Lab experiments start next week!

- To do **this week:**
  - Experiments in wks 6,7, 9, 10, **2 – 5 pm**
    - Python returns in wks 11, 12, 13, **1 – 4 pm**
  - Sign up for your first lab experiment ASAP
    - coordinate with preferred lab partner, if you have one
  - Read first chapter of the lab notes
  - Do the pre-lab reading for your first experiment and answer the questions
    - that means *before* you show up to the lab
    - your lab supervisor will check your answers early in the lab session

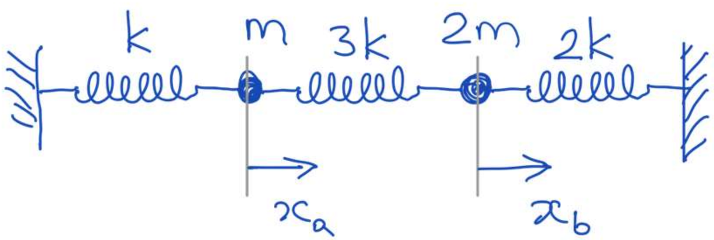
Coupled SHM – Examples

Prof David Spence

Based on slides provided by Prof David Spence

Question 1 (20 marks).

Two masses m and $2m$ lie on a horizontal frictionless surface and are connected to each other and to a pair of fixed vertical walls by springs with spring constants k, $3k$, and $2k$ as illustrated in the figure at right. The displacements of the masses from their equilibrium positions are x_a and x_b, respectively.



a) (4 marks) Write down expressions for the force on:

(i) m when $x_a = 0$

(ii) m when $x_b = 0$

(iii) $2m$ when $x_a = 0$

(iv) $2m$ when $x_b = 0$

b) (4 marks) Hence show that the displacements of the two masses from their equilibrium positions x_a and x_b obey the coupled equations:

$$\ddot{x}_a = -\frac{4k}{m}x_a + \frac{3k}{m}x_b$$

$$\ddot{x}_b = \frac{3k}{2m}x_a - \frac{5k}{2m}x_b$$

c) (4 marks) Use the substitutions $x_a = Ae^{i\omega t}$ and $x_b = Be^{i\omega t}$ to show that the normal mode frequencies are

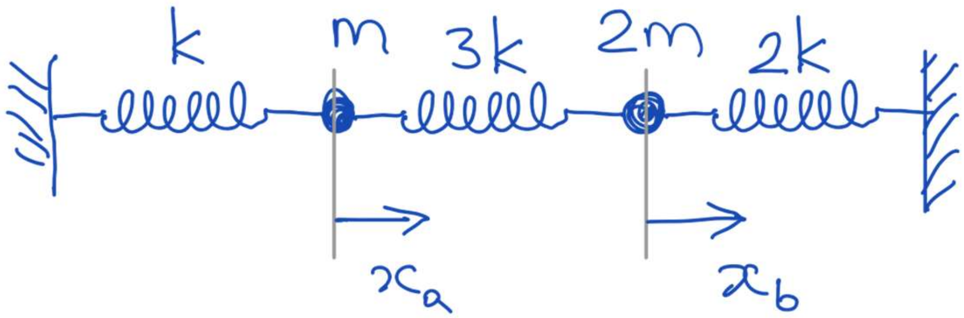
$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{11k}{2m}}$$

d) (4 marks) Calculate the amplitude ratio A/B for each of the two normal modes.

e) (4 marks) Consider the lowest-frequency mode. Explain how the springs are successively stretched and compressed during the oscillations and use this information to explain why the mode frequency is $\sqrt{k/m}$.

Question 1 (20 marks).

Two masses m and $2m$ lie on a horizontal frictionless surface and are connected to each other and to a pair of fixed vertical walls by springs with spring constants k, $3k$, and $2k$ as illustrated in the figure at right. The displacements of the masses from their equilibrium positions are x_a and x_b, respectively.



a) (4 marks) Write down expressions for the force on:

(i) m when $x_a = 0$ $3kx_b$

(ii) m when $x_b = 0$ $-4kx_a$

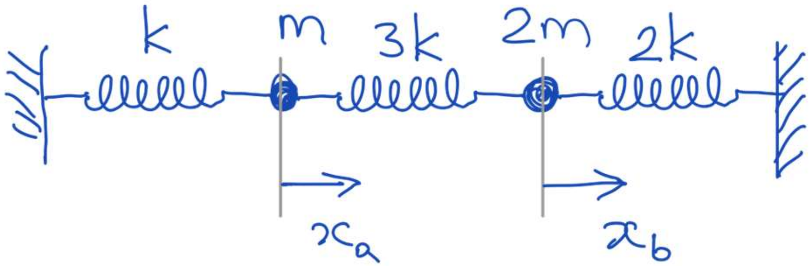
(iii) $2m$ when $x_a = 0$ $-5kx_b$

(iv) $2m$ when $x_b = 0$ $3kx_a$

b) (4 marks) Hence show that the displacements of the two masses from their equilibrium positions x_a and x_b obey the coupled equations:

From (a) (i) & (ii), $m\ddot{x}_a = 3kx_b - 4kx_a \Rightarrow$ $$\ddot{x}_a = -\frac{4k}{m}x_a + \frac{3k}{m}x_b$$

(a) (iii) & (iv), $2m\ddot{x}_b = 3kx_a - 5kx_b \Rightarrow$ $$\ddot{x}_b = \frac{3k}{2m}x_a - \frac{5k}{2m}x_b$$



$$\ddot{x}_a = -\frac{4k}{m} x_a + \frac{3k}{m} x_b$$

$$\ddot{x}_b = \frac{3k}{2m} x_a - \frac{5k}{2m} x_b$$

c) (4 marks) Use the substitutions $x_a = Ae^{i\omega t}$ and $x_b = Be^{i\omega t}$ to show that the normal mode frequencies are

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{11k}{2m}}$$

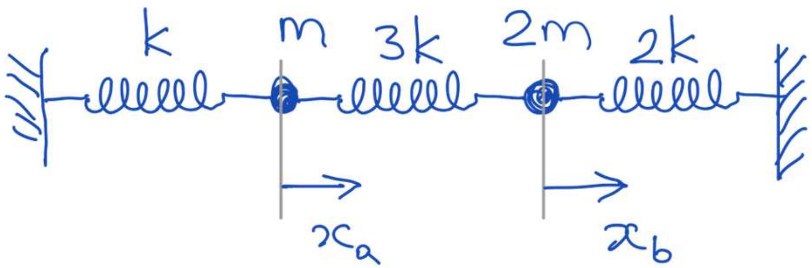
$$-\omega^2 A = -\frac{4k}{m}A + \frac{3k}{m}B$$
$$-\omega^2 B = \frac{3k}{2m}A - \frac{5k}{2m}B$$

$$\Rightarrow 0 = \begin{vmatrix} \omega^2 - \frac{4k}{m} & 3k/m \\ \frac{3k}{2m} & \omega^2 - \frac{5k}{2m} \end{vmatrix}$$

$$= \left(\omega^2 - \frac{4k}{m}\right)\left(\omega^2 - \frac{5k}{2m}\right) - \frac{3k}{m}\frac{3k}{2m} = \omega^4 - \frac{13k}{2m}\omega^2 + \frac{11}{2}\frac{k^2}{m^2}$$

$$= \left(\omega^2 - \frac{k}{m}\right)\left(\omega^2 - \frac{11k}{2m}\right)$$

$$\Rightarrow \omega_1 = \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{11k}{2m}}$$



$$\ddot{x}_a = -\frac{4k}{m}x_a + \frac{3k}{m}x_b$$

$$\ddot{x}_b = \frac{3k}{2m}x_a - \frac{5k}{2m}x_b$$

$x_a = Ae^{i\omega t}$ and $x_b = Be^{i\omega t}$

$\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{11k}{2m}}$

d) (4 marks) Calculate the amplitude ratio A/B for each of the two normal modes.

