ELEC2070 Circuits and Devices

Week 5: Complete response of 2nd order (RLC) circuits

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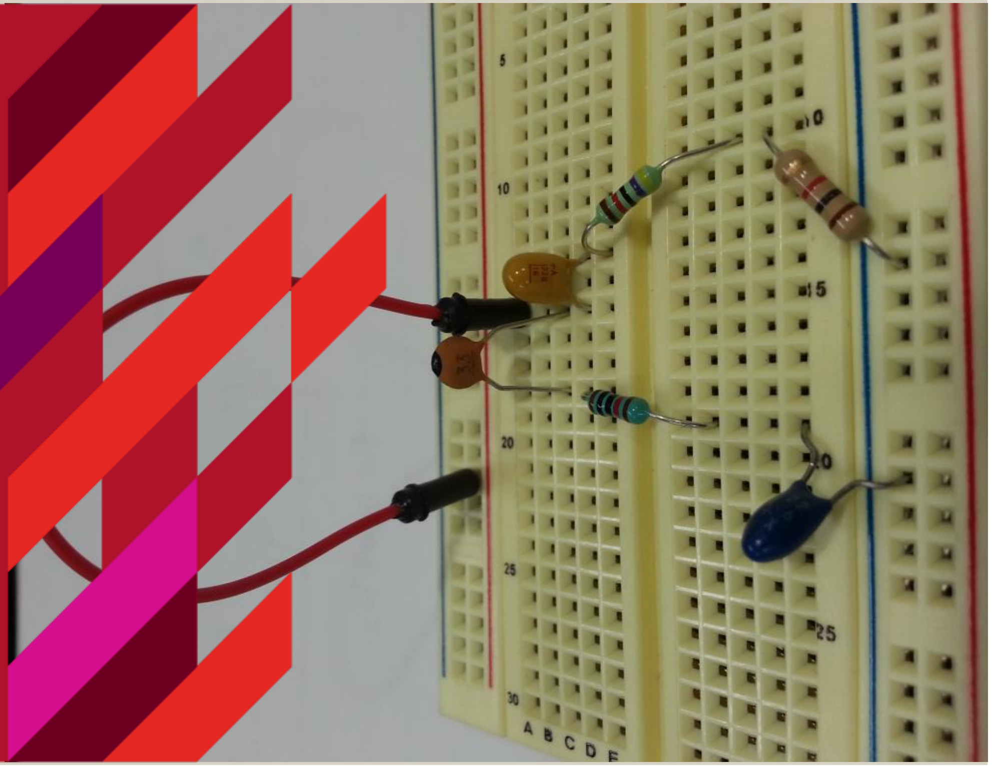


Two approaches to first order circuits

1. **Engineering approach** (useful for DC only): use the formula involving initial and final voltages (or currents) and the time constant

2. **Mathematical approach** (more general and needed for time varying sources): Find the differential equation using Kirchoff's laws and Ohm's Law

(Only the mathematical approach is used for second order circuits)

Recall: 1st order DE for 1st order circuits

General DE: $$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = y(t)$$

Time constant

Function to be solved (voltage or current)

Forcing function due to source (remember = K for a constant source)

This type of differential equation may be solved by separating the variables and integrating. (Book Section 8.3)

General Solution: $$x(t) = x_n(t) + x_f(t)$$

Complete response

Natural response

Forced response

Mathematical approach for solving first order circuits

Start with the differential equation in this form and an initial condition

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = y(t) \qquad x(t_1)$$

Find [from y(t)] the correct form of the forced response from the table

$$x_f(t)$$

Use the forced response in the differential equation to find its unknown coefficients

$$\frac{dx_f(t)}{dt} + \frac{1}{\tau}x_f(t) = y(t)$$

Forced response is fully solved

Write the correct form of the natural response, add to the forced response

$$x(t) = x_n(t) + x_f(t) = Ke^{-(t-t_0)/\tau} + x_f(t)$$

Use the initial condition with the complete response to find the unknown coefficient of the natural response

$$x(t_1) = Ke^{-(t_1-t_0)/\tau} + x_f(t_1)$$

$$t_1 \geq t_0$$

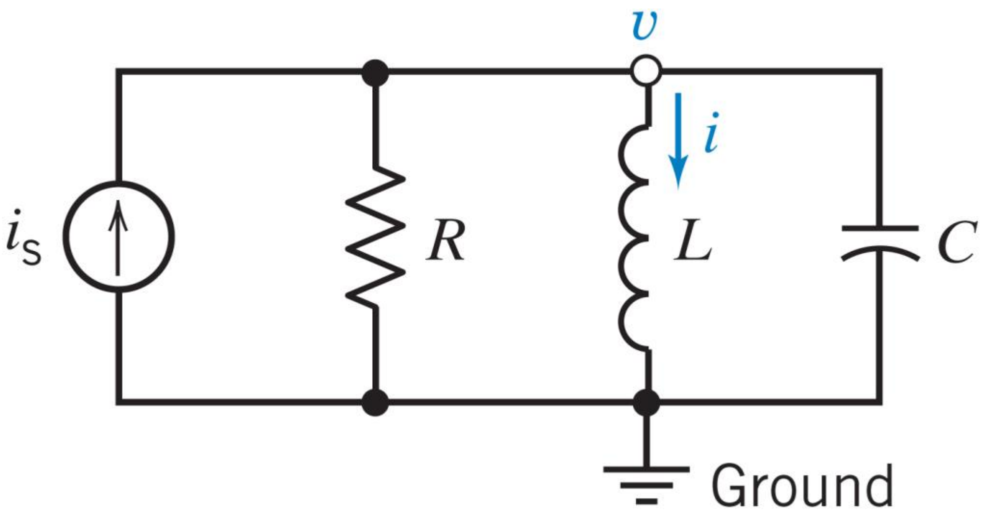
Complete response is fully solved

The differential equation for second order circuits

Chapter 9 of Dorf and Svoboda

A second order circuit

A second order circuit means the **total** number of capacitors and inductors is two



This also means that we need to solve a **second order** differential equation to find the currents and voltages

The general 2nd order differential equation for RLC circuits

$$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2\, x(t) = f(t)$$

Damping coefficient: α

Resonant frequency: ω_0

Function to be solved (voltage or current)

Forcing function (due to source)

General method for analysis of 2^{nd}-order circuits

1. **Use Kirchhoff's and Ohm's laws to write the circuit equations**

2. **Use the derivative of inductor current**

$$v_L(t) = L\frac{di_L(t)}{dt}$$

And the derivative of capacitor voltage

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

3. **Combine the equations so that the second and the first derivatives of the same variable appear in a linear combination. Do not use integrals! We want differential equations.**

