Edge-Adaptive Image Interpolation with Contour Stencils

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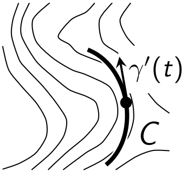
TV along Curves

Let u be an image. For C a smooth simple curve, define

$$\|u\|_{\mathrm{TV}(C)} = \int_0^T \left| \tfrac{\partial}{\partial t} u\big(\gamma(t)\big) \right| \, dt, \qquad \gamma : [0, T] \to C.$$

Strategy: Find approximate contours of u by finding curves C such that $\|u\|_{\mathrm{TV}(C)}$ is small.

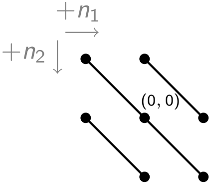


Contour Stencils

A *contour stencil* is a function $\mathcal{S} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ describing edges between pixels, and TV is estimated as

$$(\mathcal{S} \star [u])(k) := \sum_{m,n \in \mathbb{Z}^2} \mathcal{S}(m,n) \, |u_{k+m} - u_{k+n}| \approx \|u\|_{\mathrm{TV}(C)}$$

where $\mathcal{S}$ describes edges that approximate C.



$$\begin{aligned}(\mathcal{S} \star [u])(i,j) = \big(&|u_{i,j-1} - u_{i+1,j}| \\ &+ |u_{i-1,j-1} - u_{i,j}| + |u_{i,j} - u_{i+1,j+1}| \\ &+ |u_{i-1,j} - u_{i,j+1}|\big).\end{aligned}$$

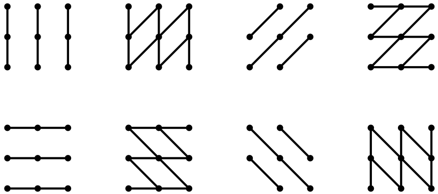
Contour Stencils

Estimate the contours locally by finding a stencil with low TV,

$$\mathcal{S}^\star(k) = \arg\min_{\mathcal{S}\in\Sigma} (\mathcal{S} \star [u])(k)$$

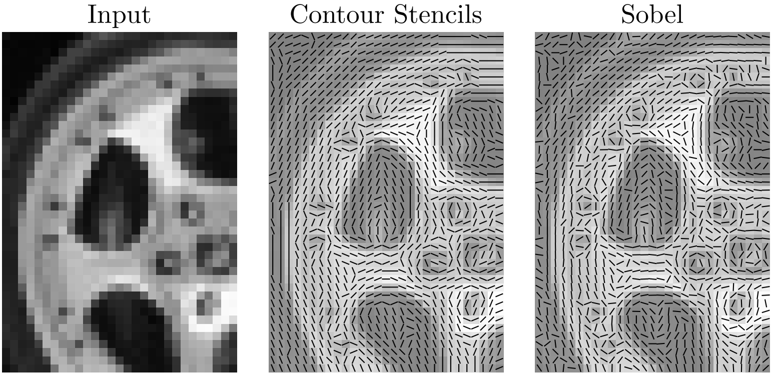
where Σ is a set of candidate stencils.

$\Sigma = \Big\{$



$\Big\}$

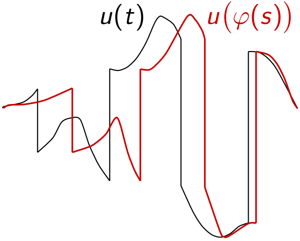
Contour Stencils



For each pixel, $\mathcal{S}^\star(k)$ is determined to estimate the local contour orientation.

Why TV?

Total variation is invariant under diffeomorphisms on space.
Consider the change of variables



$$t = \varphi(s)$$
$$dt = \varphi'(s)\, ds$$

and suppose that $\varphi'(s) > 0$, then

$$\begin{aligned}\int_0^T |u'(t)|\ dt &= \int_{\varphi^{-1}(0)}^{\varphi^{-1}(T)} \left|u'\big(\varphi(s)\big)\right| \varphi'(s)\, ds \\ &= \int_{\varphi^{-1}(0)}^{\varphi^{-1}(T)} \left|\tfrac{\partial}{\partial s} u\big(\varphi(s)\big)\right|\ ds.\end{aligned}$$

