Control Systems

Chapter 4: Time Response

Highlights

Original slides courtesy of Dr Branislav Hredzak

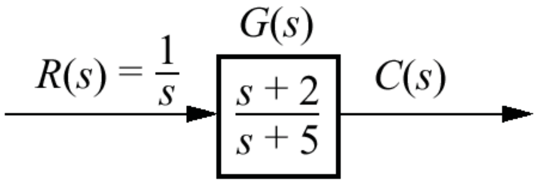
Time Response

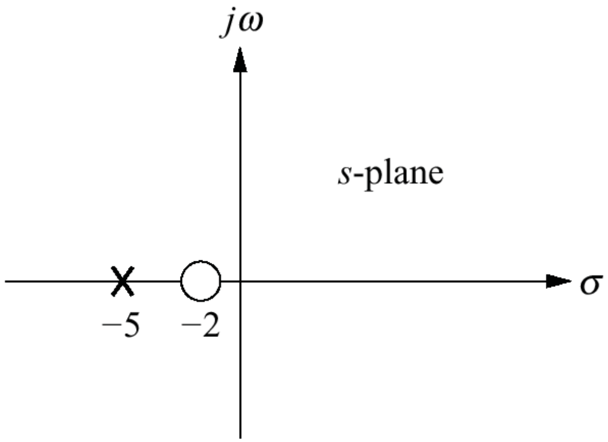
- How to find the time response from the transfer function
- How to use poles and zeros to determine the response of a control system
- How to describe quantitatively the transient response of first- and second order systems
- How to approximate higher-order systems as first or second order
- How to find the time response from the state-space representation

Poles, Zeros, and System Response

- The output response of a system is the sum of two responses: the ***forced response*** and the ***natural response*** (Chapter 1).
- The ***poles*** of a transfer function are the values of the Laplace transform variable, *s*, that cause the transfer function to become infinite.
- The ***zeros*** of a transfer function are the values of the Laplace transform variable, *s*, that cause the transfer function to become zero.

Poles and Zeros of a First-Order System (1)





Unit step response is given by:

$$C(s) = \frac{(s+2)}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}$$

$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s \to 0} = \frac{2}{5}$$

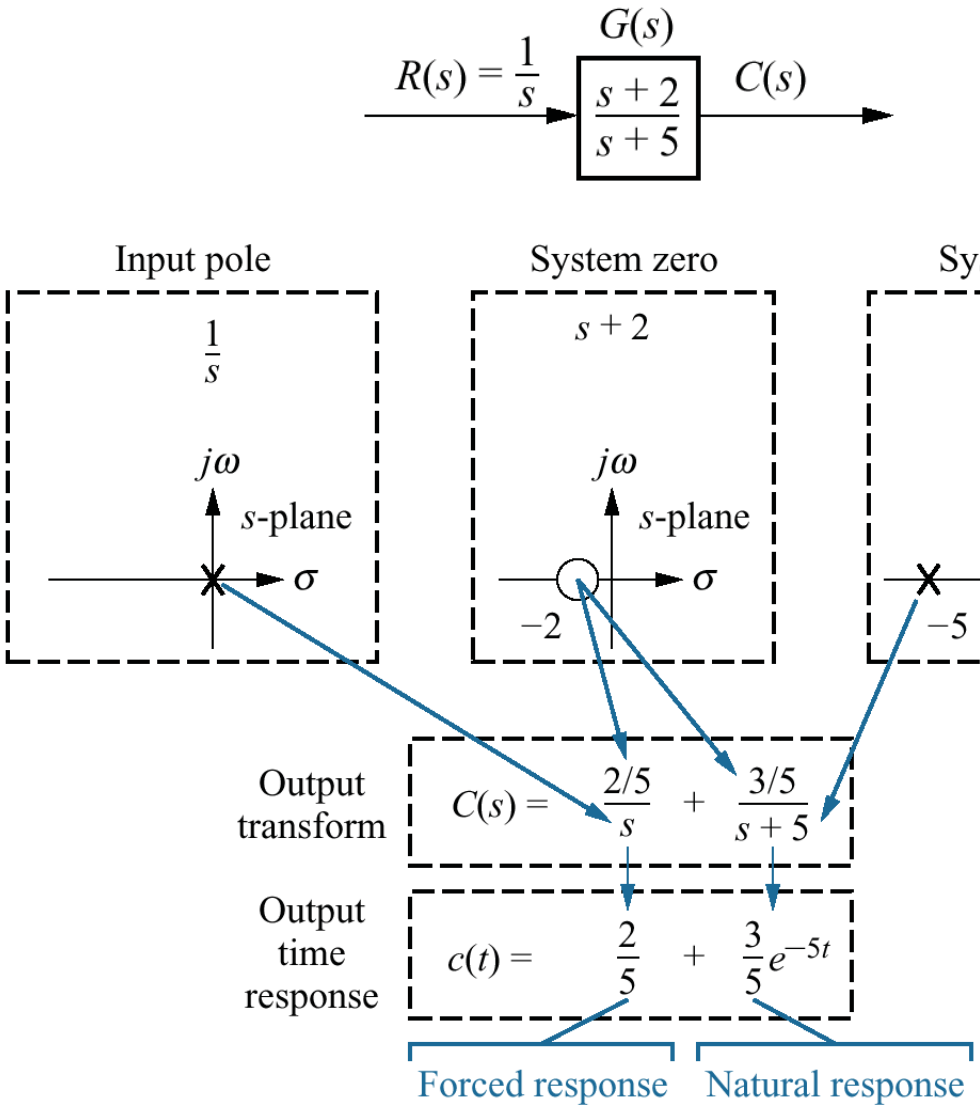
$$B = \left.\frac{(s+2)}{s}\right|_{s \to -5} = \frac{3}{5}$$

$$C(s) = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$c(t) = \frac{2}{5} + \frac{3}{5}e^{-5t}$$

$$\mathcal{L}^{-1}\left[\frac{2/5}{s}\right] = \frac{2}{5} \cdot u(t) \qquad \mathcal{L}^{-1}\left[\frac{3/5}{s+5}\right] = \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] = \frac{3}{5}e^{-5t}$$

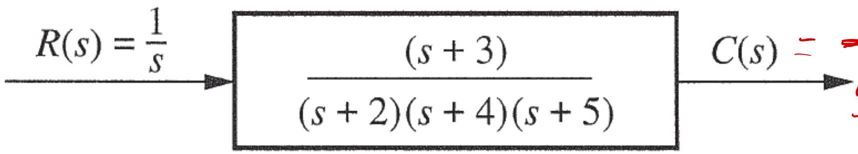
Poles and Zeros of a First-Order System (2)



$$A = \left.\frac{(s+2)}{(s+5)}\right|_{s \to 0} = \frac{2}{5}$$

$$B = \left.\frac{(s+2)}{s}\right|_{s \to -5} = \frac{3}{5}$$

Example: Given the system below, specify the natural and forced parts of the output $c(t)$



$$= \frac{(s+3)}{s(s+2)(s+4)(s+5)}$$

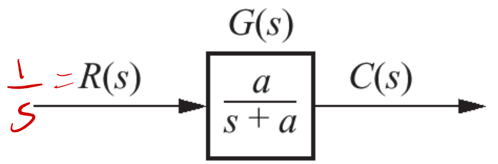
$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\substack{\text{Forced}\\ \text{response}}} + \underbrace{\frac{K_2}{(s+2)} + \frac{K_3}{(s+4)} + \frac{K_4}{(s+5)}}_{\substack{\text{Natural}\\ \text{response}}}$$

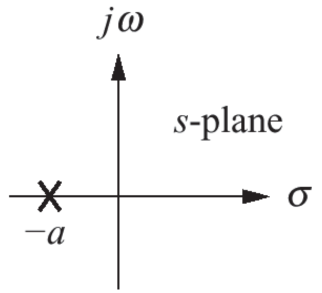
$$c(t) \equiv \underbrace{K_1}_{\substack{\text{Forced}\\ \text{response}}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\substack{\text{Natural}\\ \text{response}}}$$

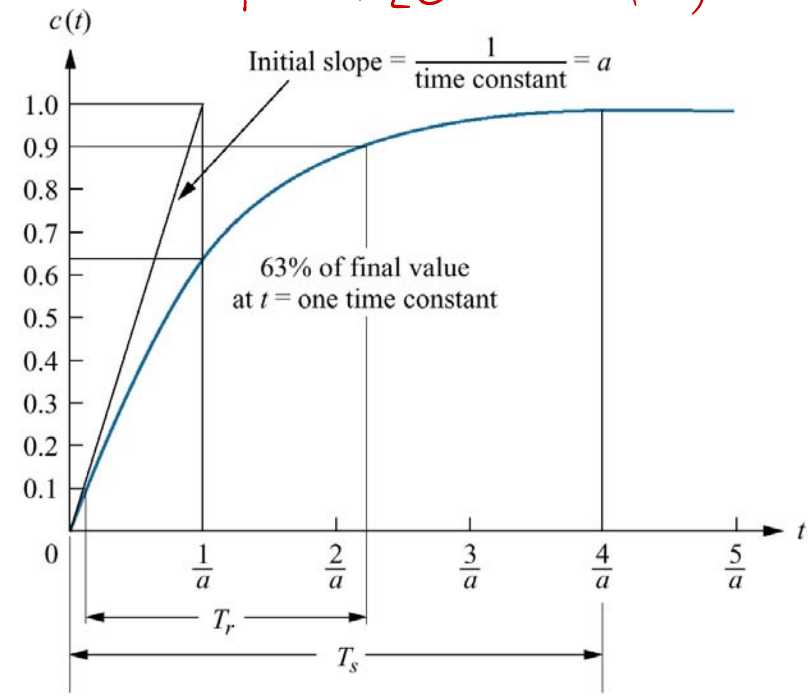
First-Order Systems

$$\frac{K_1}{s} + \frac{K_2}{s+a} \quad (s)$$

$$K_1 + K_2 e^{-at} \quad (t)$$







Step response: $C(s) = R(s)G(s) = \dfrac{a}{s(s+a)}$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

When $t = 1/a$, $\quad e^{-at}\big|_{t=1/a} = e^{-1} = 0.37$

$$c(t)\big|_{t=1/a} = 1 - e^{-at}\big|_{t=1/a} = 1 - 0.37 = 0.63$$

First-Order Systems

Time constant of the system: $1/a$, it is the time it takes for the step response to rise to 63% of its final value