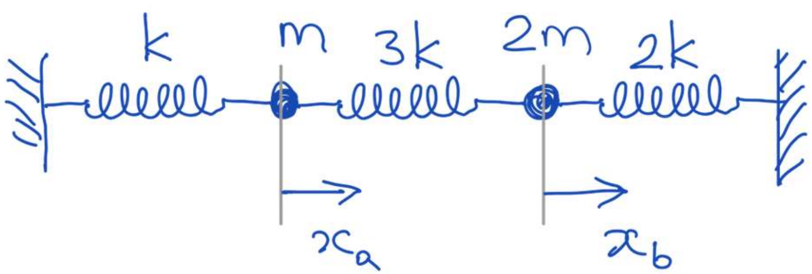
$$-\omega^2 A = -\frac{4k}{m}A + \frac{3k}{m}B$$

$$-\omega^2 B = \frac{3k}{2m}A - \frac{5k}{2m}B$$

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \frac{A}{B} = \frac{-3k/m}{\omega_1^2 - 4k/m} = \frac{-3k/m}{-3k/m} = +1$$

$$\omega_2 = \sqrt{\frac{11k}{2m}} \qquad \frac{A}{B} = \frac{-3k/m}{\omega_2^2 - 4k/m} = \frac{-3k/m}{3/2\,k/m} = -2$$

Example – Exam 2019



$$\ddot{x}_a = -\frac{4k}{m}x_a + \frac{3k}{m}x_b$$

$$\ddot{x}_b = \frac{3k}{2m}x_a - \frac{5k}{2m}x_b$$

$x_a = Ae^{i\omega t}$ and $x_b = Be^{i\omega t}$

$\omega_1 = \sqrt{\frac{k}{m}}$ $\quad \frac{A}{B} = +1$

$\omega_2 = \sqrt{\frac{11k}{2m}}$ $\quad \frac{A}{B} = -2$

e) (4 marks) Consider the lowest-frequency mode. Explain how the springs are successively stretched and compressed during the oscillations and use this information to explain why the mode frequency is $\sqrt{k/m}$.

For ω_1 $A = +B \Rightarrow x_a = x_b$. Central spring is not stretched at all

$\therefore$ masses do not feel each others displacements

$\Rightarrow$ m and 2m behave as if isolated oscillators on springs

$\Rightarrow \omega = \sqrt{\frac{k}{m}}$ and $\sqrt{\frac{2k}{2m}}$, respectively.

The loaded string

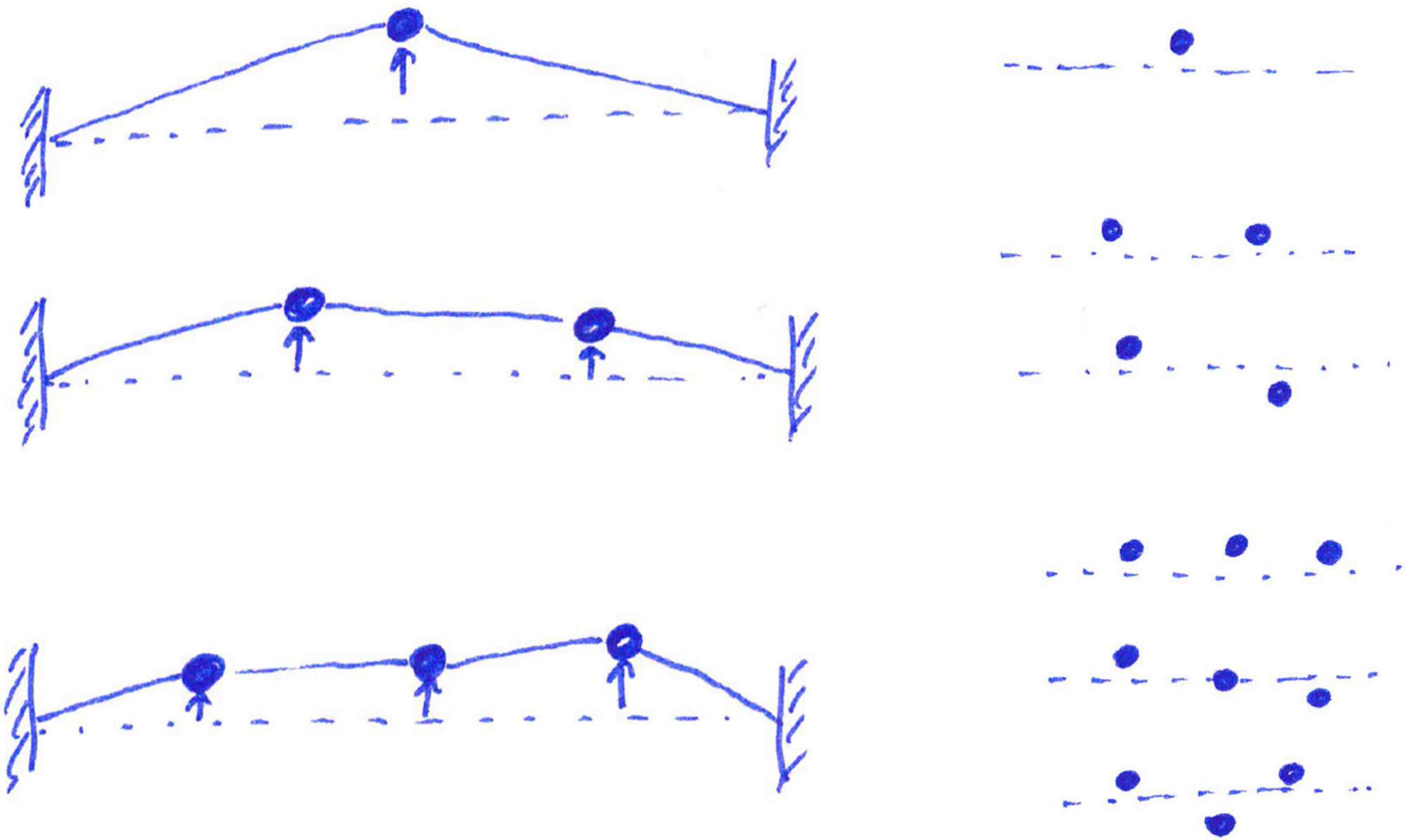
Prof David Spence

Recall: Properties of normal modes

- For a normal mode
  - Oscillation is independent of other normal modes coordinates
  - Energy is not transferred to other normal modes
  - All oscillators have a fixed phase relation
  - All oscillators have fixed amplitude ratio
  - All oscillators share the same frequency
- N oscillators → N normal modes
- Lowest frequency mode has all oscillators in phase
- Highest frequency mode has adjacent oscillators out of phase