Interpolation Problem

Given discrete image v and point spread function $h(x, y)$, find function $u(x, y)$ such that

$$v_{i,j} = (h * u)(i, j) \quad \text{for all } i, j.$$

Input v

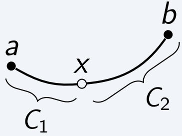
Interpolation u

Edge Directed Interpolation

Theorem (Edge Directed Interpolation)

Consider approximating $u(x)$ by

$$\hat{u}(x) = (1-\lambda)u(a) + \lambda u(b)$$



and let $C = C_1 \cup C_2$ be a curve passing through a, x, and b. Then the approximation error is bounded by

$$|\hat{u}(x) - u(x)| \leq \max\{|1-\lambda|, |\lambda|\} \|u\|_{\mathrm{TV}(C)}.$$

Choosing the stencil with the smallest TV minimizes the estimated interpolation error:

$$|\hat{u}(x) - u(x)| \leq \|u\|_{\mathrm{TV}(C)} \approx \frac{1}{|\mathcal{S}^\star|}(\mathcal{S}^\star \star [u])(k) = \min_{\mathcal{S}\in\Sigma} \frac{1}{\mathcal{S}}(\mathcal{S}\star[u])(k).$$

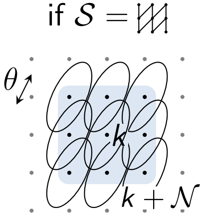
Contour Stencil Windowed Zooming

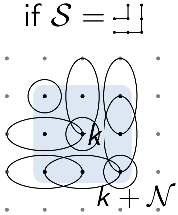
Local reconstructions:

$$u_k(x) = v_k + \sum_{n \in \mathcal{N}} c_n \, \varphi^n_{\mathcal{S}^\star(k)}(x - n),$$

where

- v_k : kth pixel of input image
- $\mathcal{N}$: neighborhood
- $\varphi^n_{\mathcal{S}^\star(k)}$: function oriented with the best-fitting stencil $\mathcal{S}^\star(k)$
- c_n : coefficients such that $(h * u_k)(m) = v_{k+m}$ for $m \in \mathcal{N}$





Contour Stencil Windowed Zooming

Combine local reconstructions with overlapping windows

$$u(x) = \sum_{k \in \mathbb{Z}^2} w(x-k) u_k(x-k),$$

where w satisfies

$\sum_k w(x-k) \equiv 1$	*s.t. method reproduces constants*
$w(k) = 0$ for $k \notin \mathcal{N}$	*s.t.* $\downarrow(h * u) \approx v$
w compact support	*for computational efficiency*

Example: Cubic B-spline

$$w(x) = B(x_1)B(x_2)$$
$$B(t) = \left(1 - |t| + \tfrac{1}{6}|t|^3 - \tfrac{1}{3}\big|1-|t|\big|^3\right)^+$$

Original Image (332×300)

Input Image (83×75)

Estimated Orientations

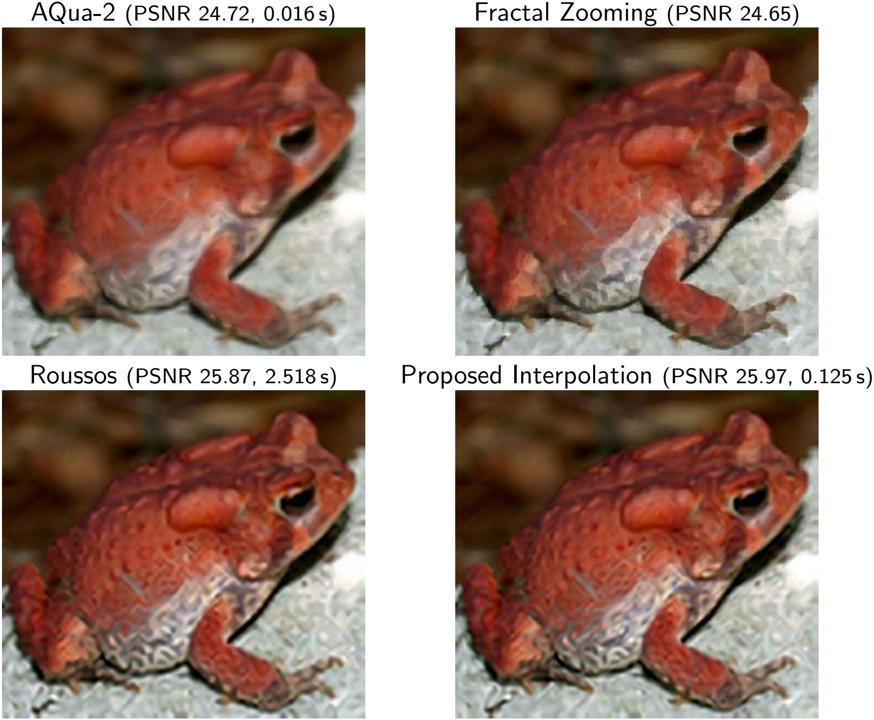
Proposed Interpolation (PSNR 25.97, 0.125 s)

