CHAPTER 10

The Arbitrage Pricing Theory

Introduction

The CAPM is based on a single-index model in which all security returns depend only on the market return. In Chapter 8, we examined this model, as well as the multi-index model in which all security returns are assumed to be a function of more than one factor. Just as the CAPM can be viewed as an extension of the single-index model, the arbitrage pricing theory (APT) can be viewed as an extension of the multi-index model. In fact, the contribution of the APT to the finance literature is in showing how an equilibrium model can be extended from one determining factor to multiple factors. The APT also differs from the CAPM in that it is more general. That is, it is based on fewer assumptions than the CAPM, and unlike the CAPM, which considers systematic risk as the determining factor, APT does not specify what factors determine a security's equilibrium returns, leaving this to empirical inquiry. What APT does do is delineate how arbitrage determines equilibrium returns. In this chapter, we examine APT.

Derivation of the Model

In Chapter 9, the SML was derived by demonstrating that in the absence of arbitrage all securities must be on a line in return and beta space. The APT applies this same arbitrage argument to establish the equilibrium state for cases in which investment returns are determined by a number of factors. The model starts by assuming that the returns on any security or portfolio are linearly related to a common set of factors (F_j). That is:

$$r_i = \alpha_i + b_{i1}F_1 + b_{i2}F_2 + \cdots + b_{in}F_n + \varepsilon_i$$

Such factors could be the market return, an industry return, aggregate output, inflation, or the difference between long-term and short-term bond rates. As we noted above, the model does not specify the factors, it only requires that there be a common set of factors determining investment returns. The APT model also assumes that the standard regression assumptions hold ($E(\epsilon_i) = 0$; $V(\epsilon_i)$ is constant over observations; $\text{cov}(\epsilon_i F_j) = 0$), and that there is no correlation between the error terms for different securities and portfolios and no correlation between factors: $\text{cov}(F_j F_k) = 0$.

According to the APT, if we assume that investors hold portfolios such that they can diversify away unsystematic risk (i.e., the residual error, ϵ), then the only relevant characteristics or attributes needed by an investor in evaluating a security or portfolio would be its expected return and its sensitivities to each common factor, b_{ij}. If all securities are described in terms of their expected returns and b_{ij}'s, then an equilibrium relationship (analogous to the SML) can be obtained by generating the portfolio expected return and portfolio b_{ij} combinations obtained from a set of selected securities.

To see how the equilibrium relation is obtained, assume that all security returns are determined by just two factors:

$$r_i = \alpha_i + b_{i1}F_1 + b_{i2}F_2 + \varepsilon_i$$

Assuming that investors can diversify away ϵ, there would be only three security attributes that investors would be concerned with in making their investment decisions: the security's $E(r_i)$, its sensitivity to the first factor, b_{i1}, and its sensitivity to the second factor, b_{i2}. If securities can be evaluated in terms of only three attributes, one can take any three securities (or portfolios) and generate all of the equilibrium expected return, b_1 and b_2 combinations available in the market. For example, suppose we select stocks A, B, and C with the following expected returns, b_{i1}'s, and b_{i2}'s:

Security	$E(r_i)$	b_{i1}	b_{i2}
A	8%	0.5	0.3
B	6%	0.25	0.5
C	4%	0.15	0.1

Any portfolio formed with these stocks would have expected returns, b_1, and b_2 values equal to:

$$\begin{aligned} E(R_p) &= w_A E(r_A) + w_B E(r_B) + w_C E(R_C) \\ b_{p1} &= w_A b_{A1} + w_B b_{B1} + w_C b_{C1} \\ b_{p2} &= w_A b_{A2} + w_B b_{B2} + w_C b_{C2} \end{aligned}$$

In addition, any portfolio formed with these stocks would also have to lie on a plane in $E(R_p)$, b_{p1}, and b_{p2} space. The plane showing all combinations of the portfolio return, b_{p1}, and b_{p2} formed with stocks A, B, and C is shown in Exhibit 10.1. The equation of any plane in $E(R_p)$, b_{p1}, and b_{p2} space is:

