LAST LECTURES: LAPLACE T.F.

(= GENERALISED F.T.)

INVERSE LT.F.

GEOMETRIC INTERPRETATIONS OF POLES & ZEROS

THIS LECTURE: PROPERTIES OF L.T.

LTI SYSTEMS & L.T.

PROPERTIES OF LAPLACE TRANSFORM (P682)

CONSIDER $x(t) \xleftrightarrow{\mathcal{L}} X(s)$ WITH ROC $= R$

LINEARITY:

$$a\, x_1(t) + b\, x_2(t) \xleftrightarrow{\mathcal{L}} a\, X_1(s) + b\, X_2(s)$$

WITH ROC $= R_1 \cap R_2$
(at least!)
(ex $a = -b$ & $x_1 = x_2$)

TIME SHIFT:

$$x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s) \qquad \text{ROC} = R$$

CONVOLUTION:

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)\, X_2(s)$$

WITH ROC $= R_1 \cap R_2$
(at least!)

DIFFERENTIATION:

$$\frac{d\,x(t)}{dt} \xleftrightarrow{\mathcal{L}} s\, X(s) \qquad \text{ROC CONTAINS } R$$

(ROC LARGER IF POLE AT S=0 IS CANCELLED)

INTEGRATION

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)$$

ROC CONTAINS

$$R \cap \{ \mathrm{Re}\{s\} > 0 \}$$

INITIAL VALUE THEOREM

$$x(0+) = \lim_{s \to \infty} s\,X(s)$$

FINAL VALUE THEOREM

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\,X(s)$$

FOR $x(t) = 0,\ t<0$

& NO SINGULARITIES AT $t=0$

EXAMPLE: FIND THE INITIAL VALUE OF THE SIGNAL

$$x(t) = e^{-2t}\, u(t) + e^{-t}(\cos 3t)\, u(t)$$

ALSO, FIND ITS L.T. & CONFIRM THE INITIAL VALUE THEOREM.

ANSWER

$$x(0^{+}) = \lim_{\substack{t \to 0 \\ \text{from} \\ \text{above}}} e^{-2t} + e^{-t}\cos(3t)$$

$$= 2$$

NOW, TO FIND THE L.T., COULD TRY TO DIRECTLY SOLVE L.T. INTEGRAL, OR, USE LINEAR PROPERTY

$$x(t) = e^{-2t}u(t) + \tfrac{1}{2}e^{-(1-3j)t}u(t) + \tfrac{1}{2}e^{-(1+3j)t}u(t)$$

$$\frac{1}{s+2} \; ; \; Re(s) > -2$$

$$\frac{1}{2}\cdot\frac{1}{s+(1-3j)} \; ; \; Re(s) > -1$$

$$\frac{1}{2}\cdot\frac{1}{s+(1+3j)} \; ; \; Re(s) > -1$$

$$\therefore X(s) = \frac{2s^2 + 5s + 12}{(s^2+2s+10)(s+2)} \quad ; \; Re\{s\} > -1$$

Now, $$\lim_{s \to \infty} s X(s) = \lim_{s \to \infty} \frac{2s^3 + 5s^2 + 12s}{s^3 + 4s^2 + 14s + 20}$$

$$= 2$$

ANALYSIS AND CHARACTERIZATION OF LTI SYSTEMS WITH THE LAPLACE TRANSFORM (p693)

$Y(s) = H(s)\,X(s)$ ie convolution in the time domain.

RECALL THAT IF $s = j\omega$ THEN IT'S JUST THE FOURIER TRANSFORM!

CAUSALITY

$h(t) = 0$ FOR $t < 0$

$\therefore$ $h(t)$ RIGHT SIDED

$\therefore$ ROC MUST BE A RIGHT HALF PLANE
BUT ONLY HOLDS IN REVERSE IF $H(s)$ RATIONAL!!

EXAMPLES:

1. $h(t) = e^{-t} u(t)$

$$\Rightarrow H(s) = \underbrace{\frac{1}{s+1}}_{\text{RATIONAL}} \quad , \; Re\{s\} > -1$$

RATIONAL $\Rightarrow$ CAUSAL

EXAMPLE 2 : $h(t) = e^{-|t|}$ NOT CAUSAL

$$H(s) = \frac{-2}{s^2-1} \; ; \quad -1 < Re(s) < 1$$

3 : $$H(s) = \frac{e^{s}}{s+1}$$ NOT RATIONAL!

$Re(s) > -1$

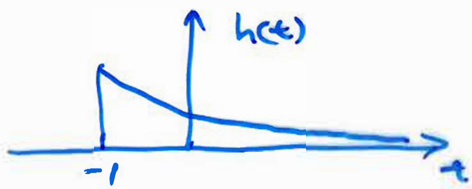
$h(t)$ IS RIGHT SIDED , BUT

SINCE $e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$; $Re(s) > -1$

THEN $e^{-(t+1)}u(t+1) \longleftrightarrow \frac{e^{s}}{s+1}$; $Re(s) > -1$

$\Rightarrow$ $h(t)$ IS NOT ZERO FOR ALL $t<0$

$\therefore$ NOT CAUSAL



STABILITY:

RECALL: LTI SYSTEM IS STABLE IFF $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$\Rightarrow$ F.T. CONVERGES

i.e. A STABLE LTI HAS A F.T.

ALSO RECALL: F.T. = L.T. EVALUATED ALONG $j\omega$-AXIS

$\therefore$ $\boxed{\text{LTI SYSTEM IS STABLE IFF R.O.C. OF } H(s) \text{ INCLUDES } j\omega \text{ AXIS.}}$

EXAMPLE: (p 695)

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

(a) IF YOU ARE TOLD THAT THIS SYSTEM IS STABLE, WHAT IS THE IMPULSE RESPONSE?

