Homework 1

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目录

1 凸集

a. 证明若函数 f 二次可微，则 f 为凸函数的充要条件为：

- dom f 为凸集
- $\nabla^2 f(x) \succeq 0$, for all $x \in$dom f

证明. First, consider the case $n = 1$. We show that a twice differentiable function $f : R \to R$ is convex if and only if $f''(x) \geq 0$ for all $x \in$ dom f. Assume first that f is convex and $x, y \in$ dom f. Since dom f is convex, $z = x + t(y - x) \in$ dom f for all $0 < t \leq 1$. Due to the first order convexity condition, we have

$$f(x) \geq f(z) + f'(z)(x - z)$$
$$f(z) \geq f(x) + f'(x)(z - x)$$

The sum of the above two inequalities gives $f(x) + f(z) \geq f(z) + f(x) + (x - z)(f'(z) - f'(x))$, i.e. $(z - x)(f'(z) - f'(x)) \geq 0$. Divide both sides by $(z - x)^2$ and we obtain

$$\frac{f'(z) - f'(x)}{z - x} \geq 0$$

and taking the limit as $t \to 0$ yields $f''(x) \geq 0$.

To show sufficiency, assume that the function satisfies $f''(x) \geq 0$ for all $x \in$ dom f. By the mean value version of Taylor’s theorem we have

$$f(y) = f(x) + f'(x)(y - x) + f''(z)(y - x)^2, \text{ for some } z \text{ between } y \text{ and } x.$$

Since $f''(z) \geq 0$, it turns out that $f(y) \geq f(x) + f'(x)(y - x)$. Thus, f is convex.

Now we can prove the general case, with $f : R^n \to R$. Let $x, y \in R^n$ and consider f restricted to the line passing through them, i.e. the function defined by $g(t) = f(x + t(y - x))$, so $g'(t) = \nabla f(x + t(y - x))^\top (y - x)$ and $g''(t) = (y - x)^\top \nabla^2 f(x + t(y - x))(y - x)$

First assume f is convex, which implies g is convex. Due to the first order convexity condition we have

$$g(t) \geq g(0) + g'(0)t$$
$$g(0) \geq g(t) + g'(t)(-t).$$

Similarly, the sum of the above two inequalities gives that $(g'(t) - g'(0))t \geq 0$. Divide both sides by t^2 and we obtain

$$\frac{g'(t) - g'(0)}{t} \geq 0$$

and taking the limit as $t \to 0$ yields $g''(0) \geq 0$. Notice that $g''(0) = (y - x)^\top \nabla^2 f(x)(y - x) \geq 0$ holds for all $y \in$ dom f, which implies $\nabla^2 f(x)$ is semi positive definite. Thus, $\nabla^2 f(x) \succeq 0$.

Now assume that $\nabla^2 f(x) \succeq 0$ for all $x \in$ dom f. By the mean value version of Taylor's theorem we have

$$g(1) = g(0) + g'(0) + \frac{1}{2} g''(t), \text{ for some } t \text{ between } 0 \text{ and } 1.$$

Because $\nabla^2 f(x) \succeq 0$ for all $x \in$ dom f, $g''(t) = (y - x)^\top \nabla^2 f(x + t(y - x))(y - x) \geq 0$. And then $g(1) \geq g(0) + g'(0)$, which is equivalent to $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$. Hence, f is convex. □

b. 闭集与凸集

b.1 证明多面体 $\{x : Ax \leq b\}, A \in \mathbb{R}^{mn}, b \in \mathbb{R}^m$ 为闭的凸集

证明. First, show that the polyhedron is closed. Assume a sequence of points $x_n \in \{x : Ax \leq b\}$. If this sequence has a limit point x, it is easy to see that $Ax = \lim_{n\to\infty} Ax_n \leq b$ implying x is also in the polyhedron. Therefore, the polyhedron $\{x : Ax \leq b\}$ is closed.

Next, show the convexity of polyhedron. If $x, y \in \{x : Ax \leq b\}$, $Ax \leq b$ and $Ay \leq b$. Denote $z = \theta x + (1-\theta)y$ where $\theta \in [0, 1]$.

$$Az = \theta Ax + (1-\theta)Ay \leq \theta b + (1-\theta)b = b.$$

Thus, $z \in \{x : Ax \leq b\}$ and $\{x : Ax \leq b\}$ is a convex set.

