mathoverflow

Proofs without words

Asked 13 years, 8 months ago Modified 12 days ago Viewed 182k times

397

Can you give examples of proofs without words? In particular, can you give examples of proofs without words for *non-trivial* results?

(One could ask if this is of interest to mathematicians, and I would say yes, in so far as the kind of little gems that usually fall under the title of 'proofs without words' is quite capable of providing the aesthetic rush we all so professionally appreciate. That is why we will sometimes stubbornly stare at one of these mathematical autostereograms with determination until we joyously *see* it.)

(I'll provide an answer as an example of what I have in mind in a second)

reference-request big-list examples intuition alternative-proof

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edited Dec 23, 2020 at 8:56

community wiki
8 revs, 6 users 57%
Mariano Suárez-Álvarez

13 I hope I am not alone in being (usually) unable to appreciate "proof by picture"... – Suvrit Jul 8, 2011 at 21:14

20 @Suvrit: I hope I am not alone in being most often unable to appreciate "proof by word" until I've read it at least twenty times and wrestled with it for many days per page! – Selene Routley Jul 9, 2011 at 12:11

72 My opinion is that almost every proof-without-words is improved by a few well-chosen words. – Joel David Hamkins Feb 12, 2012 at 0:47

18 @goblin, I am afraid that you have completely misunderstood the concept. The idea is pictures which have the rather amazing capability of immediately suggesting on the mind of the viewer the idea of a proof. How on earth you managed to get from the rather well-known idea involved in this question to «proofs without logic» is a mystery to me. – Mariano Suárez-Álvarez Jan 23, 2015 at 3:55

11 If you cannot tell the difference between a proof-tree and a proof without words in the tradition of, say, the AMM Monthly, then that is clearly a limitation of yours. I would rather you start a meta thread, or a blog, instead of further polluting this thread with what is clearly rather orthogonal chatter. – Mariano Suárez-Álvarez Jan 23, 2015 at 22:52

84 Answers

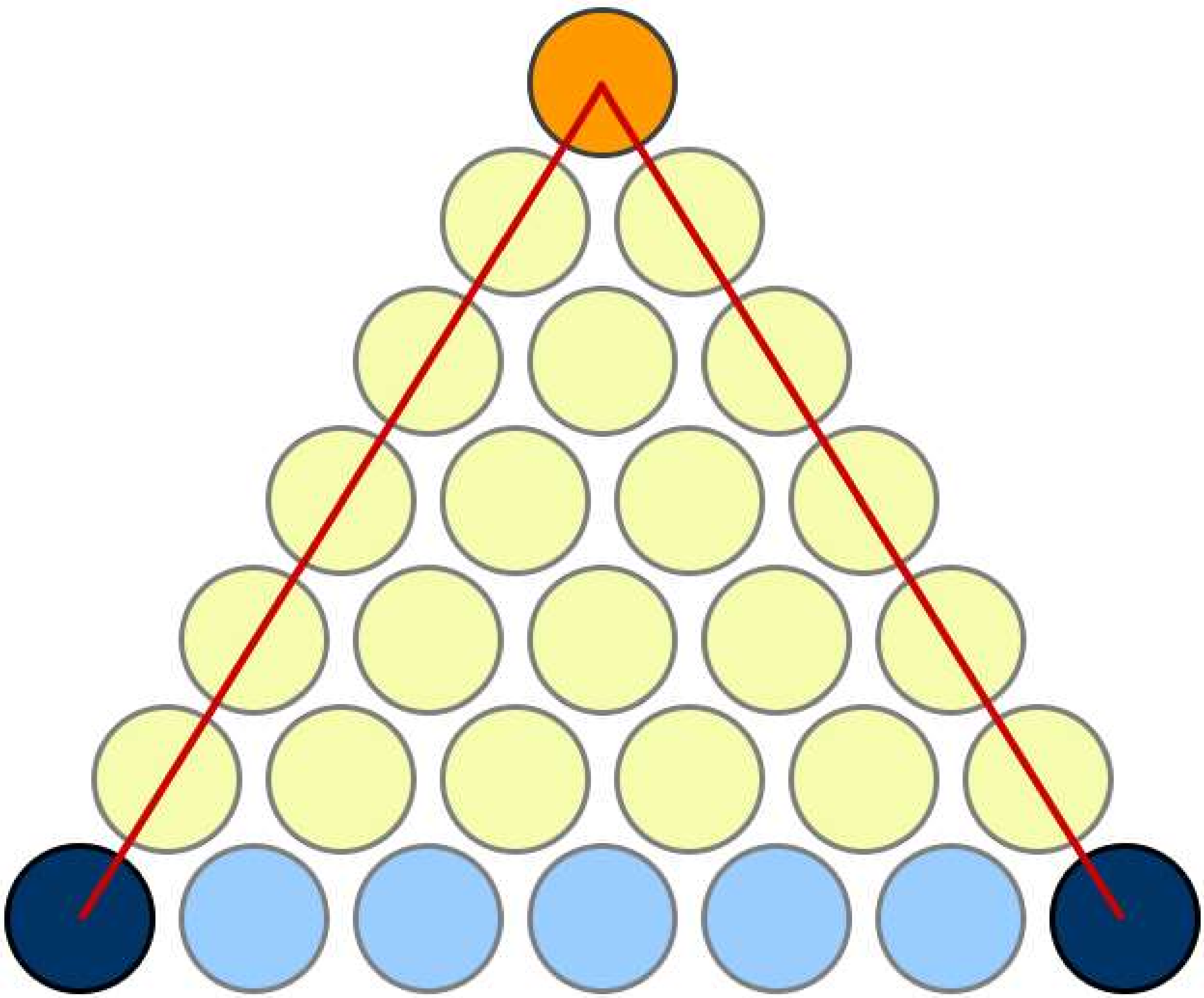
Sorted by: Highest score (default)

1 2 3 Next

552

A proof of the identity

$$1 + 2 + \cdots + (n-1) = \binom{n}{2}$$



(Adapted from an entry I saw at Wolfram Demonstrations, see also the original faster animation)

This proof was discovered by Loren Larson, professor emeritus at St. Olaf College. He included it along with a number of other, more standard, proofs, in "A Discrete Look at 1+2+...+n," published in 1985 in The College Mathematics Journal (vol. 16, no. 5, pp. 369-382, DOI: 10.1080/07468342.1985.11972910, JSTOR).

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edited Jul 28, 2018 at 7:26

community wiki
6 revs, 6 users 64%
Mariano Suárez-Álvarez

30 @Johann, people who thing that mathematics is about deducing theorems from axioms have such a mistaken idea of what the mathematical activity is thar their judgment is more or less irrelevant :D – Mariano Suárez-Álvarez Jun 29, 2010 at 13:05

16 @Johann: some days of the week, I am such a person; and from that point of view, the picture is a beautifully clear encoding of a certain bijection, and the formal construction of the bijection itself is a very beautiful proof. No beauty is destroyed!// I strongly believe that a proof with a clear intuition should *also* be clear as a formal proof. If not, either (usually) our formalism isn't as good as it could be, or (occasionally) our intuition really is overlooking some non-trivial subtleties. – Peter LeFanu Lumsdaine Jun 29, 2010 at 14:37

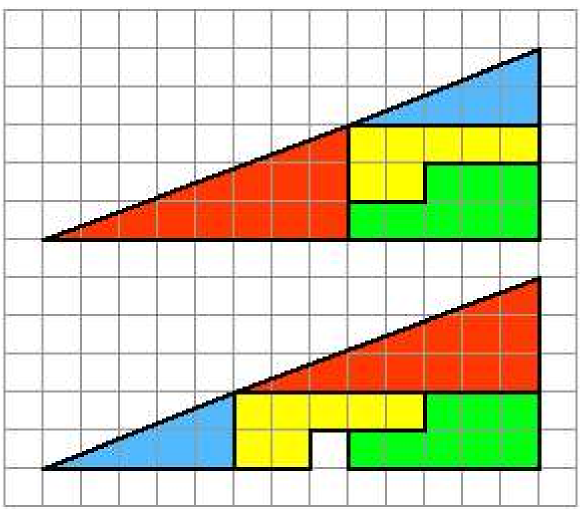
80 Am I the only one who doesn't understand this "proof" at all? – mathreader Oct 17, 2010 at 17:07

59 @mathreader - the yellow dots are the sum of the first n numbers. Choosing two of the n+1 blue dots uniquely specifies a yellow dot in a bijective fashion. – Steven Gubkin Nov 11, 2010 at 13:40

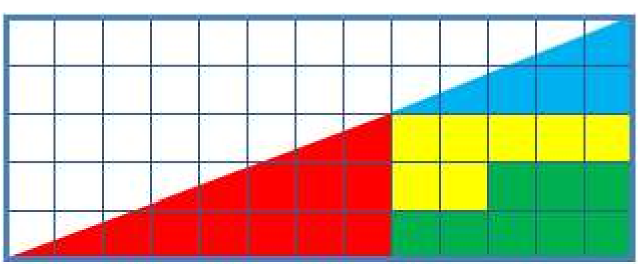
60 This beautiful proof warrants proper attribution. It was discovered by Loren Larson, professor emeritus at St. Olaf College. He included it along with a number of other, more standard, proofs, in "A Discrete Look at 1+2+...+n," published in 1985 in The College Mathematics Journal (vol. 16, no. 5, pp. 369-382). – Barry Cipra Oct 15, 2011 at 2:17

243

Because I think proof by picture is potentially dangerous, I'll present a link to the standard proof that 32.5 = 31.5:



An animation of the above is:



(This work has been released into the public domain by its author, Trekky0623 at English Wikipedia. This applies worldwide.)

There does not seem to be any necessity for the particular 'path in the relevant configuration space' that was used by the author of the above animated gif. This may be seen as an argument *against* including an animation.

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edited Sep 26, 2017 at 7:20

community wiki
3 revs, 3 users 48%
Russell O'Connor

102 I think it is just as easy to introduce some kind of logical gap in a written proof as in a graphical one. – Steven Gubkin Mar 7, 2010 at 23:41

62 @Steven: I think there is some truth to your claim, but I don't agree fully. First, we may notice that most proofs rely much more on writing than on pictures, and so mathematicians have developed a better radar for "written gaps". Second, there is a very strong sense in which written proofs may be formalized and checked by computer. Picture proofs, unless they share quite a bit of the "discrete" character of written proofs, usually are not amenable to such treatment. (And the notions of discreteness I can think of pretty much ensure that the picture proof could be turned into words.) – Pietro May 15, 2010 at 20:22

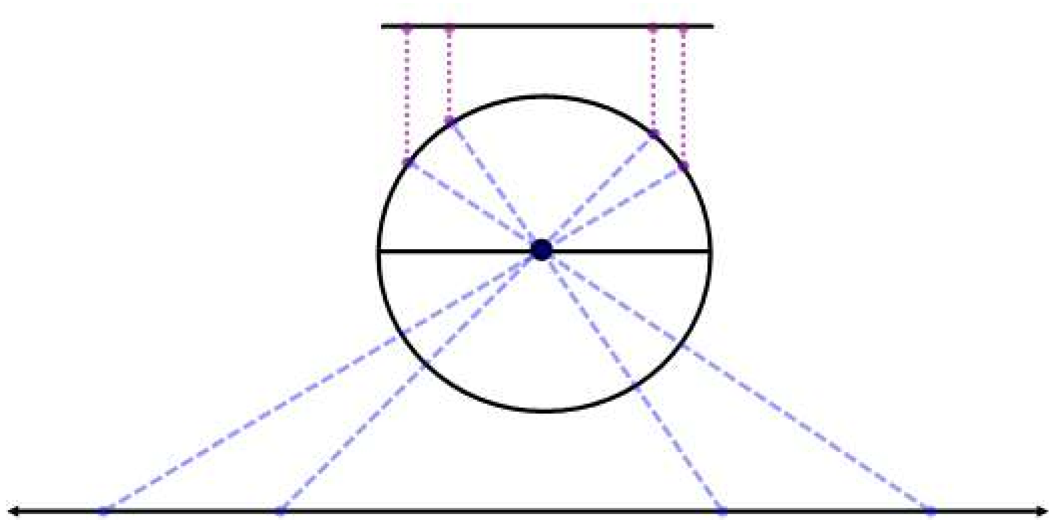
17 @Pietro: “there is a very strong sense in which written proofs may be formalised”? Formalisation is a highly non-trivial task, and typically depends on quite a lot of mathematical background. What affects the difficulty is not whether the proof is written or graphical, but whether it’s detailed or highly abstracted. Formalising a good proof-by-picture is no harder than formalising a high-level written proof. Insofar as there’s a difference, I’d say it’s just that written proofs *can* be made detailed enough that formalising them is straightforward, whereas picture proofs perhaps can’t. – Peter LeFanu Lumsdaine Nov 29, 2010 at 1:04

8 +1 , Here is the wiki page for this. en.wikipedia.org/wiki/Missing_square_puzzle – PKumar Oct 25, 2014 at 7:17

3 It might be noted that the success of the illusion partly depends on the fact this uses Fibonacci numbers (it is a coincidence I guess that the next newest answer is also about Fibonacci numbers!). – Todd Trimble ♦ Jan 23, 2015 at 2:53

208

The cardinality of the real number line is the same as that of a finite open interval of the real number line.



