Session 6: Practical Exercise

The following facts about the chi-squared distribution (see Appendix to Inference 2) will be useful in the later parts of Question 1.

i) If $X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$ then $\sum_{i=1}^{n} X_i^2 \sim \chi_n^2$

ii) If $X_m^2 \sim \chi_m^2$ then $E[X_m^2] = m$ and $Var[X_m^2] = 2m$

iii) As $m \to \infty$, $X_m^2 \sim N(E[X_m^2], Var[X_m^2])$

Question 1

A drug can be taken in two ways: by taking a pill orally or by injection. Both ways are known to lead to the same average concentration of the drug in the bloodstream after 24 hours, and this known concentration is 3 μg/l.

However, it is postulated that the *variance* of concentration is 1 if the drug is taken orally, whereas the variance is known to be only ¼ if it is injected.

Accordingly an experiment is carried out in which the drug is taken orally, and the following data are observed:

24h concentration	2.54, 0.93, 2.75, 4.51, 3.71, 1.62, 3.01, 4.13, 2.08, 3.33

Assume these data are $\overset{iid}{\sim} N(3, \sigma^2)$

(a) Show that the best test statistic for testing the hypothesis

$$H_0: \sigma^2 = 1/4 \text{ vs } H_1: \sigma^2 = 1$$

is $\sum_{i=1}^{10}(x_i - 3)^2$.

(b) Is this test statistic uniformly most powerful for testing against the composite alternative hypothesis $H_1: \sigma^2 > 1/4$?

(c) Under H_0, what is the distribution of $4\sum_{i=1}^{10}(X_i - 3)^2$? Use this distribution to define a 5% rejection region for H_0. [You don't need to find a numerical threshold – just show how to obtain such a threshold.]

(d) Write down the Normal approximation to the distribution of four times the test statistic, $4\sum_{i=1}^{10}(X_i - 3)^2$, under H_0.

(e) Use this Normal approximation to test the hypothesis

$$H_0: \sigma^2 = 1/4 \text{ vs } H_1: \sigma^2 > 1/4$$

using the data in the table above. What do you conclude?

Question 2 (*Optional*)

If you have time, show that $E[X_1^2] = 1$ and $Var[X_1^2] = 2$, and hence facts (ii) above.