In conclusion, $\{x : Ax \leq b\}$ is a closed convex set. □

b.2 举例: $\mathbb{R}^2$ 上的闭集的凸包不一定是闭的

Example: The convex hull of closed set

$$\left\{(x, y) : y \geq \frac{1}{1+x^2}\right\}$$

is an open set $\{(x, y) : y > 0\}$.

b.3 若 $A \in \mathbb{R}^{m\times n}, S \in \mathbb{R}^m$ 为凸集, 证明集合 S 在 A 下的原像 $\{x \in \mathbb{R}^n : Ax \in S\}$ 是凸集

证明. Assume $x,\ y \in \{x \in \mathbb{R}^n : Ax \in S\}$. Let $z = \theta x + (1-\theta)y$ where $\theta \in [0, 1]$. Since S is convex and Ax, Ay are both in S, $Az = \theta Ax + (1-\theta)Ay \in S$. So $z \in \{x : Ax \in S\}$ and $\{x \in \mathbb{R}^n : Ax \in S\}$ is convex. □

b.4 若 $A \in \mathbb{R}^{m\times n}, S \in \mathbb{R}^n$ 为凸集, 证明集合 S 在 A 下的像 $\{Ax : x \in S\}$ 是凸集

证明. Assume $x,\ y \in \{Ax : x \in S\}$. There exist $x_0, y_0 \in S$ such that $x = Ax_0, y = Ay_0$. Write $z = \theta x + (1-\theta)y, \theta \in [0, 1]$. We can see that $z = \theta Ax_0 + (1-\theta)Ay_0 = A(\theta x_0 + (1-\theta)y_0)$. Since S is convex, $\theta x_0 + (1-\theta)y_0 \in S$. As a result, $z \in \{Ax : x \in S\}$ and $\{Ax : x \in S\}$ is convex. □

b.5 举例: 存在 $A \in \mathbb{R}^{m\times n}$ 及闭凸集 $S \subseteq \mathbb{R}^n$, 使得 $A(S)$ 不是闭集

Example: Let $S = \left\{(x, y)^\top : xy \geq 1, x > 0\right\}$ and $A = (1, 0)$. Obviously, S is a closed convex set. But $A(S) = \{x : x > 0\}$ is not closed.

c. 多面体

c.1 证明若 $P \subseteq \mathbb{R}^n$ 为多面体, 则 $A(P)$ 为多面体, 提示: 可使用以下事实:

$$P \subseteq \mathbb{R}^{m+n} \text{ 为多面体} \Rightarrow \{x \in \mathbb{R}^n : (x, y) \in P \text{ for some } y \in \mathbb{R}^m\} \text{ 是多面体}$$

证明. Write $P = \{x \in \mathbb{R}^n : Bx \leq b\}$, where $B \in \mathbb{R}^{k\times n}$. Set $Q = \{(x, y) \in \mathbb{R}^{m+n} : Bx \leq b, y = Ax\}$. Q can be expressed as $\{z \in \mathbb{R}^{m+n} : Cz \leq c\}$, where

$$C = \begin{pmatrix} B & 0 \\ A & -1_m \\ -A & 1_m \end{pmatrix}$$

and $c = (b, 0_m, 0_m)$. So Q is also a polyhedron. Use the hint and we can obtain $A(P) = \{y \in \mathbb{R}^m : (x, y) \in Q \text{ for some } x \in \mathbb{R}^n\}$ is a polyhedron. □

c.2 证明若 $Q \subseteq \mathbb{R}^m$ 为多面体, $A \in \mathbb{R}^{m\times n}$, 则 $A^{-1}(Q)$ 为多面体.