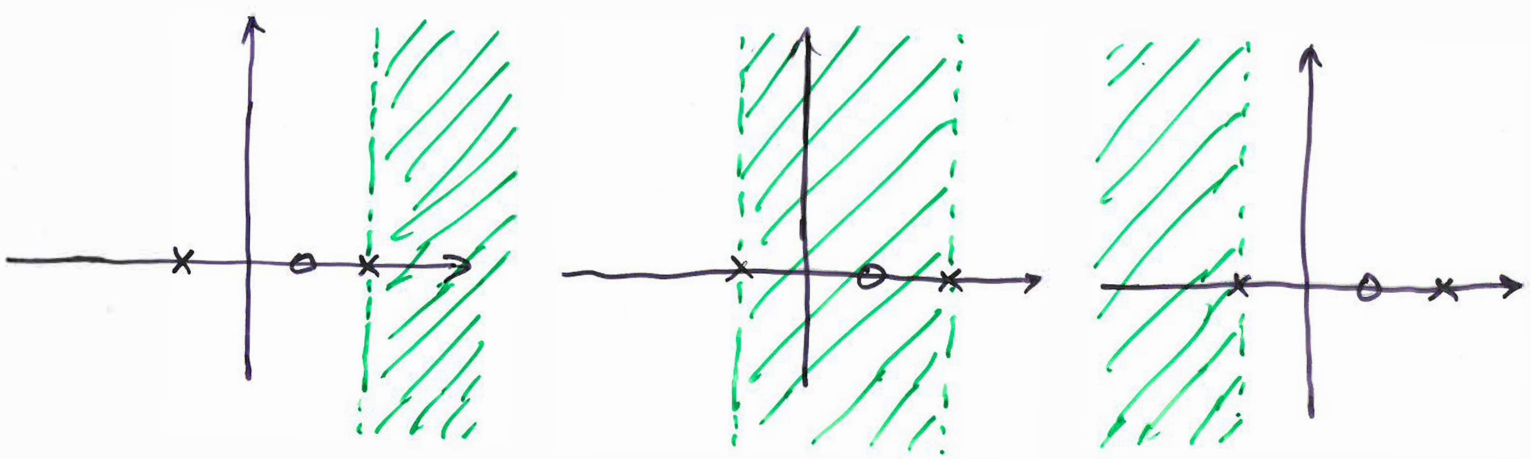
(b) WHAT IF YOU WERE TOLD IT WAS CAUSAL?

ANSWER

$$H(s) = \frac{2/3}{s+1} + \frac{1/3}{s-2}$$

ZERO AT $s=1$

POLES AT $s=-1,2$



ANSWER (a) STABLE REQUIRES R.O.C. INCLUDES $j\omega$ AXIS.

$$\Rightarrow \quad H(s) = \frac{2/3}{s+1} + \frac{1/3}{s-2} \quad ; \quad -1 < Re\{s\} < 2$$

$$\Rightarrow \quad h(t) = \frac{2}{3} e^{-t} u(t) - \frac{1}{3} e^{2t} u(-t)$$

ANSWER (b) CAUSAL REQUIRES R.O.C. IS A RIGHT HALF PLANE

$$\Rightarrow \quad Re\{s\} > 2$$

$$\Rightarrow \quad h(t) = \frac{2}{3} e^{-t} u(t) + \frac{1}{3} e^{2t} u(t)$$

IMPORTANT OBSERVATION:

> A CAUSAL SYSTEM WITH RATIONAL $H(s)$
> IS STABLE IFF ALL POLES OF $H(s)$
> HAVE $Re\{s\} < 0$

L.T.I. SYSTEMS DESCRIBED BY D.E.s (p 698)

EXAMPLE:

$$\frac{dy(t)}{dt} + 3\,y(t) = x(t)$$

L.T. $\Rightarrow$ $\quad s\,Y(s) + 3\,Y(s) = X(s)$

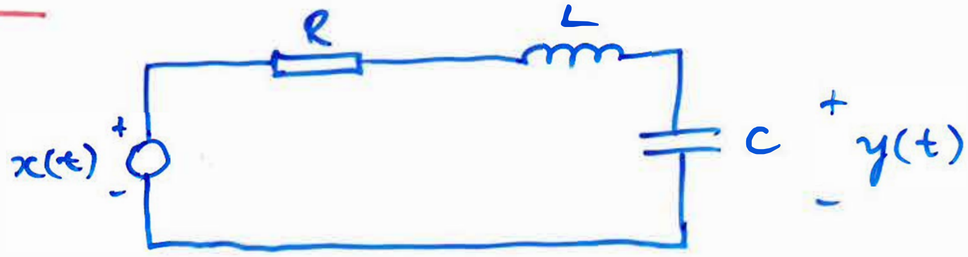
$$\therefore \quad H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

R.O.C. IS NOT SPECIFIED BY THE DIFF. EQ^{n} !!!

BUT, IF YOU ALSO KNOW IT IS CAUSAL THEN YOU KNOW $Re\{s\} > -3$ (NOTE: THIS CASE IS ALSO STABLE)

$$\Rightarrow \quad h(t) = e^{-3t} u(t)$$

EXAMPLE:



$$x(t) = i(t)R + L\frac{di(t)}{dt} + y(t)$$

$$\& \quad i(t) = C\frac{dy(t)}{dt}$$

$$\Rightarrow \quad x(t) = RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t)$$

$$\Rightarrow \quad H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}$$

NOW, SINCE R, L & C ARE ALL +ve $\rightarrow$ POLES HAVE ALL -ve REAL PTS

$\Rightarrow$ SYSTEM IS ALWAYS STABLE

Q. WHAT IF YOU REMOVE THE RESISTOR?