Unpacking Complex Numbers

Macquarie University

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MACQUARIE UNIVERSITY | FACULTY OF SCIENCE

Outline

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- Introduction
- Formats
- Format Conversions
- Operations
- Properties

Introduction

What is a Complex Number?

- A number with 2 parts
  - Real Part
  - Imaginary Part

Why Use Complex Numbers?

- Succinct method to simultaneously describe and manipulate magnitude and phase
  - Very useful in
    - Electrical engineering
      - E.g. transfer functions
    - Telecomunications engineering
    - Physics

The Imaginary Multiplier

- Rafael Bombelli, Italy, 1572
  - Used to find the roots of 3’rd order polynomials
- Useage expanded over 300 years
- $i = \sqrt{-1}$
- $i^2 = -1$
- In electrical engineering, symbol j is used
  - Since symbol i is usually reserved to signify current

Formats

The Different Formats

- Rectangular format
- Polar format
- Exponential format

Rectangular Format

Also called 'Cartesian Format'

Components

Imaginary Multiplier

$= \sqrt{-1}$

$$z = a + ib$$

Complex Number

Real Part

Imaginary Part

Alternative Notation

- Physics
  - Z = a + ib
  - Z = a + bi
- Electrical Engineering
  - We use j because i is used to represent current
  - Z = a + jb
  - Z = a + bj

- Matlab
  - Code
    - `z = a + i*b;`
    - `z = a + b*i;`
    - `z = a + j*b;`
    - `z = a + b*j;`
  - Output
    - a + bi

NOTE: In Matlab, i and j are reserved for the imaginary multiplier. Don't use them for variable names.

Graphical Representation

- Plot the two parts on a Cartesian plane
- Real part on the x-axis (horizontal)
- Imaginary part on the y-axis (vertical)

Taking the Real Part

- Let the complex number be: $z = a + ib$
- Remove the imaginary part b to leave only the real part a remaining
- Maths
  - $a = \mathrm{Re}\{Z\}$
  - $a = \Re\{Z\}$
- Matlab
  - a = real(Z)

Taking the Imaginary Part

- Let the complex number be: $z = a + ib$
- Remove the real part a
- Find the real number b which is multiplied by the imaginary multiplier i
- Maths
  - $b = \mathrm{Im}\{Z\}$
  - $b = \Im\{Z\}$
- Matlab
  - b = imag(Z)

Polar Format

Parts

Angle symbol

$$z = r \angle \theta$$

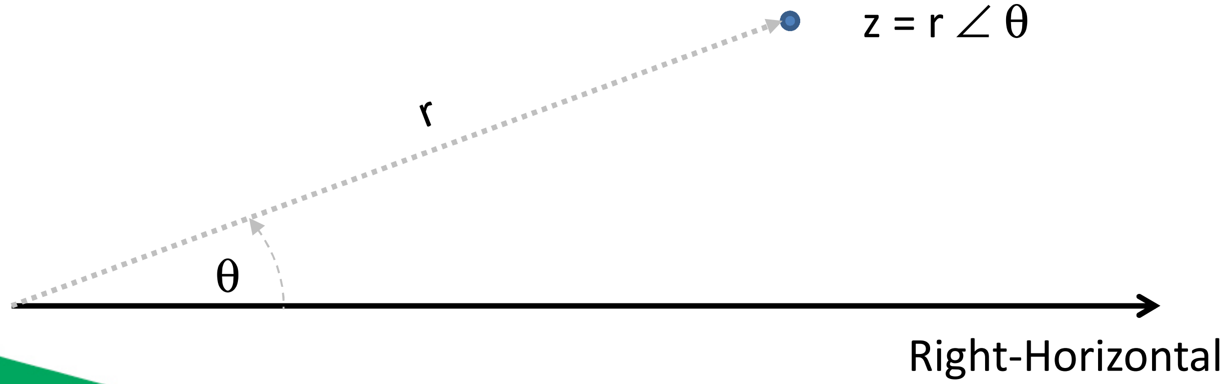
Complex Number

Magnitude

Phase

Graphical Representation

- Plot the two parts on a Polar plane
- Magnitude r is the length
- Phase θ is the rotation anti-clockwise from right-horizontal



Alternative Names

- Magnitude
  - Modulus
  - Absolute value
  - Gain
  - Amplitude
- Phase
  - Angle
  - Argument

