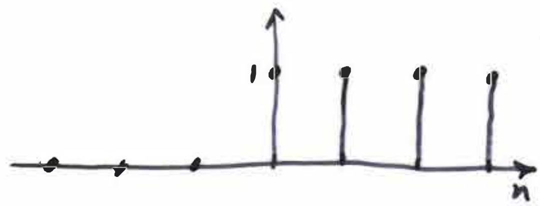
UNIT IMPULSE & STEP FUNCTIONS (p30)

D.T.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



DIFFERENCE EQUATIONS (cf. DIFFERENTIAL EQNS IN C.T.)

eg.

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

OR

$$= \sum_{k=0}^{\infty} \delta[n-k]$$

SAMPLING PROPERTY

$$x[n]\,\delta[n] = x[0]\,\delta[n]$$

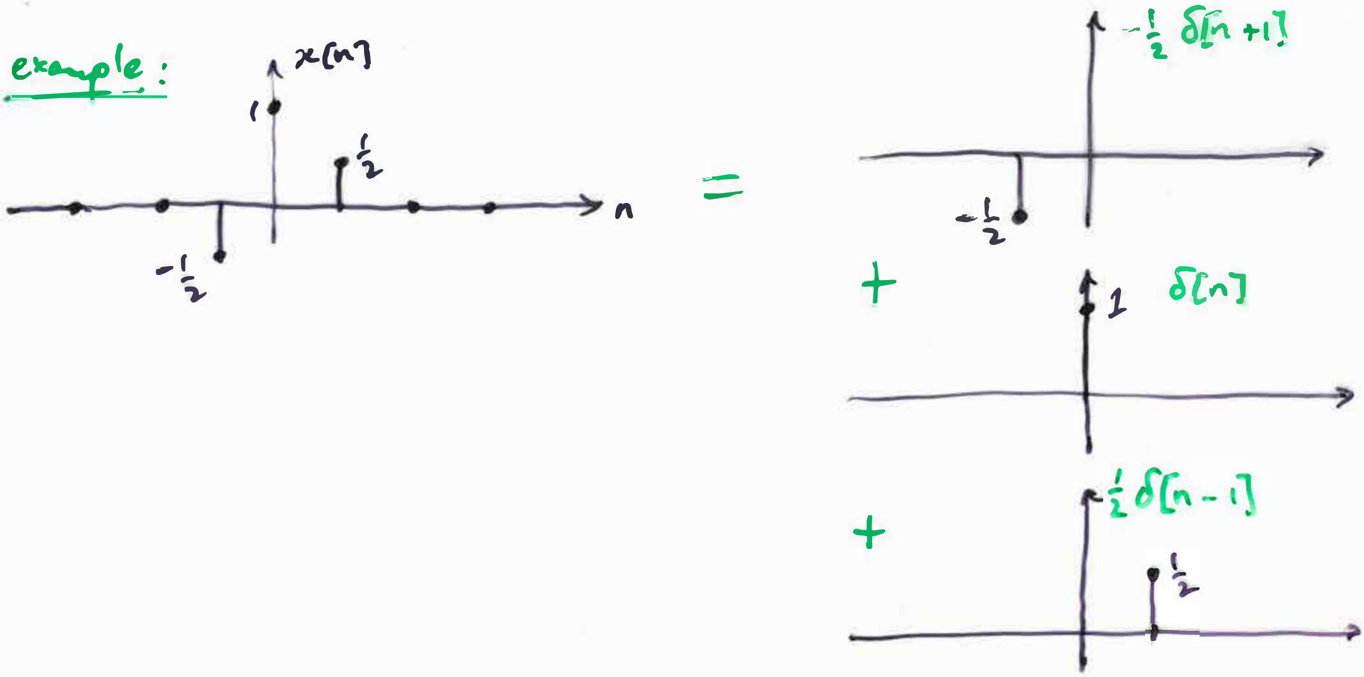
$$x[n]\,\delta[n-n_0] = x[n_0]\,\delta[n-n_0]$$

LINEAR TIME-INVARIANT SYSTEMS (p 74)
(L.T.I)

