SUMMARY: F.T. (D.T. OR C.T.) $\rightarrow$ $H(j\omega)$
$H(e^{j\omega})$

LAPLACE TRANSFORM $\rightarrow$ $H(s)$ C.T. ONLY
("GENERALISED" F.T. IN C.T.)

REMAINING: Z TRANSFORM ("GENERALIZED" F.T. IN D.T.)

THE Z TRANSFORM (p 741)

GIVEN $x[n]$,

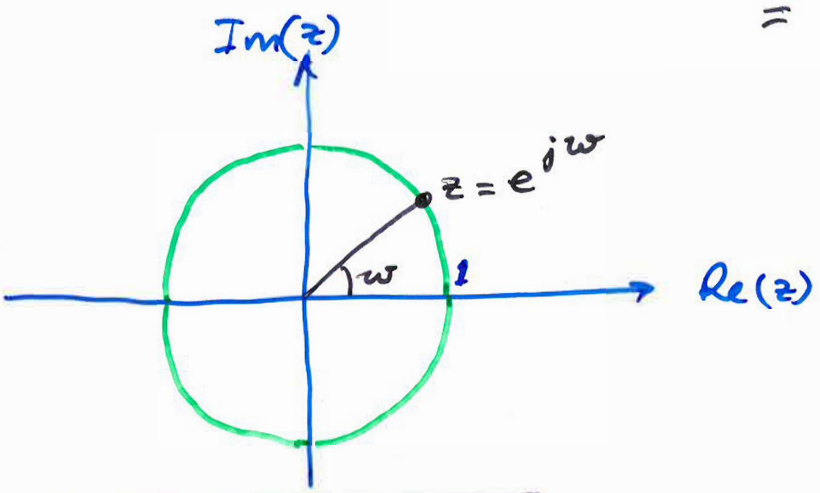
$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

IF $z = e^{j\omega}$ $\Rightarrow$ $X(e^{j\omega})$ = D.T. F.T. OF $x[n]$

MORE GENERALLY,

IF $z = r e^{j\omega}$ $\Rightarrow$ $X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, r^{-n} e^{-j\omega n}$

= F.T. OF $(x[n]\, r^{-n})$



* $$\boxed{X(z)\Big|_{\text{unit circle}} = X(e^{j\omega})}$$

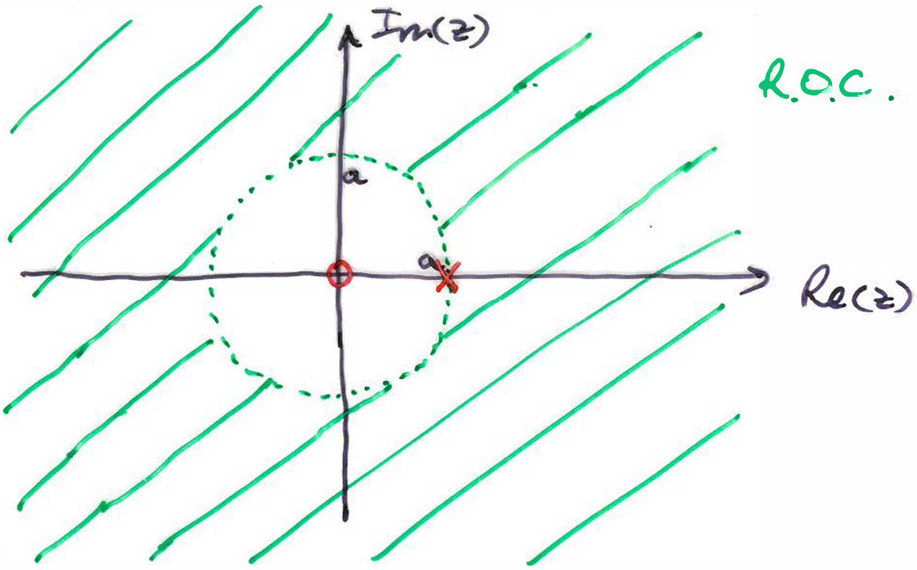
EXAMPLE: (p 743)