Cubic B-spline (PSNR 25.92, 0.011 s)

Fourier (PSNR 25.34, 0.062 s)

TV Minimization (PSNR 25.73, 0.784 s)

Proposed Interpolation (PSNR 25.97, 0.125 s)


AQua-2 (PSNR 24.72, 0.016 s)
Fractal Zooming (PSNR 24.65)
Roussos (PSNR 25.87, 2.518 s)
Proposed Interpolation (PSNR 25.97, 0.125 s)

Zooming Comparison

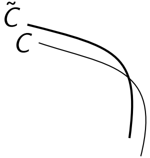
Average PSNR on the Kodak Image Suite

Zoom Factor	$2\times$	$3\times$	$4\times$
AQua-2	25.06	22.48	21.35
Fractal Zooming	29.00	27.20	25.25
Contour Stencils	29.87	27.77	25.93
Roussos	30.56	27.97	26.19

Computation Time (s) vs. Output Image Size

Image Size	128×128	256×256	512×512
AQua-2	0.0048	0.017	0.068
Contour Stencils	0.025	0.088	0.34
Roussos	0.23	2.22	8.64

Analysis of Contour Stencils



$$\tilde{C} \approx C \quad \Longrightarrow \quad \|u\|_{\mathrm{TV}(\tilde{C})} \approx \|u\|_{\mathrm{TV}(C)}$$

Curve Perturbation

Let C and $\tilde{C}$ be smooth curves parameterized by $\gamma : [0, T] \to C$ and $\tilde{\gamma} : [0, T] \to \tilde{C}$. Then if u is twice continuously differentiable,

$$\begin{aligned} &\big| \|u\|_{\mathrm{TV}(\tilde{C})} - \|u\|_{\mathrm{TV}(C)} \big| \\ &\quad \leq \big\| |\nabla u| \big\|_\infty \big\| |\tilde{\gamma}' - \gamma'| \big\|_1 \\ &\qquad + \|\nabla^2 u\|_\infty \big(\tfrac{|\tilde{C}| + |C|}{2} + \tfrac{1}{4} \big\| |\tilde{\gamma}' - \gamma'| \big\|_1 \big) \big\| |\tilde{\gamma} - \gamma| \big\|_\infty . \end{aligned}$$

Analysis of Contour Stencils

$$u \in C^2 \quad \Longrightarrow \quad \text{discrete TV is first-order accurate}$$

TV Discretization

Suppose $u \in C^2[0, T]$ and $0 = t_0 < t_1 < \cdots < t_N = T$, and define $h_i = t_i - t_{i-1}$. Then

$$\|u\|_{\mathrm{TV}} - \tfrac{1}{3} T \tfrac{h_{max}^2}{h_{avg}} \|u''\|_\infty \leq \sum_{i=1}^{N} |u(t_i) - u(t_{i-1})| \leq \|u\|_{\mathrm{TV}} .$$

Analysis of Contour Stencils

Let $\mathcal{S}^{\star 2}(k)$ denote the second best-fitting stencil and define the separation $\operatorname{sep}_v(k)$ between the first and second best,

$$\begin{aligned} \mathcal{S}^{\star 2}(k) &:= \operatorname*{arg\,min}_{\mathcal{S}\in\Sigma\setminus\mathcal{S}^\star(k)} (\mathcal{S}\star[v])(k), \\ \operatorname{sep}_v(k) &:= \big(\mathcal{S}^{\star 2}(k)\star[v]\big)(k) - \big(\mathcal{S}^{\star}(k)\star[v]\big)(k). \end{aligned}$$

Stability of the Best-Fitting Stencil

Suppose that for two images v and $\tilde{v}$

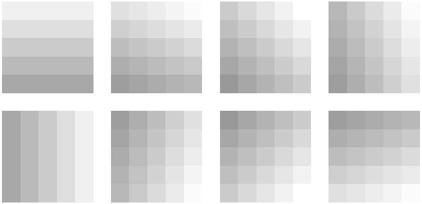
$$\begin{aligned} &\operatorname{sep}_v(k) > 2M\,\|v-\tilde{v}\|_2\,, \\ &\qquad M = \max_{\mathcal{S}\in\Sigma}\Bigg[\sum_{m\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}\big|\mathcal{S}(m,n)+\mathcal{S}(n,m)\big|\Big)^2\Bigg]^{1/2}. \end{aligned}$$

Then they have the same best-fitting stencil at k.