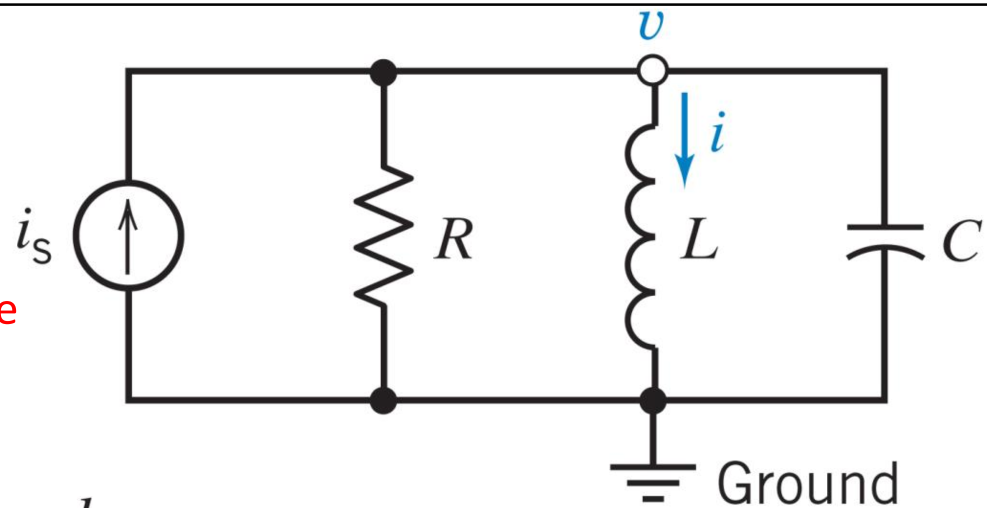
How do we get the 2nd-order DE?

When the two storage elements are connected to an equivalent circuit (Nortons or Thévenin) we can use the so-called Direct Method

In parallel arrangement

We want to find the current through the inductor, **i** (as a function of time)

Since we want the **current** we use Norton's equivalent circuit.



Apply KCL at top node: $$\frac{v}{R} + i + C\frac{dv}{dt} = i_s$$

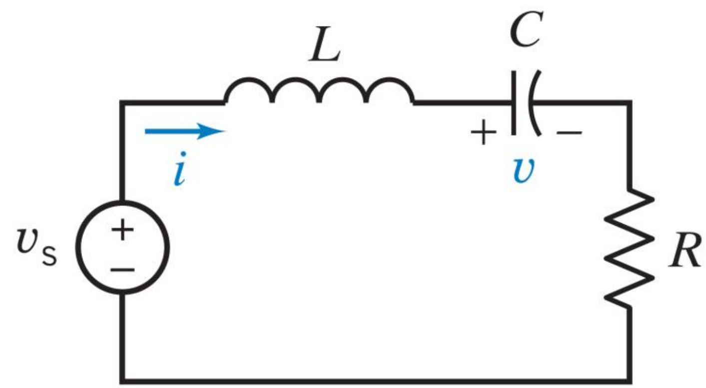
But the voltage across the inductor, resistor and capacitor is: $v = L\frac{di}{dt}$ (voltage the same for ALL elements)

Hence: $$\frac{L}{R}\frac{di}{dt} + i + CL\frac{d^2 i}{dt^2} = i_s$$ Which is a second-order DE!

Solve for i(t) and use $v = L\frac{di}{dt}$ to find v

In series arrangement

We want to find the voltage across the capacitor, **v** (as a function of time)



Since we want the **voltage**, we use Thévenin equivalent circuit.

Apply KVL around the loop: $L\dfrac{di}{dt} + v + Ri = v_s$ Rearrange: $\dfrac{di}{dt} + \dfrac{v}{L} + \dfrac{R}{L}i = \dfrac{v_s}{L}$

But: $$C\frac{dv}{dt} = i$$ (current the same for ALL elements)

Hence: $$C\frac{d^2v}{dt^2} + \frac{v}{L} + \frac{RC}{L}\frac{dv}{dt} = \frac{v_s}{L}$$

Or: $$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{v_s}{LC}$$

Solve for v(t) and use $C\dfrac{dv}{dt} = i$ to find i

General problem solving method for 2^{nd}-order circuits

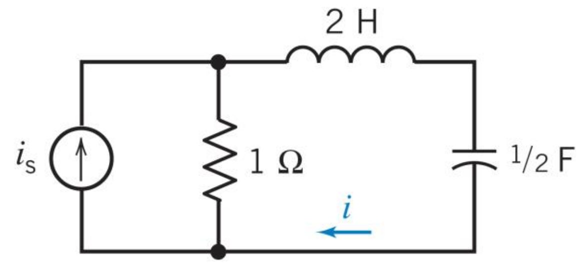
The direct method – used with equivalent circuits

Step 1	Identify the first and second variables, x_1 and x_2. These variables are capacitor voltages and/or inductor currents.
Step 2	Write one first-order differential equation, obtaining $\frac{dx_1}{dt} = f(x_1, x_2)$.
Step 3	Obtain an additional first-order differential equation in terms of the second variable so that $\frac{dx_2}{dt} = Kx_1$ or $x_1 = \frac{1}{K}\frac{dx_2}{dt}$.
Step 4	Substitute the equation of step 3 into the equation of step 2, thus obtaining a second-order differential equation in terms of x_2.

Remember – the direct method is useful when the inductor(s) and/or capacitor(s) are connected to an equivalent circuit (Norton's or Thévenin)

Exercise 9.2-1

Find the differential equation for ***i***



We see an inductor + capacitor connected to Norton's → use direct method

What we know:

Call capacitor voltage $\boldsymbol{v}$ → $\boxed{i = C\frac{dv}{dt} \quad (1)}$

(now we need an expression for v as a function of i)

KCL: Resistor current $= i_s - i$ → Resistor voltage $= R(i_s - i)$

Resistor || (cap. + ind.) Therefore **same** voltage: $R(i_s - i) = L\frac{di}{dt} + v$ → $\boxed{Ri_s - Ri - L\frac{di}{dt} = v \quad (2)}$

Substitute (2) in (1)

$$i = C\frac{d}{dt}\left(Ri_s - Ri - L\frac{di}{dt}\right) = CR\frac{di_s}{dt} - CR\frac{di}{dt} - CL\frac{d^2i}{dt^2}$$

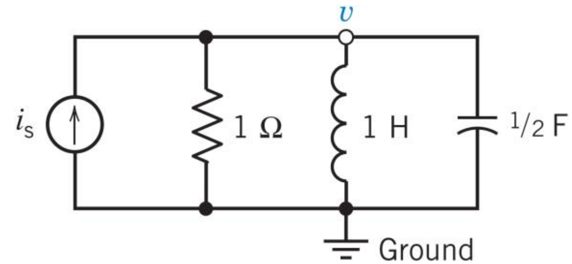
Rearrange

$$CL\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = CR\frac{di_s}{dt} \rightarrow \boxed{\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{CL}i = \frac{R}{L}\frac{di_s}{dt}}$$

Standard (Normalised) Form

Exercise 9.2-2

Find the differential equation for v



We see an inductor + capacitor connected to Norton's → use direct method

What we know:

Call inductor current i ➡ $v = L\dfrac{di}{dt}$ (1)

(now we need an expression for i)

Capacitor current $= C\dfrac{dv}{dt}$ Resistor current $= \dfrac{v}{R}$

Apply KCL at top node: $i = i_s - C\dfrac{dv}{dt} - \dfrac{v}{R}$ (2)

Substitute (2) in (1)

$$v = L\frac{d}{dt}\left(i_s - C\frac{dv}{dt} - \frac{v}{R}\right) = L\frac{di_s}{dt} - LC\frac{d^2v}{dt^2} - \frac{L}{R}\frac{dv}{dt}$$