Exponential Format

Parts

Complex multiplier

$$z = r\, e^{i\theta}$$

Complex Number

Magnitude

Euler's Number

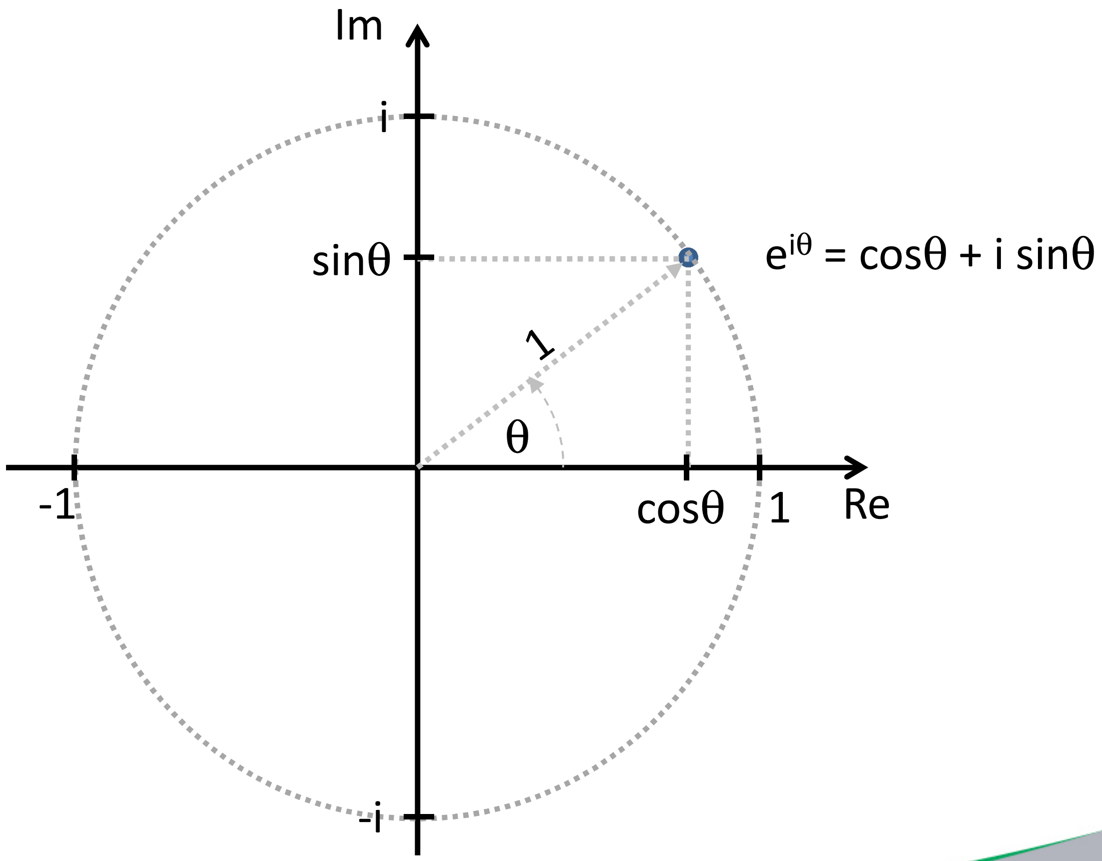
Phase

Euler's Identity

- $e^{i\theta} = \cos\theta + i \sin\theta$

$$= 1 \angle \theta$$

θ (rad)	$e^{i\theta}$
0	+1
$\pi/2$	+i
π	-1
$3\pi/2$	-i
2π	+1
$\pi/4$	$1/\sqrt{2} + i\, 1/\sqrt{2}$

