Error correlation schemes for fully correlated quantum channels protecting both quantum and classical information

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Abstract

We study efficient quantum error correction schemes for the fully correlated channel on an n-qubit system with error operators that assume the form $\sigma_x^{\otimes n}$, $\sigma_y^{\otimes n}$, $\sigma_z^{\otimes n}$. In particular, when $n = 2k+1$ is odd, we describe a quantum error correction scheme using one arbitrary qubit σ to protect the data state ρ in the $2k$-qubit system such that only $3k$ CNOT gates (each with one control bit and one target bit) are needed to encode the n-qubits. The inverse operation of the CNOT gates will produce $\tilde{\sigma} \otimes \rho$, so a partial trace operation can recover ρ. When $n = 2k + 2$ is even, we describe a hybrid quantum error correction scheme that protects a $2k$-qubit state ρ and 2 classical bits encoded as $\sigma \in \{|ij\rangle\langle ij| : i, j \in \{0, 1\}\}$; the encoding can be done by $3k + 2$ CNOT gates and a Hadamard gate on one qubit, and the inverse operation will be the decoding operation producing $\sigma \otimes \rho$. The scheme was implemented using Matlab, Mathematica and the IBM's quantum computing framework `qiskit`.

1 Introduction

In quantum information processing, information is stored and processed with a quantum system. In the mathematical setting, quantum states are represented as density matrices, i.e., complex positive semi-definite matrices with trace one. Denote by D_N the set of density matrices in the set M_N of $N \times N$ complex matrices. A qubit will be represented as a matrix in D_2, and a quantum state of an n-qubit system will be a matrix in D_N with $N = 2^n$. A quantum channel on an n-qubit system is a trace preserving completely positive linear map $\mathcal{E} : M_N \to M_N$ that admits the operator sum representation

$$\mathcal{E}(\rho) = \sum_{j=1}^{r} F_j \rho F_j^\dagger \qquad \text{for all } \rho \in M_N,$$

where $\sum_{j=1}^{r} F_j^\dagger F_j = I_N$. The matrices $F_1, \ldots, F_r$ are sometimes called the error operators of the quantum channel $\mathcal{E}$, which is the source of the corruption of the quantum states corresponding to decoherence and other quantum effect on the quantum state ρ. To protect the information stored in the quantum state ρ, one can use quantum error correction schemes to encode the quantum state ρ with some auxiliary qubits ρ so that one can recover the quantum state ρ after the encoded quantum state go through the quantum channel.

In [6], the authors considered the fully correlated channel $\mathcal{E}$ on an n-qubit system with error operators that assume the form $X_n = \sigma_x^{\otimes n}$, $Y_n = \sigma_y^{\otimes n}$, $Z_n = \sigma_z^{\otimes n}$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are the Pauli matrices. So, the quantum channel $\mathcal{E} : M_N \to M_N$ with $N = 2^n$ has the form

$$\mathcal{E}(\rho) = p_0\rho + p_1 X_n \rho X_n^\dagger + p_2 Y_n \rho Y_n^\dagger + p_2 Z_n \rho Z_n^\dagger \qquad \text{for all } \rho \in M_N, \tag{1}$$

where p_0, p_1, p_2, p_3 are nonnegative numbers summing up to one. It was shown that if n is odd, one can use a single qubit $\sigma \in D_2$ to protect an $(n-1)$-qubit data state; if n is even, one can use a two-qubit σ to protect $(n-2)$-qubit data state. Encoding and decoding can be performed without measurement. So, for such a fully correlated channel, if one would like to protect $2k$-qubit data states, only one additional qubit is needed, and using two additional qubits to protect the $2k$-qubit data state seems to be a waste of resources.

We will show that one can actually design a hybrid quantum correction scheme for an $n = 2k+2$ qubit fully correlated channel to send $2k$-qubit quantum states together with two classical bits. The study of simultaneous transmission of both quantum and classical information over a quantum channel was initiated in [1] and followed up by other researchers, [2, 3, 4].

In this paper, we will present efficient error correction schemes for the fully correlated channel. When $n = 2k+1$ is odd, we describe a quantum error correction scheme using one arbitrary qubit σ to protect the data state ρ in the $2k$-qubit system such that only $3k$ CNOT gates (each has one control bit and one target bit) are needed to encode the n-qubits. The inverse operation of the CNOT gates will produce $\tilde{\sigma} \otimes \rho$, so a partial trace operation can recover ρ. When $n = 2k+2$ is even, we describe a hybrid quantum error correction scheme that protects a state ρ in the $2k$-qubit system, and two classical bit encoded as $\sigma \in \{|ij\rangle\langle ij| : i, j \in \{0, 1\}\}$; the encoding can be done by $3k+2$ CNOT gates and a Hadamard gate on a qubit, and the inverse operation will be the decoding operation producing $\sigma \otimes \rho$. The program was implemented using the IBM's quantum computing framework `qiskit` [9].

We state the results and prove them in the next section. Then we illustrate our schemes and depict the circuit diagrams. Furthermore, we implement and demonstrate our schemes using Matlab, Mathematica, and IBM's online quantum computer IBM Q 5 Yorktown. The last section is devoted to summary and discussions.

2 The quantum error and hybrid error correction schemes

In this section, we show that one can recursively construct the encoding matrix $P_n \in M_{2^n}$ for the n-qubit fully correlated channel $\mathcal{E}$ defined in 1. Moreover, we will show that P_n can be decomposed as simple CNOT gates (with one control qubit and one target qubit) and Hadamard gates (on one qubit). Once we find P_n, we can encode $A \mapsto P_n A P_n^\dagger$ and decode $B \mapsto P_n^\dagger B P_n$. We will give a

full description of the construction, number of CNOT gates required, and the encoding / decoding procedures in Section 2.5. The circuit diagrams for encoding and decoding will be shown in Section 2.6.

