Session 5: Solutions for Exercises and Practical

Exercise 5.2.1

Suppose $K \sim Bin(n, \pi)$. Show that S = $\sqrt{p(1-p)/n}$ where $p = k/n$, and derive an approximate supported range for π.

$$\Pr(K = k) = \pi^k (1-\pi)^{n-k} \binom{n}{k} \Rightarrow L(\pi \mid k) = \pi^k (1-\pi)^{n-k} \binom{n}{k}$$

$\Rightarrow \ell(\pi) = k \log \pi + (n-k)\log(1-\pi)$ ignoring terms not in π.

We will obtain $S^2 = -\left.\dfrac{1}{\ell''(\pi)}\right|_{\pi=\hat{\pi}}$. Recall $\hat{\pi} = k/n$. Substitute after deriving ℓ□″.

$$\ell'(\pi) = \frac{k}{\pi} - \frac{n-k}{1-\pi} \Rightarrow \ell''(\pi) = \frac{-k}{\pi^2} - \frac{n-k}{(1-\pi)^2}$$

$$\Rightarrow \frac{1}{S^2} = -\left.\ell''(\pi)\right|_{\pi=\hat{\pi}} = \left.\frac{k}{\pi^2} + \frac{n-k}{(1-\pi)^2}\right|_{\pi=\hat{\pi}} = \frac{k}{p^2} + \frac{n-k}{(1-p)^2}$$

$k=np$

$$\Rightarrow -\ell''(\hat{\pi}) = \frac{np}{p^2} + \frac{n-np}{(1-p)^2} = \frac{n}{p} + \frac{n(1-p)}{(1-p)^2} = \frac{n}{p} + \frac{n}{1-p} = \frac{n(1-p)+np}{p(1-p)}$$

$$\Rightarrow \frac{1}{S^2} = -l''(\hat{\pi}) = \frac{n}{p(1-p)} \Rightarrow S^2 = \frac{1}{-l''(\hat{\pi})} = \frac{p(1-p)}{n} \Rightarrow S = \sqrt{\frac{p(1-p)}{n}}$$

$\Rightarrow$approximate 95% CI for $\pi = p \pm 1.96\sqrt{\dfrac{p(1-p)}{n}}$

Exercise 5.3.1

1.Range π is [0,1] 2. Range $\log\pi$ is $(-\infty,0]$ 3.Range $\pi/(1-\pi)$ is $[0,\infty)$

4.Range $\log\{\pi/(1-\pi)\}$ is $(-\infty, \infty)$

Exercise 5.3.2

Show that based on observing k events out of n subjects, and a binomial likelihood, the standard error of the MLE of the log risk parameter $\log(\pi)$ is given by $S=\sqrt{\frac{1}{k}-\frac{1}{n}}$.

S will be given by: $S^2 = \left.\dfrac{1}{-\ell''(\beta)}\right|_{\beta=\hat{\beta}}$ where β is log transformation of binomial proportion π.

Parameter transformation:

$\beta = log(\pi) \Rightarrow \pi = e^{\beta}$.

Binomial log-likelihood: $\ell(\pi) = k \log \pi + (n-k)\log(1-\pi)$

$$\Rightarrow \ell(\beta) = k\beta + (n-k)\,log(1-e^{\beta})$$

$$\Rightarrow \ell'(\beta) = k + \frac{(n-k)(-e^{\beta})}{1-e^{\beta}} = k - \frac{(n-k)e^{\beta}}{1-e^{\beta}}$$

$$\Rightarrow \ell''(\beta) = -\frac{(1-e^{\beta})(n-k)e^{\beta} - e^{\beta}(n-k)(-e^{\beta})}{(1-e^{\beta})^2}$$

$$= -\frac{(1-e^{\beta})(n-k)e^{\beta} + e^{2\beta}(n-k)}{(1-e^{\beta})^2}$$

$$= -(n-k)\frac{(1-e^{\beta})e^{\beta} + e^{2\beta}}{(1-e^{\beta})^2} = -(n-k)\frac{e^{\beta} - e^{2\beta} + e^{2\beta}}{(1-e^{\beta})^2} = -(n-k)\frac{e^{\beta}}{(1-e^{\beta})^2}$$

$$\Rightarrow S^2 = \left.\frac{1}{-\ell''(\beta)}\right|_{\beta=\hat{\beta}} = \frac{(1-e^{\hat{\beta}})^2}{(n-k)e^{\hat{\beta}}}.$$

Now, by transformation invariance of MLE, $\hat{\beta} = log(\hat{\pi}) = log\left(\frac{k}{n}\right)$, so $e^{\hat{\beta}} = k/n$,

$$\Rightarrow S^2 = \frac{(1-k/n)^2}{(n-k)k/n} = \frac{\left(\frac{n-k}{n}\right)^2}{\frac{n-k}{n}k} = \frac{n-k}{nk} = \frac{1}{k} - \frac{1}{n} \quad \Rightarrow S = \sqrt{\frac{1}{k} - \frac{1}{n}}$$ as required.

