Fourier Transform

Week 5

Fourier Transform

$f(t)$ time-domain signal $\xrightarrow{\mathcal{F}}$ $F(\omega)$ frequency domain signal

$F(\omega)$ is a signal in the frequency domain ie its a function of frequency.

What does a function of ω mean?

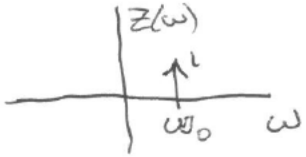
As an example of a function of ω, consider first $y(t) = \exp(j\omega t)$ which is a function of t, for a fixed value of ω.

For a fixed value of t,

$Z(\omega) = \exp(j\omega t)$ is a function of ω.

Another example is $Z(\omega) = \delta(\omega - \omega_0)$

$Z(\omega)$ here is a unit impulse signal in the frequency domain. The spike is located at ω_0 rad/sec.



We can integrate $Z(\omega)$ just as before:

eg $\int_{-\infty}^{\infty} Z(\omega)d\omega = 1$

$\int_{\omega_0-1}^{\omega_0+1} Z(\omega)d\omega = 1$

$\int_{\omega_0+1}^{\omega_0+2} Z(\omega)d\omega = 0$

Lets start with the inverse Fourier Transform equation:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(j\omega t)\,d\omega$$

This takes $F(\omega)$ & tells us how to get $f(t)$

$$\underset{\text{frequency domain}}{F(\omega)} \xrightarrow{\mathcal{F}^{-1}} \underset{\text{time domain}}{f(t)}$$

$\mathcal{F}^{-1}$: inverse Fourier Transform

In general, $F(\omega) \in \mathbb{C}$ for all ω

& we can write $F(\omega) = |F(\omega)|\exp(j\phi(\omega))$

so $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|\exp(j(\omega t + \phi(\omega)))\,d\omega$

For fixed ω,

$|F(\omega)|\exp(j(\omega t + \phi(\omega)))$ is a (time-domain) complex sinusoid, with magnitude $|F(\omega)|$, frequency ω rad/sec, & phase $\phi(\omega)$

The integral equation represents $f(t)$ as a "continuous summation" of all these sinusoids!

The Fourier Transform goes the other way.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt$$

This equation tells you how to get $F(\omega)$ from $f(t)$

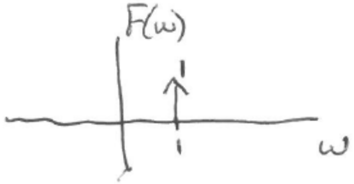
ie $\underset{\text{time domain}}{f(t)} \xrightarrow{\mathcal{F}} \underset{\text{frequency domain}}{F(\omega)}$

$\mathcal{F}$: Fourier Transform

For our first examples, lets start with $F(\omega)$ and use the inverse Fourier Transform equation to derive $f(t)$.

Examples:

- $F(\omega) = \delta(\omega - 1)$



$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - 1) \exp(j\omega t)\, d\omega$$

$$= \frac{1}{2\pi} \exp(jt)$$

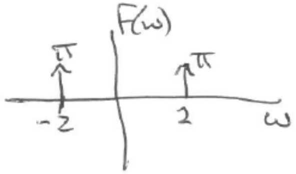
this is a complex sinusoid of frequency 1 rad/sec.

Both the Fourier Transform, & its inverse, are linear operations on signals. This means:

- $F(\omega) = 2\pi\, \delta(\omega - 1) \xrightarrow[\mathcal{F}^{-1}]{} f(t) = \exp(jt)$

- $F(\omega) = 2\pi\, \delta(\omega - \omega_0) \xrightarrow[\mathcal{F}^{-1}]{} f(t) = \exp(j\omega_0 t)$

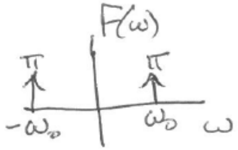
- $F(\omega) = \pi\, \delta(\omega + 2) + \pi\, \delta(\omega - 2)$



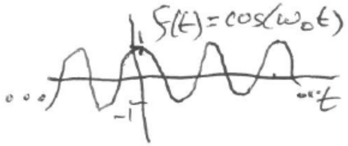
$$f(t) = \tfrac{1}{2} \exp(-j2t) + \tfrac{1}{2} \exp(j2t)$$

$$= \cos(2t)$$

- $F(\omega) = \pi\, \delta(\omega + \omega_0) + \pi\, \delta(\omega - \omega_0)$