D.T. SYSTEMS – CONVOLUTION SUM

ANY $x[n]$ CAN BE CONSTRUCTED FROM UNIT IMPULSES

example:



ie. $x[n] = -\frac{1}{2}\,\delta[n+1] + \delta[n] + \frac{1}{2}\,\delta[n-1]$

$= x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1]$

$= \sum_{k=-1}^{1} x[k]\,\delta[n-k]$

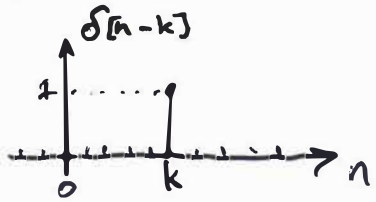
IN GENERAL:

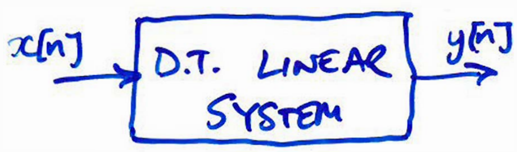
$$\boxed{x[n] = \sum_{k=-\infty}^{\infty} \underbrace{x[k]}_{\text{coefficients}}\ \underbrace{\delta[n-k]}_{\text{basis functions}}}$$

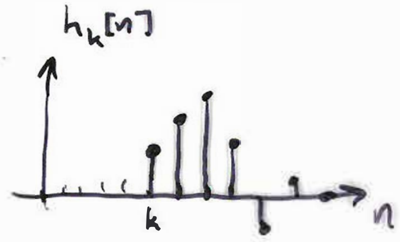
CONVOLUTIONAL SUM REPRESENTATION OF A SYSTEM

(p 77)

IMPULSE RESPONSE







ALSO, BECAUSE LINEAR

$$\underbrace{x[k]}_{\text{number}}\ \underbrace{\delta[n-k]}_{\text{function}} \longrightarrow \underbrace{x[k]}_{\text{number}}\ \underbrace{h_k[n]}_{\text{function}}$$

- Recall that in general notation, any $x[n]$ f^{n} can be represented as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$\Rightarrow$ FOR LTI SYSTEMS

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

- Also, since T.I.

$$h_1[n] = h_0[n-1]$$
$$h_2[n] = h_0[n-2]$$
$$h_3[n] = h_0[n-3]$$

$\Rightarrow$

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] = x[n] \circledast h[n]}$$

where $h[n] = h_0[n]$

PROPERTIES

- $h[n]$ FULLY CHARACTERISES A LTI SYSTEM

- $x[n] * h[n] = h[n] * x[n]$

- FINITE IMPULSE RESPONSE (FIR)

  IF

$$h[n] = h[0]\,\delta[n] + h[1]\,\delta[n-1] + h[2]\,\delta[n-2] + \dots + h[M]\,\delta[n-M]$$

for M FINITE.

