ELEC2070 Circuits and Devices

Week 11: Damped sinusoidal forcing function and the Laplace Transform

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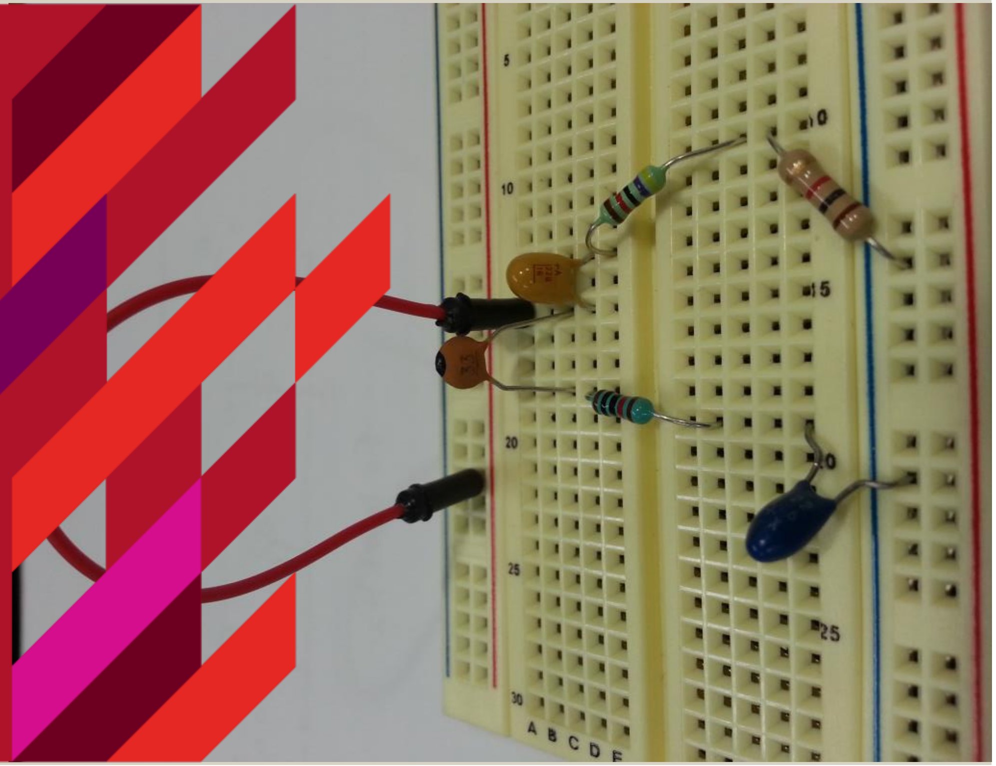


The “Big Ideas” in Chap. 14

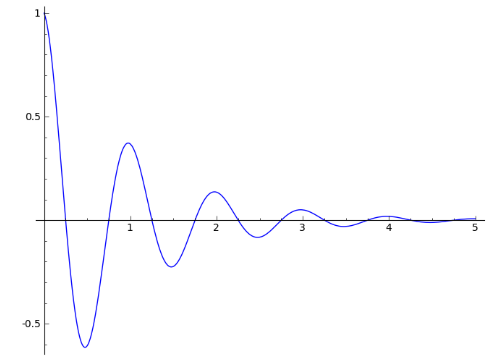
- The Laplace Transform can help us solve circuits, especially circuits containing capacitors and inductors which usually involve differential equations
- The Laplace transform allows circuits to be solved algebraically
- The Laplace transform converts from the time domain into the frequency domain
- Once we have solved the algebraic equations, we go back to the time domain using the Inverse Laplace Transform
- The Laplace Transform is very powerful, it allows any input forcing function to a circuit! (Remember, we have been studying DC and steady state sinusoidal inputs)
- The Laplace Transform is especially useful for forcing functions (inputs) that are pulses.

The damped sinusoidal forcing function

Damped sinusoidal forcing function

Recall that damped sinusoidal forcing function for voltage is given by:

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$



Making use of Euler's identity, this becomes:

$$v(t) = \mathrm{Re}\{V_m e^{\sigma t} e^{j(\omega t + \theta)}\}$$ or $$v(t) = \mathrm{Re}\{V_m e^{\sigma t} e^{j(-\omega t - \theta)}\}$$

Collecting factors and involving the complex frequency, we get

$$\mathrm{s} = \sigma + j\omega$$

$$v(t) = \mathrm{Re}\{V_m e^{j\theta} e^{st}\}$$

This equation involves a coefficient containing the amplitude and phase angle multiplied by an exponential involving the complex frequency.

The forced response

Assume that we seek the forced response which is a current going through a branch in a circuit, this current must be given by:

$$i(t) = I_m e^{\sigma t} \cos(\omega t + \phi) \qquad \text{or} \qquad i(t) = \mathrm{Re}\{I_m e^{j\phi} e^{st}\}$$

Note that it must have the SAME functional form, and we note

The complex frequency of the forcing function and the response are identical

The real part of the forcing function (σ) gives rise to the real part of the response. The imaginary part of the forcing function (ω) gives rise to the imaginary part of the response.

Usually the Re{} factor is removed but it MUST be understood that it must be reinserted if we want the time domain response.

Summary

Given the time domain forcing function:

$$v(t) = \mathrm{Re}\{V_m e^{j\theta} e^{st}\}$$

We have the forcing function in the complex frequency domain:

$$V_m e^{j\theta} e^{st}$$

The resulting forced response is:

$$I_m e^{j\phi} e^{st}$$

And the time domain response is:

$$i(t) = \mathrm{Re}\{I_m e^{j\phi} e^{st}\}$$

The solution to our circuit problem is finding the response amplitude I_m and the phase angle ϕ

The unit so far

We can summarise the unit so far using our new forcing function: $v(t) = \mathrm{Re}\{V_m e^{j\theta} e^{st}\}$

Module 1: Involved DC forcing functions of the form $V_0 e^0$

Module 2: We looked at the transient response (also with a DC forcing function) of RL and RC circuits with the response behaving with a $V_0 e^{\sigma t}$ type relationship. We found for a second order RLC circuit that was underdamped, the behaviour was in fact a $V_0 e^{\sigma t} e^{j\omega t}$ or $V_0 e^{\sigma t} \cos(\omega t + \theta)$ type relationship.

