Homework #5 — Due 11/27/18

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Problem 1

Part 1

$max\ b'u \implies min\ -b'u$

$A'u \leq p \implies -A'u \geq -p$

$$\begin{array}{lr} min & -b'u \\ subject\ to & -A'u \geq -p \\ & u \geq 0 \end{array}$$

Part 2

$$\begin{array}{lr} max & -p'x \\ subject\ to & -A''x \leq -p \\ & x \geq 0 \end{array} \implies \begin{array}{lr} min & p'x \\ subject\ to & Ax \geq p \\ & x \geq 0 \end{array}$$

Part 3

The dual of the dual is the primal problem.

Problem 2

$$\begin{array}{llllll} min & -12x_1 + 10x_2 + 2x_3 & & & & \\ subject\ to & & -4x_1 + & x_2 - & 8x_3 & \geq 1 \\ & & -x_1 + & x_2 + & 12x_3 & \geq 3 \\ & & & & x_1,\ x_2,\ x_3 & \geq 0 \end{array}$$

This has a dual linear program of the form:

$$\begin{array}{llll} max & u_1 + 3u_2 & & \\ subject\ to & -4u_1 - & u_2 & \leq -12 \\ & u_1 + & u_2 & \leq 10 \\ & -8u_1 + & 12u_2 & \leq 2 \\ & & x_1,\ x_2,\ x_3 & \geq 0 \end{array}$$

Let:

$$A = \begin{bmatrix} -4 & -1 \\ 1 & 1 \\ -8 & 12 \end{bmatrix}, \quad b = \begin{bmatrix} -12 \\ 10 \\ 2 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Since $x_2,\ x_3 \geq 0$:

$$A_2 \cdot u = b_2, \qquad A_3 \cdot u = b_3$$

Solving this system yields u = $\begin{bmatrix} 59/10 \\ 41/10 \end{bmatrix}$.

Checking the (4.7) relationships:

$$Ax \geq b \implies for\ our\ setup:\ A'x \geq p \implies \begin{bmatrix} -4 & 1 & -8 \\ -1 & 1 & 12 \end{bmatrix} \begin{bmatrix} 0 \\ 1.8 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1.8 \\ 0.1 \end{bmatrix} \geq 0, \qquad u = \begin{bmatrix} 59/10 \\ 41/10 \end{bmatrix} \geq 0$$

$$A'u \leq p \implies for\ our\ setup:\ -Au \geq -b \implies -\begin{bmatrix} -4 & -1 \\ 1 & 1 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} 59/10 \\ 41/10 \end{bmatrix} = \begin{bmatrix} 27.7 \\ -10 \\ -2 \end{bmatrix} \geq \begin{bmatrix} 12 \\ -10 \\ -2 \end{bmatrix}$$

Finally checking the (4.8) conditions:

$$u(Ax-b) \implies for\ our\ setup:\ u'(A'x-p) = \begin{bmatrix} \frac{59}{10} & \frac{41}{10} \end{bmatrix} \left(\begin{bmatrix} -4 & 1 & -8 \\ -1 & 1 & 12 \end{bmatrix} \begin{bmatrix} 0 \\ 1.8 \\ 0.1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) = 0$$

$$x(A'u+p) \implies \mathit{for\ our\ setup:}\ x'(-Au+b) = \begin{bmatrix}0 & 1.8 & 0.1\end{bmatrix}(-\begin{bmatrix}-4 & -1\\ 1 & 1\\ -8 & 12\end{bmatrix}\begin{bmatrix}59/10\\ 41/10\end{bmatrix} + \begin{bmatrix}-12\\ 10\\ 2\end{bmatrix}) = 0$$

All of the conditions in equations (4.7) and (4.8) have been satisfied, therefore the solution x = (0, 1.8, 0.1)' solves the linear program.

Problem 3

```
A = [1, 2; 3, 4];
e = [1; 1];

T = totbl(A, e, e);
T = dualbl(T);
T = ljx(T, 2, 2);
T = ljx(T, 1, 2);
```

		$u_3 =$	$u_1 =$	w =
		x_1	x_3	1
$-u_2$	$x_4 =$	1.0000	2.0000	1.0000
$-u_4$	$x_2 =$	-0.5000	0.5000	0.5000
1	z =	0.5000	0.5000	0.5000

The value of the game is $\frac{1}{0.5} = 2$, with optimal strategies $x^* = \begin{bmatrix}0\\1\end{bmatrix}$ and $y^* = \begin{bmatrix}1\\0\end{bmatrix}$.