Computer based exercises

Question 1

Part (a)

```
. clear
. range pi 0.2 0.6 41
obs was 0, now 41
. gen logL=40*log(pi)+60*log(1-pi)
```

By the way, Stata log() is the same as ln()

```
. egen logLmax=max( logL)
. gen llr= logL- logLmax
. twoway (line llr pi)
```

Part (b)

M = $\hat{\pi}$ =0.4

```
. gen quad=-(0.4-pi)^2/(2*0.1^2)
```

By trial and error, $S\approx 0.05$. Theoretically, $S = \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{100}}$ =0.049.

```
. twoway (line llr quad pi)
. drop quad
. gen quad=-(0.4-pi)^2/(2*0.049^2)
. twoway (line llr quad pi)
. twoway (line llr quad pi,yline(-1.92))
```

Likelihood ratio confidence intervals from true log-likelihood and from quadratic approximation are very similar.

Question 2

Part (a)

```
. bloglik 40 60
Most likely value for param    0.40000

cut-point -1.921
Likelihood based limits for param  0.30741  0.49766
Approx quadratic limits for param  0.30398  0.49602
```

Part (b)

```
. bloglik 4 6
Most likely value for param    0.40000

cut-point -1.921
Likelihood based limits for param  0.14562  0.70004
Approx quadratic limits for param  0.09634  0.70366
************
```

```
. bloglik 400 600
Most likely value for param    0.40000

cut-point -1.921
Likelihood based limits for param  0.36992  0.43059
Approx quadratic limits for param  0.36963  0.43037
***************
. bloglik 4000 6000
Most likely value for param    0.40000

cut-point -1.921
Likelihood based limits for param  0.39042  0.40963
Approx quadratic limits for param  0.39040  0.40960
***************
```

Quadratic approximation improves as *k* and *n* increase.

Part (c)

```
. bloglik 1 99
Most likely value for param    0.01000

cut-point -1.921
Likelihood based limits for param  0.00057  0.04329
Approx quadratic limits for param  -0.00950  0.02950
```

Note problem of negative confidence interval limits for π because the quadratic approximation is inappropriate when $k = 1$.

```
. bloglik 10 990
Most likely value for param    0.01000

cut-point -1.921
Likelihood based limits for param  0.00502  0.01747
Approx quadratic limits for param  0.00383  0.01617
```

Improvement in quadratic approximation depends more on *k* than on *n*.

```
. bloglik 100 9900
Most likely value for param    0.01000

cut-point -1.921
Likelihood based limits for param  0.00817  0.01208
Approx quadratic limits for param  0.00805  0.01195
*********

. bloglik 99 1
Most likely value for param    0.99000

cut-point -1.921
Likelihood based limits for param  0.95671  0.99943
Approx quadratic limits for param  0.97050  1.00950
```

Here the inappropriate quadratic approximation gives values of $\pi>1$.
For 99 out of 100 subjects the log-likelihood is a reflection (around $\pi = 0.5$) of that fr 1 out of 100.

Question 3
Part (a)

From Example 5.3.1, MLE of log λ = log(8/160) = -2.996; and SE of this MLE is $1/\sqrt{8}$ = 0.3536.

Part (b)

```
. clear
. range lambda 0.02 0.1 81
obs was 0, now 81
. gen logL=8*log(lambda)-160*lambda
. egen logLmax=max( logL)
. gen llr= logL- logLmax
. gen log_lambda=log(lambda)
. twoway (line llr log_lambda)
```

Plotting vs log(λ) corresponds just to change in the x-axis scale.

```
. gen quad=-(-2.996-log_lambda)^2/(2*0.3536^2)
```

```
. twoway (line llr quad log_lambda,yline(-1.92))
```

The quadratic approximation is reasonable.

Part (c)

95% likelihood ratio confidence interval for log(λ) is approximately (ie using the quadratic approximation) -2.996 $\pm$ 1.96 $\times$ 0.3536, which gives (-3.69, -2.30); so the approximate likelihood ratio confidence interval for λ is (0.025, 0.10). Note that this is not symmetric about $\hat{\lambda}$ = 8/160 = 0.05.