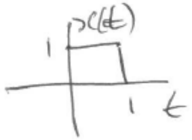
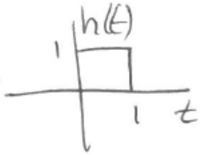
Convolution

Week 4

Convolution of two square pulses

convolution of  with 

$$x(t) = h(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w.} \end{cases}$$

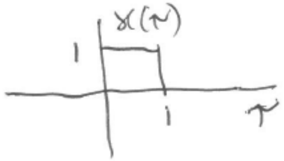
$$y(t) = x(t) * h(t).$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

We want to compute $y(t)$ at some fixed time t

integrate $x(\tau)h(t-\tau)$ with τ the dummy variable of integration

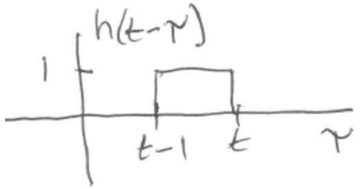
lets think about $x(\tau)$ & $h(t-\tau)$ as functions of τ



what about $h(t-\tau)$?

$$h(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases}$$

so $$h(t-\tau) = \begin{cases} 1 & 0 \le t-\tau < 1 \\ 0 & \text{o.w.} \end{cases}$$

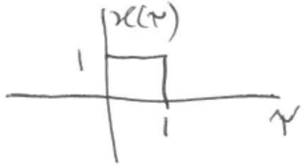


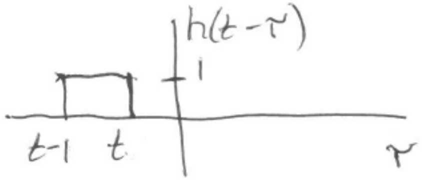
$$= \begin{cases} 1 & t-1 < \tau < t \\ 0 & \text{o.w.} \end{cases}$$

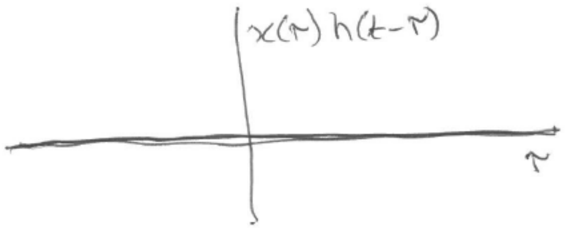
What does $x(\tau)h(t-\tau)$ look like as a function of τ?

Answer: it depends on the value of t

eg. when $t<0$:







product function is zero for all values of τ

$$\therefore \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = 0$$

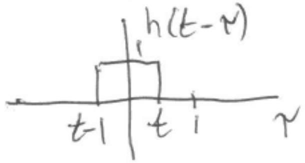
in this case

Think of trying to evaluate $y(t)$ for different values of t.

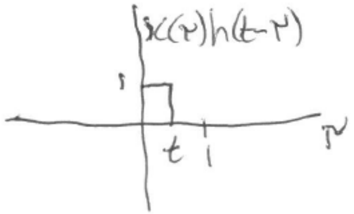
For $t<0$, $y(t)=0$

As t increases, square pulse $h(t-\tau)$ moves to the right.

When $0<t<1$ what happens?



product function $x(\tau)h(t-\tau)$ is no longer identically zero (for ALL τ)

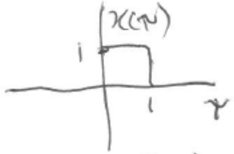


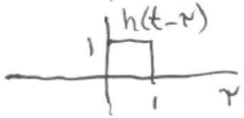
area under this function $= t$

So $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = t$

So, when $0<t<1$, $y(t)=t$

When $t=1$ what happens?

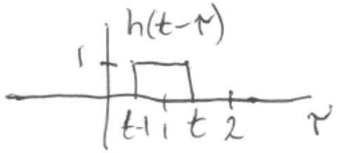


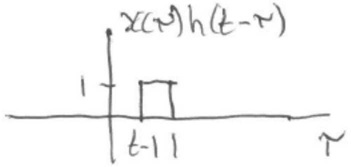


area under this function $=1$

So, $y(1)=1$

what happens for $1<t<2$?

