Practical 1 Solutions

Question 1

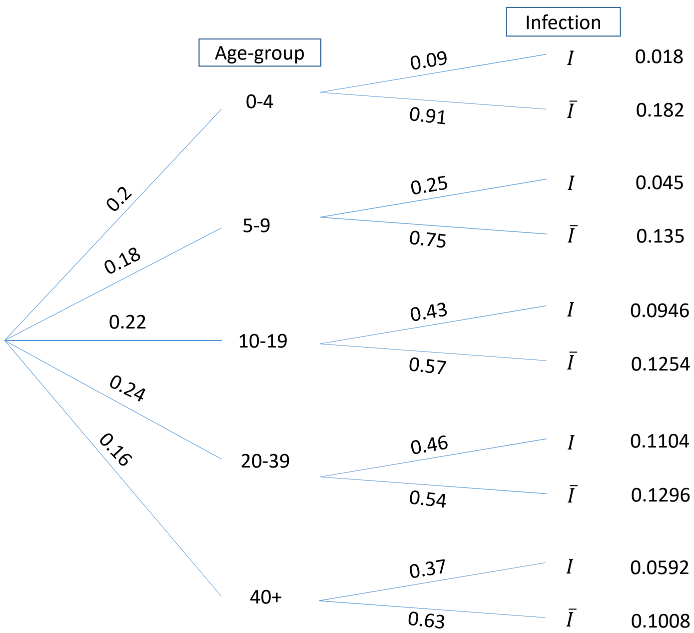
(a) $\Omega = \{0, 1, 2, 3, 4\}$

(b) $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$

(c) $\Omega = \{1, 2, 3, 4\}$

Question 2

Denote infected with hookworm as I, with the absence of infection denoted by $\bar{I}$. The probability tree looks as follows:



(a) $P(10-19) = 0.22$

(b) $P(10-19 \cap I) = 0.22 \times 0.43 = 0.0946$

(c) $P(10-39) = P(10-19 \cup 20-39) = P(10-19) + P(20-39) = 0.22 + 0.24 = 0.46$

(d) $P(10-39 \cap I) = P(10-19 \cap I) + P(20-39 \cap I)$

$$= P(10-19)P(I|10-19) + P(20-39)P(I|20-39)$$

$$= 0.22 \times 0.43 + 0.24 \times 0.46 = 0.0946 + 0.1104 = 0.205$$

(e) $P(10-39 \cap \bar{I}) = 0.22 \times 0.57 + 0.24 \times 0.54 = 0.255$

(f) $P(I) = P(I|0-4)P(0-4) + P(I|5-9)P(5-9) + \ldots + P(I|40+)P(40+) = 0.3272$

(g) The sample space is reduced to events occurring in age group 10-19 and age group 20-39.

$P(I|10-39) = P(I \cap 10-39)/P(10-39) = 0.205/0.46 = 0.446$

Question 3

(a) If being a male doctor and being qualified in England were independent, then

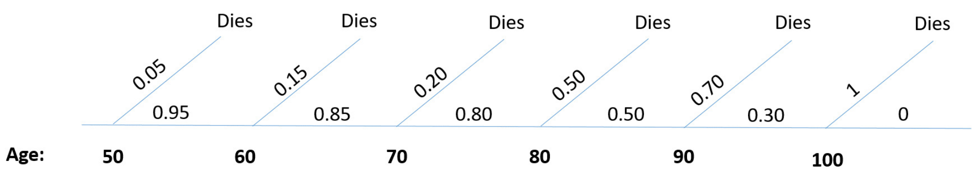
$$P(\text{male} \cap \text{qualified in England}) = 0.8 \times 0.6 = 0.48.$$

(b) Since these are not disjoint events, we have that $P(\text{male} \cup \text{qualified in England}) = P(\text{male}) + P(\text{England}) - P(\text{male} \cap \text{England}) = 0.8 + 0.6 - 0.48 = 0.92$.

(c) The proportion of males among the doctors qualified outside England and the proportion of males among doctors qualified in England may not be the same. The events of being a male doctor and being qualified in England may therefore not be independent. Possible reasons for this dependence are that there may be a difference in proportion of women entering medical school between English schools and schools outside England, or there may be a gender-dependent selection process for doctors coming to England.

Question 4

(a) The probability tree can be drawn as follows:



(b) $P(\text{Survive until 80}) = 0.95 \times 0.85 \times 0.80 = 0.65$

(c) $P(\text{Die between 70 and 90})$

$= P(\text{Survive to 70 \& Die between 70–80}) + P(\text{Survive to 80 \& Die between 80–90})$

