Section 1.1.2

1 graph g of order n with maximal number of edges if the complete graph k_n, which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

2 To show a contradiction, suppose otherwise. We know there are an even number of odd degree vertices, implying that there can be no odd degree vertices. But the max degree of a graph is n-1 (connected to every other vertex), so the degree is between 0 and n-1. Excluding the odd degrees, there are not enough unique numbers to cover all the vertices.

3 For later.

4 If no such path exists, the two odd vertics are on separate connected components A and B. Consider A by itself, it is a connected graph, but it has an odd number of vertices with an odd degree, a contradiction .

5 a)

b)

6 We prove this by induction on the length of the odd closed walk.

Base Case: The length 3 odd closed walk is just a length 3 cycle.

Suppose odd closed walks with lengths up to 2n - 1 contain odd cycles. Let $W = v_1, v_2, ..., v_{2n+1} = v_1$ be a length $2n+1$ closed walk. If no vertices in the walk repeat, then we are done, the odd walk is an odd cycle. Otherwise, let l be the smallest number not 1 such that v_l repeats, and let $v_l = v_k$ where $l < k$. Then we have two closed walks in W, v_1, ..., $v_l = v_k$, ..., $v_{2n+1} = v_1$ and v_l, ..., $v_k = v_l$. The lengths of these two walks must add up to $2n+1$, thus one of them must be an odd length closed walk, which by the inductive hypothesis must contain an odd length cycle.

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9 My guess is that it is a complete graph K_n with every edge to a single vertex missing, which has $\binom{n-1}{2} = \frac{(n-2)(n-1)}{2}$

10 (Not sure whether this proof is legitimate/rigorous).

Using induction we prove the the minimum number of edges needed to have a connected graph is $n-1$ (a tree).

Base Case: K_1 has 0 edges (certainly the minimum number), and is connected.

Suppose G of order n requires minimum $n-1$ edges to be connected. Consider such a connected graph with $n-1$ edges, and consider $G+v$, G with an additional vertex. If we add no edges,

11 $\Rightarrow$ Suppose $e = uv$ is a bridge of G and consider $G-e$. Since e is a bridge, $G-e$ is disconnected, i.e. no uv path exists in $G-e$, implying that e is not part of any cycle in G.

$\Leftarrow$ Suppose $e = uv$ is not a bridge. Then $G-e$ is still connected, i.e. there exists a uv path. This path $+e$ is a cycle.

12 a)

b)

c)

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14 Suppose G has no cycles and is connected. Consider $v \in V$ such that $deg(v) > 1$ and $x, y \in N(v)$. Graph $G - \{v\}$ is disconnected since no xy path exists since if there was, this path in addition to path x, v, y would form a cycle in G. If $order = 2$, the graph does not have a cycle.

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