$$E(R_p) = \lambda_0 + \lambda_1 b_{p1} + \lambda_2 b_{p2}$$

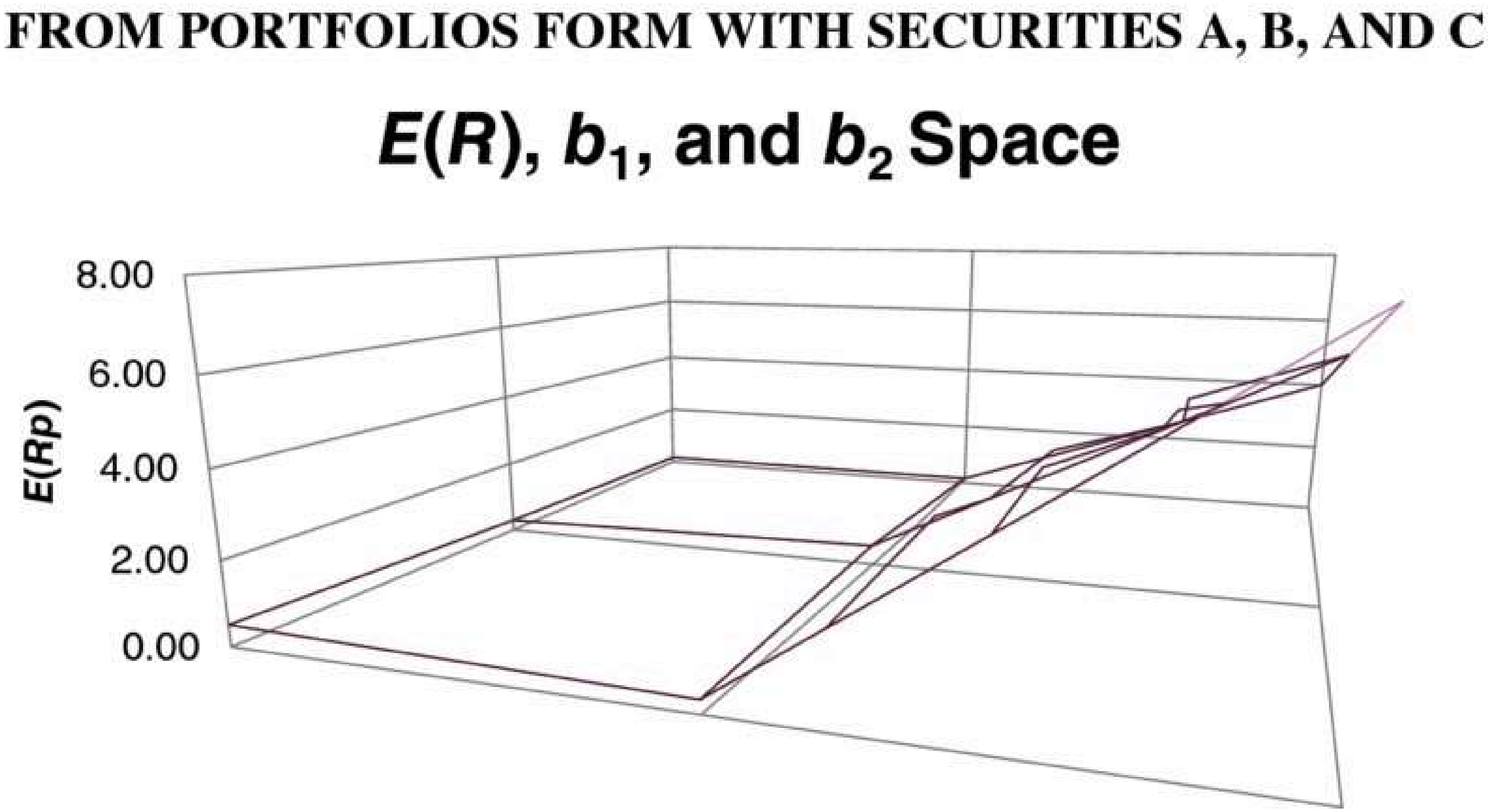


EXHIBIT 10.1 APT Plane

Mathematically, the equation of the plane formed with stocks A, B, and C is derived by first substituting the $E(r_i)$, b_{i1}, and b_{i2} values of our three stocks into the above equation for the plane to form three equations with three unknowns, λ_0, λ_1, and λ_3:

$$8\% = \lambda_0 + 0.5\lambda_1 + 0.3\lambda_2$$
$$6\% = \lambda_0 + 0.25\lambda_1 + 0.5\lambda_2$$
$$4\% = \lambda_0 + 0.15\lambda_1 + 0.1\lambda_2$$

The three equations can then be solved simultaneously for λ_0, λ_1, and λ_3 using matrix algebra. The matrix solution is shown in Exhibit 10.2 (for a primer on matrix algebra, see supplemental Appendix B). Solving this equation system simultaneously for these coefficients, we obtain λ_0 = 2.25, λ_1 = 10, and λ_2 = 2.5. The equation for the plane shown in Exhibit 10.1 is therefore:

$$E(R_p) = 2.25 + 10b_{p1} + 2.5b_{p2}$$

The unknowns in an equation system can be expressed in matrix form with the unknowns solved using either the inverse matrix approach or Cramer's rule. Given the equation system:

$$8\% = \lambda_0 + 0.5\lambda_1 + 0.3\lambda_2$$

$$6\% = \lambda_0 + 0.25\lambda_1 + 0.5\lambda_2$$

$$4\% = \lambda_0 + 0.15\lambda_1 + 0.1\lambda_2$$

Expressing the system in matrix form: $\mathbf{E} = \mathbf{A}\ \boldsymbol{\lambda}$

$$\underset{\mathbf{E}}{\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}} = \underset{\mathbf{A}}{\begin{pmatrix} 1 & 0.5 & 0.3 \\ 1 & 0.25 & 0.5 \\ 1 & 0.15 & 0.1 \end{pmatrix}} \underset{\boldsymbol{\lambda}}{\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}}$$

Using the inverse matrix approach to solve for $\boldsymbol{\lambda}$:
$\boldsymbol{\lambda} = \mathbf{A}^{-1}\ \mathbf{E}$

$$\underset{\boldsymbol{\lambda}}{\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}} = \underset{\mathbf{A}^{-1}}{\begin{pmatrix} -0.4167 & -0.04147 & 1.4583 \\ 3.333 & -1.6667 & -1.6667 \\ -0.8333 & 2.9167 & -2.0833 \end{pmatrix}} \underset{\mathbf{E}}{\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}}$$