证明. Write $Q = \{x \in \mathbb{R}^m : Bx \le b\}$, where $B \in \mathbb{R}^{k\times m}$. Set $P = \{(x, y) \in \mathbb{R}^{m+n} : Bx \le b, Ay = x\}$. P can be expressed as $\{z \in \mathbb{R}^{m+n} : Cz \le c\}$, where

$$C = \begin{pmatrix} B & 0 \\ -1_m & A \\ 1_m & -A \end{pmatrix}$$

and $c = (b, 0_m, 0_m)$. So P is also a polyhedron. Use the hint and we can obtain $A^{-1}(Q) = \{y \in \mathbb{R}^m : (x, y) \in P \text{ for some } x \in \mathbb{R}^m\}$ is a polyhedron. □

2 凸函数

a. 证明熵函数:

$$f(x) = -\sum_{i=1}^{n} x_i \log(x_i),\ \operatorname{dom} f = \left\{x \in \mathbb{R}^n_{++} : \sum_{i=1}^{n} x_i = 1\right\}$$

是严格凹的.

证明. Clearly, dom f is convex. The first order derivative of $f(x)$ is $\nabla_i f(x) = -\log x_i - 1$ and the second order derivative is

$$\nabla^2_{ij} f(x) = -\frac{1}{x_i} I(i = j).$$

So the Hessian matrix of $f(x)$ is diagonal and each diagonal element is negative. Thus, $\nabla^2 f(x)$ is negative definite and the entropy function $f(x)$ is strictly concave. □

b. 若 f 为二次可微函数且 dom f 为凸集, 证明 f 为凸函数的充要条件为:

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \ge 0, \forall x, y$$

这被称为梯度 ∇f 的单调性.

证明. Set $g(t) = f(x+t(y-x))$, so $g'(t) = \nabla f(x+t(y-x))^\top (y-x)$ and $g''(t) = (y-x)^\top \nabla^2 f(x+t(y-x))(y-x)$. First, show necessity. Assume f is convex, which implies g is convex. Due to the second-order convexity condition, $g''(t) \ge 0$ for all $t \in [0, 1]$. So the first order derivative $g'(t)$ is non-decreasing in its domain and we have $g'(1) \ge g'(0)$, i.e.

$$\begin{aligned} &\nabla f(y)^\top (y - x) \ge \nabla f(x)^\top (y - x) \\ \Rightarrow &(\nabla f(x) - \nabla f(y))^\top (x - y) \ge 0. \end{aligned}$$

Next, show sufficiency. Assume

$$(\nabla f(x) - \nabla f(y))^\top (x - y) \ge 0, \forall x, y.$$

Since f is twice differentiable, g is also twice differentiable. By the assumption, we have

$$(\nabla f(x+t(y-x))-\nabla f(x))^\top (t(y-x)) \geq 0.$$

Notice that

$$t(g'(t)-g'(0)) = (\nabla f(x+t(y-x))-\nabla f(x))^\top (t(y-x)).$$

So $t(g'(t)-g'(0)) \geq 0$ for $\forall\ t \in [0,1]$. Divide both sides by t^2 and we obtain

$$\frac{g'(t)-g'(0)}{t} \geq 0$$

and taking the limit as $t \to 0$ yields $g''(0) \geq 0$. $g''(0) = (y-x)^\top \nabla^2 f(x)(y-x) \geq 0$ for $\forall\ x, y \in \mathrm{dom}\ f$ implying that $\nabla^2 f(x) \succeq 0$ for $\forall\ x \in \mathrm{dom}\ f$. Hence, f is convex. □

c. 举例: 严格凸函数并不一定能达到其最小值.

Example: $f(x) = 1/x$, $\mathrm{dom}\ f = \{x \in \mathbb{R} : x > 0\}$ is strictly convex. But the minimal value is not attainable.

d. 函数 $f : \mathbb{R}^n \to \mathbb{R}$ 被称为强制的 (coercive), 如果当 $\|x\|_2 \to \infty$ 时, 有 $f \to \infty$. 强制函数的一个关键事实为其可以达到极小值. 证明一个二次可微的强凸函数是强制的 (coercive), 并因此可达到极小值.