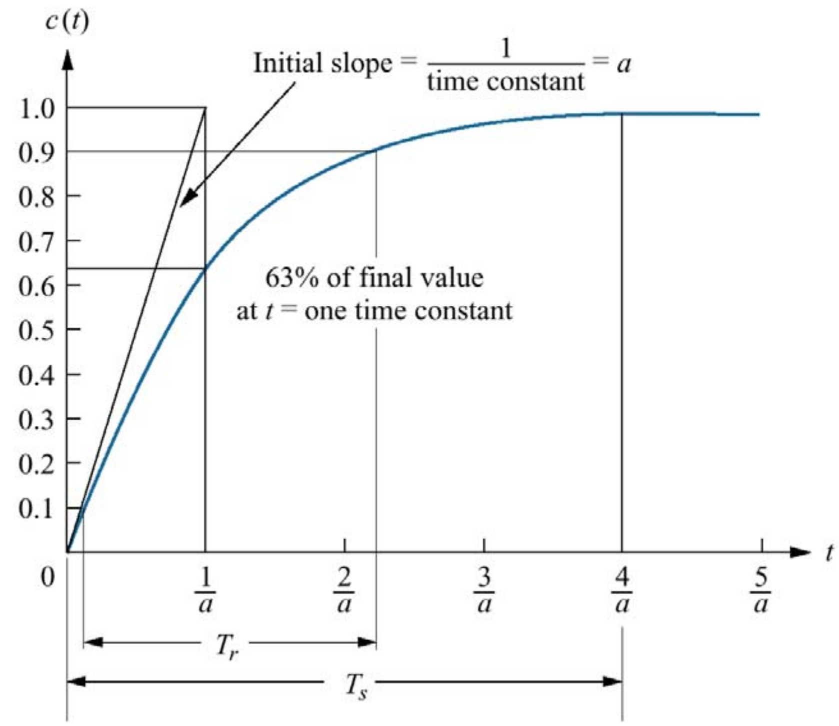
Rise time, T_r : time for the waveform to go from 0.1 to 0.9 of its final value

$$c(t) = 1 - e^{-at}$$

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

Settling time, T_s : time for the response to reach, and stay within, 2% of its final value

$$T_s = \frac{4}{a}$$



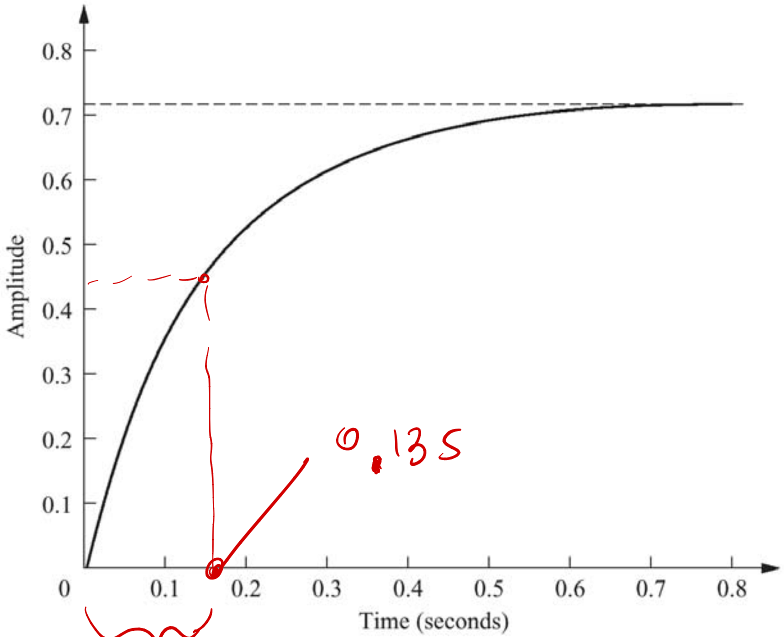
First-Order Transfer Functions via Testing

With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

Let first order system be $G(s) = K/(s+a)$

$G(s) = \frac{a}{s+a}$

and the step response be $C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{(s+a)}$



$T_c = \frac{1}{a} \Rightarrow a = \frac{1}{0.13}$

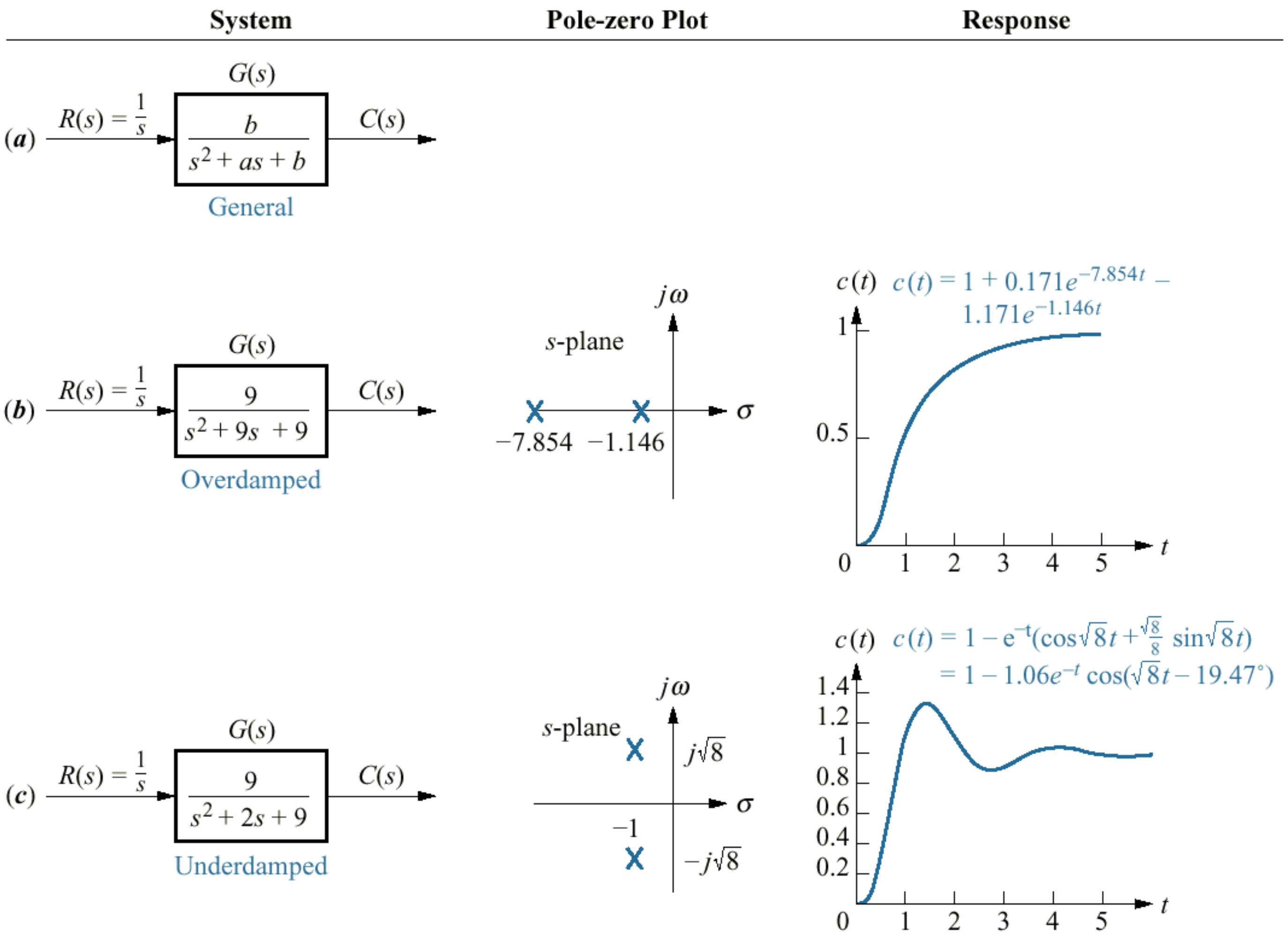
1. Final value = 0.72 (Measure)
2. 63 % of final value = 0.63×0.72 = 0.45
3. Curve reaches 0.45 at 0.13 s, hence $a = 1/0.13 = 7.7$
4. Steady state value $K/a = 0.72$, hence $K = 5.54$

