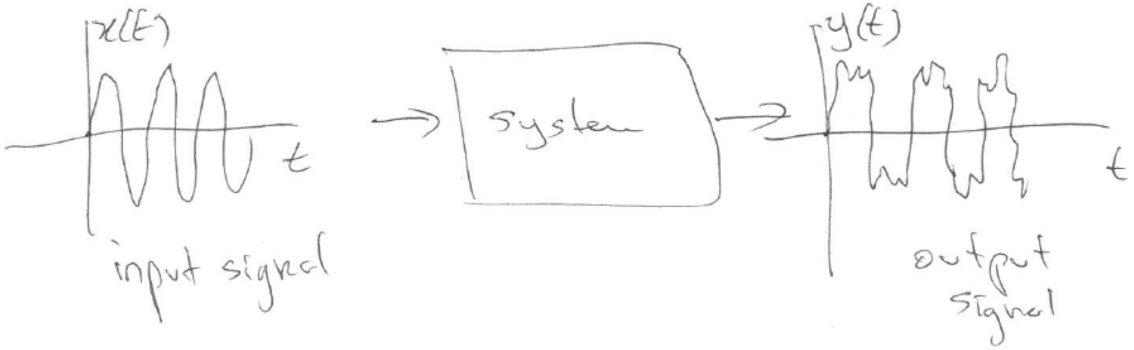
Linear Time-invariant Systems and Impulses + intro to convolution

SIGNALS AND SYSTEMS

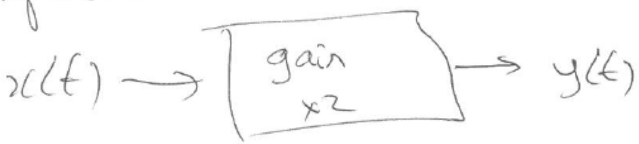
Week 3

Signals and Systems

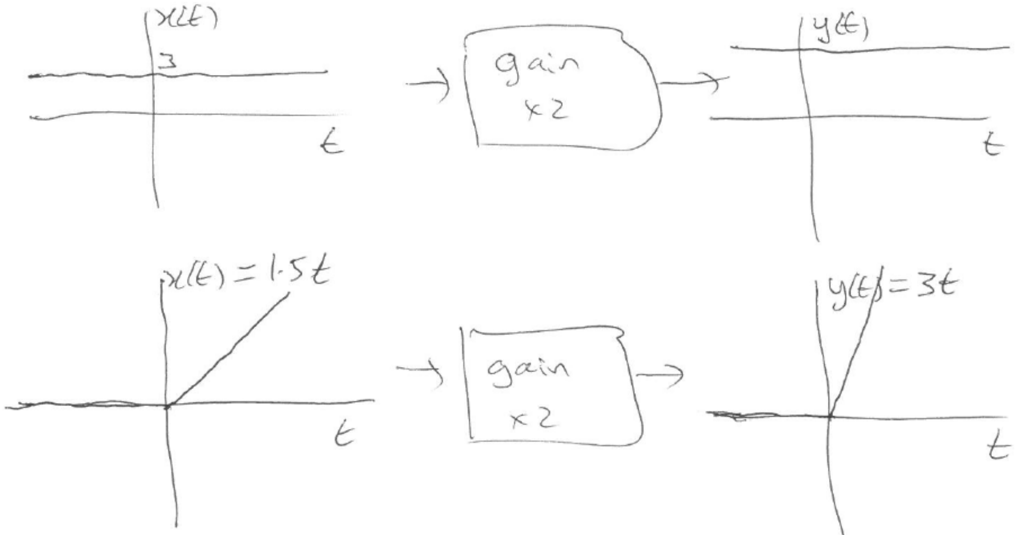


examples:

1 amplifier



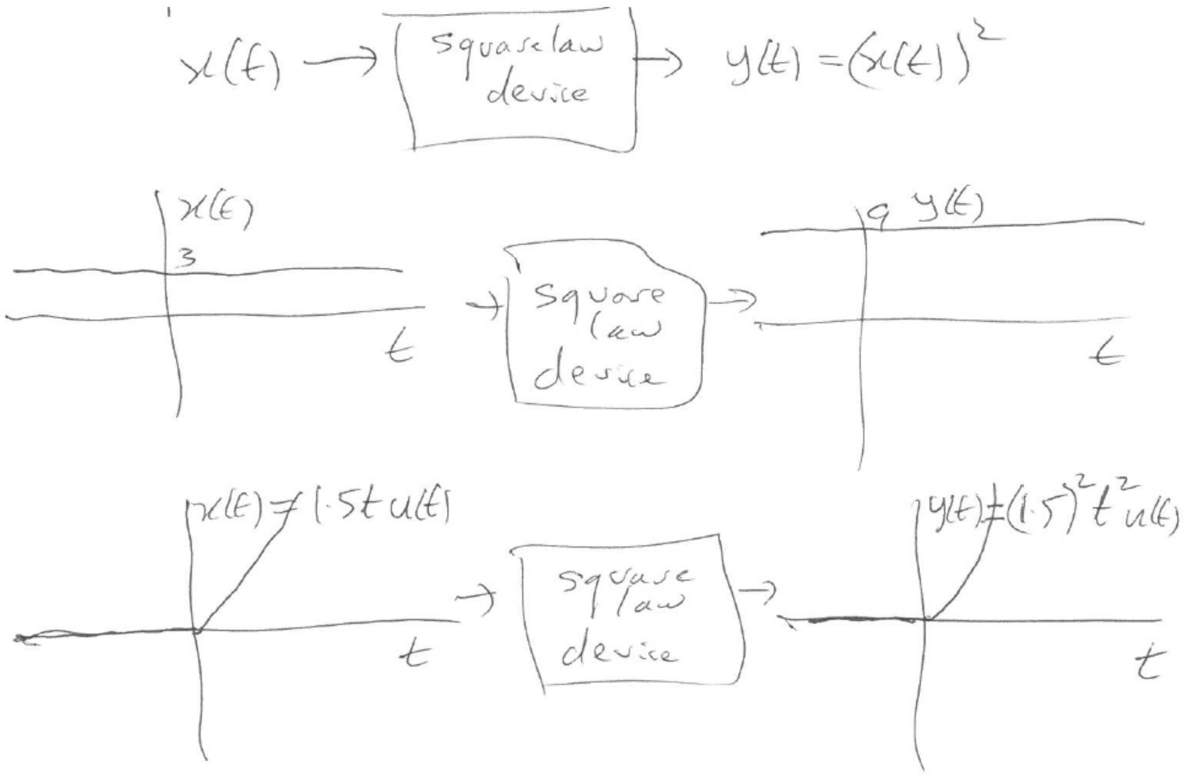
1. Gain



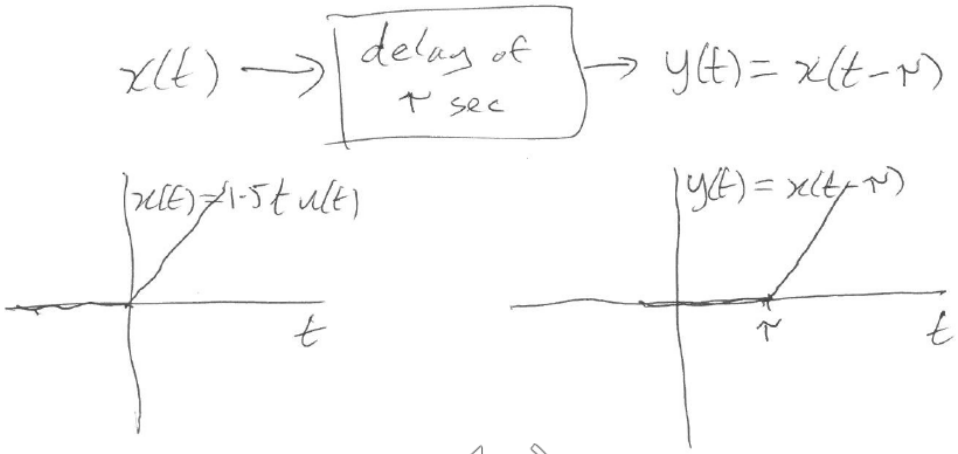
$x(t) = 1.5t\,u(t)$

$y(t) = 3t\,u(t)$

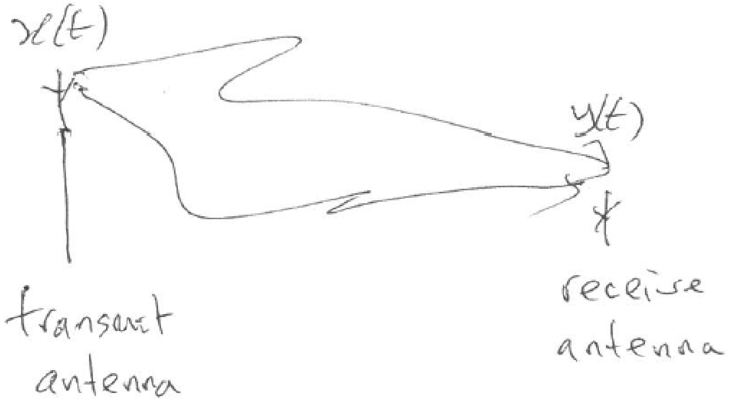
2. Square law device



3. Delay



4. Two path radio channel



$$y(t) = h_1 x(t-\tau_1) + h_2 x(t-\tau_2)$$

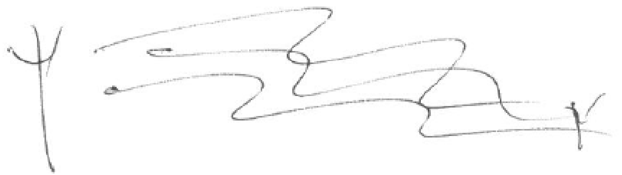
gain of path 1

delay of path 1

gain of path 2

delay of path 2.