$\xrightarrow[\mathcal{F}^{-1}]{} f(t) = \cos(\omega_0 t)$



In summary

$$f(t) = \cos(2t) \xrightarrow{\mathcal{F}} F(\omega) = \pi\delta(\omega+2) + \pi\delta(\omega-2)$$

$$f(t) = \cos(\omega_0 t) \xrightarrow{\mathcal{F}} F(\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$

– $f(t) = \cos(\omega_0 t + \phi_0)$

$$= \tfrac{1}{2}\exp(-j\phi_0)\exp(-j\omega_0 t) + \tfrac{1}{2}\exp(j\phi_0)\exp(j\omega_0 t)$$

By linearity of $\mathcal{F}$:

$$F(\omega) = \pi\exp(-j\phi_0)\,\delta(\omega+\omega_0) + \pi\exp(j\phi_0)\,\delta(\omega-\omega_0)$$

– $f(t) = \cos(\omega_0 t + \phi_0) \xrightarrow{\mathcal{F}} F(\omega) = \pi\exp(-j\phi_0)\,\delta(\omega+\omega_0)$
$+\pi\exp(j\phi_0)\,\delta(\omega-\omega_0)$

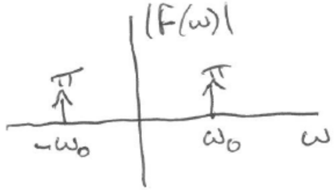
$|F(\omega)|$ is the magnitude at frequency ω

$\arg F(\omega)$ is the phase at frequency ω

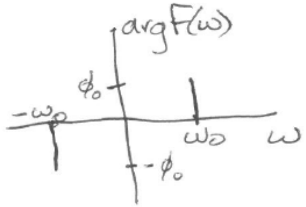
In this example,

$$|F(\omega)| = \pi\,\delta(\omega+\omega_0) + \pi\,\delta(\omega-\omega_0)$$

$$\arg F(\omega) = \begin{cases} 0 & \omega \neq -\omega_0, +\omega_0 \\ -\phi_0 & \omega = -\omega_0 \\ \phi_0 & \omega = +\omega_0 \end{cases}$$



magnitude spectrum

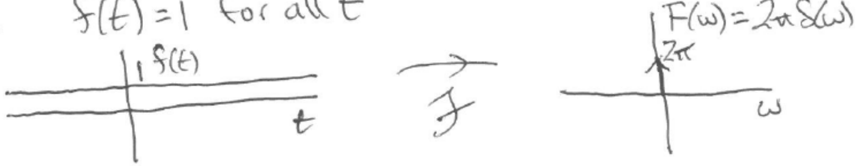


phase spectrum

– $F(\omega) = 2\pi\delta(\omega) \xrightarrow{\mathcal{F}^{-1}} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega)\exp(j\omega t)\,d\omega$

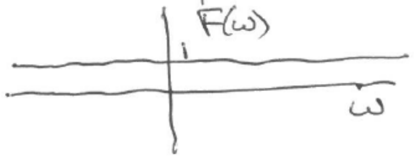
$$= \exp(j0t) = 1$$

So $f(t) = 1$ for all t



- $f(t) = \delta(t) \longrightarrow F(\omega) = \int_{-\infty}^{\infty} \delta(t) \exp(-j\omega t)\,dt$

$= \exp(-j0\omega) = 1$

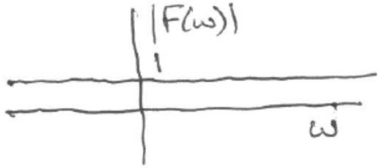


- $f(t) = \delta(t - t_0) \longrightarrow F(\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) \exp(-j\omega t)\,dt$

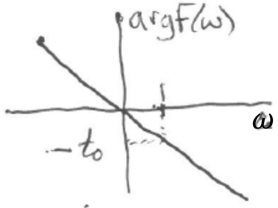
$= \exp(-j\omega t_0)$

$\Rightarrow |F(\omega)| = 1$ for all ω

$\arg F(\omega) = -\omega t_0$ for all ω



magnitude spectrum



phase spectrum

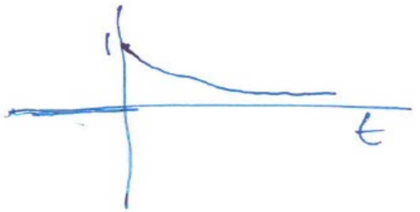
Fourier Transform was designed to be applied to signals with finite energy (ie NOT for signals like sinusoids or impulses!)