eg: AVERAGING FILTER WITH WEIGHTS

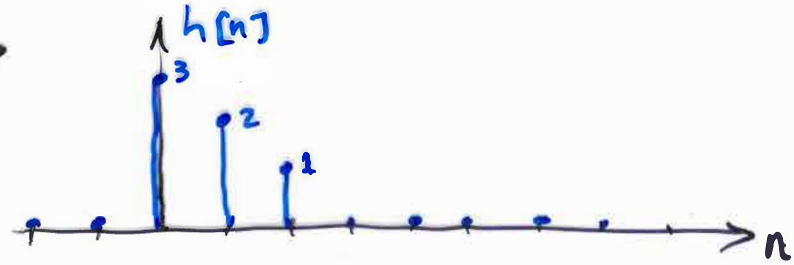
$$y[n] = 3x[n] + 2\,x[n-1] + x[n-2]$$

TEST WITH $x[n] = \delta[n]$

$$\Rightarrow y[n] = 3\,\delta[n] + 2\,\delta[n-1] + \delta[n-2]$$

$\dots\; y[-2]=0,\ y[-1]=0,\ y[0]=3,\ y[1]=2,\ y[2]=1,\ y[3]=0$

$\Rightarrow$



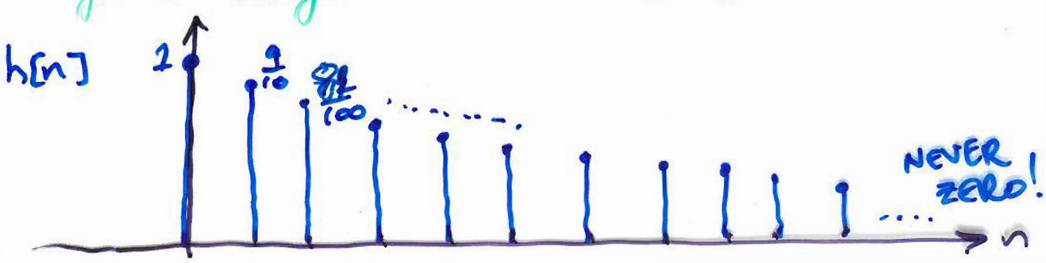
ONLY 3 NON ZERO COEFF'S $\therefore$ FIR.

- INFINITE IMPULSE RESPONSE (IIR)

  IF $h[n]$ HAS ∞ NUMBER OF NON ZERO ELEMENTS

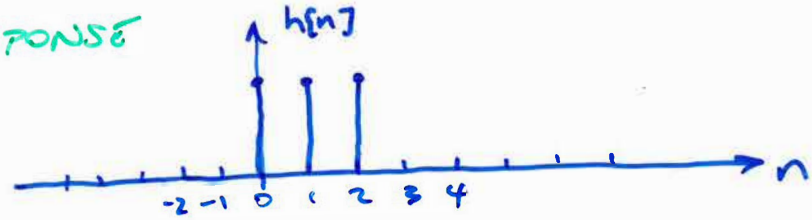
eg:

$$y[n] = \frac{9}{10}\,y[n-1] + x[n]$$

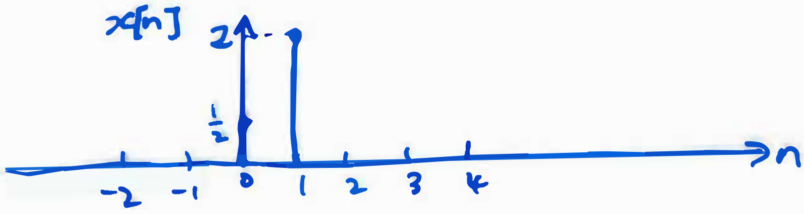


CALCULATING A CONVOLUTION (p80)

EXAMPLE: SUPPOSE A LTI SYSTEM HAS UNIT IMPULSE RESPONSE



WHAT IS THE OUTPUT WHEN THE INPUT IS



ANSWER:

METHOD 1: APPLY $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$ EQUATION

$$y[-1] = x[0]\,h[-1-0] + x[1]\,h[-1-1]$$
$$= \tfrac{1}{2} \cdot 0 + 2 \cdot 0 = 0$$

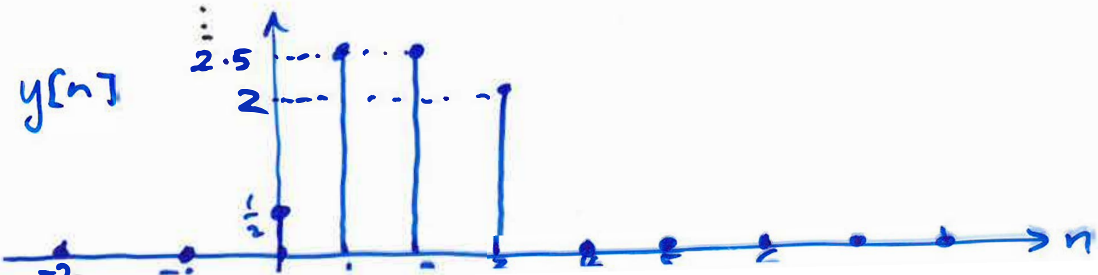
$$y[0] = x[0]\,h[0-0] + x[1]\,h[0-1]$$
$$= \tfrac{1}{2} \cdot 1 + 2 \cdot 0 = \tfrac{1}{2}$$