$K = 0.72 \times 7.7$

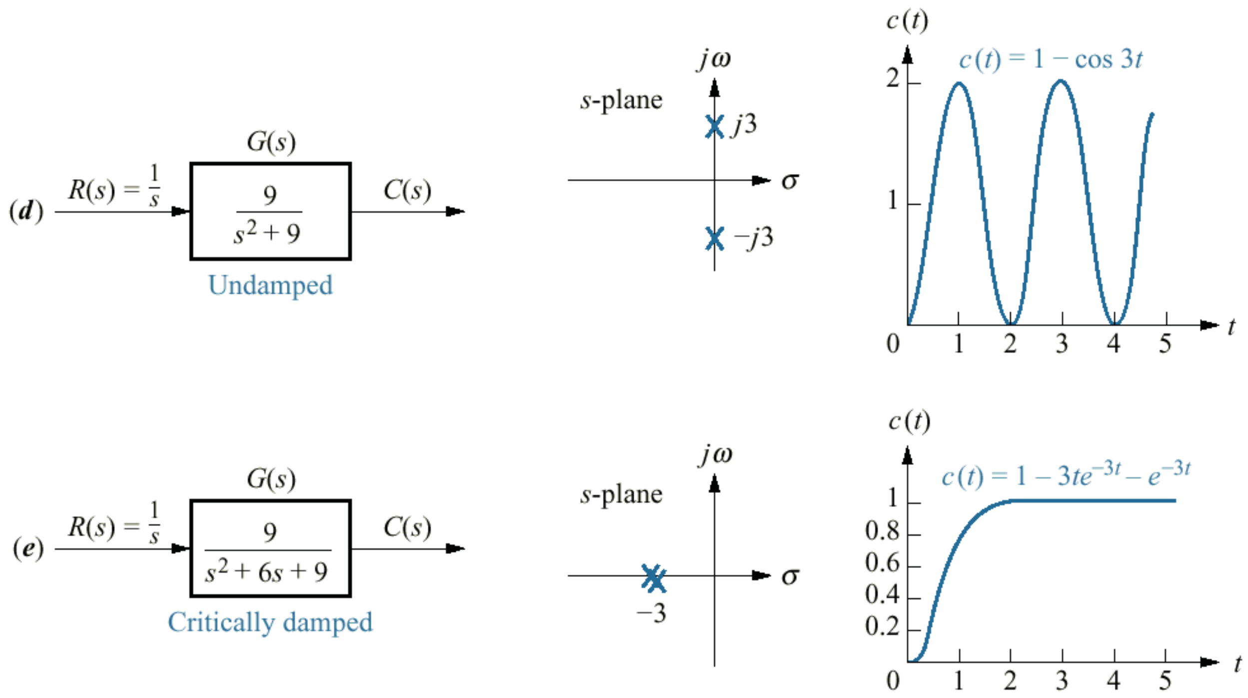
$$\boxed{G(s) = \frac{5.54}{s+7.7}}$$

Second-Order Systems: Introduction

- a second-order system exhibits a wide range of responses



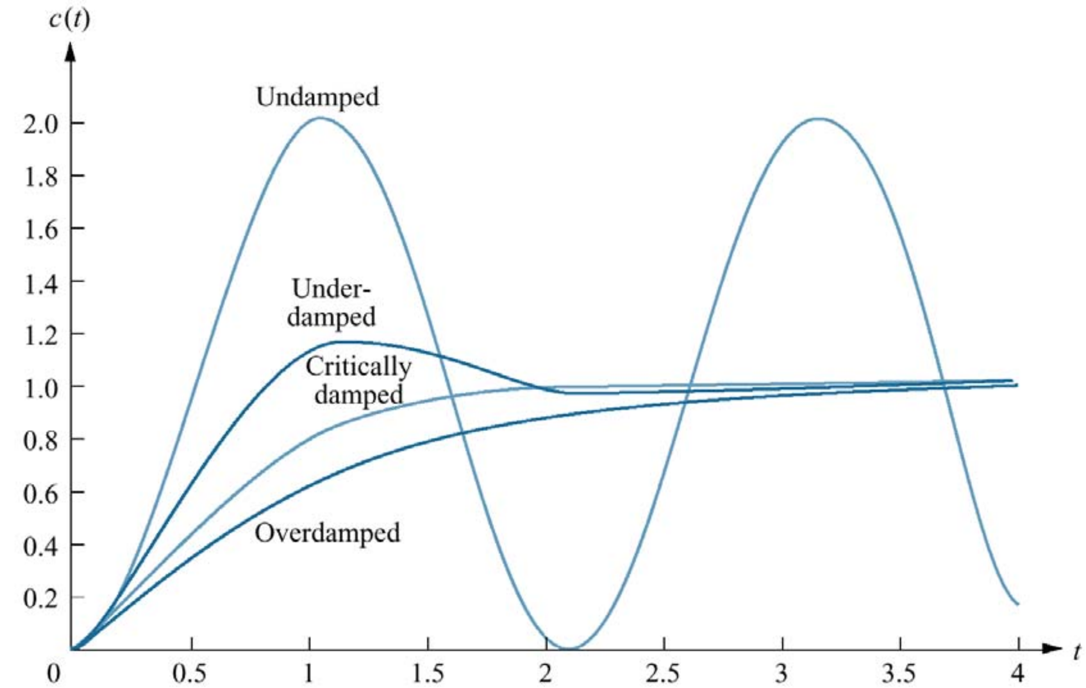
Second-Order Systems: Introduction



Summary of Second-Order Systems (1)

There are four possible responses (when $a \geq 0$ and $b > 0$):

1. Overdamped responses
2. Underdamped responses
3. Undamped response
4. Critically damped responses



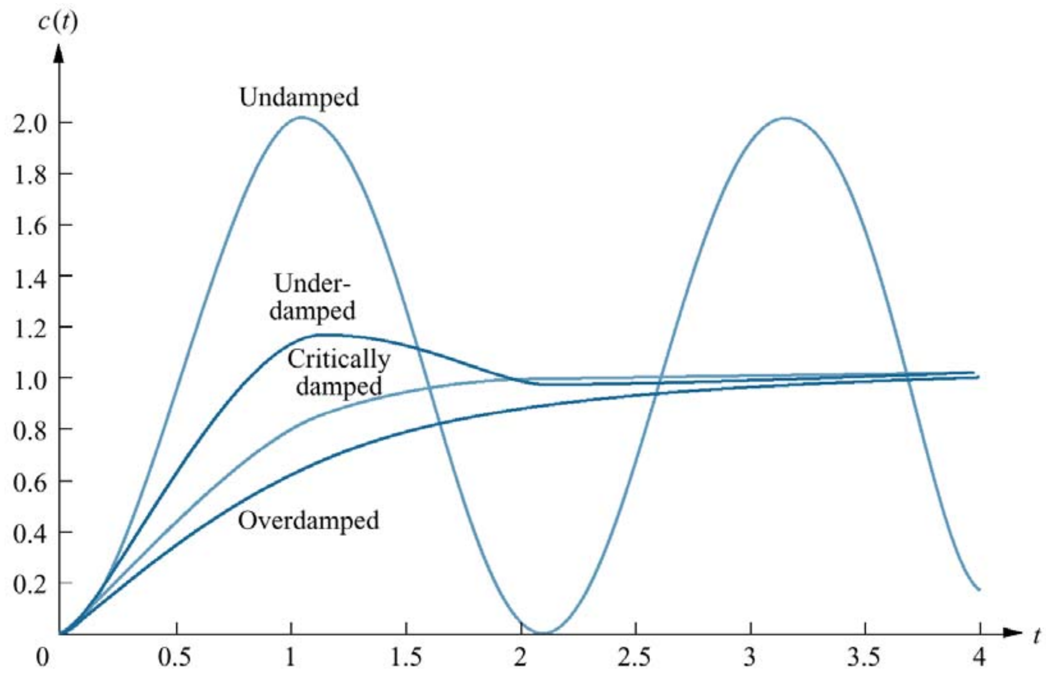
Summary of Second-Order Systems (2)

1. *Overdamped responses*

Poles: Two real at $-\sigma_1, -\sigma_2$ $\sigma_1 \neq \sigma_2$ $\sigma_1, \sigma_2 > 0$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$



Summary of Second-Order Systems (3)

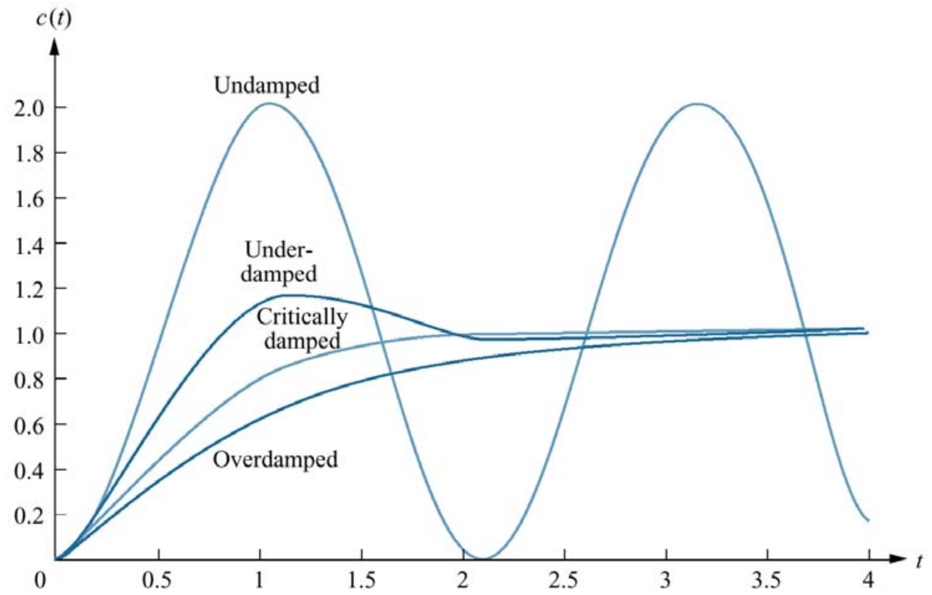
2. *Underdamped responses*

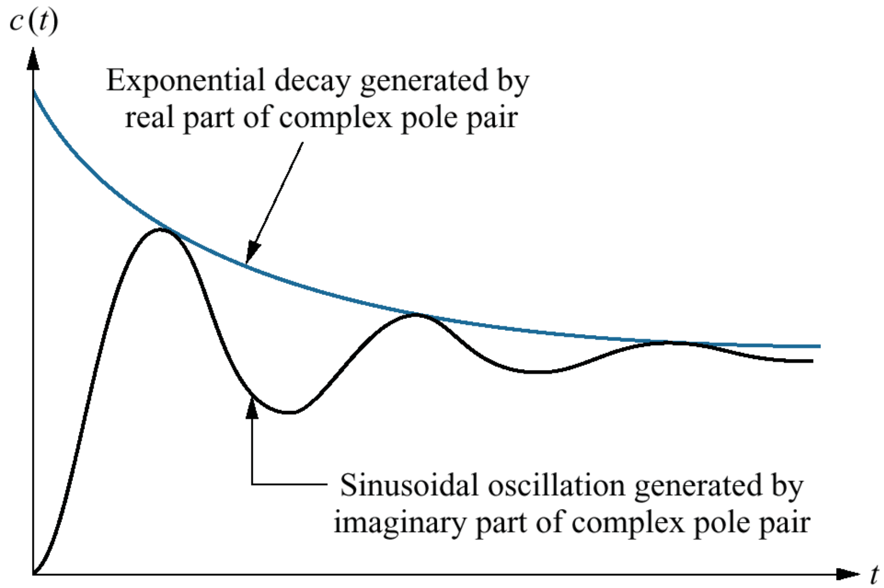
Poles: Two complex at $-\sigma_d \pm j\omega_d$ $\quad \sigma_d > 0$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

