Fourier Transform Properties:
- duality
- convolution
-differentiation
-multiplication

Audio Book Week 6

Duality Property

time domain	$\xleftrightarrow{\mathcal{F}}$ frequency domain
$x(t)$	$X(\omega)$
$y(t) = \frac{1}{2\pi} X(-t)$	$Y(\omega) = x(\omega)$

example:

$x(t) = \exp(j\omega_0 t)$ $\qquad$ $X(\omega) = 2\pi\,\delta(\omega - \omega_0)$

replace ω with $-t$ in $X(\omega)$ above

$$X(-t) = 2\pi\,\delta(-t - \omega_0)$$

$$\Rightarrow y(t) = \delta(-t-\omega_0) = \delta(-(t+\omega_0))$$

$$= \delta(t+\omega_0)$$

since $\delta(t) = \delta(-t)$

$$= \delta(t-(-\omega_0))$$

interpret $-\omega_0$ as a time – a delay (in sec)

& take Fourier Transform:

$$Y(\omega) = \exp(-j\omega(-\omega_0))$$

frequency in rad/sec $\quad$ delay in sec

$$= \exp(j\omega_0\omega)$$

We see that indeed, $Y(\omega) = x(\omega)$

as predicted above.

ω_0 is just a dummy variable, we can replace $-\omega_0$ with $+t_0$

	time $\xrightarrow{\mathcal{F}}$	frequency
a)	$\delta(t-(-\omega_0))$	$\exp(j\omega_0\omega)$
b)	$\delta(t-t_0)$	$\exp(-j\omega t_0)$

We already knew the result in b).

Another example

$x(t) = 1 \qquad X(\omega) = 2\pi\,\delta(\omega)$

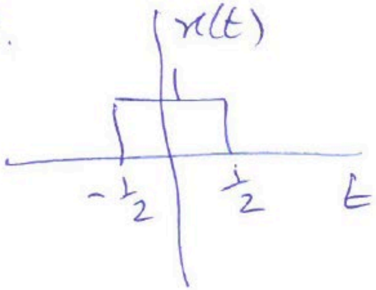
To apply duality result $\omega \to -t$ in $X(\omega)$

$$y(t) = \frac{1}{(2\pi)} X(-t) = \delta(-t) = \delta(t)$$

Take Fourier Transform: $Y(\omega) = 1 = x(\omega)$

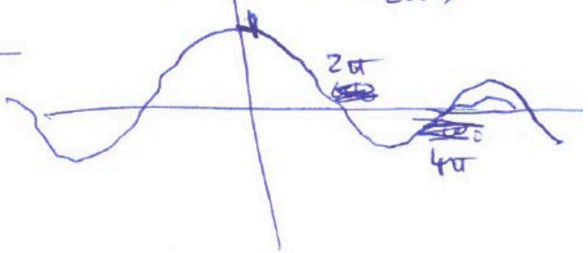
So again, indeed, $Y(\omega) = x(\omega)$

c.



