CM-AMI 推导

1. 2-D constellation case

Suppose AWGN with $N_o = 1$ for derivation simplicity.

$$
\begin{aligned}
C &= I(X;Y) \\
&= \sum_{x \in X} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x,y) \log \frac{p(y \mid x)}{p(y)} dy_I dy_Q \\
&= \sum_{x \in X} p(x) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(y \mid x) \log \frac{p(y \mid x)}{\sum_{x' \in X} p(y \mid x') p(x')} dy_I dy_Q \\
&= \sum_{x \in X} p(x) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\pi} e^{-|y-x|^2} \log \frac{e^{-|y-x|^2}}{\sum_{x' \in X} e^{-|y-x'|^2} p(x')} dy_I dy_Q \\
&= -\frac{1}{\pi} \sum_{x \in X} p(x) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-|y-x|^2} \log \frac{\sum_{x' \in X} e^{-|y-x'|^2} p(x')}{e^{-|y-x|^2}} dy_I dy_Q \\
&= -\frac{1}{\pi} \sum_{x \in X} p(x) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t_I^2} e^{-t_Q^2} \log \sum_{x' \in X} e^{|t|^2 - |x-x'+t|^2} p(x') dt_I dt_Q \\
&= -\frac{1}{\pi} \sum_{x \in X} p(x) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t_I^2} e^{-t_Q^2} \log \sum_{x' \in X} e^{|t_I|^2 + |t_Q|^2 - |x-x'+t_I+i \cdot t_Q|^2} p(x') dt_I dt_Q
\end{aligned} \tag{1-1}
$$

When the constellation symbols are taken with equal probability, the above formula reduces to

$$
\begin{aligned}
C &= I(X;Y) \\
&= m - \frac{1}{M\pi} \sum_{x \in X} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t_I^2} e^{-t_Q^2} \log \sum_{x' \in X} e^{|t_I|^2 + |t_Q|^2 - |x-x'+t_I+i \cdot t_Q|^2} dt_I dt_Q
\end{aligned} \tag{1-2}
$$

where $m = \log_2 M$ represents the number of coded bits per constellation symbol.

Notice the calculation of average symbol energy when the constellation symbols are not taken with equal probability. $\bar{E}_s = \sum_{s \in \chi} p(s) |s|^2$

———

2. 1-D constellation case

Suppose AWGN with $N_o = 1$ for derivation simplicity.

$$
\begin{aligned}
C &= I(X;Y) \\
&= \sum_{x\in X} \int_{-\infty}^{+\infty} p(x,y) \log \frac{p(y \mid x)}{p(y)} dy \\
&= \sum_{x\in X} p(x) \int_{-\infty}^{+\infty} p(y \mid x) \log \frac{p(y \mid x)}{\sum_{x'\in X} p(y \mid x') p(x')} dy \\
&= \sum_{x\in X} p(x) \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-|y-x|^2} \log \frac{e^{-|y-x|^2}}{\sum_{x'\in X} e^{-|y-x'|^2} p(x')} dy \\
&= -\frac{1}{\sqrt{\pi}} \sum_{x\in X} p(x) \int_{-\infty}^{+\infty} e^{-|y-x|^2} \log \frac{\sum_{x'\in X} e^{-|y-x'|^2} p(x')}{e^{-|y-x|^2}} dy \\
&= -\frac{1}{\sqrt{\pi}} \sum_{x\in X} p(x) \int_{-\infty}^{+\infty} e^{-|t|^2} \log \sum_{x'\in X} e^{|t|^2-|x-x'+t|^2} p(x') dt \\
&= -\frac{1}{\sqrt{\pi}} \sum_{x\in X} p(x) \int_{-\infty}^{+\infty} e^{-t^2} \log \sum_{x'\in X} e^{t^2-(x-x'+t)^2} p(x') dt
\end{aligned} \tag{2-1}
$$

When the constellation symbols are taken with equal probability, the above formula reduces to

$$
\begin{aligned}
C &= I(X;Y) \\
&= m - \frac{1}{M\sqrt{\pi}} \sum_{x\in X} \int_{-\infty}^{+\infty} e^{-t^2} \log \sum_{x'\in X} e^{t^2-(x-x'+t)^2} dt
\end{aligned} \tag{2-2}
$$

where $m = \log_2 M$ represents the number of coded bits per constellation symbol.

Notice the calculation of average symbol energy when the constellation symbols are not taken with equal probability. $\bar{E}_s = \sum_{s\in\chi} p(s)|s|^2$

——

3. 1-D case is a special 2-D case

$$
\begin{aligned}
C &= I(X;Y) \\
&= -\frac{1}{\pi} \sum_{x\in X} p(x) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-t_I^2} e^{-t_Q^2} \log \sum_{x'\in X} e^{|t_I|^2+|t_Q|^2-|x-x'+t_I+i\cdot t_Q|^2} p(x') dt_I dt_Q
\end{aligned} \tag{3-1}
$$

When the constellation is 1-D, i.e. x and x' is real, equation (3-1) can be rearranged.

$$
\begin{aligned}
C &= I(X;Y) \\
&= -\frac{1}{\pi}\sum_{x\in X} p(x)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-t_I^2}e^{-t_Q^2}\log\sum_{x'\in X} e^{|t_I|^2+|t_Q|^2-|x-x'+t_I|^2+|i\cdot t_Q|^2}p(x')dt_I dt_Q \\
&= -\frac{1}{\pi}\sum_{x\in X} p(x)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-t_I^2}e^{-t_Q^2}\log\sum_{x'\in X} e^{|t_I|^2-|x-x'+t_I|^2}p(x')dt_I dt_Q \\
&= -\frac{1}{\sqrt{\pi}}\sum_{x\in X} p(x)\int_{-\infty}^{+\infty} e^{-t_I^2}\log\sum_{x'\in X} e^{|t_I|^2-|x-x'+t_I|^2}p(x')dt_I \int_{-\infty}^{+\infty}\frac{1}{\sqrt{\pi}}e^{-t_Q^2}dt_Q \\
&= -\frac{1}{\sqrt{\pi}}\sum_{x\in X} p(x)\int_{-\infty}^{+\infty} e^{-t_I^2}\log\sum_{x'\in X} e^{|t_I|^2-|x-x'+t_I|^2}p(x')dt_I
\end{aligned}
\tag{3-2}
$$

Equation (3-2) is just an equivalent form of equation (2-1).

4. Relationship between several form of signal to noise ratio

$$
SNR = \frac{\sigma_s^2}{\sigma_n^2} = \begin{cases} \frac{E_s}{N_o}\cdot 2 & \text{1-D constellation} \\ \\ \frac{E_s}{N_o} & \text{2-D constellation} \end{cases}
\tag{4-1}
$$

$$
\frac{E_s}{N_o} = \frac{E_b * \rho}{N_o}
\tag{4-2}
$$

where ρ is the data rate in bits per symbol (or channel use) and

$$
\rho = m\cdot R
\tag{4-3}
$$

where m is the number of coded bits per constellation symbol and R is the code rate of FEC.