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}} \quad ; |a z^{-1}| < 1$$

$$= \frac{z}{z - a} \quad ; \quad |z| > |a|$$



o : ZERO

× : POLE IS ON THE CIRCLE S = a

EXAMPLE: (p 744)

$$x[n] = -a^n u[-n-1]$$

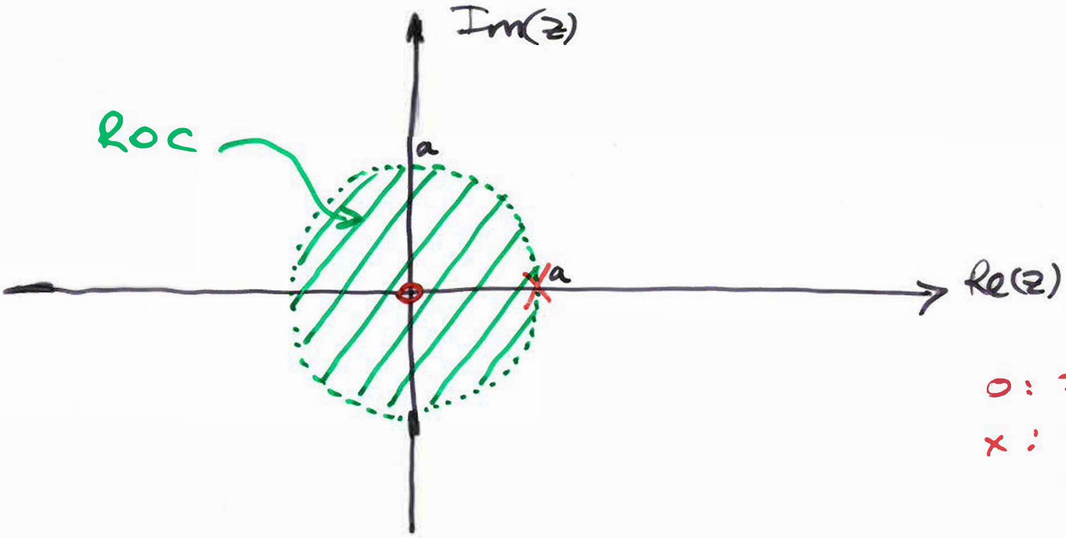
$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (a z^{-1})^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

$$\therefore\ X(z) = 1 - \frac{1}{1 - a^{-1}z} \quad \text{if } |a^{-1}z| < 1$$

$$= \frac{z}{z-a} \quad ; \quad |z| < |a|$$



O : ZERO

X : POLE IS ON THE CIRCLE S = a

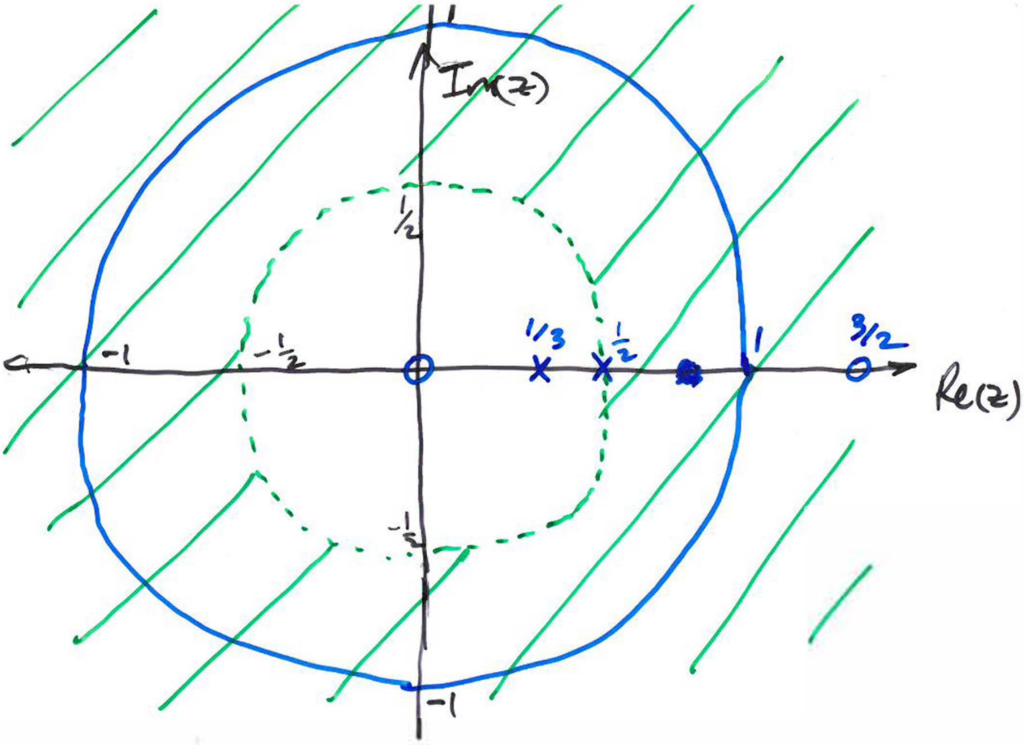
EXAMPLE : (p 745)