Rearrange

$$LC\frac{d^2v}{dt^2} + \frac{L}{R}\frac{dv}{dt} + v = L\frac{di_s}{dt}$$

➡

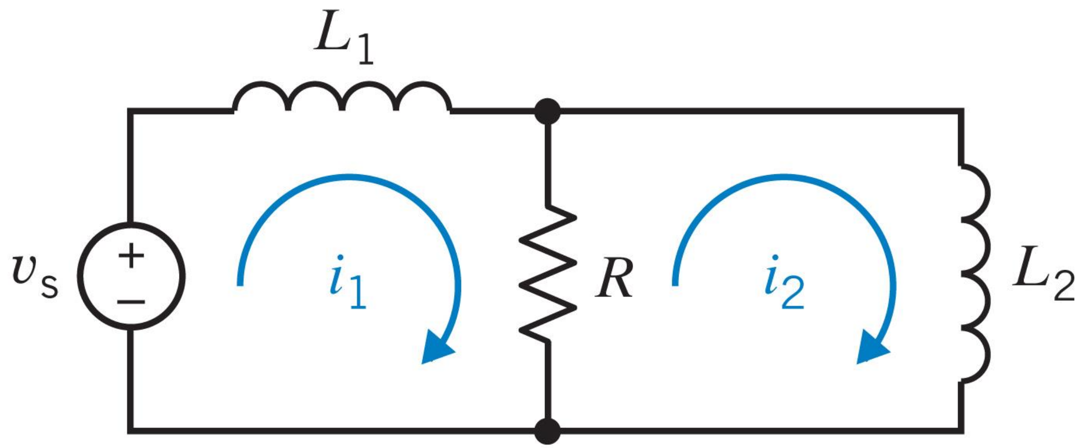
$$\boxed{\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{C}\frac{di_s}{dt}}$$

Standard Form

Other types of circuits?

Using the direct method can be complicated for complicated circuits

For example:



Two other methods are introduced in the textbook (Dorf):

1. Differential Operator Method

2. State Variable Method (we will skip this in ELEC2070)

Differential Operators

Will convert the differential equations into algebraic equations

Operator " s "

$$s = \frac{d}{dt}$$

$$s\, x(t) = \frac{dx(t)}{dt} \qquad s^2\, x(t) = \frac{d^2x(t)}{dt^2}$$

For any order of differentiation: $$s^n\, x(t) = \frac{d^n x(t)}{dt^n}$$

No differentiation: $s^0\, x(t) = x(t)$

Integration: $$\frac{1}{s} x(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

For any order of integration: $$s^{-n}\, x(t) = \frac{d^{-n} x(t)}{dt^{-n}}$$

Finding differential equations using s-operators is now easier

Inductor voltage: $v_L(t) = L\frac{di_L(t)}{dt} = sL\, i_L(t)$

Capacitor voltage: $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau = \frac{1}{sC}\, i_C\,(t)$

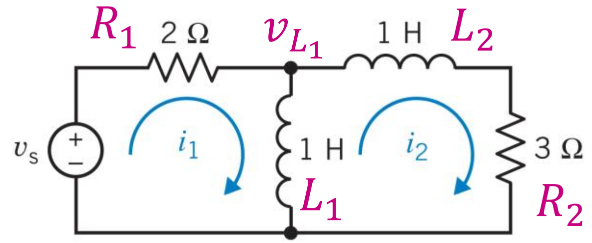
Voltages and currents of energy storage elements are now related in an algebraic form (similar to resistors)

Resistor voltage: $v_R(t) = R\, i_R(t)$

Very helpful when solving second (or higher) order circuits !

Example 9.2-1

Find the differential equation for $\boldsymbol{i_2}$



Using the **s** operator, the inductor voltages become: $v_{L_2} = sL_2\, i_2 \quad v_{L_1} = sL_1(i_1 - i_2)$

The two GENERAL mesh equations are now:

$$v_s = R_1\, i_1 + s\, L_1(\, i_1 - i_2) \quad (1)$$

$$0 = -s\, L_1(\, i_1 - i_2) + s\, L_2\, i_2 + R_2\, i_2 \quad (2)$$

Place in the values for resistances and inductances and create the matrix representation:

$$\begin{bmatrix} 2+s & -s \\ -s & 2s+3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} v_s \\ 0 \end{bmatrix}$$

The currents can be found using Cramer’s Rule

Cramer's Rule

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows:

$$\boldsymbol{Ax = b}$$

where the $n \times n$ matrix A has a nonzero determinant, and the vector

$$\boldsymbol{x = (x_1, x_2, x_3, \ldots, x_n)}$$

is the column vector of the variables.

Then the theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$$\boldsymbol{x_i = \frac{det(A_i)}{det(A)}}$$

where $\boldsymbol{A_i}$ is the matrix formed by replacing the i-th column of A by the column vector b.

Applying Cramer's Rule

Cramer's rule means that we need to find FIRST the determinant of: $\begin{vmatrix} 2+s & -s \\ -s & 2s+3 \end{vmatrix}$

Or: $D = (2+s)(2s+3) - s^2$

To find i_2 we need to find the determinant of: $D_{i_2} = \begin{vmatrix} 2+s & v_s \\ -s & 0 \end{vmatrix}$

AND then $i_2 = \frac{D_{i_2}}{D} = \frac{sv_s}{(2+s)(2s+3)-s^2}$

Note $i_1 = \frac{D_{i_1}}{D}$

Hence $i_2 = \frac{sv_s}{4s+6+2s^2+3s-s^2}$

or $(s^2 + 7s + 6)i_2 = sv_s$ $\Rightarrow$ $\left[\boldsymbol{s}^2 + 7\boldsymbol{s} + 6\right] i_2 = \boldsymbol{s} v_s$

Convert back to differential equation form:

$$\frac{d^2}{dt^2} i_2 + 7\frac{d}{dt} i_2 + 6i_2 = \frac{d}{dt} v_s$$

Solving the differential equations for second order circuits

This will determine the Complete Response

Complete response

Must be solved using the mathematical approach

The general form of a (linear) second order differential equation:

$$a_2 \frac{d^2}{dt^2} x(t) + a_1 \frac{d}{dt} x(t) + a_0 x(t) = f(t)$$

Function to be solved (voltage or current)

Forcing function (due to the source)

The values for a_i (constants) and the forcing function are known

The same differential equation with s-operators:

$$a_2 s^2 x(t) + a_1 s x(t) + a_0 x(t) = f(t)$$

Has a general solution:

$$x(t) = x_n(t) + x_f(t)$$

Complete response

Natural response

Forced response

Natural response

For Second Order Circuits

The characteristic equation

The natural response is obtained by zeroing the forcing function

IN GENERAL: $$a_2 s^2 x_n(t) + a_1 s x_n(t) + a_0 x_n(t) = 0$$

Solution to this characteristic equation is the natural response, $x_n(t)$.

