Session 10: Solutions for Exercises and Practical

Exercise 10.1.1

1. From notes,

$$l_p(\theta) = k_1 \log\theta - k\log(p_0 + \theta p_1) \quad \Rightarrow \quad l_p'(\theta) = \frac{k_1}{\theta} - \frac{kp_1}{p_0 + \theta p_1}$$

Setting $l_p' = 0$ gives

$$\frac{k_1}{\hat\theta} = \frac{kp_1}{p_0 + \hat\theta p_1}$$

$$k_1\left(p_0 + \hat\theta p_1\right) = kp_1\hat\theta$$

$$\hat\theta(kp_1 - k_1p_1) = k_1p_0$$

$$\hat\theta = \frac{k_1p_0}{(k-k_1)p_1} = \frac{k_1p_0}{k_0p_1} = \frac{k_1/p_1}{k_0/p_0}$$

as required. By invariance of MLE, $\hat\theta = \frac{\hat\lambda_1}{\hat\lambda_0} = \frac{k_1/p_1}{k_0/p_0}$.

2. Now we require the variance of $\log\theta$. Let $\beta = \log\theta$, so $\theta = e^\beta$. In other words, we require S^2, where $S^2 = -1/l''(\hat\beta)$. Rewriting the profile log-likelihood in terms of β:

$$l(\beta) = k_1\beta - k\log\left(p_0 + e^\beta p_1\right)$$

$$l'(\beta) = k_1 - \frac{ke^\beta p_1}{p_0 + e^\beta p_1}$$

$$l''(\beta) = -\frac{ke^\beta p_1}{p_0 + e^\beta p_1} + k\left(\frac{e^\beta p_1}{p_0 + e^\beta p_1}\right)^2 = \frac{k\left(e^\beta p_1\right)^2 - ke^\beta p_1\left(p_0 + e^\beta p_1\right)}{(p_0 + e^\beta p_1)^2}$$
$$= \frac{ke^\beta p_1\left(e^\beta p_1 - p_0 - e^\beta p_1\right)}{(p_0 + e^\beta p_1)^2} = -\frac{ke^\beta p_0p_1}{(p_0 + e^\beta p_1)^2}$$

$$-\frac{1}{l''(\hat\beta)} = \frac{\left(p_0 + e^{\hat\beta}p_1\right)^2}{ke^{\hat\beta}p_0p_1}$$

Since $\hat\beta = \log\hat\theta$ by invariance of the MLE, $\hat\beta = \log\left(\frac{k_1/p_1}{k_0/p_0}\right)$ and $e^{\hat\beta} = \frac{k_1/p_1}{k_0/p_0}$

$$-\frac{1}{l''(\hat\beta)} = \frac{\left(p_0 + \frac{k_1/p_1}{k_0/p_0}p_1\right)^2}{k\,\frac{k_1/p_1}{k_0/p_0}\,p_0p_1} = \frac{\left(p_0 + \frac{k_1}{k_0}p_0\right)^2}{k\,\frac{k_1}{k_0}\,p_0^2} = \frac{p_0^2\left(1 + \frac{k_1}{k_0}\right)^2}{k\,\frac{k_1}{k_0}\,p_0^2} = \frac{\left(1 + \frac{k_1}{k_0}\right)^2}{k\,\frac{k_1}{k_0}}$$
$$= \frac{\frac{1}{k_0^2}(k_0 + k_1)^2}{k\,\frac{k_1}{k_0}} = \frac{\frac{1}{k_0}k^2}{k\,k_1} = \frac{k}{k_0\,k_1} = \frac{k_0 + k_1}{k_0\,k_1} = \frac{1}{k_1} + \frac{1}{k_0}$$

So $S^2 = \frac{1}{k_0} + \frac{1}{k_1}$ as required.

Practical Exercise

a) [See solutions to Session 7 Practical Question 2c) for similar]

$$L(\lambda_x|\underline{x}) = \prod_{i=1}^{n} \lambda_x e^{-\lambda_x x_i}$$

$$\Rightarrow \quad l(\lambda_x) = \sum_{i=1}^{n} \log(\lambda_x e^{-\lambda_x x_i}) = \sum_{i=1}^{n} \log \lambda_x - \sum_{i=1}^{n} \lambda_x x_i = n \log \lambda_x - \lambda_x \sum_{i=1}^{n} x_i$$
$$= n \log \lambda_x - \lambda_x n\bar{x}$$

$$l'(\lambda_x) = \frac{n}{\lambda_x} - n\bar{x}$$

$$l'(\hat{\lambda}_x) = 0 \quad \Rightarrow \quad \frac{n}{\hat{\lambda}_x} = n\bar{x} \quad \Rightarrow \quad \hat{\lambda}_x = \frac{1}{\bar{x}}$$

Check this is a maximum:

$$l''(\lambda_x) = -\frac{n}{\lambda_x^2} \quad \Rightarrow \quad l''(\hat{\lambda}_x) = -n\bar{x}^2 < 0$$

since all x_i are greater than 0.