Contour Stencil Design

Let $\{f^1, \ldots, f^J\}$, $f^j : \mathbb{Z}^2 \to \mathbb{R}$, be a set of image features.

Example: $\quad f^j(x) = x_1 \sin \frac{\pi}{8} j - x_2 \cos \frac{\pi}{8} j, \quad j = 0, \ldots, 7$



We want to design stencils $\mathcal{S}^1, \ldots, \mathcal{S}^J$ that distinguish between these features.

Contour Stencil Design

We want stencil $\mathcal{S}^j$ to be the best-fitting stencil on f^j,

$$\textbf{Want:} \quad \tfrac{1}{|\mathcal{S}^j|}(\mathcal{S}^j \star [f^j])(0) < \tfrac{1}{|\mathcal{S}^i|}(\mathcal{S}^i \star [f^j])(0) \quad \text{for all } i \neq j.$$

Ignoring the $\frac{1}{|\mathcal{S}|}$ normalizations, this condition becomes

$$\begin{aligned} & (\mathcal{S}^j \star [f^j])(0) < (\mathcal{S}^i \star [f^j])(0) \\ \Longrightarrow \qquad & \big((\mathcal{S}^j - \mathcal{S}^i) \star [f^j]\big)(0) < 0. \end{aligned}$$

We can try to satisfy this condition by minimizing

$$\begin{aligned} & \min_{\mathcal{S}^1,\ldots,\mathcal{S}^J} \sum_{i=1}^{J} \sum_{j=1}^{J} \big((\mathcal{S}^j - \mathcal{S}^i) \star [f^j]\big)(0) + \gamma \sum_{j=1}^{J} \|\mathcal{S}^j\|_1 \\ & \text{s.t. } 0 \leq \mathcal{S}^j(m,n) \leq 1 \end{aligned}$$

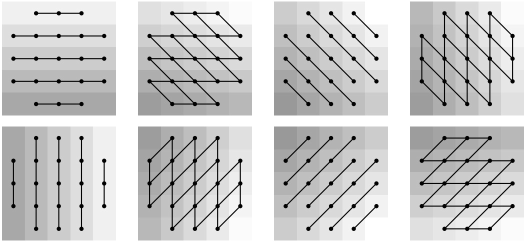
Contour Stencil Design

$$\min_{\mathcal{S}^1,\ldots,\mathcal{S}^J} \sum_{i=1}^{J}\sum_{j=1}^{J}\big((\mathcal{S}^j-\mathcal{S}^i)\star[f^j]\big)(0)+\gamma\sum_{j=1}^{J}\|\mathcal{S}^j\|_1$$

$$\text{s.t. } 0\le \mathcal{S}^j(m,n)\le 1$$

The minimization has closed-form solution

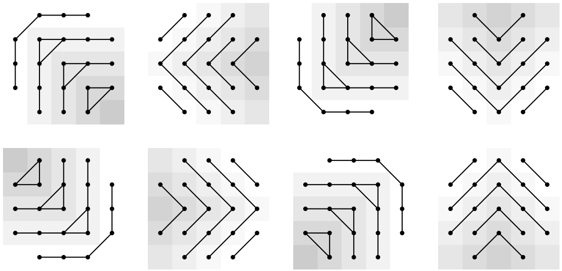
$$\mathcal{S}^j(m,n)=\begin{cases}1 & \text{if } |f^j_m-f^j_n|<\frac{1}{J}\sum_{i=1}^{J}|f^i_m-f^i_n|-\frac{\gamma}{J},\\ 0 & \text{if } |f^j_m-f^j_n|>\frac{1}{J}\sum_{i=1}^{J}|f^i_m-f^i_n|-\frac{\gamma}{J}.\end{cases}$$