We will depict an n-qubit vector state as $|q_{n-1}\dots q_0\rangle$. Let C_{ij} be the CNOT gate where the ith qubit controls the target jth qubit. E For example, for $(q_2,q_1,q_0)\in\{(0,0,0),(0,0,1),\dots,(1,1,1)\}$,

$$C_{02}|q_2q_1q_0\rangle=\begin{cases}|q_2\oplus 1,q_1,q_0\rangle & \text{if } q_0=1,\\ |q_2q_1q_0\rangle & \text{otherwise.}\end{cases}$$

Denote $H=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\in M_2$ as the Hadamard gate. Also, we use $e_1,\dots,e_n$ to denote the columns of I_n.

2.1 Two-qubit encoding/decoding operator

Let $P_2=C_{01}(I_2\otimes H)C_{01}^t\in M_4$, where C_{01} is the controlled not gate using the q_0-bit to control the q_1-bit for the two qubit state $|q_1q_0\rangle$. So, $P_2=C_{01}(I\otimes H)C_{01}$, and in the matrix form $C_{01}=Q=[e_1\ e_4\ e_3\ e_2]$. We readily verify the following.

Proposition 2.1 *Let $P_2=C_{01}(I_2\otimes H)C_{01}^t\in M_4$. Then*

$$(P_2^\dagger X_2P_2,P_2^\dagger Y_2P_2,P_2^\dagger Z_2P_2)=(D_X,D_Y,D_Z)\tag{2}$$

with $D_X=\operatorname{diag}(1,-1,1,-1),D_Y=(-1,-1,1,1),D_Z=(1,-1,-1,1)$. Consequently,

$$P_2^\dagger(\mathcal{E}(P_2(\sigma)P_2^\dagger)P_2=\sigma$$

whenever $\sigma=|q_1q_0\rangle\langle|q_1q_0|$ with $|q_1q_0\rangle\in\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$.

2.2 Three-qubit encoding/decoding operator

Proposition 2.2 *Let $P_3=C_{10}C_{02}C_{21}\in M_8$, where C_{ij} use the $|q_i\rangle$ to control the $|q_j\rangle$ in $|q_2q_1q_0\rangle$. Then*

$$(P_3^\dagger X_3P_3,P_3^\dagger Y_3P_3,P_3^\dagger Z_3P_3)=(X_1\otimes I_4,-Y_1\otimes I_4,Z_1\otimes I_4).$$

Consequently, for any $\sigma\in D_2$ and $\rho\in D_4$, we have

$$P_3^\dagger(\mathcal{E}(P_3(\sigma\otimes\rho)P_3^\dagger))P_3=\tilde{\sigma}\otimes\rho,$$

where

$$\tilde{\sigma}=p_0\sigma+p_1X_1\sigma X_1^\dagger+p_2Y_1\sigma Y_1^\dagger+p_3Z_1\sigma Z_1^\dagger.$$

Moreover, if P_3 is a product of m CNOT gates (each with one control bit and one target bit) on a 3-qubit system, then $m\geq 3$.

Proof. One readily verify the first two statements. For the last assertion, if we list the columns of I_8 and P_3 in binary form, we have

$$I_8 = [e_1\ e_2\ e_3\ e_4\ e_5\ e_6\ e_7\ e_8] = [|000\rangle\ |001\rangle\ |010\rangle\ |011\rangle\ |100\rangle\ |101\rangle\ |110\rangle\ |111\rangle],$$

$$P_3 = [e_1\ e_6\ e_4\ e_7\ e_8\ e_3\ e_5\ e_2] = [|000\rangle\ |101\rangle\ |011\rangle\ |110\rangle\ |111\rangle\ |010\rangle\ |100\rangle\ |001\rangle].$$

Since there are collectively 12 mismatched positions out of the 24 positions in the binary form of the 8 columns of the matrices I_8 to P_3, and every CNOT gate will change 4 out of the 24 positions, we see that expressing P_3 as the product of 3 CNOT gates is optimal. □

2.3 n-qubit encoding / decoding operator for odd $n \geq 5$

Proposition 2.3 *Let $n = 2k+1$ be an odd integer with $k \geq 2$, Let $P_n = (I_4 \otimes P_{n-2})(P_3 \otimes I_{2^{n-3}})$, which can be written as a product of $3k$ CNOT gates (each as one control bit and one target bit). Then*

$$(P_n^\dagger X_n P_n, P_n^\dagger Y_n P_n, P_n^\dagger Z_n P_n) = (X_1 \otimes I_{2^{n-1}}, (-1)^k Y_1 \otimes I_{2^{n-1}}, Z_1 \otimes I_{2^{n-1}}).$$

Consequently, for any $\sigma \in D_2$ and $\rho \in D_{2^{n-1}}$, we have

$$P_3^\dagger(\mathcal{E}(P_3(\sigma \otimes \rho)P_3^\dagger))P_3 = \tilde{\sigma} \otimes \rho,$$

where

$$\tilde{\sigma} = p_0\sigma + p_1 X_1 \sigma X_1^\dagger + p_2 Y_1 \sigma Y_1^\dagger + p_3 Z_1 \sigma Z_1^\dagger.$$

Proof. By Proposition 2.2, P_3 is a product of 3 CNOT gates. By the recursive construction, when k increases 1 we need 3 more CNOT gates. So, P_n can be written as a product of $3k$ CNOT gates.