Natural response must be an exponential (since an exponential is proportional is ALL its derivatives)

$$x_n(t) = Ae^{st}$$

Substitute: $$a_2 A s^2 e^{st} + a_1 A s e^{st} + a_0 A e^{st} = 0$$

Since $x_n(t) = Ae^{st}$ then: $$a_2 s^2 x_n(t) + a_1 s x_n(t) + a_0 x_n(t) = 0$$

Or: $$(a_2 s^2 + a_1 s + a_0) x_n(t) = 0$$

$x_n(t) = 0$ is trivial therefore we seek the solution to: $a_2 s^2 + a_1 s + a_0 \ = 0$

This is the **CHARACTERISTIC EQUATION**

Finding the natural response

The solution to the characteristic equation: $a_2 s^2 + a_1 s + a_0 = 0$

is: $$s_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

The natural response can now be given by:

$$x_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

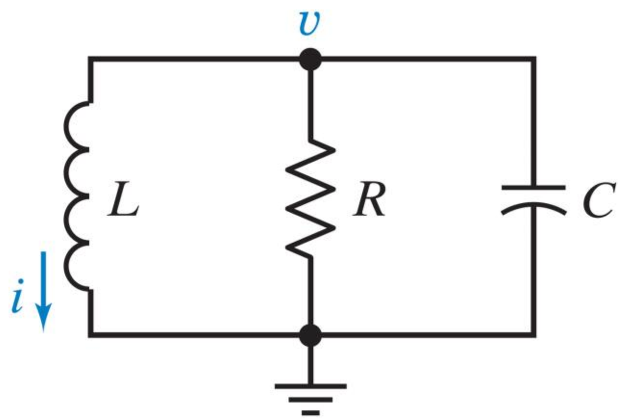
Where A_1 and A_2 are obtained from the initial conditions.

The roots s_1 and s_2 are known as the **NATURAL FREQUENCIES**.

The reciprocal magnitude of the natural frequencies are the **TIME CONSTANTS**.

Natural response of an unforced (source free) circuit: parallel RLC circuit

An unforced parallel RLC circuit would look like this:



For ALL RLC circuits we need to solve: $$a_2 \frac{d^2}{dt^2} x(t) + a_1 \frac{d}{dt} x(t) + a_0 x(t) = f(t)$$

This can be simplified to (this form is relevant to RLC circuits): $$\frac{d^2}{dt^2} x(t) + 2\alpha \frac{d}{dt} x(t) + \omega_0^2 x(t) = f(t)$$

Since f(t) = 0 (for the natural response) the general equation is: $$\frac{d^2}{dt^2} x(t) + 2\alpha \frac{d}{dt} x(t) + \omega_0^2 x(t) = 0$$

General RLC equation

The characteristic equation for the general RLC equation (for the natural response) is

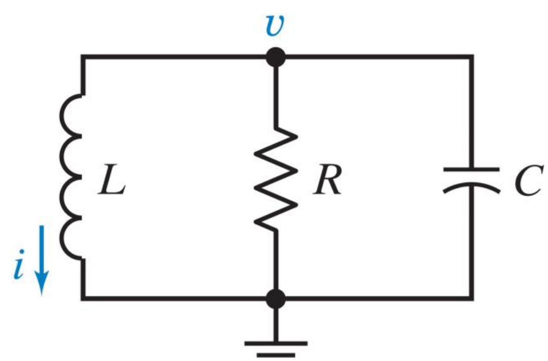
$$s^2 + 2\alpha s + \omega_0^2 = 0$$

These are called NATURAL FEQUENCIES

The roots of this characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Okay, back to our circuit:



Apply KCL: $$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + i(0) + C\frac{dv}{dt} = 0$$

Parallel RLC circuit (with f(t) = 0)

Differentiating both sides: $$C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v = 0$$

Now we have an equation for the voltage

Divide by C: $$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

Using s operator $$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Comparing with the **general equation** we have: $$\alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0^2 = \frac{1}{LC}$$

The roots of the characteristic equation are therefore:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \quad \text{and} \quad s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

The solution for the natural response

The solution therefore becomes: $$v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

With the natural frequencies given by:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

For our parallel RLC: $$\alpha = \frac{1}{2RC} \quad \text{and} \quad \omega_0^2 = \frac{1}{LC}$$

We define the **DAMPED RESONANT FREQUENCY** as: $$w_d = \sqrt{w_0^2 - \alpha^2}$$

The natural frequencies now become: $s_1 = -\alpha + jw_d \; and \; s_2 = -\alpha - jw_d$

Natural response: Summary

For Second Order Electric Circuits

The general equation: $\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = 0$

Damping coefficient: α

Resonant frequency: ω_0

Damped resonant frequency: $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

Characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

Roots (natural frequencies): $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$= -\alpha \pm j\omega_d$

Damping

The roots of the characteristic equation assume three possible conditions:

a. 2 real and distinct roots when: $\alpha^2 > \omega_0^2$

b. 2 real and equal roots when: $\alpha^2 = \omega_0^2$

c. 2 complex roots when: $\alpha^2 < \omega_0^2$

overdamped
$\alpha^2 > \omega_0^2$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

critically damped
$\alpha^2 = \omega_0^2$

$$s_1 = s_2 = -\alpha$$

underdamped
$\alpha^2 < \omega_0^2$

$$s_1 = -\alpha + j\omega_d$$
$$s_2 = -\alpha - j\omega_d$$

The complete response

Second Order Circuits

You will be given these equations in the exam!

General differential equation: $$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$$

Characteristic equation (for the natural response): $$s^2 + 2\alpha s + \omega_0^2 = 0$$

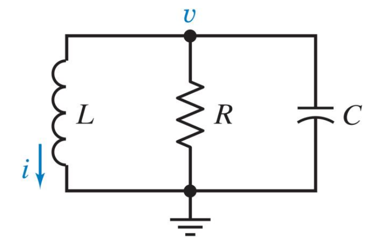
CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

	INPUT, $f(t)$	FORCED RESPONSE, x_f
Constant	K	A
Ramp	$K\,t$	$A + Bt$
Sinusoid	$K \cos \omega t$, $K \sin \omega t$, or $K \cos(\omega t + \theta)$	$A \cos \omega t + B \sin \omega t$
Exponential	Ke^{-bt}	Ae^{-bt}

Example 9.4-1

Example of an **overdamped** circuit

Let us put some element values into our RLC circuit:



Want: $v_n(t) = ?$

For $t > 0$

Now we are given : $R = \frac{2}{3}\ \Omega \quad L = 1\ \mathrm{H} \quad C = \frac{1}{2}\ \mathrm{F} \quad v(0) = 10\ \mathrm{V} \quad i(0) = 2\ \mathrm{A}$

Here the initial conditions are: $v(0) = 10\ \mathrm{V} \quad i(0) = 2\ \mathrm{A}$

Remember our characteristic equation: $s^2 + 2\alpha s + \omega_0^2 = 0$

Or for the actual circuit above: $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$

Our characteristic equation becomes: $s^2 + 3s + 2 = 0$ $\alpha = 1.5 \quad \omega_0^2 = 2$

Note $\alpha^2 > \omega_0^2$ since $(1.5)^2 > 2$, therefore overdamped

Example 9.4-1

The characteristic equation has roots: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

Hence: $s_1 = -1$ and $s_2 = -2$

The natural response is (since OVERDAMPED, look up from table): $v_n(t) = A_1 e^{-t} + A_2 e^{-2t}$

There is <u>no source</u> in this circuit:
Complete response is equal to natural response. $v(t) = v_n(t)$

We are given: $v(0) = 10\ \text{V}$ $i(0) = 2\ \text{A}$

Two unknowns are to be solved using the initial conditions

$v_\text{n}(0) = A_1 + A_2$ or $10 = A_1 + A_2$

This equation has 2 unknowns – need to apply both initial conditions

$i(0) = 2\ \text{A}$ -------- this one is more difficult to deal with but there is a method!