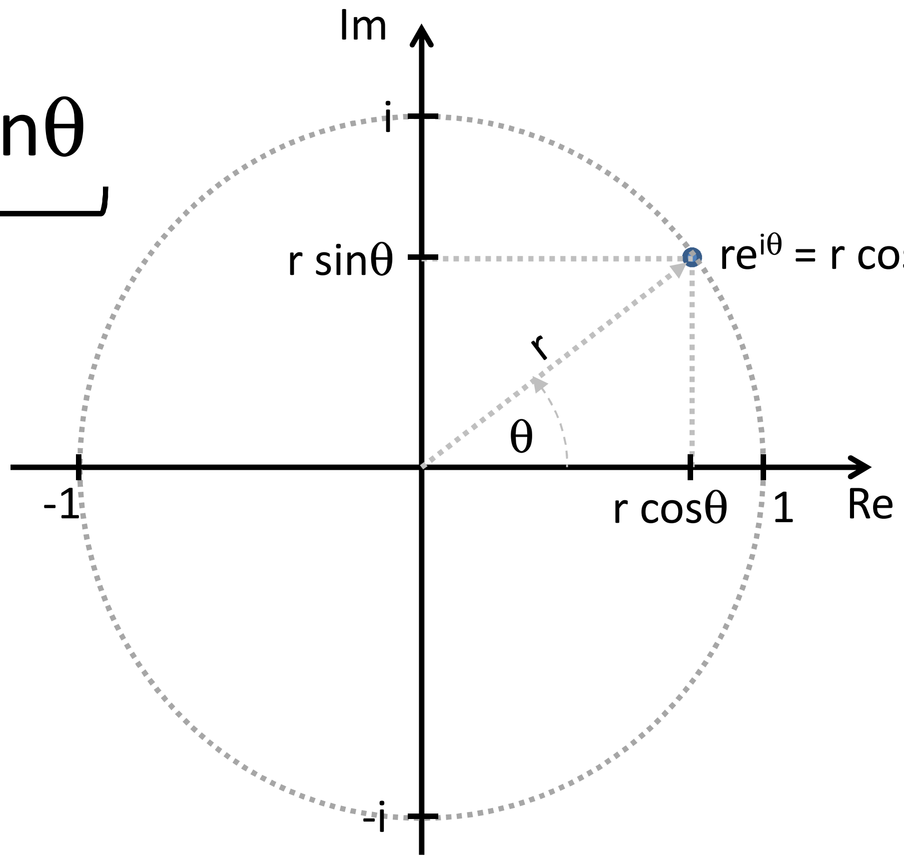


Graphical Representation

- $z = r\,e^{i\theta}$

  $= \underbrace{r\cos\theta + i\,r\sin\theta}$

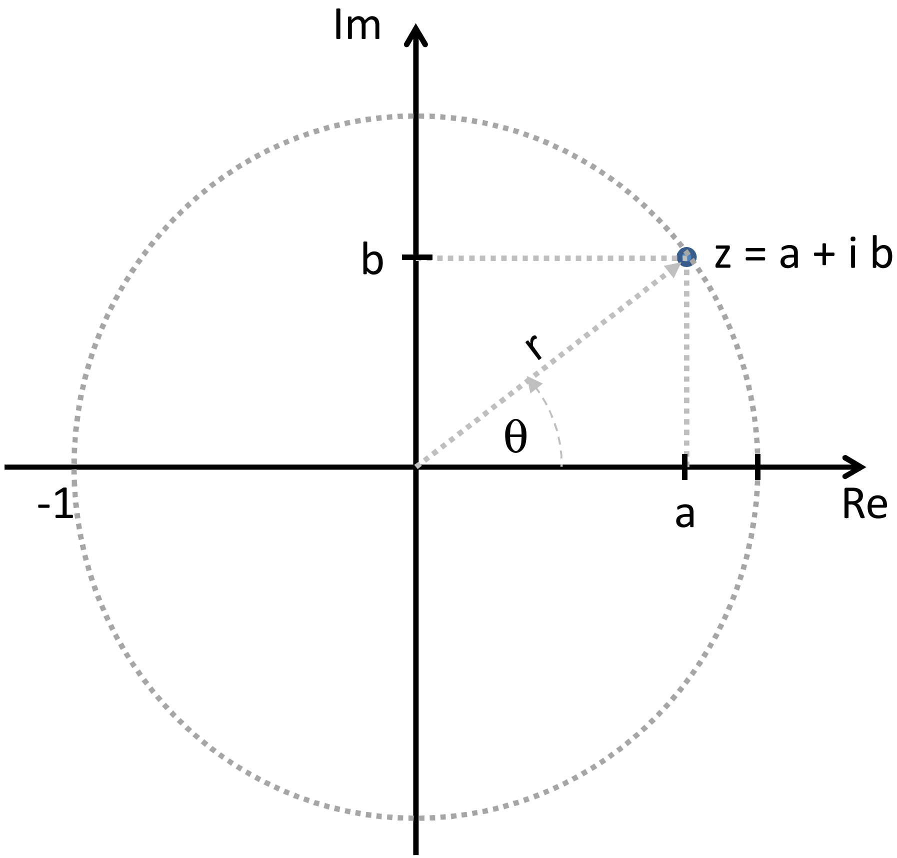
Euler's Identity multiplied by magnitude r