The other assertions can be verified readily. □

2.4 n-qubit encoding / operator for even $n \geq 4$

Proposition 2.4 *Suppose $n = 2k+2$ for $k \geq 1$, and P_{n-1} is defined as in Proposition 2.3. Let $P_n = (I_2 \otimes P_{n-1})(P_2 \otimes I_{2^{n-2}})$, which is a product of $3k+2$ CNOT gates (each has one control bit and one target bit), and 1 Hadamard gate (on one qubit). Then*

$$(P_n^\dagger X_n P_n, P_n^\dagger Y_n P_n, P_n^\dagger Z_n P_n) = (D_X \otimes I_{2^{n-2}}, (-1)^k D_Y \otimes I_{2^{n-2}}, D_Z \otimes I_{2^{n-2}})$$

$D_X = \mathrm{diag}\,(1,-1,1,-1), D_Y = (-1,-1,1,1), D_Z = (1,-1,-1,1)$. *Consequently,*

$$P_n^\dagger(\mathcal{E}(P_n(\sigma \otimes \rho)P_n^\dagger)P_n = \sigma \otimes \rho$$

whenever $\rho \in D_{n-2}$ and $\sigma = |q_1q_0\rangle\langle|q_1q_0|$ with $|q_1q_0\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

Proof. By Proposition 2.1, P_2 is a product of 2 CNOT gates and 1 Hadamard gate. By Proposition 2.3, P_{n-1} is a product of $3k$ CNOT gates. So, $P_n = (I_2 \otimes P_{n-1})(P_2 \otimes I_{2^{n-2}})$ is the product of $3k+2$ CNOT gates and a Hadamard gate.

The rest of the proposition can be verified readily. □

2.5 The encoding and decoding schemes

We summarize the results in Propositions 2.1 — 2.4 to the following.

Theorem 2.5 *Let P_2, P_3 and P_n be defined as in Sections 2.1 - 2.4.*

(a) *Suppose $n = 2k + 1 \geq 3$ is odd. Then P_n is a product of $3k$ CNOT gates (each has 1 control and 1 target bit). One can encode an $(n-1)$-qubit data state ρ using an arbitrary qubit σ by the encoding operator P_n so that*

$$\rho \mapsto P_n(\sigma \otimes \rho)P_n^\dagger.$$

After the encoded state goes through the fully correlated channel $\mathcal{E}$, one can apply the operation $B \mapsto P_n^\dagger B P_n$. Then the encoded state $P_n(\sigma \otimes \rho)P_n^\dagger$ becomes

$$\begin{aligned} &p_0(\sigma \otimes \rho) + p_1 P_n^\dagger X_n P_n(\sigma \otimes \rho)P_n^\dagger X_n^\dagger P_n + \\ &p_2 P_n^\dagger Y_n P_n(\sigma \otimes \rho)P_n^\dagger Y_n^\dagger P_n + p_3 P_n^\dagger Z_n P_n(\sigma \otimes \rho)P_n^\dagger Z_n^\dagger P_n \quad = \quad \tilde{\sigma} \otimes \rho, \end{aligned}$$

where

$$\tilde{\sigma} = p_0\sigma + p_1 X_1 \sigma X_1^\dagger + p_2 Y_1 \sigma Y_1^\dagger + p_3 Z_1 \sigma Z_1^\dagger.$$

Then, one can apply a partial trace $\mathrm{tr}_1(\tilde{\sigma} \otimes \rho)$ to recover the data state ρ.

(b) *Suppose $n = 2k + 2 \geq 4$ is even. Then P_n is a product of $3k + 2$ CNOT gates (each has 1 control and 1 target bit) with a Hadamard gate (on 1 qubit). One can encode an $(n-2)$-qubit data state ρ and two classical bits by $\sigma \in \{|ij\rangle\langle ij| : 1 \leq i, j \leq 2\}$ using the encoding operator P_n so that*

$$\rho \mapsto P_n(\sigma \otimes \rho)P_n^\dagger.$$

After the encoded state goes through the fully correlated channel $\mathcal{E}$, one can apply the operation $B \mapsto P_n^\dagger B P_n$. Then the encoded state becomes

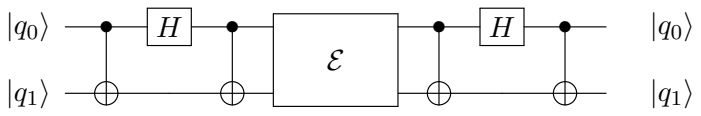
$$\begin{aligned} &p_0(\sigma \otimes \rho) + p_1 P_n^\dagger X_n P_n(\sigma \otimes \rho)P_n^\dagger X_n^\dagger P_n + \\ &p_2 P_n^\dagger Y_n P_n(\sigma \otimes \rho)P_n^\dagger Y_n^\dagger P_n + p_3 P_n^\dagger Z_n P_n(\sigma \otimes \rho)P_n^\dagger Z_n^\dagger P_n \quad = \quad \sigma \otimes \rho. \end{aligned}$$

One can apply a measurement to the first two qubits to obtain the two classical bits σ.