Applying the initial condition: i(0)

Going back to our general natural response. Take the derivative of:

$$v_{\mathrm{n}} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

And apply at t=0, we get: $$\frac{dv_{\mathrm{n}}(0)}{dt} = s_1 A_1 + s_2 A_2 \quad (1)$$

Our original KCL equation ($\frac{v}{R} + \frac{1}{L}\int_0^t v\, d\tau + i(0) + C\frac{dv}{dt} = 0$)

Becomes (at t=0): $$\frac{dv(0)}{dt} = -\frac{v(0)}{RC} - \frac{i(0)}{C} \quad (2)$$

Equating equations (1) and (2):

$$s_1 A_1 + s_2 A_2 = -\frac{v(0)}{RC} - \frac{i(0)}{C}$$

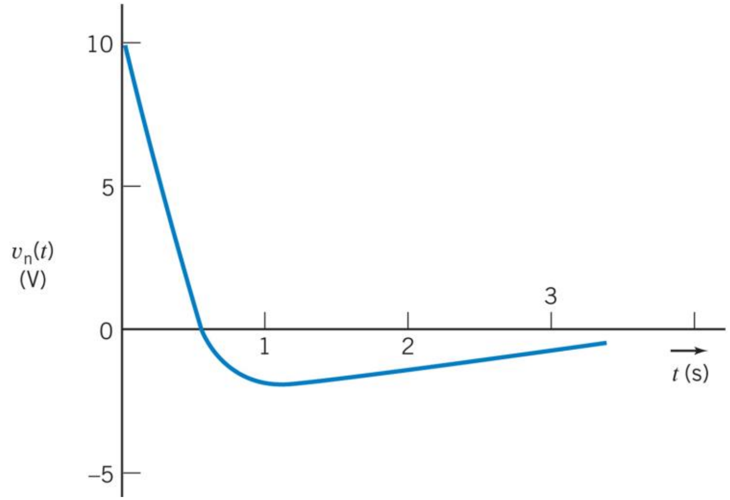
Solution

Placing the values in for s_1, s_2, v(0) and i(0) we get: $$-A_1 - 2A_2 = \frac{10}{\frac{1}{3}} - \frac{2}{1/2}$$

Now we have 2 equations for our 2 unknowns. Easy to solve!

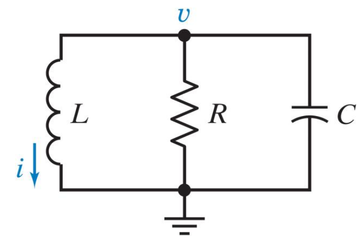
Actual solution:

$$v_n = (-14e^{-t} + 24e^{-2t})\ V$$



Example 9.4-1

Critically Damped Circuit



$v_n(t) = ?$

$t > 0$

Differential equation:

$$\frac{d^2}{dt^2} v_n(t) + \frac{1}{RC}\frac{d}{dt} v_n(t) + \frac{1}{LC} v_n(t) = 0$$

Characteristic equation is just:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

No source hence: $v(t) = v_n(t)$

New parameters:

$R = 1\,\Omega \qquad L = 1\text{ H} \qquad C = \frac{1}{4}\text{ F} \qquad v(0) = 5\text{ V} \qquad i(0) = -6\text{ A}$

Same circuit BUT different values

Using:

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$s^2 + 4s + 4 = 0 \Rightarrow \alpha = 2 \qquad \omega_0^2 = 4 \Rightarrow$ $s_1 = -2$, $s_2 = -2$

Now the natural frequencies are the same!

Use our table for the complete response

Second Order Circuits

You will be given these equations in the exam!

General differential equation: $$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$$

Characteristic equation (for the natural response): $$s^2 + 2\alpha s + \omega_0^2 = 0$$

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

	INPUT, $f(t)$	FORCED RESPONSE, x_f
Constant	K	A
Ramp	$K\,t$	$A + Bt$
Sinusoid	$K \cos \omega t$, $K \sin \omega t$, or $K \cos(\omega t + \theta)$	$A \cos \omega t + B \sin \omega t$
Exponential	Ke^{-bt}	Ae^{-bt}

Example 9.4-1

Critically Damped Circuit Continues

Since natural frequencies are the same we use: $v_n(t) = (A_1 t + A_2)e^{-2t}$

Here the solution still requires 2 unknown to be found.

Two unknowns to be solved using the 2 initial conditions

Initial conditions: $\boxed{v_n(0) = A_2 = 5 \quad (1)}$

To use i(0) we differentiate general solution: $$\frac{dv_n(t)}{dt} = -2A_1 t e^{-2t} + A_1 e^{-2t} - 2A_2 e^{-2t}$$

Apply at t=0 $$\frac{dv_n(0)}{dt} = A_1 - 2A_2$$

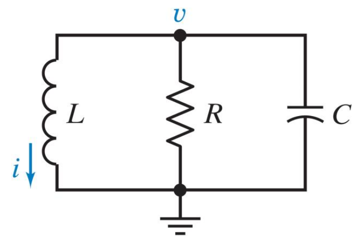
Use our original KCL equation at t=0: $$i(0) + \frac{v_n(0)}{R} + C\frac{dv_n(0)}{dt} = 0 \quad \Rightarrow \quad -6 + \frac{5}{1} + \frac{1}{4}\frac{dv_n(0)}{dt} = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_0^t v\,d\tau + i(0) + C\frac{dv}{dt} = 0$$

$\frac{dv_n(0)}{dt}$ = 4, Hence: $\boxed{\frac{d}{dt}v_n(0) = A_1 - 2A_2 = 4 \quad (2)}$

Example 9.4-1

Critically Damped Solution



$v_n(t) = ?$

$t > 0$

For these parameters: $R = 1\,\Omega \quad L = 1\text{ H} \quad C = \frac{1}{4}\text{ F} \quad v(0) = 5\text{ V} \quad i(0) = -6\text{ A}$