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edited Aug 17, 2022 at 20:55

community wiki
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Jason Dyer

6 I suppose this picture can also be adapted to obtain the stereographic projection proof that a sphere is a manifold? – Kevin H. Lin Dec 14, 2009 at 23:47

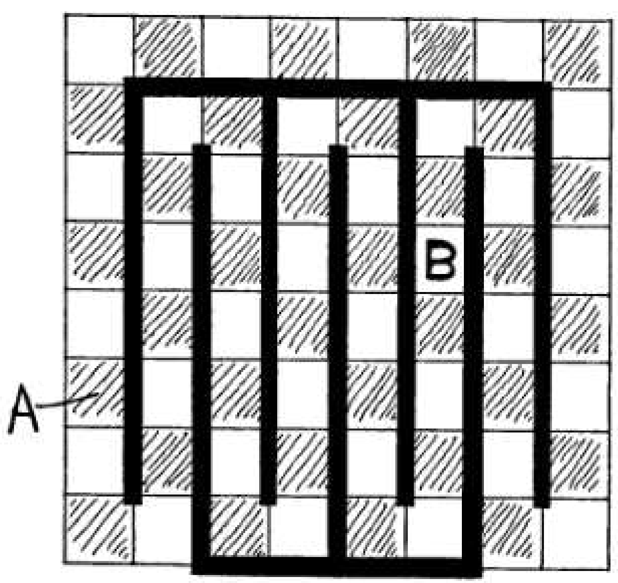
@Jason Dyer: What software did you use? Inkscape? – user577 Dec 15, 2009 at 1:48

1 I usually use Inkscape for my vector-based needs, but this was just done with my Smartboard presentation software. – Jason Dyer Dec 15, 2009 at 14:21

1 This picture shows not only that they have same cardinality but that they are homeomorphic. – Stefan Witzel Jan 23, 2020 at 7:46

189

It is known (see this other answer) that an 8x8 board in which squares at opposite corners have been removed cannot be tiled with dominoes, as the removed squares are of the same "colour". But what if two squares of *different* colours are removed? Ralph E. Gomory showed that it is always possible, no matter where the two removed squares are, and this is his proof:



(Imagine A and B are the squares removed.) The image is from *Mathematical Gems I* by Ross Honsberger.

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edited Apr 13, 2017 at 12:58

community wiki
5 revs, 3 users 62%
shreevatsa

22 What I like about this example is that there seems to be no straightforward proof *without* the picture; the crux of the proof's idea is specifically this picture. – shreevatsa May 3, 2016 at 21:21

7 Very nice. I guess the crux of the proof is that, when mn is even, $P_m \times P_n$ is a Hamiltonian bipartite graph? – bof May 4, 2016 at 4:15

Well, I'd call that a generalization, not the crux of the proof. :-) Staying concrete, for the question about the specific case of $m = n = 8$, the crux of this proof (that this graph *is* Hamiltonian) is this picture. Similar pictures can be draw whenever mn is even, sure. – shreevatsa Oct 17, 2016 at 16:37

A complete result (guessed not shown) is for m or n odd : Any mxn board with 1 square removed has a neighborhood graph that has an hamiltonian cycle. – Jérôme JEAN-CHARLES Mar 30, 2020 at 10:19

I call this the "hungry snake" proof (imagine a game of Snake and you have to get the high score. The snake would have to assume some pose like this one). The exact same idea can be used to prove a lot of boards have the same property (for general n-dimensions). – MaudPieTheRocktorate Nov 18, 2020 at 3:46

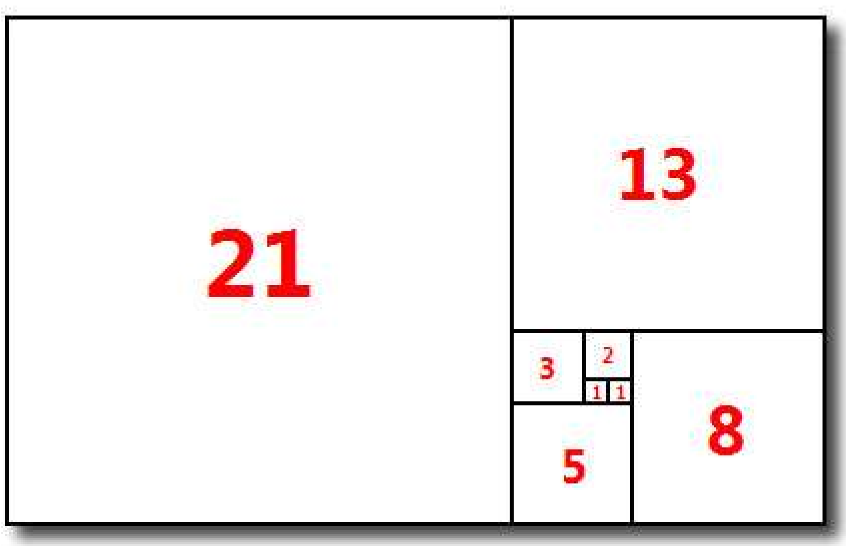
182

There are a couple of Fibonacci identities, I think. For example

$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$, with $F_0 = 1$.

By putting together squares of side F_n, one at a time, you get a rectangle of dimension $F_n F_{n+1}$: The two squares of side 1, then the square of side 2, then the square of side 3 and so on.

Here is an image I found online



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edited Dec 23, 2020 at 8:56

community wiki
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serargus

11 fantastic ! – Martin Brandenburg Apr 17, 2010 at 23:30

Really exceptional! – Koundinya Vajjha Jul 24, 2010 at 6:49

1 I think that there *is* a nice pictorial proof for this fact, but I don't think this is it. It's a proof for a specific n. To make it a general proof, the inductive step needs to be illustrated. – Max Mar 16, 2011 at 14:08

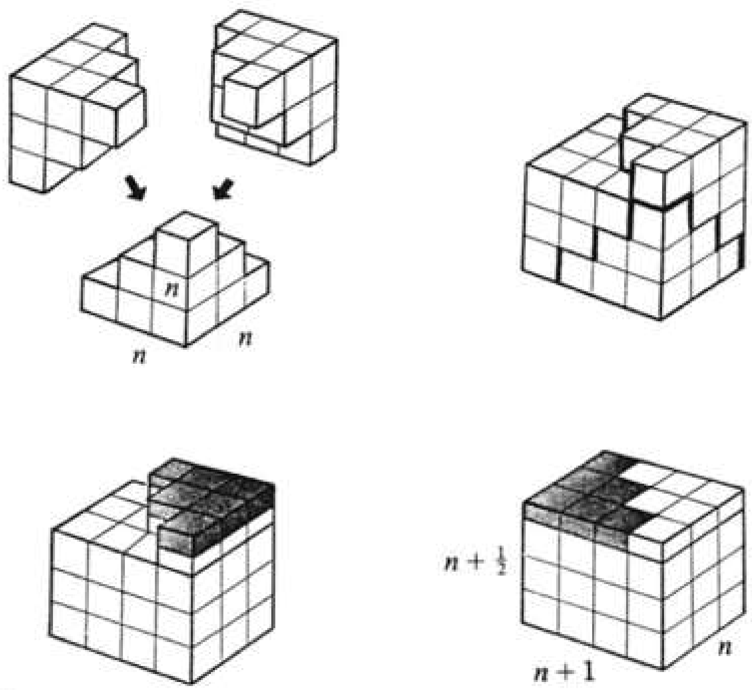
33 @Max: The inductive step is easy to figure out, since the rectangle above contains the rectangles from previous steps. – Daniel Litt Mar 16, 2011 at 20:01

147

This is elementary as well, but one of my favorite ones :)

$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$

(Author: Man-Keung Siu)



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edited Jul 27, 2017 at 11:17

community wiki
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Mike

8 There's an analogous proof that the integral of n^2 from 0 to x is x^3/3. It can be obtained from this proof by smoothing out the stepped pyramids into actual pyramids. – Michael Lugo Dec 14, 2009 at 16:47

68 I think very few people have enough spatial imagination to figure out what happens exactly in the area where the three pieces come together, or could easily depict the structure seen from the opposite end. For me the *picture* is not convincing at all (I'd rather say the formula convinces me the picture is correct than the other way round). However maybe playing with an actual model would be quite convincing. – Marc van Leeuwen Dec 12, 2011 at 13:31

5 @Mark - I think if you just think about the width of each step at each level, you will be able to see that they do all fit together. Just counting back along a given row or column shows you that it all fits. – Steven Gubkin Feb 15, 2012 at 15:10

4 A variant of Mike's construction for $\sum_{k=1}^{n} k^2$, easier to visualize (I'm going to try a proof-without-words, without pictures). Take 6 copies of each parallelepiped of size $k \times k \times 1$. Glue them together so as to make the four lateral walls of a parallelepiped of (external) size $k \times (k+1) \times (2k+1)$. Do this for k from 1 to n, forming a collection of bracelets. Insert each one in the next, like matrioskas, getting a whole parallelepiped of size $n \times (n+1) \times (2n+1)$. – Pietro Majer Apr 10, 2013 at 10:48

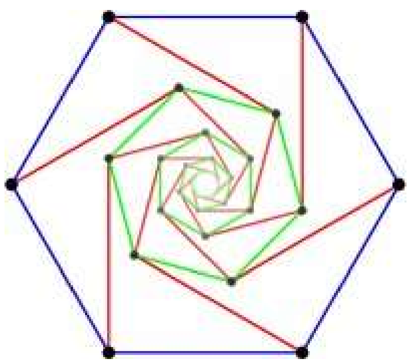
1 @Michael Lugo: The continuous version of this proof is "elementary geometry": the volume of a pyramid is one third of its height times the area of its ground surface! – nsrt Mar 18, 2014 at 13:09

136

It's a long list of wonderful answers already, but I can't resist...

Question: Is it possible to find six points on a square lattice that form the vertices of a regular hexagon?

Proof without words:



Hint: A square lattice is invariant under rotation by π/2 around any lattice point. Use reductio ad absurdum.

