Section 1.1.2

1. graph g of order n with maximal number of edges if the complete graph k_n, which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

2. To show a contradiction, suppose otherwise. We know there are an even number of odd degree vertices, implying that there can be no odd degree vertices. But the max degree of a graph is n-1 (connected to every other vertex), so the degree is between 0 and n-1. Excluding the odd degrees, there are not enough unique numbers to cover all the vertices.

3. For later

4. If no such path exists, the two odd vertics are on separate connected components A and B. Consider A by itself, it is a connected graph, but it has an odd number of vertices with an odd degree, a contradiction .

5. Consider the following algorithm

Algorithm 1 Random Traversal

i = 0
v = random vertex in G
repeat
    Mark v with i
    v = some vertex in N(v) that isn't marked
    i += 1
until all vertices in $N(v)$ is marked

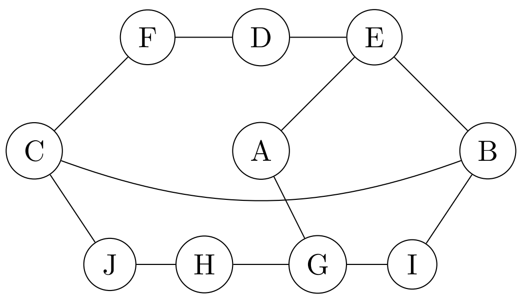
   a) Consider the vertex v that we end up after running this algorithm on G. The algorithm must have visited all vertices in $N(v)$, and are marked with a number. We have a path from the lowest marked $u \in N(v)$ to v, and since $\delta(G) \geq k$, this path is of length at least k.

   b) Once again consider the path from the lowest marked $u \in N(v)$ to v. This path in addition to the edge uv creates a cycle, and by above the path is at least k long, including uv the cycle is at least $k + 1$ long.

6. We prove this by induction on the length of the odd closed walk.

   Base Case: The length 3 odd closed walk is just a length 3 cycle.

   Suppose odd closed walks with lengths up to 2n - 1 contain odd cycles. Let $W = v_1, v_2, ..., v_{2n+1} = v_1$ be a length $2n + 1$ closed walk. If no vertices in the walk repeat, then we are done, the odd walk is an odd cycle. Otherwise, let l be the smallest number not 1 such that v_l repeats, and let $v_l = v_k$ where $l < k$. Then we have two closed walks in W, v_1, ..., $v_l = v_k$, ..., $v_{2n+1} = v_1$ and v_l, ..., $v_k = v_l$. The lengths of these two walks must add up to $2n + 1$, thus one of them must be an odd length closed walk, which by the inductive hypothesis must contain an odd length cycle.

7. 

8. Let $P_1 = v_1, v_2, ..., v_n$ and $P_2 = u_1, u_2, ..., u_n$. Since the graph is connected, there exists a path $P_c = w_1(= u_1), ..., w_m(= v_1)$ between u_1 and v_1. Consider $v_i = w_{i'}$ where i' is the lowest index such that $w_{i'} \in P_2$, and $w_{j'} = u_j$ where j' is the largest index such that $j < i$ and $w_{j'} \in P_1$. Notice that once excluding the first and last vertices, the path $w_{j'}, ..., w_{i'}$ does not contain vertices in P_1 and P_2. Let P_{1-max} the larger of $v_1, ..., v_i$ and $v_i, ..., v_n$, likewise P_{2-max} the larger of $u_1, ..., u_j$ and $u_j, ..., u_n$. Since for $n = k_1 + k_2 = p_1 + p_2$, we have $n \leq max(k_1, k_2) + max(p_1, p_2)$, the path $P_{1-max} + P_{2-max} + v_i, ..., v_j$ is longer than P_1 and P_2, which is a contradiction.

9. My guess is that it is a complete graph K_n with every edge to a single vertex missing, which has $\binom{n-1}{2} = \frac{(n-2)(n-1)}{2}$ edges.

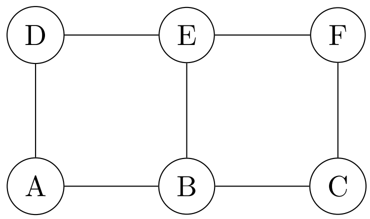
10. Using induction we prove the the minimum number of edges needed to have a connected graph is $n - 1$ (a tree).

    Base Case: K_1 has 0 edges (certainly the minimum number), and is connected.

    Suppose G of order n requires minimum $n - 1$ edges to be connected. Consider such a connected graph with $n - 1$ edges, and consider $G + v$, G with an additional vertex. If we add no edges, then the new graph is disconnected, so we must add one edge. Since G is connected by the inductive hypothesis, adding an edge between any $u \in G$ and v will make $G + v$ connected, since for ux path where $x \in G$, the path $+v$ is vx path.

11. ⇒ Suppose $e = uv$ is a bridge of G and consider $G - e$. Since e is a bridge, $G - e$ is disconnected, i.e. no uv path exists in $G - e$, implying that e is not part of any cycle in G.

    ⇐ Suppose $e = uv$ is not a bridge. Then $G - e$ is still connected, i.e. there exists a uv path. This path $+e$ is a cycle.

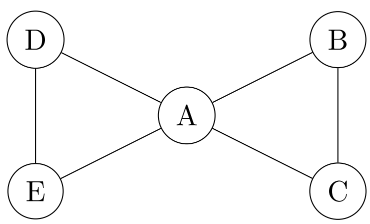
12. a) 

    This graph has no bridges and has more than one cycle.

    b) We give a counter example of the contrapositive.

    This graph has a bridge, but no cut vertex.

    c) 

    This graph has no bridges and has A as a cut vertex.

13. Consider the graph shown in 12a. It is connected, and every vertex is part of one cycle, but is not 2-connected since deleting A create more connected components.

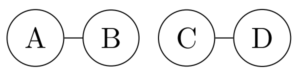
14. Suppose G has no cycles and is connected. Consider $v \in V$ such that $deg(v) > 1$ and $x, y \in N(v)$. Graph $G - \{v\}$ is disconnected since no xy path exists since if there was, this path in addition to path x, v, y would form a cycle in G. If $order = 2$, the graph does not have a cycle.

15. a) Consider $v \in G$ such that $\delta(G) = |N(v)|$. Then $G - N(v)$ is a disconnected graph, thus $\kappa(G) \leq \delta(G)$.

    b) For $\delta(G) = n - 1$, this is true by definition.
    Suppose $\delta(G) = n - 2$. Observe that every vertex is connected to at least all but one other vertex in the graph. We want to show that deleting any $n - 3$ nodes of G still results in a connected graph.

Consider G with any $n-3$ nodes deleted, we refer to the remaining three as $0, 1, 2$. Since $\delta(G) = n-2$, 0 is connected to at least one other node, say 1. By the same reasoning, 2 must be connected to at least one other node, which has to be either 0 or 1. in either case, 0, 1 and 2 are connected.

16. a) Suppose $\delta(G) \geq \frac{n-1}{2}$ but G was not connected. Then it must have more than one connected component. The smallest connected component has $\leq n/2$ vertices, implying that $\delta(G) < \frac{n-1}{2}$, a contradiction.

    b) 

    $|V(G)| = 4$, $\delta(G) = 1 \geq \frac{4-2}{2} = 1$ and G is not connected.