Module 3: We used phasors to help us solve AC circuits. Here the forcing (and response) function was of the form $V_0 e^{j\theta} e^{j\omega t}$. *The phasor in this case is* $V_0 e^{j\theta}$

Module 4: Now we are entirely in the frequency domain.

The frequency domain analysis means we drop all terms in our current and voltage relations that contain *t*.

The frequency domain method

The basic steps in the frequency domain analysis of circuits is:

1. Characterise the circuit using node or mesh analysis. This will create differential equations or integrodifferential (containing integrals and differentials) equations.
2. The forcing functions will be in complex form, so will the responses. These are substituted into the equations and the differentiations and integrations performed. (Note that since we now have exponentials only, the integrals and differentials will also be exponentials.)
3. Each term will contain the factor e^{st} . The term e^{st} can be factored out leaving an equation containing the amplitudes and phase angles ONLY. Remember, to get back to the time domain, the term e^{st} must be reinserted.

This process converts from the time domain to the frequency domain.

The Laplace Transform

The Laplace Transform

In the real word, most functions are not pure sinusoidal or pure exponential. For example, when square waves or pulses are applied to a circuit, the responses are not related to these forcing functions in a simple way.

For **non** pure sinusoidal or exponential forcing functions we are not able to simply drop terms containing *t* and work entirely in the frequency domain.

There is a solution:

Most functions of time can be decomposed into an infinite summation of exponentials (each with a unique complex frequency in their exponents).

With each exponential we can drop terms containing t and work entirely in the frequency domain (for each exponential).

Of course this will mean an infinite number of exponentials in order to accurately represent the forcing (and response) functions of time.

Definition

This is also known as the one sided Laplace Transform

The Laplace Transform is defined as:

$$F(s) = \mathscr{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

where s is the **complex frequency**:

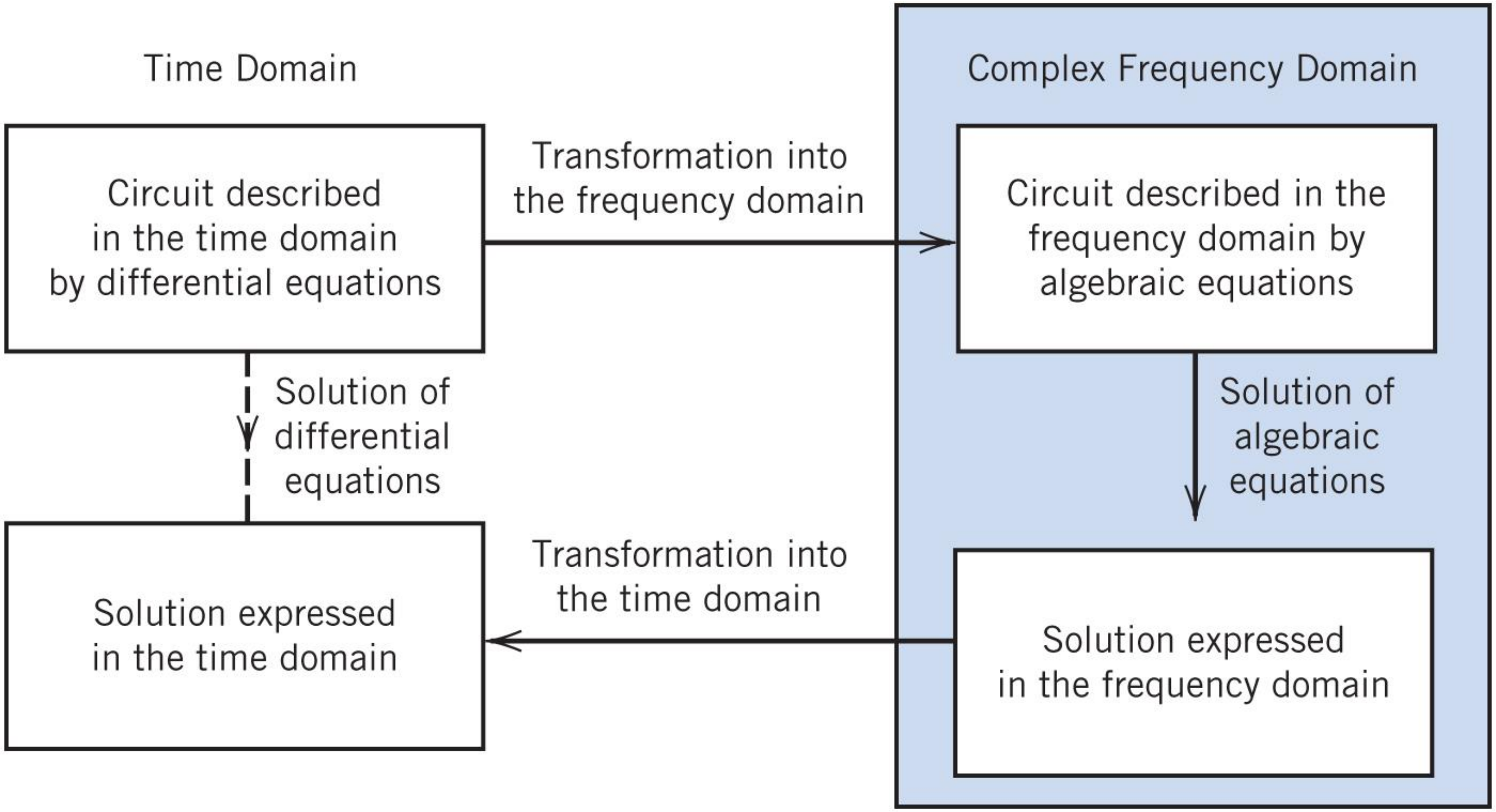
$$s = \sigma + j\omega$$

The lower limit is taken to be 0^- which allows for discontinuities in f(t) at t=0.