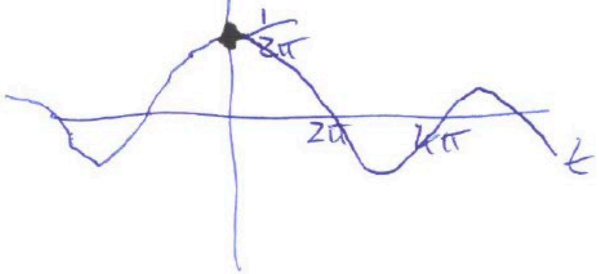
$\xrightarrow{\mathcal{F}}$

$X(\omega) = \text{sinc}\left(\frac{\omega}{2\pi}\right)$

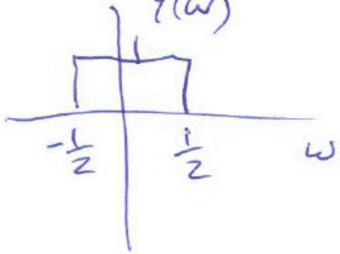


$x(t) = \text{rect}(t)$

$y(t) = \frac{1}{2\pi}\text{sinc}\left(\frac{t}{2\pi}\right)$



$\rightarrow$



$$y(t) = \frac{1}{2\pi} X(-t)$$

$$= \frac{1}{2\pi}\text{sinc}\left(\frac{t}{2\pi}\right)$$

$\rightarrow$

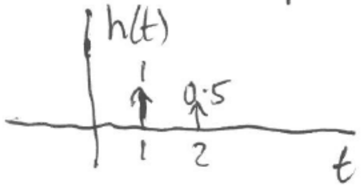
$$Y(\omega) = x(\omega)$$

$$= \text{rect}(\omega)$$

Convolution Property

time domain	$\overset{\mathcal{F}}{\longleftrightarrow}$ frequency domain
$x(t)$	$X(\omega)$
$h(t)$	$H(\omega)$
$y(t)=x(t)*h(t)$	$Y(\omega)=X(\omega)\cdot H(\omega)$

example: two path channel



(impulse response)

$x(t) \longrightarrow \boxed{\text{channel}} \longrightarrow y(t) = x(t)*h(t)$

$$= x(t)*\left(\delta(t-1)+0.5\,\delta(t-2)\right)$$

$$= x(t-1)+0.5\,x(t-2)$$

Suppose $x(t)=\cos(\frac{\pi}{2}t)$

time	frequency
$x(t)=\cos(\frac{\pi}{2}t)$	$X(\omega)=\pi\delta(\omega+\frac{\pi}{2})+\pi\delta(\omega-\frac{\pi}{2})$
$h(t)=\delta(t-1)+0.5\,\delta(t-2)$	$H(\omega)=\exp(-j\omega)+0.5\exp(-j\omega 2)$
$y(t)=x(t)*h(t)$	$Y(\omega)=X(\omega)\cdot H(\omega)$

$$Y(\omega)=X(\omega)\cdot H(\omega)$$

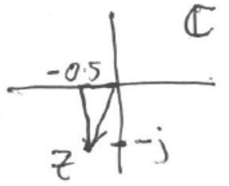
$$=\left(\pi\delta(\omega+\tfrac{\pi}{2})+\pi\delta(\omega-\tfrac{\pi}{2})\right)\cdot\left(\exp(-j\omega)+0.5\exp(-j\omega 2)\right)$$

$$=\pi\left(\exp(+j\tfrac{\pi}{2})+0.5\exp(+j\pi)\right)\delta(\omega+\tfrac{\pi}{2})$$

$$+\pi\left(\exp(-j\tfrac{\pi}{2})+0.5\exp(-j\pi)\right)\delta(\omega-\tfrac{\pi}{2})$$

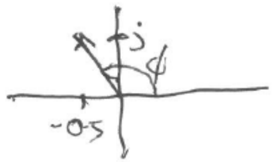
$$=\pi(j-0.5)\,\delta(\omega+\tfrac{\pi}{2})+\pi(-j-0.5)\,\delta(\omega-\tfrac{\pi}{2})$$

What is $-0.5 - j$?



let $z = -0.5 - j$

$z^* = -0.5 + j = j - 0.5$



so $z^* = \frac{\sqrt{5}}{2} \exp(j\phi)$ $\phi = \pi - \tan^{-1}(2)$

$= 2.03$ radians

$z = \frac{\sqrt{5}}{2} \exp(-j\phi)$

So $Y(\omega) = \pi \frac{\sqrt{5}}{2} \exp(j\phi)\, \delta(\omega + \frac{\pi}{2}) + \pi \frac{\sqrt{5}}{2} \exp(-j\phi)\, \delta(\omega - \frac{\pi}{2})$

Take inverse Fourier Transform to get $y(t)$:

$y = \frac{\sqrt{5}}{2} \cos(\frac{\pi}{2} t - \phi)$

$= \frac{\sqrt{5}}{2} \cos(\frac{\pi}{2} t - 2.03)$

Since input is a sinusoid of frequency $\frac{\pi}{2}$ rad/sec & phase = 0

We can write down output if we know $H(\frac{\pi}{2})$

$H(\omega) = \exp(-j\omega) + 0.5 \exp(-2j\omega)$

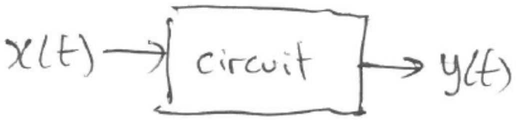
$\Rightarrow H(\frac{\pi}{2}) = -j - 0.5$

$= \frac{\sqrt{5}}{2} \exp(-j\, 2.03)$

So $y(t) = \frac{\sqrt{5}}{2} \cos(\pi t - 2.03)$

Example: Linear circuit

$$\frac{dy}{dt} + 2y(t) = 2x(t)$$



One way to solve is to solve with $x(t) = \delta(t)$

$$\frac{dy}{dt} + 2y(t) = 2\delta(t) \Rightarrow y(t) = 2\exp(-2t)\,u(t)$$

This is the impulse response $h(t) = 2\exp(-2t)\,u(t)$

The solution for any other input signal $x(t)$ can be obtained by <u>convolution</u>

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= 2\int_{-\infty}^{t} x(\tau)\exp(-2(t-\tau))\,d\tau$$

We can also obtain the solution using Fourier Transforms

$x(t)$ $\xrightarrow{\mathcal{F}}$	$X(\omega)$
$h(t)$	$H(\omega) = \frac{2}{j\omega+2}$
$y(t)$	$Y(\omega) = X(\omega) \cdot H(\omega)$ $= X(\omega) \cdot \frac{2}{j\omega+2}$

We find $Y(\omega) = X(\omega) \cdot \frac{2}{j\omega+2}$

& get $y(t)$ by taking inverse Fourier Transform.

Particularly good method when $x(t)$ is sinusoidal.

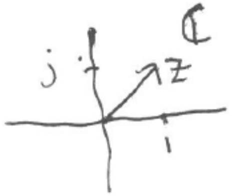
example:

time	frequency
$x(t) = \cos(2t)$	$X(\omega) = \pi\delta(\omega+2) + \pi\delta(\omega-2)$
$h(t) = 2\exp(-2t)\,u(t)$	$H(\omega) = \frac{2}{j\omega+2}$
$y(t) = x(t) * h(t)$	$Y(\omega) = X(\omega)\cdot H(\omega)$

$$Y(\omega) = \left(\pi\delta(\omega+2) + \pi\delta(\omega-2)\right)\cdot\frac{2}{j\omega+2}$$

$$= \pi H(-2)\,\delta(\omega+2) + \pi H(2)\,\delta(\omega-2)$$

$$H(-2) = \frac{2}{-2j+2} = \frac{1}{1-j} \qquad H(2) = \frac{2}{2j+2} = \frac{1}{1+j}$$



$$z = \sqrt{2}\exp\left(j\tfrac{\pi}{4}\right)$$

$$\Rightarrow H(2) = \frac{1}{\sqrt{2}}\exp\left(-j\tfrac{\pi}{4}\right)$$

$$\& \; H(-2) = H(2)^* = \frac{1}{\sqrt{2}}\exp\left(j\tfrac{\pi}{4}\right)$$

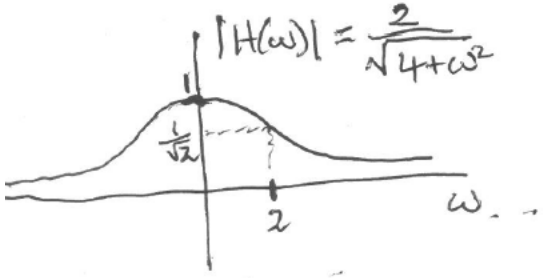
so $$Y(\omega) = \frac{\pi}{\sqrt{2}}\exp\left(j\tfrac{\pi}{4}\right)\delta(\omega+2) + \frac{\pi}{\sqrt{2}}\exp\left(-j\tfrac{\pi}{4}\right)\delta(\omega-2)$$