b) Joint log-likelihood is sum of log-likelihoods from the independent groups:

$$l\left(\lambda_x, \lambda_y \middle| \underline{x}, \underline{y}\right) = n \log \lambda_x - \lambda_x n\bar{x} + n \log \lambda_y - \lambda_y n\bar{y}$$

Let $\theta = \lambda_x/\lambda_y$ and $r = \bar{x}/\bar{y}$. Use this to reparameterise the joint log-likelihood in terms of $\theta, \lambda_y, r, \bar{y}$:

$$l(\theta, \lambda_y) = n \log(\theta\lambda_y) - \theta\lambda_y n r\bar{y} + n \log \lambda_y - \lambda_y n\bar{y}$$
$$= n \log \theta + 2n \log \lambda_y - \lambda_y n\bar{y}(\theta r + 1)$$

Now partially differentiate with respect to λ_y:

$$\frac{\partial l}{\partial \lambda_y} = \frac{2n}{\lambda_y} - n\bar{y}(\theta r + 1)$$

and equate to 0 to get $\hat{\lambda}_y(\theta)$:

$$\frac{2n}{\hat{\lambda}_y} = n\bar{y}(\theta r + 1) \quad \Rightarrow \quad \hat{\lambda}_y(\theta) = \frac{2n}{n\bar{y}(\theta r + 1)} = \frac{2}{\bar{y}(\theta r + 1)}$$

c) The profile log-likelihood is now obtained by substituting $\hat{\lambda}_y(\theta)$ for λ_y in the joint log-likelihood $l(\theta, \lambda_y)$:

$$l_p(\theta) = l\left(\theta, \hat{\lambda}_y(\theta)\right) = n \log \theta + 2n \log \hat{\lambda}_y(\theta) - \hat{\lambda}_y(\theta) n\bar{y}(\theta r + 1)$$
$$= n \log \theta + 2n \log\left(\frac{2}{\bar{y}(\theta r + 1)}\right) - \frac{2}{\bar{y}(\theta r + 1)} n\bar{y}(\theta r + 1)$$
$$= n \log \theta + 2n \log 2 - 2n \log(\bar{y}) - 2n \log(\theta r + 1) - 2n$$

Ignoring terms not in θ:

$$l_p(\theta) = n \log \theta - 2n \log(\theta r + 1)$$

We now obtain $\hat{\theta}$ by solving $l_p'(\hat{\theta}) = 0$:

$$l_p'(\hat{\theta}) = \frac{n}{\hat{\theta}} - \frac{2nr}{\hat{\theta}r+1} = 0 \quad \Longrightarrow \quad \frac{n}{\hat{\theta}} = \frac{2nr}{\hat{\theta}r+1} \quad \Longrightarrow \quad \hat{\theta}r + 1 = 2r\hat{\theta}$$

$$\hat{\theta}r = 1 \quad \Longrightarrow \quad \hat{\theta} = \frac{1}{r}$$

(d) $\theta = \lambda_x/\lambda_y$ so by invariance of the MLE:

$$\hat{\theta} = \frac{\hat{\lambda}_x}{\hat{\lambda}_y} = \frac{1/\bar{x}}{1/\bar{y}} = \frac{\bar{y}}{\bar{x}} = \frac{1}{r}$$

(e) $H_0: \theta = \theta_0 = 1$ vs $H_1: \theta \neq 1$. The profile log-likelihood ratio statistic is then $-2\, pllr(\theta_0)$:

$$H_0: -2pllr(\theta_0) = -2\left(l_p(\theta_0) - l_p(\hat{\theta})\right) \dot{\sim} \chi_1^2$$

Using the profile log-likelihood from (c), with appropriate substitutions:

$$l_p(\theta_0) = n \log \theta_0 - 2n \log(\theta_0 r + 1) = n \log 1 - 2n \log(r+1) = -2n \log(r+1)$$

$$\begin{aligned} l_p(\hat{\theta}) &= n \log \hat{\theta} - 2n \log(\hat{\theta}r + 1) = n \log \frac{1}{r} - 2n \log\left(\frac{1}{r} r + 1\right) = -n \log r - 2n \log 2 \\ &= -n \log r - n \log 2^2 = -n \log(4r) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad pllr(\theta_0) &= l_p(\theta_0) - l_p(\hat{\theta}) = -2n \log(r+1) + n \log(4r) \\ &= n \log\left(\frac{4r}{(r+1)^2}\right) \end{aligned}$$

$$\Longrightarrow \quad -2pllr(\theta_0) = -2n \log\left(\frac{4r}{(r+1)^2}\right) = 2n \log\left(\frac{(r+1)^2}{4r}\right)$$

When $n = 16$ and $r = 2$, the statistic evaluates to $32 \log(9/8) = 3.77$, which when referred to $\chi^2_{1,0.95} = 3.84$ means we do not reject the null hypothesis at the 5% level.
However, $p = 0.052$, which at the 5% level is borderline evidence that Stage II lung cancer patients have a worse survival prognosis than Stage I patients.