Corner-Shaped Stencils

Example: Corner-shaped stencils designed from the features

$$f^j(x) = \max\{x_1 \cos \tfrac{\pi}{4}j - x_2 \sin \tfrac{\pi}{4}j, \\ x_1 \sin \tfrac{\pi}{4}j + x_2 \cos \tfrac{\pi}{4}j\}.$$

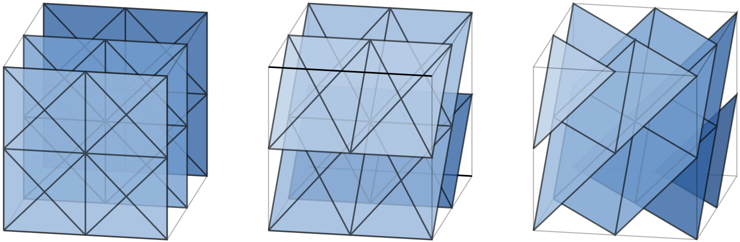


3D Stencils

In d dimensions, a stencil $\mathcal{S} : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is applied at voxel $k \in \mathbb{Z}^d$ as

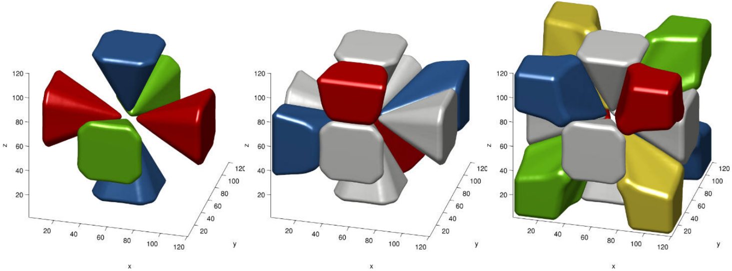
$$(\mathcal{S} \star [u])(k) := \sum_{m,n \in \mathbb{Z}^d} \mathcal{S}(m, n) \left| u_{k+m} - u_{k+n} \right| .$$

Small TV detects *isosurfaces*. Some 3D stencils:



3D Stencils

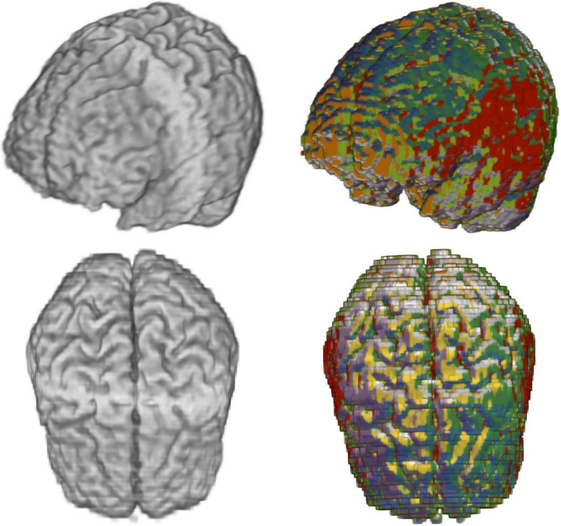
Example: Stencils applied to $u_{i,j,k} = \sqrt{i^2 + j^2 + k^2}$



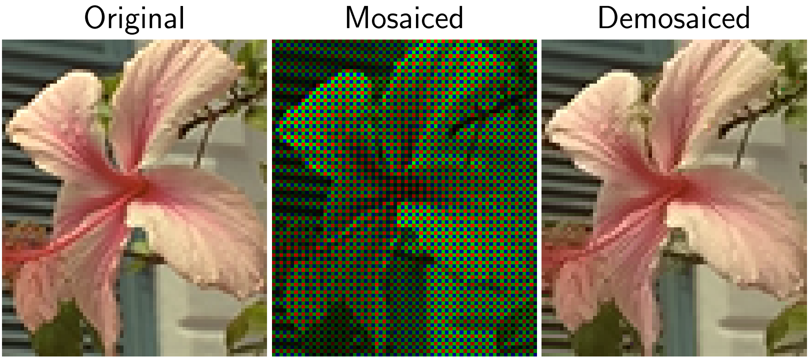
The results are visualized by assigning a color to the region of space having a particular best-fitting stencil.

