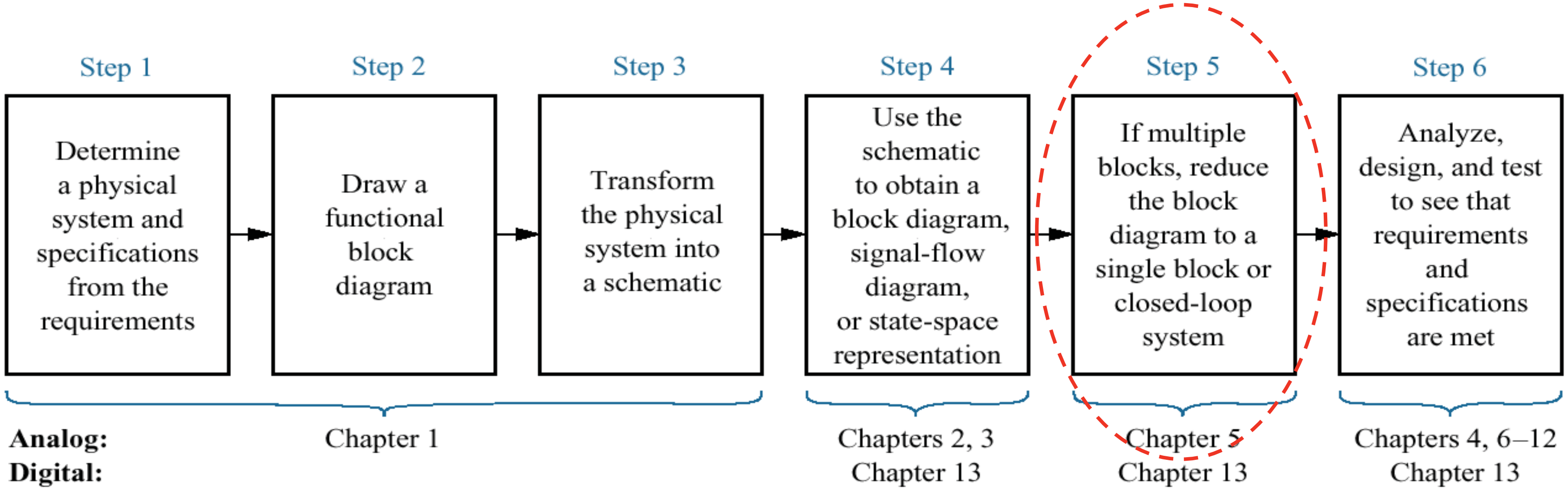
Control Systems

Chapter 5: Reduction of Multiple Subsystems

Highlights

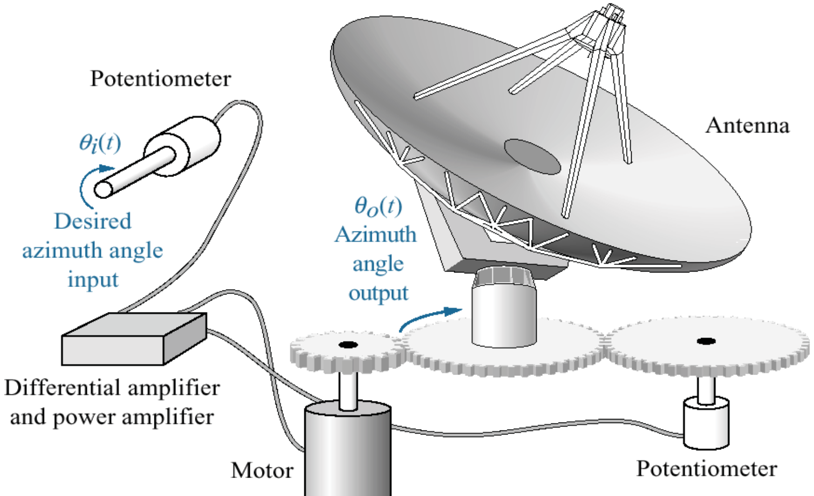
Original slides courtesy of Dr Branislav Hredzak

From Chapter 1: The Design Process

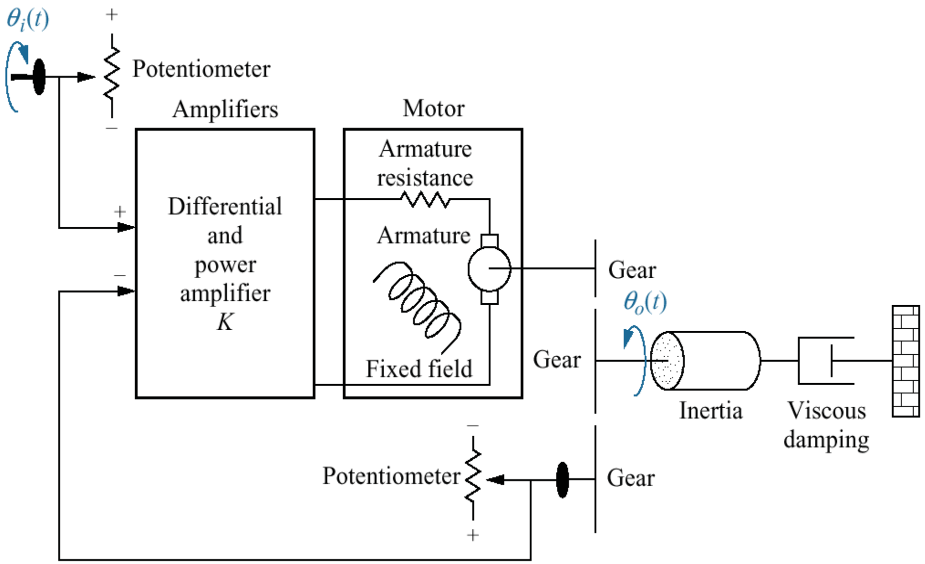


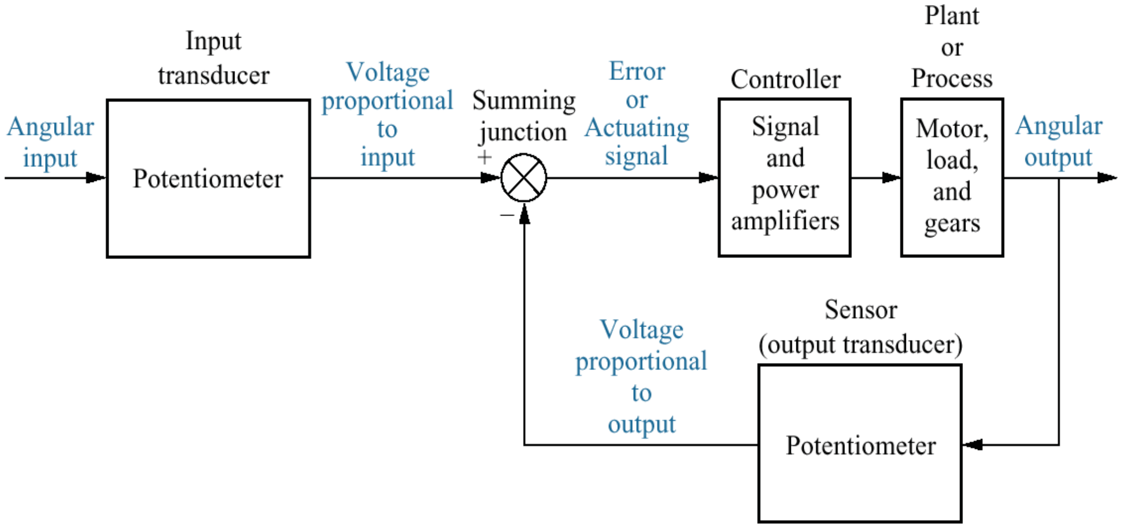
The control system design process

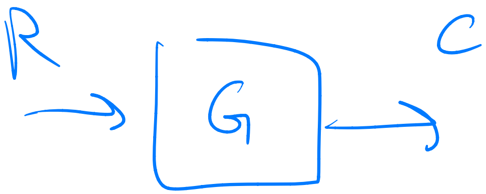
Antenna Azimuth



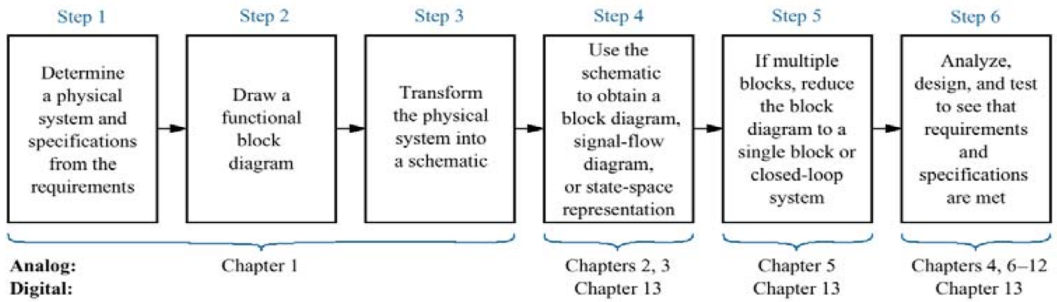
(b)





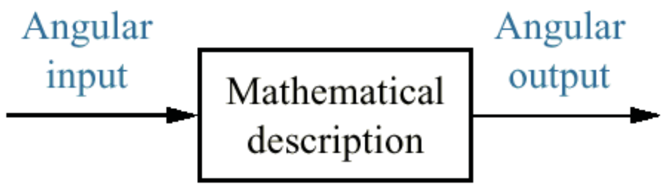


The Design Process (Antenna Azimuth)



Step 5: Reduce the Block Diagram

- In order **to evaluate** the system response we **reduce** the large system's block diagram **to a single block** with a mathematical description that represents the system from its input to its output



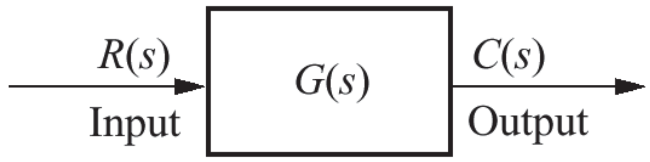
Equivalent block diagram for the antenna azimuth position control system

Reduction of Multiple Subsystems

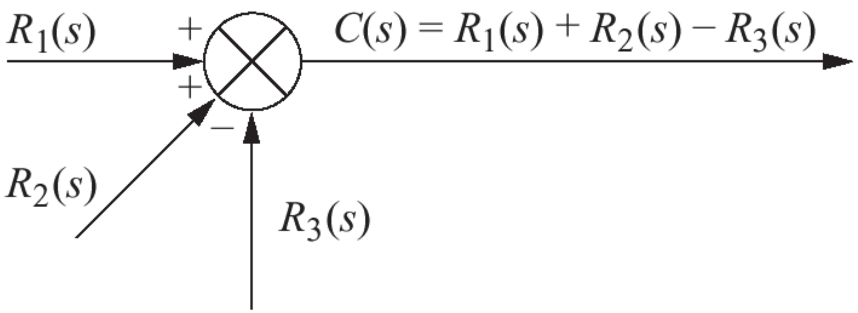
- How to reduce a block diagram of multiple subsystems to a single block representing the transfer function from input to output
- How to analyze and design transient response for a system consisting of multiple subsystems
- How to represent in state space a system consisting of multiple subsystems
- How to convert to alternate representations of a system in state space

Block diagrams (1)

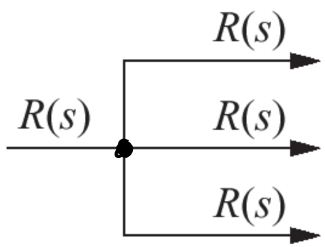
Signals



System



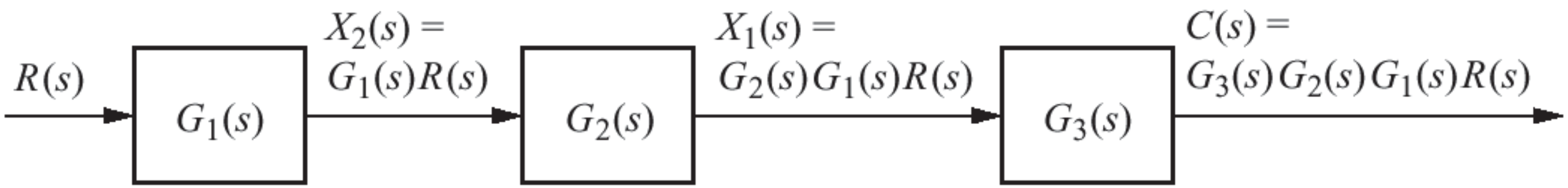
Summing junction



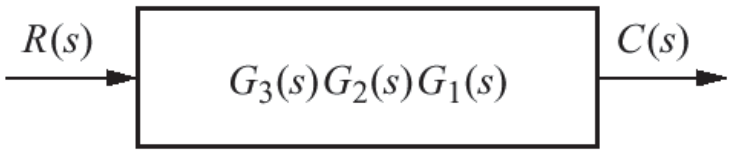
Pickoff point

Block diagrams (2)