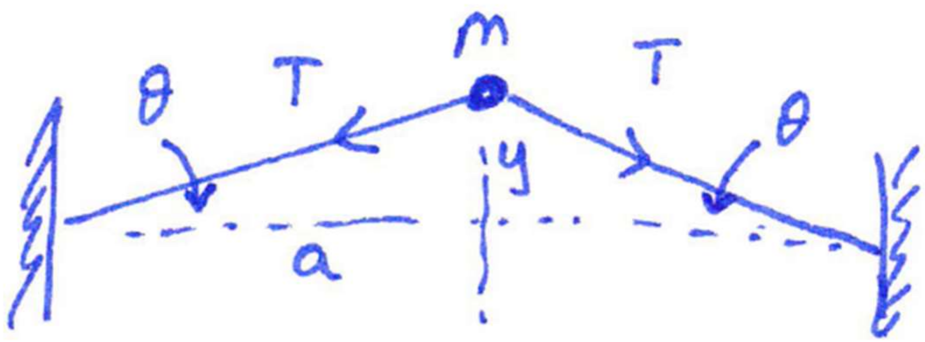
Let’s look at a ‘loaded string’

- Light string under tension, loaded with point masses



One mass

- Light string
- Point mass
- Constant tension
- Small deviation

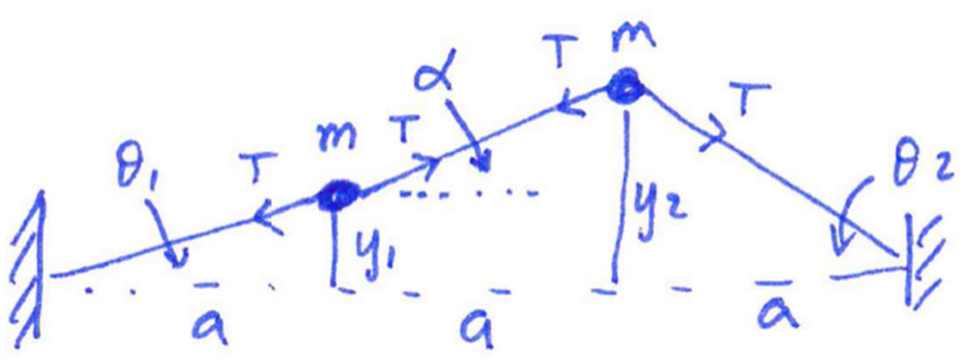


$$F = m\ddot{y} = -2T\sin\theta \approx -2T\tan\theta \approx -2Ty/a$$

$\theta \ll 1$

$$\ddot{y} + \frac{2T}{ma}y = 0$$

SHM with frequency $\omega^2 = 2T/(ma)$



left mass: $m\ddot{y}_1 = -T\sin\theta_1 + T\sin\alpha \approx -T(y_1/a - (y_2 - y_1)/a)$

$$\ddot{y}_1 + (T/ma)(2y_1 - y_2) = 0$$

right mass: $$\ddot{y}_2 + (T/ma)(2y_2 - y_1) = 0$$

Add: $$\ddot{y}_1 + \ddot{y}_2 + (T/ma)(y_1 + y_2) = 0$$

Subtract: $$\ddot{y}_1 - \ddot{y}_2 + (3T/ma)(y_1 - y_2) = 0$$

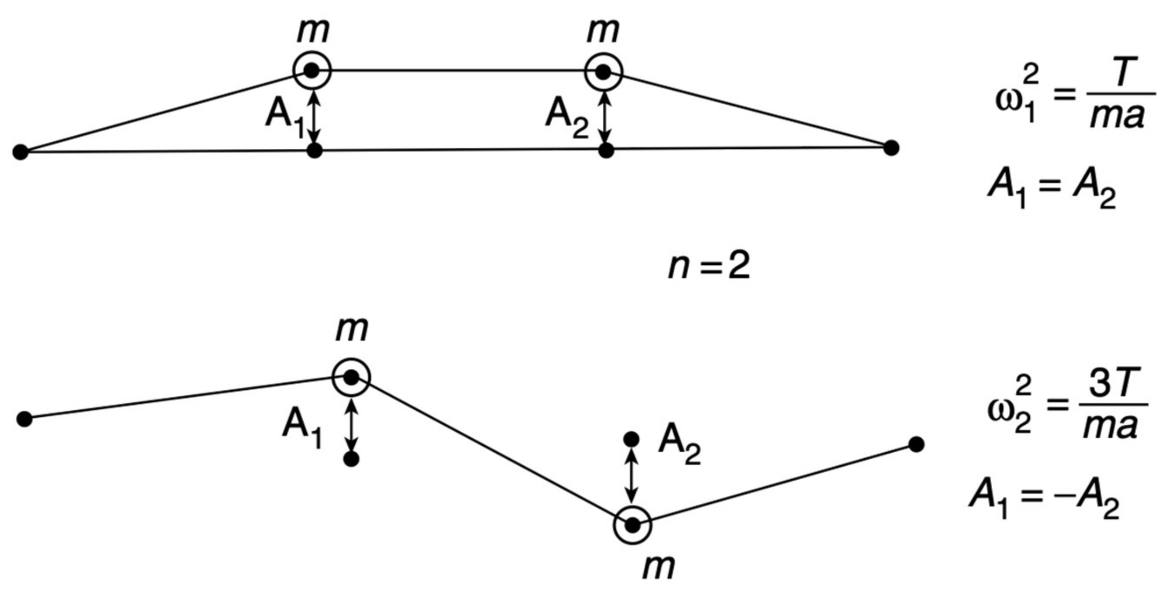
Two masses have two normal modes*

$$\ddot{y}_1 + \ddot{y}_2 + (T/ma)(y_1 + y_2) = 0$$

$$\ddot{y}_1 - \ddot{y}_2 + (3T/ma)(y_1 - y_2) = 0$$

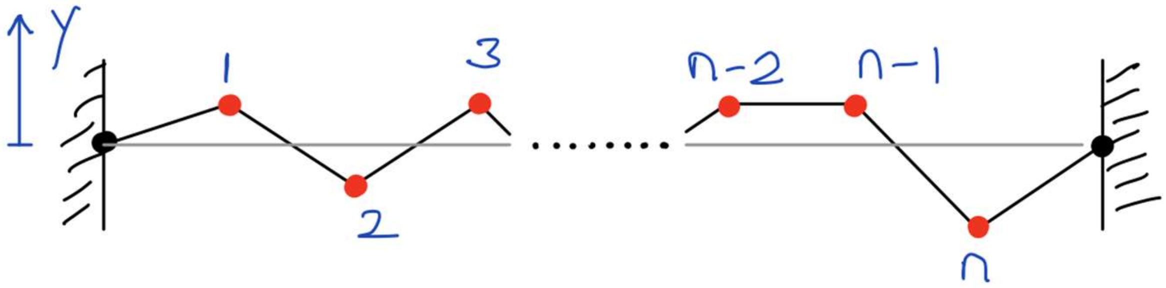
Two normal modes: $Y_1 = y_1 + y_2$ with $\omega_1^2 = T/ma$ and $A_1 = A_2$

