Session 4: Solutions for Exercises and Practical

Exercise 4.1.1

Suppose y is a single observation from random variable $Y \sim N(\mu, \tau^2)$, where τ^2 is **known**. Show that the log-likelihood for μ is $-\frac{1}{2}\left(\frac{y-\mu}{\tau}\right)^2$, calculate the maximum likelihood estimate $\hat{\mu}$, and hence write down $llr(\mu)$.

$$f(y\mid\mu) = \frac{1}{\sqrt{2\pi\tau^2}}\exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\tau}\right)^2\right) \Rightarrow L(\mu\mid y) = \frac{1}{\sqrt{2\pi\tau^2}}\exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\tau}\right)^2\right)$$

$\Rightarrow \ \ell(\mu) = \log(L) = \log\left(\frac{1}{\sqrt{2\pi\tau^2}}\right) - \frac{1}{2}\left(\frac{y-\mu}{\tau}\right)^2 = -\frac{1}{2}\left(\frac{y-\mu}{\tau}\right)^2$, omitting terms not in μ.

$$\Rightarrow \ell'(\mu) = 2.-\frac{1}{2}\left(\frac{y-\mu}{\tau}\right).\frac{-1}{\tau} = \frac{y-\mu}{\tau^2}.$$

$\ell'(\mu) = 0 \Rightarrow \frac{y-\hat{\mu}}{\tau^2} = 0 \Rightarrow \hat{\mu} = y$. To confirm this is maximum, note simply that $\ell(\mu)$ is a quadratic, which has only one turning point; a glance shows the function has a maximum value of 0 at $\ell(\hat{\mu} = y)$. As a result in this case the log-likelihood ratio, $llr(\mu)$, is equal to the log-likelihood:

$$\ell(\hat{\mu}) = 0 \Rightarrow llr(\mu) = \ell(\mu) - \ell(\hat{\mu}) = \ell(\mu)$$

Computer based exercises

Question 1

Part (a)

```
clear
. range pi 0.01 0.99 99
obs was 0, now 99
. gen L=pi^3*(1-pi)^7
. twoway line L pi

. list L pi
     +----------------+
     |        L    pi |
     |----------------|
  1. | 9.32e-07   .01 |
  2. | 6.95e-06   .02 |
  3. | .0000218   .03 |
  4. | .0000481   .04 |
  5. | .0000873   .05 |
     |----------------|
  6. | .0001401   .06 |
  7. | .0002064   .07 |
  8. | .0002856   .08 |
  9. | .0003767   .09 |
 10. | .0004783    .1 |
     |----------------|
 11. | .0005887   .11 |
 12. | .0007062   .12 |
```

Output suppressed

```
 45. | .0013873   .45 |
     |----------------|
 46. | .0013033   .46 |
```

```
 47. | .0012196   .47 |
 48. |  .001137   .48 |
 49. | .0010558   .49 |
 50. | .0009766    .5 |
     |----------------|
 51. | .0008997   .51 |
 52. | .0008255   .52 |
```

Output suppressed

```
. dis  .0004783 / .0009766
.48976039
```

So 0.5 is better supported by the data.

Part (b)

```
. egen Lmax=max(L)
. gen LR=L/Lmax
. twoway line LR pi
. gen logLR=log(LR)
. twoway line logLR pi
. twoway line logLR pi if logLR>-4
. twoway line logLR pi if logLR>-4, yline(-1.92)
. twoway line logLR pi if logLR>-4,yline(-1.92) yaxis(1 2) ylabel(-1.92,angle(horizontal) axis(2))
```

Part (c)

```
. list pi  logLR
     +-----------------+
     |  pi       logLR |
     |-----------------|
  1. | .01    -7.77722 |
  2. | .02   -5.768845 |
  3. | .03   -4.624245 |
  4. | .04   -3.833739 |
  5. | .05   -3.237607 |
     |-----------------|
  6. | .06   -2.764717 |
  7. | .07   -2.377132 |
  8. | .08   -2.052214 |
  9. | .09   -1.775369 |
 10. |  .1   -1.536636 |
     |-----------------|
```

Output suppressed

```
 58. | .58   -1.598042 |
 59. | .59   -1.715442 |
 60. |  .6   -1.837869 |
     |-----------------|
 61. | .61   -1.965506 |
 62. | .62   -2.098553 |
 63. | .63   -2.237229 |
```

Output suppressed

95% limits: from between π=0.08 to 0.09, for lower limit, to between 0.6 and 0.61. (To identify more accurately we would need a finer grid.)

Question 2

Part (a)

```
. gen logL_30=30*log(pi)+70*log(1-pi)
. egen logL_30max=max(logL_30)
. gen logLR_30= logL_30- logL_30max

. twoway (line logLR pi if logLR>-4, yaxis(1 2)) /*
> */(line logLR_30 pi if logLR_30>-4, yaxis(1 2))/*
> */, yline(-1.92) ylabel(-1.92,  angle(horizontal) axis(2))
```

Part (b)

The MLEs are the same for both (0.3, as you would expect); however, the log-likelihood ratio has a much tighter curvature when n=100 than when n=10, giving a narrow likelihood ratio confidence interval: this indicates that we have much more precise information about the true value of π : the curvature of the log-likelihood reflects the amount of information the data provide regarding the parameter.

Notice also that the log-likelihood ratio is much more nearly quadratic in shape when n=100.

Question 3

```
. clear
. range lambda 0.010 0.100 91
obs was 0, now 91
. gen L=lambda^8*exp(-lambda*160)
. gen logL=log(L)
. egen logL_max=max(logL)
. gen logLR= logL- logL_max

.  twoway  (line  logLR  lambda  if  logLR>-4,  yaxis(1  2)),  yline(-1.92)  ylabel(-1.92,
angle(horizontal) axis(2))

.  * visually  we  estimate  the  95%  confidence  interval  between  about  lambda=  0.025  to
0.095
. list  logLR lambda

     +--------------------+
     |     logLR   lambda |
     |--------------------|
  1. | -6.475502      .01 |
  2. |  -5.87302     .011 |
  3. | -5.336929     .012 |
  4. |  -4.85659     .013 |
  5. | -4.423725     .014 |
     |--------------------|
  6. | -4.031784     .015 |
  7. | -3.675472     .016 |
  8. | -3.350475     .017 |
  9. | -3.053211     .018 |
 10. | -2.780674     .019 |
     |--------------------|
 11. | -2.530327      .02 |
 12. | -2.300003     .021 |
 13. | -2.087845     .022 |
 14. | -1.892231     .023 |
Output suppressed
82. | -1.769306     .091 |
 83. | -1.841877     .092 |
 84. | -1.915386     .093 |
 85. | -1.989826     .094 |
     |--------------------|
 86. |  -2.06517     .095 |
```

So more accurately we can give the 95% confidence limits as between lambda=0.022 and 0.023, and 0.093.