Format Conversions

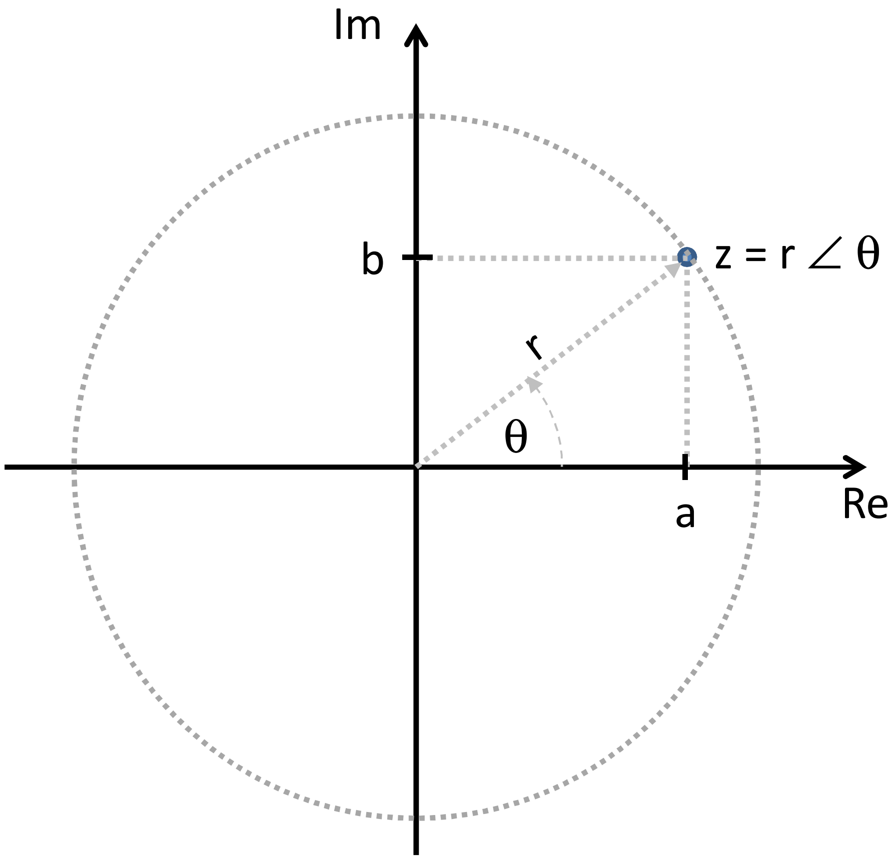
Rectangular to Polar

- From Rectangular : a + i b
- To Polar : $r \angle \theta$
- Pythagoras
  - Hypotenuse of right-angle triangle
  - $r = \sqrt{(a^2 + b^2)}$
    $= \sqrt{(Re\{z\}^2 + Im\{z\}^2)}$
- Trigonometry
  - $\theta = \tan^{-1}[b/a]$
    $= \tan^{-1}[Im\{z\}/Re\{z\}]$
- Matlab
  - z = a + i*b
  - r = sqrt(a^2 + b^2); % Or ...
  - r = abs(z);
  - theta = atan2(b,a); % Or ...
  - theta = angle(z);