5. n-path radio channel

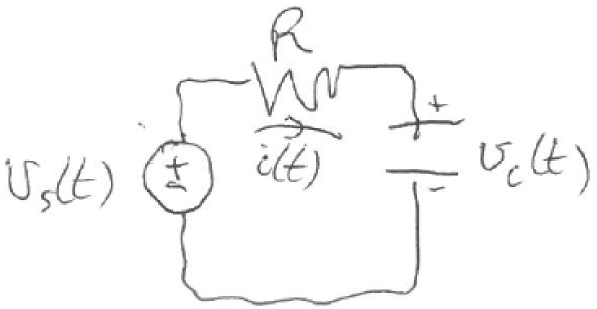


$$y(t) = \sum_{i=1}^{n} h_i x(t-\tau_i)$$

6. Continuum of paths

$$x(t) \to \boxed{\text{radio channel}} \to y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

7. Electrical Circuit



$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c}{dt}$$

$$\Rightarrow C\frac{dv_c}{dt} = \frac{1}{R}v_s(t) - \frac{1}{R}v_c(t)$$

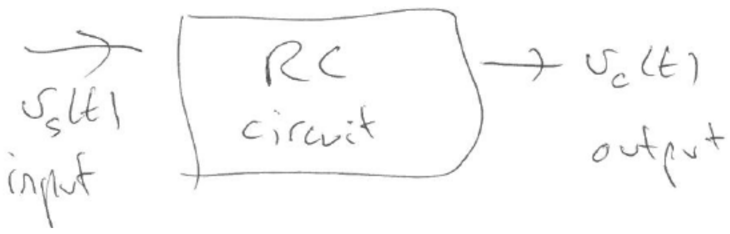
$$\Rightarrow \frac{dv_c}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

eg $RC = \frac{1}{2}$

$$\frac{dv_c}{dt} + 2v_c(t) = 2v_s(t)$$

initial conditions $v_s(t) = 0$ for all $t < 0$

$v_c(t) = 0$ for all $t < 0$



All above systems (except square-law device) are linear, time-invariant systems

Linear Systems

if system is linear

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

$$\Rightarrow \quad \alpha x(t) \longrightarrow \boxed{\text{system}} \longrightarrow \alpha y(t)$$

for any constant α

More generally,

$$x_1(t) \longrightarrow \boxed{\text{system}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{system}} \longrightarrow y_2(t)$$

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for any input signals $x_1(t)$ & $x_2(t)$

& constants α_1, & α_2

then system is linear.

Examples:
1. Amplifier

$$x(t) \rightarrow \boxed{\text{gain} \ \times 2} \rightarrow y(t) = 2x(t)$$

$$x_1(t) \rightarrow \boxed{\text{gain} \ \times 2} \rightarrow y_1(t) = 2x_1(t)$$

$$x_2(t) \rightarrow \boxed{\text{gain} \ \times 2} \rightarrow y_2(t) = 2x_2(t)$$

let α_1, α_2 be arbitrary constants

$$x(t) = \alpha_1 x_1(t) + \alpha_2 x(t) \rightarrow \boxed{\text{gain} \ \times 2} \rightarrow y(t)$$

$$\begin{aligned} y(t) &= 2x(t) \\ &= 2(\alpha_1 x_1(t) + \alpha_2 x_2(t)) \\ &= \alpha_1 (2x_1(t)) + \alpha_2 (2x_2(t)) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

So amplifier system is linear

example 6:

$$x(t) \rightarrow \boxed{\text{radio channel}} \rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) \, d\tau$$

exercise: show it <u>is</u> a linear system

Non-examples:

1. Affine System

$$x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t) = 2x(t) + 3$$

$$x_1(t) = \alpha x(t) \rightarrow \boxed{\text{system}} \rightarrow y_1(t) = 2x_1(t) + 3$$

$$= 2(\alpha x(t)) + 3$$

$$= 2\alpha x(t) + 3$$

$$\alpha y(t) = 2\alpha x(t) + 3\alpha \neq y_1(t)$$

unless $\alpha = 1$

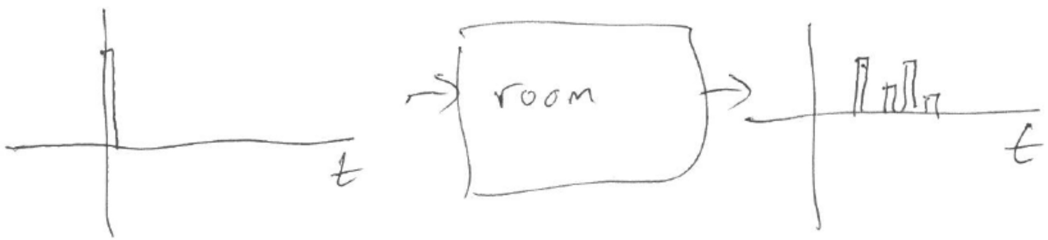
So its not a linear system

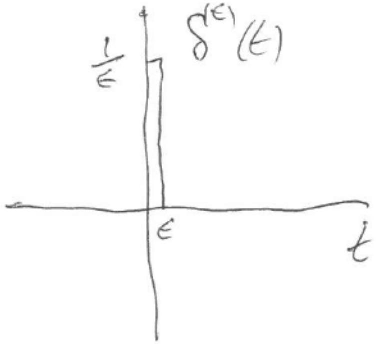
2. Square law device

$$x(t) \rightarrow \boxed{\text{square law device}} \rightarrow y(t) = (x(t))^2$$

exercise to show its not linear

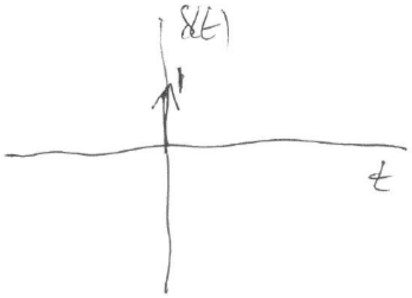
Unit impulse signal





$\delta^{(\epsilon)}(t)$ is a short pulse of duration ϵ seconds

$$\int_{-\infty}^{\infty} \delta^{(\epsilon)}(t)\,dt = 1$$

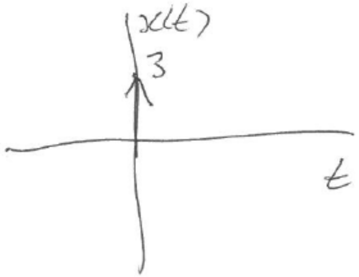


$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{everywhere else } (t \neq 0) \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

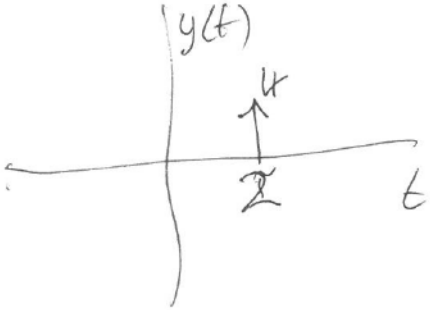
We can think of it as the limit of $\delta^{(\epsilon)}(t)$ as $\epsilon \downarrow 0$

Magnitude of impulse and integration



$x(t) = 3\,\delta(t)$

$$\int_{-\infty}^{\infty} x(t)\,dt = 3$$



$y(t) = 4\,\delta(t-2)$

$$\int_{-\infty}^{\infty} y(t)\,dt = 4$$

$$\int_{-\infty}^{1} y(t)\,dt = 0$$

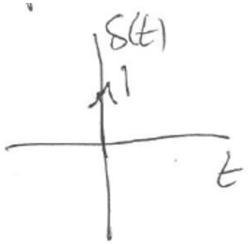
$$\int_{1}^{3} y(t)\,dt = 4$$

etc.

Integration Properties of signals made up of Impulses (ie of delta functions)

We now want to recall some integration properties of the unit δ-function:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

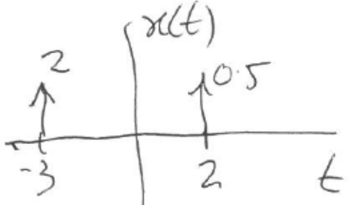


$$\int_{-\infty}^{-1} \delta(t)\,dt = 0 \qquad \int_{0.5}^{\infty} \delta(t)\,dt = 0$$

$$\int_{-0.3}^{0.5} \delta(t)\,dt = 1$$

Here is a signal with two delta functions:

$$x(t) = 2\delta(t+3) + 0.5\,\delta(t-2)$$



We can also integrate this signal, *x(t)*, over different intervals

$$\int_{-\infty}^{-4} x(t)\,dt = 0 \qquad \int_{-\infty}^{0} x(t)\,dt = 2$$

$$\int_{1}^{3} x(t)\,dt = 0.5 \qquad \int_{2.1}^{\infty} x(t)\,dt = 0$$

$$\int_{-\infty}^{\infty} x(t)\,dt = 2.5$$

The rule is to add up the magnitudes of the impulses that lie in the range of integration.