We get $y(t)$ by taking inverse Fourier Transform

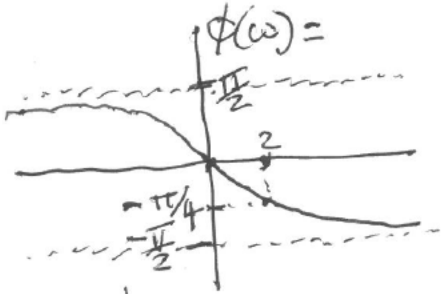
$$y(t) = \frac{1}{\sqrt{2}}\cos\left(2t - \frac{\pi}{4}\right)$$

Note that $H(2)$ provides the gain & phase shift of the system at frequency $\omega = 2$ rad/sec

$$|H(2)| = \frac{1}{\sqrt{2}} \qquad \arg H(2) = -\frac{\pi}{4} \text{ radians}$$



magnitude response



phase response

$$H(\omega) = |H(\omega)| \exp(j\phi(\omega))$$

$$|H(\omega)| = \frac{2}{\sqrt{4+\omega^2}} \qquad \phi(\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$$

$H(\omega)$ is the system response.

In general, for any LTI system

time domain view

$$x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t) = x(t) * h(t)$$

$h(t)$ is the impulse response

frequency domain view

$$X(\omega) \rightarrow \boxed{\text{system}} \rightarrow Y(\omega) = X(\omega) \cdot H(\omega)$$

$H(\omega)$ is the system response

Differentiation Property

time domain	$\xrightarrow{\mathcal{F}}$ frequency domain
$x(t)$	$X(\omega)$
$\frac{d}{dt}x(t)$	$(j\omega)X(\omega)$

example: Revisit $\frac{dy}{dt} + 2y(t) = 2x(t)$

using differentiation property

Take Fourier Transforms to obtain an algebraic equation

$$(j\omega)Y(\omega) + 2Y(\omega) = 2X(\omega)$$

$$\Rightarrow \frac{Y(\omega)}{X(\omega)} = \frac{2}{2+j\omega}$$

$$\Rightarrow H(\omega) = \frac{2}{2+j\omega}$$ as before.

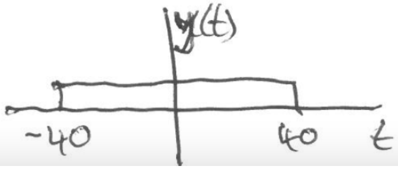
Multiplication Property

time domain	$\xrightarrow{\mathcal{F}}$ frequency domain
$x(t)$	$X(\omega)$
$y(t)$	$Y(\omega)$
$z(t) = x(t)\cdot y(t)$	$Z(\omega) = \frac{1}{2\pi}X(\omega) * Y(\omega)$

example: $x(t) = \cos(\frac{\pi}{2}t)$

I want to limit it to the time interval $(-40, 40)$

so I multiply $x(t)$ by $y(t) = \text{rect}(\frac{t}{80})$



$$z(t) = \cos\left(\tfrac{\pi}{2}t\right) \cdot \mathrm{rect}\left(\tfrac{t}{80}\right)$$

$$Z(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$= \frac{1}{2\pi}\left(\pi\delta\left(\omega+\tfrac{\pi}{2}\right) + \pi\delta\left(\omega-\tfrac{\pi}{2}\right)\right) * Y(\omega)$$