ω_d … damped frequency of oscillation





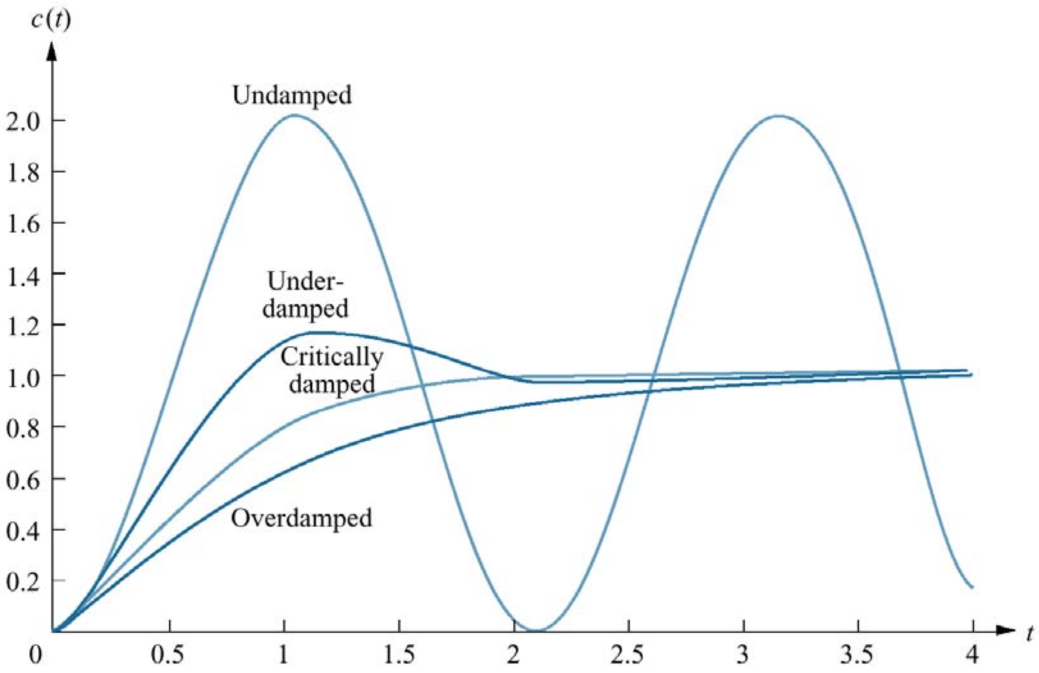
Summary of Second-Order Systems (4)

3. *Undamped responses*

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$



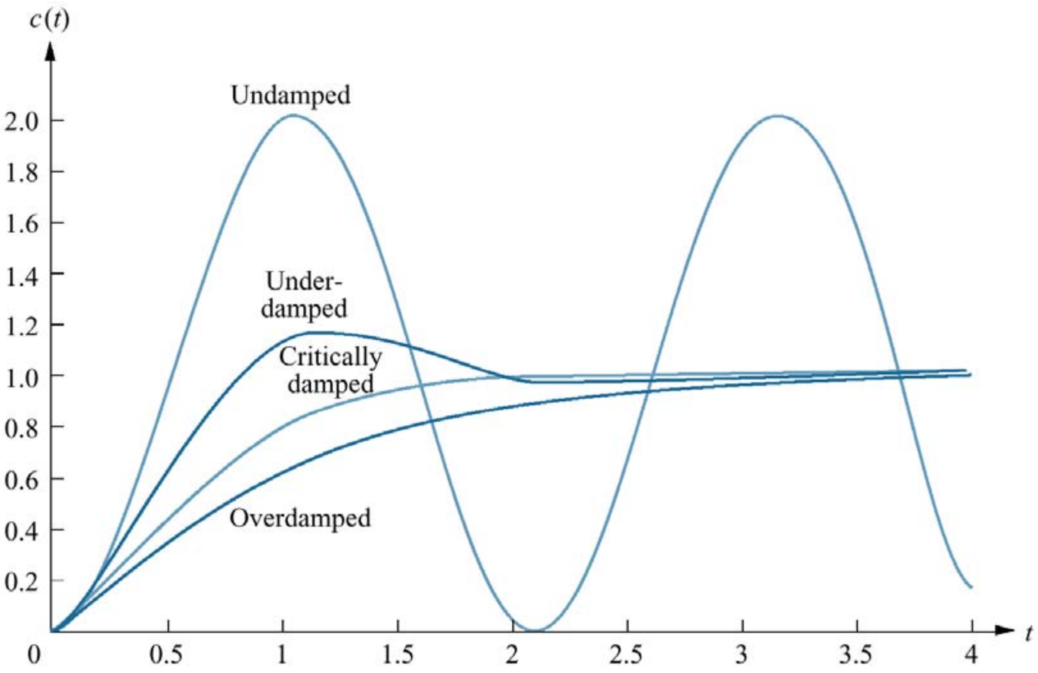
Summary of Second-Order Systems (5)

4. *Critically damped responses*

Poles: Two real at $-\sigma_1$ $\sigma_1 > 0$

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$



The General Second-Order System (1)

Natural Frequency, ω_n
- the frequency of oscillation of the system without damping.

Damping Ratio, ζ

$$\boxed{\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \frac{1}{2\pi}\frac{\text{Natural period (seconds)}}{\text{Exponential time constant}}}$$

$$\omega_n = 2\pi f_n = \frac{2\pi}{T_n}$$

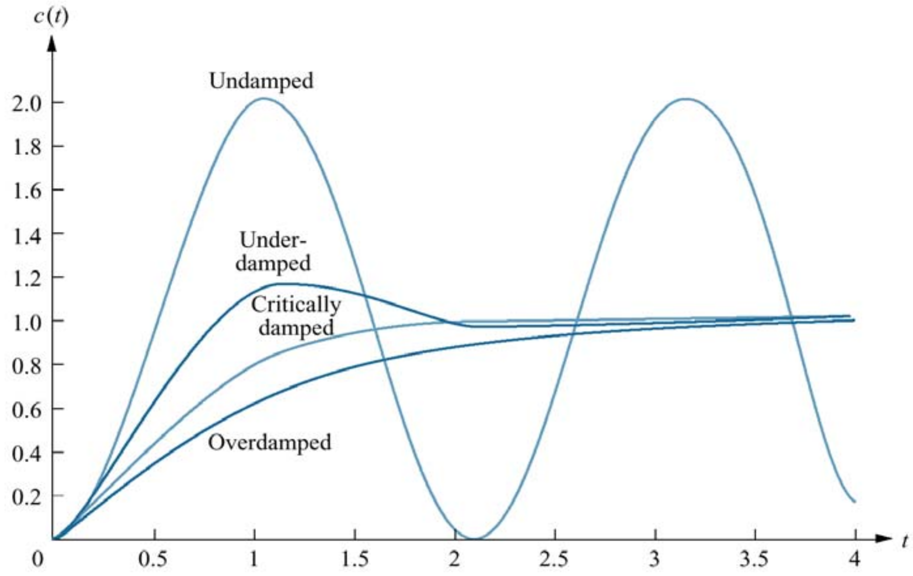
Derivation of the second order system in terms of ω_n and ζ

Consider $G(s) = \dfrac{b}{s^2 + as + b}$

Let us determine what is the term a and the term b in terms of ω_n and ζ.

$$G(s) = \frac{b}{s^2 + as + b}$$

Without damping (a =0) the poles are on the imaginary axis and $G(s) = \dfrac{b}{s^2 + b}$



Then $\omega_n = \sqrt{b}$

$$b = \omega_n{}^2$$

Now, let us assume an *underdamped system*. The real part of the complex pole is $\sigma = - a/2$ and it determines the exponential decay frequency.

Hence $\zeta = \dfrac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}} = \dfrac{|\sigma|}{\omega_n} = \dfrac{a/2}{\omega_n} \longrightarrow a = 2\zeta\omega_n$

$$\begin{cases} b = \omega_n^2 \\ a = 2\xi\omega_n \end{cases}$$

Then, the general second-order transfer function is

$$\boxed{G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}}$$

and the poles of this transfer function are:

$$s_{1,2} = \frac{-2\xi\omega_n \pm \sqrt{(\xi\omega_n)^2 - 4\omega_n^2}}{2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

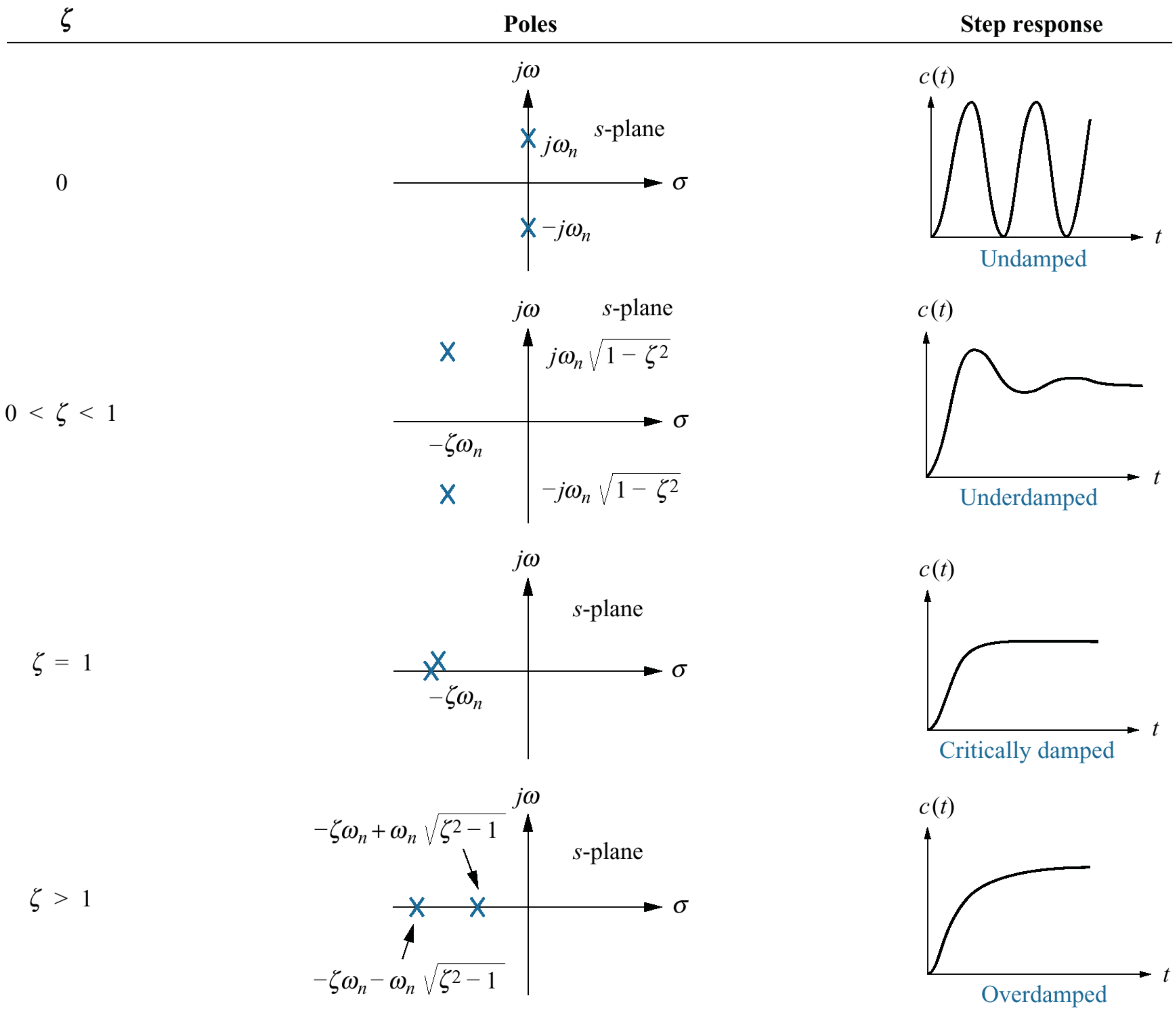
$$G(s) = \frac{b}{s^2 + as + b}$$

$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm \frac{\sqrt{a^2-4b}}{2} = \underbrace{-\frac{a}{2}}_{\mathbb{R}e(s_{1,2})} \pm j(\ldots)$$

$$c_n(t) = A e^{-\sigma t}(\cos \omega_d t + \phi)$$

$$\sigma = \frac{a}{2}$$

$$\boxed{s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$



Underdamped Second-Order Systems (1)