Credit: I learned that proof from György Elekes during the Conjecture and Proof course in the Budapest Semesters in Mathematics, after constructing a proof of my own that used entirely too many words and made very laboured use of the fact that $\sqrt{3}$ is irrational. The picture here is my own creation (using Asymptote).

Follow-up: Can you find four points on a hexagonal lattice that form the vertices of a square? The proof is similar but not immediate.

Share Cite Improve this answer Follow

edited Mar 10, 2017 at 9:42

community wiki
4 revs
Vaughn Climenhaga

8 Why would you resist? – Mariano Suárez-Álvarez May 20, 2010 at 17:41

6 +1 for the "Conjecture & Proof" shout-out. Best, course, ever! – Kevin O'Bryant Nov 10, 2010 at 23:18

1 Igen, nagyon jó. – Douglas Zare Feb 7, 2011 at 5:08

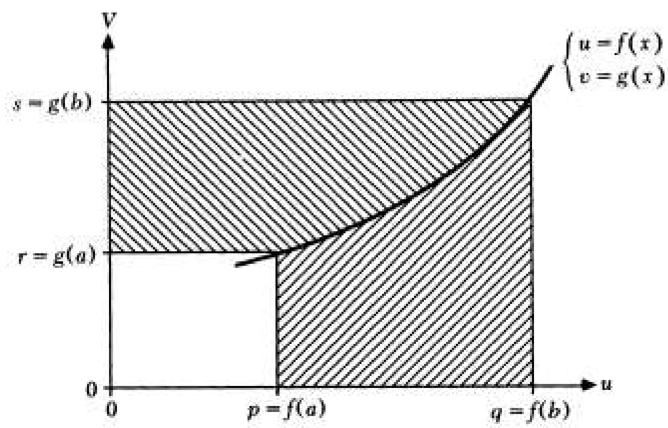
1 I really like this image and proof. The same idea works for other regular polygons, and I made images for pentagons, heptagons and so on, which you can find on my blog post at jdh.hamkins.org/no-regular-polygons-in-the-integer-lattice. (The code accept n as input, and makes the image for an n-gon.) – Joel David Hamkins Dec 4, 2016 at 16:24

3 And here is my post on the hexagonal lattice: the only regular polygons to be found are triangles and hexagons. jdh.hamkins.org/no-regular-polygons-in-the-hexagonal-lattice – Joel David Hamkins Dec 9, 2016 at 3:48

124

This might be trivial but integration by parts has a nice proof without words:

Proof without Words: Integration by Parts



Area ▧ + Area ▨ $= qs - pr$

$$\int_r^s u\,dv + \int_p^q v\,du = uv\Big|_{(p,r)}^{(q,s)}$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

—Roger B. Nelsen
Lewis and Clark College
Portland, OR 97219

(Got from: Roger B. Nelsen, Proof without Words: Integration by Parts, Mathematics Magazine, Vol. 64, No. 2 (Apr., 1991), p. 130; the original link is https://www.maa.org/sites/default/files/Roger_B04151._Nelsen.pdf).

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edited Dec 23, 2020 at 8:54

community wiki
5 revs, 5 users 47%
Daniel Parry

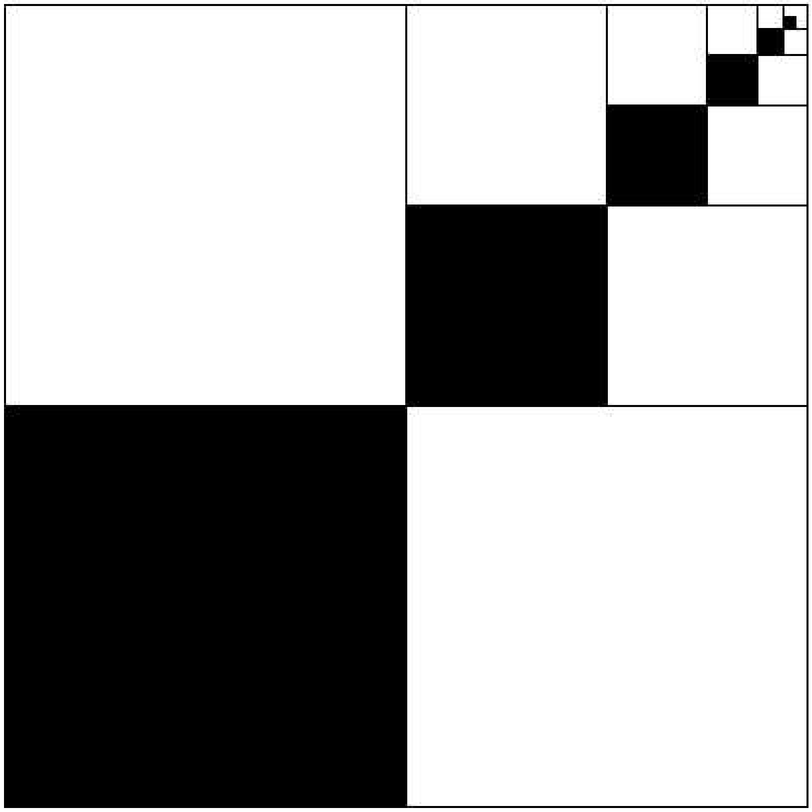
3 @Daniel, I've turned the PDF into a PNG, and inserted the relevant part. I did keep the URL to the PDF for reference. Thanks, by the way! – Mariano Suárez-Álvarez Feb 7, 2011 at 2:55

10 The same picture also gives an interesting formula for the integral of an inverse function! – Matt Noonan Jun 29, 2011 at 0:57

6 I guess this proof works only when f and g are both increasing? – Greg Martin Nov 19, 2015 at 19:16

1 @GregMartin : The way this drawn assumes that $f(b) > f(a)$ and $g(b) > g(a)$, but that's all (and you can draw similar pictures for the other possibilities). If they're not increasing the whole way, then there will be stuff outside the shaded regions, but it will count both positively and negatively and so cancel. Thus, the shaded regions' areas do equal the stated integrals. – Toby Bartels Dec 7, 2020 at 18:34

122



This visual proof of

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}=\frac{1}{3}$$

is from http://www.cecm.sfu.ca/~loki/Papers/Numbers/ (Visible Structures in Number Theory, by Peter Borwein and Loki Jorgenson, The American Mathematical Monthly, vol. 108, no. 5, 2002, pp. 897-910).

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edited Jul 27, 2017 at 11:25

community wiki
4 revs, 2 users 91%
Zurab Silagadze

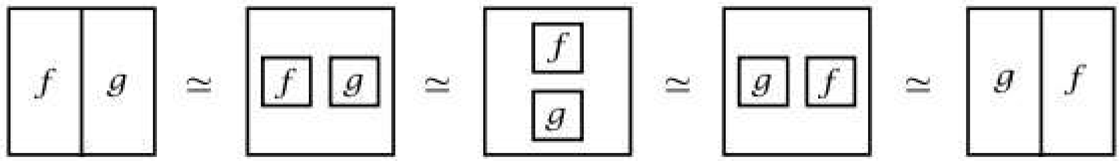
13 This proof is actually known to Archimedes and used in his *Quadrature of the Parabola* en.wikipedia.org/wiki/1/4_%2B_1/16_%2B_1/64_%2B_1/... – Machinato Jul 18, 2016 at 12:48

1 Visual recursion. Awesome. – Manuel Bärenz Mar 6, 2017 at 17:46

Pierre Arnoux made a nice video about the geometric series, which has a version (with colours and animation) of this picture: youtu.be/6KQiTJLBwEw The video has words though! – Samuel Lelièvre Sep 12, 2018 at 8:52

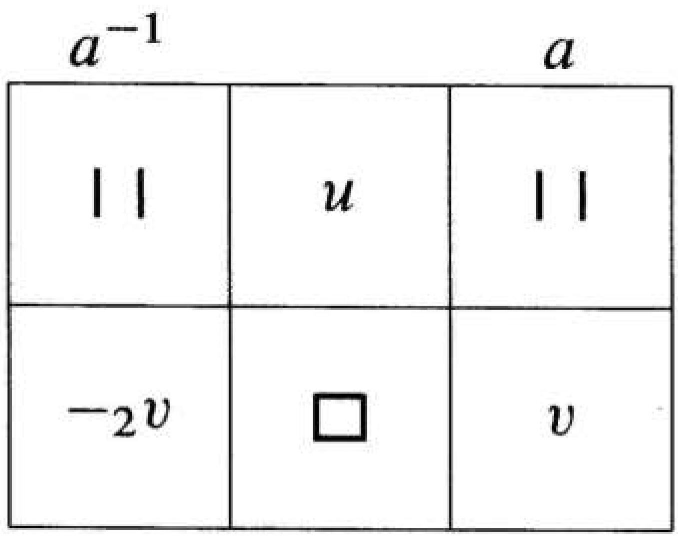
114

There's a picture proof in the *Princeton Companion*, or alternatively on p. 340 of Hatcher, of the fact that the higher homotopy groups are abelian. Actually, here's a screenshot of the one in Hatcher (hopefully fair-use!):



Here f and g are mappings (with basepoint) of S^n into some space for $n > 1$; the picture shows a homotopy between $f + g$ and $g + f$.

The above diagrams show an application of the interchange law, a more general expression of the Eckmann-Hilton argument, for double categories or groupoids. Here is a more general picture



which shows that the interchange law for a double groupoid implies the second rule $v^{-1}uv = u^{\delta v}$, where in the picture $a = \delta v$, for the crossed module associated to a double groupoid, taken from the book advertised here. There are many 2-dimensional rewriting arguments which are essential to the results of this book.

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edited Dec 23, 2020 at 8:55

community wiki
5 revs, 4 users 32%
Harrison Brown

6 Page 340 of Hatcher's book: math.cornell.edu/~hatcher/AT/AT.pdf – Dan Piponi Dec 14, 2009 at 18:27

14 This is sometimes called the Eckmann-Hilton argument: en.wikipedia.org/wiki/Eckmann%E2%80%93Hilton_argument – Kevin H. Lin Dec 14, 2009 at 20:46

3 I've heard that term, but I've never quite understood how the diagram is supposed to prove the more general abstract nonsense theorem. But if you can explain it, that's what community wiki's for! :D – Harrison Brown Dec 14, 2009 at 20:53

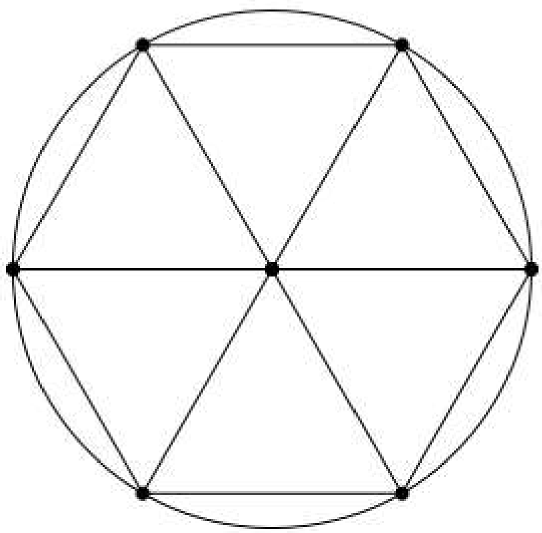
5 There are lots of places on the web where this is explained nicely: youtube.com/watch?v=Rjdo-RWQVIY , math.ucr.edu/home/baez/week258.html , ncatlab.org/nlab/show/Eckmann-Hilton+argument , etc....

