ELEC2040 Signals and Systems

Practical Week 6: Fourier Transform Properties

Q1 Let $x(t) = 3\cos(200t)$ and $y(t) = (x(t))^2$.

a. Use the multiplication property of the Fourier Transform to find $Y(\omega)$, the Fourier Transform of $y(t)$.

b. Plot $Y(\omega)$.

c. Take the inverse Fourier Transform of $Y(\omega)$ to obtain a formula for $y(t)$.

Q2 The signal $x(t) = \cos(2\pi t)$ is windowed by a rectangular time function $w(t)$ that extends from -20 to 20. The windowed signal is $y(t) = w(t)x(t)$, where $w(t) = \text{rect}\left(\frac{t}{4}\right)$.

a. Draw $y(t)$

b. Find and draw $X(\omega)$, the Fourier Transform of $x(t)$

c. Find and draw $W(\omega)$, the Fourier Transform of $w(t)$

d. Find and draw $Y(\omega)$ the Fourier Transform of $y(t)$

Q3 Find the Fourier Transform, $X(\omega)$, of $x(t) = \cos(2\pi t)\,\text{sinc}(3t)$. Draw $X(\omega)$ as well.

Q4 A LTI system has impulse response $h(t) = \text{rect}(t-4)$. A band-limited pulse, $x(t) = 4\text{sinc}(4t)$ is the input to the LTI, and $y(t)$ is the corresponding output.

a. Draw $h(t)$ and $x(t)$ (on separate graphs)

b. Use the convolution property to find the Fourier Transform, $Y(\omega)$, of $y(t)$, and plot the amplitude and phase spectrum of this output signal.

Q5 Consider the following ordinary differential equation:

$$\frac{d}{dt}y(t) + 3y(t) = 2x(t)$$

where $x(t)$ is the input and $y(t)$ is the output.

(a) Use the differentiation property of the Fourier Transform to find the system response, $H(\omega)$, of this linear system.

(b) Find the impulse response, $h(t)$, of this linear system.

(c) Find the Fourier Transform, $Y(\omega)$, of the output signal, $y(t)$, if the input signal, $x(t)$, is given by $x(t) = \cos(t)$.

(d) Take the inverse Fourier Transform of $Y(\omega)$ to obtain the output signal, $y(t)$. To do this, you will need to find the phase shift, but you don't need to find this explicitly: you can express it using the arctan function, for example.

Q6 A LTI system has impulse response $h(t) = \exp(-(t-2))u(t-2)$. A time-limited pulse, $x(t) = \mathrm{rect}(t - 1.5)$ is the input to the LTI, and $y(t)$ is the corresponding output.
Note this is the $y(t)$ you computed via convolution in Practical Week 4, Q4.

Use the convolution property of the Fourier Transform to find the Fourier Transform, $Y(\omega)$, of $y(t)$.

Q7 Consider a multipath-path channel where the output, $y(t)$, depends on the input, $x(t)$, according to the equation

$$y(t) = x\left(t - \frac{\pi}{8}\right) + \frac{1}{\sqrt{2}}\left(x\left(t - \frac{\pi}{4}\right)\right) + \frac{1}{2}\left(x\left(t - \frac{5\pi}{8}\right)\right).$$

(a) Compute the impulse response, $h(t)$, of the channel.

(b) Compute the Fourier Transform, $H(\omega)$, of $h(t)$.

(c) With input signal $x(t) = \exp\left(j\left(2t + \frac{\pi}{3}\right)\right)$, use the convolution property to show that the output can be written

$$y(t) = A_1 \exp\left(j\left(2t + \frac{\pi}{3} + B_1\right)\right),$$

where A_1 is the gain, and B_1 is the phase shift. Compute the values of the gain and phase shift.

(d) With input signal $x(t) = \exp\left(-j\left(2t + \frac{\pi}{3}\right)\right)$, show that the output can be written

$$y(t) = A_2 \exp\left(-j\left(2t + \frac{\pi}{3} - B_2\right)\right),$$

where A_2 is the gain, and B_2 is the phase shift. Compute the values of the gain and phase shift.

(e) With input signal $x(t) = \cos\left(2t + \frac{\pi}{3}\right)$, show that the output can be written

$$y(t) = A_3 \cos\left(2t + \frac{\pi}{3} + B_3\right),$$

where A_3 is the gain, and B_3 is the phase shift. Compute the values of the gain and phase shift.