$$y[1] = x[0]\,h[1-0] + x[1]\,h[1-1]$$
$$= \tfrac{1}{2} \cdot 1 + 2 \cdot 1 = 2\tfrac{1}{2}$$

$$y[2] = x[0]\,h[2-0] + x[1]\,h[2-1]$$
$$= \tfrac{1}{2} \cdot 1 + 2 \cdot 1 = 2\tfrac{1}{2}$$

$$y[3] = x[0]\,h[3-0] + x[1]\,h[3-1] = 2$$

$$y[4] = x[0]\,h[4-0] + x[1]\,h[4-1] = 0$$



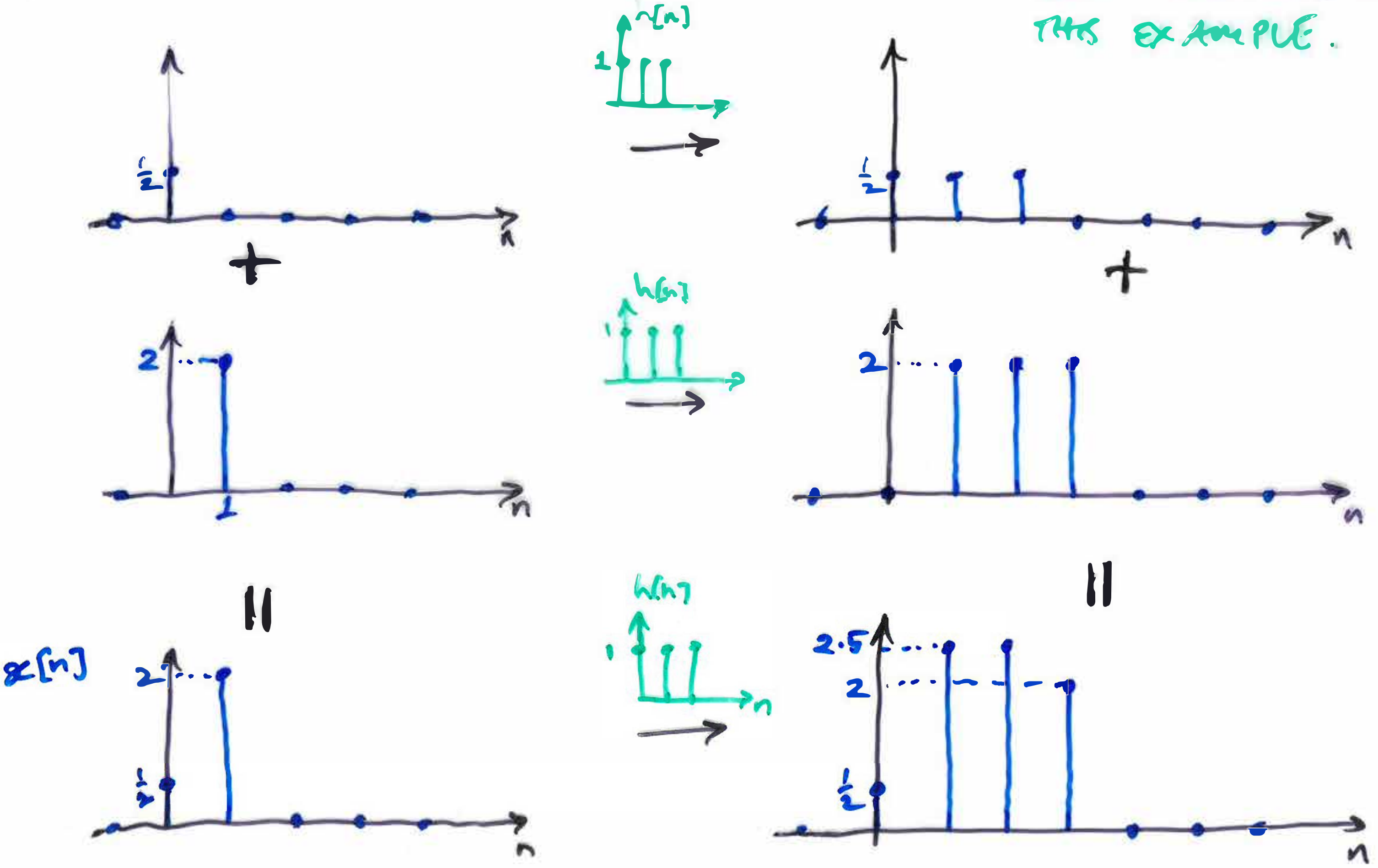
METHOD 2: GRAPHICAL

THINK OF $x[n]$ AS TWO IMPULSES SUMMED

$$\frac{1}{2}\delta[n] \quad \& \quad 2\,\delta[n-1]$$

SINCE LTI SYSTEM, OUTPUT WILL BE THE SUM OF 2 RESPONSES