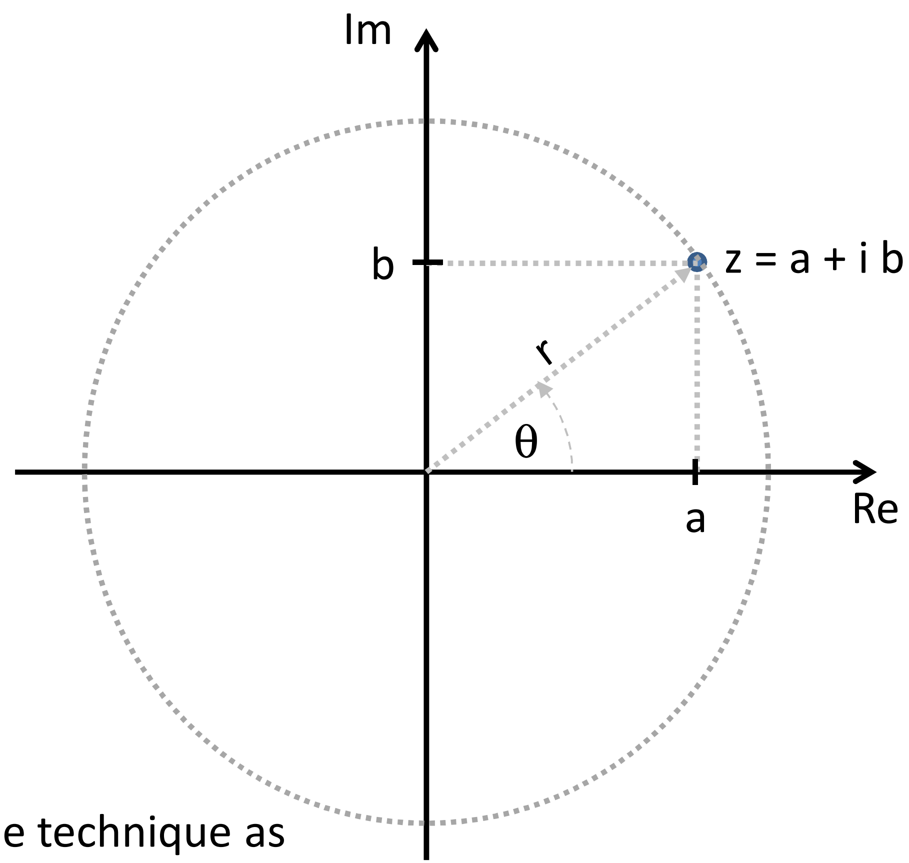
Polar to Rectangular

- From Polar : $r \angle \theta$
- To Rectangular : $a + i\,b$
- Trigonometry
  - $a = r \cos[\theta]$
  - $b = r \sin[\theta]$
- Matlab
  - a = real(z);
  - b = imag(z);



Rectangular to Exponential

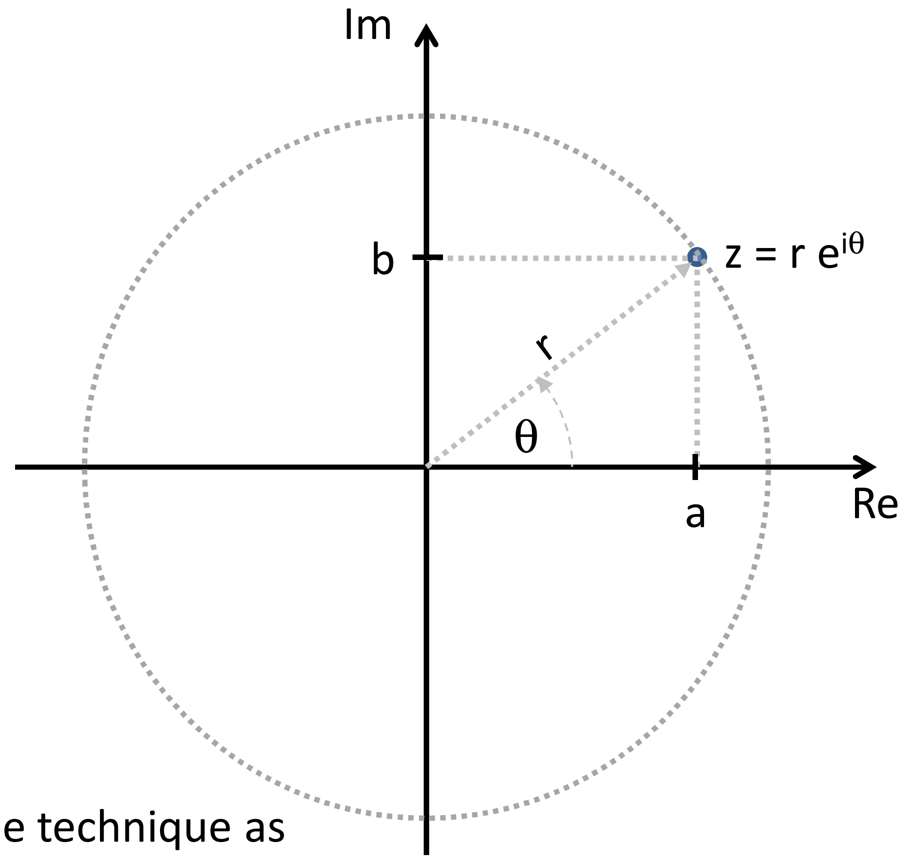
- From Rectangular : $a + i\,b$
- To Exponential : $r\,e^{i\theta}$
- Pythagoras
  - Hypotenuse of right-angle triangle
  - $r = \sqrt{(a^2 + b^2)}$
    $= \sqrt{(\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2)}$
- Trigonometry
  - $\theta = \tan^{-1}[b/a]$
    $= \tan^{-1}[\mathrm{Im}\{z\}/\mathrm{Re}\{z\}]$
- Matlab
  - z = a + i*b;
  - r = sqrt(a^2 + b^2); % Or ...
  - r = abs(z);
  - theta = atan2(b,a); % Or ...
  - theta = angle(z);