– Kevin H. Lin Dec 14, 2009 at 23:50

The "more general picture" seems to be broken. – Gerry Myerson Mar 18, 2014 at 22:15

107

$$2\pi > 6$$



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edited Jun 17, 2014 at 2:52

community wiki
2 revs, 2 users 62%
muad

13 And similarly one proves that $\pi < 4$ by inscribing a circle in a square. – Michael Hardy Nov 16, 2010 at 21:46

27 At first I was thrown off by this, because I was looking at area and not circumference. The area of an inscribed regular *12-sided* polygon in the unit circle is also 3. – Todd Trimble ♦ Mar 12, 2011 at 22:07

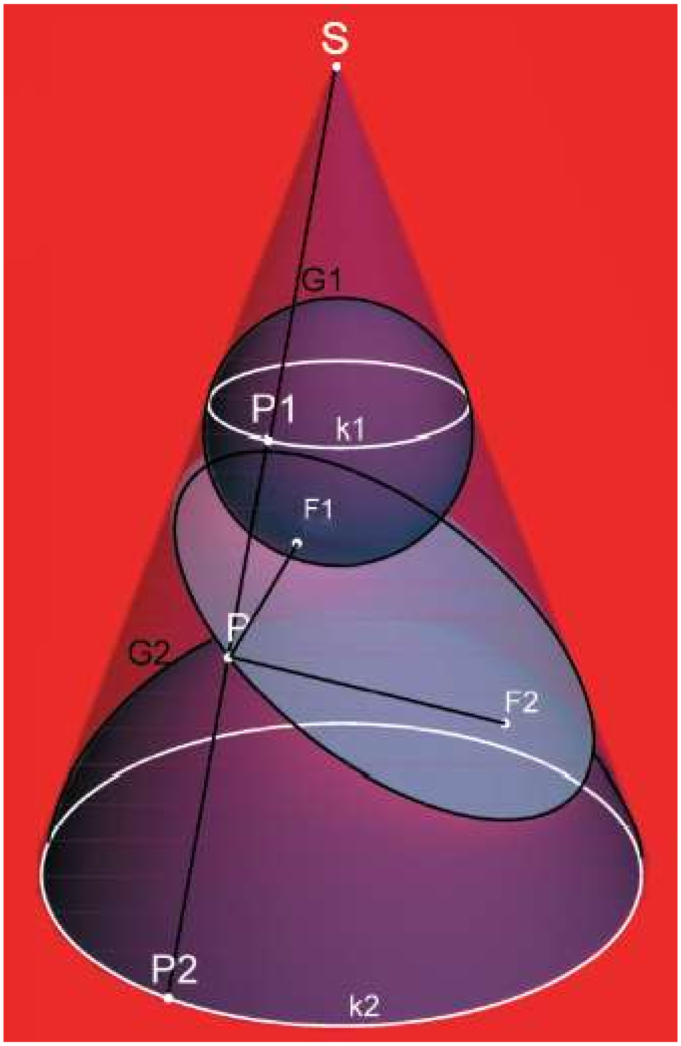
jstor.org/stable/10.5951/mathteacher.105.8.0632 – Benjamin Dickman Dec 2, 2012 at 8:48

Helpful to remember that pi is ratio of circumference of the circle to diameter. – Talespin_Kit Dec 21, 2016 at 15:54

9 Another case for τ. – Daniel R. Collins Jul 25, 2018 at 16:53

106

I'm partial to the proof using Dandelin spheres that (certain) cross sections of cones are ellipses, where an ellipse is defined as the locus of points whose total distance to two foci is constant. It's particularly nice because it explains the foci geometrically, as well as the focus-directrix property with some more work.



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edited Jun 17, 2014 at 2:47

community wiki
2 revs, 2 users 80%
aorq

1 Yes, this one is beautiful. – Andrés E. Caicedo May 15, 2010 at 18:47

3 How does the picture explain the invariance of the total distance to two foci? I don't see it ; I haven't done geometry in a while though, I'm guessing it's some triviality... refresh my memory please? :) – Patrick Da Silva Jul 28, 2013 at 18:35

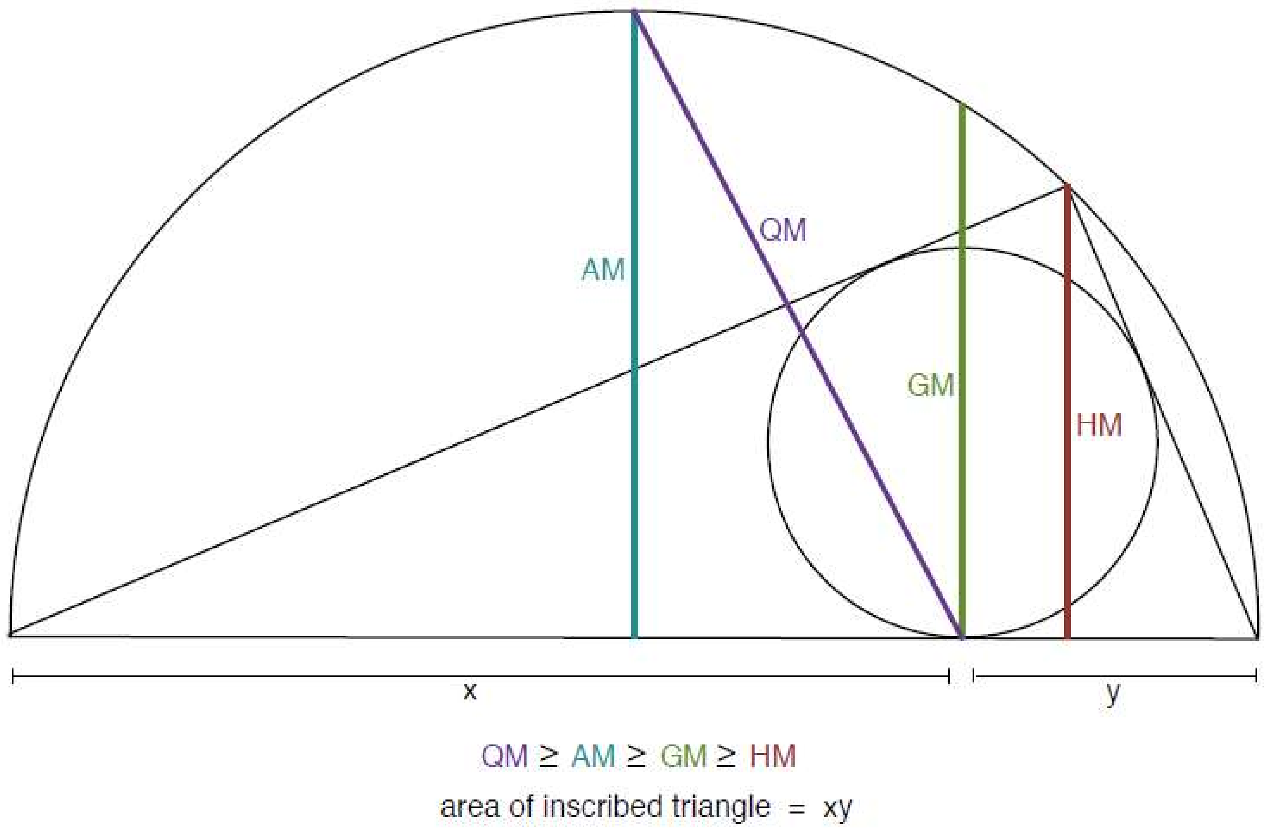
10 @PatrickDaSilva: $PF1 = PP1$ because tangents to a circle/sphere have equal length. The total distance is thus equal to $PF1 + PF2 = PP1 + PP2 = P1P2$, which is constant. – aorq Jul 29, 2013 at 8:53

1 I've learned this and related proofs from Hilbert and Cohn-Vossen (but these proofs still originated mostly with Dadelin). – Włodzimierz Holsztyński Nov 9, 2013 at 3:21

3 I was confused by the perspective. In case anyone is having the same problem: the perspective is from a point that is (below the apex S of the cone but) *above* the base of the cone (circle $k2$) and the bottom half-sphere ($G2$). I mistakenly thought we were looking up through $k2$ into the inside of the cone -- I think this is because in my browser, at least, the the circle $k2$ gets *thicker* when it passes behind the ellipse and should if anything get *thinner*. It's generally a nice drawing, though, and a nice proof. – Tim Campion ♦ Mar 5, 2014 at 22:49

95

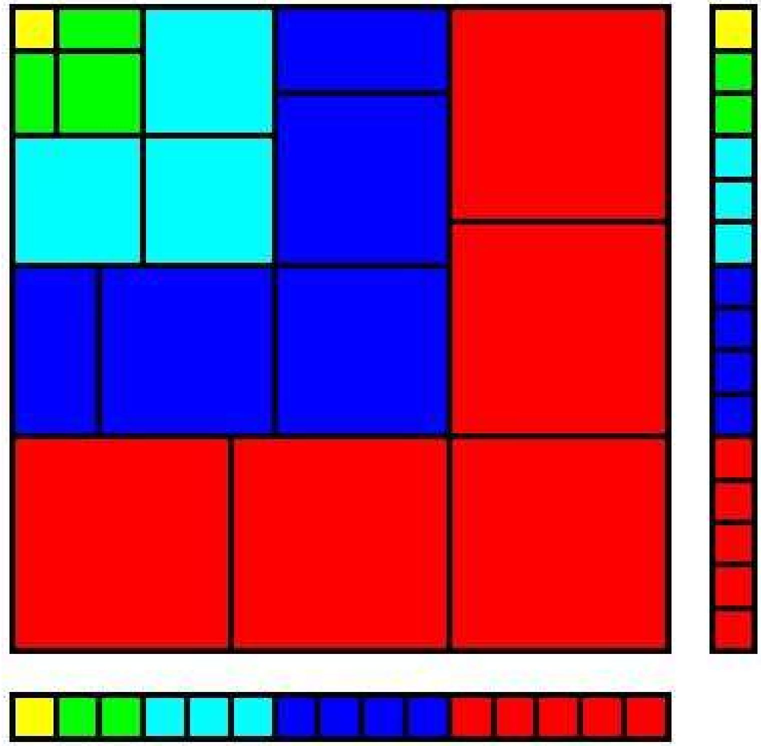
Means inequalities:



The image was sent to me by James M. Lawrence, grazie! See also page 53 of "Proofs without words: exercises in visual thinking, Volume 2" for a very different layout of the same 4 inequalities.

Another one exists involving the sum

$$1^3 + 2^3 + \cdots + n^3 :$$



The second image is due to Brian Sears (Wayback Machine)

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edited Dec 23, 2020 at 8:54

community wiki
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Dave Pritchard

2 I used the second proof (involving sum of cubes) in my class today after proving it by induction. A few were quite inspired by it! – Somnath Basu Feb 24, 2012 at 18:42

2 2nd proof: It would be nicer if the small strips were above and to the left of the big square. – Günter Rote Feb 25, 2013 at 22:56

4 The first proof could use some words. How is HM constructed? What is the small circle for? How does one prove that those segments have the claimed lengths? – Federico Poloni Apr 19, 2014 at 16:04

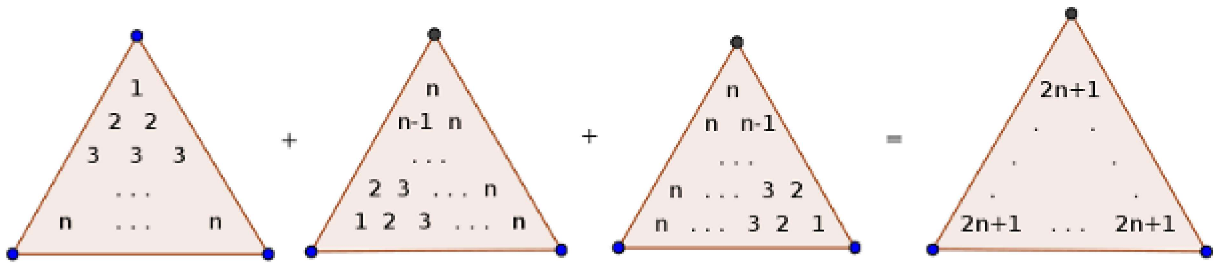
1 For completeness, since a link is still missing: Mariano Suárez-Álvarez has given a beautiful improved version of the second image here: math.stackexchange.com/q/61483 – Peter Heinig Feb 24, 2018 at 13:20

1 I didn't hear about "Means inequalities" until today and I don't know what this is useful for, but I have re-created the first image in GeoGebra, so you can drag around the point E to get a better feel for the line lengths: geogebra.org/classic/ndrfsstq – Fabian Röling Feb 16, 2020 at 2:26

88

Another proof of the sum of the first n squares, relying on the knowledge of the formula for the sum of the first n numbers:

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$



This one has a similar flavor to the fabled proof by Gauss of the sum of the first n numbers. It's a good follow up for students after Gauss's proof.