2 IS BECAUSE $x[n]$ IS MADE UP OF 2 DELTA FUNCTIONS IN THIS EXAMPLE.



NOTE THE SLIDING EFFECT !!

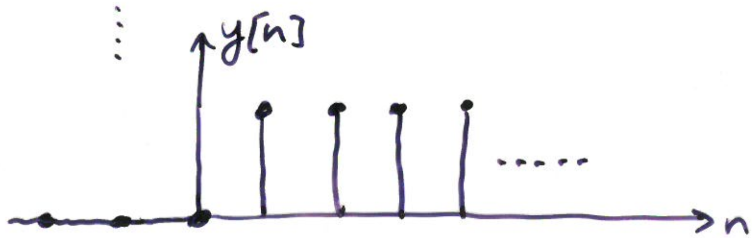
EXAMPLE: WHAT IS THE IMPULSE RESPONSE OF THIS IIR SYSTEM:

$$y[n+1] - y[n] = x[n] \quad ; \quad y[n] = 0 \text{ for } n<0$$

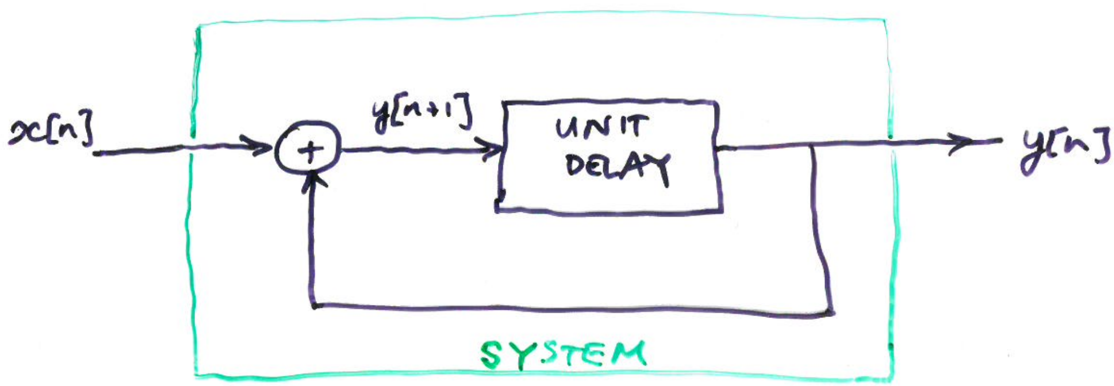
ALSO, DRAW A BLOCK DIAGRAM OF THE SIGNAL FLOW

ANSWER: CHOOSE $x[n] = \delta[n]$

$$y[0] = y[-1] + x[-1] = 0$$
$$y[1] = y[0] + x[0] = 1$$
$$y[2] = y[1] + x[1] = 1$$
$$\vdots$$



BLOCK DIAGRAM



THIS IS AN INTEGRATOR !!