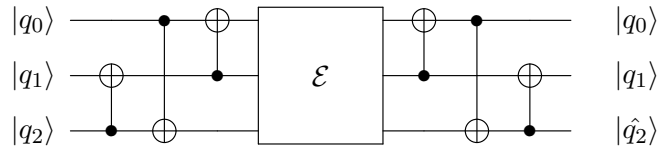
2.6 Circuit diagrams

We can set the n quantum state as $|q_n \cdots q_1\rangle$. Using our scheme, the circuit diagram are depicted in the following.

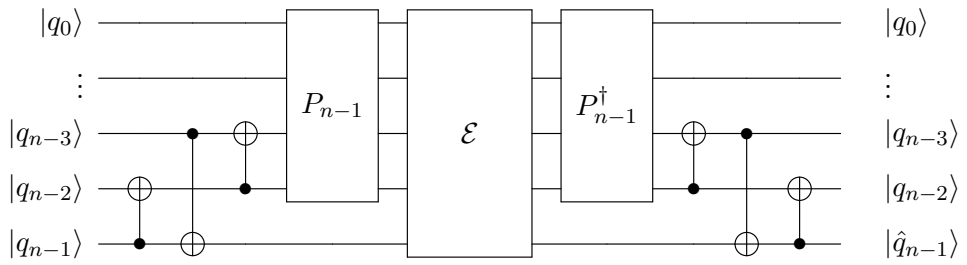
For $n = 2$, if $|q_1 q_0\rangle \in \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, then circuit diagram will be:



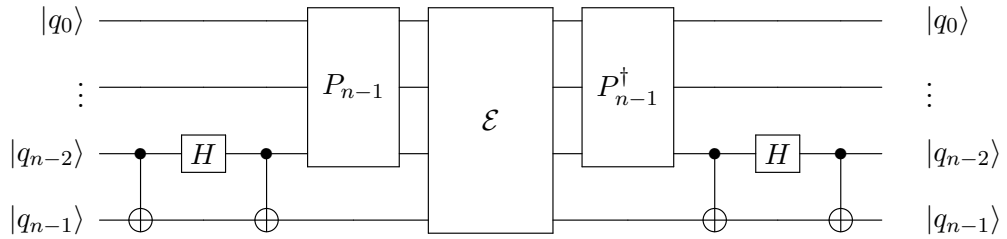
For $n = 3$,



For odd n, the circuit diagram will be



For even n, if $|q_{n-1}q_{n-2}\rangle \in \{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$, then the circuit diagram will look like:



3 Concluding remarks and future work

We obtain an efficient error correction scheme for fully correlated quantum channels protecting both quantum and classical information. The scheme was implemented using Matlab, Mathematica and the IBM's quantum computing framework `qiskit`.

Several remarks are in order concerning the implementation.

In Matlab, the we generate the encoding operation P_n for $n \geq 2$, and check the properties described in Propositions 2.1 – 2.4. Note that when the dimension is high, the matrices P_n, X_n, etc. are too big to display, and there are too many entries to check (though most of them are zeros). So, we just use the command `norm(Pn'*Zn*AA*Zn*Pn - kron(S,R))` to compute the norm of the matrix $P_n^\dagger Z_n P_n(\sigma \otimes \rho) P_n^\dagger Z_n P_n - \sigma \otimes \rho$ to confirm that it gives zero (up to machine error). In fact, in the odd case, when we do not use the Hadamard gate to do encoding and decoding, the norm values of the relevant matrices are exactly 0; in the even case, when the Hadamard gate is used (once in encoding and once in decoding), the norm value of matrices will yield a number at the order of the machine error.

In Mathematica, the computation of the norm of the matrix $P_n^\dagger \mathcal{E}(P_n(\sigma \otimes \rho)P_n^\dagger)P_n - \tilde{\sigma} \otimes \rho$ is always exactly 0 even if the Hadamard gate is used in the even case. Here $\tilde{\sigma} = p_0\sigma + p_1 X\sigma X^\dagger +$

$p_2 Y\sigma Y^\dagger + p_3 Z\sigma Z^\dagger \in D_2$ if n is odd and $\sigma = \tilde{\sigma} \in D_4$ is one of the 4 classical binary bits if n is even. This is due to Mathematica being an algebraic solver versus Matlab and Python.

In the IBM quantum computer setting, it is interesting to note that for $U \in \{I_2, X, Y, Z\}$, when we apply the encoding scheme to 3-qubit $P_3|q_2q_1q_0\rangle$, then the error operator $U^{\otimes 3}P_3|q_2q_1q_0\rangle$, and then the operator $P_n^\dagger U^{\otimes 3}P_3|q_2q_1q_0\rangle$, the second qubit always attracts more error compared with the expected output $|q_2q_1Uq_0\rangle$, even for the case when $U = I_2$. (See the figure in Appendix C.2.) It is curious to know why such a noise pattern will happen to our scheme.

We may see whether 4-qubit, 5-qubit case also produce some funny noise pattern.

For future research, we plan to extend the techniques to more general quantum channels such as the fully correlated quantum channels on n-qubits with general noise of the form $U^{\otimes n}$, where $U \in M_2$ is unitary, or a non-classical

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Below we present the formulation and emperical verification of the encoding and decoding schemes. All code is available here:

`https://github.com/slyles1001/QECC`

A Matlab results

We write a Matlab program to generate the matrices X_n, Y_n, Z_n, P_n, etc., and demonstrate our quantum error correction scheme described in Section 2.