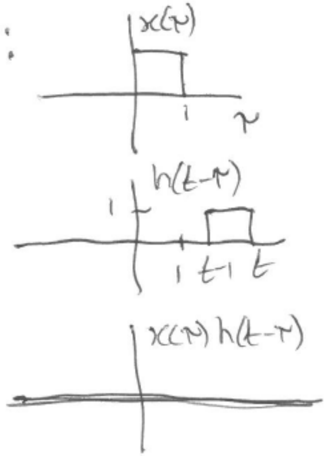




area under this function $= 1-(t-1)$
$= 2-t$

So, when $1<t<2$, $y(t)=2-t$

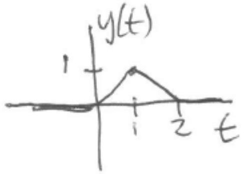
For $t>2$:



area $= 0$

so $y(t) = 0$

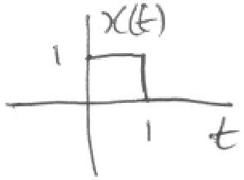
$$\therefore \quad y(t) = \begin{cases} 0 & t<0 \\ t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & t \geq 2 \end{cases}$$



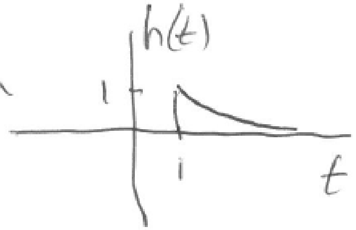
$\boxed{\text{This is convolution of } x(t) \text{ with } h(t)}$

Convolution of square pulse with exponential

convolution of



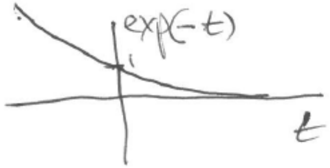
with



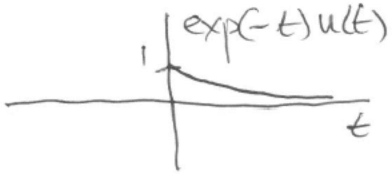
decaying exponential

equation for h(t):

start with

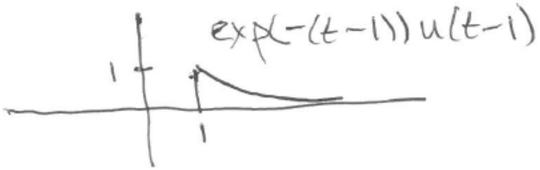


chop off values for $t<0$



now delay signal by 1 sec

recall: $t \rightarrow t-1$



So $h(t) = \exp(-(t-1))\,u(t-1)$

convolution:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

what is $h(t-\tau)$ as a function of τ?

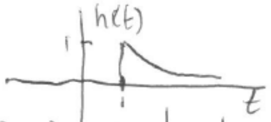
replace t with $t-\tau$ in equation ~~for~~ $h(t)$

$$h(t) = \exp(-(t-1))\, u(t-1)$$

$$h(t-\tau) = \exp(-((t-\tau)-1))\, u((t-\tau)-1)$$

$$= \exp(-((t-1)-\tau))\, u((t-1)-\tau)$$

lets draw $h(t-\tau)$.



note: $h(1) = 1$ ($h(t)$ jumps to 1 at $t=1$)

so when $\tau = t-1$, $h(t-\tau) = 1$ also

(* why? $h(t-\tau) = h(t-(t-1)) = h(1) = 1$ *)

when $\tau > t-1$, $u((t-1)-\tau) = 0$

(* why? when $\tau > t-1$, $(t-1)-\tau < 0$

so $u((t-1)-\tau) = 0$ *)

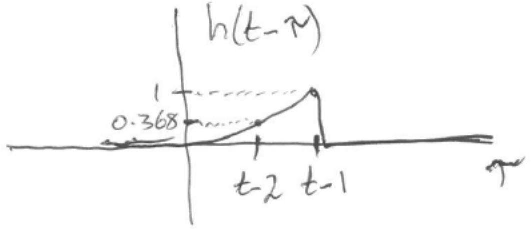
so $h(t-\tau) = 0$ in this case.

