# Session 9: Solutions

### Exercise 9.2.1

$$H_0: \phi = \phi_0: W_\phi = \left(\frac{\phi_0 - \hat{\phi}_0}{S}\right)^2 \dot{\sim} \chi_1^2 \text{ where } \frac{1}{S^2} = -l''(\hat{\phi})$$
$$\Rightarrow W_\phi = (\phi_0 - \hat{\phi})^2[-l''(\hat{\phi})] \dot{\sim} \chi_1^2$$

Separately, $H_0: \theta = \theta_0 \Rightarrow W_\theta \dot{\sim} \chi_1^2$ and $H_0: \lambda = \lambda_0 \Rightarrow W_\lambda \dot{\sim} \chi_1^2$, with both test statistics being distributed asymptotically as $\chi_1^2$ if the respective null hypotheses are true.
Therefore $H_0: \theta = \theta_0, \lambda = \lambda_0 \Rightarrow W_\theta + W_\lambda \dot{\sim} \chi_2^2$, since we are adding two independent $\chi_1^2$ variables. So:

$$H_0: \theta = \theta_0, \lambda = \lambda_0 \Rightarrow W_\theta + W_\lambda = (\theta_0 - \hat{\theta})^2[-l''(\hat{\theta})] + (\lambda_0 - \hat{\lambda})^2[-l''(\hat{\lambda})] \dot{\sim} \chi_2^2$$

### Exercise 9.3

$$H_0: \theta = \theta_0, \lambda = \lambda_0 \Rightarrow W = (\theta_0 - \hat{\theta}, \lambda_0 - \hat{\lambda}) \begin{pmatrix} -l''(\hat{\theta}) & 0 \\ 0 & -l''(\hat{\lambda}) \end{pmatrix} \begin{pmatrix} \theta_0 - \hat{\theta} \\ \lambda_0 - \hat{\lambda} \end{pmatrix} \dot{\sim} \chi_2^2$$

Multiply out the matrices and vectors, starting at the right hand-side. We are multiplying a $2 \times 2$ matrix by a $2 \times 1$ column vector, so will end up with a $2 \times 1$ column vector. Move across the top row of the matrix and down the column vector for the first entry, then across the bottom row of the matrix and down the column vector for the second entry:

$$W = (\theta_0 - \hat{\theta}, \lambda_0 - \hat{\lambda}) \begin{pmatrix} -l''(\hat{\theta})(\theta_0 - \hat{\theta}) + 0 \times (\lambda_0 - \hat{\lambda}) \\ 0 \times (\theta_0 - \hat{\theta}) - l''(\hat{\lambda}) \times (\lambda_0 - \hat{\lambda}) \end{pmatrix} = (\theta_0 - \hat{\theta}, \lambda_0 - \hat{\lambda}) \begin{pmatrix} -l''(\hat{\theta})(\theta_0 - \hat{\theta}) \\ -l''(\hat{\lambda})(\lambda_0 - \hat{\lambda}) \end{pmatrix}$$

Now multiply the $1 \times 2$ row vector by the $2 \times 1$ column vector, which will give us a $1 \times 1$ matrix (i.e. a scalar) – move across the row vector and down the column vector:

$$W = (\theta_0 - \hat{\theta})^2[-l''(\hat{\theta})] + (\lambda_0 - \hat{\lambda})^2[-l''(\hat{\lambda})] \dot{\sim} \chi_2^2$$

as in exercise 9.2.1.

### Practical Exercise

<u>Model:</u>
Let random variable $K_i$ be the number of heart attacks in group $i$; with $i = 0$ for group with no previous heart disease, $i = 1$ for group with previous heart disease.

We assume $K_i \sim Po(\mu_i)$; and Poisson mean $\mu_i = \lambda_i p_i$, where $\lambda_i$ is the rate parameter in group $i$, and $p_i$ is the number of person-years at risk in group $i$.