We briefly describe our program in the following. For an integer $n > 1$, the following commands will generate the encoding matrix P_n:

```
if mod(n,2) == 1
       P = eye(8); P3 = P(:,[1,6,4,7,8,3,5,2]); Pn = P3;
          k = (n-1)/2;
       for j = 2:k
            Pn = kron(eye(4),Pn)*kron(P3,eye(2^(2*j-2)));
   end
else
     H = [1 1; 1 -1]/sqrt(2); C01 = [1 0 0 0; 0 0 0 1; 0 0 1 0; 0 1 0 0];
                    P2 = C01*kron(eye(2),H)*C01;
   if n == 2
              Pn = P2;
   else
       P = eye(8); P3 = P(:,[1,6,4,7,8,3,5,2]); Pn = P3; k = (n-2)/2;
     for j = 2:k
           Pn = kron(eye(4),Pn)*kron(P3,eye(2^(2*j-2)));
     end
           Pn = kron(eye(2),Pn)*kron(P2,eye(2^n/4));
   end
end
```

Then one can test the encoding and decoding schemes. First, set up the error operators X_n, Y_n, Z_n.

```
%%%   Set up the error operators for the channel
     X = [0 1; 1 0];  Y = [0 -i; i 0];   Z = [1 0; 0 -1];
        Xn=X; Yn = Y; Zn = Z;
   for j = 2:n
        Xn = kron(X,Xn); Yn = kron(Y,Yn); Zn = kron(Z,Zn);
   end
```

Suppose $n = 2k + 1$ is odd. The following commands check

$$(P_n^\dagger X_n P_n, P_n^\dagger Y_n P_n, P_n^\dagger Z_n P_n) = (X \otimes I, (-1)^k Y \otimes I, Z \otimes I). \tag{3}$$

The output 0,0,0 will confirm the equality.

```
%%%  Check (Pn'XnPn, Pn'YnPn, Pn'ZnPn)
    II = eye(2^(n-1));  norm(Pn'*Xn*Pn -kron(X,II)),
    norm(Pn'*Yn*Pn - (-1)^k*kron(Y,II)), norm(Pn'*Zn*Pn -kron(Z,II))
```

Next, we verify the error correction scheme for random input $\sigma \otimes \rho$ with $\sigma \in D_2$ and $\rho \in D_4$. The output 0,0,0 will confirm the scheme works.

```
%%% Generate random S in D_2, S in D_{2k}
   S = rand(2,2) + i*rand(2,2) - rand(1,1)*(1+i)*eye(2); S = S*S'; S = S/trace(S);
   K = 2^(n-1);
   R = rand(K,K) + i*rand(K,K) - rand(1,1)*(1+i)*eye(K); R = R*R'; R = R/trace(R);
%  Encode kron(S,R) and compared with the decoded state for each error operator.
   A = kron(S,R); AA = Pn*A*Pn';
   norm( Pn'*Xn*AA*Xn*Pn - kron(X*S*X',R)), norm(Pn'*Yn*AA*Yn*Pn - kron(Y*S*Y',R))
   norm( Pn'*Zn*AA*Zn*Pn - kron(Z*S*Z',R))
```

Suppose n is even. The following commands check

$$(P_n^\dagger X_n P_n, P_n^\dagger Y_n P_n, P_n^\dagger Z_n P_n) = (D_Z \otimes I_{2^{n-2}}, D_Y \otimes I_{2^{n-2}}, D_Z \otimes I_{2^{n-2}}). \tag{4}$$

The output 0,0,0 will confirm the equality.

```
    Dx = diag([1 -1 1 -1]); Dy = diag([-1 -1 1 1]); Dz = diag([1 -1 -1 1]);
    II = eye(2^n/4);  norm(Pn'*Xn*Pn - kron(Dx,II));
     norm(Pn'*Yn*Pn - (-1)^k*kron(Dy,II)), norm(Pn'*Zn*Pn - kron(Dz,II))
```

Then we verify our error correction scheme that for any $\sigma \in \{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$ and $\rho \in D_{2^{n-2}}$, the encoding and decoding yield $\sigma \otimes \rho$. Again, the output 0,0,0 will confirm the scheme works.

```
%%%  Set up the classical bits in D_4, and arbitrary qubits in D_{2k}
   K = 2^(n-2);
   R = rand(K,K) + i*rand(K,K) - rand(1,1)*(1+i)*eye(K); R = R*R'; R = R/trace(R);
   b0 = [1 0; 0 0]; b1 = [0 0; 0 1];
   b00 = kron(b0,b0); b01 = kron(b0,b1); b10 = kron(b1,b0); b11 = kron(b1,b1);
%
   S = b00; A = kron(S,R); AA = Pn*A*Pn';  norm(Pn'*Xn*AA*Xn*Pn - kron(S,R)),
       norm(Pn'*Yn*AA*Yn*Pn - kron(S,R)), norm(Pn'*Zn*AA*Zn*Pn - kron(S,R))
```

```
%
   S = b01; A = kron(S,R); AA = Pn*A*Pn'; norm(Pn'*Xn*AA*Xn*Pn - kron(S,R)),
       norm(Pn'*Yn*AA*Yn*Pn - kron(S,R)), norm(Pn'*Zn*AA*Zn*Pn - kron(S,R))
%
   S = b10; A = kron(S,R); AA = Pn*A*Pn'; norm(Pn'*Xn*AA*Xn*Pn - kron(S,R)),
       norm(Pn'*Yn*AA*Yn*Pn - kron(S,R)), norm(Pn'*Zn*AA*Zn*Pn - kron(S,R))
%
   S = b11; A = kron(S,R); AA = Pn*A*Pn'; norm(Pn'*Xn*AA*Xn*Pn - kron(S,R)),
       norm(Pn'*Yn*AA*Yn*Pn - kron(S,R)), norm(Pn'*Zn*AA*Zn*Pn - kron(S,R))
```

B Mathematica results

We write a Mathematica program to generate the matrices X_n, Y_n, Z_n, P_n, etc., and demonstrate our quantum error correction scheme described in Section 2.