Problem 4

```
A = [1, 0, 0; 0, 2, 0; 0, 0, 3];
e = [1; 1; 1];

T = totbl(A, e, e);
T = dualbl(T);
T = ljx(T, 1, 1);
T = ljx(T, 2, 2);
T = ljx(T, 3, 3);
```

		$u_1 =$	$u_2 =$	$u_1 =$	w =
		x_4	x_5	x_6	1
$-u_4$	$x_1 =$	1.0000	-0.0000	0.0000	1.0000
$-u_5$	$x_2 =$	0.0000	0.5000	-0.0000	0.5000
$-u_6$	$x_3 =$	-0.0000	-0.0000	0.3333	0.3333
1	z =	1.0000	0.5000	0.3333	1.8333

The value of the game is $\frac{1}{11/6} = \frac{6}{11}$, with optimal strategies $x^* = \begin{bmatrix}6/11\\3/11\\2/11\end{bmatrix}$ and $y^* = \begin{bmatrix}6/11\\3/11\\2/11\end{bmatrix}$.

Problem 5

```
A = [0, 2, -1; -2, 0, 1; 1, -1 0];
e = [1; 1; 1];

T = totbl(A + 5, e, e);
T = dualbl(T);
T = ljx(T, 1, 2);
T = ljx(T, 2, 3);
T = ljx(T, 3, 1);
```

		$u_3 =$	$u_1 =$	$u_2 =$	w =
		x_6	x_4	x_5	1
$-u_5$	$x_2 =$	-0.2250	0.2625	0.0125	0.0500
$-u_6$	$x_3 =$	0.0500	-0.2250	0.2750	0.1000
$-u_4$	$x_1 =$	0.2750	0.0125	-0.2375	0.0500
1	z =	0.1000	0.0500	0.0500	0.2000

The value of the game is $\frac{1}{0.2} - 5 = 0$, with optimal strategies $x^* = \begin{bmatrix}1/4\\1/4\\2/4\end{bmatrix}$ and $y^* = \begin{bmatrix}1/4\\1/4\\2/4\end{bmatrix}$.

Problem 6

$y^{*'}Ax^* = \frac{1}{\theta} \implies \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix} \begin{bmatrix}0 & 1 & -1 & 0\\ 1 & 0 & -1 & -2\end{bmatrix} \begin{bmatrix}1/2\\ 1/2\\ 0\\ 0\end{bmatrix} = 0.5 \implies \theta = 2$, and the value of the game is $\frac{1}{2}$.

Note $x^* = \frac{1}{\theta}\bar{x}$ and $y^* = \frac{1}{\theta}\bar{y} \implies \bar{x} = 2\begin{bmatrix}1/2\\ 1/2\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 0\\ 0\end{bmatrix}$ and $\bar{y} = 2\begin{bmatrix}1/2\\ 1/2\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}$

Checking the KKT conditions:

$$\bar{x}'(-A'\bar{y} + p) = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}(-\begin{bmatrix}0 & 1\\ 1 & 0\\ -1 & -1\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\\ 1\end{bmatrix}) = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}(\begin{bmatrix}0\\ 0\\ 3\\ 3\end{bmatrix}) = 0$$

$$\bar{y}'(A\bar{x} - b) = \begin{bmatrix}1 & 1\end{bmatrix}(\begin{bmatrix}0 & 1 & -1 & 0\\ 1 & 0 & -1 & -2\end{bmatrix}\begin{bmatrix}1\\ 1\\ 0\\ 0\end{bmatrix} - \begin{bmatrix}1\\ 1\end{bmatrix}) = \begin{bmatrix}1 & 1\end{bmatrix}(\begin{bmatrix}0\\ 0\end{bmatrix}) = 0$$

Since the KKT conditions are satisfied, then the optimal strategies are $x^* = \begin{bmatrix}1/2\\ 1/2\\ 0\\ 0\end{bmatrix}$ and $y^* = \begin{bmatrix}1/2\\ 1/2\end{bmatrix}$, and the value of the game is $\frac{1}{2}$.

Problem 7

The reason the prisoner's dilemma is not a zero-sum game is because both players are able to cooperate to minimize their sentences, and both are able to lose (being sentenced to 6 years each). If this were a zero-sum game, the gain of one prisoner would have to match the loss of the other. For example, one would have to be sentenced while the other would have to be set free with no way for both to be set free or sentenced.