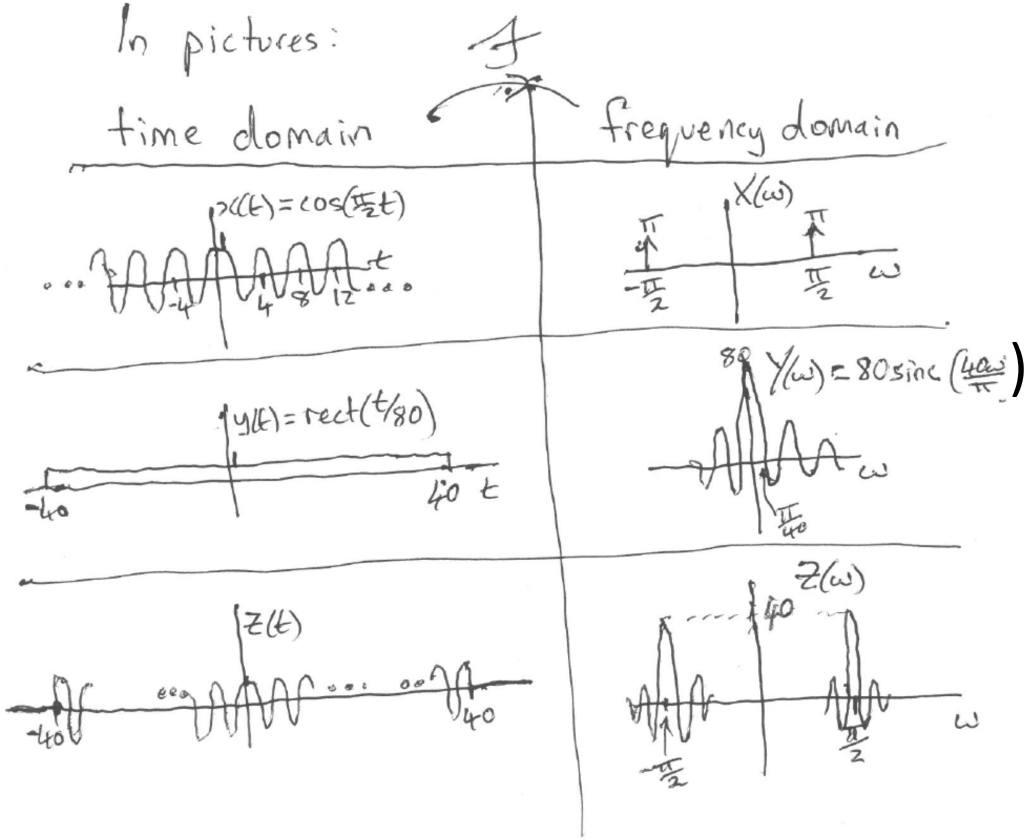
$$Y(\omega) = 80\,\mathrm{sinc}\left(\frac{80\omega}{2\pi}\right) = 80\,\mathrm{sinc}\left(\frac{40\omega}{\pi}\right)$$

$$\Rightarrow Z(\omega) = \left(\tfrac{1}{2}\delta\left(\omega+\tfrac{\pi}{2}\right) + \tfrac{1}{2}\delta\left(\omega-\tfrac{\pi}{2}\right)\right) * 80\,\mathrm{sinc}\left(\frac{40\omega}{\pi}\right)$$

$$= 40\,\mathrm{sinc}\left(\frac{40(\omega+\frac{\pi}{2})}{\pi}\right) + 40\,\mathrm{sinc}\left(\frac{40(\omega-\frac{\pi}{2})}{\pi}\right)$$

$$= 40\,\mathrm{sinc}\left(\frac{40\omega}{\pi} + 20\right) + 40\,\mathrm{sinc}\left(\frac{40\omega}{\pi} - 20\right)$$

In pictures:



This time limiting operation is called windowing.

Picture on RHS shows effect in frequency domain.

More advanced topics
(Not for examination)

- ❑ Fourier Transform of unit step function
- ❑ Integration Property of Fourier Transform
- ❑ Fourier Transforms of Periodic Signals
- ❑ Fourier Transform of a train of impulses

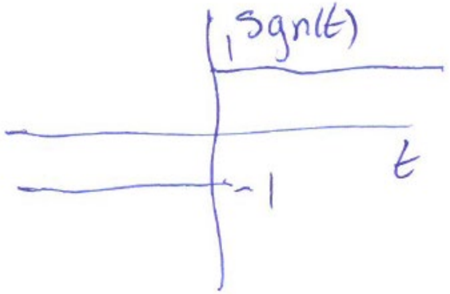
Fourier Transform of unit step function

There is also an integration property of the Fourier Transform.