Note that f(t) is not considered for t<0, or we can say f(t)=0 for t<0

The Laplace Transform converts the time domain function f(t) to a corresponding frequency domain representative, F(s).

The process of using the transform



All functions that are physically possible have a Laplace Transform.

The Inverse Laplace Transform

The Inverse Laplace Transform is defined by the complex inversion integral:

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}ds$$

This is a contour integration in the complex plane. We will not be doing these!!

A table of transforms and inverse transforms are used in practise.

Since the Inverse Laplace Transform is the inverse of the Laplace Transform, i.e.,
F(s)= $\mathscr{L}$[f(t)] and f(t)=$\mathscr{L}^{-1}$ [F(s)]

then

$$f(t) \leftrightarrow F(s)$$

These comprise a **Laplace Transform pair**

Examples of the Laplace Transform of some important functions, f(t)

Example 1

Find the Laplace transform of $f(t) = e^{-at}$

$$F(s) = L[f(t)] = \int_0^{\infty} e^{-at} e^{-st} dt$$

This gives:

$$F(s) = \int_0^{\infty} e^{-(s+a)t} dt$$

$$= \left. \frac{-e^{-(s+a)t}}{s+a} \right|_0^{\infty}$$

$$= \frac{1}{s+a}$$

Example 2

Compute the Laplace transform of the function $f(t) = 2u(t-3)$.

In order to find the one-sided Laplace transform of $f(t) = 2u(t-3)$, we must evaluate the integral

$$\begin{aligned}\mathbf{F}(\mathbf{s}) &= \int_{0^-}^{\infty} e^{-\mathbf{s}t} f(t)\, dt \\ &= \int_{0^-}^{\infty} e^{-\mathbf{s}t} 2u(t-3)\, dt \\ &= 2\int_{3}^{\infty} e^{-\mathbf{s}t}\, dt\end{aligned}$$

Simplifying, we find

$$\mathbf{F}(\mathbf{s}) = \frac{-2}{\mathbf{s}} e^{-\mathbf{s}t}\Big|_{3}^{\infty} = \frac{-2}{\mathbf{s}}(0 - e^{-3\mathbf{s}}) = \frac{2}{\mathbf{s}} e^{-3\mathbf{s}}$$

Properties: Linearity

Linearity means that: $a_1 f_1(t) + a_2 f_2(t) \quad \leftrightarrow \quad a_1 F_1(s) + a_2 F_2(s)$

Example: Find the Laplace transform of sin(ωt)

Using the Euler identity: $\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$

We want $F(s) = L[\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)]$

Or $F(s) = \frac{1}{2j}\int_0^\infty \left(e^{j\omega t} - e^{-j\omega t}\right)e^{-st}dt = \frac{1}{2j}\int_0^\infty e^{j\omega t}e^{-st}dt - \frac{1}{2j}\int_0^\infty e^{-j\omega t}e^{-st}dt$

Using our previous example:

$F(s) = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$ for t>0

Differentiation

Now we want to find F(s)= $\mathcal{L}$ [df(t)/dt]

We need to use:

$$\int u dv = uv - \int v du$$

Or $\quad \mathrm{F(s)} = \int_0^\infty \frac{df}{dt} e^{-st} dt$

Since we will be integrating by parts, set u = e^{-st} and $dv = \left(\frac{df}{dt}\right) dt = df$

Then du = $-se^{-st}$ and v = f(t)

Integrating by parts we get: F(s) = $e^{-st}\mathrm{f(t)} - \int_0^\infty f(t) \times -se^{-st}\, dt$

Or F(s) = $e^{-st}\mathrm{f(t)}|_0^\infty + \mathrm{s}\int_0^\infty f(t)e^{-st}\, dt = sF(s) - f(0)$

$$\boxed{\frac{df}{dt} \leftrightarrow sF(s) - f(0^-)}$$

Example

Find the Laplace Transform of cos(ωt)

We know that $\cos(\omega t) = \frac{1}{\omega}\frac{d}{dt}(\sin(\omega t))$

Then using linearity: $\mathcal{L}[\cos(\omega t)]=\frac{1}{\omega}\mathcal{L}[\frac{d}{dt}(\sin(\omega t))]$

Using our new identity: $\mathcal{L}[\frac{d}{dt}(\sin(\omega t))] =s\mathcal{L}[(\sin(\omega t)] - \sin(\omega 0)$

$$= s\frac{\omega}{s^2+\omega^2}$$

Therefore: $\mathcal{L}[\cos(\omega t)] = \frac{1}{\omega} s \frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2}$

Table of Laplace Transforms

$f(t)$ for $t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
e^{-at}	$\dfrac{1}{s+a}$
t	$\dfrac{1}{s^2}$
t^n	$\dfrac{n!}{s^{n+1}}$
$e^{-at}t^n$	$\dfrac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

Important Properties

PROPERTY	$f(t),\ t > 0$	$F(s) = \mathscr{L}[f(t)u(t)]$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time scaling	$f(at)$, where $a > 0$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time integration	$\int_0^t f(\tau)d\tau$	$\frac{1}{s} F(s)$
Time differentiation	$\frac{df(t)}{dt}$ $\frac{d^2 f(t)}{dt^2}$ $\frac{d^n f(t)}{dt^n}$	$sF(s) - f(0^-)$ $s^2 F(s) - \left(sf(0^-) + \frac{df(0^-)}{dt}\right)$ $s^n F(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1} f(0^-)}{dt^{k-1}}$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time convolution	$f_1(t)^* f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\lambda) d\lambda$
Frequency differentiation	$tf(t)$	$-\frac{dF(s)}{ds}$
Initial value	$f(0^+)$	$\lim_{s \to \infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s \to 0} sF(s)$

Example 14.2-4

Find the Laplace transform of $5 - 5e^{-2t}(1 + 2t)$.