$$x[n] = 7\left(\tfrac{1}{3}\right)^n u[n] - 6\left(\tfrac{1}{2}\right)^n u[n]$$

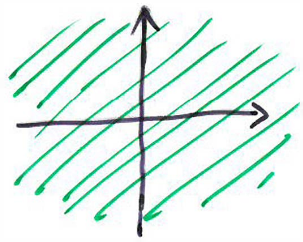
$$\left(\tfrac{1}{3}\right)^n u[n] \longleftrightarrow \frac{z}{z - 1/3} \quad ; \quad |z| > \frac{1}{3}$$

$$\left(\tfrac{1}{2}\right)^n u[n] \longleftrightarrow \frac{z}{z - \frac{1}{2}} \quad ; \quad |z| > \frac{1}{2}$$

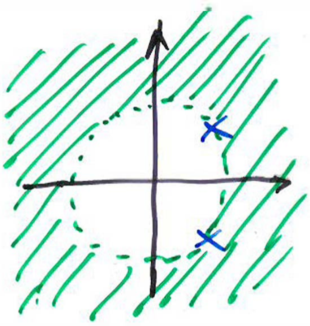
$$\therefore\ X(z) = \frac{7z}{z - 1/3} - \frac{6z}{z - 1/2} \quad ; \quad |z| > 1/2$$



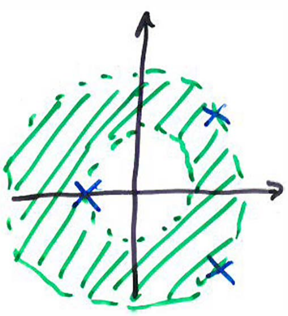
ROC



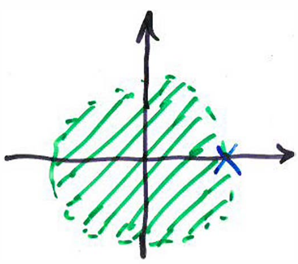
x[n] FINITE DURATION



x[n] RIGHT SIDED



x[n] TWO SIDED



x[n] LEFT SIDED

EXAMPLE: IF $x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]\, z^{-n} = z^{0} = 1$$

R.O.C. = ENTIRE z-PLANE

EXAMPLE: IF $x[n] = \delta[n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-1]\, z^{-n}$$