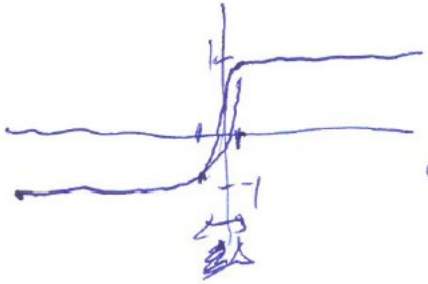
We will derive this by first deriving the Fourier Transform of the unit step function.

Even before that, we focus on a related signal, the *sgn(t)* signal.

The *sgn(t)* signal takes one of two values: +1 and -1 as depicted here:



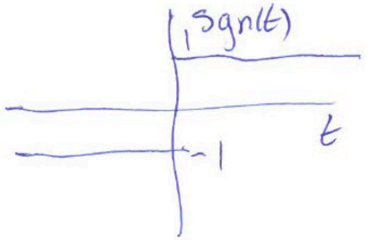
The *sgn(t)* signal can be obtained as the limit of a sequence of continuous signals:

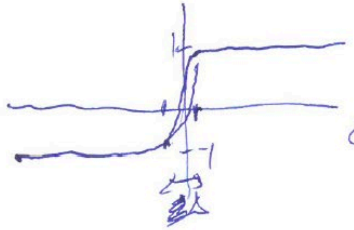


taking limit as $\Delta \downarrow 0$

$\leftarrow$ & we get sgn(t)

Fourier Transform of *sgn(t)*





taking limit as $\Delta \downarrow 0$

$\leftarrow$ & we get sgn(t)

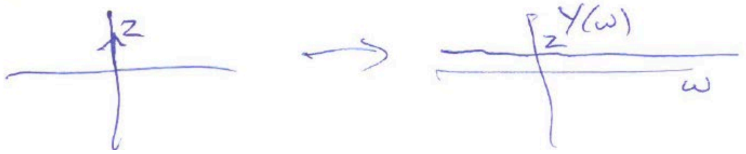
$$\Rightarrow \frac{d}{dt}\,\mathrm{sgn}(t) = 2\,\delta(t)$$

$$x(t) = \mathrm{sgn}(t) \xrightarrow{\mathcal{F}} X(\omega)$$

$$y(t) = \frac{d}{dt}\,\mathrm{sgn}(t) \xrightarrow{\mathcal{F}} (j\omega)\,X(\omega) = Y(\omega)$$

$$\| \quad 2\,\delta(t)$$

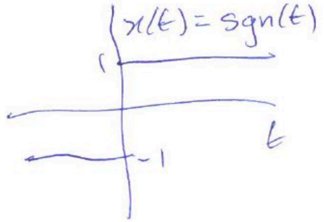
$$y(t) = 2\,\delta(t) \xrightarrow{\mathcal{F}} Y(\omega) = 2$$



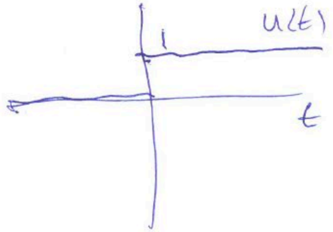
Fourier Transform of unit step function

$$\Rightarrow (j\omega)\,X(\omega) = 2$$

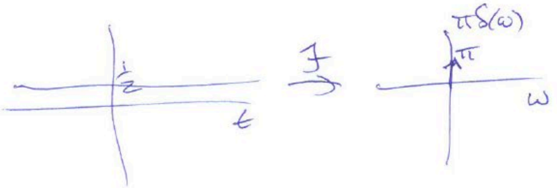
$$\Rightarrow X(\omega) = \frac{2}{j\omega}$$



$$\xrightarrow{\mathcal{F}} X(\omega) = \frac{2}{j\omega}$$



$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$



$$u(t) \xrightarrow{\mathcal{F}} U(\omega)$$

$$U(\omega) = \pi\,\delta(\omega) + \frac{1}{2}\,\frac{2}{j\omega}$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega}$$