证明. Assume f is m-strongly convex, i.e. $g(x) = f(x) - m/2\|x\|_2^2$ is convex for some $m > 0$. By the first-order convexity condition,

$$\begin{aligned} g(y) \geq& g(x) + \nabla g(x)^\top (y-x) \\ f(y) - \frac{m}{2}\|y\|_2^2 \geq& f(x) - \frac{m}{2}\|x\|_2^2 + (\nabla f(x) - mx)^\top (y-x). \end{aligned}$$

Let $\alpha = \nabla f(x) - mx$ and $K_x = f(x) - m/2\|x\|_2^2 - \alpha^\top x$. Then

$$f(y) \geq \frac{m}{2}\|y\|_2^2 + \alpha^\top y + K_x.$$

By Cauchy-Schwartz inequality, $\alpha^\top y \geq -\|\alpha\|_2 \|y\|_2$. Hence,

$$f(y) \geq \frac{m}{2}\|y\|_2^2 - \|\alpha\|_2\|y\|_2 + K_x.$$

The right hand side goes to infinity when $\|y\|_2 \to \infty$. So f is coercive. □

e. 证明在有界多面体上的凸函数的最大值一定在其中一个顶点上. 提示: 已知一个有界的多面体可以被表示为其顶点的凸组合.

证明. Denote $a_i,\ i = 1, \ldots, n$ as vertices of the bounded polyhedron P and

$$P = \left\{ x \in \mathbb{R}^m : x = \sum_{i=1}^n \lambda_i a_i, \lambda_i \geq 0 \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}.$$

Let f be the convex function defined in P. Assume that x_0 is the point at which f attains its maximal value. There exist $\lambda_{i0} \geq 0$ such that $\sum_{i=1}^n \lambda_{i0} = 1$ and $x_0 = \sum_{i=1}^n \lambda_{i0} a_i$. Since f is convex,

$$f(x_0) = f(\sum_{i=1}^n \lambda_{i0} a_i) \leq \sum_{i=1}^n \lambda_{i0} f(a_i) \leq \sum_{i=1}^n \lambda_{i0} f(x_0) = f(x_0),$$

hence the equality holds here. Because $f(a_i) \leq f(x_0)$ for all $1 \leq i \leq n$, $f(a_i) = f(x_0)$ for those i with $\lambda_i \neq 0$ otherwise the equality doesn't hold. Since there is at least one such i, the claim follows. □

3 带 l_2 惩罚的部分优化问题

考虑问题

$$\min_{\beta,\sigma\geq 0} f(\beta)+\frac{\lambda}{2}\sum_{i=1}^{n} g(\beta_i,\sigma_i), \tag{1}$$

其中 f 为定义在 $\mathbb{R}^n$ 上的凸函数, $\lambda \geq 0$, 且

$$g(x,y)=\begin{cases} x^2/y+y & \text{if } y>0; \\ 0 & \text{if } x=0, y=0; \\ \infty & \text{else.} \end{cases}$$

a. 证明 g 是凸函数, 即上述问题为凸优化问题. (后面我们可根据此进行部分优化, 且部分优化后的函数也是凸函数)

证明. First, the domain of g is convex. When $y>0$, $g(x,y)=P(x^2,(0,1)(x,y)^\top)+(0,1)(x,y)^\top$, where P is the perspective transformation. The affine mapping $(0,1)(x,y)^\top$ and x^2 are both convex, so $P(x^2,(0,1)(x,y)^\top)$ is also convex. And the sum of two convex function is still convex implying $g(x,y)$ is convex, i.e.

$$\min_{y\geq 0} g(x,y)=2|x|.$$

□

b. 证明:

$$\min_{y\geq 0} g(x,y)=2|x|.$$

证明. Given x, the first order partial derivative of $g(x,y)$ is

$$\frac{\partial}{\partial y} g(x,y)=1-\frac{x^2}{y^2}.$$

Let $\partial g(x,y)/\partial y=0$ and we have $y=|x|$ since $y\geq 0$. In question a., we've known $g(x,y)$ is convex, so it attains its minimum at $y=|x|$ when x is fixed. □

c. 证明 (1) 中对于 $\sigma \geq 0$ 的优化可得 ℓ_1 惩罚问题

$$\min_{\beta} f(\beta)+\lambda\|\beta\|_1.$$

证明. We can optimize over σ first and then minimize the question over β. The result in question b. gives that $g(\beta_i,\sigma_i)$ attains the minimum $2|\beta_i|$ when $\sigma_i=|\beta_i|$. Thus,

$$\min_{\beta,\sigma\geq 0} f(\beta)+\frac{\lambda}{2}\sum_{i=1}^{n} g(\beta_i,\sigma_i),$$

is equivalent to

$$\min_{\beta} f(\beta)+\frac{\lambda}{2}\sum_{i=1}^{n}|\beta_i| = \min_{\beta} f(\beta)+\lambda\|\beta\|_1.$$