- represents a common model for physical problems

Step response: $$C(s) = \frac{\omega_n{}^2}{s(s^2 + 2\zeta\omega_n s + \omega_n{}^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

Expanding by partial fraction we obtain ($0 < \zeta < 1$):

$$C(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n{}^2(1-\zeta^2)}$$

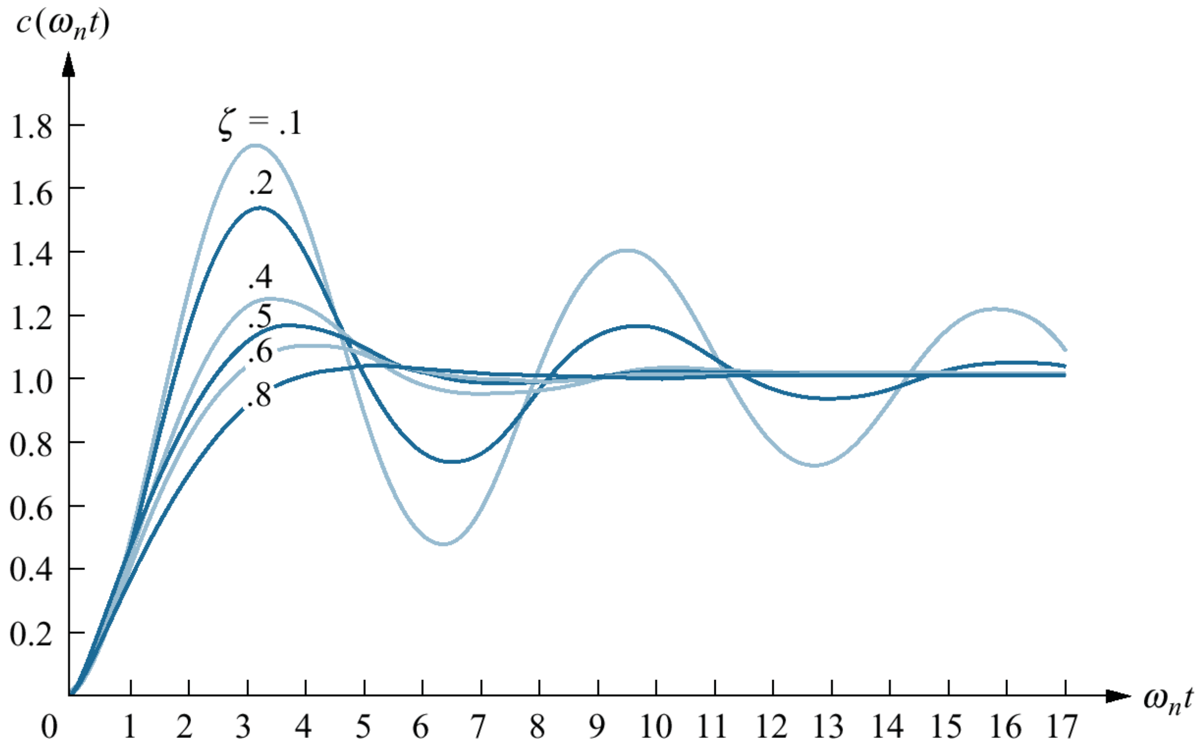
Then in time domain:

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right)$$

$$\text{where } \phi = \tan^{-1}(\zeta/\sqrt{1-\zeta^2})$$

Underdamped Second-Order Systems (2)



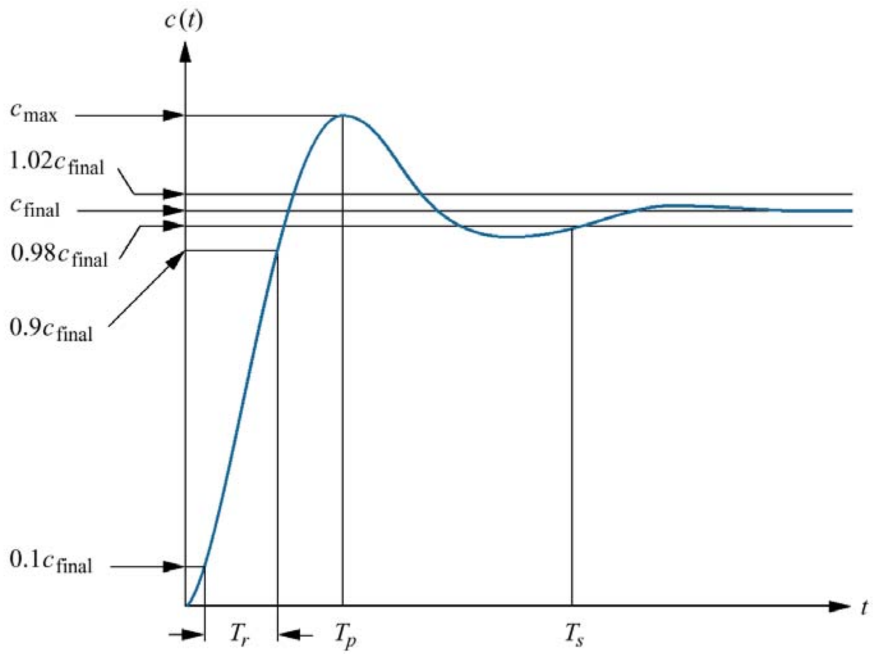
$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos \omega_n \sqrt{1-\zeta^2}t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_n \sqrt{1-\zeta^2}t\right)$$

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}t - \phi\right)$$

Underdamped Second-Order Systems (3)

Specifications:

1. *Rise time*, T_r. The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
2. *Peak time*, T_p. The time required to reach the first, or maximum, peak.
3. *Percent overshoot*, *%OS*. The amount that the waveform overshoots the steady-state, or final, value at the peak time, expressed as a percentage of the steady-state value.
4. *Settling time*, T_s. The time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.

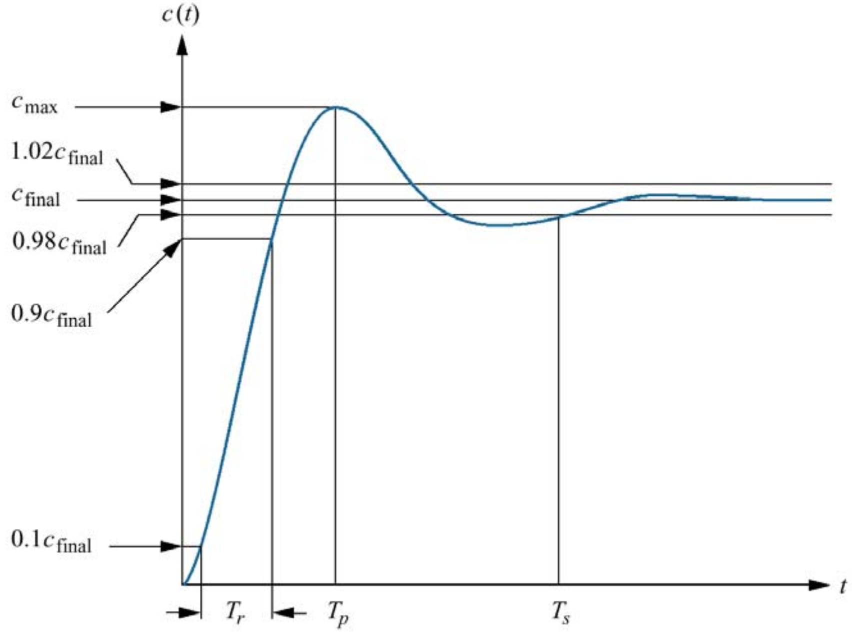


Underdamped Second-Order Systems (4)

Evaluation of T_p,

T_p is found by differentiating $c(t)$ and finding the first zero crossing after $t = 0$.