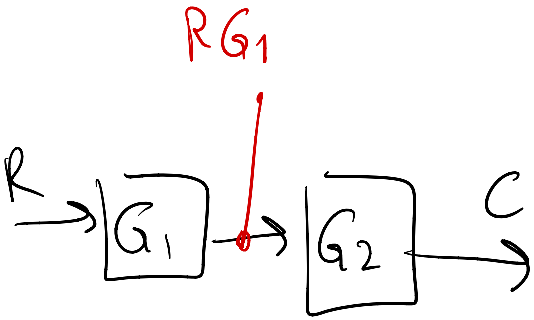
Cascaded subsystems



(a)



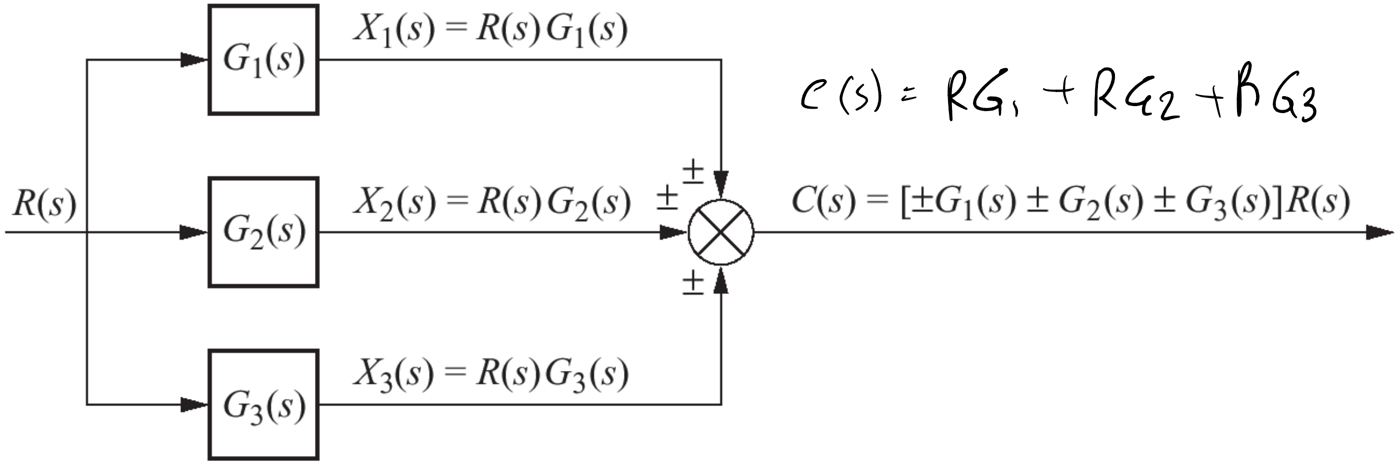
(b)



$C = RG_1 \cdot G_2 = R(G_1 \cdot G_2)$

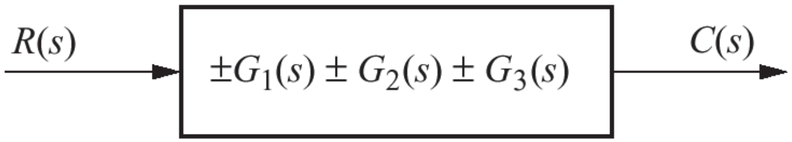
Block diagrams (3)

Parallel subsystems



(*a*)

$C(s) = RG_1 + RG_2 + RG_3$



(*b*)

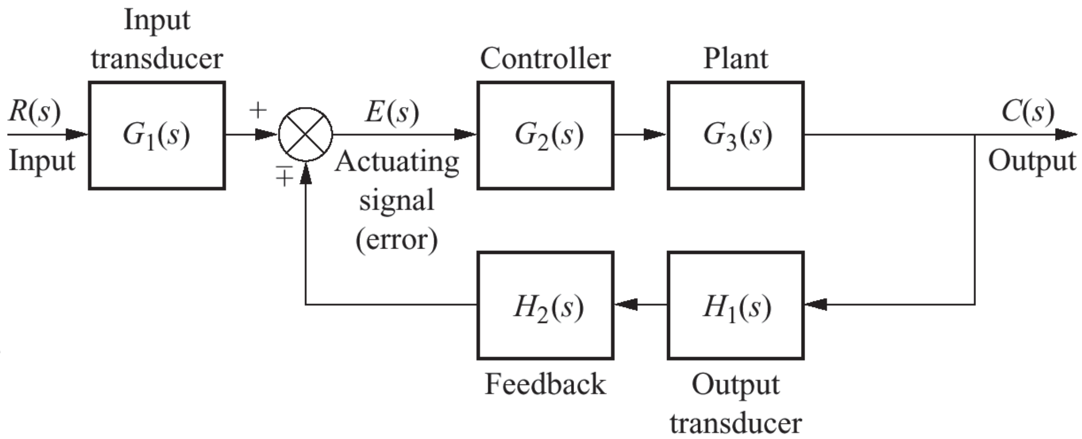
Block diagrams (4)

Feedback systems

$$E(s) = R(s) \mp C(s)H(s)$$

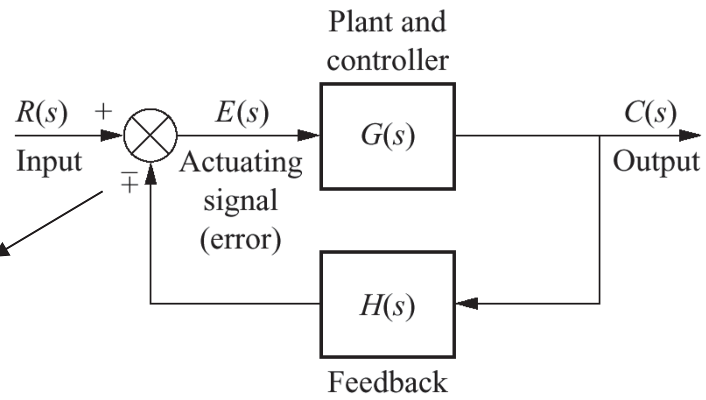
$$C(s) = E(s)G(s) \rightarrow E(s) = \frac{C(s)}{G(s)}$$

$$\boxed{G_e(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}}$$



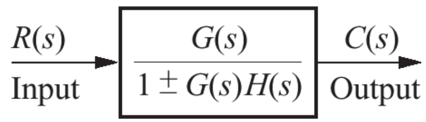
(*a*)

“-” negative feedback
“+” positive feedback

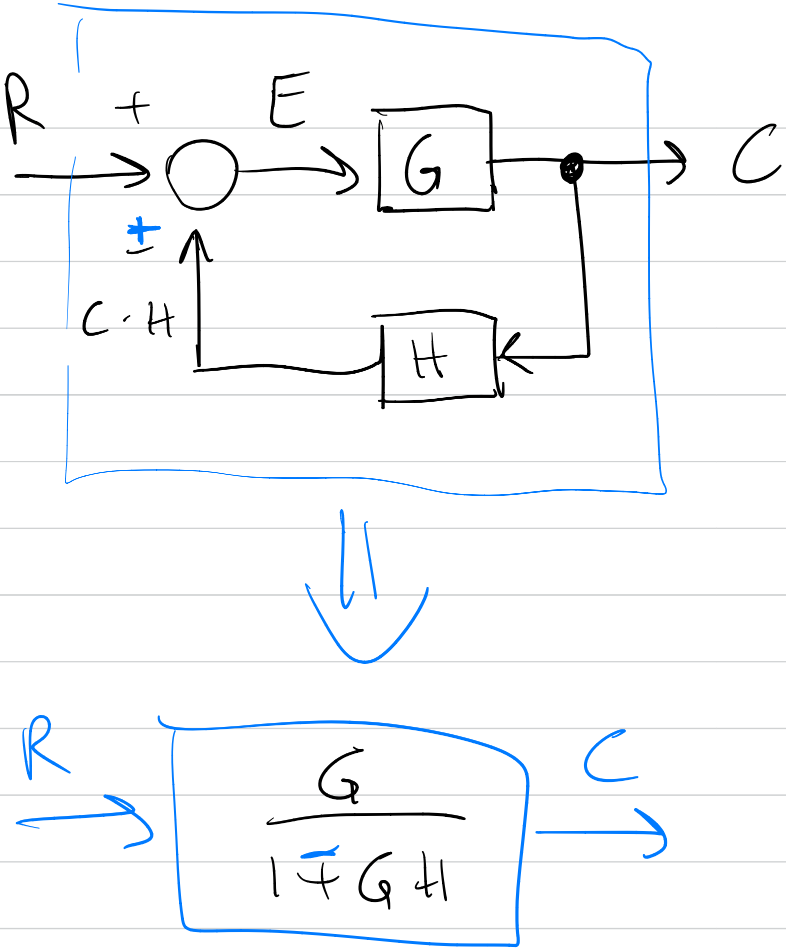


(*b*)

*$G(s)H(s)$ … **open-loop transfer function (loop gain)***



(*c*)



$$C = E \cdot G = (R \pm CH)G = RG \mp CHG$$

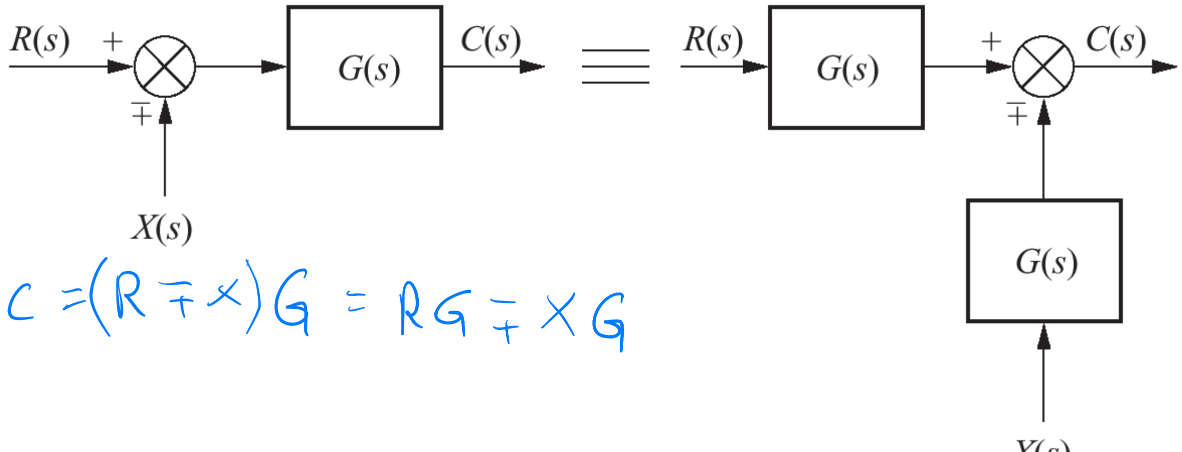
$$E = R \pm CH$$

$$C(1 + GH) = RG$$

$$\boxed{\frac{C}{R} = \frac{G}{1 \pm GH}}$$

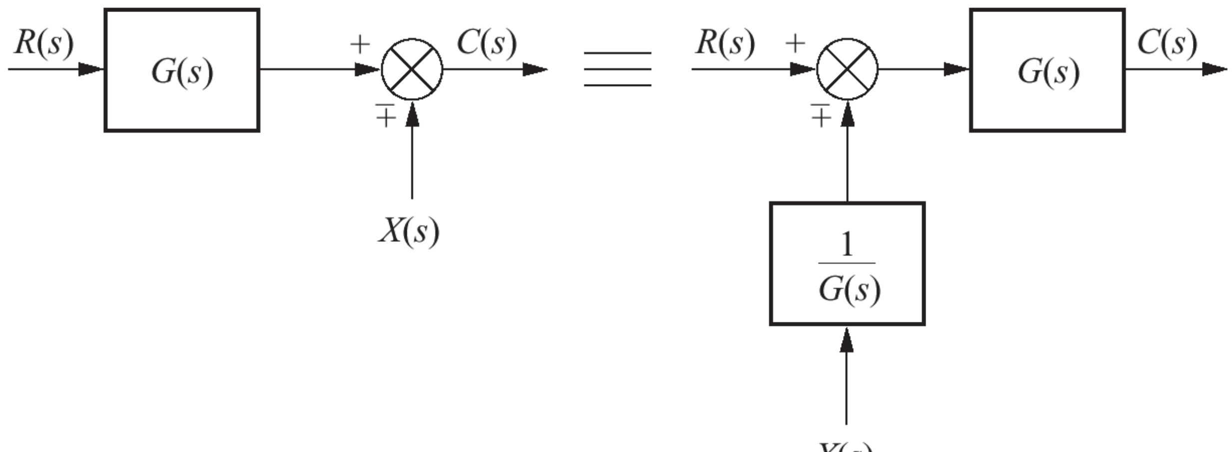
Moving Blocks to Create Familiar Forms (1)