Multiplying $\mathbf{A}^{-1}\ \mathbf{E}$:

$$\underset{\boldsymbol{\lambda}}{\begin{pmatrix} \lambda_0 \\ \lambda_0 \\ \lambda_0 \end{pmatrix}} = \underset{\mathbf{A}^{-1}\mathbf{E}}{\begin{pmatrix} (-0.4167)(8) + (-0.04167)(6) + (1.4583)(4) = 2.25 \\ (3.333)((8) + (-1.6667)(6) + (-1.6667)(4) = 10 \\ (-0.8333)(8) + (2.9167)(6) + (-2.0833)(4) = 2.5 \end{pmatrix}}$$

Or in algebraic form:

$$\lambda_0 = (-0.4167)(8) + (-0.04167)(6) + (1.4583)(4) = 2.25$$

$$\lambda_1 = (3.333)((8) + (-1.6667)(6) + (-1.6667)(4) = 10$$

$$\lambda_2 = (-0.8333)(8) + (2.9167)(6) + (-2.0833)(4) = 2.5$$

The equation for the plane is therefore:

$$E(R_p) = \lambda_0 + \lambda_1 b_{p1} + \lambda_2 b_{p2}$$

$$E(R_p) = 2.25 + 10b_{p1} + 2.5b_{p2}$$

Solving Equations Simultaneously in Excel with Matrix Multiplication Commands:

1. Form a coefficient matrix **A** in Excel (e.g., in cells A1:C3)
2. Generate the inverse matrix, $\mathbf{A}^{-1}$, by highlighting the cells for the matrix (e.g., E1:G3), entering

2. Generate the inverse matrix, A^{-1}, by highlighting the cells for the matrix (e.g., E1:G3), entering the command "=minverse ((a1:a3), and then pressing CTrl + shift + Enter.
3. Generate the value for the λ vector by creating a column vector for E (e.g., cells B5:B7) and then multiplying the $\mathbf{A}^{-1}$ matrix by vector **E**. In Excel, the product matrix is generated by highlighting an Excel column (E5:E7), entering the command: "=mmult (E1:G3, B5:B7)," and then pressing CTrl + shift + Enter.

	A	B	C	D	E	F	G
1	1	0.5	0.3		−0.41667	−0.04167	1.458333
2	1	0.25	0.5		3.333333	−1.66667	−1.66667
3	1	0.15	0.1		−0.83333	2.916667	−2.08333
4		E			$\lambda = A^{-1}E$		
5		8			2.25		
6		6			10		
7		4			2.5		
8							

A^{-1}: The inverse matrix, A^{-1}, is generated by highlighting the cells for the matrix (e.g., E1:G3), entering the command "=minverse ((A1:A3), and then pressing CTrl + shift + Enter.

$\lambda = A^{-1}E$: The product matrix is generated by highlighting (E5:E7), entering the command: "=mmult (E1:G3, B5:B7)", and then pressing CTrl + shift + Enter.

EXHIBIT 10.2 Solving for λ_0, λ_1, and λ_3 Using Matrix Algebra

Any portfolio formed with stocks A, B, and C would lie on the plane in Exhibit 10.1. For example, an equally allocated portfolio formed with stocks A, B, and C would yield an expected portfolio return of 6 percent, a sensitivity to the first factor of $b_{p1} = 0.3$, and a sensitivity to the second factor of $b_{p2} = 0.3$, and the portfolio's coordinate $(E(R_p), b_{p1}, b_{p2}) = (6\%, 0.3, 0.3)$ would lie on the plane:

$$\begin{aligned} E(R_p) &= (1/3)(0.08) + (1/3)(0.06) + (1/3)(0.04) = 0.06 \\ b_{p1} &= (1/3)(0.5) + (1/3)(0.25) + (1/3)(0.15) = 0.3 \\ b_{p2} &= (1/3)(0.3) + (1/3)(0.5) + (1/3)(0.10) = 0.3 \end{aligned}$$

$$\begin{aligned} E(R_p) &= 2.25 + 10b_{p1} + 2.5b_{p2} \\ E(R_p) &= 2.25 + 10(0.3) + 2.5(0.3) = 0.06 \end{aligned}$$

Finally, using the same arbitrage argument we used to derive the SML, we can establish that in equilibrium any security or portfolio would also have to be on the plane defined by stocks A, B, and C. For example, consider a stock D with attributes of $b_{D1} = 0.3$, $b_{D2} = 0.3$, and $E(r_D) = 4\%$. Stock D's $E(r)$, b_1, and b_2

coordinate is below the plane formed with A, B, and C. In this case, arbitragers would be able to profit by going short in stock D and long in an equally allocated portfolio of A, B, and C, with the portfolio investment financed from the proceeds from the short position. This arbitrage portfolio would yield a positive cash flow at the end of the period, with no risk and no investment—a “free lunch.”