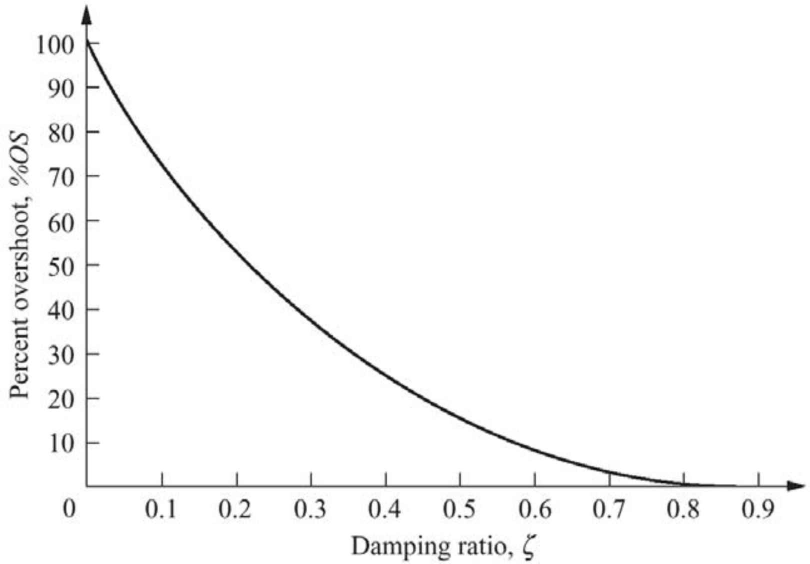
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{W_d}$$



Evaluation of % OS

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



$S_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

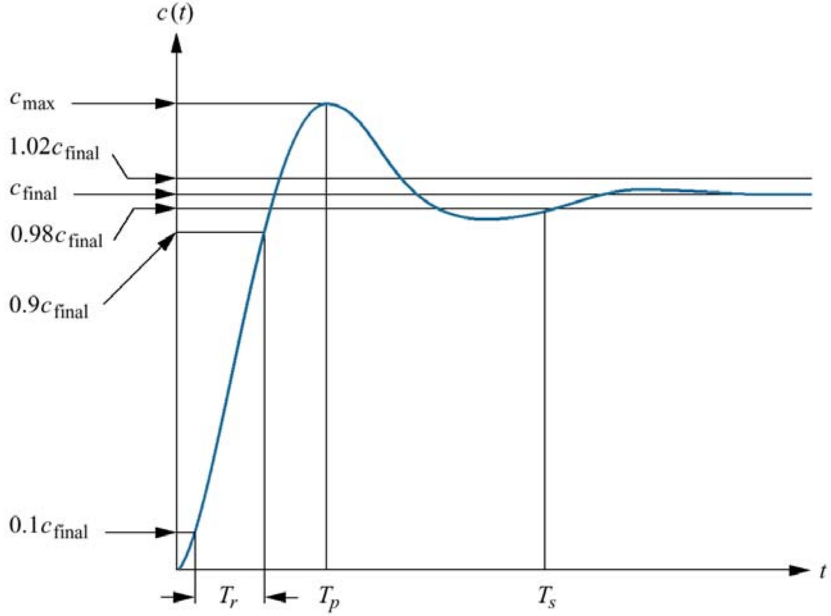
Underdamped Second-Order Systems (5)

Evaluation of T_s

$$e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}} = 0.02$$

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

$$T_s \approx \frac{4}{\zeta\omega_n}$$



Evaluation of T_r

- cannot be found analytically

Problem: Find T_p, $\%OS$, T_s, and T_r for the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

From transfer function:
$\omega_n = \sqrt{100} = 10$
$\xi = 15/(2\omega_n) = 0.75.$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.475$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 2.838$$

$$T_s \approx \frac{4}{\zeta\omega_n} = 0.533$$

Normalised rise time corresponding to ξ

$$T_r \approx \frac{2.3}{\omega_n} = 0.23$$

Underdamped Second-Order Systems (6)

- now we will evaluate peak time, settling time and overshoot **in terms of the pole location**

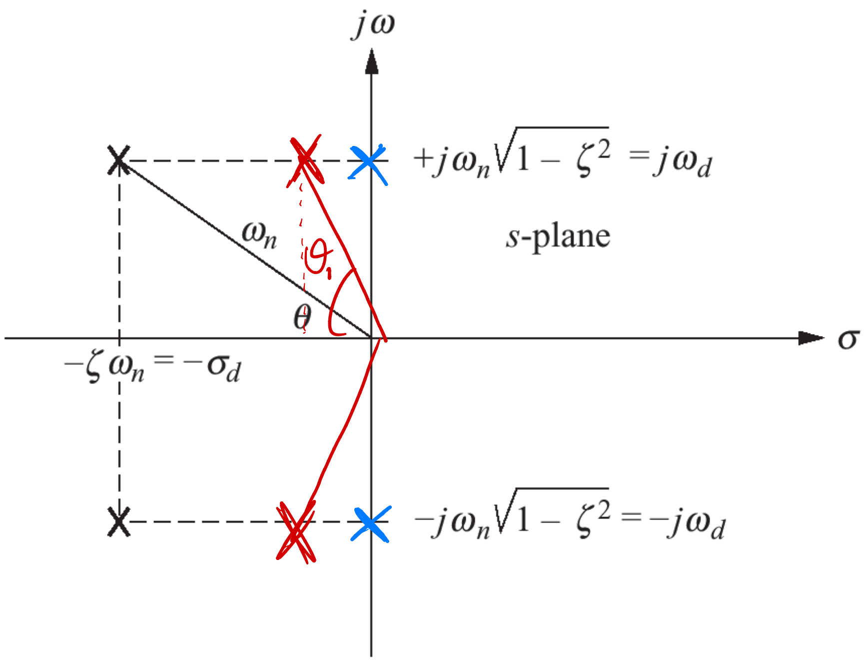
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

From the Pythagorean theorem: $\cos\theta = \zeta$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$



ω_d … *damped frequency of oscillation*
σ_d … *exponential damping frequency*

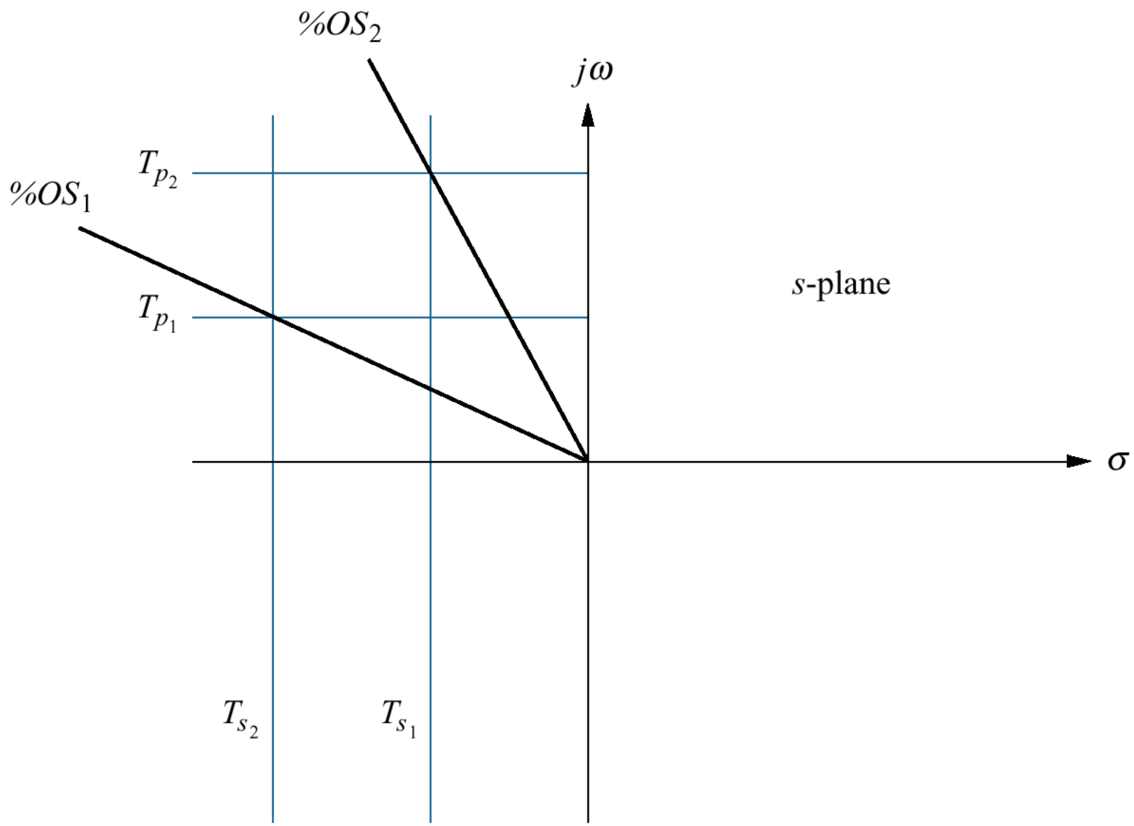
Underdamped Second-Order Systems (7)

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

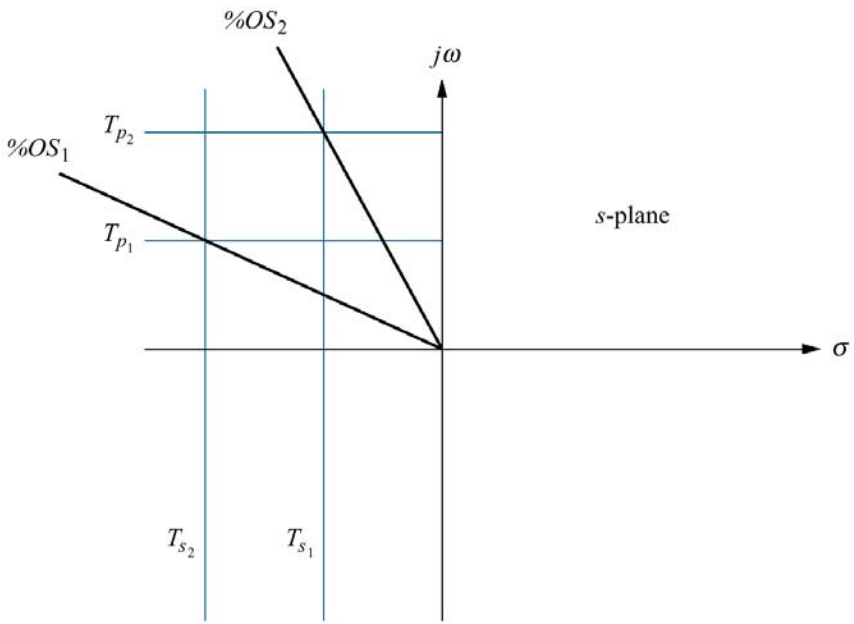
$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