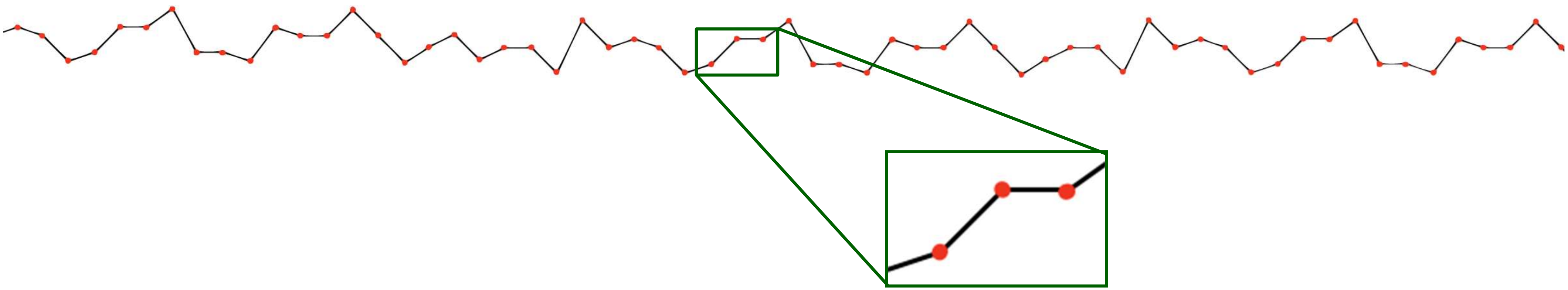
$Y_2 = y_1 - y_2$ with $\omega_2^2 = 3T/ma$ and $A_1 = -A_2$



 *no matrices were mistreated in finding the normal modes

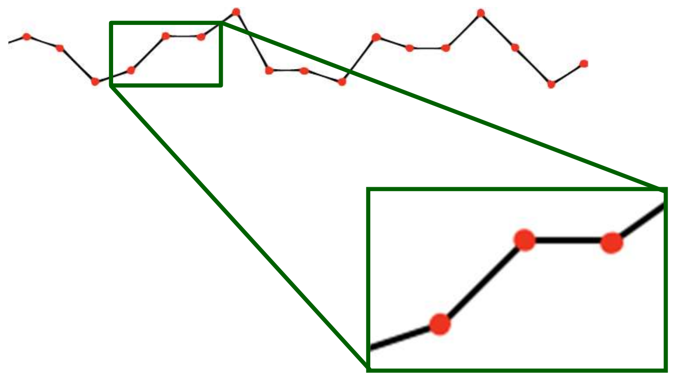
Generalise to n masses (yikes!)

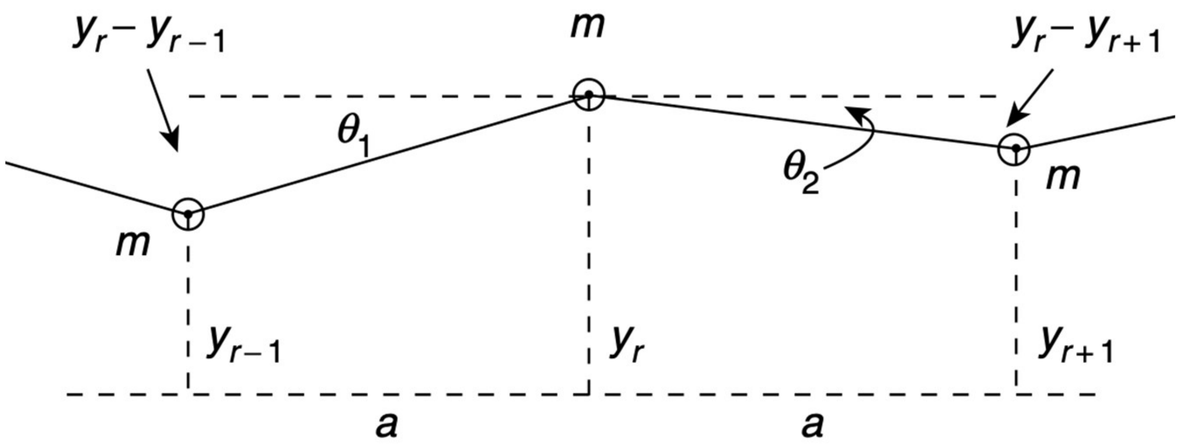




look at the r^{th} mass: $\ddot{y}_r$ depends only on the positions y_r y_{r-1} and y_{r+1}

Generalise to n masses (yikes!)





$$m\ddot{y}_r = -T\sin\theta_1 - T\sin\theta_2$$

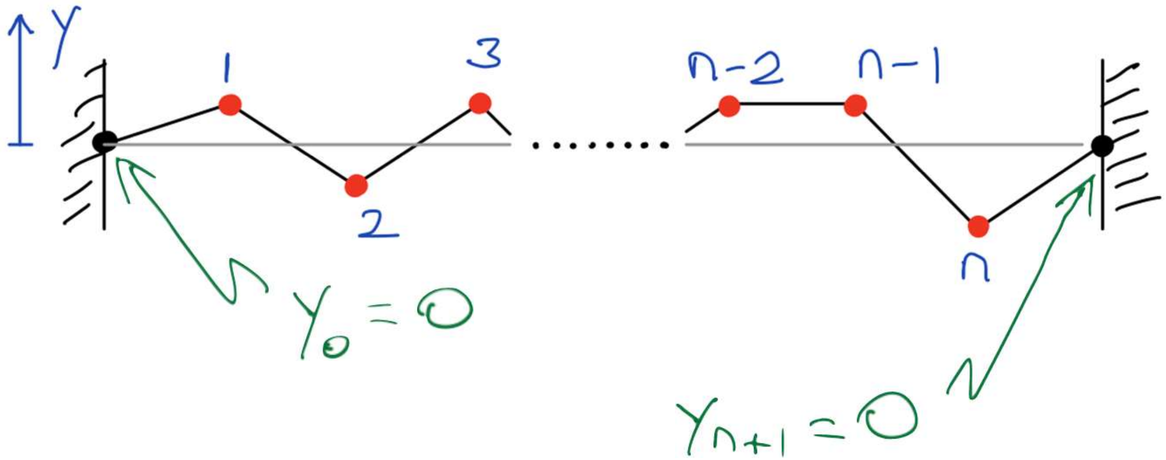
$$\approx -T[(y_r - y_{r-1})/a + (y_r - y_{r+1})/a)]$$