	Investment	**Expected Return**	b_{i1}	b_{i2}
Short Security D	−$100	−$4.00	−0.3	−0.3
Long Portfolio of A,B,C	$100	$6.00	0.3	0.3
Arbitrage Portfolio	0	$2.00	0	0

With this opportunity, arbitrageurs would pursue this strategy, causing the price of stock D to fall and its expected return to increase (and possibly the prices of A, B, and C to increase and their returns to fall) until stock D was on the plane formed with securities A, B, and C. Moreover, this same arbitrage argument can be applied to any security or investment with an expected return, b_1, and b_2 combinations not on the plane formed with A, B, and C. The plane in Exhibit 10.1 therefore depicts the equilibrium return, b_1, and b_2 combinations for any security or portfolio, not just for portfolios formed with securities A, B, and C. Thus, if all securities are determined by just two factors, then in equilibrium all investments must be on a plane in return, b_1, and b_2 space, with the equilibrium return for any security or portfolio being determined by the investment's attributes b_{i1} and b_{i2}; that is:

$$E(r_i)^* = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

λ Coefficient Values

The λ values defining the equilibrium return relation were obtained by solving the three-equation system simultaneously. Alternatively, these coefficients can be found by examining particular portfolios. For example, consider a portfolio formed with zero sensitivity to both factors, $b_{p1} = b_{p2} = 0$. Such a portfolio would have an equilibrium return equal to λ_0.[1] With securities A, B, and C, this portfolio can be formed with an allocation of $w_A = -0.41667$, $w_B = -0.04167$, and $w_C = 1.458333$ and would yield an expected return of 2.25 percent, which is equal to the λ_0 value we calculated. Since investors can eliminate systematic risk with this portfolio, this portfolio would be riskless. Using the arbitrage argument, the rate on a riskless security, in equilibrium, would have to be equal to the rate on the portfolio formed with zero sensitivity to the factors; thus, a priori the intercept should be equal to the risk-free rate:

$$\lambda_0 = R_f$$

Next, consider a portfolio formed with unit sensitivity to the first factor and zero sensitivity to the second: $b_{p1} = 1$ and $b_{p2} = 0$. With securities A, B, and C, this portfolio is formed with an allocation of $w_A = 2.92$, $w_B = -1.71$, and $w_C = -0.21$ and would yield an expected return of 12.25 percent. If the value of the first factor is $E(R_1)$, then in equilibrium the coefficient λ_1 would be equal to $E(R_1) - R_f$; that is, solving for λ_1 for a portfolio with unit sensitivity to the first factor and zero to the second and with a return in equilibrium of $E(R_1)$, we obtain:

$$\begin{aligned} E(r_i)^* &= \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} \\ E(R_1) &= R_f + \lambda_1(1) + \lambda_2(0) \\ \lambda_1 &= E(R_1) - R_f \end{aligned}$$

This is consistent with the λ_1 value we solved for in our equation for the plane:

$$\lambda_1 = E(R_1) - R_f = 12.25\% - 2.25\% = 10\%$$

If the first factor, in turn, were the market return, then in equilibrium a portfolio formed with unit sensitivity to the market and zero sensitivity to the other factor would have to have a return equal the market return and λ_1 would have to be equal to $E(R^M) - R_f$.

Finally, consider a portfolio formed with unit sensitivity to the second factor and zero sensitivity to the first: $b_{p2} = 1$ and $b_{p1} = 0$. With securities A, B, and C, this portfolio is formed with an allocation of $w_A = -1.25$, $w_B = 2.875$, and $w_C = -0.625$ and would yield an expected return of 4.75 percent. If the value of the second factor is $E(R_2)$, then in equilibrium the coefficient λ_2 would be equal to $E(R_2) - R_f$:

$$E(r_i)^* = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$
$$E(R_2) = R_f + \lambda_1(0) + \lambda_2(1)$$
$$\lambda_2 = E(R_2) - R_f$$

This result is also consistent with our example, where the λ_2 value we solved for in our equation for the plane was 2.5 percent:

$$\lambda_2 = E(R_2) - R_f = 4.75\% - 2.25\% = 2.5\%$$

Thus, if the second factor were the expected return on the average bond investment, $E(R_B)$ (e.g., bond index return), then in equilibrium a portfolio formed with unit sensitivity to the bond market return and zero sensitivity to the other factor would have to have a return equal the bond market return and λ_2 would have to be equal to $E(R_B) - R_f$.

Given the a priori values of the coefficients, the equilibrium relation for any investment *i* can alternatively be defined as

$$E(r_i)^* = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$
$$E(r_i)^* = R_f + [E(R_1) - R_f]b_{i1} + [E(R_2) - R_f]b_{i2}$$

As noted, this equation is consistent with our numerical example:

$$E(R_p) = \lambda_0 + \lambda_1 b_{p1} + \lambda_2 b_{p2}$$

$$E_i^* = 2.25 + 10b_{p1} + 2.5b_{p2}$$

It should be highlighted that the equilibrium APT equation is similar in form to the SML equation, which is also defined in terms of risk-free rate, factor (market) premium, and the investment's sensitivity to the factor.