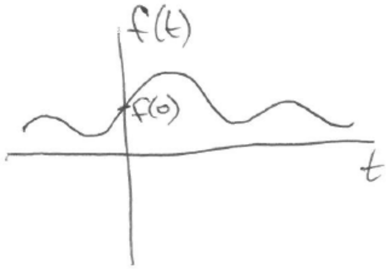
Since we are looking at integrating the unit impulse signal, there is a simple connection between the unit impulse signal and the unit step function, which is obtained by integration: $u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

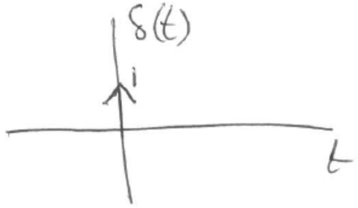
why? when $t<0$ $\int_{-\infty}^{t} \delta(\tau)\,d\tau = 0$

when $t>0$ $\int_{-\infty}^{t} \delta(\tau)\,d\tau = 1$

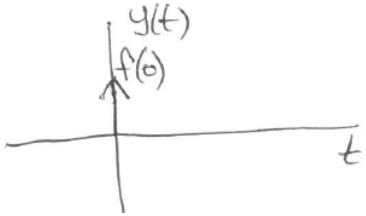
Some other Properties of unit impulses

What happens if we multiply a signal by $\delta(t)$?



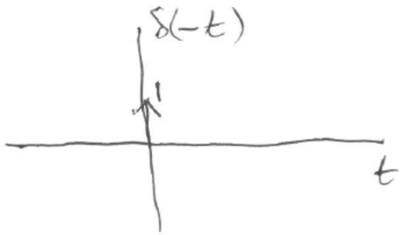


$y(t) = f(t)\delta(t)$



What happens if we multiply a signal by $\delta(t - t_0)$?

What does $\delta(-t)$ look like ?

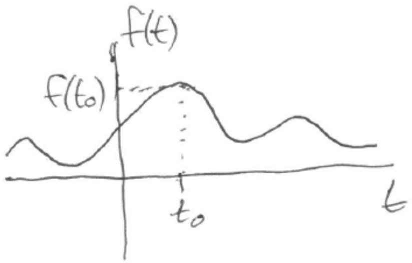


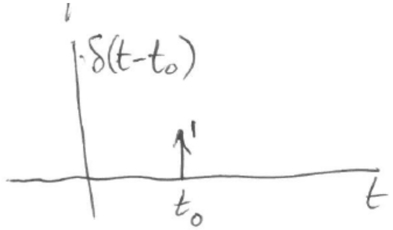
i.e. $\delta(-t) = \delta(t)$

Since $\delta(-t) = \delta(t)$, we have:

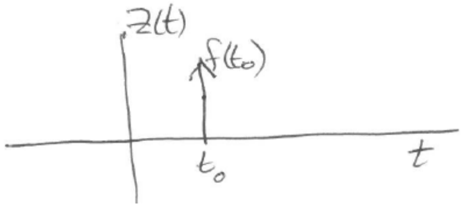
$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = \int_{-\infty}^{\infty} f(t)\delta(-t)\,dt$$

(useful later)