NOTE: Same technique as rectangular to polar

Exponential to Rectangular

- From Exponential : $r\ e^{i\theta}$
- To Rectangular : $a + i\ b$
- Trigonometry
  - $a = r \cos[\theta]$
  - $b = r \sin[\theta]$
- Matlab
  - a = real(z);
  - b = imag(z);

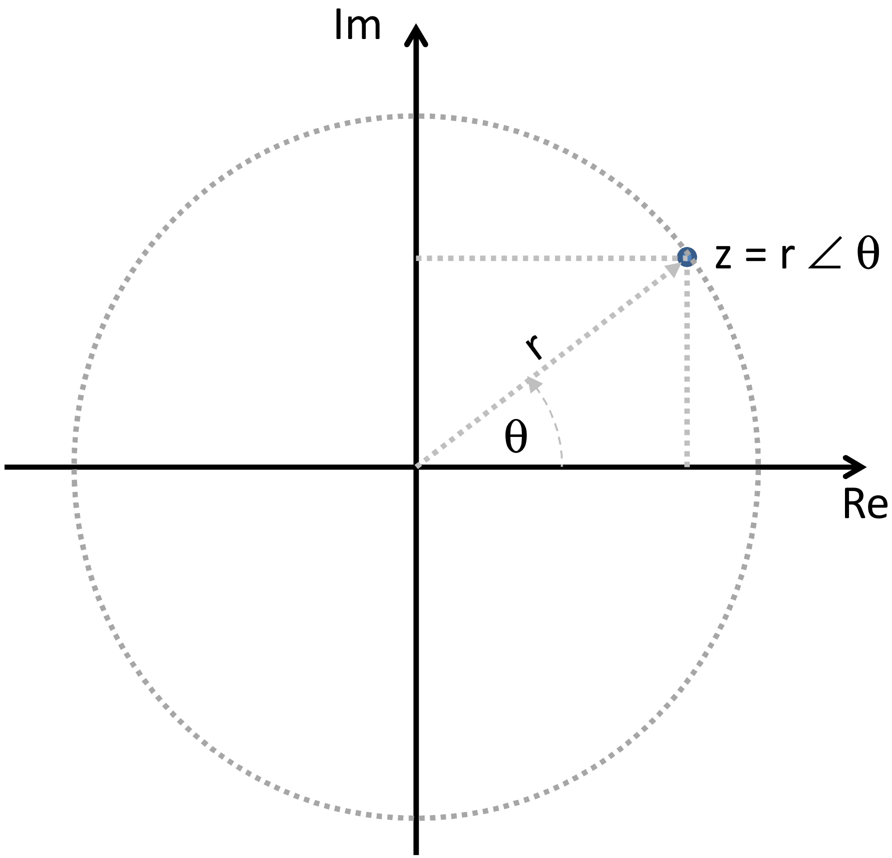


NOTE: Same technique as polar to rectangular

Polar to Exponential

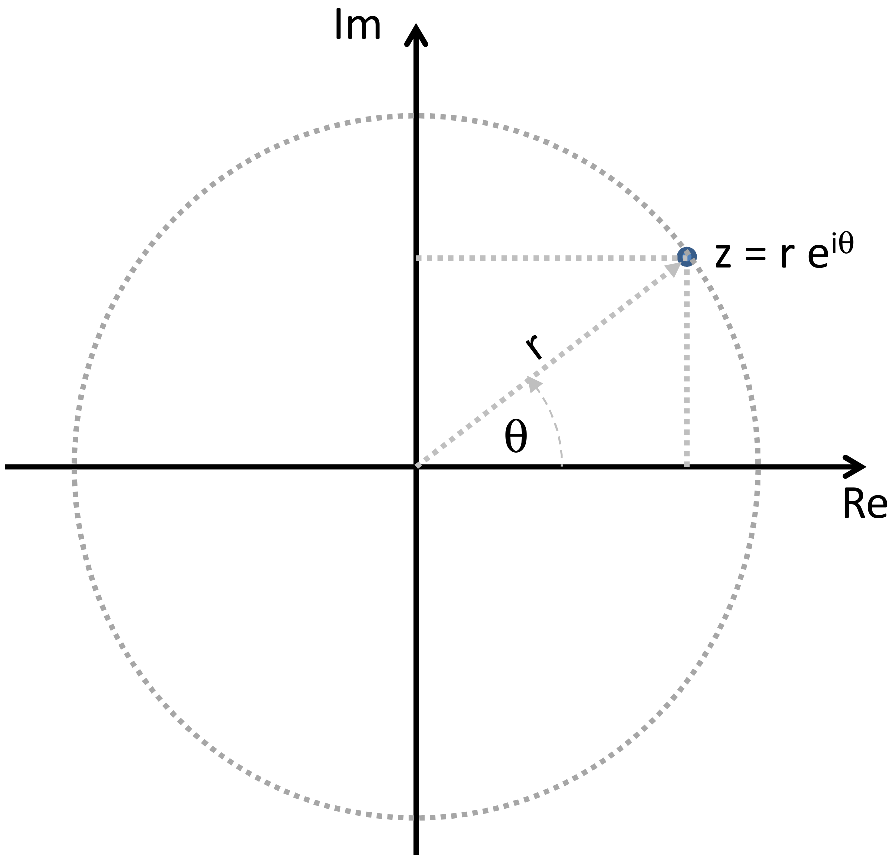
- From Polar : $r \angle \theta$
- To Exponential : $r \; e^{i\theta}$

NOTE: Polar and exponential formats have the same two arguments r and θ.