General APT Model

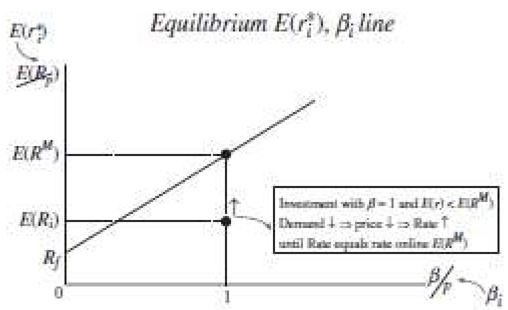
The equilibrium return in the above case is based on a two-factor model. The equilibrium relation was obtained by generating a plane from the return, b_1, and b_2 characteristics of three securities, then using an arbitrage argument to establish that in equilibrium all securities should lie on that plane. This methodology is the same approach we used in Chapter 9 to derive the SML. For the SML, we had a single-factor model, with the one factor being the market return, and we generated a line from the return and beta combinations of two investments; we then used an arbitrage argument to establish that in equilibrium all securities would be on that line. See Exhibit 10.3 for a summary comparison of CAPM and APT. In general, the APT is defined by the multifactor model (sometimes referred to as the multifactor return-generating process) and the resulting equilibrium return relation:

$$
\begin{aligned}
r_i &= \alpha_i + b_{i1}F_1 + b_{i2}F_2 + \cdots + b_{in}F_n + \varepsilon_i \\
E(r_i)^* &= \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_2 b_{i2} \\
E(r_i)^* &= R_f + [E(R_1) - R_f]b_{i1} + [E(R_2) - R_f]b_{i2} + \cdots \\
&\quad + [E(R_n) - R_f]b_{in}
\end{aligned}
$$

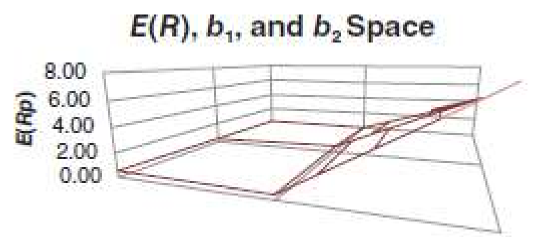
CAPM	APT
Based on Single-Index Model: Assumes all investments (*i*) are related to the market: $r_i = \alpha_i + \beta_i R^M + \epsilon_i$ The explanatory factor is the market return, R^M	**Based on Multi-Index Model** Assumes all investments (*i*) are related to a common set of factors: $r_i = \alpha_i + b_{i1}F_1 + b_{i2}F_2 + \cdots + b_{in}F_n + \epsilon_i$ Common set of explanatory factors are not specified
ϵ_i is Diversified Away Investors make their investment decisions in a portfolio context. With a portfolio, investors can diversify away unsystematic risk, ϵ_i. Thus, a security's undiversified or unsystematic risk would not affect the security's equilibrium price and return.	**ϵ_i is Diversified Away** Investors make their investment decisions in a portfolio context. With a portfolio, investors can diversify away unsystematic risk, ϵ_i. Thus, a security's undiversified or unsystematic risk would not affect the security's equilibrium price and return.

CAPM	APT
Equilibrium Return The equilibrium return on an investment is determined by its systematic risk as measured by its beta and not by unsystematic factors that investors can diversify away.	**Equilibrium Return** The equilibrium return on an investment is determined by its sensitivity to each factor, b_1, b_2,...b_n and not by unsystematic factors that investors can diversify away.
Equilibrium Return Relation: With unsystematic risk diversified away, the only relevant factors for investors in evaluating any security would be its expected rate of return, $E(r_i)$, and its β_i. If all securities and investments are explained by just two factors, then all of the equilibrium return and beta combinations available in the market can be generated from the return and beta combinations formed from a portfolio of the market portfolio with an expected return of $E(R^M)$ and beta of one and the risk-free	**Equilibrium Return Relation:** With unsystematic risk diversified away, the only relevant factors for investors in evaluating any security would be its expected rate of return, $E(r_i)$, and its sensitivity to each factor b_1, b_2,... b_n. If all securities and investments are explained by two factors, for example, then all of the equilibrium return and b_1 and $b_2$sensitivity combinations available in the market can be generated from the return and b_1 and b_2 combinations formed from a portfolio of three investments. Arbitrage would ensure that all equilibrium return and b_1 and b_2 combinations would be on a plane in $E(r)$, b_1 and b_2 space. For two factors, the equilibrium relation is depicted by

CAPM	**APT**
security with a rate of R_f and beta of zero. Arbitrage would ensure that all equilibrium return and beta combinations would be on a line in $E(r)$, β space. The equilibrium relation is depicted by the SML and SML equation:	a plane and the equation for the plane. $E(r_i)^* = R_f + [E(R_1) - R_f]b_{i1} + [E(R_2) - R_f]b_{i2}$



$$E(r_i)^* = R_f + [R(R^M) - R_f]\beta_i$$



For n-securities, the equilibrium relation is

$$E(r_i)^* = R_f + [E(R_1) - R_f]b_{i1} + [E(R_2) - R_f]b_{i2} + \cdots + [E(R_n) - R_f]b_{in}$$

Empirical Relation:	**Empirical Relation:**
Empirically the SML is estimated by using a first-pass and second-pass methodology. In the first pass, time-series regressions are used to estimate the betas of securities (or portfolios):	Empirically APT is estimated by using a first-pass and second-pass methodology: In the first pass, time-series regressions are used to estimate the factor sensitivities of securities (or portfolios), $b_1, b_2, \dots b_n$ $r_i = c_0 + c_1F_{i1} + c_2F_{i2} + c_3F_{i3} + \cdots + c_nF_{in} + \epsilon_i$ In the second pass, a cross-sectional regression is used