$$\ddot{y}_r + (T/\,ma)(2y_r - y_{r+1} - y_{r-1}) = 0 \qquad r = 1, 2, \ldots, n$$

note that the first and last masses ($r = 1$ and $r = n$) are special

n masses

$$\ddot{y}_r + (T/ma)(2y_r - y_{r+1} - y_{r-1}) = 0 \qquad r = 1, 2, \ldots, n$$



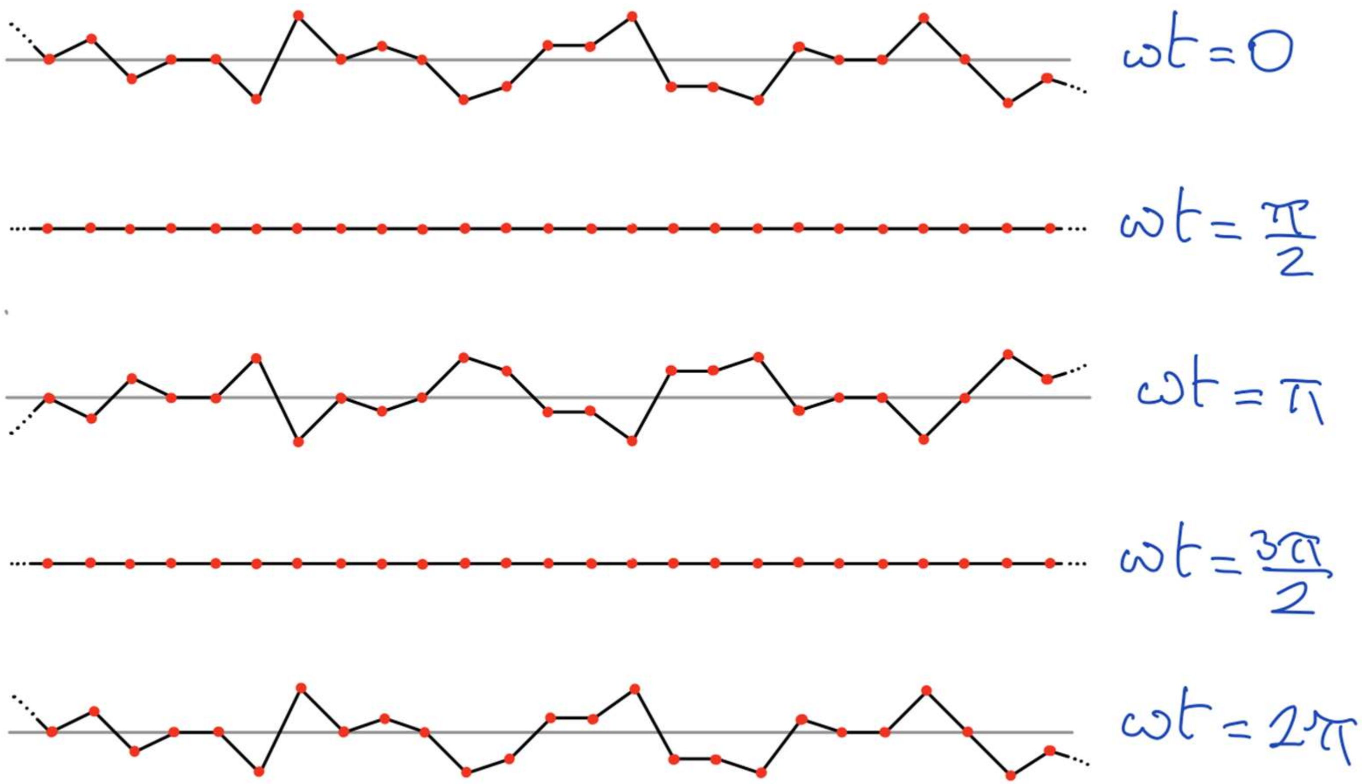
first and last masses depend on the left and right endpoints, y_0 and y_{n+1}

Boundary conditions: $y_0 = y_{n+1} = 0$

$$\ddot{y}_r + (T/ma)(2y_r - y_{r+1} - y_{r-1}) = 0 \qquad y_0 = y_{n+1} = 0$$

$r = 1, 2, \ldots., n$

Look for normal modes: $y_r = A_r e^{i\omega t}, y_{r+1} = A_{r+1} e^{i\omega t}, y_{r-1} = A_{r-1} e^{i\omega t}$

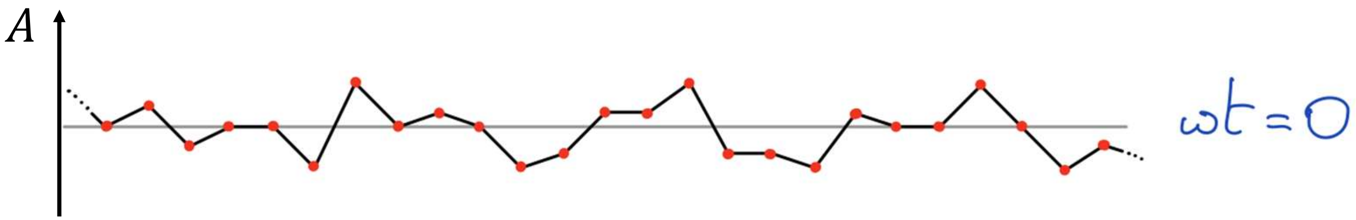


n masses

$$\ddot{y}_r + (T/\,ma)(2y_r - y_{r+1} - y_{r-1}) = 0 \qquad y_0 = y_{n+1} = 0$$

$$r = 1, 2, \ldots., n$$

Look for normal modes: $y_r = A_r e^{i\omega t}, y_{r+1} = A_{r+1} e^{i\omega t}, y_{r-1} = A_{r-1} e^{i\omega t}$



$$-\omega^2 A_r + T/(ma)(2A_r - A_{r+1} - A_{r-1}) = 0$$

$$\boxed{-A_{r-1} + \left(2 - \frac{\omega^2 ma}{T}\right) A_r - A_{r+1} = 0}$$

n equations for $r = 1$ to $r = n$, with $A_0 = A_{n+1} = 0$

$$-A_{r-1} + \left(2 - \frac{\omega^2 ma}{T}\right)A_r - A_{r+1} = 0$$

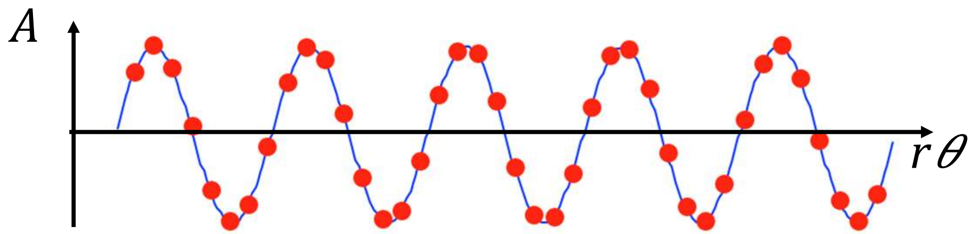
$$\frac{A_{r+1} + A_{r-1}}{A_r} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \text{ with } \omega_0^2 = T/ma$$

