

Section 1.1.2

- 1 Graph G of order n with maximal number of edges is the complete graph K_n , which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.
- 2 To show a contradiction, suppose otherwise. We know there are an even number of odd degree vertices, implying that there can be no odd degree vertices. But the max degree of a graph is $n-1$ (connected to every other vertex), so the degree is between 0 and $n-1$. Excluding the odd degrees, there are not enough unique numbers to cover all the vertices.
- 3 For later.
- 4 If no such path exists, the two odd vertices are on separate connected components A and B. Consider A by itself, it is a connected graph, but it has an odd number of vertices with an odd degree, a contradiction .
- 5 a)
b)
- 6 We prove this by induction on the length of the odd closed walk.
Base Case: The length 3 odd closed walk is just a length 3 cycle.
Suppose odd closed walks with lengths up to $2n - 1$ contain odd cycles.
Let $W = v_1, v_2, \dots, v_{2n+1} = v_1$ be a length $2n + 1$ closed walk. If no vertices in the walk repeat, then we are done, the odd walk is an odd cycle. Otherwise, let l be the smallest number not 1 such that v_l repeats, and let $v_l = v_k$ where $l < k$. Then we have two closed walks in W , $v_1, \dots, v_l = v_k, \dots, v_{2n+1} = v_1$ and $v_l, \dots, v_k = v_l$. The lengths of these two walks must add up to $2n + 1$, thus one of them must be an odd length closed walk, which by the inductive hypothesis must contain an odd length cycle.
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- 11 \rightarrow Suppose $e = uv$ is a bridge of G and consider $G - e$. Since e is a bridge, $G - e$ is disconnected, i.e. no uv path exists in $G - e$, implying that e is not part of any cycle in G .

← Suppose $e = uv$ is not a bridge. Then $G - e$ is still connected, i.e. there exists a uv path. This path $+e$ is a cycle.

12 a)

b)

c)

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14 Suppose G has no cycles and is connected. Consider $v \in V$ such that $\deg(v) > 1$ and $x, y \in N(v)$. Graph $G - \{v\}$ is disconnected since no xy path exists since if there was, this path in addition to path x, v, y would form a cycle in G . If $order = 2$, the graph does not have a cycle.

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