Session 7: Practical Exercise

Question 1

In Inference 4 (section 4.2) we showed that for the Normal mean parameter, with known variance,

$$llr(\mu|\underline{x}) = l(\mu|\underline{x}) = -\frac{1}{2}\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$$

$$\Rightarrow \quad -2llr(\mu|\underline{x}) = -2\, l(\mu|\underline{x}) = \left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2$$

Suppose $X_1, \dots, X_n$ represents a random sample from a Normal distribution with unknown mean μ but known variance 1. Obtain the likelihood ratio test and Wald test statistics for the hypothesis test $H_0{:}\, \mu = \mu_0$ vs $H_1{:}\, \mu \neq \mu_0$, and confirm they are identical. Now derive the score statistic for this hypothesis test and show that it is identical to the others. Do you understand why?

Question 2

Based on previous experience, patients with a certain type of cancer have survival times which are exponentially distributed with mean survival time $1/\beta_0$. It is thought that a new treatment will still give exponentially distributed survival times, but with a different mean survival time. Recall that the exponential density is $f(x|\beta) = \beta \exp(-\beta x)$, where $\beta, x > 0$.

(a) [optional] Show that the mean of the exponential distribution is $1/\beta$.
Hint: you will have to use integration by parts:

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

(b) Write down the model and null hypothesis, when survival times are random variables $X_1, \dots, X_n$.

(c) Derive the maximum likelihood estimator for β, and a log-likelihood ratio test statistic to assess whether there is an effect of the new treatment on survival time.

(d) Derive the score and Wald test statistics for evaluating the same issue. Compare them with the log-likelihood ratio test statistic.

(e) Given the following survival times, in years, for five patients: 0.5, 1, 1.25, 1.5, 0.75, test the null hypothesis $H_0{:}\, \beta = 0.5$ against the alternative $H_0{:}\, \beta \neq 0.5$, by computing all three test statistics, and the resulting p-values. What do you conclude?

Question 3

The random variables $X_1, \dots, X_n$ are independent and have identical uniform distributions on the range $[0, \alpha]$. Derive and sketch the likelihood function for the parameter α. Without performing any calculations, would you envisage any problem with testing hypotheses about α?