$C = RG \mp XG$



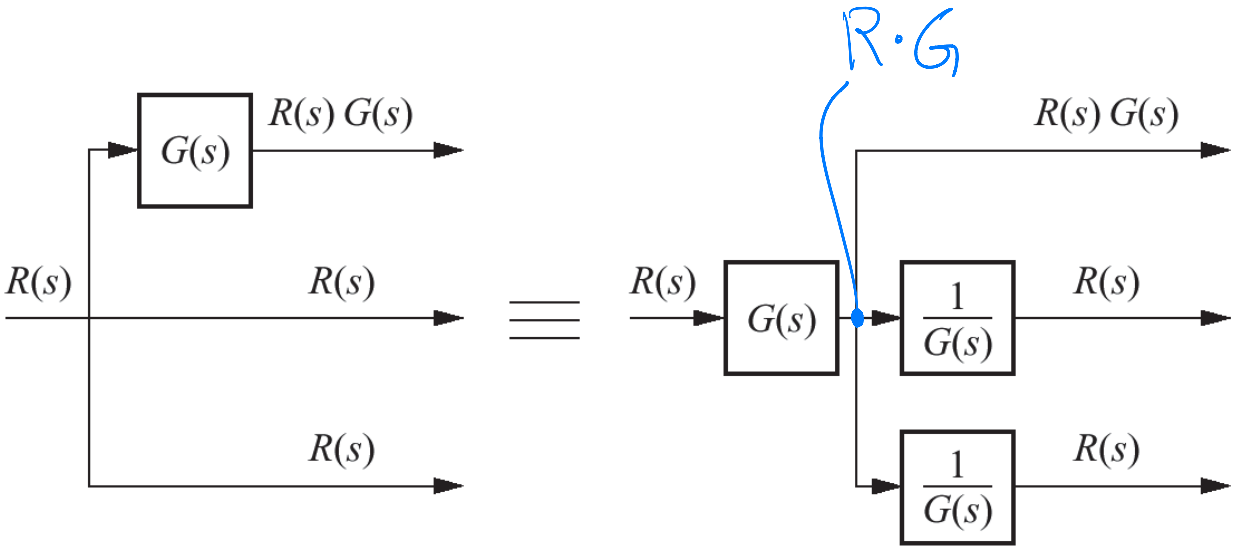
$C = (R \mp X)G = RG \mp XG$

(a)

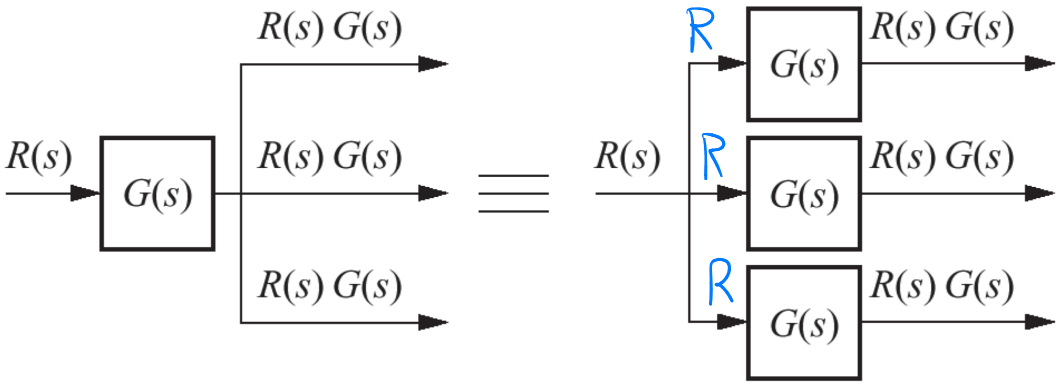


(b)

Moving Blocks to Create Familiar Forms (2)

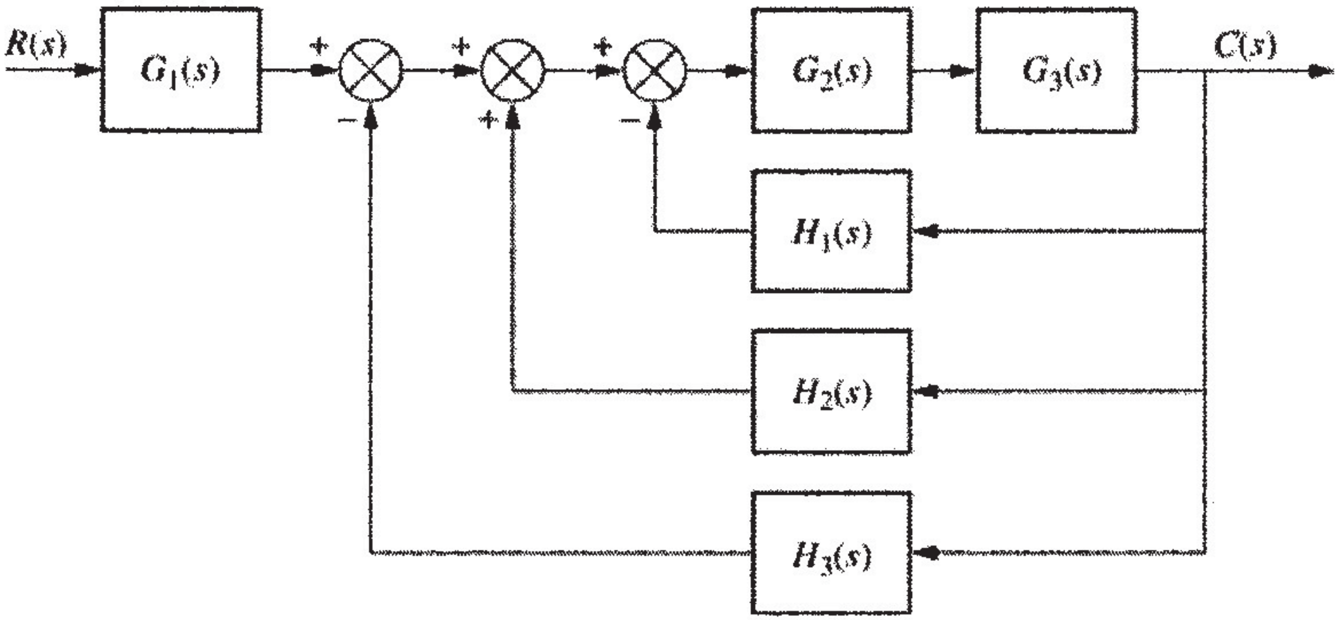


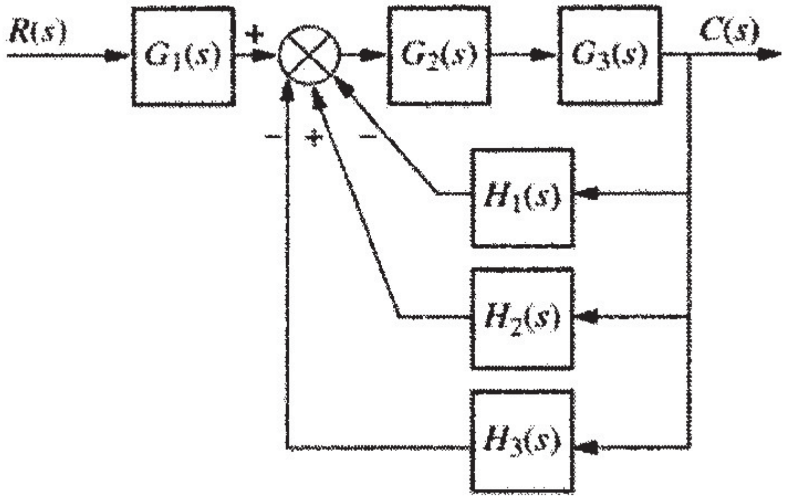
(a)



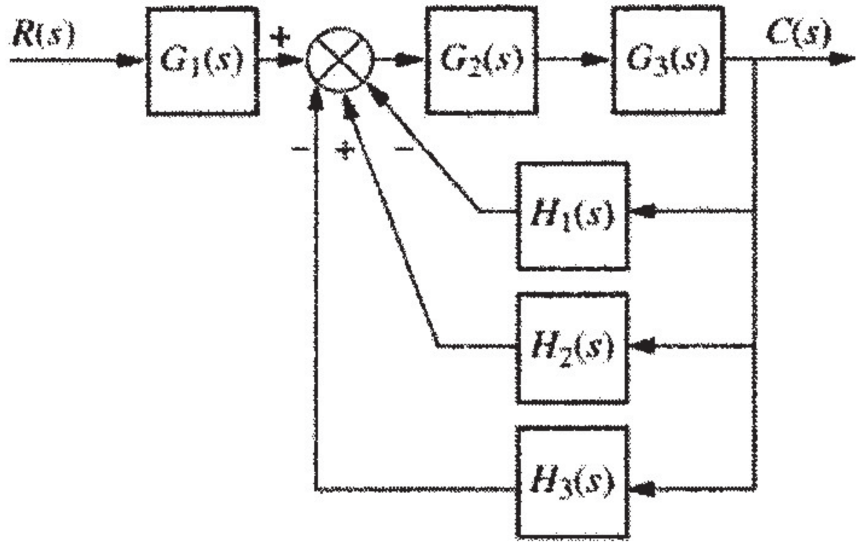
(b)

Example 5.1 Reduce the following system to a single transfer function

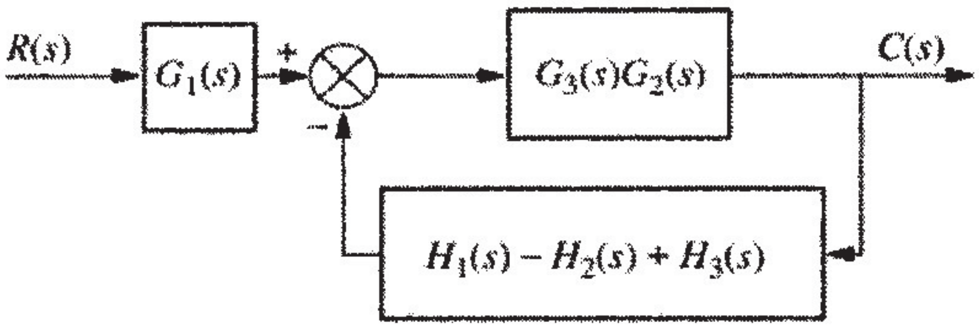




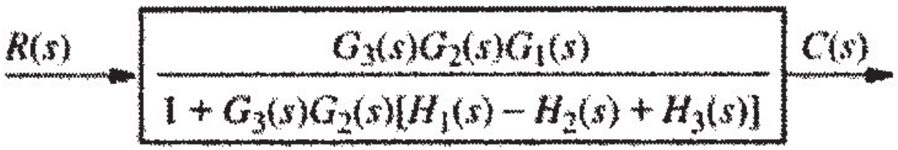
(a)


R(s)
G1(s)
+
G2(s)
G3(s)
C(s)
−
+
−
H1(s)
H2(s)
H3(s)

(a)