□

4 Lipschitz 梯度与强凸性

令 f 为二次连续可微的凸函数

a. 证明以下命题等价:

i. ∇f 为 L-Lipschitz 函数

ii. 对任意 x, y, $(\nabla f(x) - \nabla f(y))^\top (x-y) \le L\|x-y\|_2^2$

iii. 对任意 x, $\nabla^2 f(x) \preceq LI$

iv. 对任意 x, y, $f(y) \le f(x) + \nabla f(x)^\top (y-x) + \frac{L}{2}\|y-x\|_2^2$

循环证明 $i \Rightarrow ii,\ ii \Rightarrow iii,\ iii \Rightarrow iv,\ iv \Rightarrow ii,\ iii \Rightarrow i.$

$(i \Rightarrow ii)$. Since ∇f is L-Lipschitz function,

$$\|\nabla f(x) - \nabla f(y)\|_2 \le L\|x-y\|_2 \text{ for all } x, y.$$

Multiply both sides by $\|x-y\|_2$ and we obtain

$$\|\nabla f(x) - \nabla f(y)\|_2 \|x-y\|_2 \le L\|x-y\|_2^2.$$

Cauchy-Schwarz inequality guarantees

$$\|\nabla f(x) - \nabla f(y)\|_2 \|x-y\|_2 \ge (\nabla f(x) - \nabla f(y))^\top (x-y).$$

Thus, ii. holds.

$(ii \Rightarrow iii)$. Let $x = y + t\alpha$, where $t \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$, $\|\alpha\|_2 = 1$. Apply Taylor's expansion and we have

$$\nabla f(x) = \nabla f(y) + \nabla^2 f(y)(x-y) + o(\|x-y\|).$$

According to result in (ii),

$$\begin{aligned} &(\nabla f(x) - \nabla f(y))^\top (x-y) = (\nabla^2 f(y)(x-y))^\top (x-y) + o(\|x-y\|_2^2) \le L\|x-y\|_2^2 \\ \Rightarrow\ & t^2 \alpha^\top \nabla^2 f(y) \alpha + o(t^2\|\alpha\|_2^2) \le L(t^2\|\alpha\|_1^2) \\ \Rightarrow\ & \alpha^\top \nabla^2 f(y)\alpha + o(1) \le L \qquad \text{(since } \|\alpha\|_2 = 1\text{)}. \end{aligned}$$

Let α be the eigenvector of the maximal eigenvalue of $\nabla^2 f(y)$ and taking $t \to 0$ gives $\lambda_{\max}\nabla^2 f(y) \le L$. Hence, $\nabla^2 f(x) \preceq LI$.

$(iii \Rightarrow iv)$. By mean value version of Taylor's theorem,

$$f(y) = f(x) + \nabla f(x)^\top (y-x) + \frac{1}{2}(y-x)^\top \nabla^2 f(\xi)(y-x), \text{ where } \xi \text{ between } x \text{ and } y.$$

Since $\nabla^2 f(\xi) \succeq LI$ confirmed by (iii),

$$f(y) \le f(x) + \nabla f(x)^\top (y-x) + \frac{L}{2}\|y-x\|_2^2.$$

$(iv \Rightarrow ii)$. From (iv), we have

$$f(y) \le f(x) + \nabla f(x)^\top (y-x) + \frac{L}{2}\|y-x\|_2^2,$$
$$f(x) \le f(y) + \nabla f(y)^\top (x-y) + \frac{L}{2}\|x-y\|_2^2.$$

The sum of the two inequalities gives

$$\begin{aligned} &f(y) + f(x) \le f(x) + f(y) + (\nabla f(y) - \nabla f(x))^\top (x-y) + L\|x-y\|_2^2 \\ \Rightarrow &(\nabla f(x) - \nabla f(y))^\top (x-y) \le L\|x-y\|_2^2. \end{aligned}$$

$(iii \Rightarrow i)$. Apply the mean value version of Taylor's theorem and then

$$\nabla f(x) - \nabla f(y) = \nabla^2 f(\xi)(x-y), \quad \text{where } \xi \text{ between } x \text{ and } y.$$