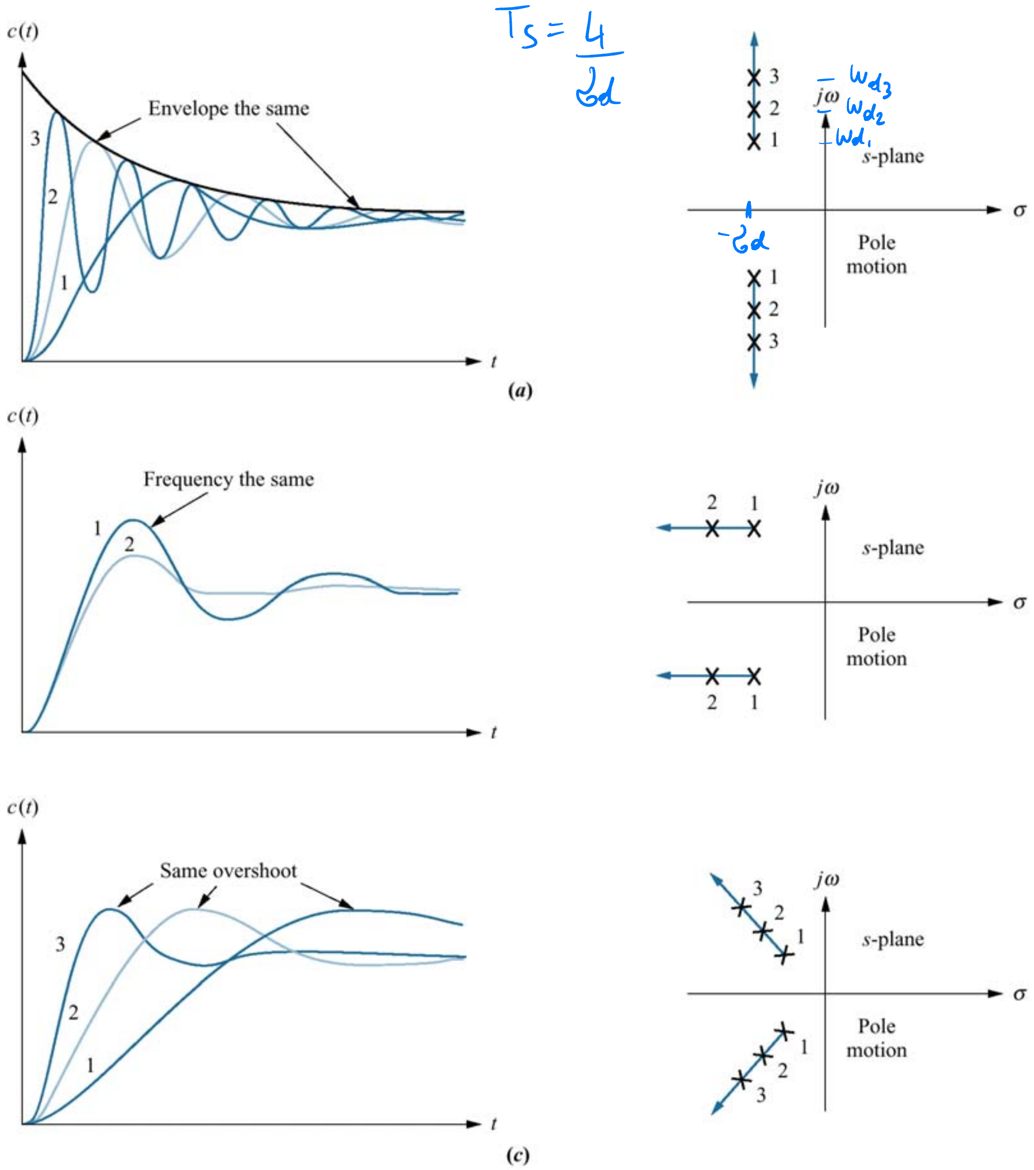
$$\cos\theta = \zeta$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100$$

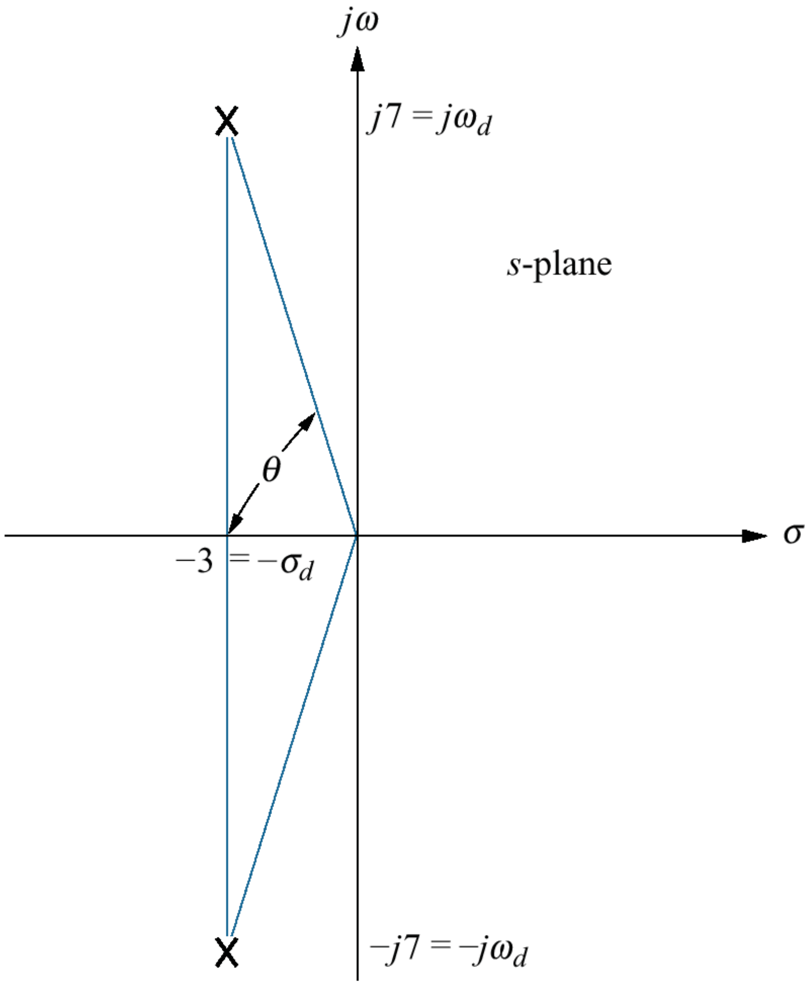


Underdamped Second-Order Systems (8)





Problem: Given the plot, find ζ, ω_n, T_p, $\%OS$ and T_s



Solution:

$$\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$$

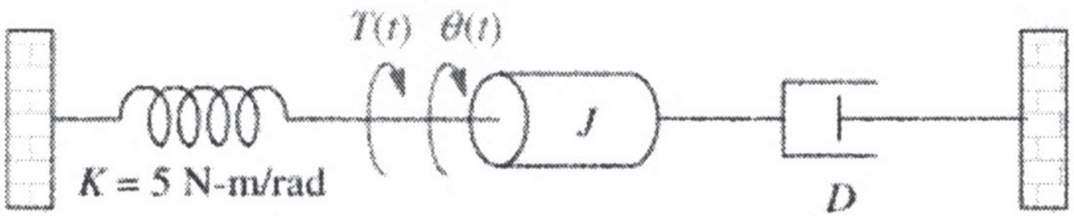
$$\omega_n = \sqrt{7^2 + 3^2} = 7.62$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second}$$

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\%$$

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds}$$

Example 4.7 Find J and D to yield a 20% overshoot and $T_s = 2$ s for step input torque $T(t)$



$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$2\zeta\omega_n = \frac{D}{J}$$

$$\%OS = 20 \Rightarrow \zeta = 0.456$$

$$T_s = 2 \approx \frac{4}{\zeta\omega_n} \Rightarrow \sqrt{\frac{K}{J}} = \frac{2}{\zeta} \Rightarrow J/K = 0.052 \Rightarrow J = 0.26$$

$$D = 2J\zeta\omega_n = 2\zeta\sqrt{KJ} = 1.04$$

System Response with Additional Poles (1)

- If a system has **more than two poles or has zeros**, we cannot use the previously derived formulae to calculate the performance specifications

- However, **under certain conditions**, a system with more than two poles or with zeros **can be approximated as a second-order system** that has just two complex **dominant poles**

System Response with Additional Poles (2)

- Let us consider a system with complex poles and a real pole

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_3 = -\alpha_r$$

- Then the step response can be found using partial fraction expansion

$$C(s) = \frac{A}{s} + \frac{B(s + \zeta\omega_n) + C\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s + \alpha_r}$$

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

- Assume that $c(t) \to 1$ as $t \to \infty$ then as $\alpha_r \to \infty$

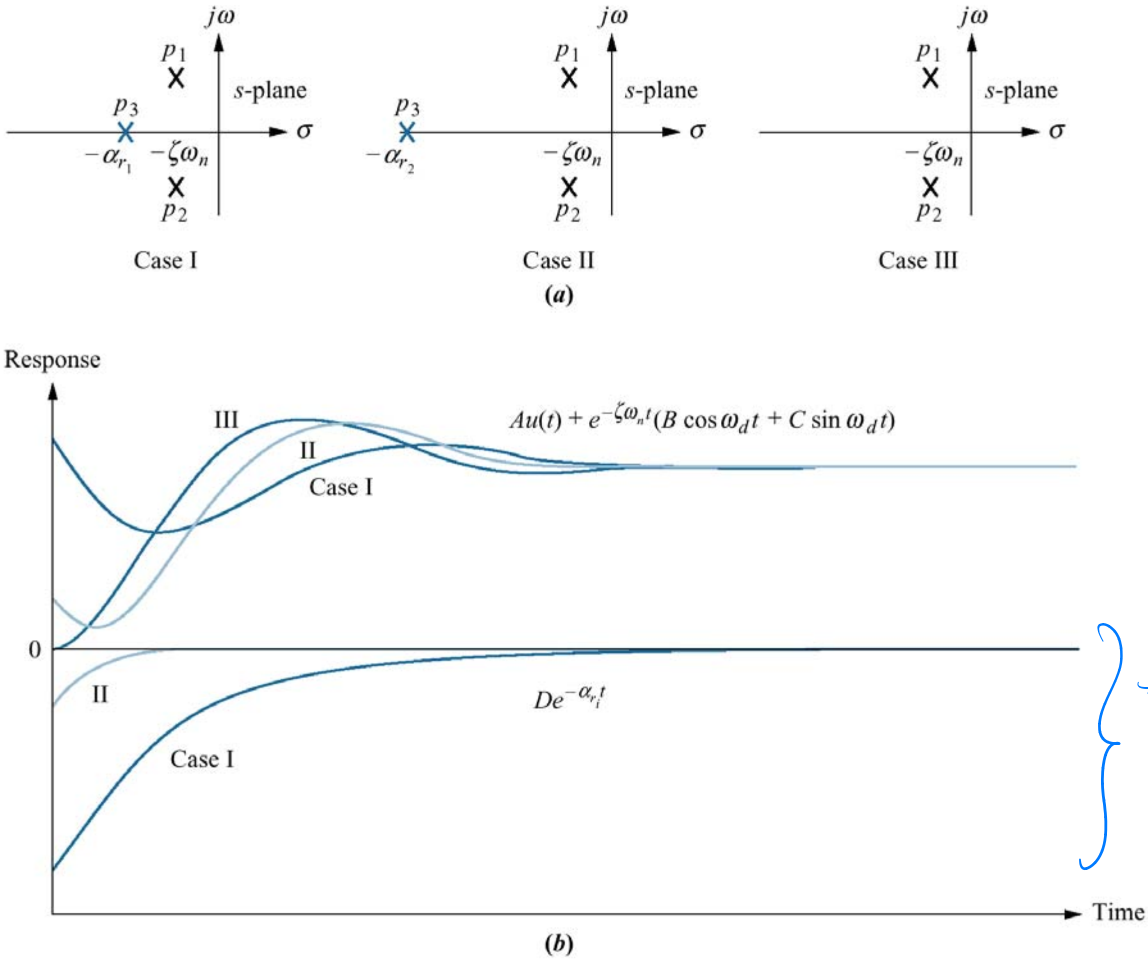
$$A = 1;\ B \to -1$$

$$C \to -\frac{\zeta}{\sqrt{1-\zeta^2}};\ D \to 0$$

$$\Rightarrow \quad C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

System Response with Additional Poles (3)

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$



third pole only

- We can assume that the exponential decay is negligible after five time constants.