CAPM	APT
$r_{it} = \alpha_i + \beta_i R^M{}_t + \epsilon_{it}$ In the second pass, a cross-sectional regression is used in which the average returns or risk premiums of the securities (or portfolios) are regressed against their betas to estimate a SML relation: $$\bar{R}_{pi} - \bar{R}_f = \gamma_0 + \gamma_1 \beta_i + \varepsilon_i$$	in which the average returns or risk premiums of the securities (or portfolios) are regressed against their sensitivity factors b_1, b_2,... b_n to estimate the APT relation: $$\bar{r}_i - \bar{R}_f = \gamma_0 + \gamma c_{11} + \gamma_2 c_2 + \gamma_3 c_{31} + \cdots + \gamma_4 c_n + \varepsilon_i$$
Explanatory Factors **Market Return**	**Explanatory Factors** **Chen, Roll, and Ross:** Unanticipated changes in inflation, the term structure of rates (differences in long-term and short-term bond yields), industrial production, and risk spreads (differences in yields on low quality and high quality bonds. **Burmeister and McElroy:** Unexpected changes in default risk, changes in term structure, inflation, changes in growth rate in real aggregate, and independent market return factor. **Fama and French:** Market return, return of a portfolio of small stocks over a portfolio of large stocks, and excess return as measured by return of a

CAPM	APT
	portfolio of high book-to-market value in excess of the return of a portfolio of low book-to-market value.

EXHIBIT 10.3 Comparison of CAPM and APT

Empirical Tests of the APT

Like the CAPM, the empirical testing of the APT lends itself to a first- and second-pass methodology. The problem in applying first- and second-test methods to the APT, however, is that unlike the CAPM where the factor is known (the market rate), the factors are not identified in the APT. Because of this problem, researchers in testing the APT have used two approaches. The first is to use a statistical approach known as *factor analysis* that does not specify specific factors; the second is to use a specified factor approach in which factors are identified.

Unspecified Approach: Factor Analysis

Factor analysis is a complex statistical procedure used to create proxy factors that explain correlations among observations from a set of variables. The procedure involves gathering data on a large number of variables and trying to statistically identify underlying forces that could explain the observed trends. For example, suppose a statistician collected information such as weight, height, eye color, hair color, and length of hair from a sample of people. Applying factor analysis to the data, the statistician might find that there is a correlation among height, weight, and hair length, but not hair color, that could be explained by some common factor. We know a priori that the factor is gender. Looking just at the data, however, the statistician would not know this. She would be able to create a proxy variable or factor based on the sample that reflects gender (e.g., an index equal to the 30th percentile of hair length, 40th percentile of weight, and 40th percentile of height). This proxy factor would not be able to identify the correlation in hair color with other characteristics. By analyzing the data further, however, the statistician might observe that 10 percent of the sample has gray hair, which she might define as an index or proxy factor that we know would describe the factor of “age over 50.” From this factor analysis, the statistician might conclude that there are two proxy factors that explain the correlations among the observations.

When factor analysis is applied to security returns, the data usually consists of time-series rates of return from a large sample of securities. Factors are then constructed that best explain the correlations of returns within the sample. After the factors have been formed, the time-series data on each security's return is often regressed against the factors (first-pass test) to estimate each security's sensitivity attributes, b_{ij}. Finally, each security's average return is regressed against its b_{ij}'s (second-pass test). In a classic study examining the validity of the APT, Roll and Ross used factor analysis techniques on 42 groups of 30 stocks over a sample period from 1962 to 1972. They found between four and six factors that were significant in explaining the correlations among investment returns. Their study provided early support for the multifactor/APT models.

Specified Factor Approach

From factor analysis, equilibrium returns are explained in terms of factors that are manufactured from the data. Studies using a specified factor approach to test the validity of the APT begin by identifying the common factors important in determining a security's equilibrium returns. Given those factors, many studies then apply a first-pass test to estimate each security's sensitivity (either its return, r_i or risk premium) to the identified factors:

$$r_i = c_0 + c_1 F_{i1} + c_2 F_{i2} + c_3 F_{i3} + \cdots + c_n F_{in} + \varepsilon_i$$

A second-pass test in which the average returns on each security (or average risk premium) are regressed against their sensitivity factors is then used to estimate the market relation:[2]

$$\bar{r}_i = \gamma_0 + \gamma c_{i1} + \gamma_2 c_{i2} + \gamma_3 c_{i3} + \cdots + \gamma_n c_{in} + \varepsilon_i$$

Sharpe Study

One of the early studies using specified factors was done by William Sharpe. In his first-pass test, he estimated the relationship between a stock's return and the market rate, long-term bond rate, and the stock's dividend yield. He estimated this relationship for over 2,000 stocks using data covering 1931 through 1979. In his second pass, Sharpe regressed the average returns of each stock against the coefficients estimated from the first-pass tests. He made several second-pass tests, varying the number of coefficients used as explanatory variables and the time periods. In comparing several regression models, Sharpe found that models with two or more coefficients had better regression results than did the linear regression model explained by just beta. This finding provided some support for a multifactor/APT model.