when $\tau < t-1$, $u((t-1)-\tau) = 1$

so $h(t-\tau) = \exp(-((t-1)-\tau))$ in this case,

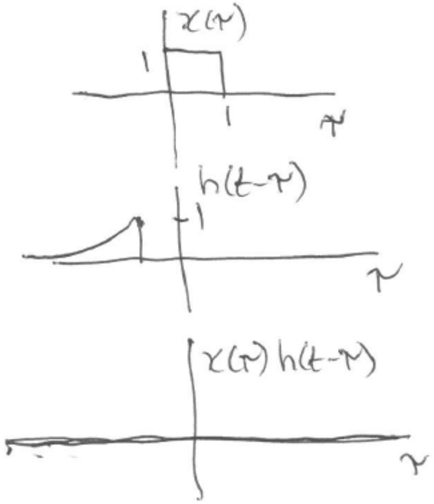
eg if $\tau = t-2$, $h(t-\tau) = \exp(-1) \approx 0.368$

(* why? $(t-1)-\tau = (t-1)-(t-2) = 1$ *)



Now lets draw $x(\tau)h(t-\tau)$ for different values of t.

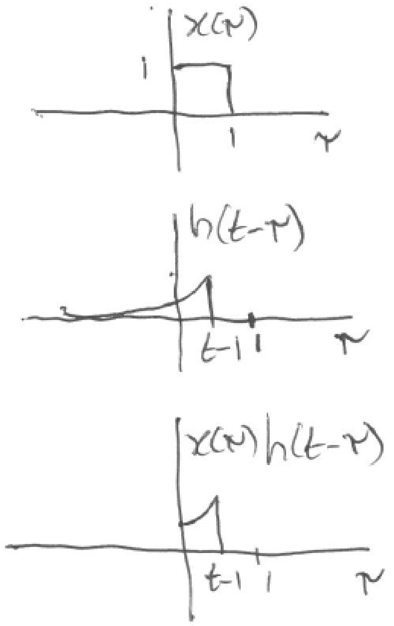
When $t<1$, we have $t-1<0$



$x(\tau)h(t-\tau)=0$ for all τ

$\therefore\ y(t)=0$

when $1<t<2$, we have $0<t-1<1$



$y(t)=$ area under this function

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$=\int_{0}^{t-1} 1.\ \exp(-((t-1)-\tau))\,d\tau$$

$$=\int_{0}^{t-1} \exp(-(t-1))\exp(\tau)\,d\tau$$

$$=\exp(-(t-1))\int_{0}^{t-1}\exp(\tau)\,d\tau$$

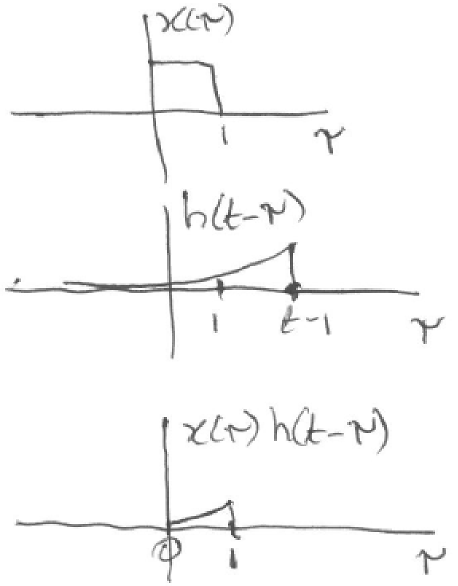
$$=\exp(-(t-1))\big[\exp(\tau)\big]_{0}^{t-1}$$