Solution

From linearity,
$$\mathcal{L}\left[5 - 5e^{-2t}(1 + 2t)\right] = 5\,\mathcal{L}[1] - 5\,\mathcal{L}\left[e^{-2t}(1 + 2t)\right]$$

Using frequency shift from Table 14.2-2 with $f(t) = 1 + 2t$ gives
$$\mathcal{L}\left[e^{-2t}(1 + 2t)\right] = \mathcal{L}\left[e^{-2t}f(t)\right] = F(s + 2)$$

where
$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[1 + 2\,t] = \mathcal{L}[1] + 2\,\mathcal{L}[t] = \frac{1}{s} + 2\left(\frac{1}{s^2}\right)$$

Next,
$$F(s + 2) = F(s)\big|_{s \leftarrow s+2}$$

That is, we must replace each s in $F(s)$ by $s + 2$ to obtain $F(s + 2)$:
$$F(s + 2) = \left(\frac{1}{s} + 2\left(\frac{1}{s^2}\right)\right)\Bigg|_{s \leftarrow s+2} = \frac{1}{s + 2} + 2\left(\frac{1}{(s + 2)^2}\right) = \frac{s + 2 + 2(1)}{(s + 2)^2} = \frac{s + 4}{s^2 + 4s + 4}$$

Putting it all together gives
$$\mathcal{L}\left[5 - 5e^{-2t}(1 + 2t)\right] = 5\left(\frac{1}{s}\right) - 5\left(\frac{s + 4}{s^2 + 4s + 4}\right) = \frac{5(s^2 + 4s + 4) - 5s(s + 4)}{s(s^2 + 4s + 4)} = \frac{20}{s(s^2 + 4s + 4)}$$

Example 14.2-5

Find the Laplace transform of $10\, e^{-4t} \cos(20t + 36.9°)$.

Solution

Table 14.2-1 has entries for $\cos(\omega t)$ and $\sin(\omega t)$ but not for $\cos(\omega t + \theta)$. We can use the trigonometric identity

$$A \cos(\omega t + \theta) = (A \cos\theta) \cos(\omega t) - (A \sin\theta) \sin(\omega t)$$

to write

$$10 \cos(20t + 36.9°) = 8 \cos(20t) - 6 \sin(20t)$$

Now use linearity to write

$$\begin{aligned}\mathscr{L}[10e^{-4t} \cos(20t + 36.9°)] &= \mathscr{L}[e^{-4t}(8 \cos(20t) - 6 \sin(20t))] \\ &= 8\,\mathscr{L}[e^{-4t} \cos(20t)] - 6\mathscr{L}[e^{-4t} \sin(20t)]\end{aligned}$$

Using frequency shifts from Table 14.2-2 with $f(t) = \cos(20t)$ gives

$$\mathscr{L}\left[e^{-4t} \cos(20t)\right] = \mathscr{L}\left[e^{-4t} f(t)\right] = F(s+4)$$

where

$$F(s) = \mathscr{L}[f(t)] = \mathscr{L}[\cos(20t)] = \frac{s}{s^2 + 20^2} = \frac{s}{s^2 + 400}$$

Next,

$$F(s+4) = F(s)|_{s \leftarrow s+4}$$

Example 14.2-5

That is, we must replace each s in $F(s)$ by $s+4$ to obtain $F(s+4)$:

$$\mathscr{L}\left[e^{-4t}\cos(20t)\right] = F(s+4) = \left.\frac{s}{s^2+400}\right|_{s\leftarrow s+4} = \frac{s+4}{(s+4)^2+400} = \frac{s+4}{s^2+8s+416}$$

Similarly,

$$\mathscr{L}\left[e^{-4t}\sin(20t)\right] = \left.\frac{20}{s^2+400}\right|_{s\leftarrow s+4} = \frac{20}{(s+4)^2+400} = \frac{20}{s^2+8s+416}$$

Putting it all together gives

$$\mathscr{L}\left[10e^{-4t}\cos(20t+36.9^\circ)\right] = 8\left(\frac{s+4}{s^2+8s+416}\right) - 6\left(\frac{20}{s^2+8s+416}\right) = \frac{8s-88}{s^2+8s+416}$$

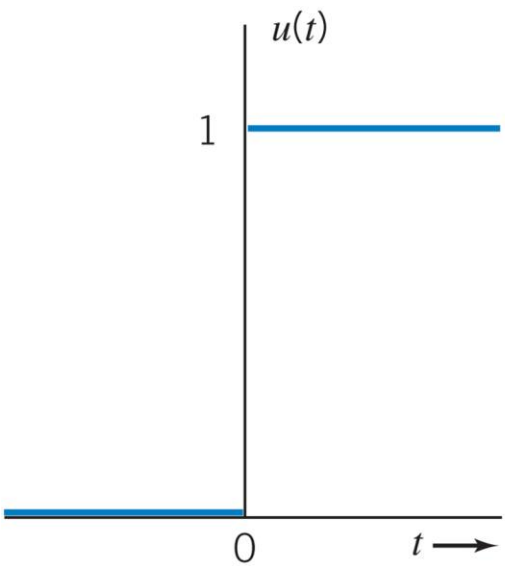
Pulse functions

The step function

The step function is given mathematically by:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

and this is represented by:



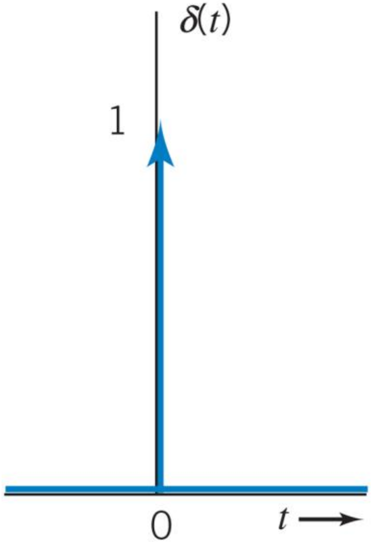
(a)

Impulse function

We define the impulse function as:

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t < 0 \\ undefined & t = 0 \\ 0 & t > 0 \end{cases}$$

Which looks like this:

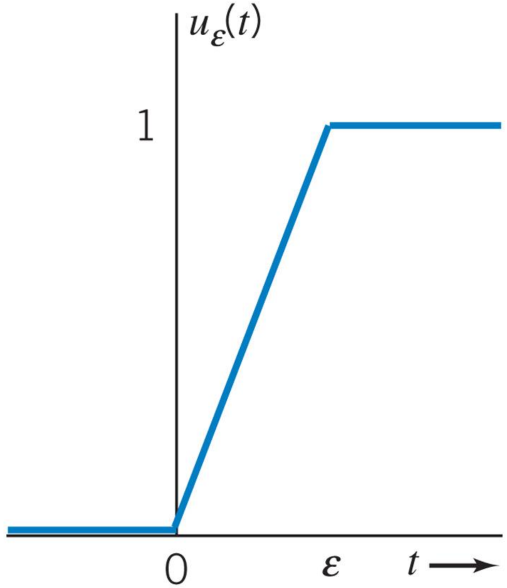


(d)

An approximation

Because the impulse function is undefined at t=0, we need an approximation to allow us to better understand the step function and impulse function for the analysis of pulses.

Consider the linear function:



(b)

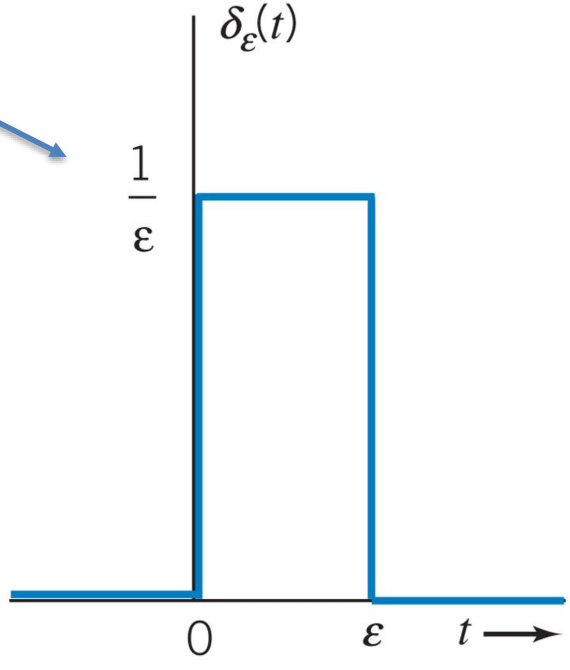
For $0 < t < \epsilon$

$$u_\epsilon(t) = \frac{1}{\epsilon} t$$

When ε = 0, we have our original step function, i.e., $\lim_{\varepsilon \to 0} u_\epsilon(t) = u(t)$.

The pulse function

We now differentiate this function $u_\varepsilon(t)$ and we get:



(c)

Note that the area under this curve is ALWAYS equal to 1.

Or $\int_{-\infty}^{\infty} \delta_\epsilon(t)dt = \int_0^\varepsilon \frac{1}{\varepsilon} dt = 1$

AND

$$\lim_{\varepsilon \to 0} \delta_\epsilon(t) = \delta(t)$$

Thus we can say the impulse function $\delta(t)$ has infinite amplitude, infinitesimal duration BUT an area equal to 1.

The impulse function is sometimes called the **delta function**.

Property of the impulse function

An important property of the impulse function is:

$$\int_{-\infty}^{+\infty} f(t)\delta(t)dt = f(0)$$

This is also called the **sifting property** of δ(t)

We can show that the area of an impulse function is 1 by letting f(t) = 1.

Which means the equation above becomes: $\int_{-\infty}^{\infty} \delta\,(t)dt = 1$
Which we have already shown before.

Now let's find the Laplace transform of the impulse function:

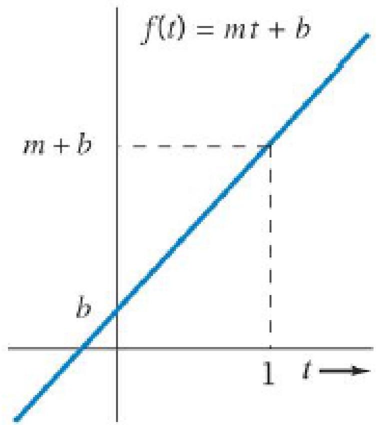
$$\mathcal{L}\,[\delta(t)] = \int_0^{\infty} \delta(t)e^{-st}dt = e^0 = 1$$

Delaying a linear function

We will need the Laplace Transform of other pulse functions.