A moment of genius: look for modes with $A_r = Ce^{i(r\theta+\phi)}$

Note that C is real

Take real value to compare with physical coordinates:

$$\mathrm{Re}[A_r] = a_r = C\cos(r\theta + \phi)$$



$$\frac{Ce^{i(r+1)\theta+i\phi} + Ce^{i(r-1)\theta+i\phi}}{Ce^{ir\theta+i\phi}} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$e^{x+y} = e^x e^y$$

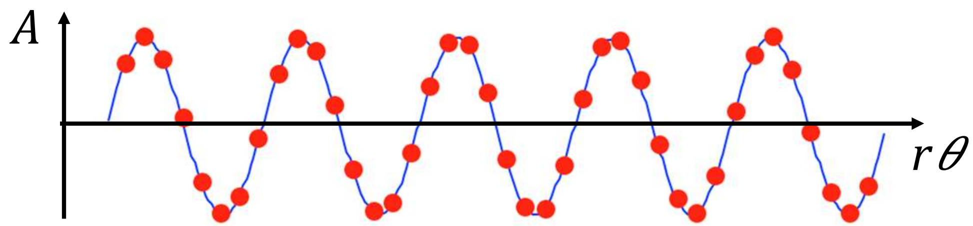
$$e^{x-y} = e^x/e^y$$

$A_r = Ce^{i(r\theta+\phi)}$ is a valid solution as long as $2\cos\theta = \frac{2\omega_0^2-\omega^2}{\omega_0^2}$

$$A_r = Ce^{i(r\theta+\phi)}$$

where $\quad 2\cos\theta = \frac{2\omega_0^2-\omega^2}{\omega_0^2}$

$$\mathrm{Re}[A_r] = a_r = C\cos(r\theta+\phi)$$



A_r = amplitude of oscillation of mass r

$y_r = A_r e^{i\omega t}$

Boundary conditions: $\quad a_0 = a_{n+1} = 0$

$$a_0 = C\cos(\phi) = 0 \quad \Rightarrow \phi = -\pi/2 \quad \Rightarrow \quad a_r = C\sin(r\theta)$$

$$a_{n+1} = C\sin([n+1]\theta) = 0 \quad \Rightarrow \quad \theta = j\pi/(n+1),\ j = 1,2,\ldots,n$$

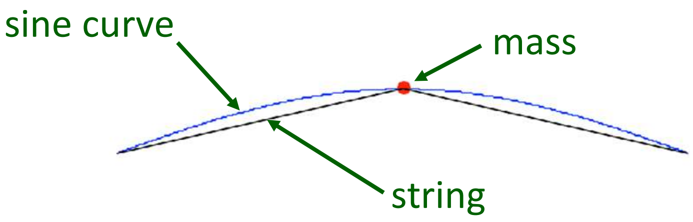
n solutions

$$\therefore \quad a_r = C\sin\left(\frac{j\pi r}{n+1}\right) \quad \text{and} \quad \omega_j^2 = 2\omega_0^2\left[1-\cos\left(\frac{j\pi}{n+1}\right)\right]$$

Masses lie on a sine curve…..

$$\therefore \quad a_r = C\sin\left(\frac{j\pi r}{n+1}\right) \quad \text{and} \quad \omega_j^2 = 2\omega_0^2\left[1-\cos\left(\frac{j\pi}{n+1}\right)\right] \qquad j = 1, 2, 3, \ldots, n$$

$n=1$ $\quad$ $j=1$



$a_1 = C\sin(\pi/2) = C$ $\qquad$ $\omega_1 = 2\omega_0$

Masses are connected by straight string segments (any curvature implies infinite acceleration!)

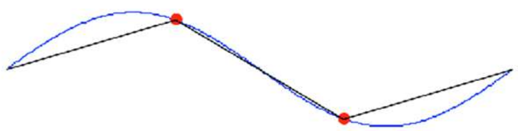
$n=2$

$j=1$

$a_1 = C\sin(\pi/3) = 0{\cdot}87C$ $\qquad$ $\omega_1 = \omega_0$

$a_2 = C\sin(2\pi/3) = 0{\cdot}87C$.

$j=2$



$a_1 = C\sin(2\pi/3) = 0{\cdot}87C$ $\qquad$ $\omega_2 = 3\omega_0$

$a_2 = C\sin(4\pi/3) = -0{\cdot}87C$

Additional notes: http://www.ectropy.info/scholar/node/the-loaded-string

They do!

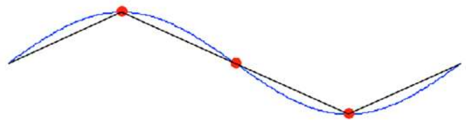
$n=3$

$j=1$

$a_1 = C\sin(\pi/4) = 0.71C$
$a_2 = C\sin(2\pi/4) = C$
$a_3 = C\sin(3\pi/4) = 0.71C$

