Section 1.3.4

1 Prufer's sequence records the neighbour of the smallest leaf and deletes the leaf, thus the only way for a leaf to be recorded is if it is the neighbour of the smallest leaf left, but then the graph at this point is K_2 and the algorithm should stop.

Vertex v is recorded if they are the neighbour of a leaf with the minimum label, and the leaf is deleted afterwards. Thus the number of times v can be in the seqeunce is $< deg(v)$. We want to show that this number is $> deg(v) - 2$. Suppose it is $\leq deg(v) - 1$. The same number of neighbours of v is removed as the number of its appearance in the sequence, thus the resulting tree must have x, y such that $x \neq y$ and $xv, yv \in E(T)$. But this resulting tree is not K_2, thus the algorithm should not have stopped.

4 Whenever a leaf is chosen, it must all have the same neighbour, implying that a star is such a tree. From 1, and knowing the Prufer sequences are $n-2$ long, we must have that a single vertex v in such a graph must have $deg(v) = n - 2 + 1 = n - 1$. Trees have $n - 1$ edges, so every edge is incident with this v, thus a star is the only graph with this property.

5 Whenever a leaf is chosen, it must have a distinct neighbour to the ones already seen in the sequence. Paths are such trees with the property. Exercise 1 implies that the maximum degree of such a tree is 2. We know every tree must have at least 2 leaves. To show that paths (P_n) are the only type of trees with this property, we show that having more than 2 leaves results in $\Delta(G) > 2$. Suppose such a tree had more than 2 leaves, say x, y and z. Since trees are connected, we know there is a $x - y$ path ($P_v = v_1, ..., v_m$) and $x - z$ path ($P_u = u_1, ..., u_k$). Let i, j be minimum indices such that $v_i = u_j$. Then v_i has degree at least 3, since $y \notin P_u$ and $z]notinP_v$ because they are leaves, $v_i \neq x$ since x is a leaf, and $v_i vi - 1, v_i v_{i+1}, v_i u_{j+1} \in E(G)$, where $v_{i+1} \neq u_{i+1}$.

6 From: *https://math.stackexchange.com/questions/666997/how-many-different-spanning-trees-of-k-n-setminus-e-are-there*

K_n contains $\binom{n}{2}$ edges, and each tree of order n contains $n - 1$ edges, meaning each spanning tree has $\frac{n-1}{\binom{n}{2}} = \frac{2}{n}$ of all edges. Equivalently, any edge belongs to $\frac{2}{n}$ of all spanning trees. $1 - \frac{2}{n}$ trees don't contain any edge e and by Cayley's theorem, $\frac{n-2}{n} n^{n-2} = (n-2)n^{n-3}$ don't contain e.

7 We apply the Matrix Tree Theorem to K_n.

For K_n, we have

$$D - A = \begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & \\ -1 & \dots & -1 & n-1 \end{bmatrix}$$

Thus,

$$cof_{1,1}(D - A) = det(\begin{bmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & \\ -1 & \dots & -1 & n-1 \end{bmatrix})$$

where the above matrix is a $(n-1)x(n-1)$ matrix. Doing appropriate row and column operations, we have

$$cof_{1,1}(D - A) = det(\begin{bmatrix} 1 & -1 & \dots & -1 \\ 0 & n & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & n \end{bmatrix})$$

and the determinant of an upper triangular matrix is the product of the diagonal entries, thus $cof_{1,1}(D - A) = n^{n-2}$.