We briefly describe our program in the following. We begin by setting up the CNOT gates, the Hadamard matrix, the Pauli matrices, D_X, D_Y, D_Z and P_2 by the following commands:

```
CNOT[n0_,h0_,k0_]:=Module[{n=n0,h=h0,k=k0},U=IdentityMatrix[2^n];
Do[cindex=IntegerDigits[i-1,2,n];
If[cindex[[n-h]]==1,cindex[[n-k]]=Mod[cindex[[n-k]]+1,2]];
s=Sum[cindex[[r]]*2^(n-r),{r,1,n}]+1;
U[[i]]=Table[KroneckerDelta[s,j],{j,1,2^n}],{i,1,2^n}];U]

H={{1,1},{1,-1}}/Sqrt[2];
x={{0,1},{1,0}};
y={{0,-I},{I,0}};
z={{1,0},{0,-1}};
DX = DiagonalMatrix[{1, -1, 1, -1}];
DY = DiagonalMatrix[{-1, -1, 1, 1}];
DZ = DiagonalMatrix[{1, -1, -1, 1}];
P2=CNOT[2,0,1].KroneckerProduct[IdentityMatrix[2],H].Transpose[CNOT[2,0,1]];
```

Then we define X_n, Y_n and Z_n recursively:

```
X[n_]:=X[n]=KroneckerProduct[X[n-1],x]; X[1]=x;
Y[n_]:=Y[n]=KroneckerProduct[Y[n-1],y]; Y[1]=y;
Z[n_]:=Z[n]=KroneckerProduct[Z[n-1],z]; Z[1]=z;
```

Then we define P_n recursively by setting $P_{2k+1} = Q[k]$ and $P_{2k} = R[k]$.

```
Q[k_]:=Q[k]=KroneckerProduct[IdentityMatrix[4],Q[k-1]].
KroneckerProduct[Q[1],IdentityMatrix[2^(2k-2)]];
```

```
Q[1]=CNOT[3,1,0].CNOT[3,0,2].CNOT[3,2,1];

R[k_]:=KroneckerProduct[IdentityMatrix[2],Q[k-1]].
KroneckerProduct[P2,IdentityMatrix[2^(2k-2)]];
```

Then the validity of the formula

$$(P_{2k+1}^t X_{2k+1} P_{2k+1}, P_{2k+1}^t Y_{2k+1} P_{2k+1}, P_{2k+1}^t Z_{2k+1} P_{2k+1}) = (X_1 \otimes I_{2^{2k}}, (-1)^k Y_1 \otimes I_{2^{2k}}, Z_1 \otimes I_{2^{2k}}).$$

can be checked by calculating $\|P_n^t X_n P_n - X_1 \otimes I_{2^{2k}}\|$, $\|P_n^t Y_n P_n - (-1)^k Y_1 \otimes I_{2^{2k}}\|$, $\|P_n^t Z_n P_n - Z_1 \otimes I_{2^{2k}}\|$ with the corresponding functions:

```
Norm[Transpose[Q[3]].X[7].Q[3] - KroneckerProduct[x,IdentityMatrix[2^6]]]
Norm[Transpose[Q[3]].Y[7].Q[3] - (-1)^3*KroneckerProduct[y,IdentityMatrix[2^6]]]
Norm[Transpose[Q[3]].Z[7].Q[3] - KroneckerProduct[z,IdentityMatrix[2^6]]]
```

Similarly, we can check the formula

$$(P_{2k}^t X_{2k} P_{2k}, P_{2k}^t Y_{2k} P_{2k}, P_{2k}^t Z_{2k} P_{2k}) = (D_X \otimes I_{2^{2k-2}}, (-1)^{k-1} D_Y \otimes I_{2^{2k-2}}, D_Z \otimes I_{2^{2k-2}}$$

by calculating (for $k = 3$) respectively as follows:

```
Norm[Transpose[R[3]].X[6].R[3] - KroneckerProduct[DX,IdentityMatrix[2^4]]]
Norm[Transpose[R[3]].Y[6].R[3] - (-1)^2*KroneckerProduct[DY,IdentityMatrix[2^4]]]
Norm[Transpose[R[3]].Z[6].R[3] - KroneckerProduct[DZ,IdentityMatrix[2^4]]]
```

We can also check that for all $n \geq 3$, $\|P_n^\dagger \mathcal{E}(P_n(\sigma \otimes \rho)P_n^\dagger)P_n - \tilde{\sigma} \otimes \rho\| = 0$.

C Python results

Similar to the other methods, we can verify the results in Python. This is convenient because IBM has provided the `qiskit` framework, and the encoding matrices can be extracted from a built circuit using the unitary backend [9].

In contrast to the matrix operations of sections A and B, the recursive scheme operates on the quantum circuit itself. For convenience, the imports have been omitted, but can be found in the full code.