Chen, Roll, and Ross

A more recent study of multifactor/APT models was done by Chen, Roll, and Ross (CRR). They argued that equilibrium returns on investments should be related to macroeconomic factors that affect the present value of an investment's future cash flows (e.g., real gross domestic product, interest rates, market risk, and inflation). In addition, CRR argued that investors respond to unanticipated changes in these factors. To test this empirically, CRR first estimated the relationship between the returns on portfolios grouped in terms of the size of their betas (Fama and MacBeth approach) and four factors measuring unanticipated changes in inflation, the term structure of rates (differences in long-term and short-term bond yields), industrial production, and risk spreads (differences in yields on low-quality and high-quality bonds). Finding a relatively strong statistical relation between the portfolio returns and the unanticipated macroeconomic factors, CRR next estimated the cross-sectional relationship between the average portfolio returns and the estimated coefficients from the time-series regressions. They found the coefficients of the macroeconomic factors to be significant. They also found that when they included the portfolio's beta as an explanatory variable along with the other macro variable coefficients, it was insignificant. Thus, like the Sharpe study, the CRR study provided evidence in support of the multifactor/APT model as an explanation of what determines equilibrium returns.

Burmeister and McElroy

Following the lead of CRR, Burmeister and McElroy (BM) also tested multifactor models using unexpected macroeconomic variables. In their time-series regressions, the monthly returns of 70 stocks were regressed against the following five variables:

1. F_1 = unexpected changes in default risk as measured by the average difference between corporate and government monthly rates (0.5 percent) minus the observed difference: $F_1 = 0.5\% - (R_{corp} - R_{gov})$.
2. F_2 = unanticipated changes in term structure as measured by the average difference between the monthly rates on long-term Treasury bonds and short-term Treasury bills (which they assumed to be zero) minus the observed differences: $F_2 = R_{LT} - R_{ST}$.
3. F_3 = unanticipated inflation as measured by the forecast inflation for the beginning of the month (using trend analysis) and the actual inflation rate at the end of the month.
4. F_4 = unanticipated changes in the growth rate in real aggregate output as measured by the forecast growth rate in real aggregate expenditures at the beginning of the month (using trend analysis) and the actual rate at the end of the month.

5. F_5 = independent market return factor. To minimize statistical bias (problems related to multicollinearity), the market return factor was measured by the error term, ϵ_M, obtained by first regressing the market risk premium ($R^M - R_f$) against the above four factors and then taking the difference between the observed market premium and the estimated one using observed values of the factors, F^{Obs}:

$$R^M - R_f = \gamma_0 + \gamma_1 F_1 + \gamma_2 F_2 + \gamma_3 F_{31} + \gamma_4 F_4 + \varepsilon_M$$

$$F_5 = \varepsilon_M = [R^M - R_f] - [\gamma_0 + \gamma_1 F_1^{\text{Obs}} + \gamma_2 F_2^{\text{Obs}} + \gamma_3 F_3^{\text{Obs}} + \gamma_1 F_4^{\text{Obs}}]$$

From the 70 regressions, BM found that 215 out of the 350 coefficients estimated were significant and that on average the five factors explained approximately 50 percent of the variation in a security's return. BM also regressed portfolio returns against the above factors. The portfolios consisted of stocks grouped into growth, cyclical, stable, utility, transportation, and finance categories. BM found the results from these regressions were better than from stock regression: More coefficients were significant and the five factors explained about 80 percent of the variation in returns. Thus, like the Sharpe and CRR studies, the BM study also supports a multifactor model.

Fama and French

Fama and French proposed a three-factor model using (1) the market return, (2) corporate capitalization or size as measured by the difference in the return of a portfolio of small stocks over a portfolio of large stocks, and excess returns as measured by the return of a portfolio of high book-to-market value stocks in excess of the return of a portfolio of low book-to-market value stocks. (Firms with high ratios of book-to-market value are likely to be in more financial stress than firms with low ratios.) The Fama-French model was tested by Davis, Fama, and French, who formed nine portfolios with different ranges in size and book-to-market value. The statistical results for each of the nine portfolios had R^2 over 0.90 and large t-statistics, providing support for the explanatory power of this three-factor model.

Summary of Empirical Models

In studies using factor analysis, equilibrium returns are explained in terms of factors that are manufactured from the data. Because the APT does not specify the determining factors, factor analysis is a proper and sufficient empirical test of the theory. In the financial and economic literature, however, there is a ubiquitous debate over whether or not models should be prespecified on the basis of theory or whether models should be generated empirically. Without theory, the empirical results are often difficult to interpret and are not as insightful.[3] The empirical studies of Sharpe; Chen, Roll, and Ross; Burmeister and McElroy and others avoid this problem by first specifying the factors and then applying first- and second-pass regression tests to empirically estimate the equilibrium relationship. A problem with this approach is that poor statistical results could be the effect of selecting the wrong factors and would therefore not necessarily invalidate the APT. Many of the studies using a specified factor approach, however, are consistent with the Roll and Ross factor analysis study in their support of the multifactor/APT model. In addition, the more recent studies using a specified factor approach provide some evidence in support of equilibrium returns being determined by macroeconomic factors.