$\omega_1 = 0.59\,\omega_0$

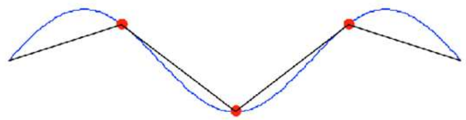
$j=2$



$a_1 = C\sin(2\pi/4) = C$
$a_2 = C\sin(4\pi/4) = 0$
$a_3 = C\sin(6\pi/4) = C$

$\omega_2 = 2\omega_0$

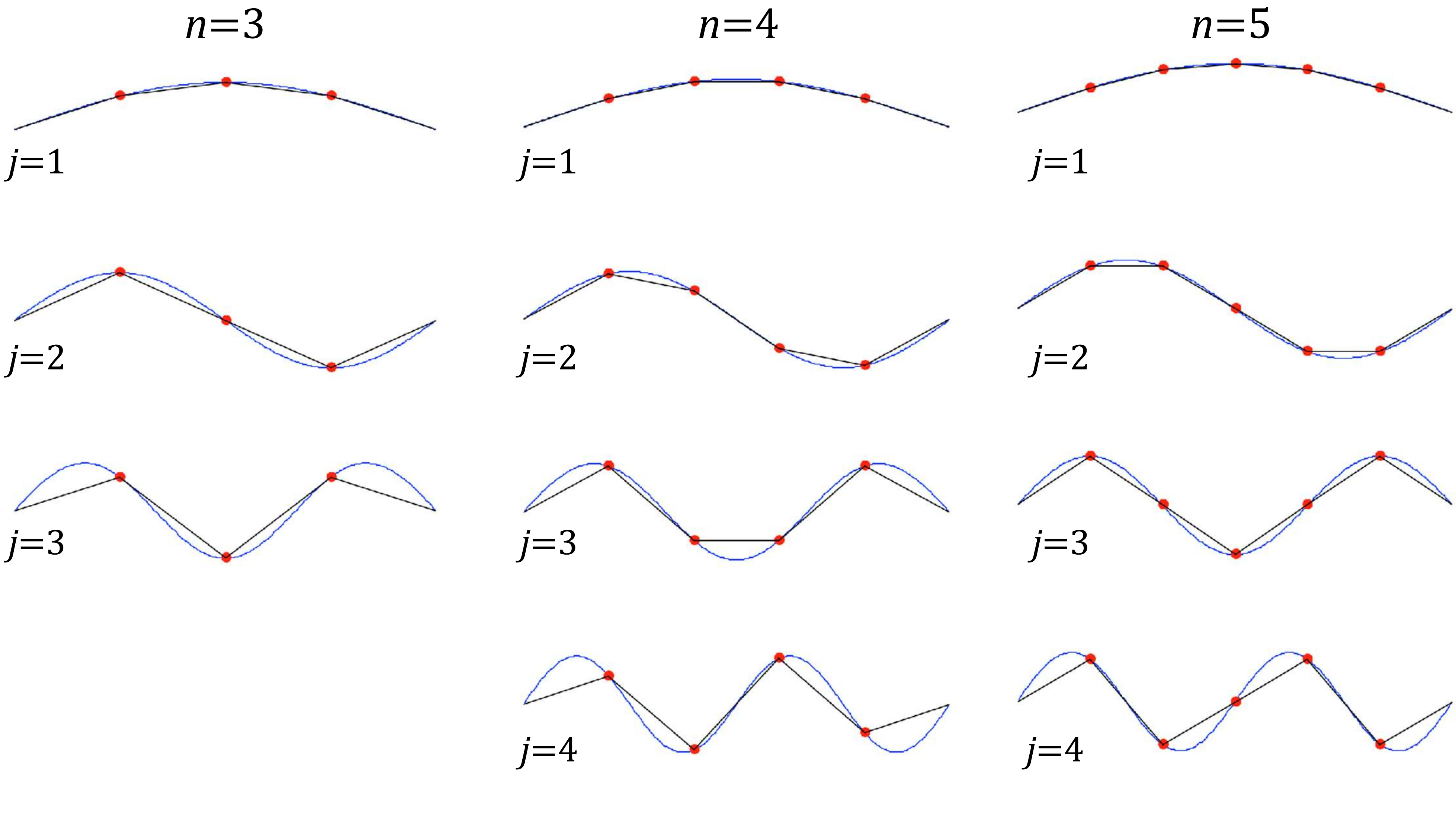
$j=3$

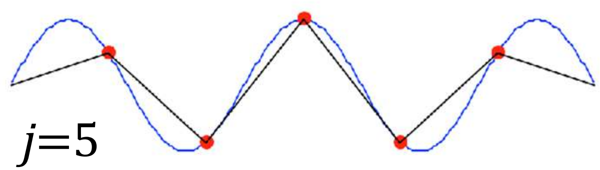


$a_1 = C\sin(3\pi/4) = 0.71C$
$a_2 = C\sin(6\pi/4) = -C$
$a_3 = C\sin(9\pi/4) = 0.71C$

$\omega_3 = 3.4\,\omega_0$

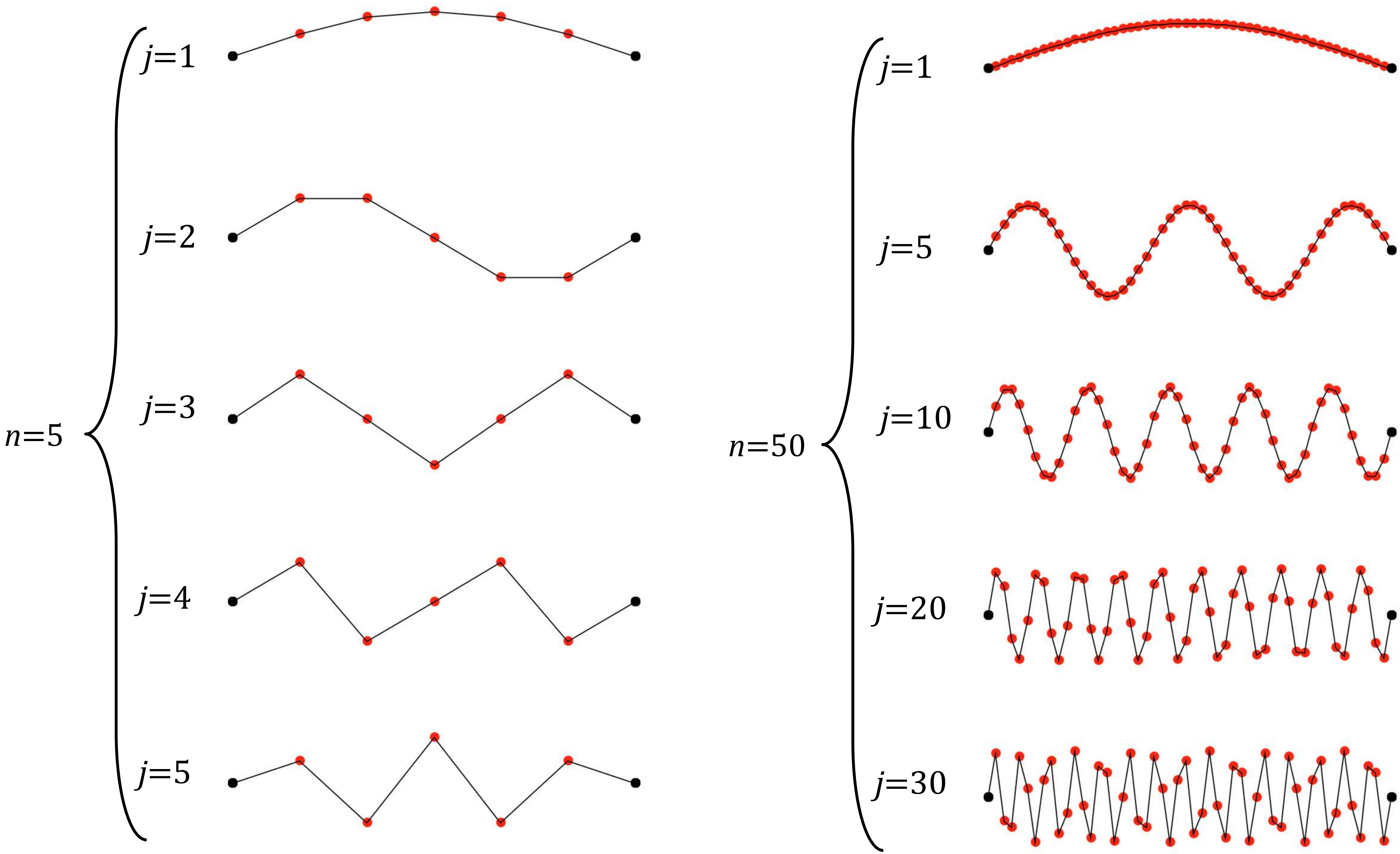
No really, they do!