Fourier Transforms of finite energy signals don't have impulses in frequency domain.

Example 1: Decaying exponential signal

example 1:

$$f(t) = \exp(-at)\,u(t) \qquad a > 0$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)\,dt$$

$$= \int_{-\infty}^{\infty} \exp(-at)\,u(t)\exp(-j\omega t)\,dt$$

$$= \int_{0}^{\infty} \exp(-at)\exp(-j\omega t)\,dt$$

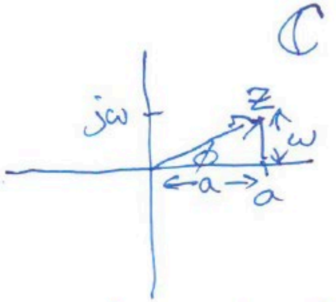
$$= \int_{0}^{\infty} \exp(-(j\omega + a)t)\,dt$$

$$= \frac{1}{-(j\omega+a)}\int_{0}^{\infty} -(j\omega+a)\exp(-(j\omega+a)t)$$

$$= \frac{1}{-(j\omega+a)}\left[\exp(-(j\omega+a)t)\right]_{0}^{\infty}$$

$$= \frac{-1}{(j\omega+a)}\left[0-1\right]$$

$$= \frac{1}{j\omega+a}$$



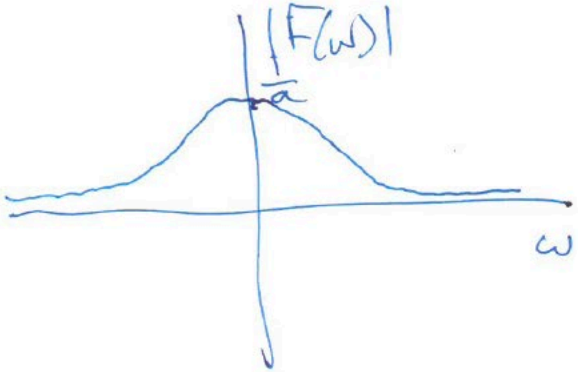
$$= \frac{1}{\sqrt{a^2+\omega^2}}\exp(-j\phi)$$

$$\phi = \arctan\left(\frac{\omega}{a}\right)$$

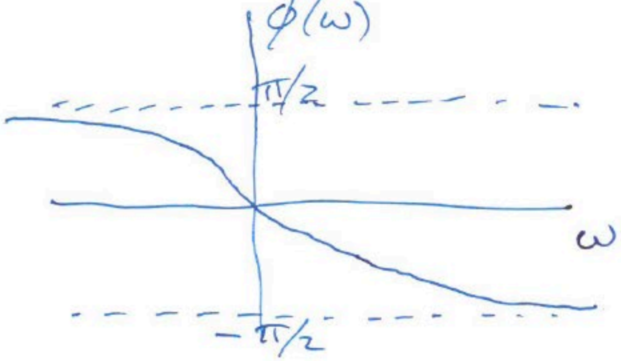
$$|Z| = \sqrt{a^2+\omega^2}$$

$$|F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\phi(\omega) = \arg(F(\omega)) = -\arctan\left(\frac{\omega}{a}\right)$$



amplitude spectrum



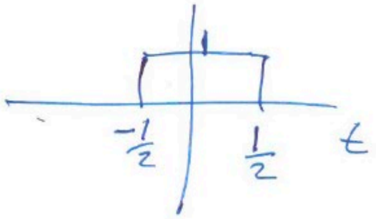
phase spectrum

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(j\omega t)\,d\omega$$

Example 2: rectangular pulse

example 2:

$x(t) = \mathrm{rect}(t)$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\,dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-j\omega t)\,dt$$

$$= \frac{1}{-j\omega}\int_{-\frac{1}{2}}^{1/2} (-j\omega)\exp(-j\omega t)\,dt$$

Rectangular pulse (ctd)

$$= -\frac{1}{j\omega}\left[\exp(-j\omega t)\right]_{-1/2}^{1/2}$$

$$= -\frac{1}{j\omega}\left[\exp(-j\omega\tfrac{1}{2}) - \exp(j\omega\tfrac{1}{2})\right]$$

$$= \frac{1}{j\omega}\left[\exp(j\omega\tfrac{1}{2}) - \exp(-j\omega\tfrac{1}{2})\right]$$

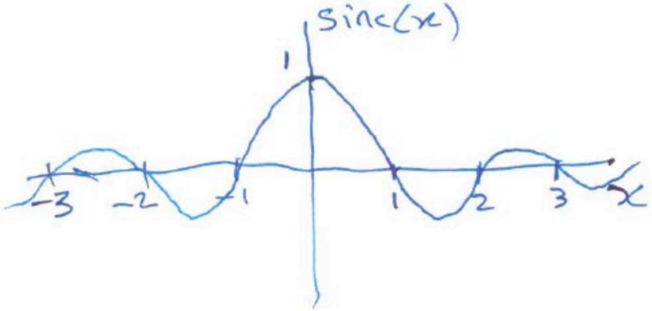
$$= \frac{2}{\omega}\left[\frac{\exp(j\omega\frac{1}{2}) - \exp(-j\omega\frac{1}{2})}{2j}\right]$$

$$= \frac{2}{\omega}\sin(\omega/2)$$

$$= \frac{\sin(\frac{\omega}{2})}{(\frac{\omega}{2})}$$

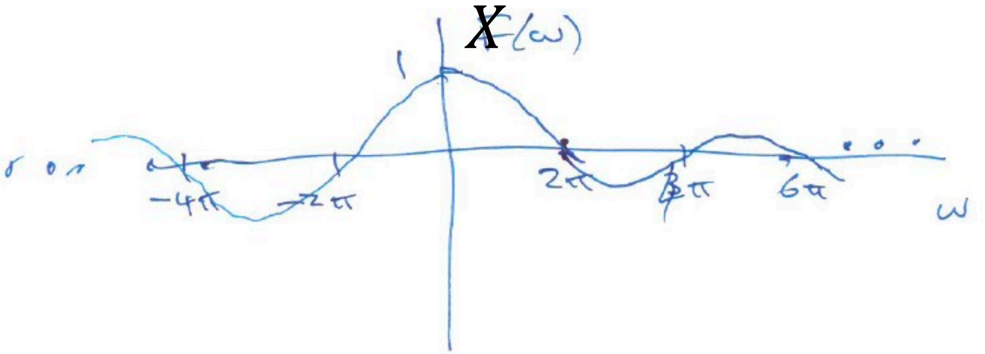
$$= \frac{\sin(\pi\frac{\omega}{2\pi})}{\pi(\frac{\omega}{2\pi})}$$

$$= \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \qquad \mathrm{sinc}(x) \equiv \frac{\sin(\pi x)}{(\pi x)}$$