$$= z^{-1} = \frac{1}{z}$$

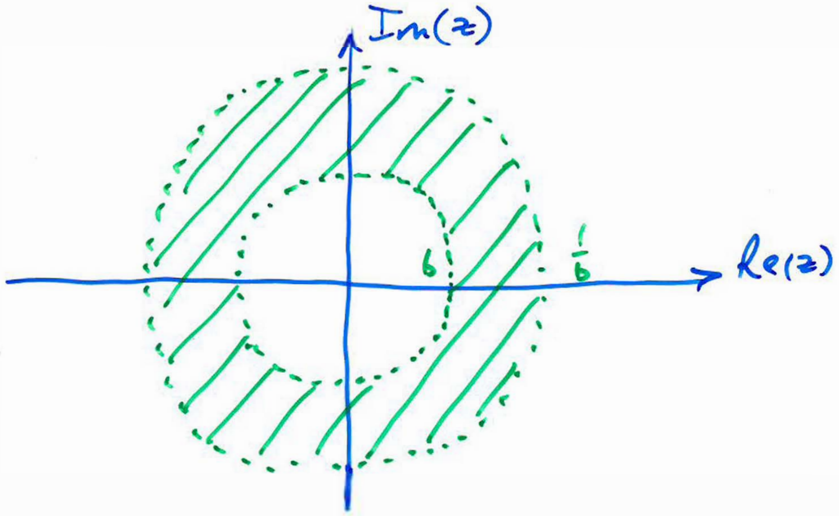
R.O.C. ENTIRE z-PLANE EXCEPT $z=0$

EXAMPLE: (p 753)

$$x[n] = b^{|n|} \qquad b>0$$

$$x[n] = \underbrace{b^{n}\, u[n]}_{\frac{1}{1-bz^{-1}},\ |z|>b} + \underbrace{b^{-n}\, u[-n-1]}_{-\frac{1}{1-b^{-1}z^{-1}},\ |z|<\frac{1}{b}}$$

$$\therefore\ X(z) = \frac{1}{1-bz^{-1}} - \frac{1}{1-b^{-1}z^{-1}}, \quad b<|z|<\frac{1}{b}$$



$\therefore$ $X(z)$ EXISTS FOR $b<1$

& DOES NOT EXIST FOR $b \geq 1$

INVERSE Z TRANSFORM (p 757)

$$\boxed{x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz}$$

$\oint$ MEANS ~~INTEGRATE~~ COUNTER CLOCKWISE AROUND A CIRCLE OF CONSTANT AMPLITUDE IN THE ROC.

EITHER DIRECTLY SOLVE THE ABOVE EQUATION, OR USE PARTIAL FRACTION EXPANSION & THE FORMULA SHEETS.

EXAMPLE: (p 758)

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{3} z^{-1}\right)} \quad ; \; |z| > \frac{1}{3}$$

INVERSE

$$= \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}} \quad ; \; |z| > \frac{1}{3}$$

$$\therefore \quad x[n] = \left(\tfrac{1}{4}\right)^n u[n] + 2\left(\tfrac{1}{3}\right)^n u[n]$$

EXAMPLE: (p 761)

$$X(z) = 4z^2 + 2 + 3z^{-1} \quad ; \quad 0 < |z| < \infty$$

FIND $x[n]$

ANSWER:

FROM DEFINITION, $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

$\therefore$ MATCH TERMS $\Rightarrow$ $x[-2] = 4$ & $x[n] = 0$ OTHERWISE

$x[0] = 2$

$x[1] = 3$

$$\therefore \quad x[n] = 4\,\delta[n+2] + 2\,\delta[n] + 3\,\delta[n-1]$$

EXAMPLE: (p 762)

$$X(z) = \log(1 + az^{-1}) \quad ; \quad |z| > |a|$$

SINCE $|az^{-1}| < 1$, CAN EXPAND AS TAYLOR SERIES

$$X(z) = az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \dots + \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