$$=\exp(-(t-1))\big[\exp((t-1))-1\big]$$

$$=1-\exp(-(t-1))$$

0.632
$=1-e^{-1}$

For $t > 2$, we have $t-1 > 1$



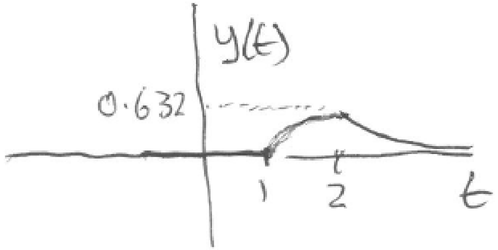
$$y(t) = \int_0^1 1 \cdot \exp(-((t-1)-\tau))\, d\tau$$

$$= \exp(-(t-1)) \int_0^1 \exp(\tau)\, d\tau$$

$$= \exp(-(t-1)) \left(\exp(1) - 1\right)$$

$$y(2) = 1 - \exp(-1) \approx 0.632$$

$y(t) \downarrow 0$ as $t \uparrow \infty$



$\boxed{\text{this is convolution of } x(t) \text{ with } h(t)}$

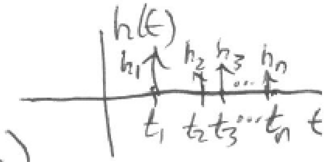
Linear Time-invariant Systems and Convolution

Recall, if we have a L.T.I. system with impulse response $h(t)$, then the output signal $y(t)$ from an input signal $x(t)$ can be computed via convolution:

$$y(t) = x(t) * h(t)$$

So, convolution is the way to think about L.T.I. systems in the time domain.

LTI~~s~~ Systems are pretty simple when $h(t)$ is just a train of impulses

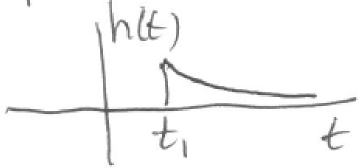


ie when $h(t) = \sum_{i=1}^{n} h_i \delta(t - t_i)$

then $y(t) = x(t) * h(t) = \sum_{i=1}^{n} h_i x(t - t_i)$

But outputs of LTI systems are harder to compute when $h(t)$ is a continuous signal as we've just seen.

eg

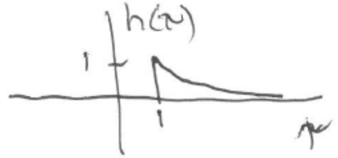


Proceed as above!

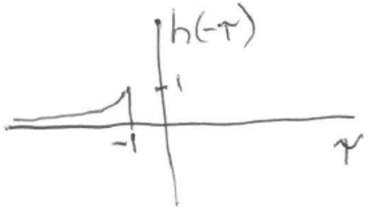
Convolution: time flip and time shift method

Optional extra: we can obtain $h(t-\tau)$ – as a function of τ – by flip & shift method

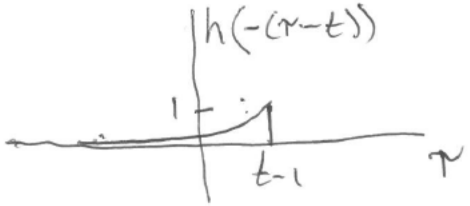
eg.



flip time direction
ie. reverse time
$\tau \to -\tau$

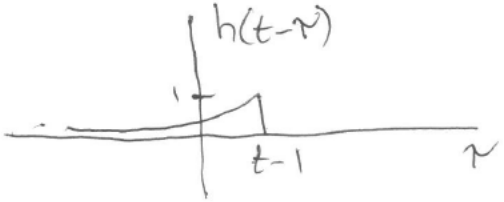


now delay signal by t sec
$\Leftrightarrow$ shift it right by t sec
$\Leftrightarrow$ $\tau \to \tau - t$



But note that $-(\tau - t) = t - \tau$

So the above graph ~~is~~ can equivalently be labelled $h(t-\tau)$



Frequency Domain View of Signals

Periodic signals can be decomposed as sums of complex sinusoids (Fourier Series)

Very simple examples are:

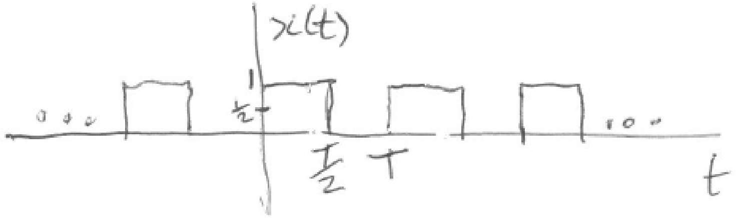
$$\cos(\omega_0 t) = \frac{1}{2}\exp(-j\omega_0 t) + \frac{1}{2}\exp(j\omega_0 t)$$

$$\sin(\omega_0 t) = \frac{1}{2j}\left(\exp(j\omega_0 t) - \exp(-j\omega_0 t)\right)$$

$$= \frac{j}{2}\exp(-j\omega_0 t) - \frac{j}{2}\exp(j\omega_0 t)$$

Surprisingly, we can do this for any periodic signal!

e.g. periodic square wave



fundamental frequency is $\omega_0 = \frac{2\pi}{T}$ rad/sec

This signal has a sinusoidal component at frequency $-\omega_0$ rad/sec & at $+\omega_0$ rad/sec

It also has sinusoidal components at $-k\omega_0$ & $+k\omega_0$ rad/sec for all integers $k = 0, 1, 2, \ldots$

These components are called harmonics.

The 1st harmonic is the fundamental frequency

All periodic signals are like this!

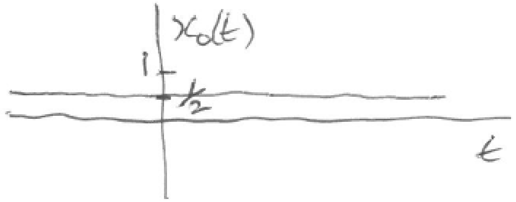
To see this, consider approximating $x(t)$ with sinusoids

periodic square wave signal

The zeroth order approximation is

$$x_0(t) = \frac{1}{2} \text{ for all } t$$

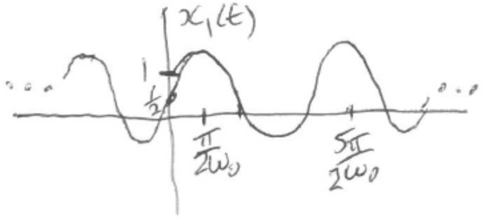
(* this is the average value of $x(t)$ *)



The 1st order approximation adds in a sinusoid

$$x_1(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t)$$

$$= \frac{j}{\pi}\exp(-j\omega_0 t) + \frac{1}{2} - \frac{j}{2}\exp(j\omega_0 t)$$

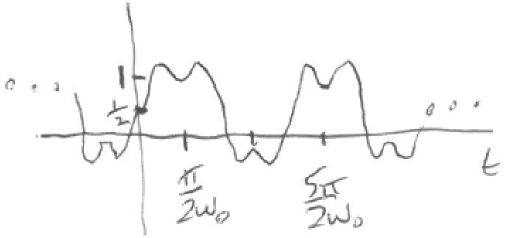


For the next level of approximation, we add in 3^{rd} harmonic (since 2^{nd} harmonic is not present in $x(t)$) ie frequencies $-3\omega_0$ & $3\omega_0$ rad/sec

$$x_3(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t)$$

$$= \frac{j}{3\pi}\exp(-j3\omega_0 t) + \frac{j}{\pi}\exp(-j\omega_0 t) + \frac{1}{2}$$

$$- \frac{j}{\pi}\exp(j\omega_0 t) - \frac{j}{3\pi}\exp(j3\omega_0 t)$$



starting to look abit like a periodic square wave!