The two groups are independent, so the log likelihood for both groups is the sum of the log-likelihood for each group.

<u>Data</u>: $k_0 = 52, p_0 = 4862; k_1 = 25, p_1 = 512.$

**(a)** Recall Poisson log-likelihood: $l(\lambda) = k \log \lambda - \lambda p$, ignoring terms not in $\lambda$.

Let $\psi = \log \lambda$, so $l(\psi) = k\psi - e^{\psi}p$.

We have two groups, so let $\psi_0 = \log \lambda_0$ and $\psi_1 = \log \lambda_1$.
Then $H_0: \psi_0 = \psi_0^* = -4.5, \psi_1 = \psi_1^* = -3$ vs $H_1: \psi_0 \neq -4.5$ **or** $\psi_1 \neq -3$

The alternative hypothesis is that at least one of the null values is not true (so either one not equal to the null value, or both not equal to the null value). Note that we are using a $*$ superscript to denote the null values (since we already have a 0 subscript).

(i) For the log-likelihood ratio test, we need to find the statistic $-2llr(\psi_0^*, \psi_1^*)$, where

$$llr(\psi_0^*, \psi_1^*) = l(\psi_0^*, \psi_1^*) - l(\hat{\psi}_0, \hat{\psi}_1)$$

Therefore we need to find $l(\psi_0^*, \psi_1^*)$ and $l(\hat{\psi}_0, \hat{\psi}_1)$.
The two groups are independent, so the log likelihood for both groups is the sum of the log-likelihood for each group:

$$l(\psi_0, \psi_1) = k_0\psi_0 - e^{\psi_0}p_0 + k_1\psi_1 - e^{\psi_1}p_1$$

$$\Rightarrow \quad \frac{\partial l}{\partial \psi_0} = k_0 - e^{\psi_0}p_0 \quad \text{and} \quad \frac{\partial l}{\partial \psi_1} = k_1 - e^{\psi_1}p_1$$

(we could work in terms of the vector $\underline{l}'(\psi_0, \psi_1)$, but there is no need at this point – we only have two parameters so it's easy to keep track of our two expressions for the partial derivatives).

We now equate partial derivatives to 0 (i.e. set $\underline{l}'(\psi_0, \psi_1) = \underline{0}$) and solve to obtain the MLEs:

$$\left.\frac{\partial l}{\partial \psi_0}\right|_{\psi_0 = \hat{\psi}_0} = 0 \quad \Rightarrow \quad e^{\hat{\psi}_0} = \frac{k_0}{p_0} \quad \Rightarrow \quad \hat{\psi}_0 = \log\frac{k_0}{p_0}$$

and similarly

$$\hat{\psi}_1 = \log\frac{k_1}{p_1}$$

We can now write down both $l(\psi_0^*, \psi_1^*)$ and $l(\hat{\psi}_0, \hat{\psi}_1)$, substituting in to $l(\psi_0, \psi_1)$ above:

$$l(\psi_0^*, \psi_1^*) = k_0\psi_0^* - e^{\psi_0^*}p_0 + k_1\psi_1^* - e^{\psi_1^*}p_1$$

$$\begin{aligned} l(\hat{\psi}_0, \hat{\psi}_1) &= k_0\hat{\psi}_0 - e^{\hat{\psi}_0}p_0 + k_1\hat{\psi}_1 - e^{\hat{\psi}_1}p_1 = k_0\log\frac{k_0}{p_0} - \frac{k_0}{p_0}p_0 + k_1\log\frac{k_1}{p_1} - \frac{k_1}{p_1}p_1 \\ &= k_0\left(\log\frac{k_0}{p_0} - 1\right) + k_1\left(\log\frac{k_1}{p_1} - 1\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad llr(\psi_0^*, \psi_1^*) &= l(\psi_0^*, \psi_1^*) - l(\hat{\psi}_0, \hat{\psi}_1) \\ &= k_0\psi_0^* - e^{\psi_0^*}p_0 + k_1\psi_1^* - e^{\psi_1^*}p_1 - k_0\left(\log\frac{k_0}{p_0} - 1\right) - k_1\left(\log\frac{k_1}{p_1} - 1\right) \\ &= k_0\left(\psi_0^* - \log\frac{k_0}{p_0} + 1\right) - e^{\psi_0^*}p_0 + k_1\left(\psi_1^* - \log\frac{k_1}{p_1} + 1\right) - e^{\psi_1^*}p_1 \end{aligned}$$