$\therefore$ MATCH TERMS $\Rightarrow$

$$x[n] = (-1)^{n+1} \frac{a^n}{n} \quad ; \quad n \geq 1$$

$$= 0 \quad ; \quad n \leq 0$$

$$\Rightarrow \quad x[n] = -\frac{(-a)^n}{n} u[n-1]$$

GEOMETRIC EVALUATION FROM POLE-ZERO PLOT

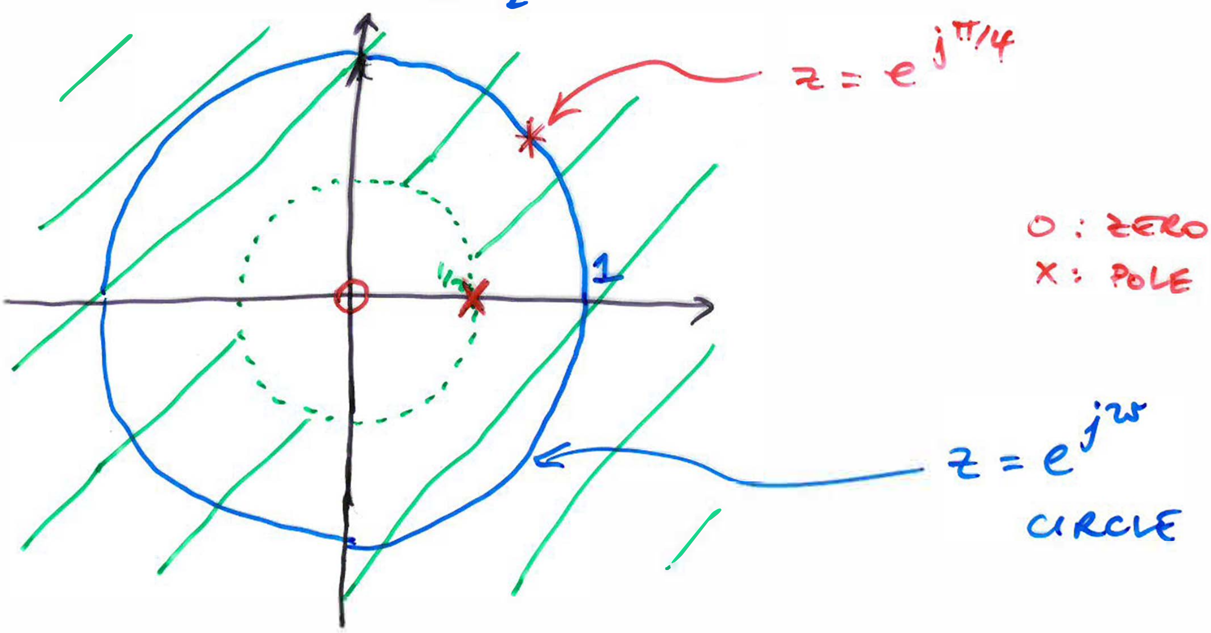
(p 763)

EXAMPLE: SUPPOSE $H(z) = \dfrac{10}{1-\frac{1}{2}z^{-1}}$; $|z| > \frac{1}{2}$

WHAT IS $H(z)$ WHEN $z = e^{j\pi/4}$?

ANSWER:

$$H(z) = \frac{10\,z}{z-\frac{1}{2}} \quad ; \quad |z| > \frac{1}{2}$$



$$|H(e^{j\pi/4})| = \left| \frac{10\, e^{j\pi/4}}{e^{j\pi/4} - \frac{1}{2}} \right|$$

$$= \frac{10}{\left(\text{DISTANCE BETWEEN THE POINT } e^{j\pi/4} \text{ AND THE POLE LOCATION}\right)}$$

OBSERVATIONS :

1. AS $\omega \uparrow$, DIST FROM POLE TO PT $\uparrow$ & MAGNITUDE OF TRANSFER FN $\downarrow$ $\therefore$ LOW PASS FILTER.

2. IF THE POLE IS MOVED TOWARDS THE ORIGIN, THE EFFECT IN 1 IS REDUCED, SO THE SYSTEM IS MORE STABLE.

PROPERTIES OF THE Z TRANSFORM (p 767)

TIME DELAY:

$$x[n-n_0] \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$$

(1 DELAY IN TIME $\longleftrightarrow$ MULTIPLY BY z^{-1})

CONVOLUTION:

$$x[n] * y[n] \overset{Z}{\longleftrightarrow} X(z)\,Y(z)$$

EXAMPLE:

$$x[n] = \delta[n+1] + \delta[n]$$

$$h[n] = \delta[n] + 2\,\delta[n-2]$$

FIND $y[n] = h[n] * x[n]$.

