Practical 3 Solutions

Question 1

(a) Each of the 26 possible first initials could be paired with each of the 26 possible second initials, i.e. $26^2 = 676$ possible pairs.

(b) There are ${}^8C_2 = 8!/(2!\,6!) = 28$ possible pairs of teams.

Looking at it another way, each of the eight teams plays against all the other teams, which gives $8 \times 7 = 56$ matches. But we are double counting because the match of A against B is the same as the match of B against A. So the total number of matches is half of this, $56/2 = 28$.

(c) The number of ways of choosing 2 males from 9 is ${}^9C_2 = 9!/(2!\,7!) = 36$.
The number of ways of choosing 2 females from 11 is ${}^{11}C_2 = 11!/(2!\,9!) = 55$.
Therefore there are $36 \times 55 = 1980$ ways of choosing 2 males from 9 and 2 females from 11 (note that we use the multiplication principle because for each of the 36 ways of choosing 2 males from 9 there are 55 ways of choosing 2 females from 11).

(d) The number of possible groups is obtained as the number of combinations of 6 people from 12, which is ${}^{12}C_6 = 12!/(6!\,6!) = 924$.

The number of possible groups in which there are equal numbers of males and females in each group is calculated as follows. There are ${}^6C_3$ ways of choosing 3 male doctors from 6 and ${}^6C_3$ ways of choosing 3 female doctors from 6. Therefore there are ${}^6C_3 \times {}^6C_3 = 400$ ways of choosing 3 men from 6 and 3 women from 6.

Therefore P(both groups will have same number of men) $= 400/924 = 0.43$. Note that we obtain the probability by dividing the number of ways of obtaining the event of interest by the set of all possible ways in the sample space (all of which are equally likely).

Question 2

(a) The probability is 0.20

(b) The probability of both obese is $0.20 \times 0.20 = 0.04$

(c) The probability of four specific children being obese, and the other not, is $0.20^4 \times 0.80^1$. There are five ways to select four out of five (${}^5C_4 = 5!/(4!\,1!) = 5$). So the probability of 4 of 5 children being obese is $5 \times 0.20^4 \times 0.80^1 = 0.0064$.

There is only one way to select 5 from 5, so the probability that 5 children are obese is $0.20^5 = 0.00032$.

The probability that 4 or 5 are obese is then $0.0064 + 0.00032 = 0.00672$.

(d) The probability that at least 1 is obese is the complement of the probability that none are obese, therefore the answer is $1 - 0.80^{10} = 0.893$.

Question 3

(a) Proportion of miscarriages among all pregnancies $= 81/280 = 0.289$.

(b) We are interested in the random variable X = number of miscarriages experienced by a woman who has had four pregnancies. The variable X represents the number of 'successes' (i.e. miscarriages) from a fixed number of trials (pregnancies), so the Binomial distribution seems a sensible choice. Our best guess at the probability of miscarriage is the proportion from part (a). Thus, our assumed distribution is:

$$X \sim Binomial(4, 0.289)$$

(c) To obtain the expected probability distribution, we use the standard Binomial probability mass function:

$$P(X = x) = {}^4C_x \, 0.289^x (1 - 0.289)^{4-x}, \qquad \text{for} \qquad x = 0, 1, 2, 3, 4.$$

(d) To obtain the expected frequency among a sample of 70 women, we simply multiply the probability by 70. This gives the table below:

No. of miscarriages	Expected probability	Expected frequency	Observed frequency
0	$1 \times 0.289^0 \times (1 - 0.289)^4 = 0.256$	$0.256 \times 70 = 17.9$	24
1	$4 \times 0.289^1 \times (1 - 0.289)^3 = 0.415$	$0.415 \times 70 = 29.1$	28
2	$6 \times 0.289^2 \times (1 - 0.289)^2 = 0.253$	$0.253 \times 70 = 17.7$	7
3	$4 \times 0.289^3 \times (1 - 0.289)^1 = 0.069$	$0.069 \times 70 = 4.8$	5
4	$1 \times 0.289^4 \times (1 - 0.289)^0 = 0.007$	$0.007 \times 70 = 0.5$	6

(e) There is poor agreement between observed and expected frequencies under the assumption of a Binomial model. This implies that the assumptions of constant probability of miscarriage and independence between pregnancies are probably invalid. It is possible that some women are more prone to miscarriage than others.

Question 4

(a) We are told the average number of accidental drownings per year in the US is 3 per 100,000. In a city of 200,000 people, μ = expected number of drownings in a year $= 2 \times 3 = 6$.

(b) The random variable of interest here is the number of accidental drownings (in this city) in a year. This is a *count* of the total number of events occurring at random in a fixed period of time, hence we model this count using the Poisson distribution.

If the random variable X denotes the number of drownings in one year in this city, then

$$X \sim Poisson(\mu = 6)$$

The probability mass function is then given by:

$$P(X = x) = \frac{\mu^x e^{-\mu}}{x!} = \frac{6^x e^{-6}}{x!}, \qquad \text{for} \qquad x = 0, 1, 2, \ldots$$

(c)

$$P(X = 0) \;=\; 6^0 e^{-6}/0! = e^{-6} = 0.002$$

(d)

$$\begin{aligned}
P(1 \leq X \leq 2) &= P(X=1) + P(X=2) \\
&= 6^1 e^{-6}/1! + 6^2 e^{-6}/2! \\
&= 0.0149 + 0.045 \\
&= 0.0599
\end{aligned}$$

(e)

$$\begin{aligned}
P(X < 3) &= P(X=0) + P(X=1) + P(X=2) \\
&= 0.002 + 0.0599 \\
&= 0.062
\end{aligned}$$

Optional: Question 5

We can derive the expectation of a Poisson random variable X with parameter μ as

$$\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x P(X=x) \\
&= \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} \\
&= 0 + \sum_{x=1}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} \\
&= \sum_{x=1}^{\infty} \frac{\mu^x}{(x-1)!} e^{-\mu} \\
&= \mu \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} e^{-\mu} \\
&= \mu \sum_{i=0}^{\infty} \frac{\mu^i}{i!} e^{-\mu} \text{ (where we made the substitution: } i = x-1) \\
&= \mu
\end{aligned}$$

To find the variance of X we first find $E(X^2)$, then use the fact that $Var(X) = E(X^2) - E(X)^2$.

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} \\
&= \mu \sum_{x=0}^{\infty} x \frac{\mu^{x-1}}{(x-1)!} e^{-\mu} \\
&= \mu \sum_{x=1}^{\infty} x \frac{\mu^{x-1}}{(x-1)!} e^{-\mu} \\
&= \mu \sum_{i=0}^{\infty} (i+1) \frac{\mu^i}{i!} e^{-\mu} \text{ (where we made the substitution: } i = x-1) \\
&= \mu \left(\sum_{i=0}^{\infty} i \frac{\mu^i}{i!} e^{-\mu} + \sum_{i=0}^{\infty} \frac{\mu^i}{i!} e^{-\mu} \right) \\
&= \mu(E(X)+1) = \mu^2 + \mu.
\end{aligned}$$

Therefore we have that

$$Var(X) = E(X^2) - E(X)^2 = \mu^2 + \mu - \mu^2 = \mu.$$

The variance of X therefore identical to the mean/expectation.