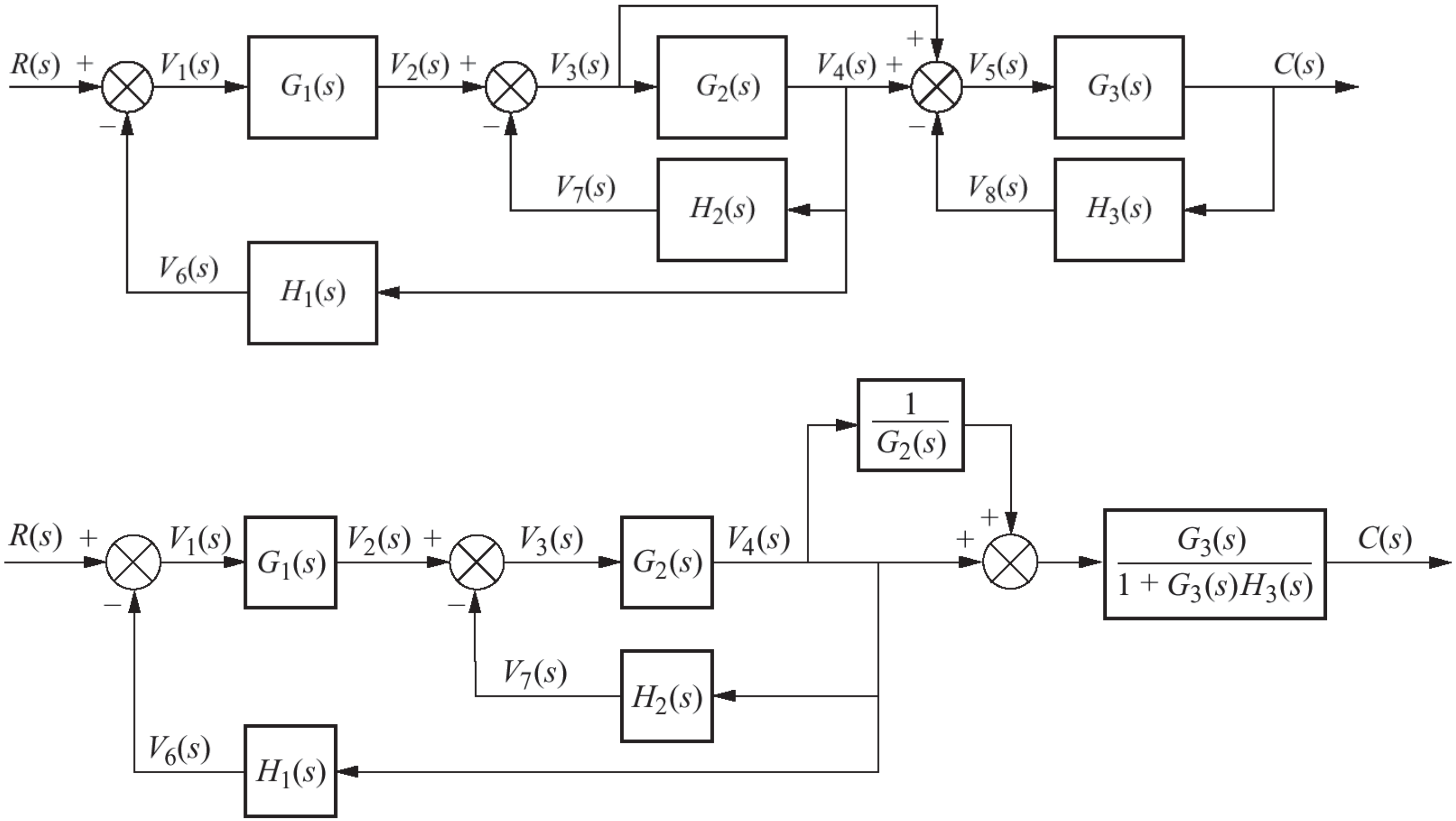

R(s)
G1(s)
+
−
G3(s)G2(s)
C(s)
H1(s) − H2(s) + H3(s)

(b)

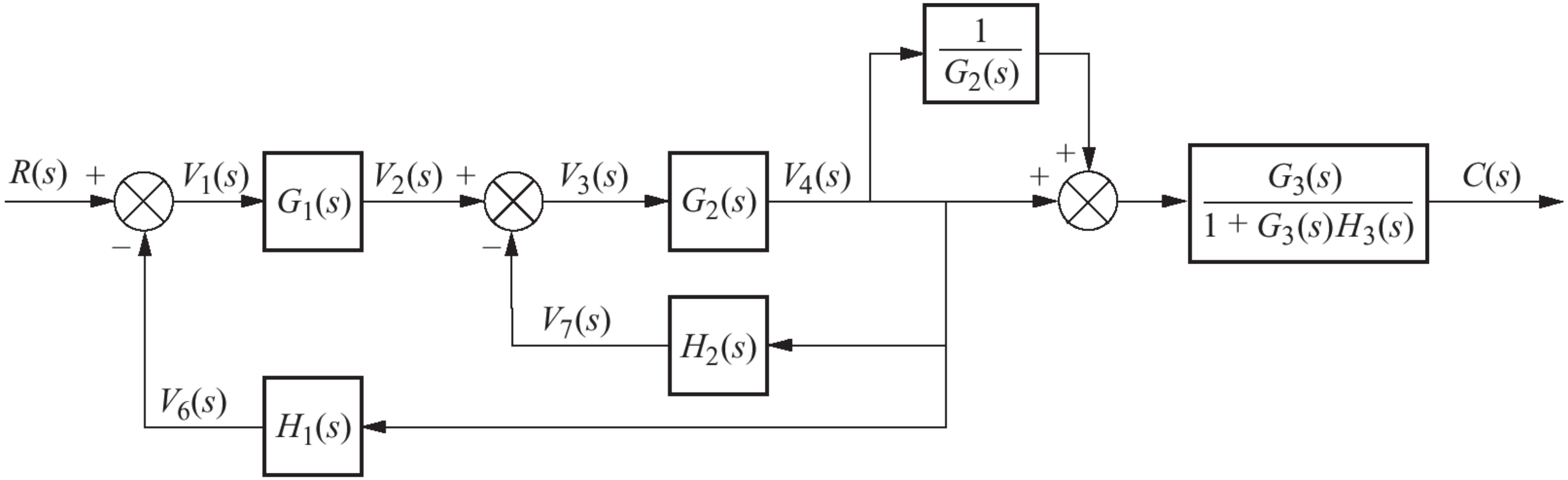

R(s)
G3(s)G2(s)G1(s)
1 + G3(s)G2(s)[H1(s) − H2(s) + H3(s)]
C(s)

(c)

Example 5.2 Reduce the following block diagram to a single transfer function

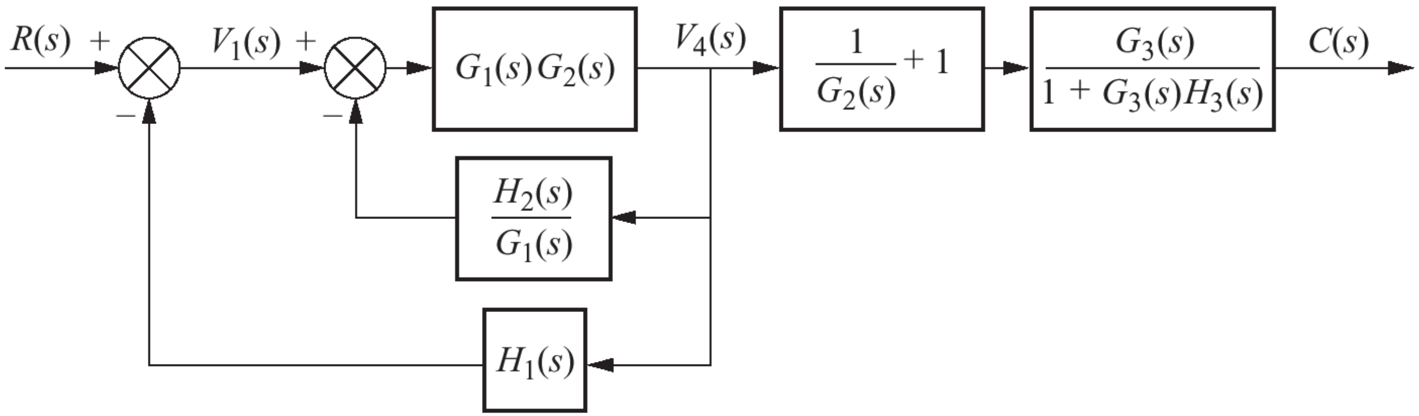


(*a*)



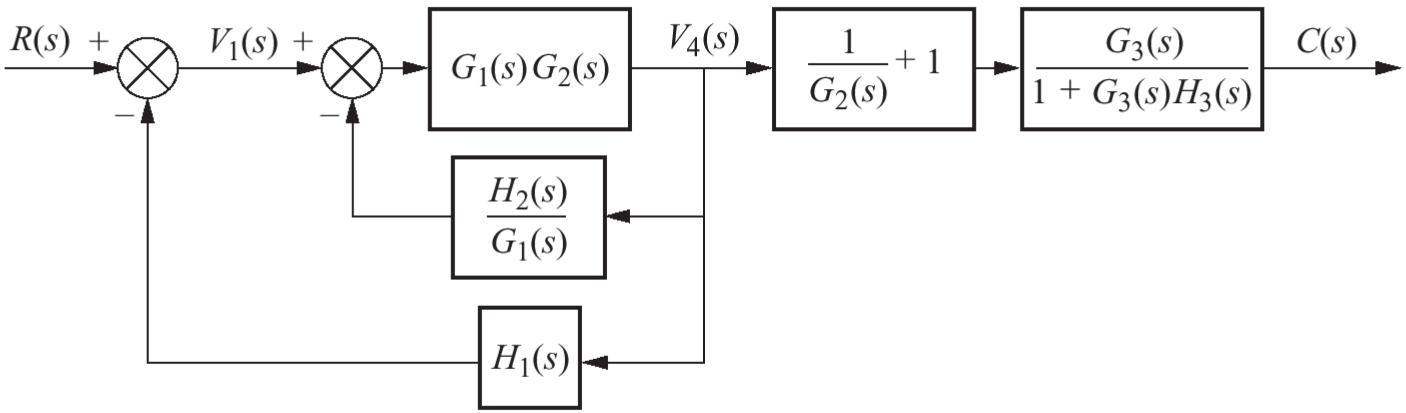
$R(s)$
$V_1(s)$
$G_1(s)$
$V_2(s)$
$V_3(s)$
$G_2(s)$
$V_4(s)$
$\frac{1}{G_2(s)}$
$\frac{G_3(s)}{1 + G_3(s)H_3(s)}$
$C(s)$
$V_7(s)$
$H_2(s)$
$V_6(s)$
$H_1(s)$

(*a*)

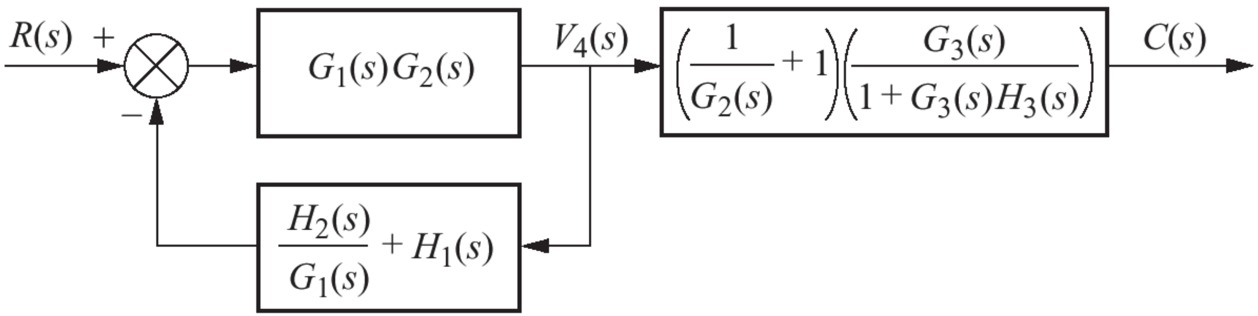


$R(s)$
$V_1(s)$
$G_1(s)G_2(s)$
$V_4(s)$
$\frac{1}{G_2(s)} + 1$
$\frac{G_3(s)}{1 + G_3(s)H_3(s)}$
$C(s)$
$\frac{H_2(s)}{G_1(s)}$
$H_1(s)$

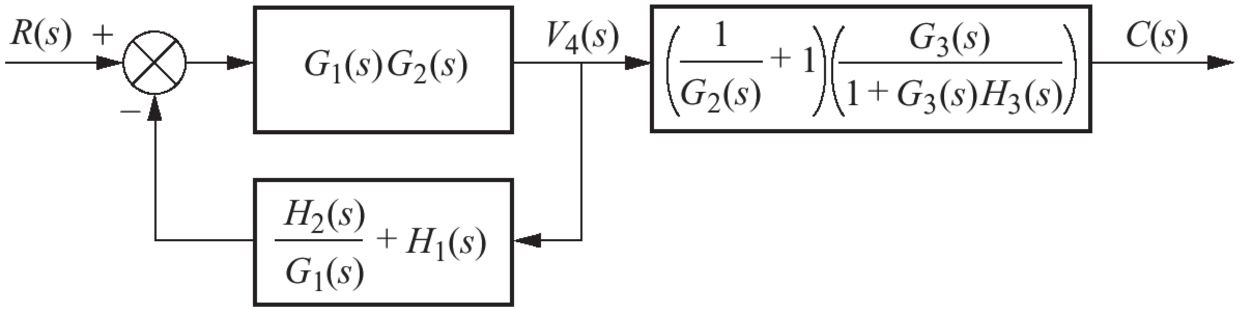
(*b*)


R(s)
+
V1(s)
+
−
−
G1(s)G2(s)
V4(s)
1/G2(s) + 1
G3(s) / (1 + G3(s)H3(s))
C(s)
H2(s)/G1(s)
H1(s)

(*b*)