ANSWER:

$$X(z) = z+1 \quad ; \ \forall z$$

$$H(z) = 1 + 2z^{-2} \quad ; \ \forall z \mid z \neq 0$$

$$\therefore \ Y(z) = (z+1)(1+2z^{-2})$$

$$= z + 1 + 2z^{-1} + 2z^{-2} \quad ; \ \forall z \text{ except } z=0$$

i.e. MATCH TERMS IN Z T.F. DEF$^{\underline{n}}$.

$$\Rightarrow y[n] = \delta[n+1] + \delta[n] + 2\delta[n-1] + 2\delta[n-2]$$

(OR YOU COULD HAVE DONE THIS VIA DIRECT CONVOLUTION)

L.T.I SYSTEMS

CAUSAL: IFF ROC OF $H(z)$ IS EXTERIOR OF A CIRCLE EXTENDING TO ∞

STABILITY: ROC MUST CONTAIN UNIT CIRCLE $|z| = 1$

DIFFERENCE EQUATIONS:

EXAMPLE: (p 779)

$$y[n] - \frac{1}{2} y[n-1] = x[n] + \frac{1}{3} x[n-1]$$

FIND THE IMPULSE RESPONSE

ANSWER:

USE $x[n-1] \overset{Z}{\longleftrightarrow} z^{-1} X(z)$

$$\Rightarrow \quad Y(z) - \frac{1}{2} z^{-1} Y(z) = X(z) + \frac{1}{3} z^{-1} X(z)$$

$$\Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

ROC ?

* TWO POSSIBLE ROC'S $|z| > \frac{1}{2}$ OR $|z| < \frac{1}{2}$

IF $|z| > \frac{1}{2}$ THEN SYSTEM IS STABLE & CAUSAL

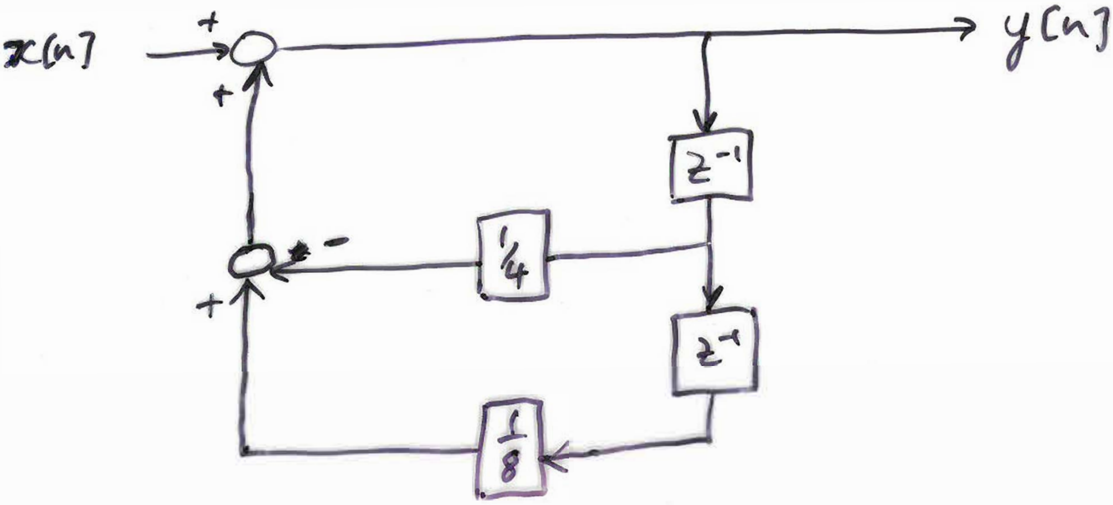
$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{3} z^{-1} \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$$\Rightarrow \quad h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

IF $|z| < \frac{1}{2}$ SYSTEM IS NOT CAUSAL & UNSTABLE

$$h[n] = \left(-\tfrac{1}{2}\right)^n u[-n-1] - \tfrac{1}{3}\left(\tfrac{1}{2}\right)^{n-1} u[-n]$$

FILTERING EXAMPLE:



$$y[n] = x[n] - \tfrac{1}{4}\, y[n-1] + \tfrac{1}{8}\, y[n-2]$$

$$\therefore\ Y(z) = X(z) - \tfrac{1}{4}\underbrace{z^{-1}}\, Y(z) + \tfrac{1}{8}\underbrace{z^{-2}}\, Y(z)$$

DELAY ELEMENTS MULTIPLY
IN THE Z-DOMAIN!

$$H(z) = \frac{1}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$= \frac{1}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

2 ZEROS AT $\emptyset$

1 POLE AT $z = -\frac{1}{2}$

1 POLE AT $z = \frac{1}{4}$