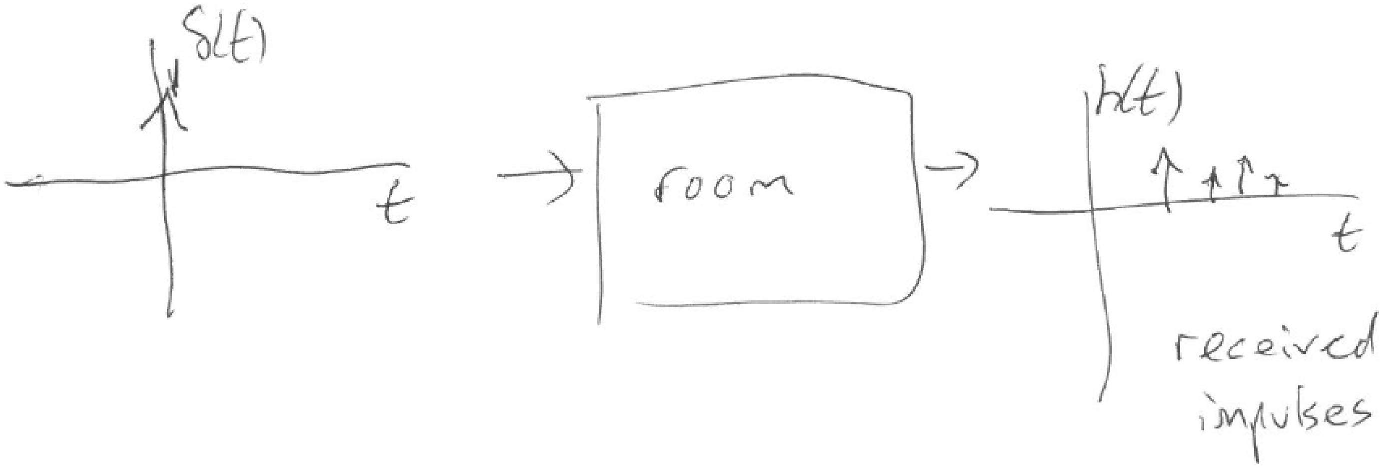




$z(t) = f(t)\delta(t - t_0)$



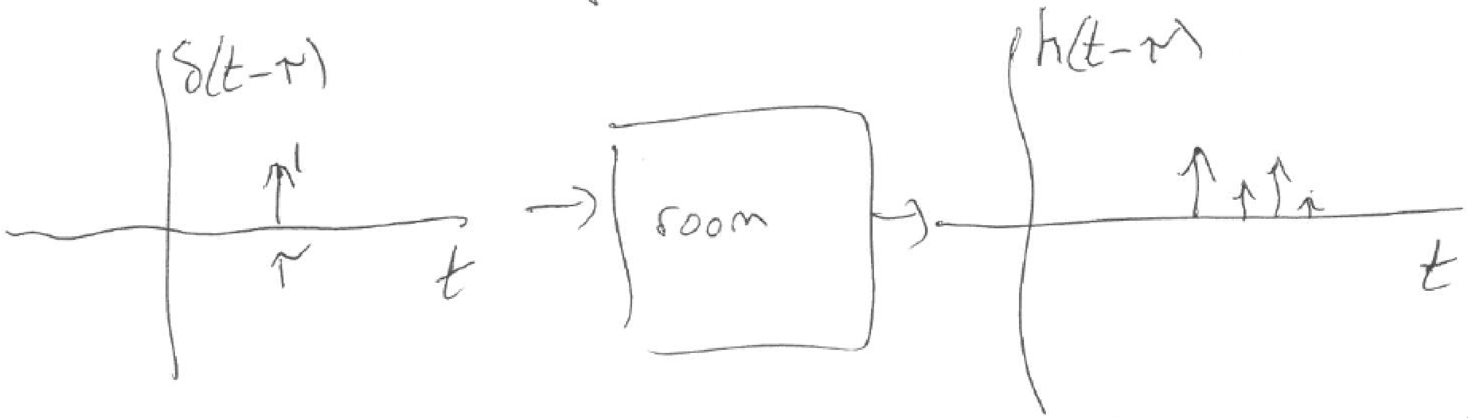
Impulse response of a system



h(t) is called the impulse response of the system

Time invariant system

The system is time-invariant if the system impulse response is the same, no matter what time we send the impulse

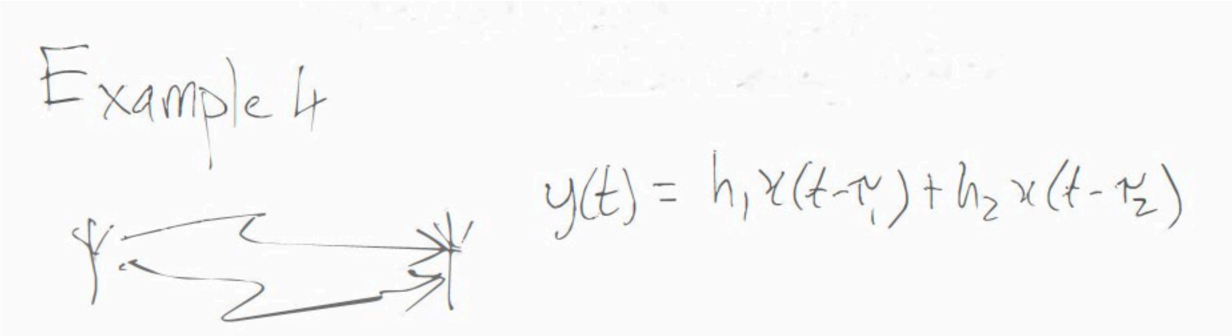


if system is time-invariant output will be $h(t-\tau)$

A linear time-invariant system (a LTI system) is one that is linear & its time invariant