Integration Property

integration property -

$$x(t) \longrightarrow X(\omega)$$

$$g(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \rightarrow \underline{G(\omega)}$$

$$g(t) = x(t) * u(t)$$

now apply convolution property

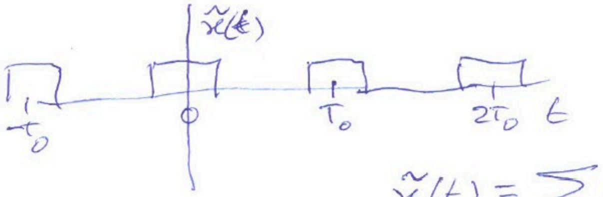
$$G(\omega) = X(\omega) \cdot U(\omega)$$

$$= X(\omega) \cdot \left(\pi \delta(\omega) + \frac{1}{j\omega}\right)$$

$$= \pi X(0)\, \delta(\omega) + \frac{X(\omega)}{j\omega}$$

Connection between Fourier Transform and Fourier Series

Suppose $x(t) \rightleftarrows X(\omega)$



$\tilde{x}(t)$ is periodic

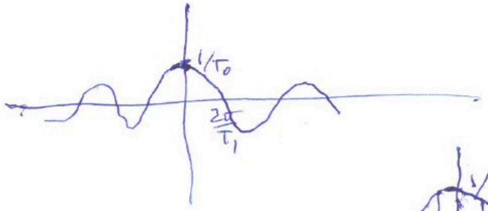
$T_0 \gg T_1$

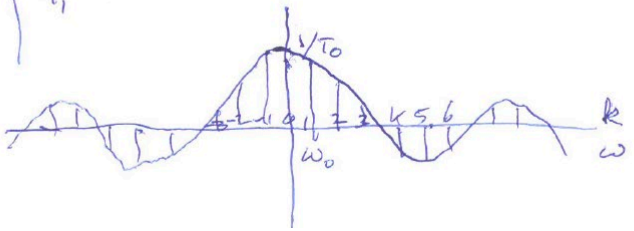
$$\tilde{x}(t) = \sum_k C_k \exp(j k \omega_0 t)$$

