Section 1.3.2

2 If T has order n, it has $n-1$ edges. Having an even number of edges then implies that there are an odd number of vertices. If all the degrees of the vertices were odd, then the sum of an odd number of odd numbers is odd, which contradicts that $\sum_{v \in V(T)} deg(v) = 2|E(T)|$.

3 Let $v \in V(T)$ be the vertex with the maximum degree. Consider $T - \{v\}$, which is a forest of δ trees. The resulting trees are either K_1, which means that it was a leaf in T, or has order greater than 1. These trees have at least 2 leaves, with at most one created from the deletion of v. Thus T has at least δ leaves.

4 Each of the k connected components must be a tree with n_i vertices such that $n = n_1 + ... + n_k$. By theorem 1.10, we have that the trees have $n_1 - 1$, $n_2 - 1$, ..., $n_k - 1$ edges, and the forest has $(n_1 - 1) + ... + (n_k - 1) = n - k$ edges.

5 $\Rightarrow$ Let G be a tree, and suppose more than one xy path exists in G for $x, y \in V(G)$. We label them $P_v = v_1(= x), ..., v_k(= y)$, $P_u = u_1(= x), ..., u_m(= y)$. Let i_1, i_2 be the smallest indices such that $v_{i_1+1} \neq u_{i_2+1}$, and j_1, j_2 be smallest indices with $i_1 < j_1$ and $i_2 < j_2$ such that $v_{j_1} = u_{j_2}$. Then G contains the cycle $v_{i_1}, ..., v_{j_1} = u_{j_2}, u_{j_2-1}, ..., u_{i_2} = v_{i_1}$.

$\Leftarrow$ Let G be a graph such that $\forall u, v \in V(G)$, there is exactly one uv path. To show a contradiction, suppose G contains a cycle $v_1, ..., v_k, v_1$. Then there are two paths from v_1 to v_k, namely $v_1, ..., v_k$, and v_1, v_k.

6 $\Rightarrow$ Let T be a tree. By definition, T does not contain any cycles. Let $u, v \in V(T)$ such that $uv \notin E(T)$. Since T is connected, there is a u to v path that forms a cycle with uv in $T + uv$. Now suppose more than one cycle exists in $T + uv$. Since there are no cycles in T, all the cycles must contain uv. Consider two such cycles, $C_w = w_1(= u), w_2, ..., w_k(= v), u$ and $C_q = q_1(= u), q_2, ..., q_m(= v), u$. (Fill in the argument that this implies that G contains a cyclce)

$\Leftarrow$ Let T be a graph that contains no cycles, and for any new edge e, the graph $T + e$ has exactly one cycle. We want to show T is connected. Consider $u, v \in V(T)$ such that $v \neq u$ and $uv \notin E(T)$. $G + uv$ contains exactly one cycle, while G did not , so the introduced cycle is $C = u, ..., v, u, ..., v$. Explicitly, there exists a u to v path that does not use uv, and this path is in G. Thus G is connected and acyclic, implying that it is a tree.

7 Suppose $u, v \in V(T)$ such that uv is not a bridge in T. Then in $T - uv$, there exists a uv path, but this uv path $+uv$ forms a cycle in T, which is a contradiction.

8 Consider a nonleaf vertex $v \in V(T)$, where T is a tree. There exists $x, y \in V(T)$ such that $xv, yv \in E(T)$. If v wasn't a cut vertex, $T - v$ contains a xy path, but this path with xv and yv forms a cycle in T, contradicting that T is a tree.

9 Consider the longest path $P = v_1, ..., v_k$ in T. We show that the end vertices of this path must be leaves. Wlog, suppose v_1 is not a leaf. Then either $v_1 v_i \in E(T)$ where $i \neq 2$, or $\exists x \in V(T)$ such that $x \notin P$. In the first case, $v_1, ..., v_i, v_1$ is a cycle in T. In the second case, $x, v_1, ..., v_k$ is a longer path in T. Both cases lead to contradictions, meaning that v_1, v_k are leaves, and every tree has at least two leaves.

10 We induct on the order of T.

Tree of order 2 is K_2, which has 2 leaves.

Suppose this was true for a tree with $2 < k$ vertices, and consider a tree T with $k+1$ vertices. From above, we know that it has at least two leaves, and let u be a leaf, with $uv \in E(T)$. $T - u$ has k leaves, and satisfies the formula. In $T - u$, either $deg(v) = 1$ or $deg(v) > 1$. In the first case, v is a leaf in T and T has the same number of leaves and the same number of vertices with degree ≥ 3, and the formula still holds. In the second case, $deg(v)$ is greater by 1 in T, and u is a new leaf, so the number of leaves in T is greater by 1 compared to the number of leaves in $T - u$. Thus the formula holds for all trees.

11 Let $|V(T)| = n$. We know that $\frac{\sum_{v \in V(G)} deg(v)}{|V(G)|} = 2|E| = 2(n-1)$, which yields

$$a = \frac{2(n-1)}{n} \Rightarrow n = \frac{2}{2-a}$$

12 Let T be a tree such that every vertex adjacent to a leaf has degree at least 3, and suppose no pairs of leaves has a common neighbour. Let k be the number of nodes that neighbour a leaf. Then,

$$2(n-1) = \sum_{v \in V(T)} deg(v) \geq k + 3k + 2(n-2k) = 2n$$