We can delay a function by τ using the time delay $t - \tau$

For example, consider the function f(t) = $mt+b$



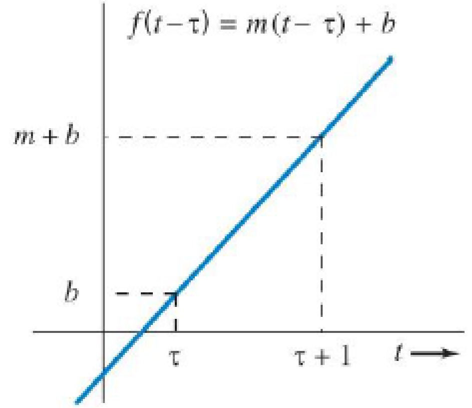
If we wish to delay this function by τ, we now have the equation:

$f(t-\tau) = m(t-\tau) + b$

or

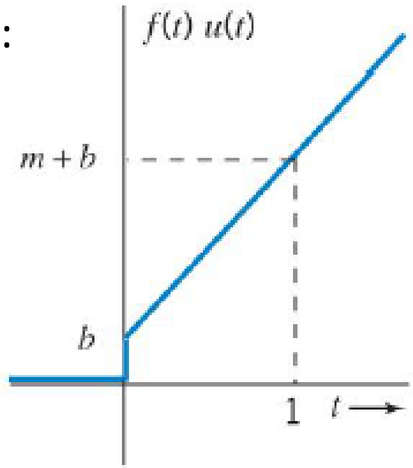
$f(t-\tau) = mt + (b - m\tau)$

Which has the same slope, but different y intercept



Using the step function

Now consider multiplying the function f(t) by the step function u(t):



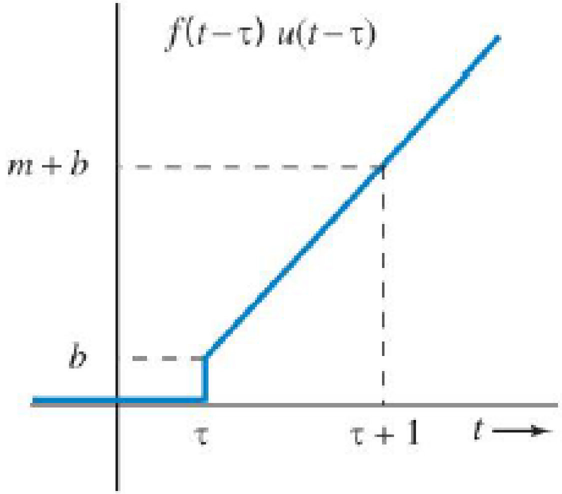
$g(t) = f(t)u(t) = (mt + b)u(t)$, this looks like this:

which is the same as f(t) when $t>0$, but is 0 for $t<0$

Now, if we wish to delay g(t) by τ, we have the function:

$g(t-\tau) = f(t - \tau)u(t - \tau) = [m(t - \tau) + b]u(t - \tau)$

This can be graphed to look like this:

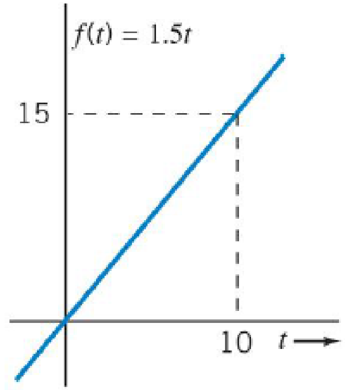


Note that $f(t - \tau)u(t - \tau)$ is different from

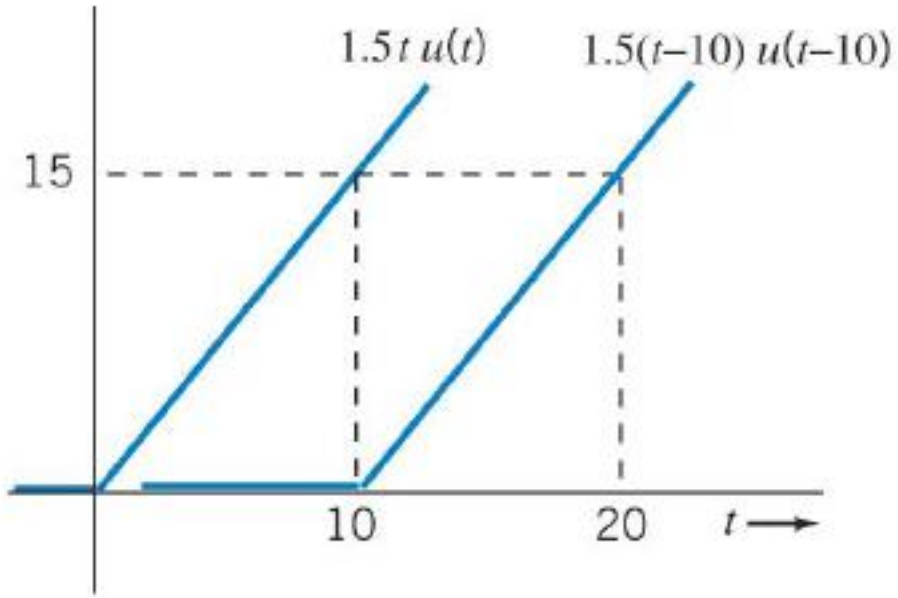
$f(t-\tau)u(t)$ OR $f(t)u(t-\tau)$

Now we can create a pulse function

Start with the function f(t)=1.5t



Multiply f(t) by the step function to give f(t)u(t)

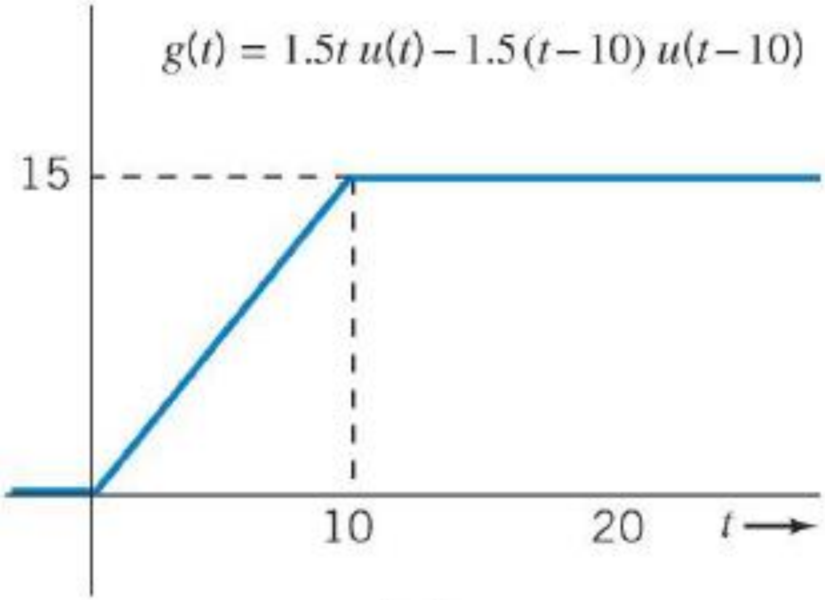


This is the delayed copy of f(t)u(t) which is given by f(t-10)u(t-10)