Example 4.8 Find the step response of the transfer functions below and compare them

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542}$$

$$T_2(s) = \frac{245.42}{(s+10)(s^2 + 4s + 24.542)}$$

$$T_3(s) = \frac{73.626}{(s+3)(s^2 + 4s + 24.542)}$$

Solution: Taking the Laplace inverse transform of $C_i(s) = T_i(s)R(s)$ for $R(s) = s^{-1}$ And $i = 1, 2, 3$ gives

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8°)$$

$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°)$$

$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°)$$

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t - 23.8^\circ)$$
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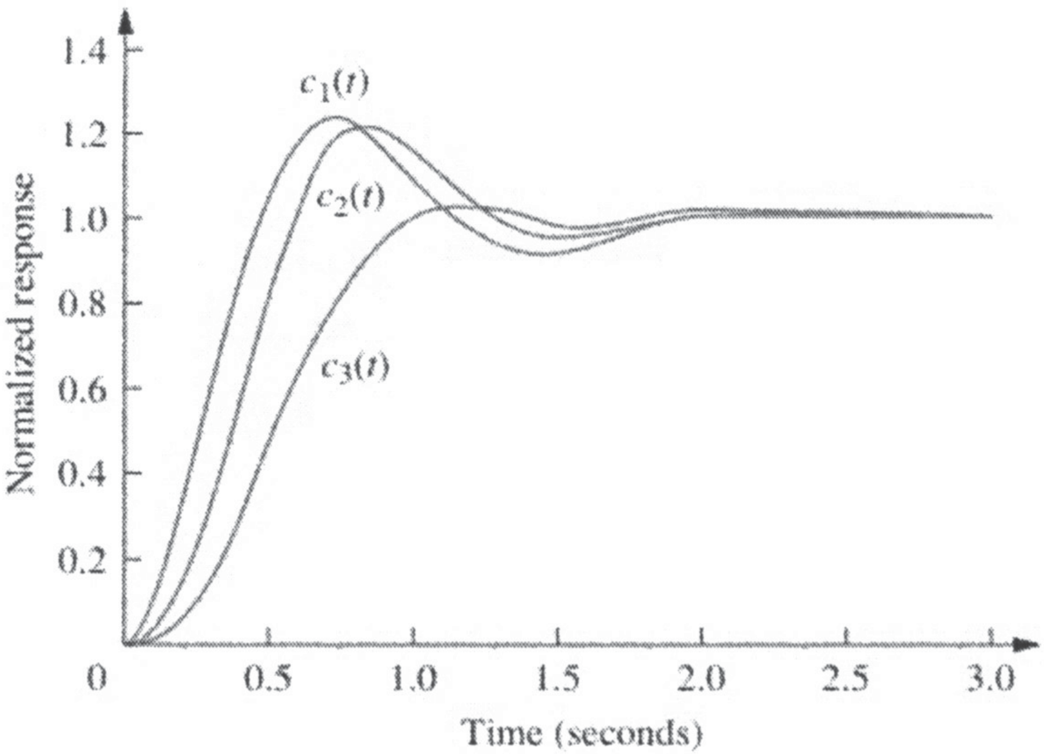


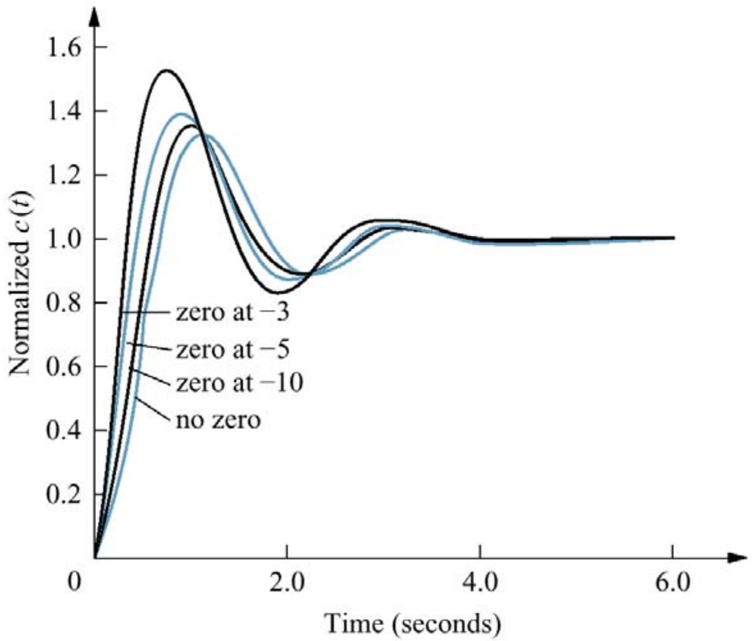
System Response with Zeros (1)

$$T(s) = \frac{(s+a)}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

$$= \frac{(-b+a)/(-b+c)}{s+b} + \frac{(-c+a)/(-c+b)}{s+c}$$

If $a >> b,c$: $\quad T(s) \approx a\left[\frac{1/(-b+c)}{s+b} + \frac{1/(-c+b)}{s+c}\right] = \frac{a}{(s+b)(s+c)}$

- **zero looks like a simple gain factor and does not change the relative amplitudes of the components of the response.**



System Response with Zeros (2)

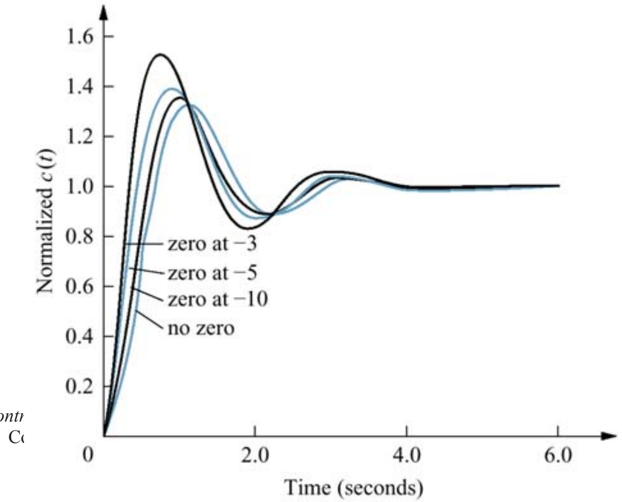
Different approach to investigate the effect of a zero:

- Let $C(s)$ be the response of a system, $T(s)$, with unity in the numerator
- If we add a zero to the transfer function, yielding $(s + a)T(s)$, the Laplace transform of the response will be

$$\boxed{(s+a)C(s) = sC(s) + aC(s)}$$

If a **is very large and positive**, then the response is approximately a scaled version of the original response

If a **is small and positive**, then the derivative term has a greater effect on the response – **more overshoot**



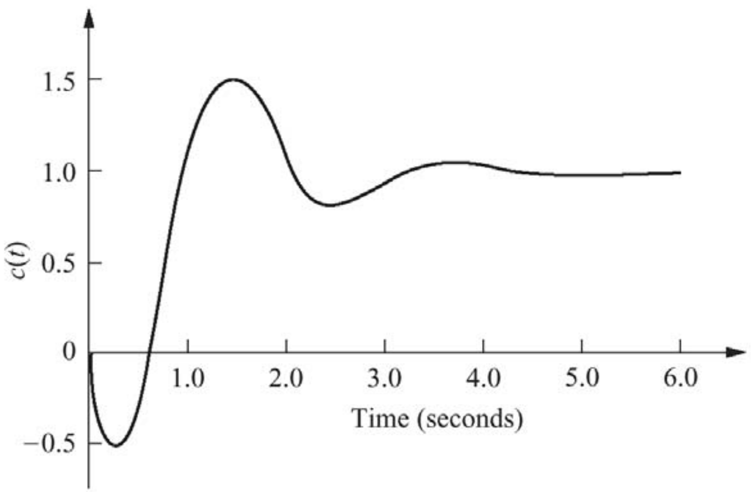
$(s+a) \rightarrow s = -a$
(zero)

System Response with Zeros (3)

$$(s+a)C(s) = sC(s) + aC(s)$$

$(s-a)$
zero is at $s = +a$

If ***a* is negative (in the right haf-plane)**, then the derivative term will be of opposite sign from the scaled response term



If the response begins to turn towards the negative direction even though the final value is positive then the system is known as ***nonminimum-phase system* - (*zero is in the right half-plane*)**.

Pole-zero cancellation

Consider:

$$T(s) = \frac{K\cancel{(s+z)}}{\cancel{(s+p_3)}(s^2+as+b)}$$

- If the zero at $-z$ is very close to the pole at $-p_3$ then it can be shown using partial fraction expansion of T(s) that the residue of the pole at –**p3 is much smaller than any of the other residues**

Example 4.7 For each of the response functions below determine whether there is an approximate pole-zero cancellation and if so, find an approximate second order response.

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

Solution

$$C_1(s) = \frac{1}{s} - \frac{3.5}{s+5} + \frac{3.5}{s+6} - \frac{1}{s+3.5}$$

No approximate cancellation

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$

$$C_2(s) = \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6} + \frac{0.033}{s+4.01}$$

Approximate cancellation, coefficient of the last term
is at least an order of magnitude smaller than the others

$$C_2(s) \approx \frac{0.87}{s} - \frac{5.3}{s+5} + \frac{4.4}{s+6}$$

$$c_2(t) \approx 0.87 - 5.3e^{-5t} + 4.4e^{-6t}$$

(Approximate overdamped second order response)