To initialize a quantum circuit, we first create qubits using `qr = QuantumRegister(n)` and classical bits for measurement with `cr = ClassicalRegister(n)`. The quantum circuit is constructed by the `qc = QuantumCircuit(qr, qc)`. To verify functionality, we can apply arbitrary unitary operations to initialize the state to a given vector. Then, the encoding scheme is defined recursively:

```
def build_circ(qc, q, E):
    n = q[-1][1] + 1 # number of qubits
```

```
def err(e, base=False):
    d = {'X':qc.x, 'Y':qc.y, 'Z':qc.z, 'I':lambda x: None} # apply errors
    d[e](q[n-1])
    if base: # if we're at q3
        d[e](q[0])
        d[e](q[1])

if n % 2 == 0: # qn is even
    qc.cx(q[n-2], q[n-1]) # encode with P_2
    qc.h(q[n-2])
    qc.cx(q[n-2], q[n-1])
    build_circ(qc, q[:-1], E) #recurse
    err(E)
    qc.cx(q[n-2], q[n-1]) # decode with P_2^T
    qc.h(q[n-2])
    qc.cx(q[n-2], q[n-1])
else: # it's odd
    qc.cx(q[n-1], q[n-2]) # encode with P_3
    qc.cx(q[n-3], q[n-1])
    qc.cx(q[n-2], q[n-3])
    if n == 3: # base case
        err(E, True)
    else:
        build_circ(qc, q[:-1], E) #recurse
        err(E)
    qc.cx(q[n-2], q[n-3]) # decode with P_3^T
    qc.cx(q[n-3], q[n-1])
    qc.cx(q[n-1], q[n-2])
return()
```

C.1 Using IBM's simulated quantum computer

We can use the IBM's `qasm` quantum computer simulator to validate our scheme. The results are identical to the above two sections, so we omit them here.

One may consider using $|q_2\rangle = U|0\rangle$ and $|q_1q_0\rangle = V|00\rangle$ for unitary $U \in M_2, V \in M_4$, and apply our scheme. One should get perfect decoding of $|q_1q_0\rangle = V|00\rangle$ with the qasm simulator. One may do simulation and get similar statistics for $n = 4, 5, 6$, etc.

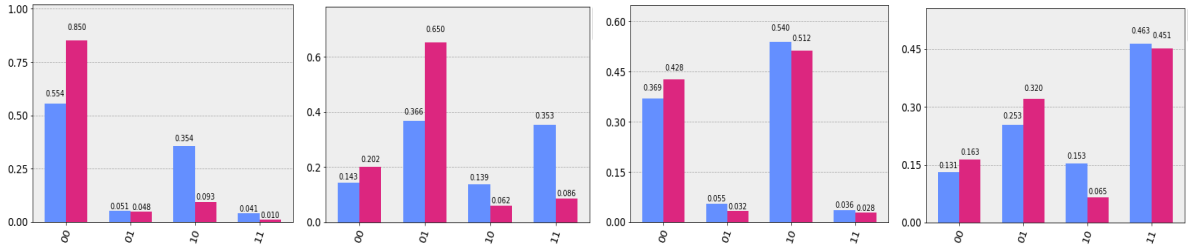

Figure 1: Graphs for Tenerife (pink) vs Yorktown (blue) on inputs rho = 00, 01, 10, 11

C.2 Using IBM's quantum computer

We can use IBM's online quantum computers to implement our scheme. We compare two different machines: Tenerife (`ibmqx4`) and Yorktown (`ibmqx2`) to show the discrepancy of error between the two. Yorktown has more than $9\times$ the gate error of Tenerife, and $2.4\times$ the readout error. This can be observed in the graphs of Figure 1.

It is interesting to observe that for inputs $00, 01$ Tenerife is significantly better at preserving states than Yorktown. However, for the inputs $10, 11$ Tenerife is in fact worse at preserving the states. Also, for both of the computers, most of the error seems to take the form of q_1 flipping. In the case of 01 on Yorktown, the output is evenly split between the correct state and the flipped bit.

We use $\sigma = 0$ but an arbitrary state sees the same results. The errors for $00, 01, 10, 11$ are X, Y, Z, I. Oddly, the run with the lowest accuracy was the run that contained no errors.

For $n = 3$, we input $|000\rangle$ and use the error operators X_3, Y_3, Z_3 in three different experiments using the following circuit (IMB version of the one in Section 2.6). The expected results states should be $|100\rangle, |100\rangle, |000\rangle$, respectively. We get the following statistics of the experiments. The expected measured state always has the highest probability. It is interesting to note that q_1 always attracts more noise.

One may consider using $|q_2\rangle = U|0\rangle$ and $|q_1 q_0\rangle = V|00\rangle$ for unitary $U \in M_2, V \in M_4$, and apply our scheme. One should high probability of decoded state: $|q_1 q_0\rangle = V|00\rangle$.