In the limit of infinite harmonics, we get the Fourier Series for $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \exp(jk\omega_0 t)$$

where
$$C_k = \begin{cases} \frac{1}{2} & k=0 \\ 0 & k \text{ even}, k\neq 0 \\ \frac{-j}{k\pi} & k \text{ odd} \end{cases}$$

C_k is the Fourier Series coefficient at harmonic k

$$C_k = |C_k|\exp(j\phi_k)$$

$|C_k|$ is the magnitude of the k^{th} harmonic

ϕ_k is the phase of the k^{th} harmonic

$$x(t) = \sum_{k=-\infty}^{\infty} |C_k|\exp(jk\omega_0 t + \phi_k)$$

Fourier Decomposition when signal is real

If $x(t)$ is a real-valued signal, then

$$|C_{-k}| = |C_k| \quad \& \quad \arg(C_{-k}) = -\arg(C_k)$$

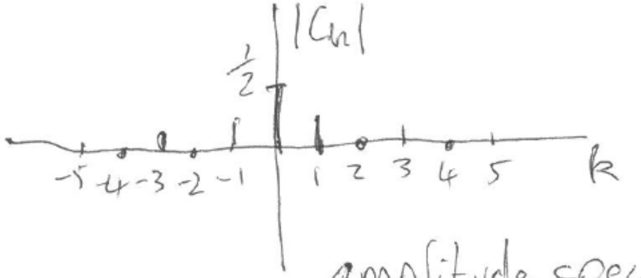
$$\Rightarrow \quad x(t) = 2 \sum_{k=-\infty}^{\infty} |C_k| \cos(k\omega_0 t + \phi_k)$$

In fact, in this case, $C_{-k} = C_k^*$
(complex conjugates)

Amplitude and Phase Spectrum

We can plot the amplitude spectrum & the phase spectrum of $x(t)$

$$|C_k| = \begin{cases} \frac{1}{2} & k=0 \\ 0 & k \text{ even}, k \neq 0 \\ \frac{1}{|k\pi|} & k \text{ odd} \end{cases}$$

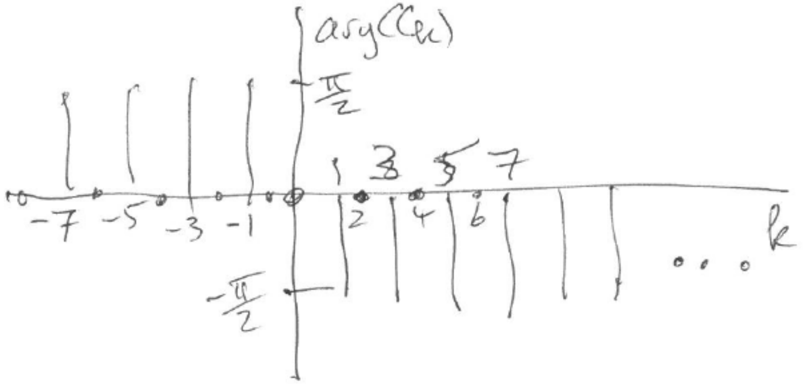


even spectrum

amplitude spectrum

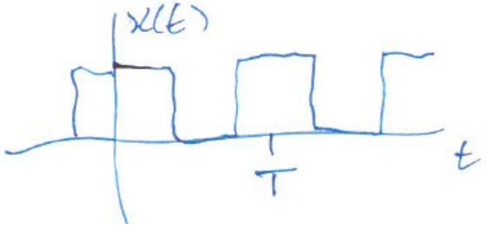
phase spectrum $\quad C_k = \frac{-j}{k\pi} \quad k$ odd

$$\arg(C_k) = \begin{cases} 0 & k \text{ even} \\ -\pi/2 & k \text{ odd}, k>0 \\ \frac{\pi}{2} & k \text{ odd}, k<0 \end{cases}$$



odd spectrum

Fourier Series: amplitude and phase



$$\omega_0 = \frac{2\pi}{T}$$

The Fourier Series expands the signal *x(t)* in terms of a basis of signals made up of complex sinusoids

Fourier series, Fourier coefficients, and time domain representation:

The coefficients of the signal with respect to that basis are the Fourier coefficients

$$x(t) = \sum_k C_k \exp(j k\omega_0 t)$$

The basis signals are all sinusoids with frequencies restricted to be harmonics of the fundamental frequency ω_0 rad/sec