Exponential to Polar

- From Exponential : $r \quad e^{i\theta}$
- To Polar : $r \angle \theta$

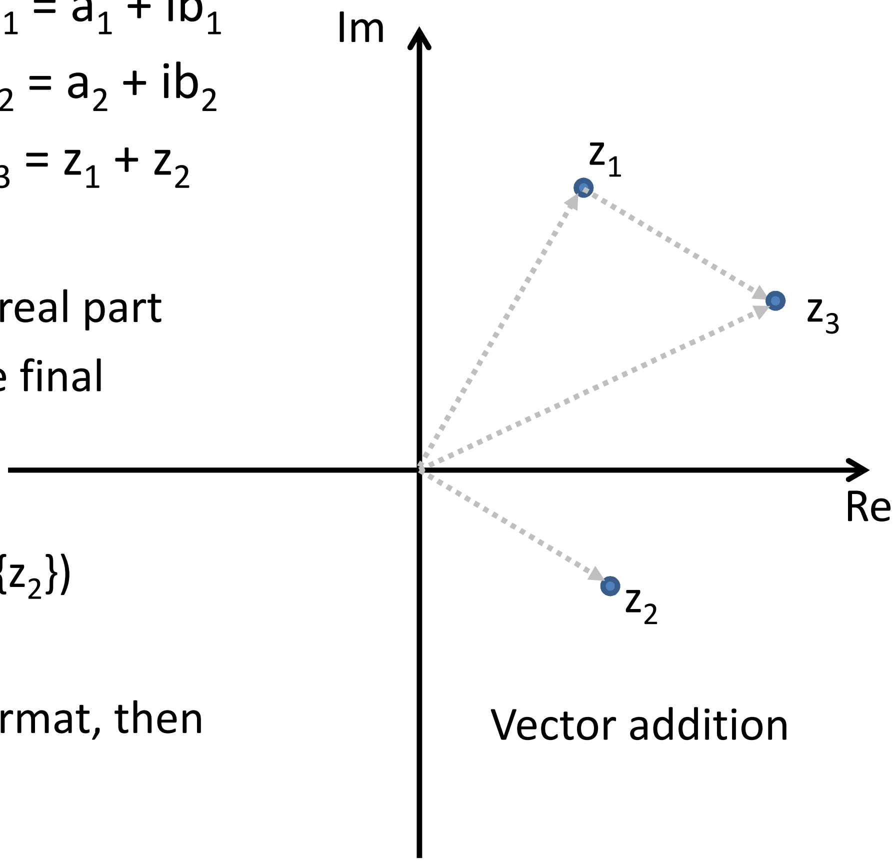


NOTE: Polar and exponential formats have the same two arguments r and θ.

Operations

Addition

- Let the 1’st complex number be: $z_1 = a_1 + ib_1$
- Let the 2’nd complex number be: $z_2 = a_2 + ib_2$
- Let the sum complex number be: $z_3 = z_1 + z_2$
- Rectangular Format
  - Sum the real parts to get the final real part
  - Sum the imaginary parts to get the final imaginary part
  - $z_3 = (a_1 + a_2) + i(b_1 + b_2)$
    $= (\mathrm{Re}\{z_1\} + \mathrm{Re}\{z_2\}) + i(\mathrm{Im}\{z_1\} + \mathrm{Im}\{z_2\})$
- Polar & exponential formats
  - Convert z_1 and z_2 to rectangular format, then add as above
- Matlab: `z3 = z1 + z2;`



Vector addition

Multiplication

- Let the 1'st complex number be: $z_1 = a_1 + ib_1 = r_1 \angle \theta_1 = r_1\, e^{i\theta_1}$
- Let the 2'nd complex number be: $z_2 = a_2 + ib_2 = r_2 \angle \theta_2 = r_2\, e^{i\theta_2}$
- Let the product complex number be: $z_3 = z_1 \times z_2 = a_3 + ib_3 = r_3 \angle \theta_3 = r_3\, e^{i\theta_3}$
- Rectangular Format
  - $z_3 = z_1 \times z_2 = (a_1 + ib_1) \times (a_2 + ib_2)$
    $= (a_1a_2 + a_1ib_2 + ib_1a_2 + i^2b_1b_2)$
    $= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$