3D Stencils

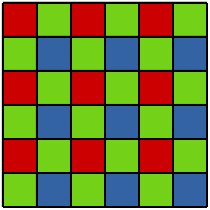
Example: Stencils applied to an MRI brain volume



Demosaicing

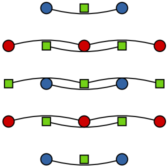


Bayer Grid

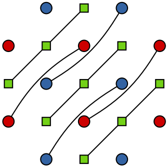


Demosaicing Stencils

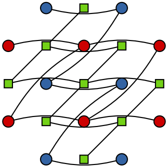
Centered on a green pixel:



Axially-oriented

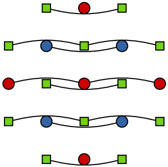


Diagonally-oriented

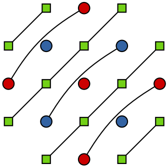


$\frac{\pi}{8}$-oriented

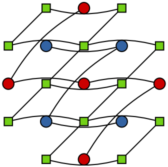
Centered on a red or blue pixel:



Axially-oriented



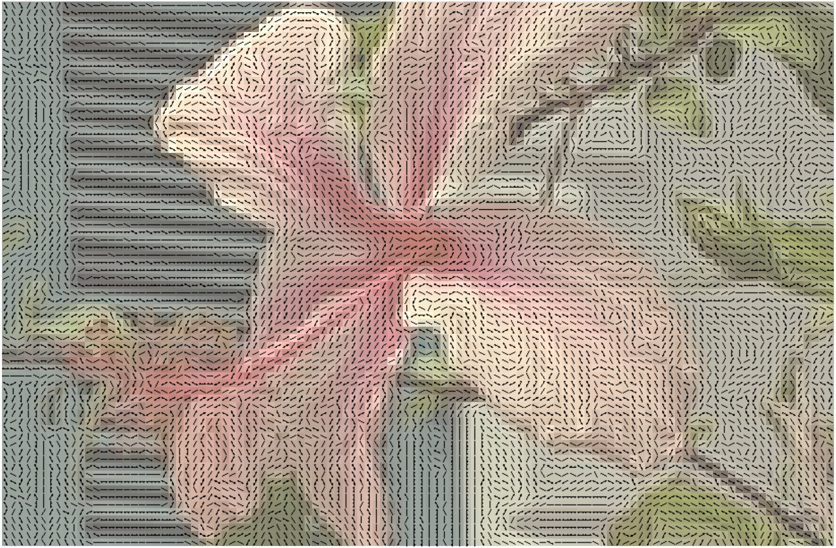
Diagonally-oriented



$\frac{\pi}{8}$-oriented

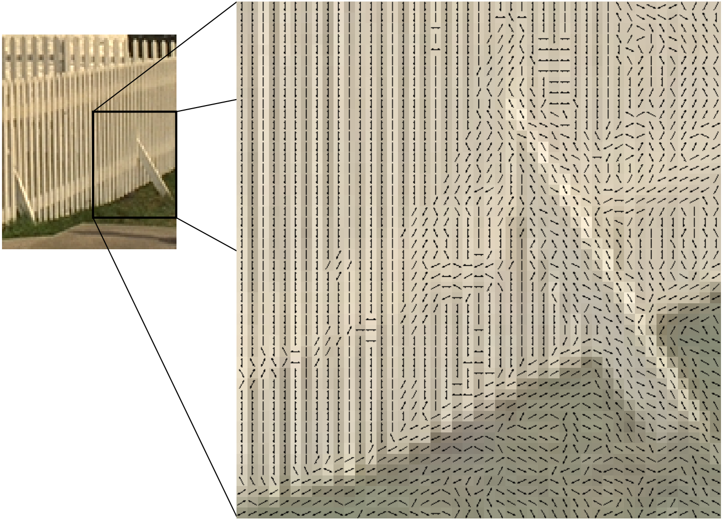
Demosaicing Stencils

Stencil orientation estimation on mosaiced data:



Demosaicing Stencils

Stencil orientation estimation on mosaiced data:



Preliminary Demosaicing Method

Let f be the given mosaiced image. We consider demosaicing by the minimization of

$$\arg\min_u \sum_{k\in\{Y,C_b,C_r\}} \sum_{n\in\Omega} \sum_{m\in\mathcal{N}(n)} w_{m,n} \left| u_m^{(k)} - u_n^{(k)} \right| + \frac{\lambda}{2} \sum_{k\in\{R,G,B\}} \sum_{n\in\Omega^{(k)}} (f_n - u_n^{(k)})^2$$