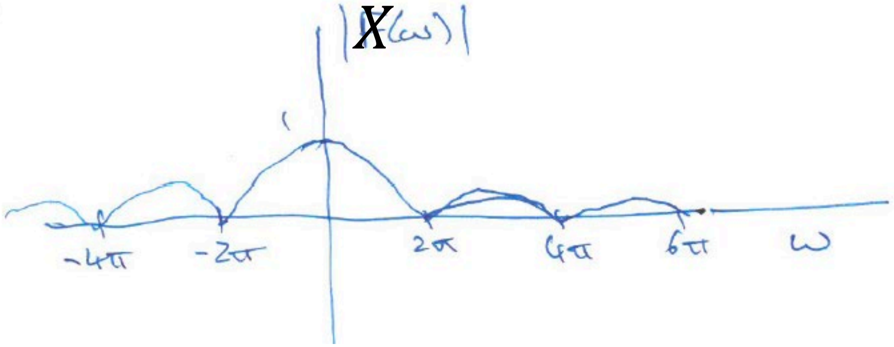


Amplitude Spectrum

$$X(\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

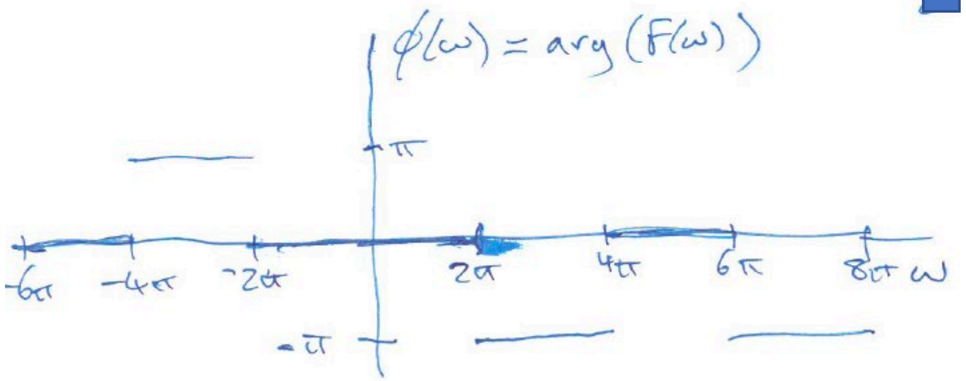


$$\mathrm{rect}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{\omega}{2\pi}\right) \exp(j\omega t)\, d\omega$$



Phase Spectrum

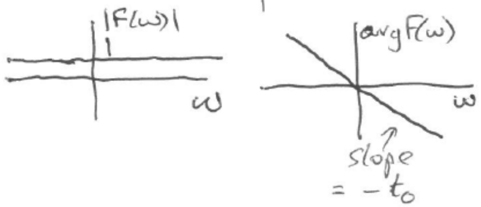
$$\mathrm{rect}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)\right| \cdot \exp(j(\omega t + \phi(\omega)))\, d\omega$$

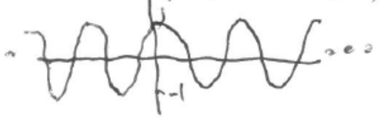


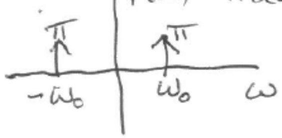
phase spectrum

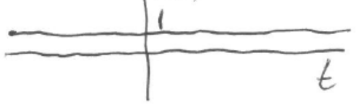
Summary so far:

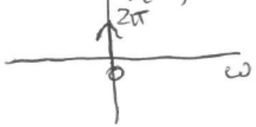
time domain	$\underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}}$	frequency domain
$f(t) = \delta(t - t_0)$		$F(\omega) = \exp(-j\omega t_0)$
$f(t) = \cos(\omega_0 t)$		$F(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$
$f(t)$		$F(\omega) = 2\pi\delta(\omega)$
$f(t) = \exp(-at)u(t)$		$F(\omega) = \dfrac{1}{j\omega + a}$
$f(t) = \mathrm{rect}(t)$		$F(\omega) = \mathrm{sinc}\left(\dfrac{\omega}{2\pi}\right)$

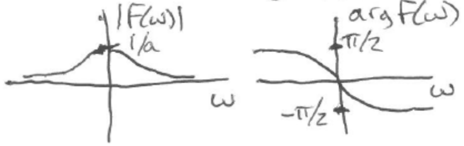












Last two examples, $f(t) = \exp(-at)ut$, and $f(t) = \text{rect}(t)$, are examples of signals with finite energy. ie $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$

This is NOT the case for $f(t) = \cos(\omega_0 t)$ for all t or $f(t) = 1$ for all t, which are power signals

It is also NOT the case for $f(t) = \delta(t - t_0)$ for all t which is an impulse signal with infinite energy.

Energy spectral density.

For an energy signal $x(t)$ with Fourier Transform $X(\omega)$, we can interpret $\frac{1}{2\pi}|X(\omega)|^2$ as the energy density at frequency ω rad/sec.

Parseval's equation states that

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Parseval's equation holds because the two domains are just different ways of looking at the same signal. We can compute the energy either in the time domain: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

or in the frequency domain: $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

Properties of the Fourier Transform

Moreover, the energy of the signal in a frequency band $[\omega_0, \omega_1]$ is $\frac{1}{2\pi}\int_{\omega_0}^{\omega_1} |X(\omega)|^2 \, d\omega$

This is the energy of the signal that we would obtain on passing $x(t)$ through a filter that passes all frequencies in the band $[\omega_0, \omega_1]$ & filters out all other frequencies.