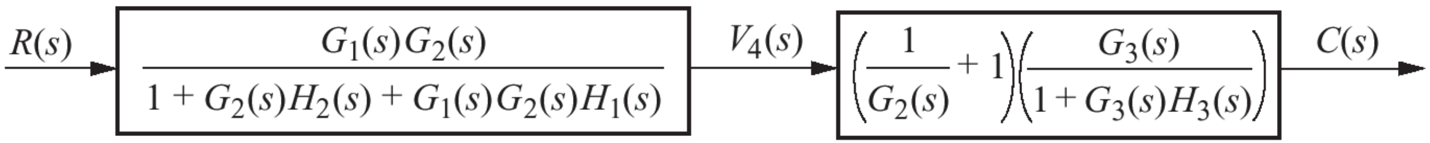

R(s)
+
−
G1(s)G2(s)
V4(s)
(1/G2(s) + 1)(G3(s)/(1 + G3(s)H3(s)))
C(s)
H2(s)/G1(s) + H1(s)

(*c*)



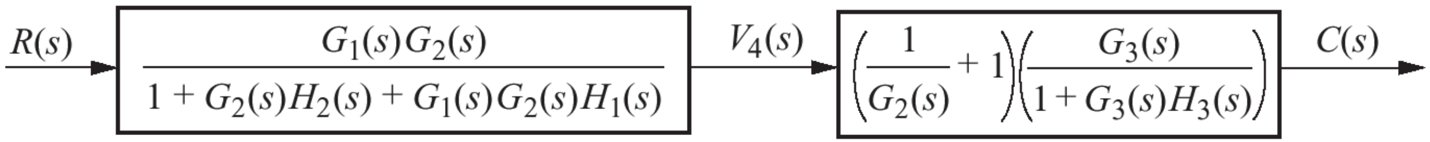
R(s)
+
−
G1(s)G2(s)
V4(s)
(1/G2(s) + 1)(G3(s)/(1 + G3(s)H3(s)))
C(s)
H2(s)/G1(s) + H1(s)

(c)

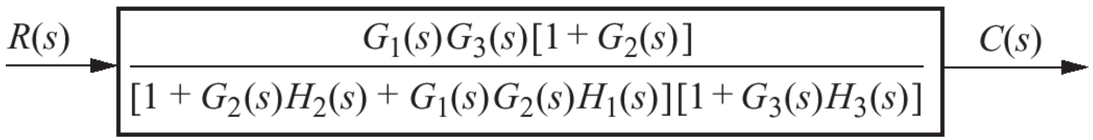


R(s)
G1(s)G2(s) / (1 + G2(s)H2(s) + G1(s)G2(s)H1(s))
V4(s)
(1/G2(s) + 1)(G3(s)/(1 + G3(s)H3(s)))
C(s)

(d)

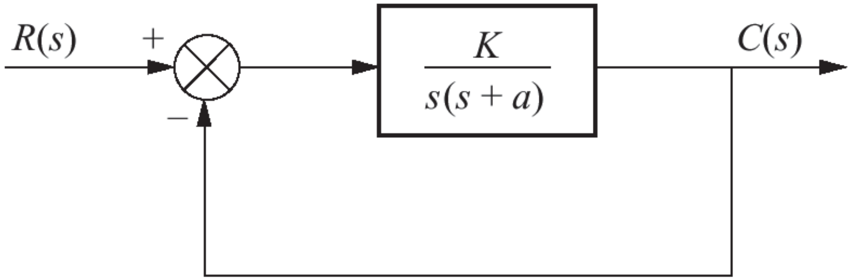

$R(s)$
$\frac{G_1(s)G_2(s)}{1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)}$
$V_4(s)$
$\left(\frac{1}{G_2(s)} + 1\right)\left(\frac{G_3(s)}{1 + G_3(s)H_3(s)}\right)$
$C(s)$

(***d***)


$R(s)$
$\frac{G_1(s)G_3(s)[1 + G_2(s)]}{[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)][1 + G_3(s)H_3(s)]}$
$C(s)$

(***e***)

Analysis and Design of Feedback Systems



K … amplifier gain

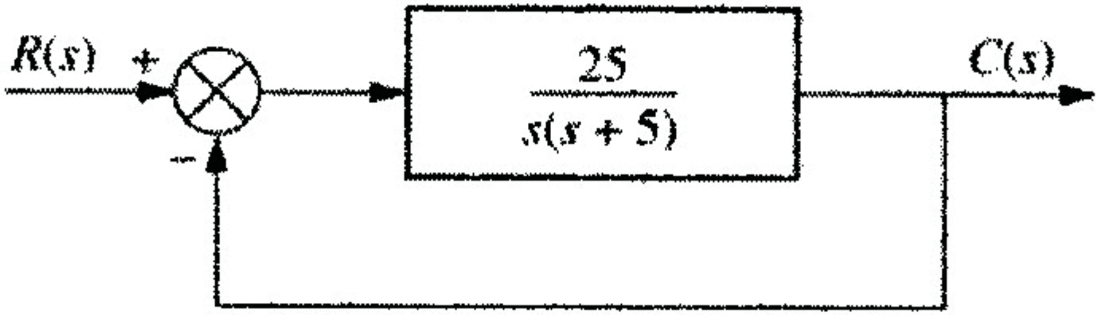
$$T(s) = \frac{K}{s^2 + as + K}$$

$K \in \left(0, \frac{a^2}{4}\right)$: $\quad s_{1,2} = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4K}}{2}$ … overdamped response

$K = \frac{a^2}{4}$: … critically damped

$K > \frac{a^2}{4}$: $\quad s_{1,2} = -\frac{a}{2} \pm j\frac{\sqrt{4K - a^2}}{2}$ … underdamped response

Example 5.3 Find T_p, $\%OS$, and T_s for the following system



Solution

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{25}{s(s+25)}}{1+\frac{25}{s(s+5)}} = \frac{25}{s^2+5s+25}$$

So,

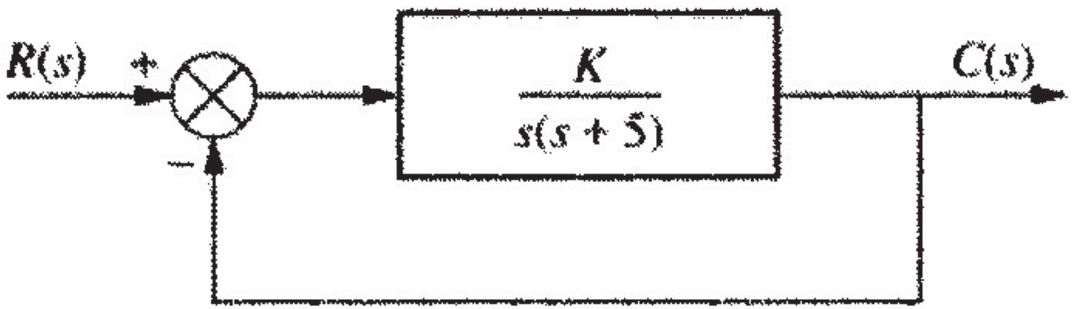
$$\omega_n = \sqrt{25} = 5$$
$$2\zeta\omega_n = 5$$
$$\zeta = 0.5$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.726 \text{ s}$$

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 = 16.303$$

$$T_s \approx \frac{4}{\zeta\omega_n} = 1.6$$

Example 5.4 Find the value of the gain K for the system below so that it responds with a 10% OS.



Solution

$$T(s) = \frac{C(s)}{R(s)} = \frac{K}{s^2 + 5s + K}$$

So,

$$2\zeta\omega_n = 5$$

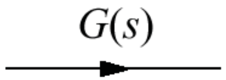
$$\omega_n = \sqrt{K}$$

$$\zeta = \frac{5}{2\sqrt{K}} \Leftrightarrow K = \frac{25}{4\zeta^2}$$

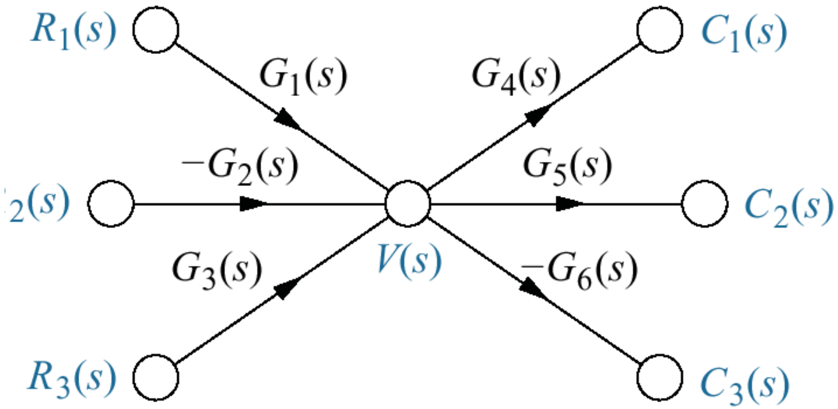
$$10\%OS \Rightarrow \zeta = 0.591 \Rightarrow K = 17.9$$

Signal-Flow Graphs

Branches – represent systems



Nodes – represent signals



$$V(s) = G_1(s)R_1(s) - G_2(s)R_2(s) + G_3(s)R_3(s)$$

$$C_1(s) = G_4(s)V(s) = G_4(s)G_1(s)R_1(s) - G_4(s)G_2(s)R_2(s) + G_4(s)G_3(s)R_3(s)$$

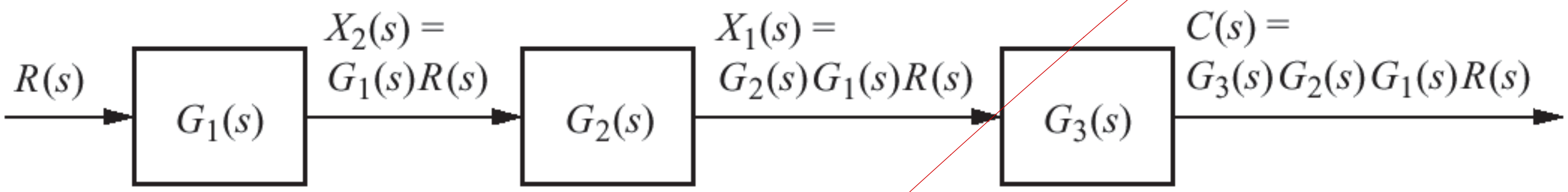
$$C_2(s) = G_5(s)V(s) = G_5(s)G_1(s)R_1(s) - G_5(s)G_2(s)R_2(s) + G_5(s)G_3(s)R_3(s)$$

$$C_3(s) = -G_6(s)V(s) = -G_6(s)G_1(s)R_1(s) + G_6(s)G_2(s)R_2(s) - G_6(s)G_3(s)R_3(s)$$

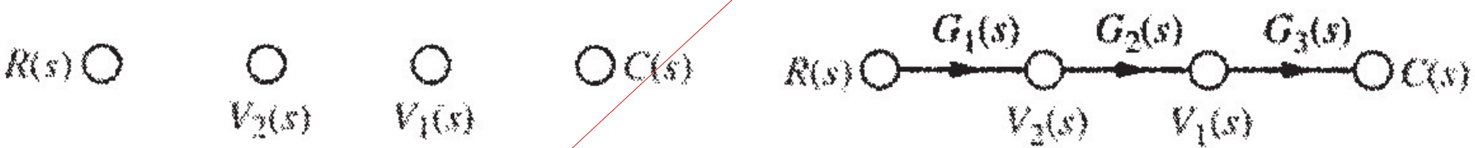
Following material is not needed

Example 5.5 Convert the cascaded, parallel, and feedback forms into signal flow graphs

Cascaded:

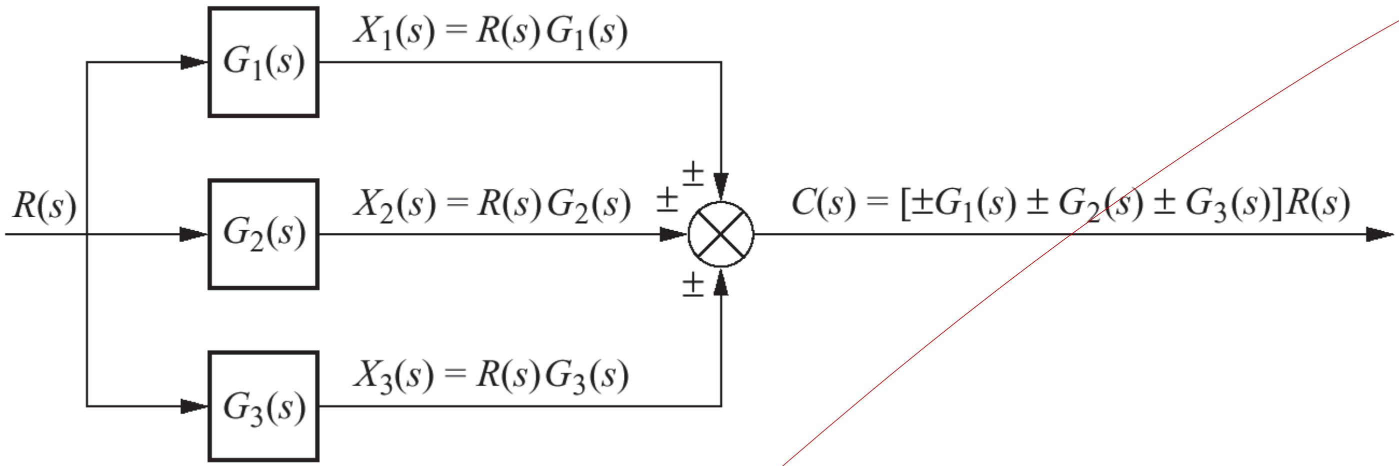


(a)