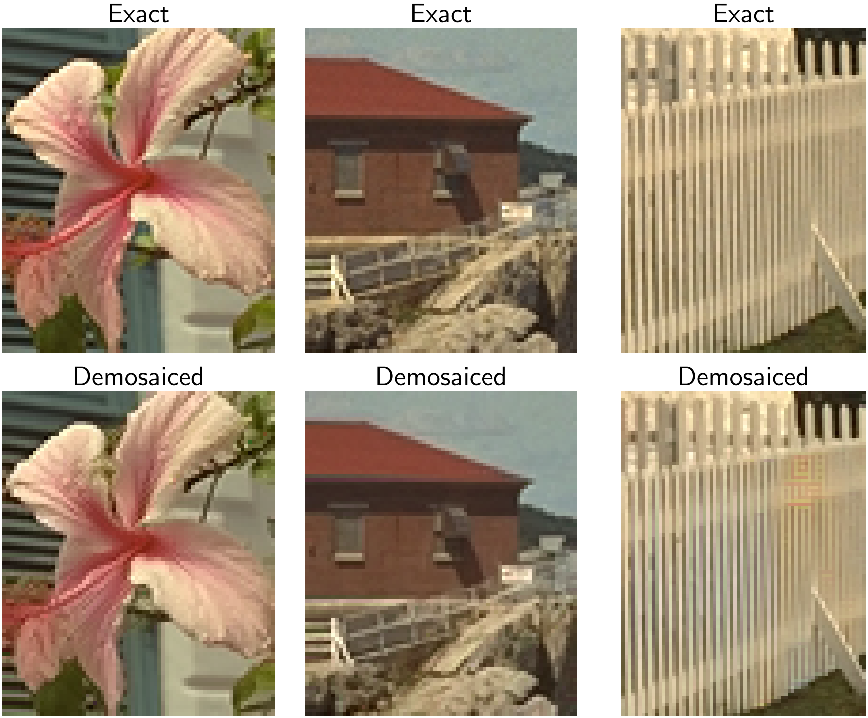
where

$w_{m,n}$: weights choosen according the best-fitting stencils

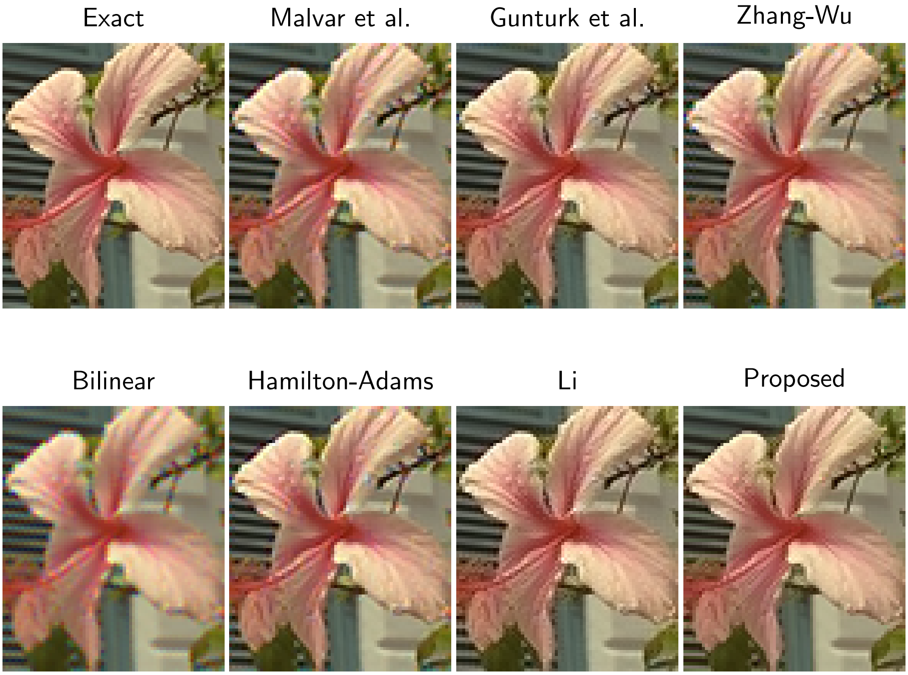
$\mathcal{N}(n)$: neighbors of pixel n

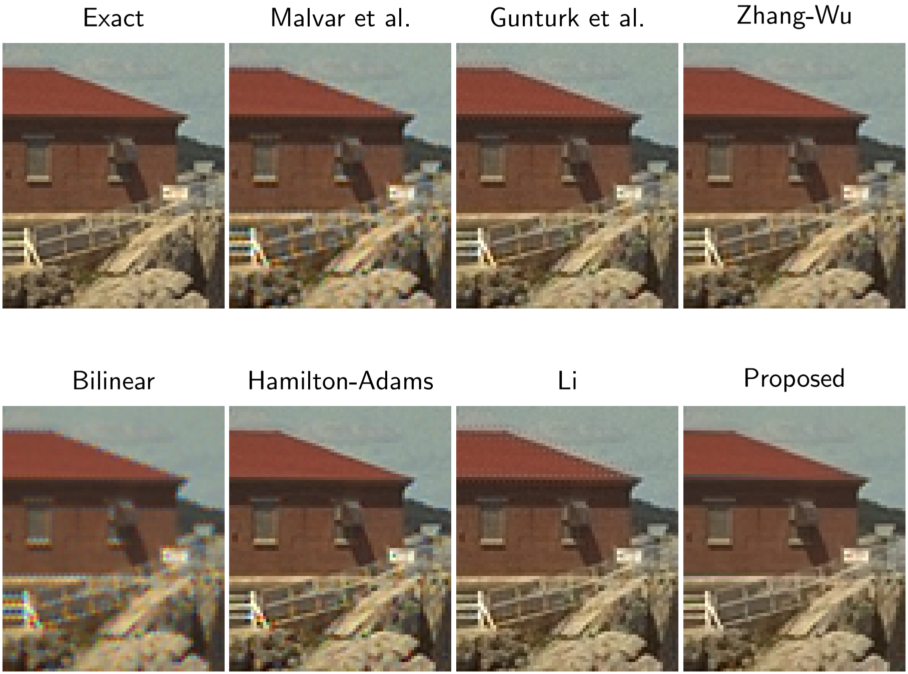
λ : fidelity parameter

$\Omega^{(k)}$: subset of Ω where kth channel is given



Exact
Exact
Exact
