \frac{10s+20}{(s^2+2s+10)(s+3)} = \frac{10(s+2)}{[(s+1)^2+3^2](s+3)}$$

$$Re\{s\} > -1$$

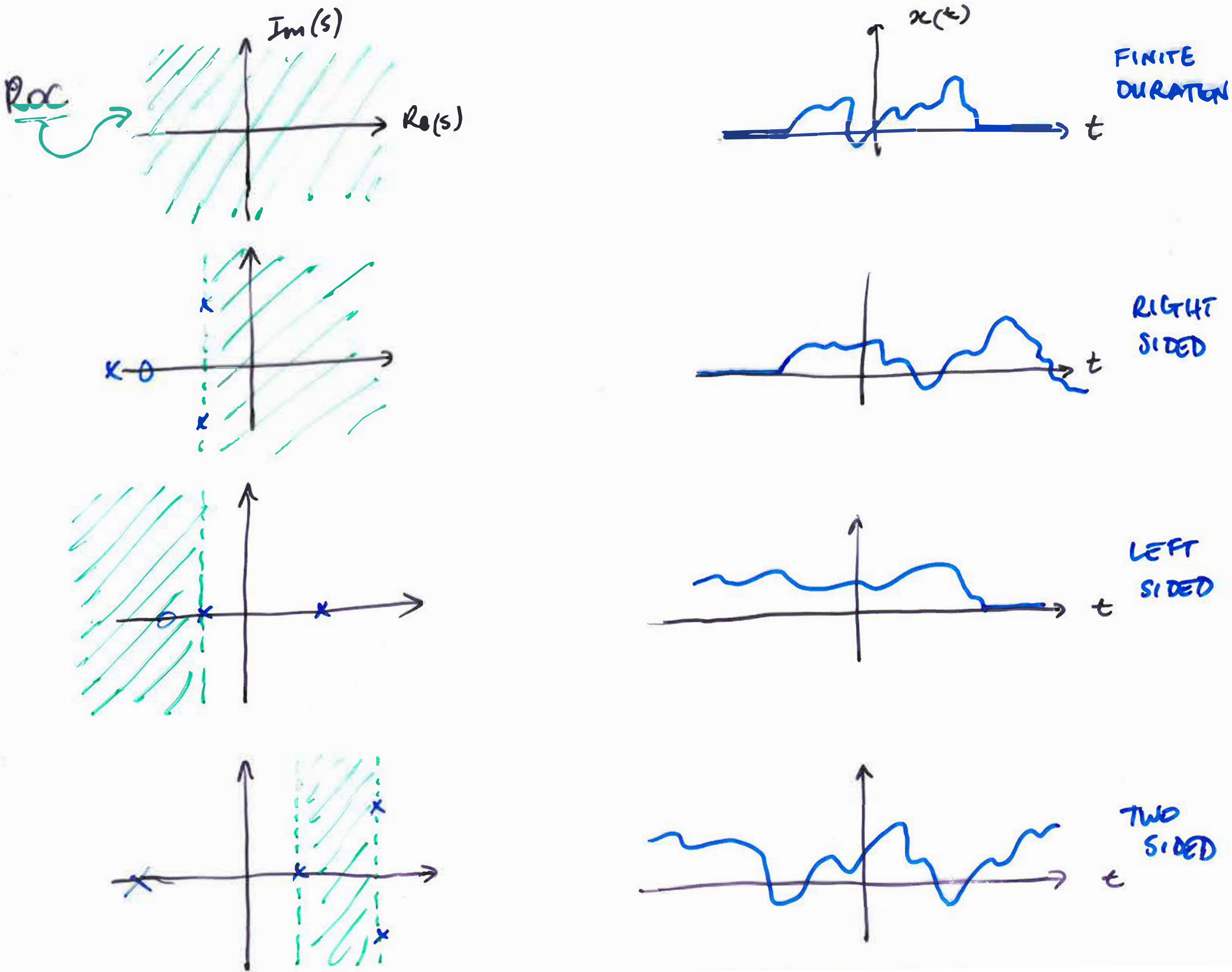
$\Rightarrow$ ZERO AT $s = -2$

POLE AT $s = -1 \pm j3$, $s = -3$



REGIONS OF CONVERGENCE (p 662)

FOR RATIONAL $X(s)$ THERE ARE 4 POSSIBILITIES



NOTE: NO POLES IN THE R.O.C.