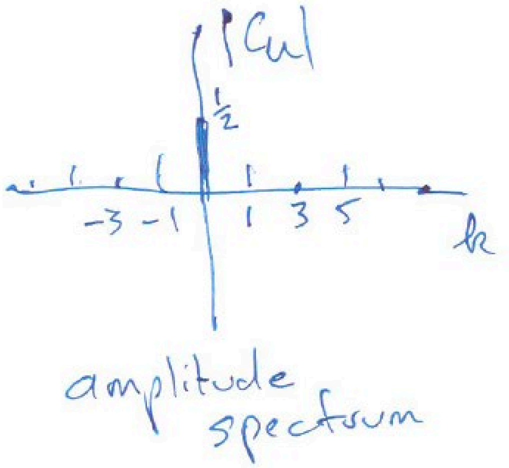
$$\exp(j k\omega_0 t)$$

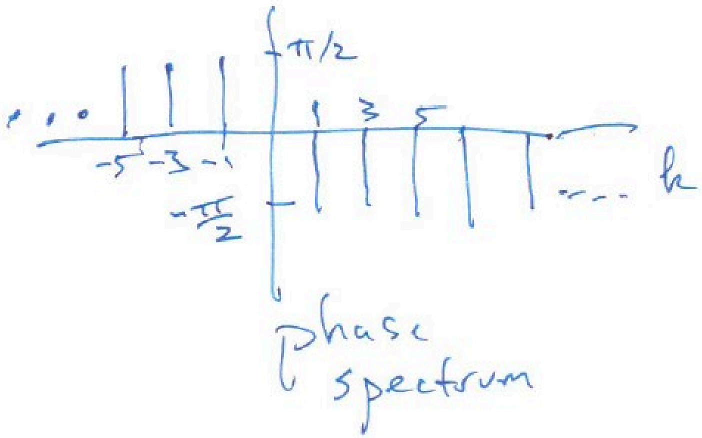
$$C_k = |C_k| \exp(j\theta_k)$$

$$x(t) = \sum_k \underbrace{|C_k|}_{\text{amplitude}} \exp(j(k\omega_0 t + \underbrace{\theta_k}_{\text{phase}}))$$

The amplitude and phase spectrum provides the amplitude and phase of the components across all the frequencies in the expansion

Amplitude and Phase Spectrum

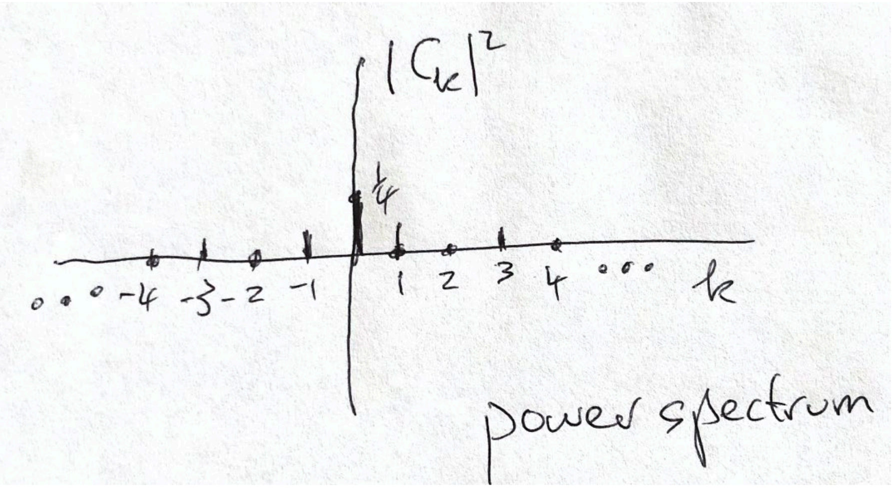




Amplitude coefficient:

$$|C_k| = \begin{cases} \frac{1}{2} & k = 0 \\ 0 & k \text{ even}, k \neq 0 \\ \frac{1}{|k|\pi} & k \text{ odd} \end{cases}$$

Power Spectrum



Power coefficient:

$$|C_k|^2 = \begin{cases} \frac{1}{4} & k = 0 \\ 0 & k \text{ even}, k \neq 0 \\ \frac{1}{(k\pi)^2} & k \text{ odd} \end{cases}$$