CONVOLUTION EXAMPLE

COMPUTE $h[n] * x[n]$ FOR $x[n] = \alpha^n u[n]$

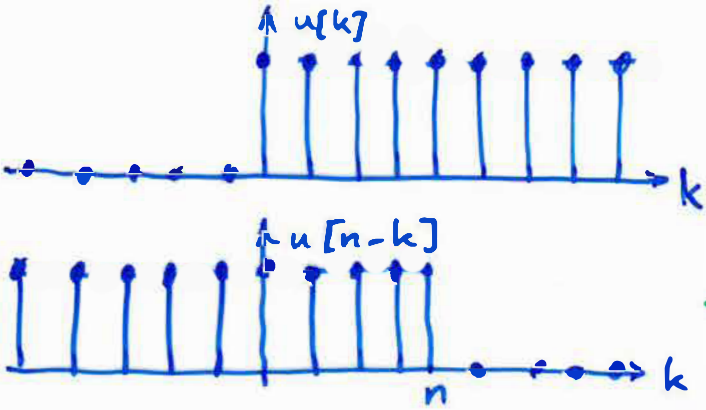
AND $h[n] = \beta^n u[n]$

$(\alpha \neq \beta)$

ANSWER: $$h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

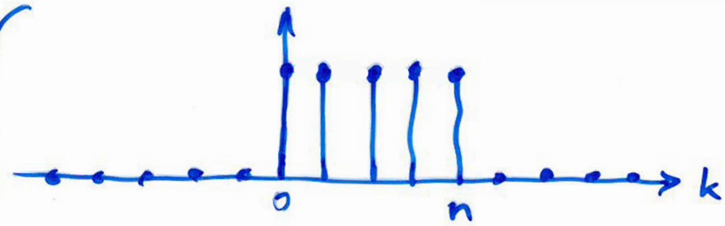
$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, \beta^{n-k} u[n-k]$$

Now, $u[k]\, u[n-k]$ is



NOTE: THIS FUNCTION SHIFTS FOR DIFFERENT VALUES OF n

=



if $n \geq 0$

0 if $n < 0$

$$\therefore\ h[n] * x[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} \quad \text{if } n \geq 0, \ \& = 0 \text{ if } n<0$$

$$= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k, \quad n \geq 0$$

$$= \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}, \quad n \geq 0$$

PROPERTIES & EXAMPLES OF DTFT

TIME SHIFT	$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$
FREQ SHIFT	$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$

RECALL THAT IN D.T. ONCE YOU HAVE SHIFTED BY 2π IN FREQ, YOU ARE BACK TO THE START.