Demosaiced
Demosaiced
Demosaiced


Exact
Malvar et al.
Gunturk et al.
Zhang-Wu
Bilinear
Hamilton-Adams
Li
Proposed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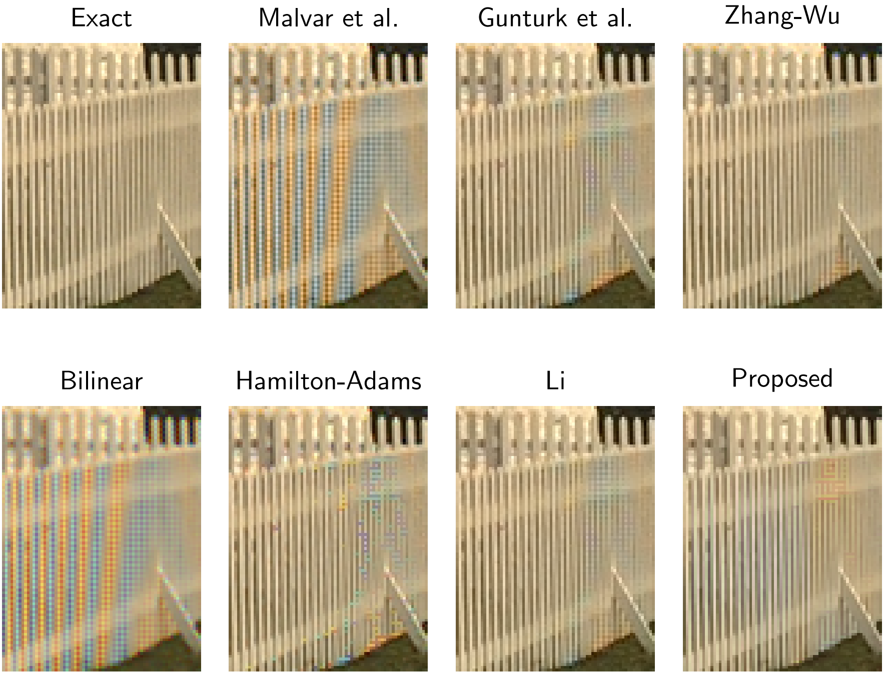
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Proposed

Thanks!

Webpage

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