There is no energy density for a signal like $x(t) = \cos(\omega_0 t)$ (or for $x(t) = 1$ for all t) but there is a power spectrum for such signals (see end of audio book in week 4)

Linearity

1. Linearity:

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$
$$y(t) \xrightarrow{\mathcal{F}} Y(\omega)$$

$$a_1 x(t) + a_2 y(t) \xrightarrow{\mathcal{F}} a_1 X(\omega) + a_2 Y(\omega)$$

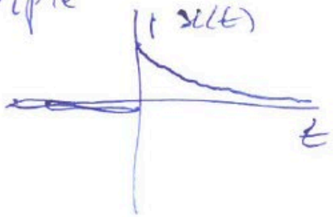
Time Reversal

2. Time reversal:

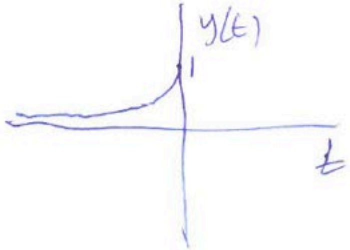
$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$y(t) = x(-t) \xrightarrow{\mathcal{F}} Y(\omega) = X(-\omega)$$

example



$$x(t) = \exp(-2t)\, u(t)$$
$$X(\omega) = \frac{1}{j\omega + 2}$$



$$y(t) = \exp(2t)\, u(-t)$$
$$Y(\omega) = \frac{1}{-j\omega + 2}$$
$$= \frac{-1}{j\omega - 2}$$

Time Scaling Property

3. Time scaling:

$$x(t) \xrightarrow{\mathcal{F}} X(\omega)$$

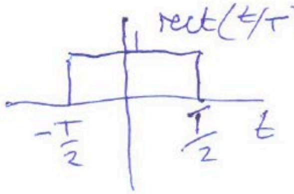
$$x(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

examples

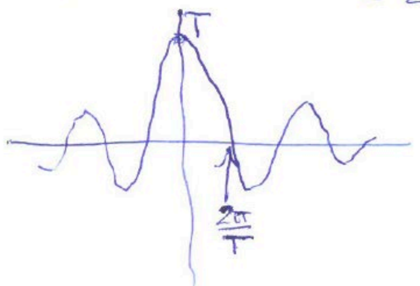
$$x(t) = \text{rect}(t) \longrightarrow X(\omega) = \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

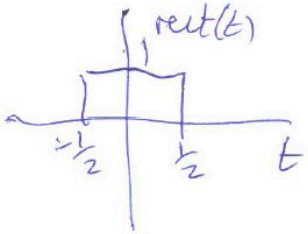
$$y(t) = \text{rect}(at) \longrightarrow Y(\omega) = \frac{1}{|a|}\text{sinc}\left(\frac{\omega}{2\pi a}\right)$$

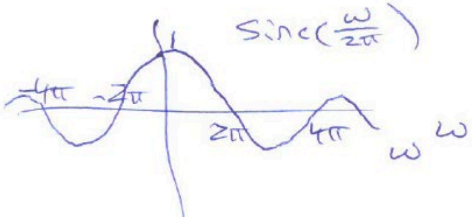
eg $a = \frac{1}{T}$

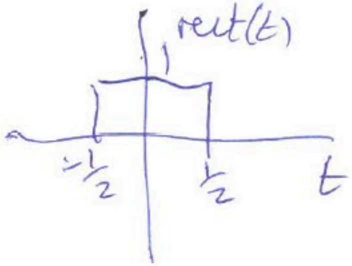


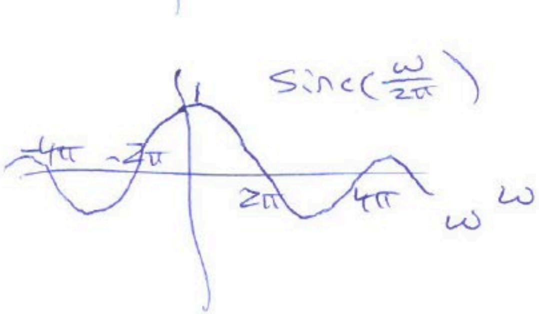
$$\longrightarrow Y(\omega) = T\,\text{sinc}\left(\frac{T\omega}{2\pi}\right)$$

