Session 3: Solutions for Exercises and Practical

Exercise 3.3.1

Columns of table correspond to the probability function $f(x\,;\theta)$, and therefore add to 1. Rows of the table correspond to the likelhood function $L(x\,;\theta)$. Hence MLE for θ is:

$$\begin{cases} 1 & \text{if } x=0 \\ 1 & \text{if } x=1 \\ 2 & \text{if } x=2 \\ 3 & \text{if } x=3 \\ 3 & \text{if } x=4 \end{cases}$$

NB Likelihood function is not a probability density function: sum for values in a row may not equal 1.

Exercise 3.4.1

Assuming the likelihood is continuous, has a unique maximum not at the boundary and is twice differentiable. Let $\hat{\theta}$ be that maximum. Then:

$$\frac{dL}{d\theta}=0, \frac{d^2L}{d\theta^2}<0 \Rightarrow \theta=\hat{\theta}$$

Let $\ell = \log L$; then

$$\frac{d\ell}{d\theta}=\frac{d\ell}{dL}.\frac{dL}{d\theta}=\frac{1}{L}.\frac{dL}{d\theta}$$

At turning point for ℓ :

$$\frac{d\ell}{d\theta}=0 \Leftrightarrow \frac{1}{L}.\frac{dL}{d\theta}=0 \Leftrightarrow$$

$$(\text{since } \frac{1}{L}\neq 0).\frac{dL}{d\theta}=0 \Leftrightarrow \theta=\hat{\theta}\text{, provided maximum of } \ell$$

$$\frac{d^2\ell}{d\theta^2}=\frac{d}{d\theta}\left(\frac{d\ell}{dL}.\frac{dL}{d\theta}\right)=\frac{d\ell}{dL}.\frac{d^2L}{d\theta^2}+\frac{dL}{d\theta}.\frac{d}{d\theta}\left(\frac{d\ell}{dL}\right)$$

$$=\frac{1}{\boldsymbol{L}}\times[\text{a value} < 0 \text{ at } \hat{\boldsymbol{\theta}}]+0[\text{at } \hat{\boldsymbol{\theta}}]$$, which is less than 0 at $\hat{\theta}$, giving the maximum.

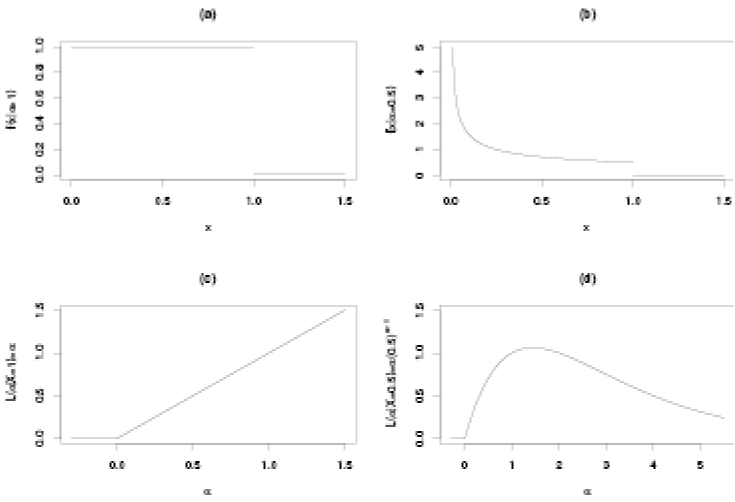
Exercise 3.7.1

1.(i) $f(x|\alpha=1) = 1$, provided $0 \le x \le 1$. This is shown in panel (a) of Figure.

1.(ii) $f(x|\alpha=0.5) = 0.5x^{-0.5}$, provided $0 \le x \le 1$. This is shown in panel (b) of Figure.

2.(i) $L(\alpha|x=1) = \alpha$, provided $\alpha > 0$. This is shown in panel (c) of Figure.

2.(ii) $L(\alpha|x=0.5) = \alpha(0.5)^{\alpha-1}$, provided $\alpha > 0$. This is shown in panel (d) of Figure.



3. Density (probability function): $X\sim f(x|\alpha) = \alpha x^{\alpha-1}$, so

$$\text{Prob}(X = x \mid \alpha) = \alpha x^{\alpha-1} \Rightarrow L(\alpha \mid x) = \alpha x^{\alpha-1}$$

$\Rightarrow \ell = \log L = \log\alpha + (\alpha-1)\log x = \log\alpha + \alpha\log x$ [ignoring -log x, not in α].

$$\Rightarrow \ell'(\alpha) = \frac{1}{\alpha} + \log x \Rightarrow \text{if } \ell'(\alpha) = 0 \Rightarrow \frac{1}{\alpha} = -\log x \Rightarrow \hat{\alpha} = \frac{-1}{\log x}$$