$= 0.95 \times 0.85 \times 0.2 + 0.95 \times 0.85 \times 0.8 \times 0.5$

$= 0.48$

Additional: Question 5

Our strategy starts from noting that $A \cup B$ can be written as the union of three disjoint sets:

$$A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B),$$

and so

$$P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B).$$

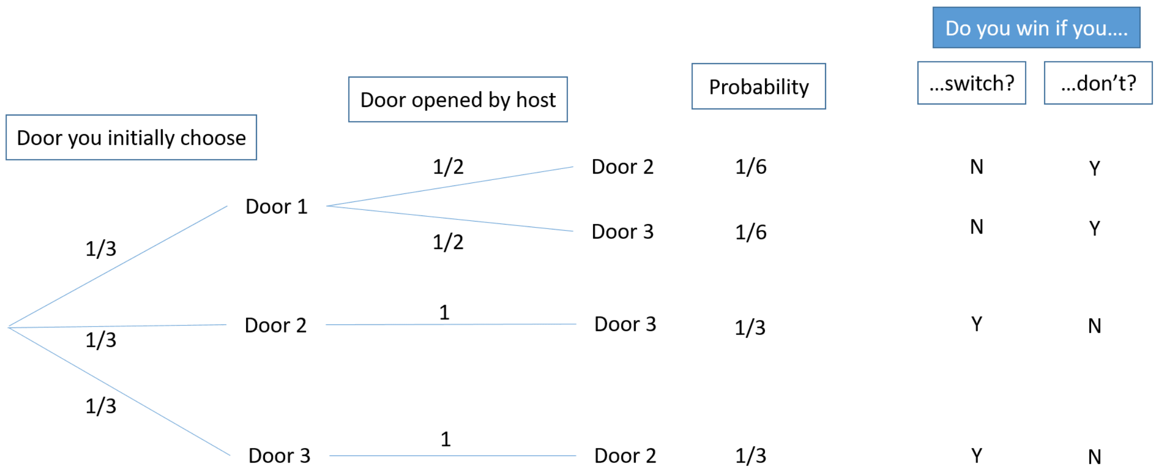
To find an expression for $P(A \cap \bar{B})$, note that we can express A as the union of two disjoint sets: $A = (A \cap \bar{B}) \cup (A \cap B)$. Therefore $P(A) = P(A \cap \bar{B}) + P(A \cap B)$, which implies $P(A \cap \bar{B}) = P(A) - P(A \cap B)$. Similarly we can show that $P(\bar{A} \cap B) = P(B) - P(A \cap B)$. Thus we have

$$\begin{aligned} P(A \cup B) &= P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B) \\ &= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

Optional: Brainteaser

There are many (equivalent) ways of tackling this problem, but in light of today's session, we will start by drawing a probability tree.

Without loss of generality, suppose the car is behind door 1. Then we can draw the following probability tree:



Make sure you understand the second column. If the car is behind door 1 and you choose door 1 then the host can choose to open either door 2 or door 3 and he will reveal a goat. But if the car is behind door 1 and you choose door 2, then the host must choose to open door 3. He has no choice: he cannot open door 1 since this will reveal a car, and he cannot open door 2 since this is the door that you have chosen.

The third column shows the joint probabilities calculated by multiplying the probabilities along the branches. The last two columns show whether or not (Y/N respectively) you would win the car if your policy is to switch, or not to switch.

We can therefore calculate

$$P\left[\text{Win the car, if your policy is to switch}\right] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$
$$P\left[\text{Win the car, if your policy is to NOT switch}\right] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Therefore, **you should always switch**!

This is a famous problem based on a real American television game show called *Let's Make a Deal*, in which the show's host was called Monty Hall. The problem was published in *Parade* magazine, along with a correct solution (similar to the one given above) and approximately 10,000 readers (including several mathematicians) wrote to the magazine claiming that the published solution was wrong. This is how these people argued:

1. Once a goat has been revealed to you, you are left with a choice of only two doors.
2. Since the car and goats were distributed at random initially, the probability that the car is behind any particular door is the same for both remaining doors: one-half.
3. Thus it makes no difference to you whether you swap or not.

The fault in this logic appears in step 2. When you make your initial choice, you have probability one-third of choosing the correct door. Whichever door you choose and whether your choice is correct or not, it is always possible (with probability one) for the host to choose one of the other two doors and reveal a goat. Therefore, you have learnt nothing about the door you initially chose and the probability that this choice is the correct one remains at one-third. However, the host's actions *have* told you a great deal about the *other two* doors. Suppose that you initially chose door 1 and that door 3 is the one opened by the host: you now know that door 3 cannot lead to the car. With probability one-third, your initial choice was correct, in which case door 2 leads to a goat, but with probability two-thirds, your initial choice was incorrect, the car is behind door 2 and the host was forced to open door 3. The probability that door 2 leads to the car in light of this new information is therefore $\frac{1}{3} \times 0 + \frac{2}{3} \times 1 = \frac{2}{3}$. Your initial probabilities of $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ have been updated to $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$ in light of the information you infer from the host's actions. The $\frac{1}{3}$ for door 3 has been 'transferred' to door 2, but your information about door 1 remains uninfluenced by his actions. Therefore, it is now better to switch to door 2.

If you still find this counter-intuitive, imagine a more extreme example in which there are 1,000 doors, 999 goats and 1 car. You choose a door at random with a very small chance of choosing the car. Now, the host opens 998 doors, all of which must reveal goats, leaving you with two choices: your initial door and the one door the host has chosen to leave closed. Imagine that you choose door 1, then he opens doors 2 to 345, leaves 346 closed, then opens doors 347 to 1000, revealing goats behind every single one of the 998 doors he opens. Are you not thinking "what's so special about door 346, I wonder?". Of course, with probability one-thousandth, your initial choice was correct and he picked door 346 completely at random. But with probability 999/1000, he had no choice in the matter at all.