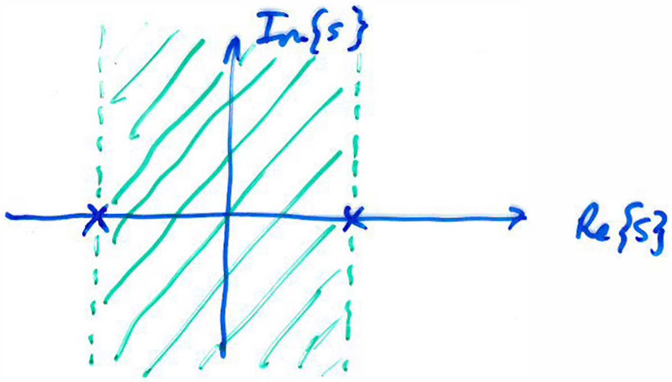
R.O.C. EXTENDS TO ∞ OR TO A POLE

EXAMPLE $\quad x(t) = e^{-b|t|}, \quad b > 0$

$$= \underbrace{e^{-bt}u(t)}_{\frac{1}{s+b},\ Re\{s\} > -b} + \underbrace{e^{bt}u(-t)}_{-\frac{1}{s-b},\ Re\{s\} < b}$$

$$\therefore \; X(s) = \frac{1}{s+b} - \frac{1}{s-b}$$

$$= -\frac{2b}{s^2-b^2}, \quad -b < Re\{s\} < b$$



ROC

INVERSE LAPLACE TRANSFORM (p 670)

RECALL THAT $X(s)$ = F.T. OF $x(t)\,e^{-\sigma t}$

$$\therefore \quad x(t)\,e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\,e^{j\omega t}\,d\omega$$

$$\Rightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)\,e^{st}\,d\omega$$

$s = \sigma + j\omega$

$\Rightarrow ds = j\,d\omega$

$$\therefore \quad \boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\,e^{st}\,ds}$$

* FOR $s = \sigma + j\omega$ IN R.O.C.

EXAMPLE: (p 671)

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

$$= \frac{A}{s+1} + \frac{B}{s+2} \quad \Rightarrow A = 1,\ B = -1$$

NOW, RECALL THAT

$$e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad Re\{s\} > -a$$

$$-e^{-at} u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad Re\{s\} < -a$$

$$\therefore \quad x(t) = e^{-t} u(t) - e^{-2t} u(t)$$

EXAMPLE:

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad Re\{s\} < -2$$

$$= \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$\therefore$ ROC'S MUST BE $Re\{s\} < -1$ & $Re\{s\} < -2$

$$\therefore \quad x(t) = -e^{-t} u(-t) + e^{-2t} u(-t)$$

EXAMPLE: $X(s)$ AS ABOVE, BUT $-2 < Re\{s\} < -1$

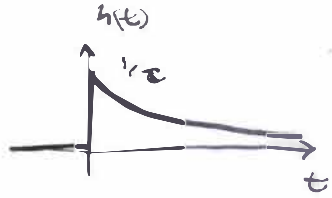
$\Rightarrow$ ROC'S MUST BE $Re\{s\} < -1$ & $Re\{s\} > -2$

$$\therefore \quad x(t) = -e^{-t} u(-t) - e^{-2t} u(t)$$

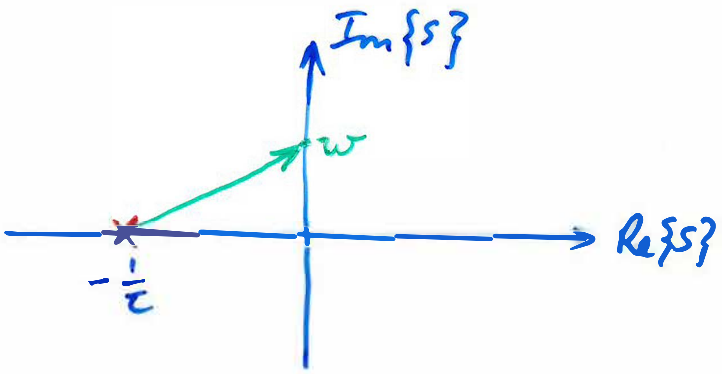
GEOMETRIC EVALUATION OF $X(s)$ – OR $X(j\omega)$

EXAMPLE: FIRST ORDER SYSTEM (p 676)

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$



$$\Rightarrow H(s) = \frac{1}{s\tau + 1}, \quad Re\{s\} > -\frac{1}{\tau}$$



RECALL:

$$F.T. = L.T. \Big|_{s=j\omega}$$

OBSERVATIONS:

① FOR $\omega = \frac{1}{\tau}$, $\left|H\left(j\frac{1}{\tau}\right)\right| = \frac{1}{\sqrt{2}}$

FOR $\omega = 0$, $|H(0)| = 1$

$\therefore$ $\omega = \frac{1}{\tau}$ IS THE 3dB POINT OF THE F.T.

② AS POLE IS MOVED FURTHER TO THE LEFT, THE SYSTEM CAN RESPOND FASTER (BIGGER B.W.)

CONCLUSIONS: POLES & ZEROS ARE EXCELLENT DESIGN TOOLS