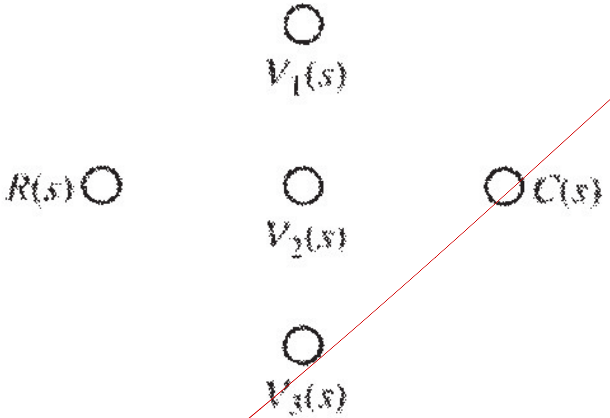


(a) ***(b)***

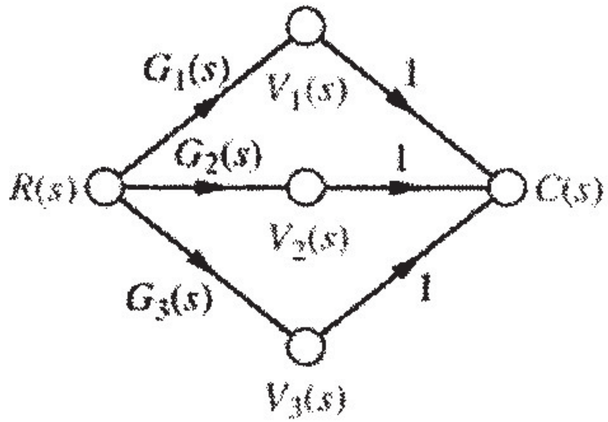
Parallel:



(*a*)

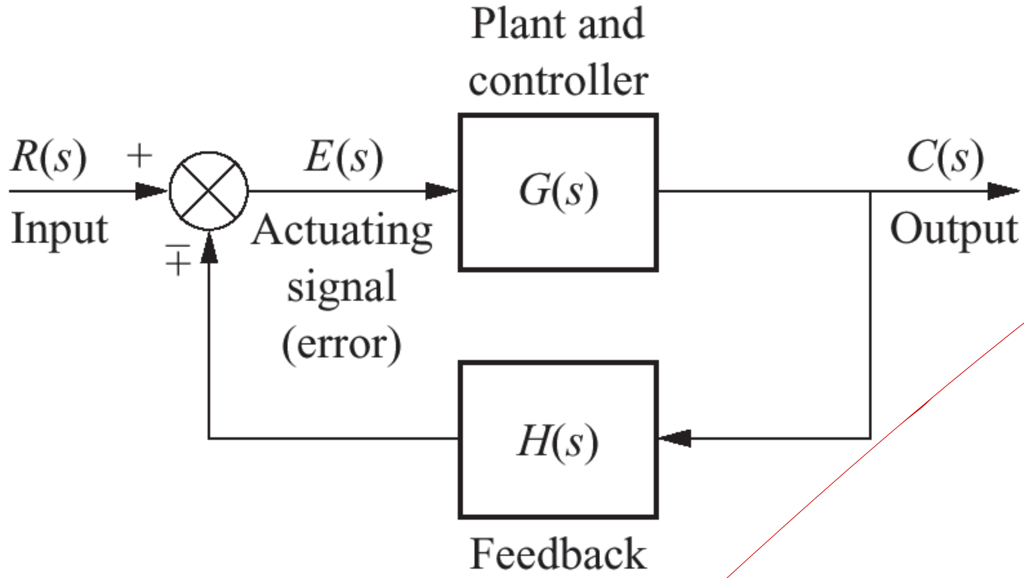


(*c*)



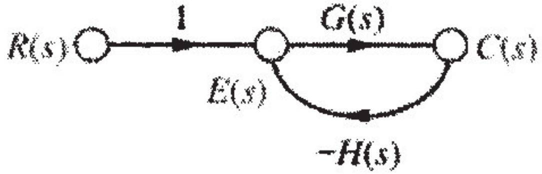
(*d*)

Feedback:



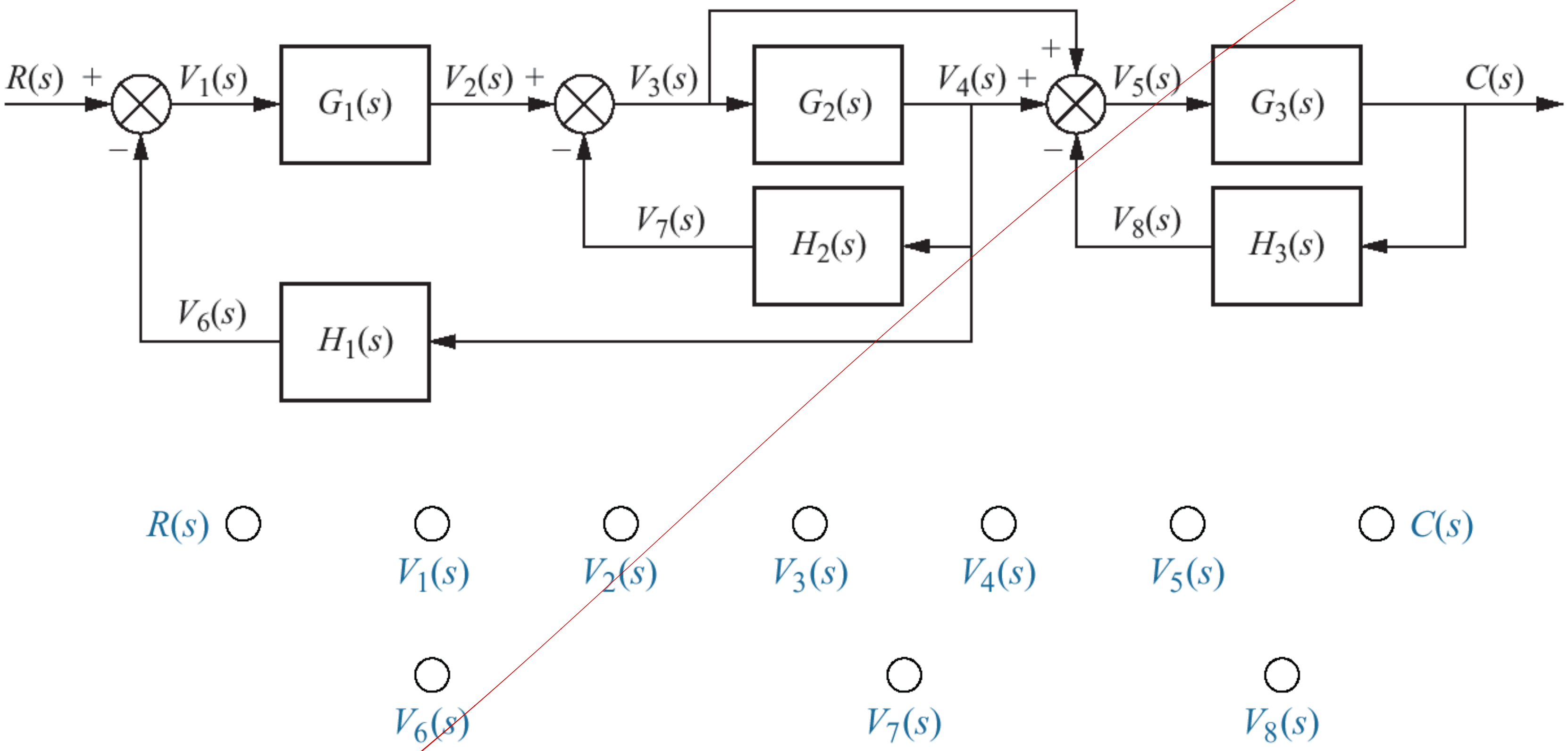
(*b*)

(*e*)

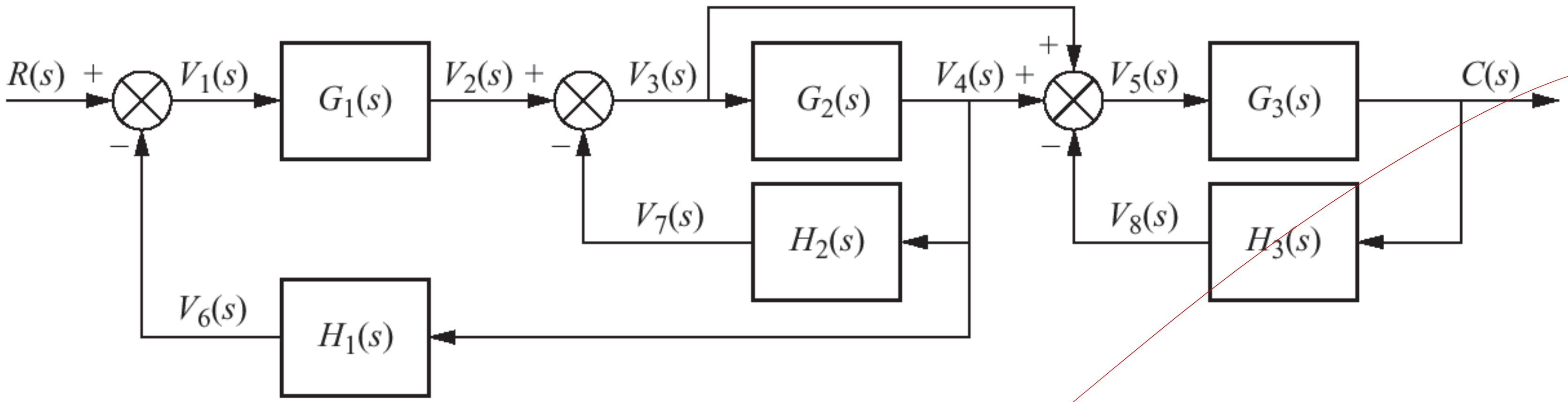


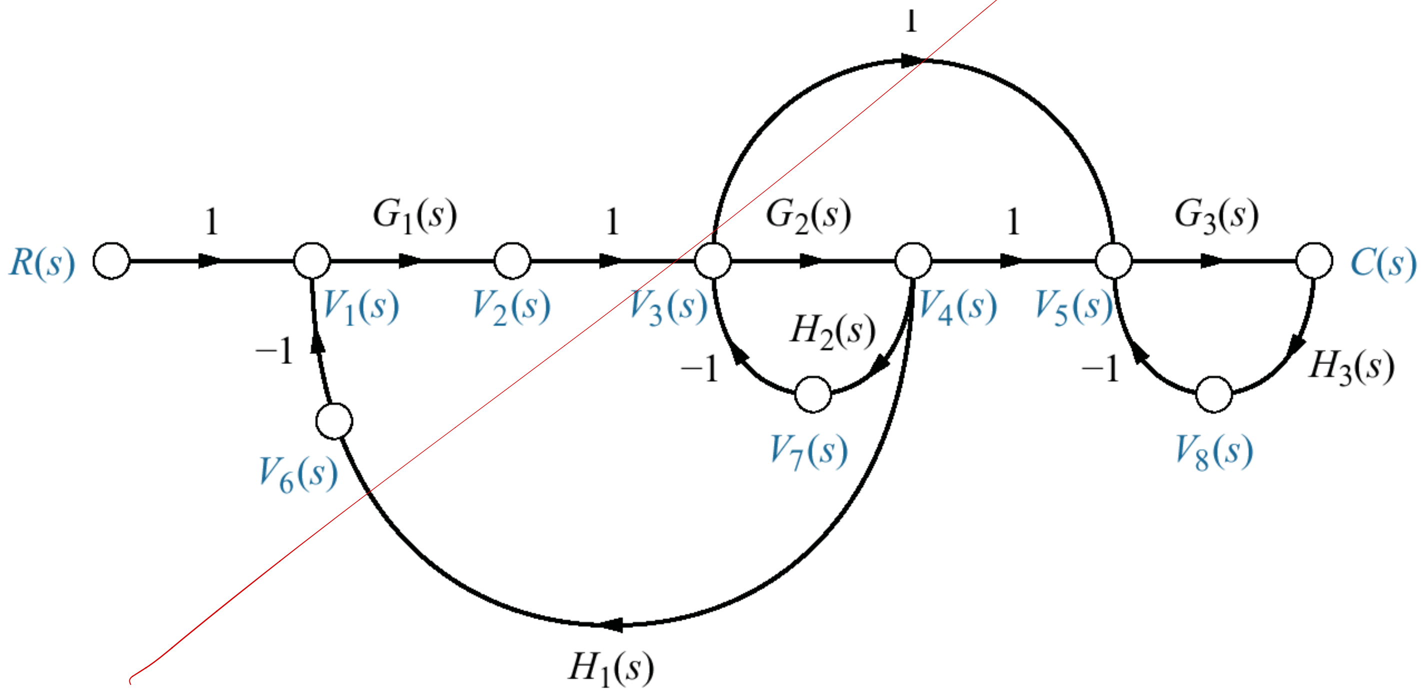
(*f*)