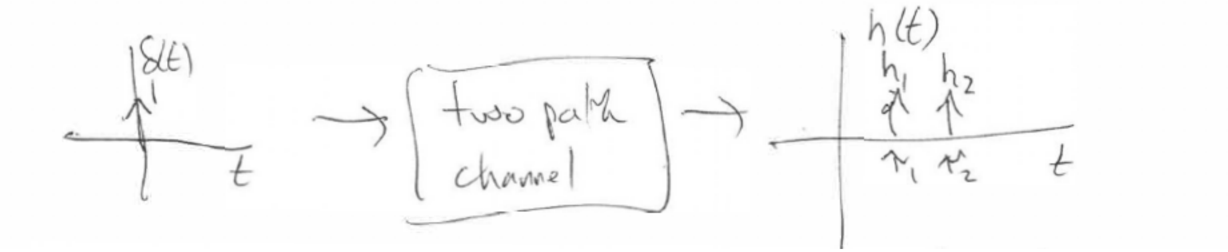
This focus from now on is on Linear, Time-invariant Systems. We call such systems LTI systems for short.

Recall the two path radio channel example:



We want to show that this system is *time invariant*. (We saw it was *linear* above).

To show this, consider the impulse response:



What happens if we delay the impulse to start at a later time (eg. at time t = τ) ?.

The picture here shows that all that happens is that the impulse response shifts to the right by τ seconds.

We can use mathematics to show that the previous picture is indeed correct.

Let the input signal be $x(t)=\delta(t-\tau)$

Remember the system equation

$$y(t) = h_1 x(t-\tau_1) + h_2 x(t-\tau_2)$$

So the output corresponding to this particular input $x(t)$ is:

$$y(t) = h_1 \delta((t-\tau)-\tau_1) + h_2 \delta((t-\tau)-\tau_2)$$

Recall the formula for $h(t)$ for the two path channel is:

$$h(t) = h_1 \delta(t-\tau_1) + h_2 \delta(t-\tau_2)$$

$$\text{So } h(t-\tau) = h_1 \delta((t-\tau)-\tau_1) + h_2 \delta((t-\tau)-\tau_2)$$

$$= y(t)$$

We conclude that the output $y(t)$ is just $h(t)$ delayed by τ seconds.

This means that the system is time invariant.

The Sifting Property of the δ-function

A very important property of the δ-function is the sifting property

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} f(0)\,\delta(t)\,dt$$
$$= f(0)$$

This first equation says that we can use the unit δ-function to *sift* out the value of the signal *f(t)* at time *t=0*.

A more general version of the sifting property of a δ-function is the following:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-\tau_1)\,dt = \int_{-\infty}^{\infty} f(\tau_1)\,\delta(t-\tau_1)\,dt = f(\tau_1)$$

This second equation says that if we multiply a signal *f(t)* by a unit impulse, with the impulse located at a particular time, say, $t=\tau_1$, and we integrate the whole thing, we *sift* out the value of the signal f(t) at that particular time ie we sift out the value $f(\tau_1)$.

In other words, we can use the unit δ-function to sift out the value of the signal *f(t)* at any *arbitrary* time $t=\tau_1$.

Convolution

Section 2.4 in Haykin and Van Veen, p 115; Section 2.2 in Oppenheim and Willsky p90

Convolution is an operation on two signals; it will be of fundamental importance to LTI Systems

If we have two signals $h(t)$ and $x(t)$ their convolution is the signal $y(t) = h(t)*x(t)$.

The equation for convolution is :

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Convolution Equation Explained

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Think of t as a fixed constant

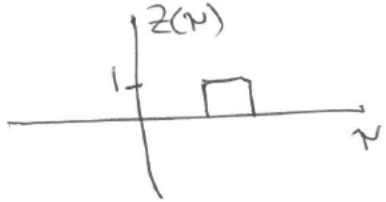
What then is $x(t-\tau)$?

Think of τ as the input variable for the signal $z(\tau) = x(t-\tau)$

τ is the time variable of $z(\tau)$

The integral above is then $\int_{-\infty}^{\infty} h(\tau)\, z(\tau)\, d\tau$

eg. suppose $z(\tau)$ is



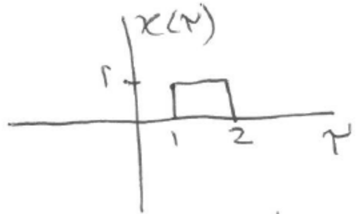
& suppose $h(\tau)$ is

then $x(t-\tau)h(\tau)$ is

& therefore $y(t)$ = area under this curve.

What does $z(\tau)$ look like?