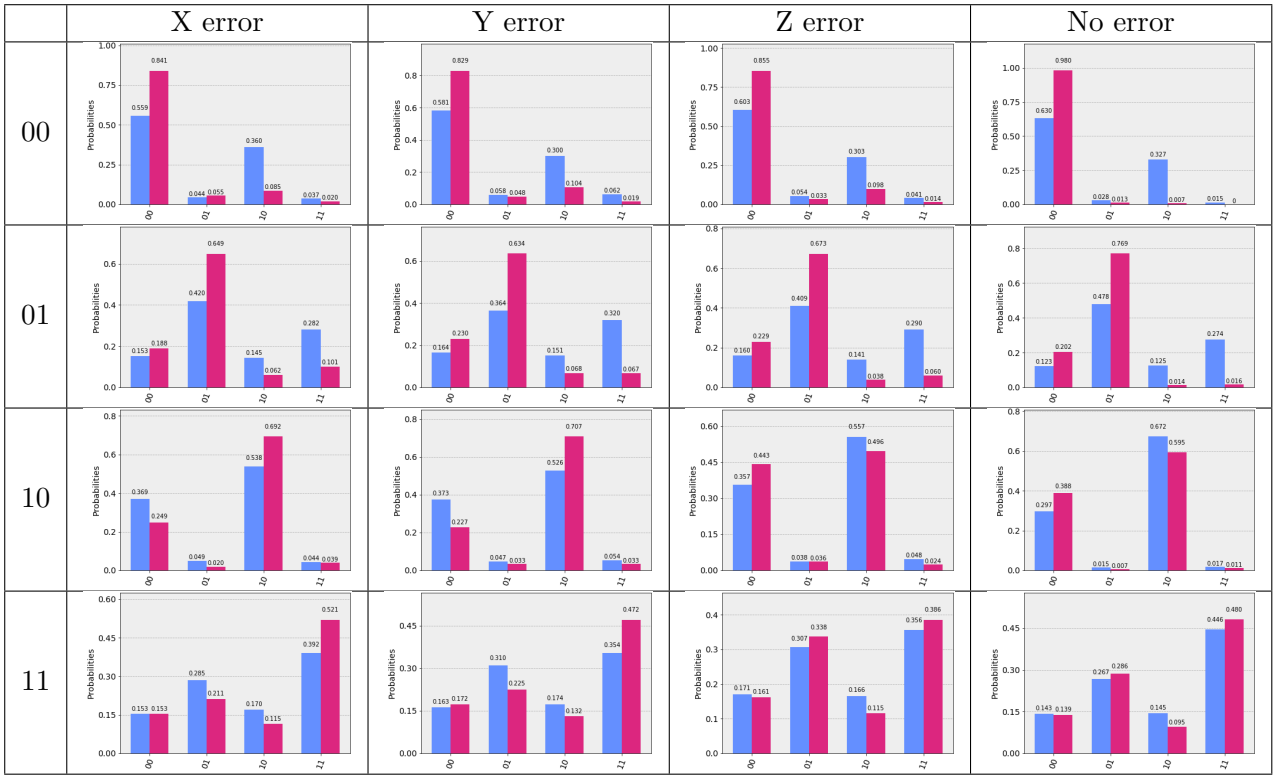

Table 1: Comparing Inputs and Errors, sigma = 0

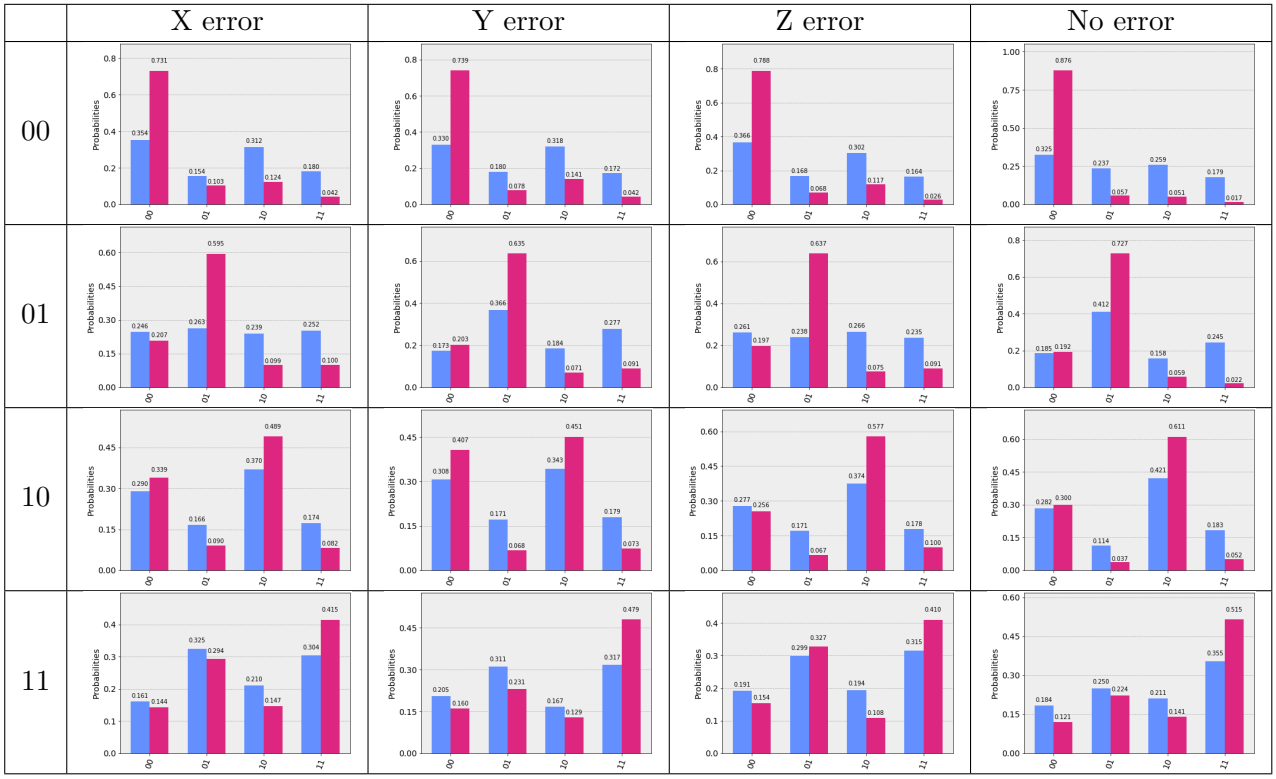

Table 2: Comparing Inputs and Errors, random sigma