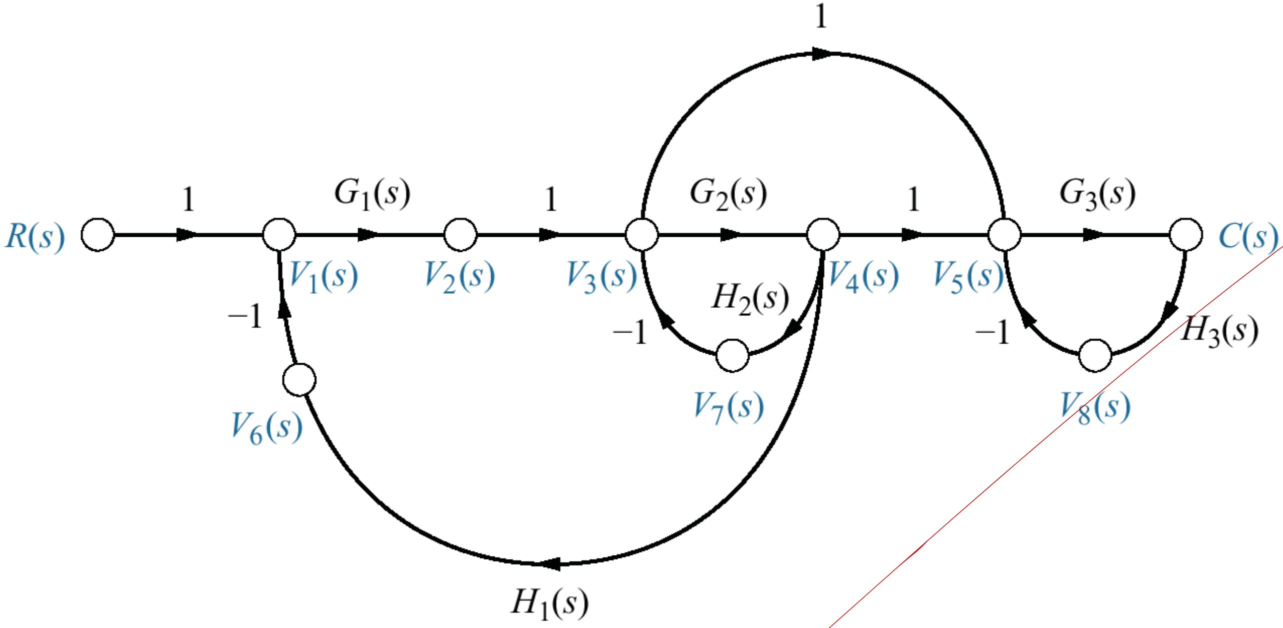
Example 5.6 Convert the block diagram from Example 5.2 to a signal-flow graph



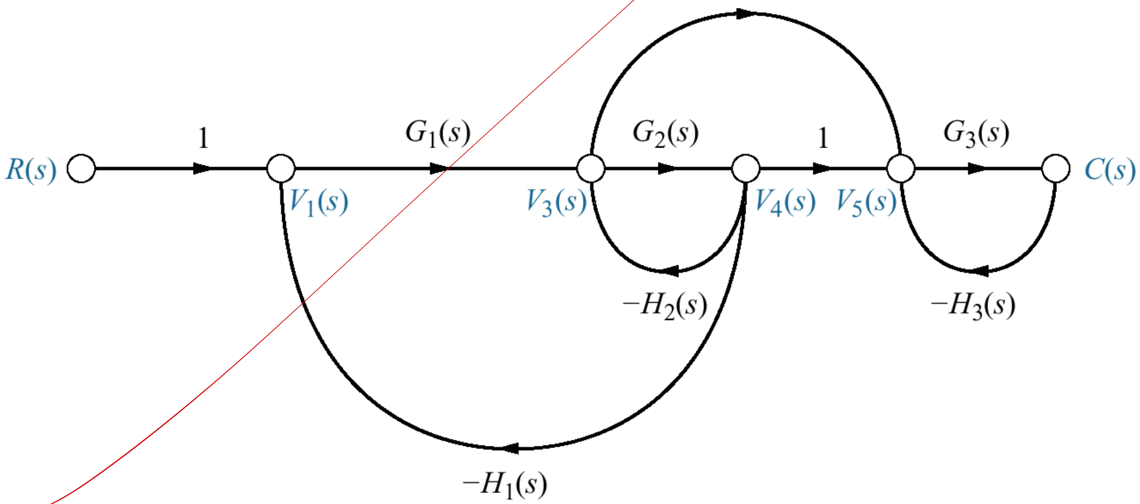
(*a*)


R(s)
+
V1(s)
G1(s)
V2(s)
+
V3(s)
G2(s)
V4(s)
+
+
V5(s)
G3(s)
C(s)
–
–
–
V7(s)
H2(s)
V8(s)
H3(s)
V6(s)
H1(s)


1
R(s)
1
G1(s)
1
G2(s)
1
G3(s)
C(s)
V1(s)
V2(s)
V3(s)
V4(s)
V5(s)
–1
H2(s)
–1
–1
H3(s)
V6(s)
V7(s)
V8(s)
H1(s)


1
$R(s)$
1
$G_1(s)$
1
$G_2(s)$
1
$G_3(s)$
$C(s)$
$V_1(s)$
$V_2(s)$
$V_3(s)$
$V_4(s)$
$V_5(s)$
-1
$H_2(s)$
-1
-1
$H_3(s)$
$V_6(s)$
$V_7(s)$
$V_8(s)$
$H_1(s)$

(*b*)


$R(s)$
1
$G_1(s)$
$G_2(s)$
1
$G_3(s)$
$C(s)$
$V_1(s)$
$V_3(s)$
$V_4(s)$
$V_5(s)$
$-H_2(s)$
$-H_3(s)$
$-H_1(s)$

(*c*)

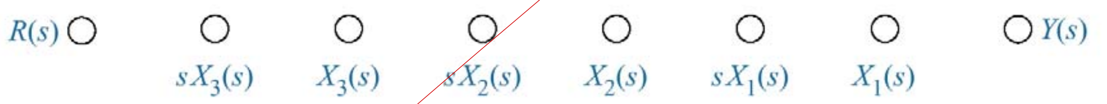
Signal-Flow Graphs of State Equations

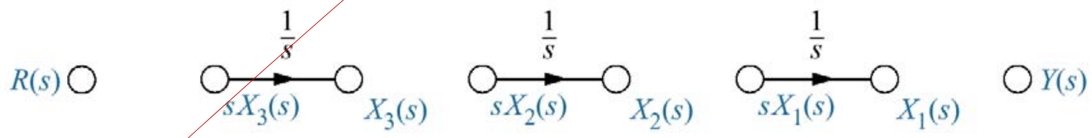
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

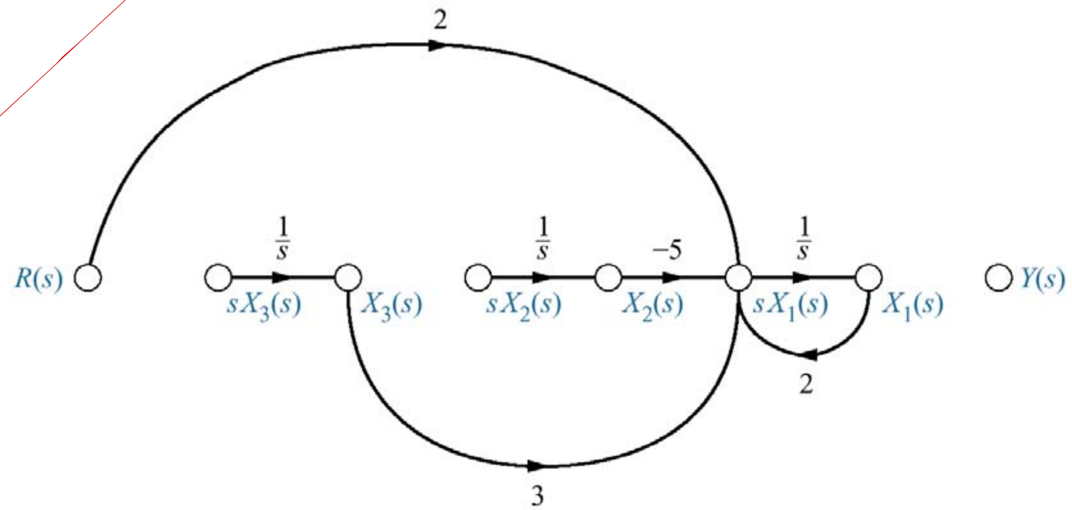
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

$$y = -4x_1 + 6x_2 + 9x_3$$

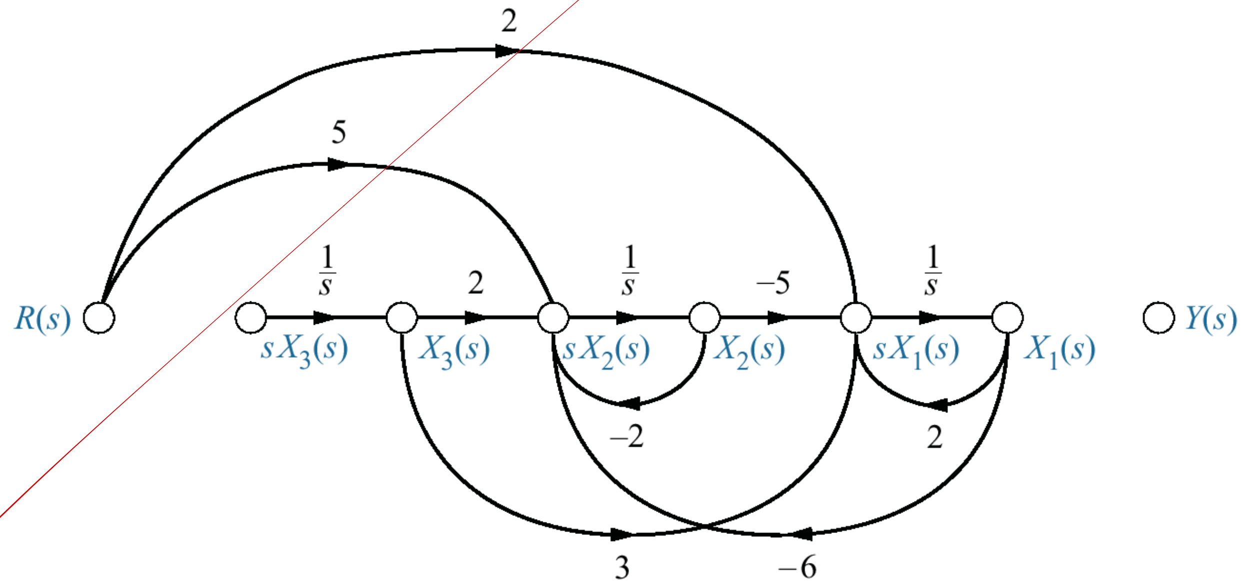






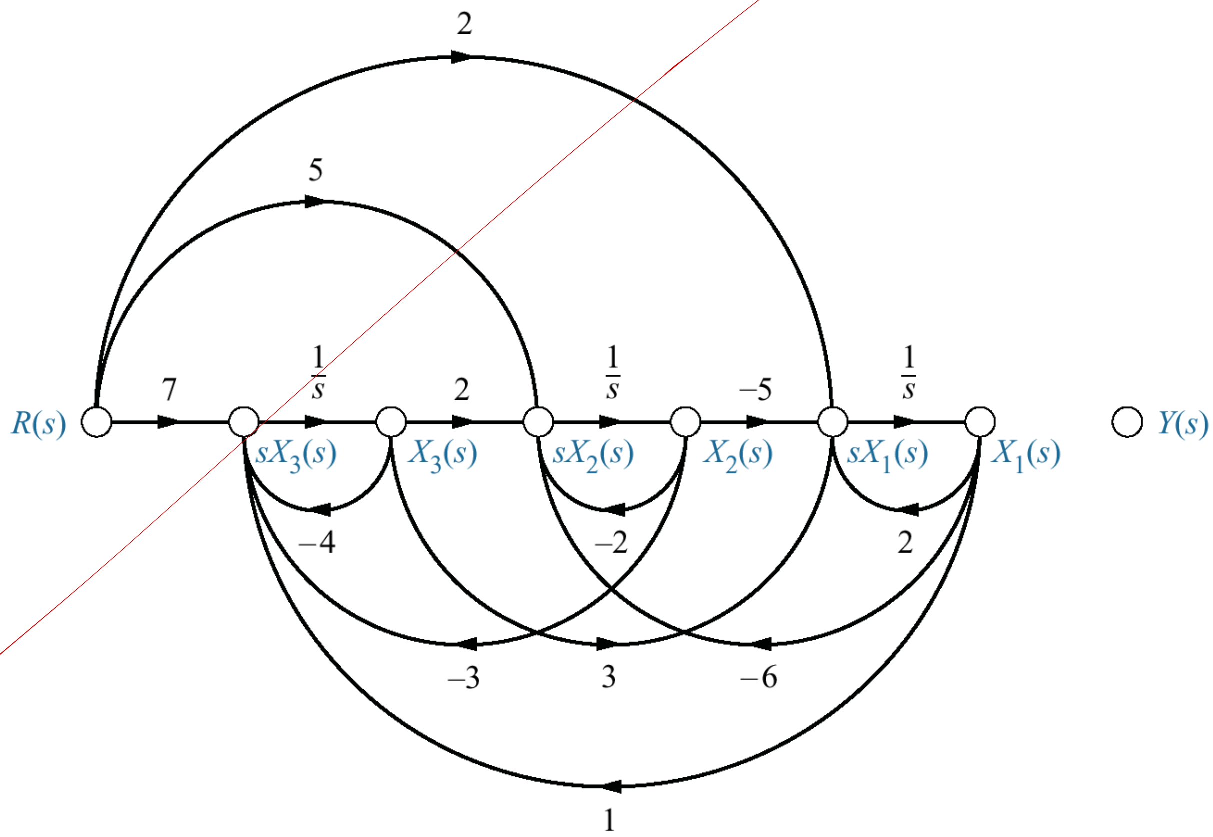
$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$