$$e^{j(\omega+2\pi)n} = e^{j\omega n} \underbrace{e^{j2\pi n}} = e^{j\omega n}$$

ALWAYS $= 1$ BECAUSE IN D.T. n IS AN INTEGER

$\therefore$ $0, 2\pi, 4\pi, 6\pi, \ldots$ REPRESENT LOW FREQS

$\pi, 3\pi, 5\pi, 7\pi, \ldots$ REPRESENT HIGH FREQS

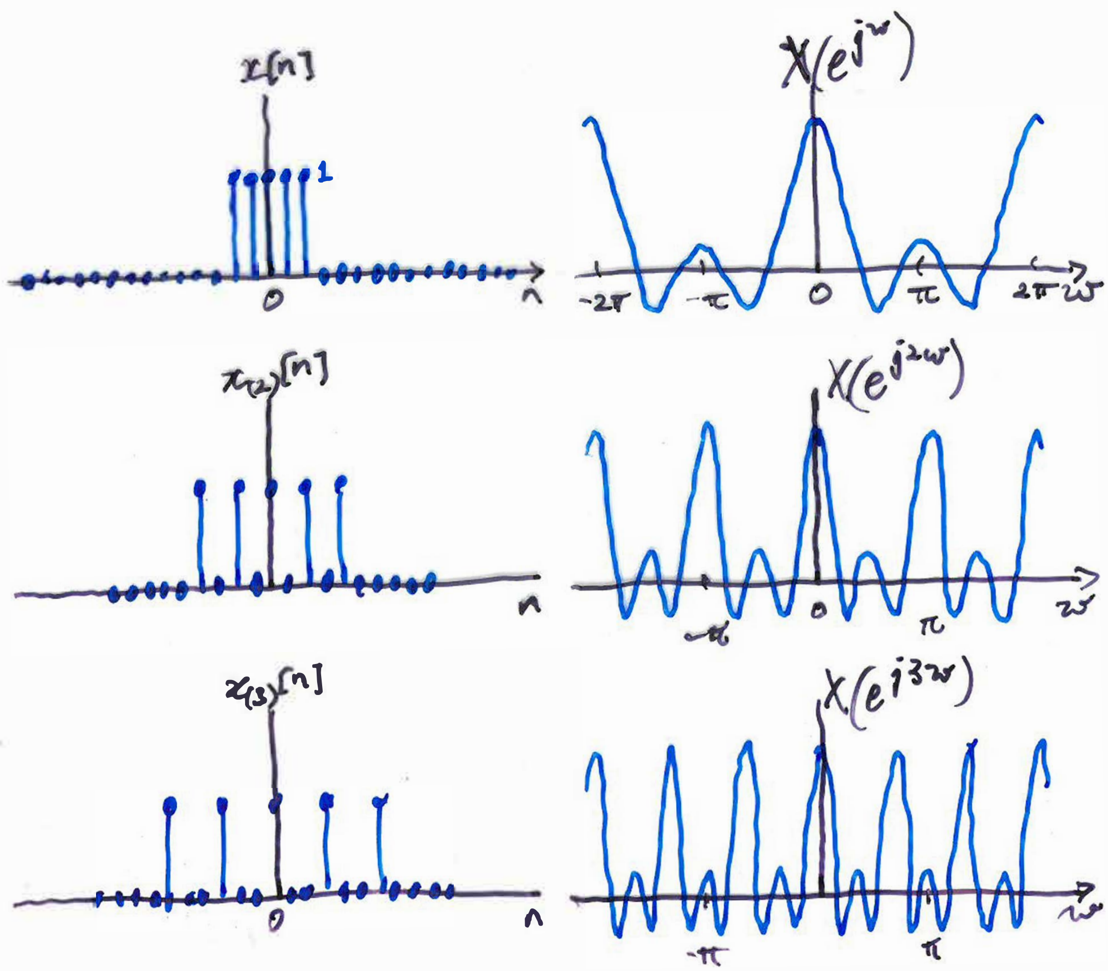
TIME EXPANSION

FOR C.T. $\quad x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$

IN D.T. IF $\quad x_{(k)}[n] = \begin{cases} x[n/k] & \text{IF } n \text{ IS A MULT OF } k \\ 0 & \text{IF } n \text{ NOT A MULTIPLE OF } k \end{cases}$

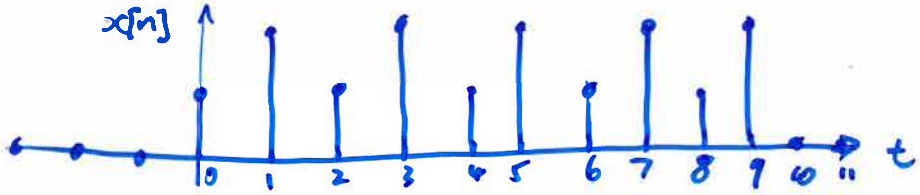
THEN $\quad \boxed{x_{(k)}[n] \xleftrightarrow{\mathcal{F}} X(e^{jk\omega})}$

EXAMPLE: (p 378)



EXAMPLE (TIME SHIFT, TIME EXPANSION & LINEARITY)

FIND THE D.T.F.T. OF



ANSWER: YOU COULD APPLY THE DTFT FORMULA

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-jwn}$$

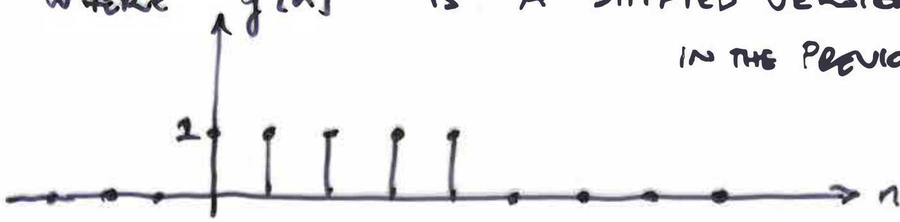
AND THEN SPEND TIME MANIPULATING ALL THE TERMS

OR

REALISE THAT

$$x[n] = y_{(2)}[n] + 2\, y_{(2)}[n-1]$$

WHERE $y[n]$ IS A SHIFTED VERSION OF THE $x[n]$ IN THE PREVIOUS EXAMPLE & LAST LECTURE



SHIFT $\Rightarrow$ $$Y(e^{jw}) = e^{-j2w}\left(\frac{\sin(5w/2)}{\sin(w/2)}\right)$$