Natural Response:

$$v_n(t) = (A_1 t + A_2)e^{-2t}$$

Initial conditions:

$$\boxed{v_n(0) = A_2 = 5 \quad (1)}$$

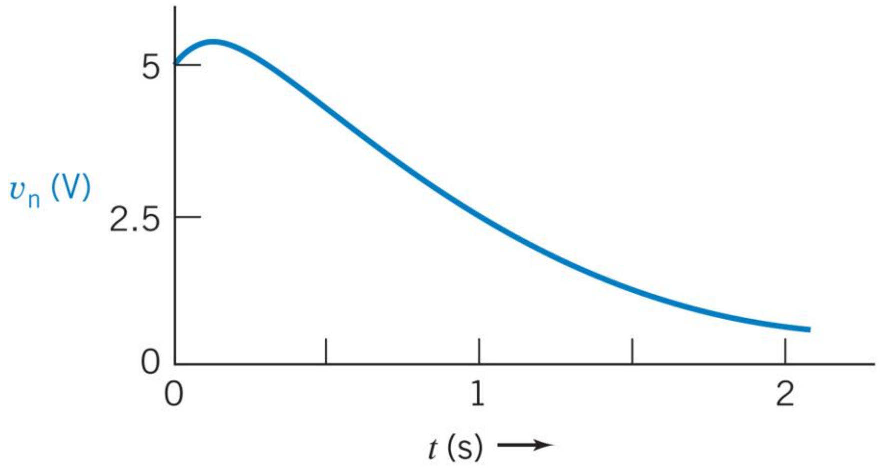
$$\boxed{\frac{d}{dt}v_n(0) = A_1 - 2A_2 = 4 \quad (2)}$$

$$A_1 = 14$$

$$A_2 = 5$$

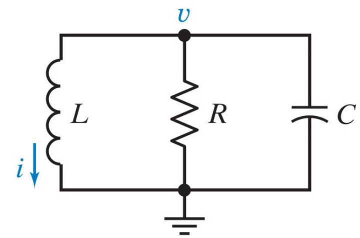
Complete Response:

$$v(t) = v_n(t) = e^{-2t}(14t + 5)\text{ V}$$



Example 9.4-1

Underdamped Circuit



$v_n(t) = ?$

$t > 0$

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Differential equation:

$$\frac{d^2}{dt^2}v_n(t) + \frac{1}{RC}\frac{d}{dt}v_n(t) + \frac{1}{LC}v_n(t) = 0$$

Characteristic equation is again:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Parameters: $R = \frac{25}{3}\,\Omega \quad L = 0.1\text{ H} \quad C = 1\text{ mF} \quad v(0) = 10\text{ V} \quad i(0) = -0.6\text{ A}$

No source: $v(t) = v_n(t)$

Using: $s^2 + 2\alpha s + \omega_0^2 = 0 \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

$\alpha^2 < \omega_0^2$ since $(60)^2 < 10^4$, therefore underdamped

We have: $s^2 + 120s + 10^4 = 0 \Rightarrow \alpha = 60 \qquad \omega_0^2 = 10^4$

We get: $\omega_d = 80$

Hence: $s_1 = -60 + j80$

$s_2 = -60 - j80$

$$v_n(t) = A_1 e^{-60t}e^{+j80t} + A_2 e^{-60t}e^{-j80t}$$

$$= e^{-60t}(A_1 e^{+j80t} + A_2 e^{-j80t})$$

We get: $v_n(t) = e^{-60t}\left[(A_1 + A_2)\cos 80t + j\,(A_1 - A_2)\,\sin 80t\right]$

$since\ e^{\pm j\omega t} = \cos\omega t \pm j\sin\omega t$

The complete response

Second Order Circuits

You will be given these equations in the exam!

General differential equation: $$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t)$$

Characteristic equation (for the natural response): $$s^2 + 2\alpha s + \omega_0^2 = 0$$

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

	INPUT, $f(t)$	FORCED RESPONSE, x_f
Constant	K	A
Ramp	$K\,t$	$A + Bt$
Sinusoid	$K \cos \omega t$, $K \sin \omega t$, or $K \cos(\omega t + \theta)$	$A \cos \omega t + B \sin \omega t$
Exponential	Ke^{-bt}	Ae^{-bt}

Example 9.4-1

Underdamped Circuit............

Now we can set $B_1 = (A_1 + A_2)\ and\ B_2 = j\ (A_1 - A_2)$

Then: $$v_n(t) = B_1 e^{-60t} \cos 80t + B_2 e^{-60t} \sin 80t$$

Two unknowns to be solved using the initial conditions

Initial conditions: $$\boxed{v_n(0) = B_1 = 10 \quad (1)}$$

To use the other initial condition we differentiate the general solution:

$$\frac{dv_n(t)}{dt} = -60B_1 e^{-60t} \cos 80t - 80B_1 e^{-60t} \sin 80t - 60B_2 e^{-60t} \sin 80t + 80B_2 e^{-60t} \cos 80t$$

Apply at t=0: $$\frac{dv_n(0)}{dt} = -60B_1 + 80B_2$$

Our original KCL (at t=0): $$i(0) + \frac{v_n(0)}{R} + C\frac{dv_n(0)}{dt} = 0 \Rightarrow -0.6 + \frac{30}{25} + 10^{-3}\frac{dv_n(0)}{dt} = 0$$ Hence: $$\frac{dv_n(0)}{dt} = -600$$

$$\boxed{\frac{d}{dt}v_n(0) = -600 = -60B_1 + 80B_2 \quad (2)}$$

Example 9.4-1

Underdamped Solution

Natural Response:

$$v_n(t) = (B_1 t + B_2)e^{-2t}$$

Initial conditions:

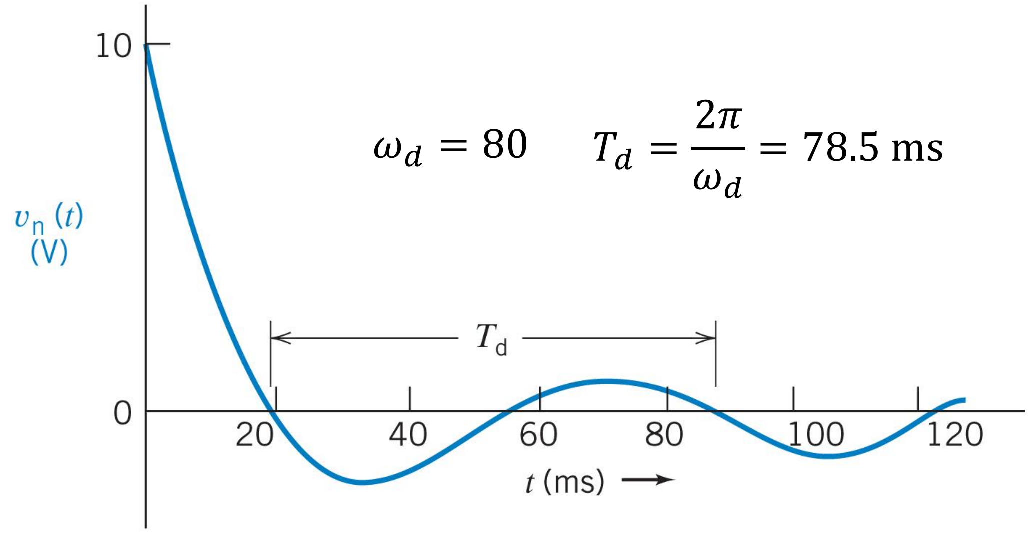
$$\boxed{v_n(0) = B_1 = 10 \quad (1)}$$

$$\boxed{\frac{d}{dt}v_n(0) = -600 = -60B_1 + 80B_2 \quad (2)} \Rightarrow B_2 = 0$$