4. Joint density (probability function) is

$$f(\underline{x}|\alpha) = \prod_{i=1}^{n} \alpha x_i^{\alpha-1}$$

so likelihood is

$$L(\alpha|\underline{x}) = \prod_{i=1}^{n} \alpha x_i^{\alpha-1}$$

so log likelihood is

$$\ell(\alpha) = \sum_{i=1}^{n} \log \alpha x_i^{\alpha-1} = \sum \log\alpha + (\alpha-1)\sum\log x_i = n\log\alpha + \alpha\sum\log x_i \text{ (+ term not in } \alpha)$$

$$\Rightarrow \ell'(\alpha) = \frac{n}{\alpha} + \sum_{i=1}^{n}\log x_i \xrightarrow[\ell'=0]{} \frac{n}{\hat{\alpha}} = -\sum_{i=1}^{n}\log x_i \Rightarrow \hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\log x_i}$$

This is a maximum since $\ell''(\alpha) = -n/\alpha^2 < 0$ given $\alpha > 0$.

5. $\hat{\alpha} = \dfrac{-n}{\log 0.2 + \log 0.3 + \log 0.7} = 0.946$ [natural logs throughout]

Below is an edited of log file obtaining this result in Stata, not analytically (ie using algebra and calculus, as above) but by numerical iteration.

```
./*
Using Stata's maximum likelihood estimation command: mlexp.
We need to tell Stata what is the log likelihood for one observation
Stata then knows to sum the log likelihoods to obtain the likelihood for
all three observations
 */
. clear
. set obs 3
. gen x=.
. * this is the data for Ex 3.7.1 q5:

. replace x=0.2 in 1
. replace x=0.3 in 2
. replace x=0.7 in 3

* now using mlexp, giving it the log-likelihood function:
. mlexp (ln({alpha})+{alpha}*ln(x)),nolog

Maximum likelihood estimation
Log likelihood = -3.1654389                     Number of obs   =          3

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /alpha |   .9463467   .5463735     1.73   0.083    -.1245257    2.017219
------------------------------------------------------------------------------
. * note that the above log likelihood at the maximum was negative.
.
. * now changing the data:
. replace x=0.7 in 1
. replace x=0.75 in 2
. replace x=0.8 in 3
.
. * with these data you will see a positive log likelihood…
. mlexp (ln({alpha})+{alpha}*ln(x)),nolog

Maximum likelihood estimation
Log likelihood =   .7222542                     Number of obs   =          3

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      /alpha |   3.458211   1.996599     1.73   0.083    -.4550512    7.371473
------------------------------------------------------------------------------
```

Computer based exercises

Question 1

```
. clear
. range pi 0.1 0.9 9
. gen L=pi^3*(1-pi)^7
. list pi L

pi L
1. .1 .0004783
2. .2 .0016777
3. .3 .0022236
4. .4 .0017916
5. .5 .0009766
6. .6 .0003539
7. .7 .000075
8. .8 6.55e-06
9. .9 7.29e-08
```

$\pi = 0.3$ gives the highest likelihood (value best supported by the data); $L(0.1)$ is not as high as $L(0.5)$, so $\pi = 0.5$ is better supported by the data.

Question 2

```
. clear
. range pi 0.01 0.99 99
. gen L=pi^3*(1-pi)^7
. twoway line L pi
. list
pi L
```
Output suppressed
```
25. .25 .0020857
26. .26 .0021357
27. .27 .0021745
28. .28 .0022019
29. .29 .0022182
```
30. .3 .0022236
```
31. .31 .0022183
32. .32 .002203
33. .33 .002178
34. .34 .0021441
35. .35 .0021018
```
Output suppressed

confirmed:

.* pi=.3 is the MLE;

.* pi=.5 is better supported by the data as its likelihood

.* is greater than pi=.1

Question 3

```
. clear
. range lambda 0.010 0.100 91
obs was 0, now 91
. gen L=lambda^8 * exp(-lambda*160)
```
[Ignoring terms not in lambda, which don’t affect the shape]
```
. twoway line L lambda
. list lambda L
lambda L
```
Output suppressed
```
36. .045 1.26e-14
37. .046 1.28e-14
38. .047 1.29e-14
39. .048 1.30e-14
40. .049 1.31e-14
```
41. .05 1.31e-14
```
42. .051 1.31e-14
43. .052 1.30e-14
44. .053 1.29e-14
45. .054 1.28e-14
```
Output suppressed

.* maximum at lambda = 0.05.
```
. gen l=log(L)

. twoway line l lambda
. list lambda l
```
Output suppressed
```
34. .043 -32.05244
35. .044 -32.02853
36. .045 -32.00874
37. .046 -31.99291
38. .047 -31.98086
39. .048 -31.97243
40. .049 -31.96748
```
41. .05 -31.96586
```
42. .051 -31.96744
43. .052 -31.97209
44. .053 -31.97971
45. .054 -31.99017
46. .055 -32.00338
47. .056 -32.01923
48. .057 -32.03763
```