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edited Jun 17, 2014 at 2:56

community wiki
2 revs, 2 users 75%
JeremyKun

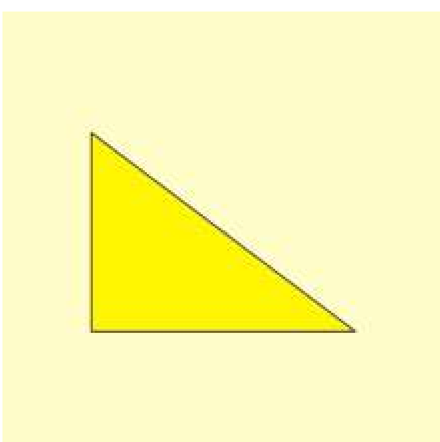
There's probably a nice three-dimensional rendition of this that doesn't require writing down all those numbers. – Michael Lugo Aug 16, 2012 at 23:03

11 +1 Superb. Is this original? If not, to whom is it attributed? – I. J. Kennedy Nov 22, 2013 at 2:05

2 @MichaelLugo the 3D rendition is elsewhere on this page mathoverflow.net/q/8851 (you commented on it), but I do prefer this version. – adl Jan 12, 2016 at 16:21

83

Wikipedia has a few nice proofs of the pythagorean theorem. Elementary, but elegant.



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edited Mar 10, 2017 at 9:42

community wiki
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Steve Flammia

27 Pythagoras' theorem is trivial? I had no idea ... Seriously, I don't necessarily think that the existence of a very simple proof implies triviality. Such proofs are, after all, not so easily discovered. Anyway, this is my favourite proof of the theorem. – Harald Hanche-Olsen Dec 14, 2009 at 20:58

4 The 20th President of the US, James Garfield, independently discovered the proof obtained by halving the right-hand diagram along a diagonal of the square of side length c. It requires you to write down an

equation, though. That's my favorite proof, but mostly because of the corollary that B. Obama isn't the first geeky POTUS. – Harrison Brown Dec 15, 2009 at 3:23

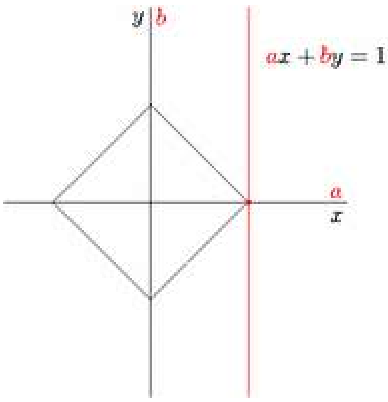
14 @HB: Um, Thomas Jefferson? – Pete L. Clark Mar 6, 2010 at 3:23

17 A typical fake proof --- a simple statement as Pythagorean theorem is proved using much more advanced theorem on existence of area... – Anton Petrunin Nov 30, 2010 at 20:26

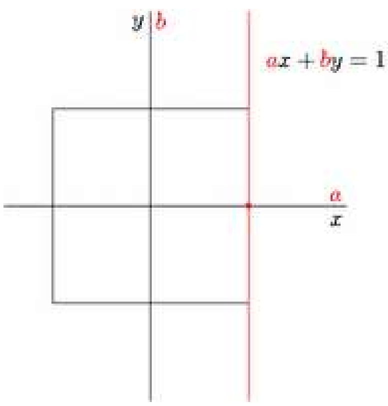
32 A typical fake refutation. You don't need to define Lebesgue measure to do manipulations in geometry. All operations can be defined geometrically if I associate a number X with the segment of length X, and define $X \mapsto X^2$ as a function, mapping a segment to a square with such side. In fact, even many of infinite summations can be done geometrically, using the obvious topology and metric on shapes. Thanks to this formalistic tradition it took 100 years of pain to get from non-trivial Lebesgue construction to much more natural motivic integration. – Anton Fetisov Nov 13, 2011 at 10:38

79

Duality between ℓ^1 and ℓ^∞ norms.



and the reverse animation



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edited Jun 17, 2014 at 2:48

community wiki
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Mariano Suárez-Alvarez

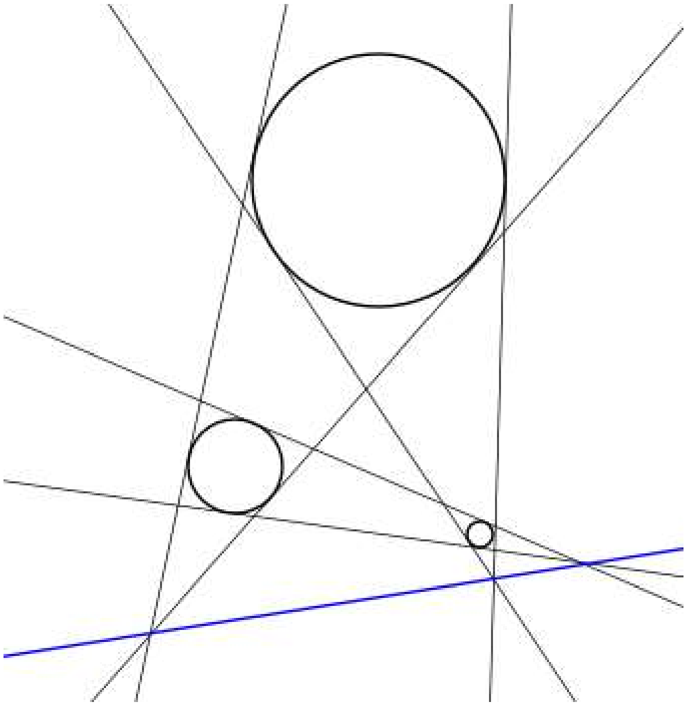
Mariano, thanks for fixing up my post! – Igor Khavkine Dec 16, 2009 at 13:56

27 I... don't quite get it. I think I need a few more words: What's the dot representing in each picture? – Harrison Brown Dec 16, 2009 at 15:01

14 The red line in xy-space satisfies the given equation. The dot gives the (a,b) coordinates of the same line in ab-space. The xy- and ab-spaces are linearly dual to each other. The resulting black and red shapes represent the unit balls in respective norms. – Igor Khavkine Dec 16, 2009 at 15:34

76

If we have 3 circles on the plane with tangent lines, we can notice they have colinear intersection!



To prove it, we can visualize the same configuration in 3D, the balls lay on a surface and rather than tangent lines we take cones: The colinearity comes from the fact that if we lay a plane ontop of this configuration it will intersect the table in a line!

This is from 'curious and interesting geometry' and the proof is attributed to John Edson Sweet. I really like this proof because it gives a vivid example of the general idea that sometimes, to solve a problem in the most simple way you need to view it as a part of some bigger whole.

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edited Aug 17, 2015 at 17:37

community wiki
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muad

28 You need to draw a 3D picture of this to get rid of the words! – Ian Agol Jun 28, 2011 at 16:39

Don't we need the cones to all have the same slope? – benblumsmith Jan 23, 2012 at 13:39

2 In this pretty solution there is another pretty geometric problem: Given three spheres there is a plane which is tangent to all three. – Rogelio Fernández-Alonso Jan 29, 2012 at 16:51

5 Where is the picture? – Patrick Da Silva Jul 28, 2013 at 18:29

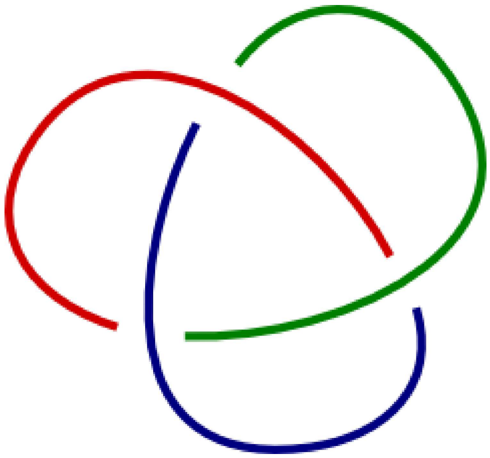
1 This was one of my favorite proofs in this list... it's a shame that imageshack took this picture off to promote their site. – KalEl Nov 7, 2014 at 22:36

I'm quite surprised no-one pointed out this one yet:

Theorem. The trefoil knot is knotted.

73

Proof.



$\square$

Some comments: a *3-colouring* of a knot diagram D is a choice of one of three colours for each arc D, such that at each crossing one sees either all three colours or one single colour. Every diagram admits at least three colourings, *i.e.* the constant ones. We'll call *nontrivial* every 3-colouring in which at least two colours (and therefore all three) actually show up. It's easy to see (one theorem, more pictures!) that Reidemeister moves preserve the property of having a nontrivial 3-colouring, and that the unknot doesn't have any nontrivial colouring.

The picture shows a (nontrivial) 3-colouring of the trefoil.

EDIT: I've made explicit what "nontrivial" meant — see comments below. Since I'm here, let me also point out that the *number* of 3-colourings is independent of the diagram, and is itself a knot invariant. It also happens to be a power of 3, and is related to the fundamental group of the knot complement (see Justin Robert's [Knot knotes] if you're interested).

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edited Jun 17, 2014 at 2:54

community wiki
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Marco Golla

This is wonderful. – Kevin H. Lin Jun 16, 2011 at 19:28

What does "nontrivial" mean? – Tom Goodwillie Jun 28, 2011 at 15:11

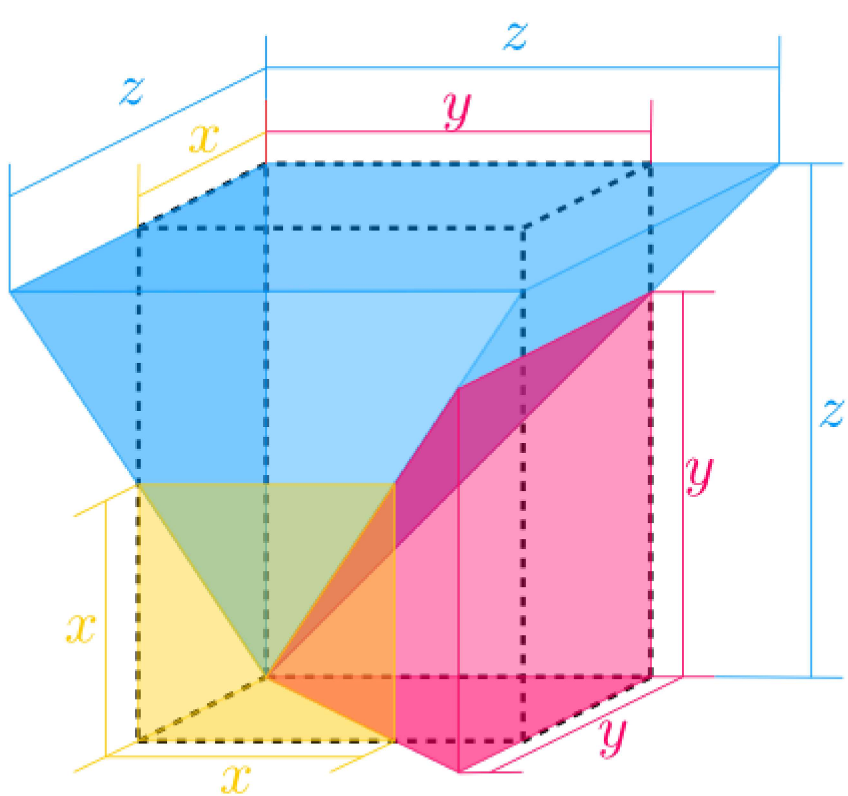
@Tom Goodwillie: I've edited, and added some remarks. Thank you. – Marco Golla Jun 28, 2011 at 20:39

70

Here's a proof of the inequality of the arithmetic and geometric means in the form

$$\frac{x_1^n}{n} + \cdots + \frac{x_n^n}{n} \geq x_1 \cdots x_n.$$

Proof for $n = 3$:



$$\frac{x^3}{3}+\frac{y^3}{3}+\frac{z^3}{3}\geq xyz$$

The "figure" for general n is similar, with n right pyramids, one with an $(n-1)$-cube of side length x_k as its base and height x_k for each $k=1,\ldots,n$.