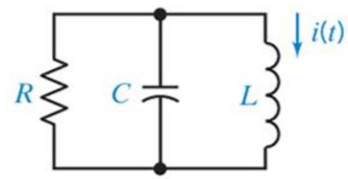
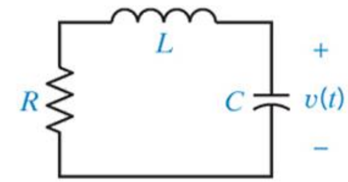
Complete Response:

$$v(t) = v_n(t) = 10e^{-60t}\cos 80t\ \mathrm{V}$$

$$\omega_d = 80 \qquad T_d = \frac{2\pi}{\omega_d} = 78.5\ \text{ms}$$



Summary RLC Circuits

Natural Response	PARALLEL *RLC*	SERIES *RLC*
Circuit		
Differential equation	$\frac{d^2}{dt^2}i(t) + \frac{1}{RC}\frac{d}{dt}i(t) + \frac{1}{LC}i(t) = 0$	$\frac{d^2}{dt^2}v(t) + \frac{R}{L}\frac{d}{dt}v(t) + \frac{1}{LC}v(t) = 0$
Characteristic equation	$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$	$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$
Damping coefficient, rad/s	$\alpha = \frac{1}{2RC}$	$\alpha = \frac{R}{2L}$
Resonant frequency, rad/s	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
Damped resonant frequency, rad/s	$\omega_d = \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$	$\omega_d = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$
Natural frequencies: overdamped case	$s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$ when $R < \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$ when $R > 2\sqrt{\frac{L}{C}}$
Natural frequencies: critically damped case	$s_1 = s_2 = -\frac{1}{2RC}$ when $R = \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1 = s_2 = -\frac{R}{2L}$ when $R = 2\sqrt{\frac{L}{C}}$
Natural frequencies: underdamped case	$s_1, s_2 = -\frac{1}{2RC} \pm j\sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$ when $R > \frac{1}{2}\sqrt{\frac{L}{C}}$	$s_1, s_2 = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ when $R < 2\sqrt{\frac{L}{C}}$

Finding Natural Response

Summary

Differential equation: $$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = 0 \qquad (1)$$

Characteristic equation: $$s^2 + 2\alpha s + \omega_0^2 = 0 \qquad (2)$$

CASE	NATURAL FREQUENCIES	NATURAL RESPONSE, x_n
Overdamped	$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$	$A_1 e^{s_1 t} + A_2 e^{s_2 t}$
Critically damped	$s_1, s_2 = -\alpha$	$(A_1 + A_2 t)e^{-\alpha t}$
Underdamped	$s_1, s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$	$(A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t}$

Find natural frequencies from (2)

Find the two unknown coefficients from two initial conditions: $x_n(0)$ *and* $\frac{dx_n(0)}{dt}$

The complex frequency plane

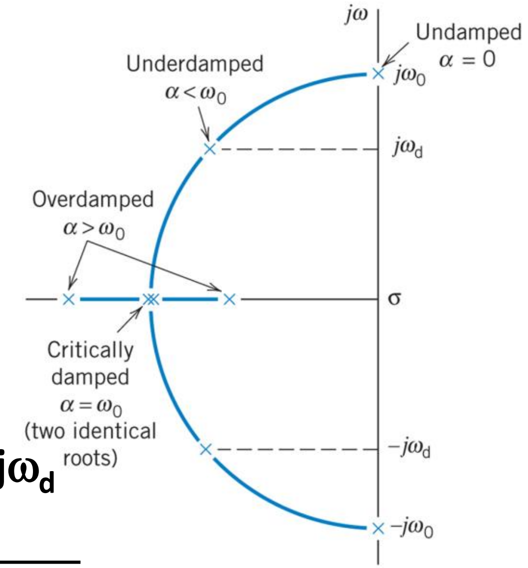
In general, the natural frequencies (i.e., $s_{1,2}$) are located on the complex plane.

With σ being the REAL axis and $j\omega$ being the imaginary axis.

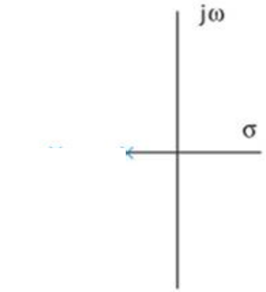
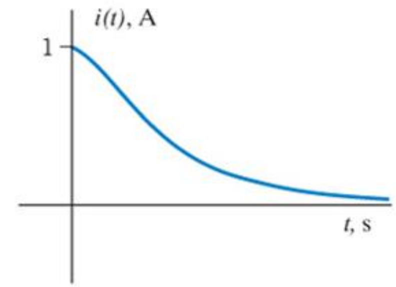
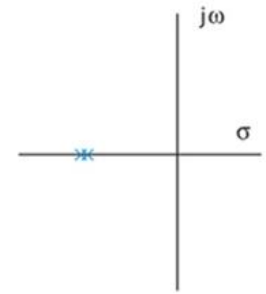
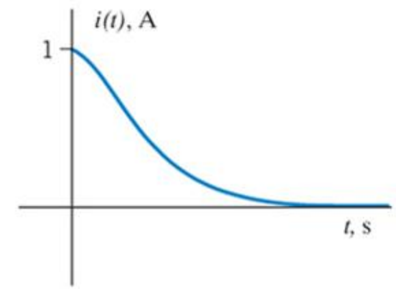
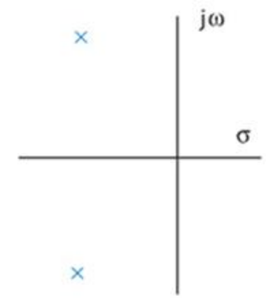
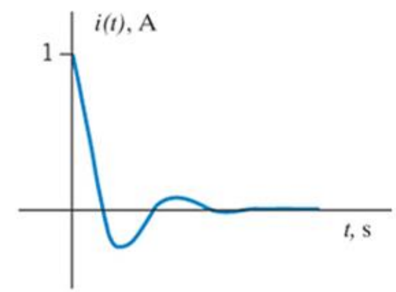
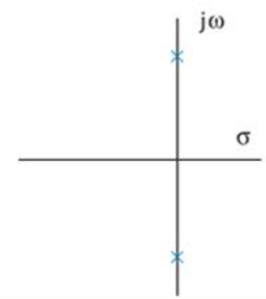
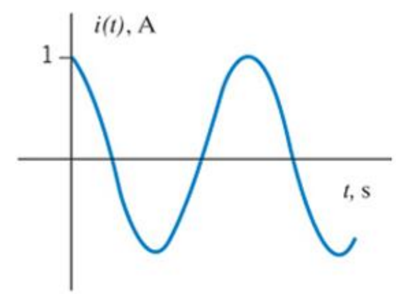
The σ - $j\omega$ plane is known as the *s*-plane or (because *s* has units of frequency) **complex frequency plane**

$$s = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm j\omega_d$$

The locations on *s*-plane



1. **Undamped: $\alpha = 0$, the 2 roots are imaginary and $s_{1,2} = \pm j\omega_d$**
2. **Underdamped: $\alpha < \omega_0$, the 2 roots are complex and $s_{1,2} = -\alpha \pm j\omega_d$**
3. **Critically damped: $\alpha = \omega_0$ the 2 roots are real and $s_{1,2} = -\alpha$**
4. **Overdamped: $\alpha > \omega_0$ the 2 roots are real and $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$**

TYPE OF RESPONSE	ROOT LOCATION	FORM OF RESPONSE
Overdamped		
Critically damped		
Underdamped		
Undamped		

*The $i(t)$ is the inductor current in the circuit shown in Figure 9.4-1 for the initial conditions $i(0) = 1$ and $v(0) = 0$.

