Practical 5

Question 1

The random variables X and Y have joint density function

$$f(x,y) = \begin{cases} 12xy(1-x) & \text{if } 0 < x < 1, \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) The marginal density of Y is given by $f(y) = 2y$, for $0 < y < 1$. Find the marginal density of X.

(b) Are X and Y independent?

(c) Find $E(X)$.

(d) Find $Var(X)$.

Question 2

Use the Central Limit Theorem to find the following probabilities:

(a) $P(X > 60)$ where $X \sim Bin(100, 0.5)$

(b) $P(30 < X < 39)$ where $X \sim Bin(48, 0.75)$

(c) $P(X \leq 20)$ where $X \sim Poisson(30)$

(d) $P(X_1 + X_2 \leq 40)$ where X_1, X_2 are iid $Poisson(30)$

Question 3

The leg lengths of 1,494 children were recorded when they were 2, 4 and 6 years old. At these ages the mean leg lengths were 85cm, 103cm, and 114cm respectively. The corresponding covariance matrix was

$$\begin{pmatrix} 22.2 & 11.8 & 13.7 \\ 11.8 & 26.3 & 21.5 \\ 13.7 & 21.5 & 29.0 \end{pmatrix}.$$

Assuming that the joint distribution of leg lengths at the three ages follows a trivariate multivariate normal distribution:

(a) Find the marginal distribution of leg length at age 2.

(b) Find the distribution of leg length age 6 conditional on leg length at age 2.

Additional: Question 4

Let X and Y be random variables with the same variance σ^2. Show that $U = X - Y$ and $V = X + Y$ are uncorrelated.

Additional: Question 5

Let X follow the standard normal distribution $N(0, 1)$. X and $Y = X^2$ are clearly not independent, since Y is the square of X. Despite this fact, prove that $Cov(X, X^2) = 0$.

Hint: It may be useful to know that $E(X^3) = 0$ when $X \sim N(0, 1)$.

Thus although X and X^2 are not independent, they have zero covariance (and hence zero correlation).

Additional: Question 6

Suppose that X_1 and X_2 follow the bivariate normal distribution. Show that if $\sigma_{12} = Cov(X_1, X_2) = 0$, then X_1 and X_2 are independent.