(I made this in Inkscape, a wonderful free-software vector drawing application. For the inequality and associated labels, I used the textext extension.)

Share Cite Improve this answer Follow

edited Oct 29, 2015 at 19:55

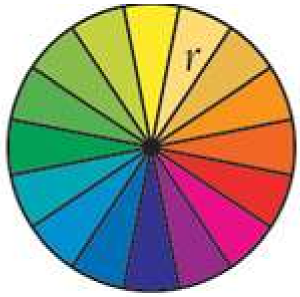
community wiki
6 revs, 5 users 83%
Darsh Ranjan

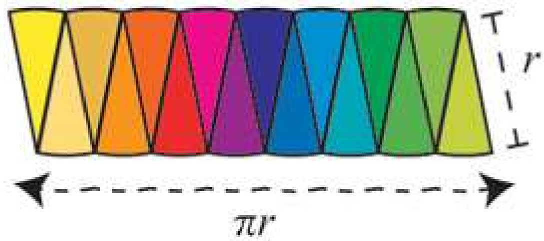
3 And what exactly is a proof about this? – darij grinberg Nov 10, 2010 at 23:40

9 The box has volume xyz and is contained in the union of the three square pyramids, which respectively have volumes x^3/3, y^3/3, and z^3/3. Thus xyz <= x^3/3 + y^3/3 + z^3/3. – Darsh Ranjan Nov 11, 2010 at 3:41

from Steven Strogatz's column: http://opinionator.blogs.nytimes.com/2010/04/04/take-it-to-the-limit/ (Wayback Machine)

61





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edited Dec 23, 2020 at 8:50

community wiki
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AndrewLMarshall

21 Nice, but that reminds me of the "proof" of $2 = \pi$ by approximating a straight line of length 2 by starting with a circle with this line as diameter, then two circles with one half of the line as diameter each, then for circles with on quarter of the line as diameter, ... One still has to find an argument that a geometric process converges at all and converges to the desired result. Both cannot be deduced purely from looking at a picture. – Johannes Hahn Nov 8, 2010 at 11:27

3 Hmm, not sure, the point behind a proof by picture is that you do "get it," i.e., you see how the argument works in its full rigor. Now, either you do or you don't, but in this case I think it's all there. With circular arcs approximating a straight line you might notice upon observation that the arc length is independent of the iterations, which immediately discounts convergence... – AndrewLMarshall Nov 10, 2010 at 6:21

1 By contrast, here you might observe that the difference between, say, how 2 circular wedges differ from their triangular counterparts in ratio, and how a wedge of twice the size differs from its triangular counterpart in ratio, does give on the order of geometric convergence. You can more or less just see that. – AndrewLMarshall Nov 10, 2010 at 6:21

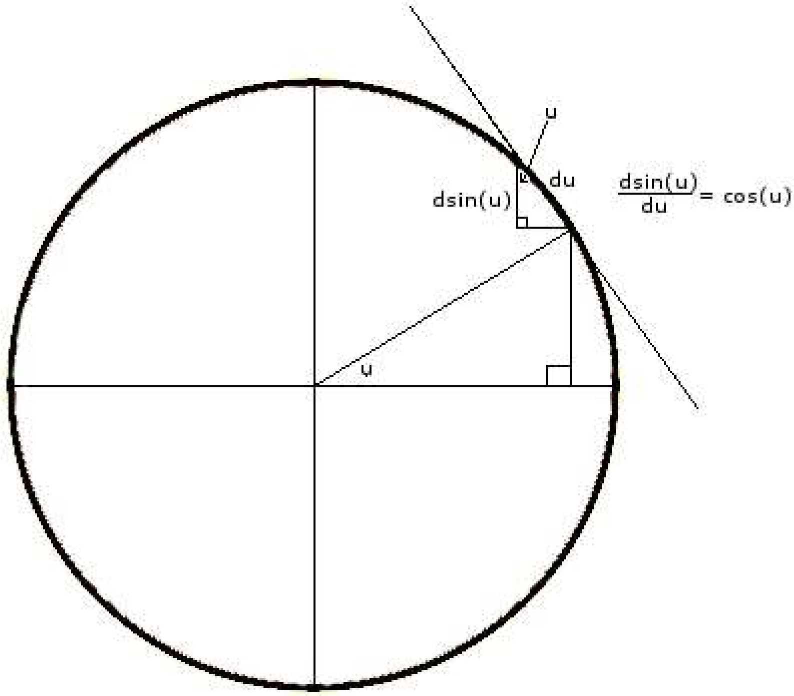
6 Wikipedia attributes this proof to Leonardo da Vinci. You can make establish rigorous convergence by using triangles that inscribe and circumscribe the wedges. – S. Carnahan ♦ Nov 11, 2010 at 3:04

2 Hah, this is actually the proof appear in my primary school textbook. (I went to primary school in China, it was like 6th or 5th year) I'm amazed by this proof, but I'm not sure many kids can remember this though. – temp May 30, 2012 at 2:40

58

Here is the very first piece of original mathematics I ever did, in high school:

The derivative of sine is cosine.



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edited Nov 19, 2015 at 17:14

community wiki
4 revs, 2 users 85%
Steven Gubkin

6 It looks like your image is no longer available... – I. J. Kennedy Nov 16, 2010 at 19:09

3 Leibniz actually did this drawing. It's very nice because you can teach it to undergrads. You can do the same with any of the trig functions and their inverses. For tangent, you can extend the hypotenuse of the above triangle until it intersects the line tangent at the point 1 (assuming this is the unit circle in the complex plane). Then you get a triangle with base 1, height tangent, and hypotenuse secant. For cosecant and cotangent, you draw a tangent line from the point i. Then through similar triangles you can differentiate all these functions and their inverses. – Phil Isett Jul 8, 2011 at 21:00

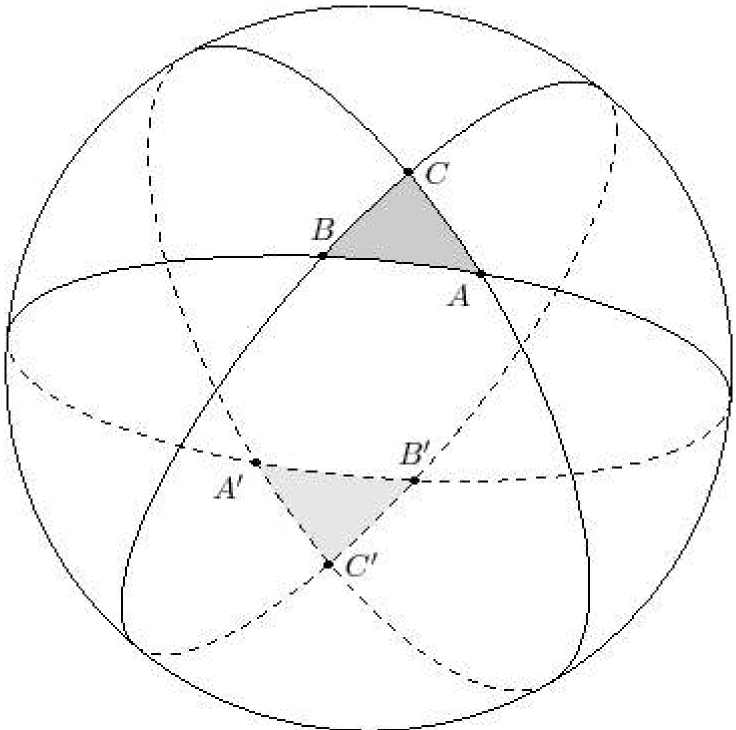
1 Yup, I have drawn pictures for all of those as well, but this always seemed like the simplest one. Never understood why this isn't in calculus books. – Steven Gubkin Jul 10, 2011 at 15:41

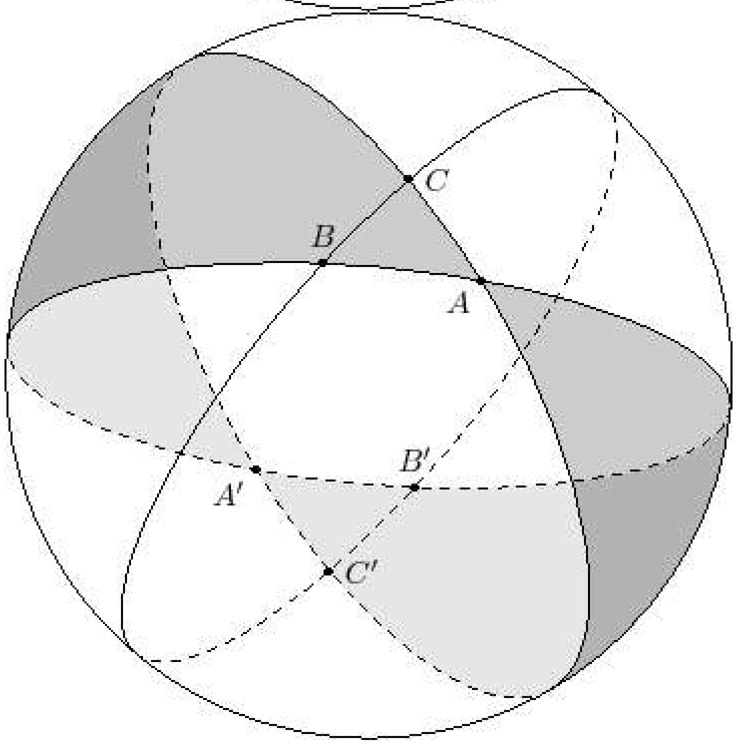
3 @StevenGubkin If you still have it, could you put the picture back in? No one can see it. – Todd Trimble ♦ Oct 20, 2015 at 16:05