Subtracting a delayed function

Subtracting the delayed copy from the original function gives:

$$g(t) = f(t)\,u(t) - f(t-10)\,u(t-10) = 1.5\,t\,u(t) - 1.5(t-10)\,u(t-10)$$

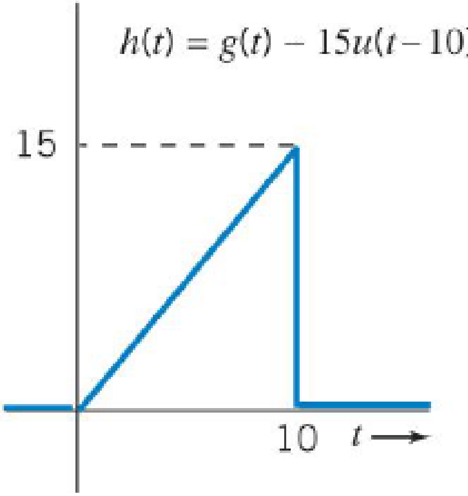


This can be easily seen by subtracting the 2 curves in the bottom figure on the previous page.

Subtracting a scaled and delayed step function

Now we can take away from this new function g(t) a SCALED step function 15u(t-10) to give:

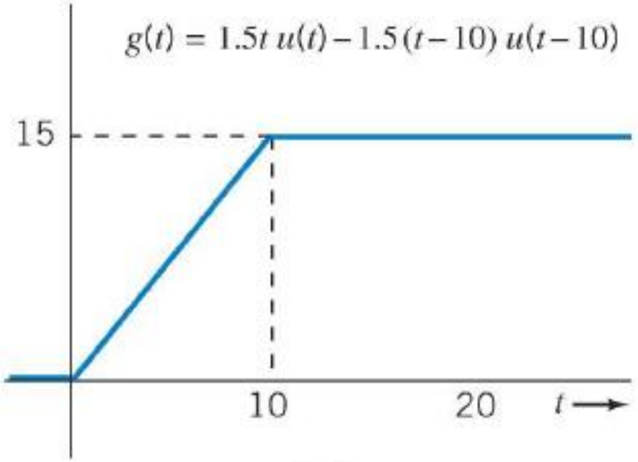
$$h(t) = g(t) - 15\,u(t-10) = 1.5\,t\,u(t) - 1.5(t-10)u(t-10) - 15u(t-10)$$



h(t) looks like a pulse now!

We can create another kind of pulse function.

Let's start with g(t) again:



Another pulse function

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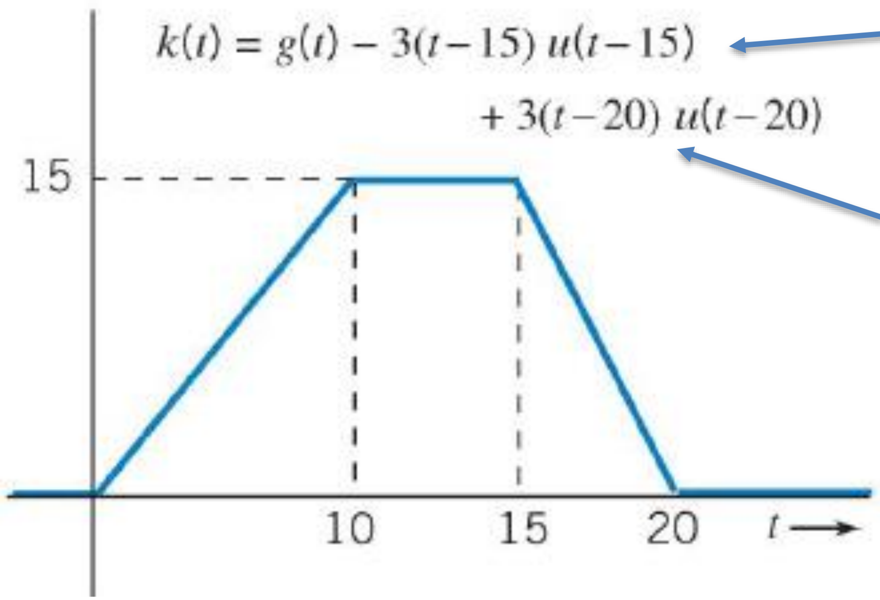
We can add new functions to g(t) to create a more symmetric kind of pulse, for example

$$k(t) = g(t) - 3.0(t - 15)\, u(t - 15) + 3.0(t - 20)\, u(t - 20)$$



This is ZERO until t=15

This is ZERO until t=20.