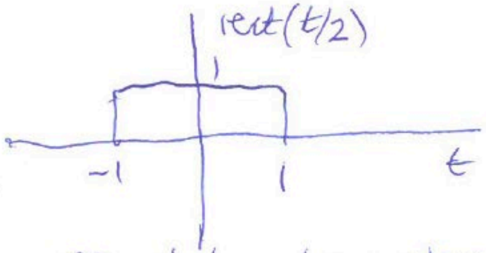


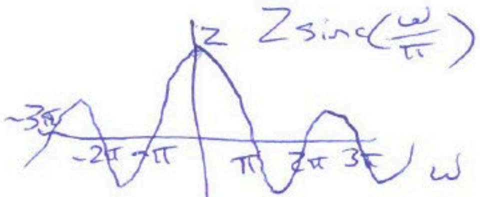






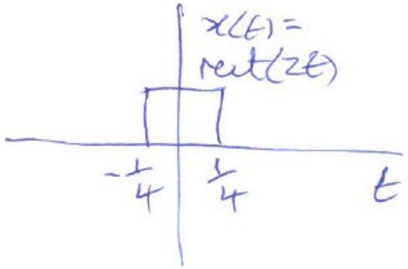




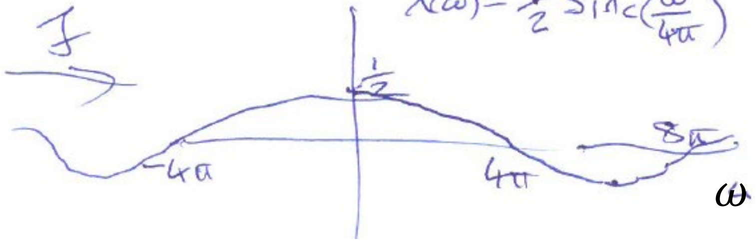


Signal has been slowed down x2

Freq. spectrum has been compressed x 2



$\xrightarrow{\mathcal{F}}$



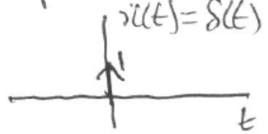
Time Shift

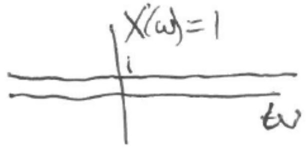
4. time-shift. $x(t) \overset{\mathcal{F}}{\rightleftarrows} X(\omega)$

$\Rightarrow x(t-t_0) \overset{\mathcal{F}}{\rightleftarrows} \exp(-j\omega t_0)X(\omega)$

example:

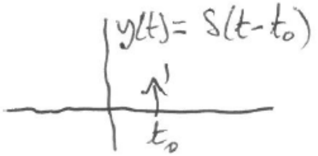
$x(t)=\delta(t) \xrightarrow{\mathcal{F}} X(\omega)=1$

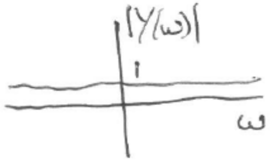


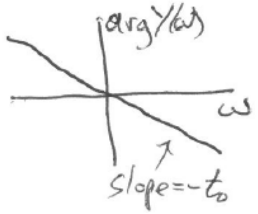


$y(t)=\delta(t-t_0) \xrightarrow{\mathcal{F}} Y(\omega)=\exp(-j\omega t_0)$

$= \exp(-j\omega t_0)X(\omega)$







Frequency Shift

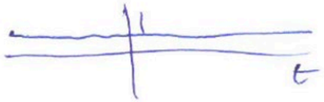
5. frequency shift

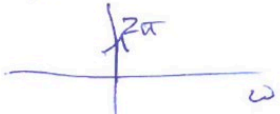
$x(t) \xrightarrow{\mathcal{F}} X(\omega)$

$x(t)\exp(j\omega_0 t) \xrightarrow{\mathcal{F}} X(\omega-\omega_0)$

example 1.

$x(t)=1 \qquad X(\omega)=2\pi\delta(\omega)$





$y(t)=\exp(j\omega_0 t)$

example 2.

$x(t) \longrightarrow X(\omega)$

$y(t)=x(t)\cos(\omega_0 t) = \frac{1}{2}x(t)\exp(j\omega_0 t) + \frac{1}{2}x(t)\exp(-j\omega_0 t)$