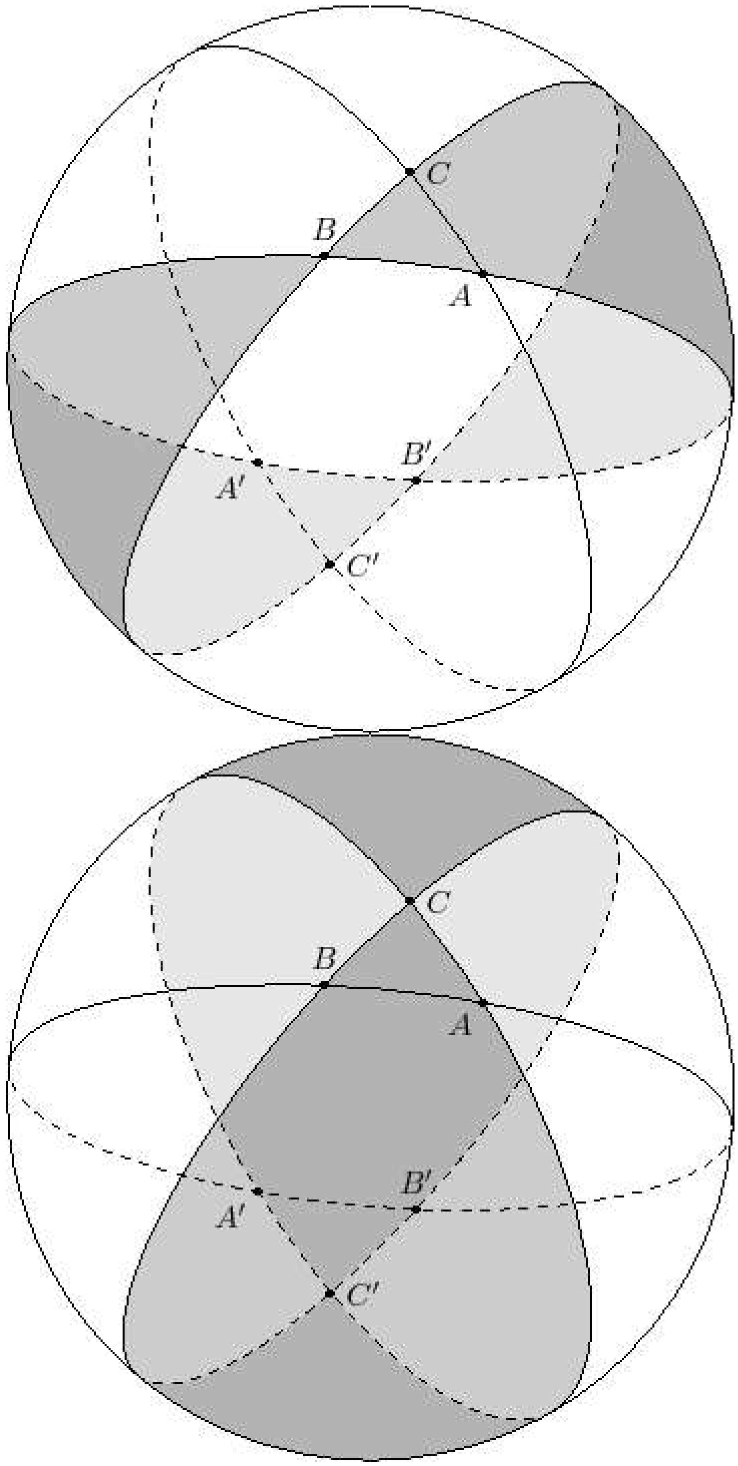
1 This post is very much related to this. – Simply Beautiful Art Nov 17, 2016 at 0:46

53

The sequence of pictures


C
B
A
B'
A'
C'


C
B
A
B'
A'
C'



proves the area formula for spherical triangles $\mathrm{area}(ABC) = A\hat{B}C + B\hat{C}A + C\hat{A}B - \pi$.

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edited Dec 6, 2019 at 14:36

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CuriousUser

3 Thomas Harriot first proved this formula in 1603, apparently by a similar argument, though I have not seen his picture(s). – John Stillwell Feb 22, 2010 at 22:31

7 Haha, I'm happy to see these illustrations useful to someone! I created them some years ago, mainly to crystalize what I saw in my minds eye after finding some simple proofs of this identity online. The words

accompanying these images can be found at planetmath.org/encyclopedia/AreaOfASphericalTriangle.html Also, original MetaPost source can be obtained from this unfortunately obscure link: images.planetmath.org:8080/cache/objects/5841/src/sph-tri.mp – Igor Khavkine Apr 26, 2010 at 20:55

Incidentally, I tried to find a similar proof for the area formula for hyperbolic triangles. Unfortunately, that did not work due to non-compactness of hyperbolic space. If anyone knows whether such a proof exists, I'd be happy to see it. – Igor Khavkine Apr 26, 2010 at 20:57

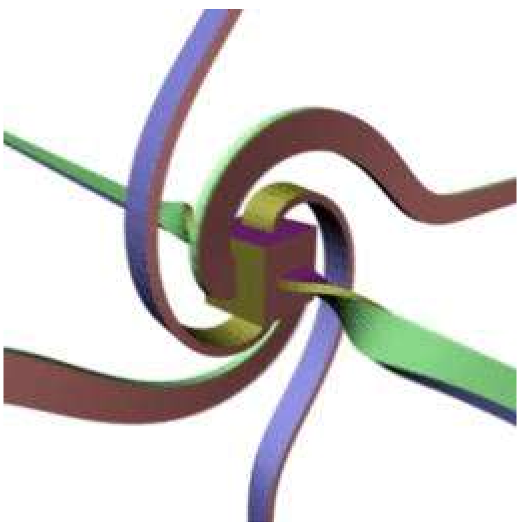
3 There is an analogous proof using the fact that although the hyperbolic plane has infinite area, a triply asymptotic triangle has finite area, so once you pick one of the two triply asymptotic triangles containing your triangle, you're in business. The relevant picture's in my answer posted separately (I posted it before I had the reputation to leave comments): mathoverflow.net/questions/8846/proofs-without-words/... – Vaughn Climenhaga May 18, 2010 at 19:04

This same proof also appears at the very opening of this paper: arxiv.org/abs/1301.0352 – Yaakov Baruch Jun 8, 2016 at 14:44

45

Late to the party, but David Lehavi and Bob Palais both mentioned the proof that $\pi_1(SO(3))$ has an element of order 2. In fact it is the only nontrivial element, and so the double cover of $SO(3)$ is simply connected.

Here's an animated illustration of that fact, courtesy of Wikipedia (here):



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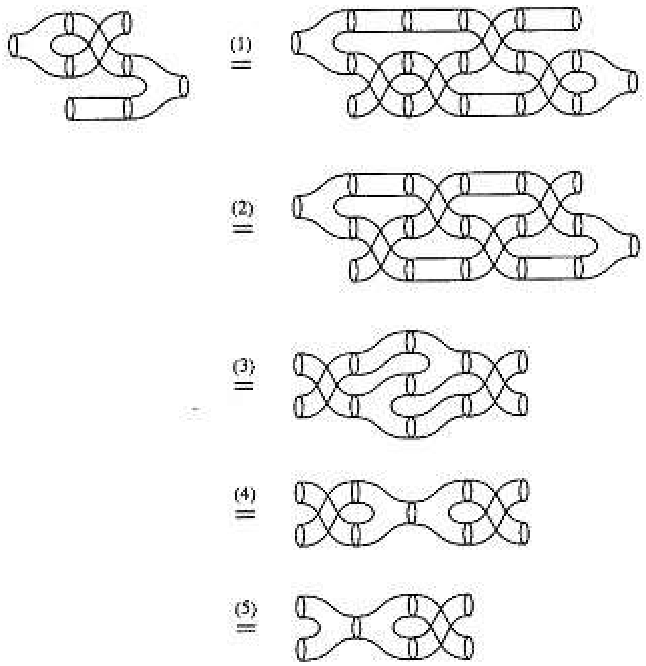
edited Dec 29, 2016 at 21:30

community wiki
3 revs
Robin Saunders

44

Algebraic manipulations in monoidal categories can also be performed in a graphical calculus. And the best part is that this is completely rigorous: a statement holds in the graphical language if and only if it holds (in the algebraic formulation). See for example Peter Selinger's "A survey of graphical languages for monoidal categories". There are many instances, for example in knot theory studied via braided categories. The following specific example comes from Joachim Kock's book "Frobenius Algebras and 2D Topological Quantum Field Theories", and proves that the comultiplication of a Frobenius algebra is cocommutative if and only if the multiplication is commutative.



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edited Dec 23, 2020 at 8:49

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Chris Heunen

The link to Selinger's paper wasn't working – Yemon Choi Feb 15, 2012 at 4:47

There are many proofs of similar flavor about 4-manifolds using the Kirby calculus. – Matt Brin Aug 15, 2012 at 14:04

Picture is dead – BlueRaja Jun 28, 2013 at 6:37

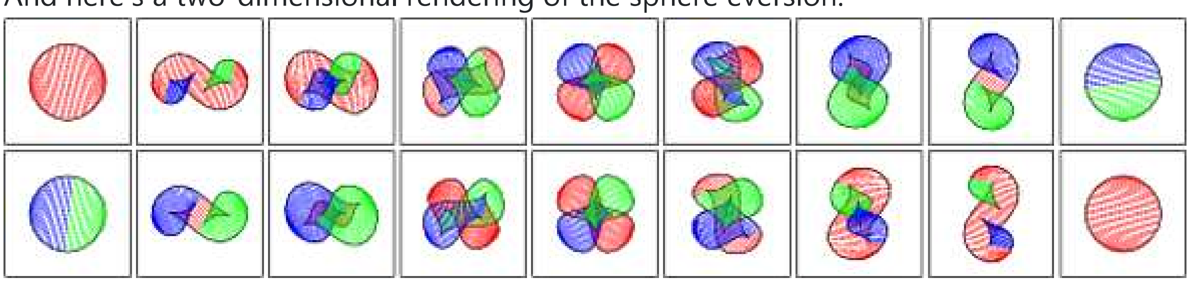
10 this proof makes me wonder what is a 'picture' and what is – Jeremy Apr 19, 2014 at 14:43

2 @LSpice ... notation. – Jeremy Nov 4, 2022 at 21:05

42

[Sphere eversion](https://en.wikipedia.org/wiki/Sphere_eversion)

And here's a two-dimensional rendering of the sphere eversion:



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edited May 21, 2021 at 21:24

community wiki

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4 revs, 4 users 40%
Ian Agol

19 As pretty as it is, that is nowhere understandable as a proof. More as an illustration. – Willie Wong Mar 11, 2010 at 16:44

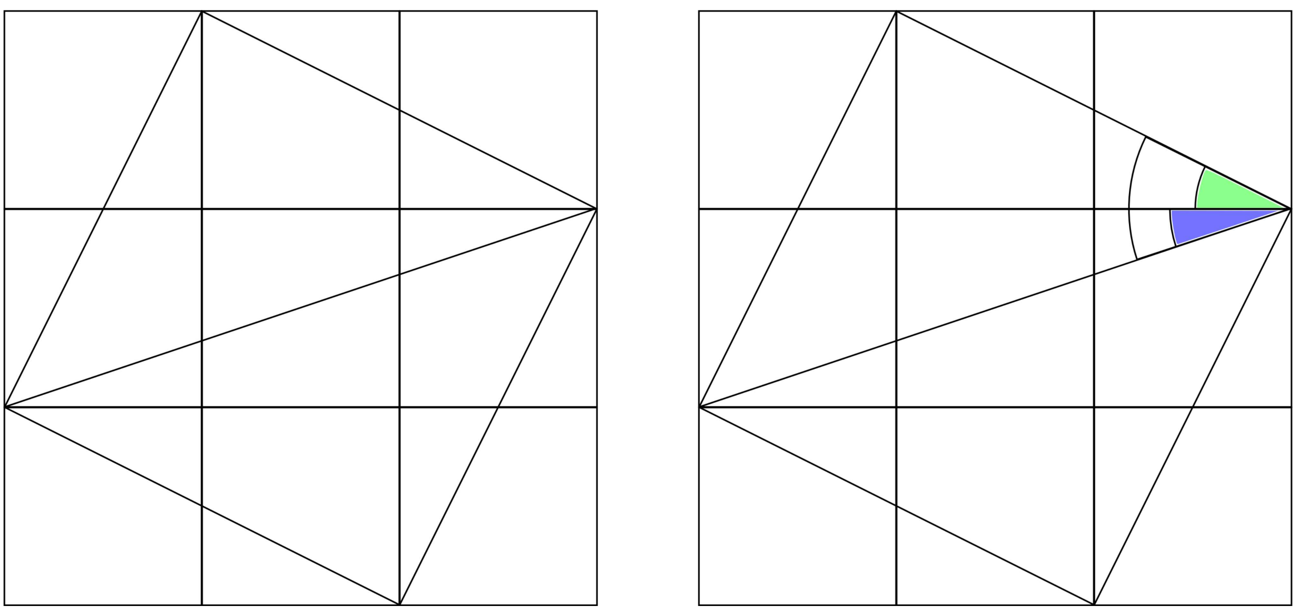
18 @Willie: Suppose someone wrote down the equations/formulas for the sphere eversion in that video. It seems to me that checking that the formulas indeed give a sphere eversion would be a rather difficult and tedious task, whereas a video animation is, although not a rigorous proof, much more immediately convincing. – Kevin H. Lin Apr 6, 2010 at 16:30