The response is the sum of 2 decaying exponentials

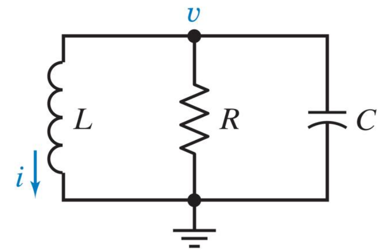
The response is the sum of 2 decaying exponentials

The response is an exponentially decaying sinusoid

$R = 0$ $\longrightarrow$ Undamped
(no resistor)

Exercise 9.4-1

Revisited



$R = 6\ \Omega$
$L = 7$ H
$C = 1/42$ F
$v(0) = 0$
$i(0) = 10$ A

Differential equation: $$\frac{d^2}{dt^2}v(t) + \frac{1}{RC}\frac{d}{dt}v(t) + \frac{1}{LC}v(t) = 0$$

Damping coefficient: $$\alpha = \frac{1}{2RC} = \frac{42}{12} = 3.5$$

Resonant frequency: $$\omega_0^2 = \frac{1}{LC} = 6 \quad \omega_0 = 2.45$$

Overdamped

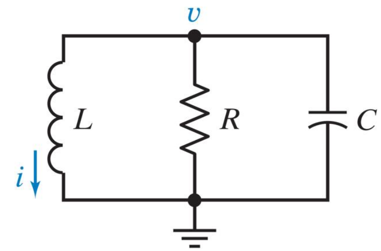
The natural frequencies:

$$s_1 = -3.5 - \sqrt{12.25 - 6} = -6$$
$$s_2 = -3.5 + 2.5 = -1$$

$$v_n(t) = A_1\ e^{-6t} + A_2\ e^{-t}$$

Exercise 9.4-1

Continues…..



$R = 6\ \Omega$
$L = 7\ \mathrm{H}$
$C = 1/42\ \mathrm{F}$
$v(0) = 0$
$i(0) = 10\ \mathrm{A}$

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No source: $v(t) = v_n(t) = A_1\, e^{-6t} + A_2\, e^{-t}$

Initial conditions: $v(0) = A_1 + A_2 = 0 \quad (1)$

KCL at t=0: $i(0) + \dfrac{v(0)}{R} + C\dfrac{dv(0)}{dt} = 0 \Rightarrow \dfrac{dv(0)}{dt} = -420$

$$\frac{dv(t)}{dt} = -6A_1\, e^{-6t} - A_2\, e^{-t} \Rightarrow -6A_1 - A_2 = -420 \quad (2)$$

Solution: $v_n(t) = 84\, e^{-6t} - 84\, e^{-t}$

Forced Response

For Second Order Circuits

Solving for Forced Response

General differential equation: $$\frac{d^2}{dt^2}x(t) + 2\alpha\frac{d}{dt}x(t) + \omega_0^2 x(t) = f(t) \quad (1)$$

Forced response must satisfy: $$\frac{d^2}{dt^2}x_f(t) + 2\alpha\frac{d}{dt}x_f(t) + \omega_0^2 x_f(t) = f(t) \quad (2)$$

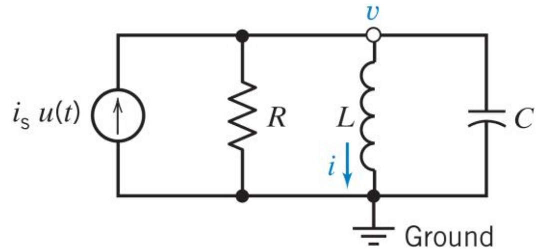
$f(t)$ FORCING FUNCTION	$x_f(t)$ ASSUMED RESPONSE
K	A
Kt	$At + B$
Kt^2	$At^2 + Bt + C$
$K \sin \omega t$	$A \sin \omega t + B \cos \omega t$
Ke^{-at}	Ae^{-at}

This table will be given

Find the coefficients from (2) when initial conditions applied, usually t=0

Example 9.7-2

Forced response to a constant input



$$R = 6\ \Omega \qquad L = 7\ \mathrm{H} \qquad C = \frac{1}{42}\ \mathrm{F}$$

$$i_s = I_0\ \mathrm{A} \qquad t > 0$$

A constant current is applied to the circuit

Apply KCL: $i + \dfrac{v}{R} + C\dfrac{dv}{dt} = i_s$

Using:

$$v = L\frac{di}{dt} \quad \Rightarrow \quad \frac{dv}{dt} = L\frac{d^2 i}{dt^2}$$

Putting in the values and $i_s = I_0$, for t>0:

Gives:

$$i_f + \frac{L}{R}\frac{di_f}{dt} + CL\frac{d^2 i_f}{dt^2} = i_s$$

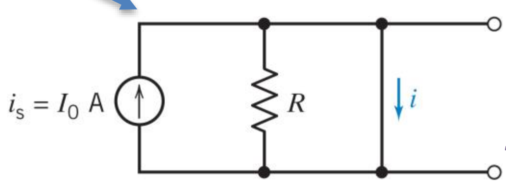
$$i_f + \frac{7}{6}\frac{di_f}{dt} + \frac{7}{42}\frac{d^2 i_f}{dt^2} = I_0$$

Standardise

$$\frac{d^2 i_f}{dt^2} + 7\frac{di_f}{dt} + 6i_f = 6I_0$$

Differential equation

The circuit in the steady state

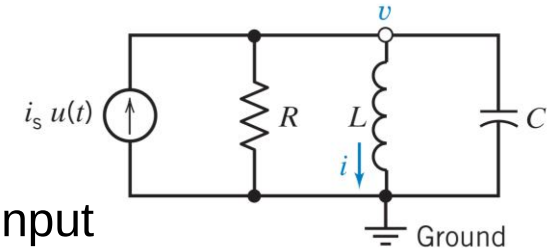


$i_f = A$ From the table! $\rightarrow$ $0 + 0 + 6A = 6I_0$

$$\boxed{i_f = I_0}$$

Example 9.7-2

Forced response to ramped constant input



Given the differential equation: $$\frac{d^2 i}{dt^2} + 9\frac{di}{dt} + 20i = 6i_s \qquad (1)$$

AND: $i_s = (6 + 2t)$ A

Find forced response i_f for $t > 0$

Get general form from table $i_f = At + B$

Substitute in (1) $$\frac{d^2}{dt^2}(At + B) + 9\frac{d}{dt}(At + B) + 20(At + B) = 6(6 + 2t\,)$$

$$9A + 20At + 20B = 36 + 12t$$

Time varying part $20At = 12t \Rightarrow 20A = 12 \Rightarrow A = 0.6$

DC part $9A + 20B = 36 \Rightarrow B = 1.53$

Solution $i_f = (0.6t + 1.53)$ A

Solving for Complete Response

$x(t) = x_n(t) + x_f(t)$

1. Find the forced response (i.e., find the coefficients) using the differential equation for x_f (Note: no initial conditions are needed)

2. Find natural frequencies (from the characteristic equation); then write the general natural response (with two unknown coefficients)

3. Add natural + forced together (i.e., the complete response with two unknown coefficients)

4. Use two initial conditions (for second order circuits) to find the remaining two unknown coefficients, for example using:

$$x(0) \text{ and } \frac{dx(0)}{dt}$$

Where *x* is the current or voltage