Control Systems

Chapter 6: Stability

Highlights

Original slides courtesy to Dr Branislav Hredzak

Stability

- How to determine the stability of a system represented as a transfer function
- How to determine system parameters to yield stability

Introduction

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

Definitions of stability for linear, time-invariant systems **using the natural response:**

1. A system is ***stable*** if the natural response approaches zero as time approaches infinity.

2. A system is ***unstable*** if the natural response approaches infinity as time approaches infinity.

3. A system is ***marginally stable*** if the natural response neither decays nor grows but remains constant or oscillates.

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

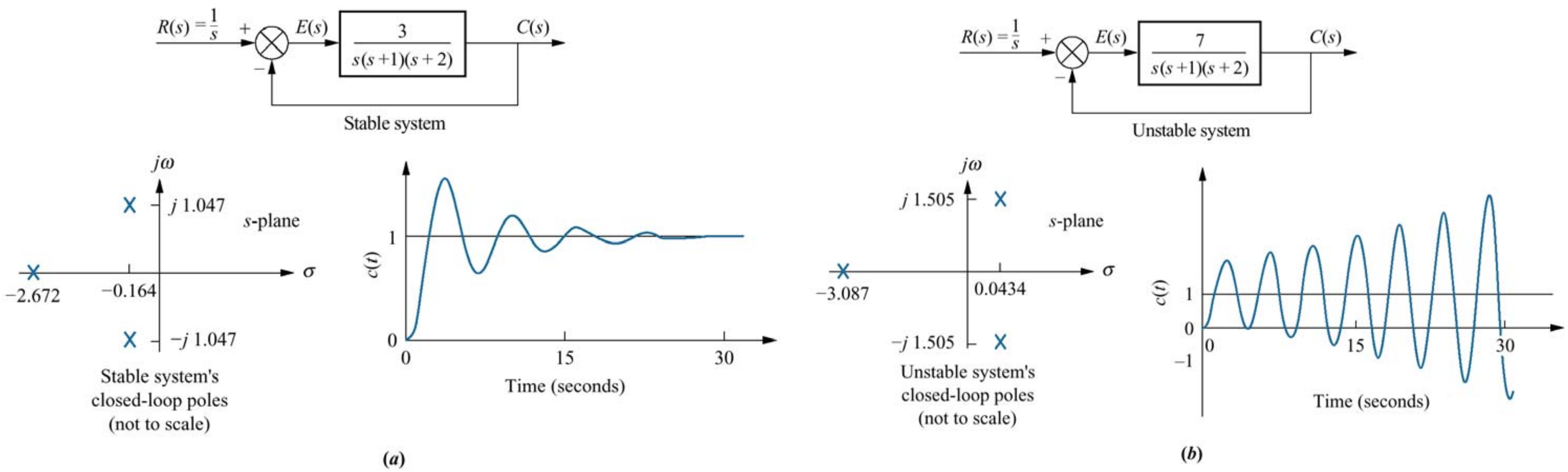
Stable systems have closed-loop transfer functions with poles only in the ***left half-plane*** **(LHP, has negative real part).**

Unstable systems have closed loop transfer functions with at least one pole in the ***right half-plane*** **(RHP, has positive real part)** and/or poles of multiplicity ***greater than one on the imaginary axis***

Marginally stable systems have closed loop transfer functions with only ***imaginary axis poles of multiplicity*** **1** and poles in the ***left half-plane***

Stable System

Unstable System



Note: These definitions assume implicitly there have been are **no cancellations of RHP and/or certain imaginary axis poles**! There are systems that after cancellation only has LHP poles left, yet is unstable for *non-zero* initial conditions.

Routh-Hurwitz Criterion

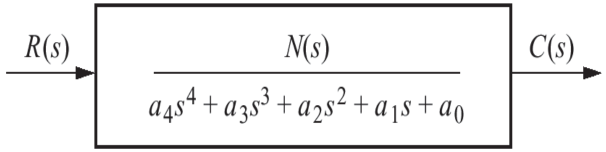
- yields stability information **without the need to solve** for the closed-loop system poles

Two steps:

(1) Generate a data table called a *Routh table*.

(2) Interpret the Routh table to tell how many closed-loop system poles are in the LHP and in the RHP.

Generating a Basic Routh Table



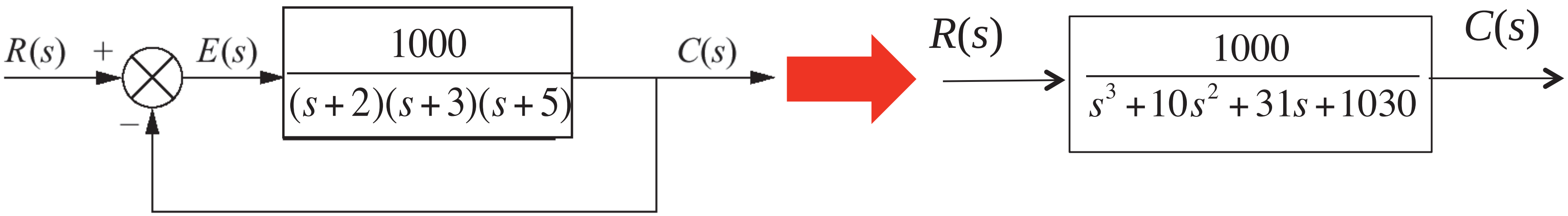
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Note: any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below.

Interpreting the Basic Routh Table

- *the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.*



s^3	1	31
s^2	$\cancel{10}$ 1	$\cancel{1030}$ 103
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

- two sign changes, hence two poles in the right half-plane. System unstable.

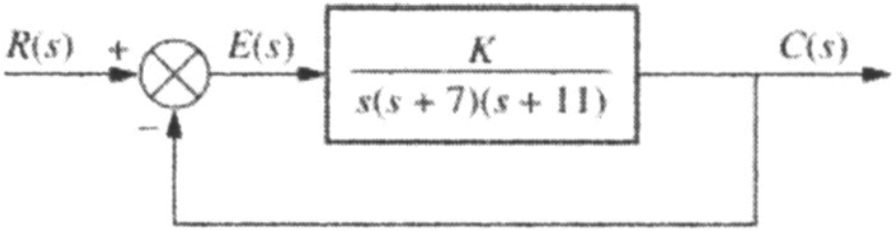
Routh-Hurwitz Criterion: Special cases

- Two special but important cases can occur in the Routh-Hurwitz stability test:
  1. The Routh table can have a zero only in the first column of a row.
  2. The Routh table can have an entire row that consists of zeros.

Keep in mind: **a zero does not have a definite sign. +0 is the same as -0. Zeros cannot be used to count sign changes. Must watch out for them in the first column.**

Routh-Hurwitz Criterion: Additional Examples

Example 6.9 Find the range of gain $K > 0$ for the system below that will cause the system to be stable, unstable, and marginally stable.



Solution

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

Routh table

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

Routh-Hurwitz Criterion: Additional Examples

s^3	1	77
s^2	18	K
s^1	$\dfrac{1386-K}{18}$	
s^0	K	

- If $K < 1386$ then there will be no sign change in the first column, so system is stable.
- If $K > 1386$ the first element of the third row becomes negative so that there will be two sign changes and the system has two RHP poles. So, in this case it is not stable.