NOW TIME EXPANSION $\Rightarrow$ $$y_2[n] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j4w}\,\frac{\sin 5w}{\sin w}$$

$$\Rightarrow \quad X(e^{jw}) = e^{-j4w}\left(1 + 2e^{-jw}\right)\frac{\sin(5w)}{\sin(w)}$$

CONVOLUTION (p 382)

$$x[n] * h[n] \xleftrightarrow{\mathcal{F}} X(e^{jw})\, H(e^{jw})$$

MULTIPLICATION (p 388)

$$x_1[n]\; x_2[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\, X_2(e^{j(w-\theta)})\, d\theta$$

PERIODIC CONVOLUTION

DIFFERENCE EQUATIONS

EXAMPLE: (p396)

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

FIND $h[n]$

ANSWER:

BY INSPECTION

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}(e^{-j\omega})^2}$$

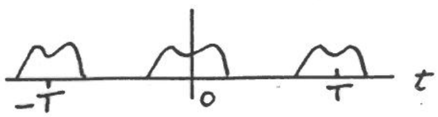
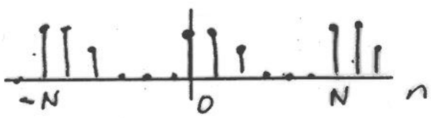
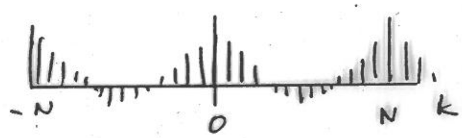
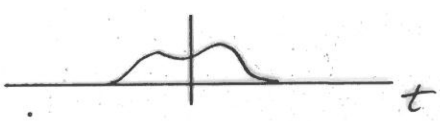
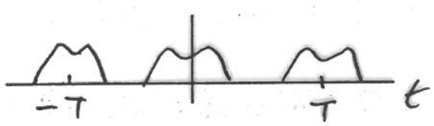
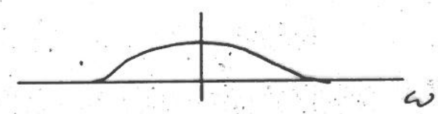
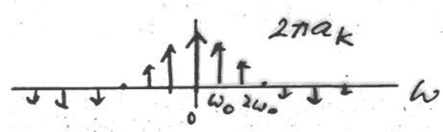
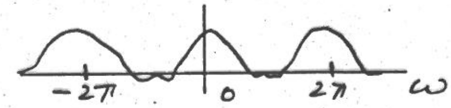
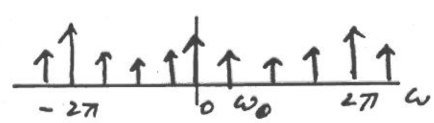
$$= \frac{A}{1 - \frac{1}{2}e^{-j\omega}} - \frac{B}{1 - \frac{1}{4}e^{-j\omega}}$$

find $A = 4$, $B = 2$

$$\therefore h[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]$$

$$= 2\left(2\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right) u[n]$$

Summary of Fourier Series and Transform

Time		Frequency
 CT, periodic with period T $\omega_0=\frac{2\pi}{T}$	**CTFS** $x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$, $a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt$	 Discete, aperiodic
 DT, periodic with period N $\omega_0=\frac{2\pi}{N}$	**DTFS** $x[n]=\sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$ $a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$	 Discete, periodic with period N
 CT, aperiodic  CT, periodic with period T, $\omega_0=\frac{2\pi}{T}$	**CTFT** $x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$ $X(j\omega)=\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$	 Cont, aperiodic  CT impulsive, aperiodic
 DT, aperiodic  DT, periodic with period N, $\omega_0=\frac{2\pi}{N}$	**DTFT** $x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$ $X(e^{j\omega})=\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$	 Cont, periodic with period 2π  CT, impulsive, periodic with period 2π