(c)



(d)

$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$
$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$
$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$
$$y = -4x_1 + 6x_2 + 9x_3$$

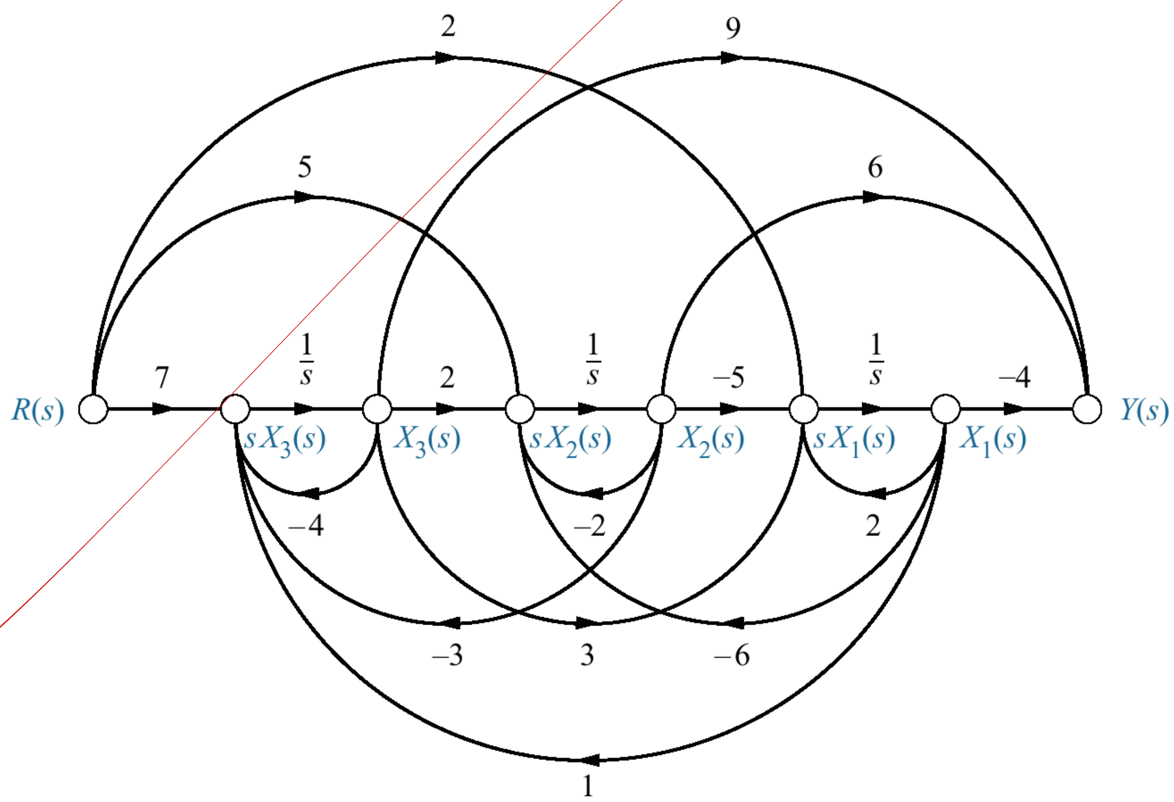


$$\dot{x}_1 = 2x_1 - 5x_2 + 3x_3 + 2r$$

$$\dot{x}_2 = -6x_1 - 2x_2 + 2x_3 + 5r$$

$$\dot{x}_3 = x_1 - 3x_2 - 4x_3 + 7r$$

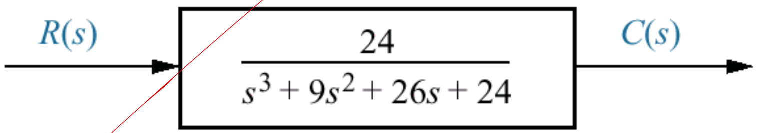
$$y = -4x_1 + 6x_2 + 9x_3$$



Alternative Representations in State Space

Phase variable Form

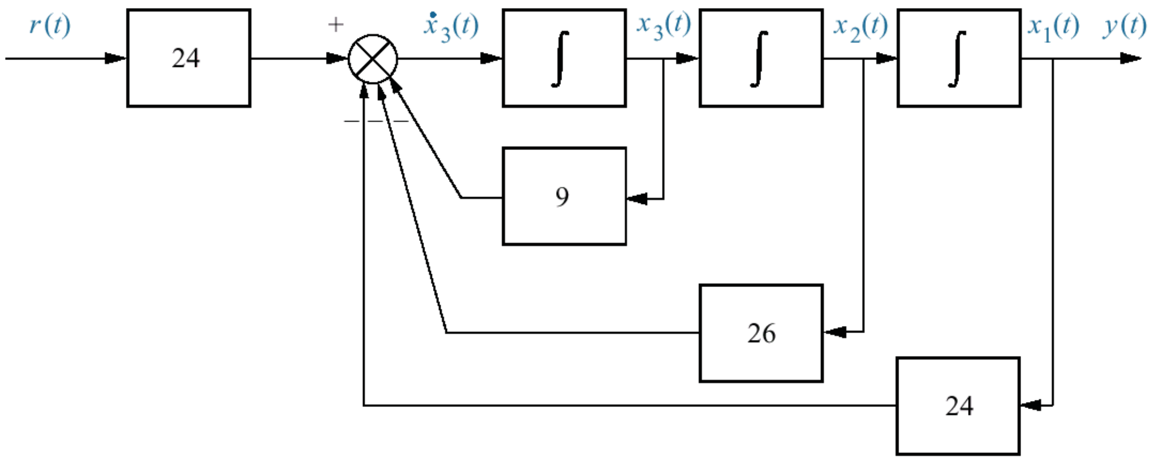
$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$



$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = 24r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

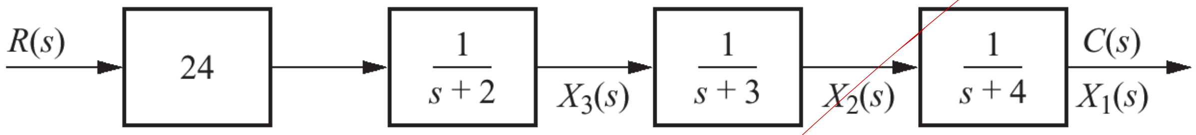
$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



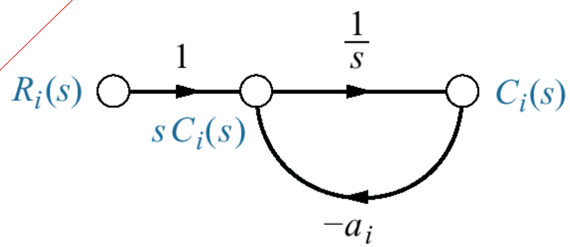
Alternative Representations in State Space

Cascade Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)} \longrightarrow \frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$



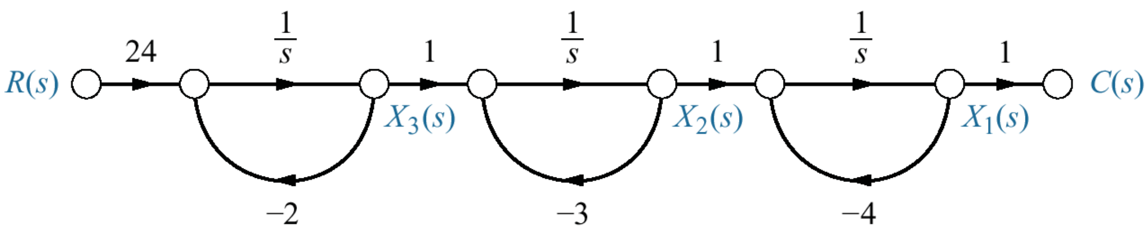
$$\frac{C_i(s)}{R_i(s)} = \frac{1}{(s + a_i)}$$

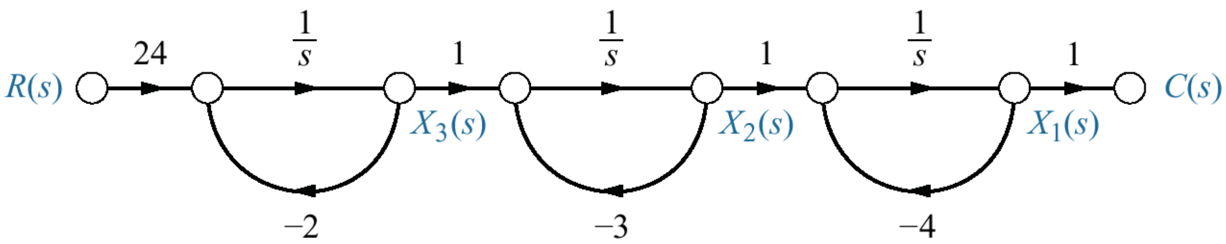


$$(s + a_i)C_i(s) = R_i(s)$$

$$\frac{dc_i(t)}{dt} + a_i c_i(t) = r_i(t)$$

$$\frac{dc_i(t)}{dt} = -a_i c_i(t) + r_i(t)$$





$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + 24r$$

$$y = c(t) = x_1$$

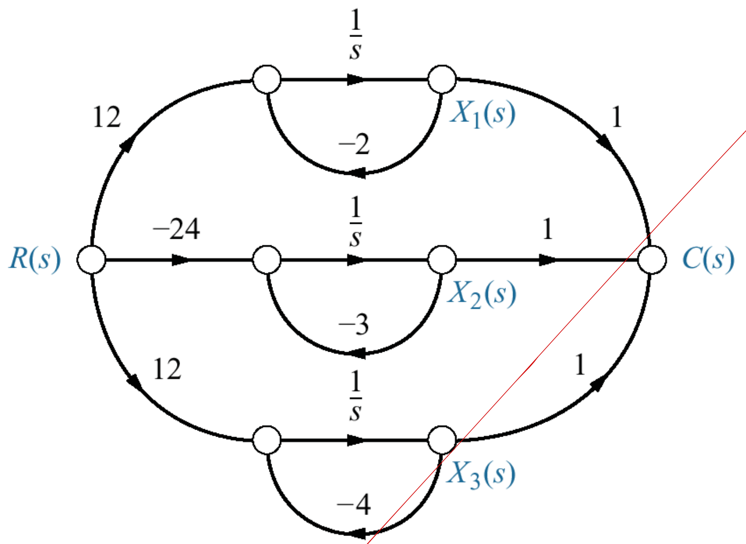
$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

Alternative Representations in State Space

Parallel Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

$$C(s) = R(s)\frac{12}{(s+2)} - R(s)\frac{24}{(s+3)} + R(s)\frac{12}{(s+4)}$$



$$\begin{aligned} \dot{x}_1 &= -2x_1 && && + 12r \\ \dot{x}_2 &= && -3x_2 && - 24r \\ \dot{x}_3 &= && && -4x_3 + 12r \end{aligned}$$

$$y = c(t) = x_1 + x_2 + x_3$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

Note that the equations are decoupled
(each state equation is function of only one state variable)

$$y = [1 \quad 1 \quad 1]\mathbf{x}$$