Taking the norm of both sides gives

$$\|\nabla f(x) - \nabla f(y)\|_2 = \|\nabla^2 f(\xi)(x-y)\|_2.$$

Since $\nabla^2 f(x) \preceq LI$ for $\forall x$, $\nabla^2 f(\xi) \preceq LI$. By Cauchy-Schwarz inequality, we have

$$\|\nabla f(x) - \nabla f(y)\|_2 = \|\nabla^2 f(\xi)(x-y)\|_2 \le \|\nabla^2 f(\xi)\|_2 \|x-y\|_2 \le L\|x-y\|_2.$$

b. 证明以下命题等价:

i. f 为 m-强凸函数
ii. 对任意 x, y, $(\nabla f(x) - \nabla f(y))^\top (x-y) \ge m\|x-y\|_2^2$
iii. 对任意 x, $\nabla^2 f(x) \succeq mI$
iv. 对任意 x, y, $f(y) \ge f(x) + \nabla f(x)^\top (y-x) + \frac{m}{2}\|y-x\|_2^2$

循环证明 $i \Rightarrow ii,\ ii \Rightarrow iii,\ iii \Rightarrow iv,\ iv \Rightarrow i$.

$(i \Rightarrow ii)$. Since f is m-strongly convex, $g(x) = f(x) - m/2\|x\|_2^2$ is convex for some $m > 0$. $\nabla g(x) = \nabla f(x) - mx$. It follows from the monotone gradient condition for convexity of $g(x)$, i.e.

$$\begin{aligned} &(\nabla g(x) - \nabla g(y))^\top (x-y) \ge 0 \\ \Rightarrow\ &(\nabla f(x) - \nabla f(y))^\top (x-y) \ge m\|x-y\|_2^2. \end{aligned}$$

$(ii \Rightarrow iii)$. Let $x = t\alpha + y$, where $t \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$, $\|\alpha\| = 1$. Use Taylor's theorem and we have

$$\nabla f(x) - \nabla f(y) = \nabla^2 f(y)(x-y) + o(\|x-y\|).$$

Then multiplying both sides by $x - y$ gives

$$\begin{aligned} &(\nabla f(x) - \nabla f(y))^\top (x-y) = (x-y)^\top \nabla^2 f(y)(x-y) + o(\|x-y\|_2^2) \ge m\|x-y\|_2^2 \\ \Rightarrow\ &t^2 \alpha^\top \nabla^2 f(y)\alpha + o(t^2\|\alpha\|_2^2) \ge mt^2\|\alpha\|_2^2 \\ \Rightarrow\ &\alpha^\top \nabla^2 f(y)\alpha + o(1) \ge m \end{aligned}$$

Let α be the eigenvector of the minimal eigenvalue of $\nabla^2 f(y)$ and taking $t \to 0$ gives $\lambda_{\min}\nabla^2 f(y) \ge m$. Hence, $\nabla^2 f(x) \succeq mI$.

$(iii \Rightarrow iv)$. Let $g(x) = f(x) - \frac{m}{2}\|x\|_2^2$. Since $\nabla^2 g(x) = \nabla^2 f(x) - mI \succeq 0$, $g(x)$ is convex. The convexity of $g(x)$ gives

$$\begin{aligned} & g(y) \geq g(x) + \nabla g(x)^\top (y - x) \\ \Leftrightarrow\ & f(y) - \frac{m}{2}\|y\|_2^2 \geq f(x) - \frac{m}{2}\|x\|_2^2 + (\nabla f(x) - mx)^\top (y - x) \\ \Leftrightarrow\ & f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{m}{2}\|x - y\|_2^2. \end{aligned}$$

$(iv \Rightarrow i)$. Let $g(x) = f(x) - \frac{m}{2}\|x\|_2^2$. $f(y) \geq f(x) + \nabla f(x)^\top (y - x) + \frac{m}{2}\|x - y\|_2^2$ implies that $g(y) \geq g(x) + \nabla g(x)^\top (y - x)$, i.e. $g(x)$ is convex. Thus, $f(x)$ is m-strongly convex.

5 实践：使用 CVXPY 解优化问题

见 “CVXPY 实践结果.pdf” 文件