Now we can substitute in our observed and null values to get our test statistic:

$$\begin{aligned} llr(\psi_0^*, \psi_1^*) = 52 \times \left(-4.5 - \log\frac{52}{4862} + 1\right) - e^{-4.5} \times 4862 + 25 \times \left(-3 - \log\frac{25}{512} + 1\right) \\ - e^{-3} \times 512 = -0.0427 \end{aligned}$$

$$\Rightarrow \quad -2\ llr(\psi_0^*, \psi_1^*) = 0.0854$$

Since under $H_0$: $-2\ llr\left(\underline{\psi}^*\right) \dot{\sim} \chi_2^2$, comparing this to a $\chi_2^2$ (95th centile 5.99) we do not reject $H_0$: there is no evidence at the 5% level against the null hypothesis.

(ii) For the Wald test we need to obtain

$$W = \left(\psi_0^* - \hat{\psi}_0, \psi_1^* - \hat{\psi}_1\right)\left(-\underline{l''}(\hat{\psi}_0, \hat{\psi}_1)\right)\begin{pmatrix}\psi_0^* - \hat{\psi}_0 \\ \psi_1^* - \hat{\psi}_1\end{pmatrix}$$

First we obtain

$$\underline{l'}(\psi_0, \psi_1) = \begin{pmatrix}k_0 - e^{\psi_0}p_0 \\ k_1 - e^{\psi_1}p_1\end{pmatrix}$$

(using results obtained in part (i)) then

$$\underline{l''}(\psi_0, \psi_1) = \begin{pmatrix}\dfrac{\partial^2 l}{\partial \psi_0^2} & \dfrac{\partial^2 l}{\partial \psi_1 \partial \psi_0} \\ \dfrac{\partial^2 l}{\partial \psi_0 \partial \psi_1} & \dfrac{\partial^2 l}{\partial \psi_1^2}\end{pmatrix} = \begin{pmatrix}-e^{\psi_0}p_0 & 0 \\ 0 & -e^{\psi_1}p_1\end{pmatrix}$$

$$\Rightarrow \quad -\underline{l''}(\hat{\psi}_0, \hat{\psi}_1) = \begin{pmatrix}e^{\hat{\psi}_0}p_0 & 0 \\ 0 & e^{\hat{\psi}_1}p_1\end{pmatrix} = \begin{pmatrix}e^{\log\frac{k_0}{p_0}}p_0 & 0 \\ 0 & e^{\log\frac{k_1}{p_1}}p_1\end{pmatrix} = \begin{pmatrix}\dfrac{k_0}{p_0}p_0 & 0 \\ 0 & \dfrac{k_1}{p_1}p_1\end{pmatrix}$$

$$= \begin{pmatrix}k_0 & 0 \\ 0 & k_1\end{pmatrix}$$

So

$$W = \left(\psi_0^* - \hat{\psi}_0, \psi_1^* - \hat{\psi}_1\right)\begin{pmatrix}k_0 & 0 \\ 0 & k_1\end{pmatrix}\begin{pmatrix}\psi_0^* - \hat{\psi}_0 \\ \psi_1^* - \hat{\psi}_1\end{pmatrix} = \left(\psi_0^* - \hat{\psi}_0, \psi_1^* - \hat{\psi}_1\right)\begin{pmatrix}k_0(\psi_0^* - \hat{\psi}_0) \\ k_1(\psi_1^* - \hat{\psi}_1)\end{pmatrix}$$

$$= k_0\left(\psi_0^* - \hat{\psi}_0\right)^2 + k_1\left(\psi_1^* - \hat{\psi}_1\right)^2 = k_0\left(\psi_0^* - \log\frac{k_0}{p_0}\right)^2 + k_1\left(\psi_1^* - \log\frac{k_1}{p_1}\right)^2$$