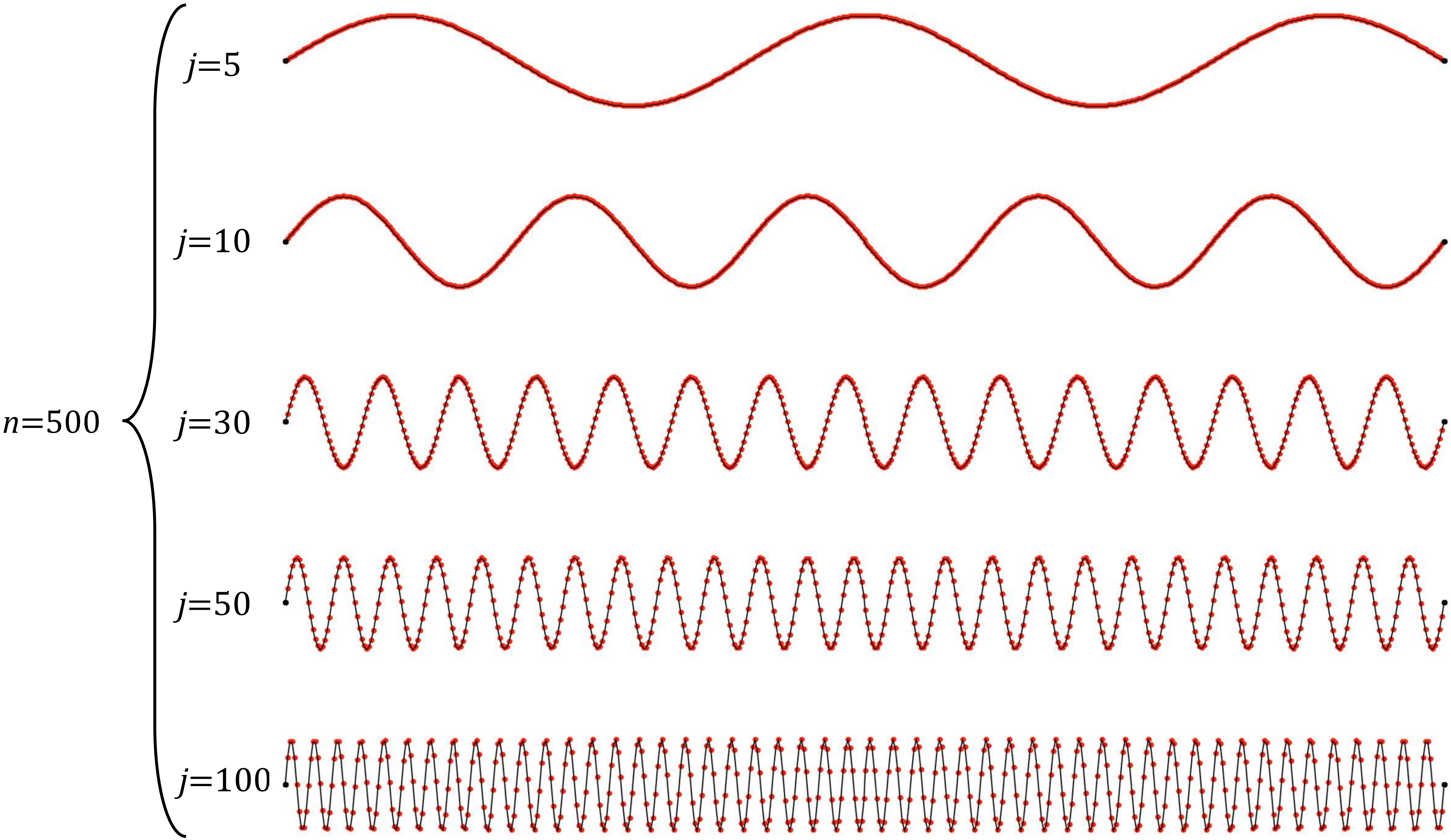


Interactive demo – http://www.falstad.com/loadedstring/

$n = 5$ vs $n = 50$



$n = 500$



See? They do!

$n =$ lots

$j=1$

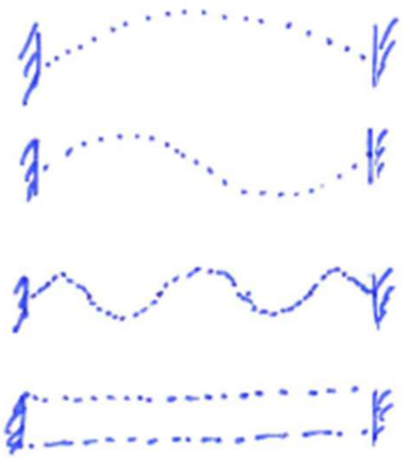
$j=2$

$\vdots$

$j=5$

$\vdots$

$j=n$



$\omega_1 \ll \omega_0$

$\omega_n \sim 4\omega_0$

$\omega_0^2 = T/ma$

Executive summary – loaded string

1. Equation of motion for mass r :

$$\ddot{y}_r + T/(ma).(2y_r - y_{r+1} - y_{r-1}) = 0$$

restoring force depends on local shape of the string

2. Seek normal modes with $y_r = A_r e^{i\omega t}$

$$\Rightarrow -A_{r-1} + \left(2 - \frac{\omega^2 ma}{T}\right) A_r - A_{r+1} = 0$$

local shape of a mode is set by frequency ω

3. Adopt a sinusoidal mode shape $A_r = Ce^{i(r\theta+\phi)}$

θ sets the wavelength of the mode

$$\Rightarrow \quad 2\cos\theta = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

θ depends on ω

$\omega_0^2 = T/ma$

4. Boundary conditions determine the allowed wavelengths

$$\Rightarrow \quad \theta = j\pi/(n+1), \; j = 1,2,\ldots,n$$

$$\therefore \quad a_r = C\sin\left(\frac{j\pi r}{n+1}\right) \quad \text{and} \quad \omega_j^2 = 2\omega_0^2\left[1 - \cos\left(\frac{j\pi}{n+1}\right)\right]$$

mode shapes and frequencies

Waves on a string

- This looks likes standing waves on a string stretched between two fixed boundaries
  - mass is distributed continuously along the string, with linear density ρ (kg/m)
- Just for laughs, we derive the equation of motion for a ``heavy'' string by approximating it as n masses on a light string
  - Set $m = \rho L/n$ and $a = L/(n+1)$ where L is the string's length
  - Let $n \longrightarrow \infty$. Hilarious!
- Then we'll derive the equation of motion using a direct approach
- Either way we end up with the *wave equation*