    $= (\mathrm{Re}\{z_1\}\mathrm{Re}\{z_2\} - \mathrm{Im}\{z_1\}\mathrm{Im}\{z_2\}) + i(\mathrm{Re}\{z_1\}\mathrm{Im}\{z_2\} + \mathrm{Re}\{z_2\}\mathrm{Im}\{z_1\})$
  - Alternative: Convert z_1 and z_2 to polar, then do polar multiplication
- Polar & exponential formats
  - $z_3 = z_1 \times z_2 = r_1\, e^{i\theta_1} r_2\, e^{i\theta_2} = (r_1\, r_2)\, e^{i(\theta_1+\theta_2)}$
  - Multiply the magnitudes, add the phases
- Matlab: $z_3 = z_1 * z_2;$

Conjugation

- Notation: Asterisk superscript or ‘overbar’
- Rectangular format
  - Change the sign of the Imaginary part
  - $z = a + ib$
  - $z^* = a - ib$
- Polar and exponential formats
  - Change the sign of the phase
  - $z = r \angle \theta = r e^{i\theta}$
  - $z^* = r \angle -\theta = r e^{-i\theta}$
  - $(z_1 z_2)^* = z_1^* z_2^*$
- Matlab
  - `z2 = conj(z1);`

Multiplicative Inverse

- Rectangular format
  - Convert to polar or exponential, then invert
- Polar and exponential formats
  - Invert the magnitude
  - Change the sign of the phase
  - $z = r \angle \theta = r\,e^{i\theta}$
  - $z^{-1} = r^{-1} \angle -\theta = r^{-1}\,e^{-i\theta}$
  - Confirmation: $zz^{-1} = r\,e^{i\theta}\,r^{-1}\,e^{-i\theta} = (r\,r^{-1})\,(e^{i\theta}\,e^{-i\theta}) = 1\,e^{-i0} = 1$
- Matlab
  - z2 = z1^-1;

Properties

- Associative
  - $z_1\,(z_2 + z_3) = z_1 z_2 + z_1 z_3$
- Commutative
  - $z_1 z_2 = z_2 z_1$