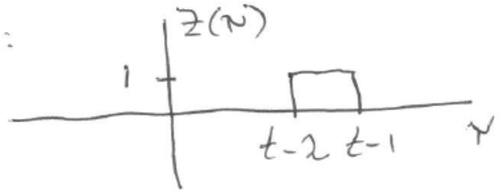
Suppose $x(\tau)$ is



& t is a fixed constant

$z(\tau) = x(t-\tau)$. What does it look like?

It looks like:



Why?

when $t-2 < \tau < t-1$

we have $1 < t-\tau < 2$

So $x(t-\tau) = 1$

when $\tau < t-2$ we have $t-\tau > 2$

$\Rightarrow x(t-\tau) = 0$

when $\tau > t-1$ we have $t-\tau < 1$

$\Rightarrow x(t-\tau) = 0$

Thus,

$$z(\tau) = x(t-\tau) = \begin{cases} 1 & t-2 < \tau < t-1 \\ 0 & \text{otherwise} \end{cases}$$

Convolution commutes; in other words, *h(t)*x(t) = x(t)*h(t),* which can be verified by showing:

$$\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

This is an exercise for the diligent student (hint: do a change of variables in the integration)

The formula for convolution should trigger a memory.

Recall the radio channel with a continuum of paths.

This channel had the input-output equation given by: $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$

In other words, *y(t)=h(t)*x(t).* Here *h(*τ*)* is the channel gain of a path of delay τ

What is the impulse response of this channel?

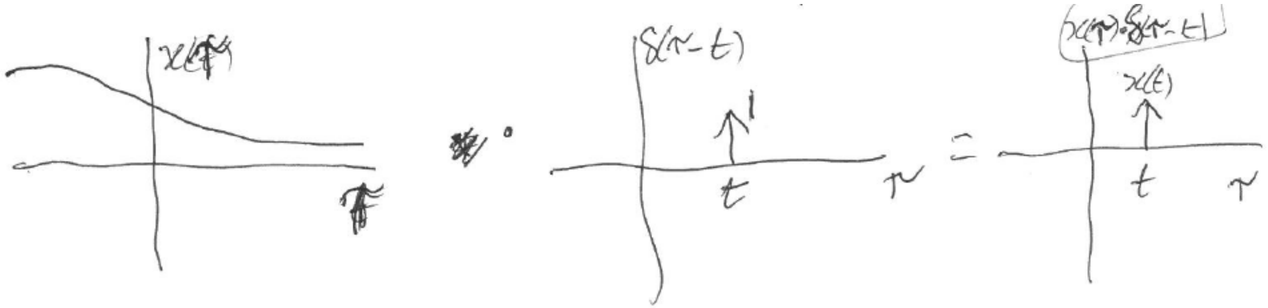
Convolution with a δ function

what is $y(t) = x(t) * \delta(t)$?

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \underline{x(\tau)\,\delta(\tau-t)}\,d\tau$$

$= x(t)$ by sifting property



This shows that convolving a signal with the unit impulse doesn't do anything to the signal. Its like multiplying a number by 1: no change to the number. Here, there is no change to the signal *x(t)*.

Radio Channel as a Convolution

Lets return to the radio channel with input-output equation: $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

and ask the question: What is the impulse response of this channel?

It will be the output, *y(t)*, when the input is given by *x(t)=δ(t),* which is *y(t)=h(t)* δ(t)*

But by the above result (previous page) about convolution with a δ function, *y(t)=h(t)* δ(t) = h(t)*

We conclude that *h(t)* is in fact the impulse response of the channel given by $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

This equation says that the output of the channel is obtained by convolving the impulse response of the channel with the input signal

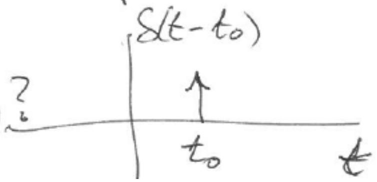
Since convolution commutes, we could equally say that the output of the channel is obtained by convolving the input signal with the impulse response of the channel, ie.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Convolution with a δ function

Another look at convolution with impulses

what is $y(t) = x(t) \overset{\text{convolution}}{*} \delta(t-t_0)$?



let $w(t) = \delta(t-t_0)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, w(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau-t_0)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta((t-t_0)-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, \delta(\tau-(\underline{t-t_0}))\, d\tau$$

$$= x(t-t_0) \quad \text{by sifting property of } \delta(t)$$

Here, we are looking at a shifted delta function, or, in other words, a delayed unit impulse (well, delayed if $t_0 > 0$).

The mathematics here shows that convolving a signal with an impulse located at a particular time, delays the signal by that amount of time, and that is all that it does.

This results includes the earlier result, which was the special case when the impulse happens at time t=0. Delaying a signal by 0 seconds doesn't do anything to a signal.

The result on this page is more general, allowing the impulse to occur at an *arbitrary* time t_0.