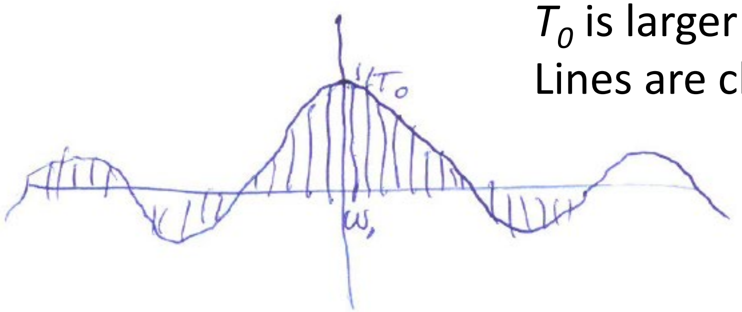
$$\omega_0 = \frac{2\pi}{T_0}$$

Its not hard to show

$$C_k = \frac{1}{T_0} X(k\omega_0)$$







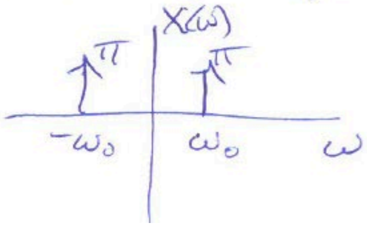
T_0 is larger here

Lines are closer together

Fourier Transform of a periodic signal

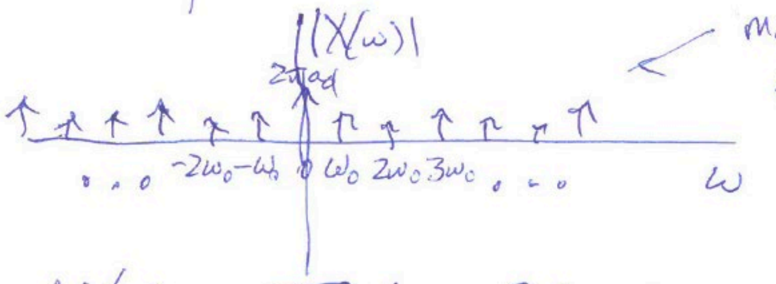
Recall that sinusoids have impulses as Fourier Transforms

$$x(t) = \cos(\omega_0 t) \qquad X(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transform of periodic signals

Lets look at an arbitrary ~~that~~ signal that consists of evenly spaced δ functions in the frequency domain



magnitude at freq $k\omega_0$ is $2\pi|a_k|$

$$X(\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0)$$

Take inverse transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\sum_k 2\pi a_k \delta(\omega - k\omega_0)\Big)\cdot\exp(j\omega t)\,d\omega$$

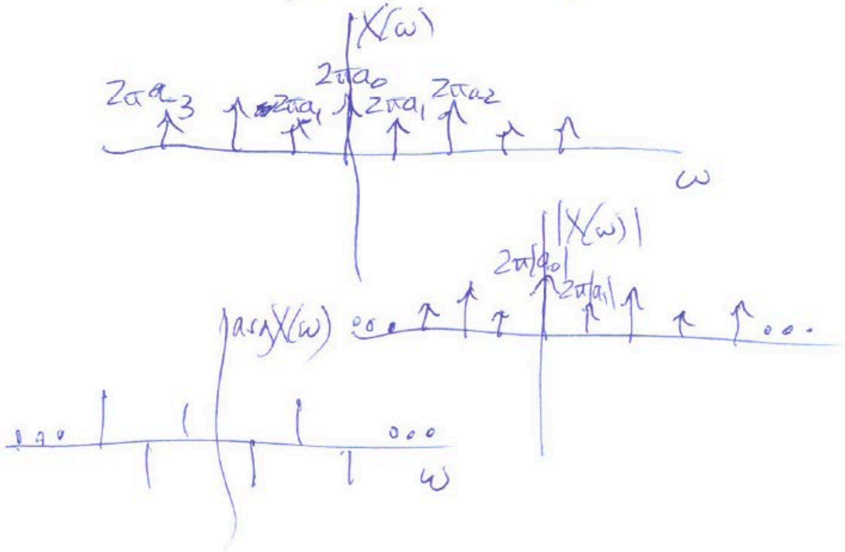
$$= \sum_k a_k \int_{-\infty}^{\infty} \delta(\omega - k\omega_0)\exp(j\omega t)\,d\omega$$

$$= \sum_k a_k \exp(jk\omega_0 t)$$

This is a periodic signal!

a_ks are Fourier Series coefficients

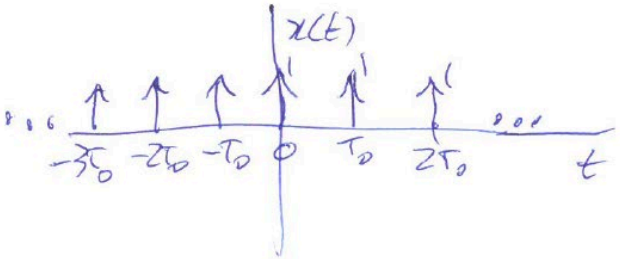
In general, a periodic signal has a spectrum consisting of evenly spaced δ functions



Train of Impulses

An important example when we do sampling is a train of impulses

$$x(t) = \sum_k \delta(t - kT_0)$$



The previous result shows that $X(\omega)$ is a train of impulses in the frequency domain

$$X(\omega) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \qquad \omega_0 = \frac{2\pi}{T_0}$$