Extra Questions:

Q8 The signal $x(t) = \sin(2\pi t)$ is windowed by a rectangular time function $w(t)$ that extends from 0.25 to 20.25 seconds. The windowed signal is denoted by $y(t)$.

a. Find $w(t)$ and draw it

b. Find $X(\omega)$, the Fourier Transform of $x(t)$

c. Find $W(\omega)$, the Fourier Transform of $w(t)$

d. Find $Y(\omega)$ the Fourier Transform of $y(t)$

e. Find approximate values for $Y(-2\pi)$ and $Y(2\pi)$

f. Sketch $|Y(\omega)|$, the magnitude spectrum of the windowed signal. You can use an approximation to obtain your sketch.

Q9

(a) Find the Fourier Transform, $F(\omega)$ of the signal $f(t) = \exp(-|t|)$.
(* This was also part (c) of Q6 on the Practical for Week 5. If you haven't done it yet, try it now. The answer to Q2 (d) on the Practical for Week 5 may also prove useful *)

(b) Use part (a) and the duality property of the Fourier Transform to find the Fourier Transform, $G(\omega)$ of the signal $g(t) = \frac{2}{1+t^2}$.

Q10 A radio signal of frequency 2 MHz is transmitted across a two path channel from a base station to a cellphone handset. The impulse response is given by

$$h(t) = \delta\left(t - \tau_0\right) + 0.5\delta\left(t - \tau_1\right),$$

where $\tau_0 = \frac{1}{12}$ μ sec, and $\tau_1 = \frac{5}{24}$ μ sec, respectively.

Use the Fourier Transform to compute the channel gain, and phase shift, of the signal. You can use Matlab to obtain a numerical answer. Compare this answer to what you obtained in Project Week 3.

Q11 The impulse response of a channel is given by $h(t) = \delta\left(t - \frac{1}{12}\right) + 0.5\delta\left(t - \frac{5}{24}\right)$, where t is in seconds.

(a) Use Matlab to plot the magnitude spectrum of the channel response across the band $(-10\pi, 10\pi)$ rad/sec.

(b) Find a frequency at which the gain of this channel is maximal. What is the maximum gain of the channel ?

(c) Find a frequency at which the gain of this channel is minimal. What is the minimum gain of the channel ?

Fourier Transforms – formulae and properties

$C_k = \frac{1}{T}\int_{<T>} f(t)e^{-jk\omega_0 t}dt$	$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$
$F(\omega) = \Im[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$	$f(t) = \Im^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$
If $F(\omega) = \Im\left(f(t)\right)$ and $G(\omega) = \Im\left(g(t)\right)$ Then: $f(t) * g(t) \quad \leftrightarrow \quad F(\omega)G(\omega)$ $f(t)g(t) \quad \leftrightarrow \quad \frac{1}{2\pi}F(\omega) * G(\omega)$	Suppose $F(\omega) = \Im\left(f(t)\right)$. Then $\Im\left(\frac{1}{2\pi}F(-t)\right) = f(\omega)$
Suppose $F(\omega) = \Im\left(f(t)\right)$. Then $\Im\left(f(at)\right) = \frac{1}{\lvert a \rvert}F(\frac{\omega}{a})$	Suppose $F(\omega) = \Im\left(f(t)\right)$. Then $\Im\left(f(t-t_0)\right) = \exp(-j\omega t_0)F(\omega)$ $\Im\left(\left(f(t)\exp(j\omega_0 t)\right)\right) = F(\omega - \omega_0)$
Suppose $F(\omega) = \Im\left(f(t)\right)$. Then $\Im\left(\frac{d}{dt}f(t)\right) = j\omega F(\omega)$	Suppose $F(\omega) = \Im\left(f(t)\right)$. Then $\Im\left(f(-t)\right) = F(-\omega)$

Table of Fourier Transforms

Time domain	Frequency Domain
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$A\,\delta(t-t_0)$	$A\exp(-j\omega t_0)$
$\exp(j\omega_0 t)$	$2\pi\delta(\omega-\omega_0)$
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$
$\beta\,\mathrm{sinc}\left(\beta t\right)$	$\mathrm{rect}\left(\frac{\omega}{2\pi\beta}\right)$
$u(t)\exp(-a\,t),\ a>0$	$\frac{1}{j\omega+a},\ a>0$

where $\mathrm{sinc}\left(x\right) = \frac{\sin\left(\pi x\right)}{\pi x}$