Substituting in our particular values:

$$W = 52 \times \left(-4.5 - \log\frac{52}{4862}\right)^2 + 25 \times \left(-3 - \log\frac{25}{512}\right)^2 = 0.08439$$

Under $H_0: W \dot{\sim} \chi_2^2$, and comparing this value to a $\chi_2^2$ we do not reject $H_0$: there is no evidence at the 5% level against the null hypothesis.

**(b)** From notes, with groups 1 and 2 and $\theta = \lambda_2/\lambda_1$:

$$l_c(\theta) = k_2 \log\theta - k\log(p_1 + \theta p_2)$$

Since we are using notation for group 0 and group 1, rewrite this for $\theta = \lambda_0/\lambda_1$ (i.e. substitute group 0 for group 2):

$$l_c(\theta) = k_0 \log\theta - k\log(p_1 + \theta p_0)$$

Note that we are comparing the rate in the unexposed group compared to the rate in the exposed group, since our null hypothesis is that the rate ratio of heart attacks in men with no previous heart disease, compared to those with previous heart disease, is 0.2.

So $\theta = \lambda_0/\lambda_1$; and we have $H_0: \theta = \theta_0 = 0.2$ vs $H_1: \theta \neq 0.2$.

For the (log-)likelihood ratio test we need to find $-2llr_c(\theta_0)$, where

$$llr_c(\theta_0) = l_c(\theta_0) - l_c(\hat{\theta})$$

To find $\hat{\theta}$, we solve $l_c' = 0$:

$$l_c'(\theta) = \frac{k_0}{\theta} - \frac{kp_0}{p_1 + \theta p_0}$$

$$l_c'(\hat{\theta}) = 0 \implies \frac{k_0}{\hat{\theta}} = \frac{kp_0}{p_1 + \hat{\theta} p_0}$$

$$k_0(p_1 + \hat{\theta} p_0) = kp_0\hat{\theta} \implies \hat{\theta}(kp_0 - k_0p_0) = k_0p_1$$

$$\implies \quad \hat{\theta} = \frac{k_0p_1}{p_0(k - k_0)} = \frac{k_0p_1}{p_0k_1} = \frac{k_0/p_0}{k_1/p_1}$$

Now we can write down $llr_c(\theta_0)$:

$$\begin{aligned} llr_c(\theta_0) &= k_0 \log \theta_0 - k \log(p_1 + \theta_0 p_0) - k_0 \log \hat{\theta} + k \log(p_1 + \hat{\theta} p_0) \\ &= k_0 \log \theta_0 - k \log(p_1 + \theta_0 p_0) - k_0 \log\left(\frac{k_0/p_0}{k_1/p_1}\right) + k \log\left(p_1 + \frac{k_0}{k_1/p_1}\right) \end{aligned}$$

Substituting in our observed values:

$$\begin{aligned} llr_c(\theta_0) = 52 \log 0.2 - (52 + 25) \log(512 + 0.2 \times 4862) - 52 \log\left(\frac{52/4862}{25/512}\right) \\ + (52 + 25) \log\left(512 + \frac{52}{25/512}\right) = -0.0705 \end{aligned}$$

$$\implies \quad -2\, llr_c(\theta_0) = 0.14$$

Under $H_0: -2llr_c(\theta_0) \dot{\sim} \chi_1^2$ (since we are only restricting one parameter in our null hypothesis), so we do not reject $H_0$: there is no evidence at the 5% level that $\theta_0 \neq 0.2$.