Note the 3(t - 20) term here cancels the -3(t - 15) term for t>20

Pulses can be constructed by adding and subtracting (delayed) functions composed of the multiplication of straight line terms and step function terms

Laplace Transform of a delayed function

The Laplace Transform of a delayed function is given by:

$$\mathscr{L}\left[f(t-\tau)u(t-\tau)\right] = \int_0^\infty f(t-\tau)u(t-\tau)e^{-st}dt = \int_\tau^\infty f(t-\tau)e^{-st}dt$$

Letting x = t - τ

And we get

$$\mathscr{L}\ \left[f(t-\tau)u(t-\tau)\right] = \int_\tau^\infty f(t-\tau)e^{-st}dt = \int_\tau^\infty f(x)e^{-s(x+\tau)}dx$$

Which can be simplified to:

$$e^{-s\tau}\int_\tau^\infty f(x)e^{-sx}dx = e^{-s\tau}F(s)$$

HENCE:

$$f(t-\tau)u(t-\tau) \leftrightarrow e^{-s\tau}F(s)$$

Example 14.3-1

Find the Laplace transforms of $g(t)$, $h(t)$ and, $k(t)$ shown in Figure 14.3-3.

Solution

After obtaining Eqs. 14.3-4, 14.3-5, and 14.3-6, we can easily determine the required Laplace transforms using $f(t-\tau)u(t-\tau) \leftrightarrow e^{-s\tau}F(s)$

$$G(s) = \mathscr{L}[g(t)] = \mathscr{L}[1.5t\,u(t)] - \mathscr{L}[1.5(t-10)\,u(t-10)]$$
$$= 1.5\left(\frac{1}{s^2}\right) - e^{-10s}\left(1.5\left(\frac{1}{s^2}\right)\right) = \frac{1.5(1-e^{-10s})}{s^2}$$

$$H(s) = \mathscr{L}[h(t)] = \mathscr{L}[g(t)] - \mathscr{L}[15\,u(t-10)] = \frac{1.5(1-e^{-10s})}{s^2} - e^{-10s}\left(\frac{15}{s}\right)$$

$$K(s) = \mathscr{L}[k(t)] = \mathscr{L}[g(t)] - \mathscr{L}[3.0(t-15)\,u(t-15)] + \mathscr{L}[3.0(t-20)\,u(t-20)]$$
$$= \frac{1.5(1-e^{-10s})}{s^2} - e^{-15s}\left(\frac{3.0}{s^2}\right) + e^{-20s}\left(\frac{3.0}{s^2}\right) = \frac{1.5(1-e^{-10s}-2e^{-15s}+2e^{-20s})}{s^2}$$

Inverse Laplace Transform

Use linearity to find f(t) given F(s)

Given the function G(s) = (7/s) – 31/(s + 17), obtain *g*(*t*).

This $\mathbf{s}$-domain function is composed of the sum of two terms, $7/\mathbf{s}$ and $-31/(\mathbf{s}+17)$. Through the linearity theorem we know that $g(t)$ will be composed of two terms as well, each the inverse Laplace transform of one of the two $\mathbf{s}$-domain terms:

$$g(t) = \mathcal{L}^{-1}\left\{\frac{7}{\mathbf{s}}\right\} - \mathcal{L}^{-1}\left\{\frac{31}{\mathbf{s}+17}\right\}$$

Let's begin with the first term. The homogeneity property of the Laplace transform allows us to write that

or linearity

$$\mathcal{L}^{-1}\left\{\frac{7}{\mathbf{s}}\right\} = 7\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}}\right\} = 7u(t)$$

Thus, we have made use of the known transform pair $u(t) \Leftrightarrow 1/\mathbf{s}$ and the homogeneity property to find this first component of $g(t)$. In a similar fashion, we find that $\mathcal{L}^{-1}\left\{\frac{31}{\mathbf{s}+17}\right\} = 31e^{-17t}u(t)$. Putting these two terms together,

$$g(t) = [7 - 31e^{-17t}]u(t)$$

Ratios of polynomials

We frequently want to find the inverse Laplace transform of a function that is represented as a ratio of polynomials in s, that is:

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

F(s) is a **rational function** in s because it is a ratio of two polynomials in s. Usually n>m, and hence F(s) is a **proper rational function**. If it is a rational function, we can decompose F(s) using a particular method (we do this in the next lecture).

POLES

The roots of the denominator are called **poles** and are found by solving this equation D(s)=0

Factoring D(s), we get:

$$D(s) = s^n + a_{n-1}s^{n-1} + \cdots \ldots . + a_1 s + a_0 = (s - p_1)(s - p_2) \ldots . . (s - p_n)$$

Poles and zeros

The poles p_i of F(s) can be real or complex. If they are complex, they occur in conjugate pairs.

That means if $p_1 = a+jb$ then $p^*_1 = a-jb$ is also a pole.

A **simple pole** is a pole that only occurs once i.e., no other pole are equal to it.

A **repeated pole** is a pole that occurs more than once.

The **multiplicity** of a repeated pole is the number of equal poles (including the original pole itself).

ZEROS

The roots of the numerator N(s) are called **zeros** of F(s).

The terms simple, repeated and multiplicity also apply to zeros.

Example

Calculate the inverse transform of $\mathbf{F}(\mathbf{s}) = 2\dfrac{\mathbf{s}+2}{\mathbf{s}}$.

Since the degree of the numerator is equal to the degree of the denominator, $\mathbf{F}(\mathbf{s})$ is *not* a rational function. Thus, we begin by performing long division:

$$\mathbf{F}(\mathbf{s}) = \mathbf{s}\overline{\smash{\big)}\,2\mathbf{s}+4\,}^{\;2} \qquad \begin{array}{r} 2\mathbf{s} \\ \hline 4 \end{array}$$

so that $\mathbf{F}(\mathbf{s}) = 2 + (4/\mathbf{s})$. By the linearity theorem,

$$\mathscr{L}^{-1}\{\mathbf{F}(\mathbf{s})\} = \mathscr{L}^{-1}\{2\} + \mathscr{L}^{-1}\left\{\frac{4}{\mathbf{s}}\right\} = 2\delta(t) + 4u(t)$$

(It should be noted that this particular function can be simplified without the process of long division; such a route was chosen to provide an example of the basic process.)