10 I just watched the video, which was excellent, but it had a lot of words in it. – Patricia Hersh Aug 19, 2012 at 0:23

2 The original picture has better image quality: andreghenriques.com/PDF/Eversion.pdf – André Henriques May 21, 2021 at 23:00

38

$$\arctan\frac{1}{3}+\arctan\frac{1}{2}=\arctan 1$$



It's easy to generalize this to

$$\arctan\frac{1}{n}+\arctan\frac{n-1}{n+1}=\arctan 1,\ \text{for } n\in\mathbb{N}$$

which can further be generalized to

$$\arctan\frac{a}{b}+\arctan\frac{b-a}{b+a}=\arctan 1,\ \text{for } a,b\in\mathbb{N}, a\le b$$

Edit: A similar result relating Fibonacci numbers to arctangents can be found here and here.

Share Cite Improve this answer edited Jun 8, 2016 at 6:45 community wiki

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2 revs
L.Z. Wong

3 It needed quite a long time for me to understand this. But, well, then it is amazing! – Gottfried Helms Oct 28, 2015 at 10:16

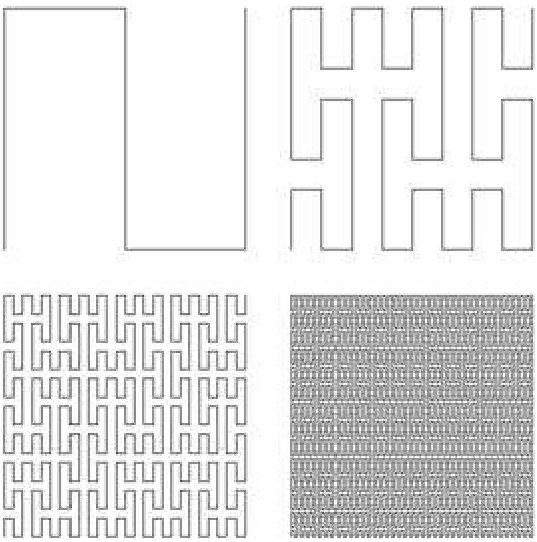
3 Very nice. It might be a bit more clear if the right triangle for arctan(1/2) and arctan(1/3) were colored in the first picture, and the triangle for arctan(1) was colored in the second picture. – Steven Gubkin Nov 17, 2016 at 2:57

37

A classic one, from the late 19th century, that surprized Peano's contemporaries.

Question : "A curve that fills a plane ? You must be kidding"

Answer :



Well, of course a formal proof was necessary, but it is still one of my favorites.

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edited Jun 17, 2014 at 3:00

community wiki
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Alexis Monnerot-Dumaine

41 How can you be sure that you're eventually covering all the points with irrational or transcendental coordinates? And giving a sequence of curves which fill more and more of the plane isn't the same as giving a single curve that does it all at once - it's not clear that such a limiting curve exists just looking at the pictures. – Michael Burge Sep 14, 2010 at 8:47

17 Existence of the limits object is something that is very often forgotten. For example most Introductions to fractals give geometric descriptions of Koch's snowflake etc. via such an iteration but don't prove that there exists a limit of this iteration. – Johannes Hahn Sep 14, 2010 at 9:22

12 Project: Fill the square one pixel at a time by following (an approximation to) this curve; then find some suitable baroque music accompaniment; then upload it to youtube. – Michael Hardy Nov 16, 2010 at 21:51

56 If you look at the picture in detail you can see that you are defining a sequence of continuous functions that converge uniformly. It's also clear from the picture that the image is dense. Therefore the limiting function exists and its image (being dense and compact) is the whole square. Of course, this proof isn't 100% visual but the non-visual part -- the basic facts about uniform convergence and compactness -- can be regarded as background knowledge. So I think it's a nice example. – gowers Apr 10, 2011 at 20:18

7 Contrary to gowers, I don't think it's clear that we are defining a sequence of continuous functions which converges uniformly. We aren't defining a sequence of functions at all, only a sequence of images of the interval under a function. If we choose the "wrong" sequence of parameterisations of this sequence of curves then we do not get a limit function. The sequence of parameterisations (which is not illustrated) is crucial to proving the existence of the limit object: without some indication of which parameterisations we must choose there is no proof. – Ian Morris Jun 26, 2014 at 7:45

35

This proof-without-words of the Pythagorean Theorem is far from a new one, but it's the first one I've ever seen 'in the wild' (this photo was snapped after finishing dinner at a Mongolian Grill restaurant):

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answered Dec 4, 2016 at 5:40

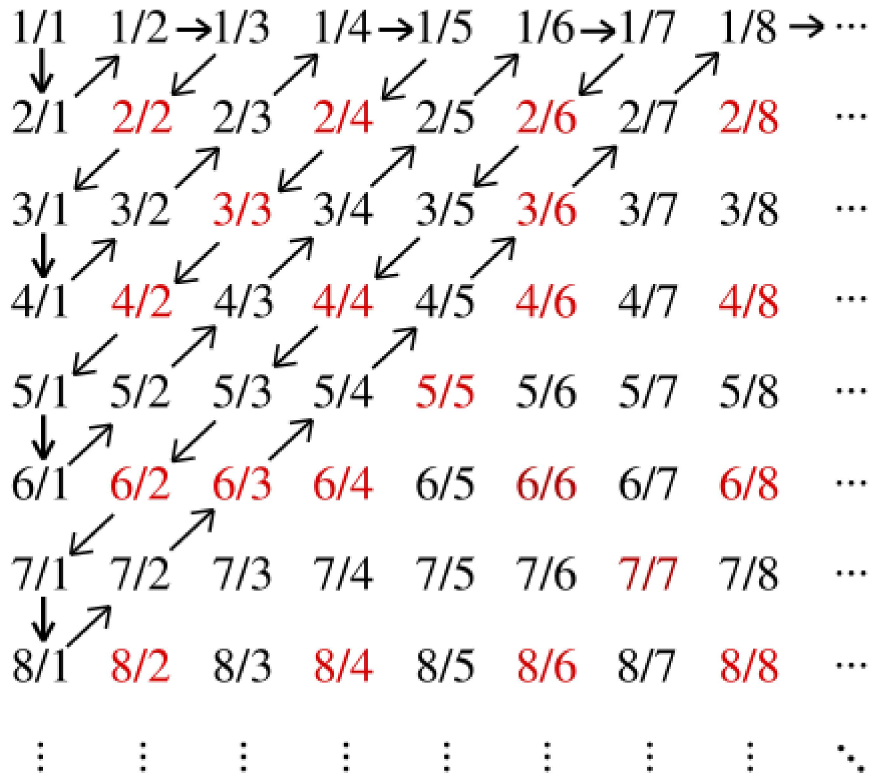
community wiki
Steven Stadnicki

31 I imagine people from the other tables, watching somebody while taking a picture of an *empty* plate!
– Pietro Majer Dec 4, 2016 at 7:31

Wait a minute -- since the triangles are tilted up toward the viewer, isn't this rather a *disproof* of the Pythagorean theorem? – Tim Campion ♦ Feb 3, 2022 at 1:31

31

I am surprised that no one had cited the "proof" that the rationals are countable yet. See, for example, this picture



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answered Mar 19, 2014 at 12:11

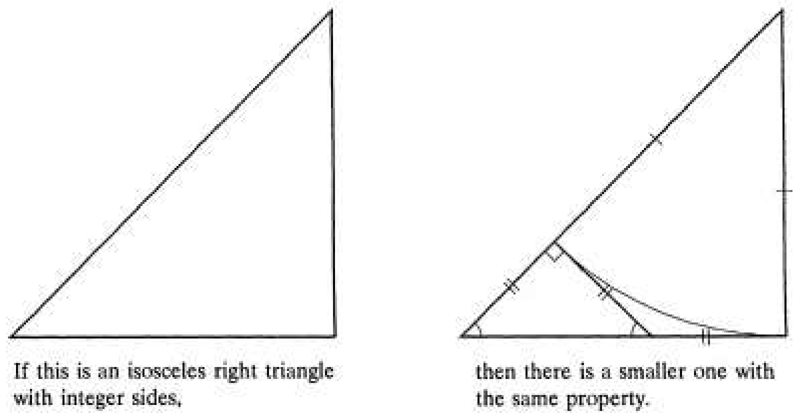
community wiki
Campello

Maybe it doesn't fit into the "non-trivial" category? – Campello Mar 19, 2014 at 12:12

9 I think that the fact that the rationals are countable qualifies as non-trivial, when put in historical perspective – Geoff Robinson Mar 19, 2014 at 19:02

31

I wonder why Apostol's proof of the irrationality of $\sqrt{2}$ - which is as visual as a proof can be (in my opinion) - has not been mentioned: One can literally *see* at a glance that it proves what it's supposed to prove: the **impossibility of a isosceles right triangle with integer side length** (by infinite descent):



Note that it's not a proof completely without words. It helps a lot to read the comments of the author:

> Each line segment in the diagram has integer length, and the three segments with double tick marks have equal lengths. (Two of them are tangents to the circle from the same point.) Therefore the smaller isosceles right triangle with hypotenuse on the horizontal base also has integer sides.

But through own thinking one could come up with this by oneself (having in mind what's to be proved).

Share Cite Improve this answer Follow

edited Jan 22, 2020 at 16:21

community wiki
3 revs
Hans-Peter Stricker

I think you mean to refer to the impossibility of an integer isosceles *right* triangle, since clearly one can have isosceles triangles with integer sides, such as an equilateral unit triangle. – Joel David Hamkins Jan 22, 2020 at 16:19

1 @JoelDavidHamkins: Happy to hear from you again after all those years;-) I made the correction. Thanks for the hint. – Hans-Peter Stricker Jan 22, 2020 at 16:21

Oh yes, I've enjoyed many of your questions on MO over the years. – Joel David Hamkins Jan 22, 2020 at 16:29

So happy to hear that! But your enjoyment could not have been greater than mine was about your answers. – Hans-Peter Stricker Jan 22, 2020 at 16:32

It might be worth noting that one doesn't need to introduce the circle here; drawing the line between the top vertex of the triangle and the interior point on the bottom edge shows that the kite-shaped quadrilateral is bilaterally symmetric (the two triangles it's split into are congruent by side-angle-side equivalence) – Steven Stadnicki Nov 2, 2020 at 18:03

1 2 3 Next