MRA: BLOOMBERG'S MULTIPLE REGRESSION SCREEN

Multiple regression analysis can be done using the Bloomberg MRA screen. On the screen, one selects a set for inputting information or editing a previous set. The dependent and independent variables are inputted by their tickers and <Equity> or Index ticker <Index>. The MRA output shows the coefficient estimates, *t*-tests, R^2, and F-Statistic.

BLOOMBERG EXCEL ADD-IN: SECOND-PASS TEST FOR MULTIFACTOR MODELS

French and Fama argue that firms with high ratios of book-to-market value are likely to be in more financial stress than firms with low ratios, and that such risk is important in explaining equilibrium returns. Altman's *Z* score is a measure of a company's likelihood of default and possibly a good proxy for financial risk.

Test of Relationship: Conduct a second-pass test of stock returns explained in terms of their betas and Altman's *Z*-scores.

Data for conducting cross-sectional analysis can be found using the Bloomberg Add-In for Excel (see Chapter 2 for a description of Bloomberg Excel Add-In). Steps to run a cross-sectional regression of returns against betas and Altman *Z*-Score are as follows:

- On the Bloomberg Add-in, click “Real-Time Current” from the “Import Data” and “Real-Time/Historical” dropdowns.
- On the Bloomberg Data Wizard Box, Step 1, click “indexes” in the “From” dropdown and the name of index (e.g., S&P 100) from the “Indexes” dropdown, and then click “Add All.” This will bring up the stocks for the index. Once loaded, click “Next.”
- On the Bloomberg Data Wizard Box, Step 2, search and then add stock returns (e.g., DDM Internal Return, betas, and Altman *Z*-Score).
- After loading variables, click “Next.”
- On the Bloomberg Data Wizard Box, Step 3, click “Finish” to export the data to Excel.
- Multiple-regression can be done in Excel by using the “Data Analysis” Add-in. See Exhibit 6.16.
- Example: The S&P 100 stocks' DDM internal rates regressed against their betas and Altman *Z*-scores and instruction on running multiple regression:

Application of APT: Factor Models

Portfolios constructed by practitioners from multifactor/APT analysis are called factor models. Three general types of multifactor models are statistical factor models, fundamental models, and macro model. Statistical factor models are based on explaining security and portfolio returns based on artificial factors created from factor analysis. Fundamental models, in turn, use a cross-sectional approach similar to the Wells Fargo model examined in Chapter 9, but with more than one explanatory variable. Macroeconomic factor models are portfolios that are constructed based on macroeconomic factors. A number of these models are rooted in the works of Chen, Roll, and Ross and Burnmeister and McElroy.

Most of the factor models are proprietary models. One published model was the Salomon-Smith-Barney's Risk Attributes Model (RAM). The model consists of four steps:

1. **Step 1:** Stock returns are explained by a set of macroeconomic variables:
   1. Investors' Confidence: $R^{\text{corp}} - R^{\text{govt}}$.
   2. Interest Rates: Δ(LT Rate – ST Rate)
   3. Inflation Shock: Actual minus expected inflation rate.
   4. Aggregate Business Fluctuations: Δ (Industrial Production).
   5. Foreign Variables: Δ(Exchange Rate).
   6. Market Factors: Residual Market Beta.
2. **Step 2:** Time-series regressions of the stock returns against the six macroeconomic variables: Salomon-Smith-Barney model regressed the returns of 3,500 stocks against the above macroeconomic factors:

$$r_i = a_i + b_{i1}F_1 + b_{i2}F_2 + \cdots + b_{i6}F_6 + \varepsilon_i$$

3. **Step 3:** Standardization of the coefficients: For each coefficient, the average coefficient and average standard deviation are computed. For example, for b_1:

$$\bar{b}_1 = \frac{\sum_{i=1}^{3,500} b_{i1}}{3,500}$$

$$\sigma_{b_1} = \left[\frac{\sum_{i=1}^{3,500} (b_{i1} - \bar{b}_1)^2}{3,500}\right]^{1/2}$$

   For stock i, its adjusted standardized coefficient is calculated as:

   If a stock has an adjusted beta equal to zero, then the stock's sensitivity to factor 1 is no different from the average. If its adjusted beta exceeds zero, then the stock has an above average responsiveness to factor 1. Finally, if the adjusted beta is less than zero, then the stock has a below average responsiveness to factor one.

4. **Step 4:** For each stock, its score, S_i, is determined. The score is obtained by multiplying the stock's adjusted coefficients by an estimate of the macroeconomic factors, then summing the products:

5. **Step 5:** A portfolio is constructed consisting of the stocks with the highest scores.

Conclusion

Since its introduction in the 1960s, the CAPM has been used by analysts and academics to determine required returns on stock investment and solve capital budgeting problems. Today, many practitioners estimate betas, or use estimated betas found in several investment publications, to determine the required returns on capital investment projects and stocks they are evaluating. The newer multifactor/APT model has not been used as extensively as the CAPM. However, it can be applied in many of the same ways as the CAPM: determining required returns for valuation, estimating abnormal returns, and defining investment and portfolio strategies. One of the advantages of the multifactor/APT model over the CAPM is that one can generate portfolios with more than one attribute. For example, a pension fund manager wanting a portfolio that is both invariant to inflation and highly correlated with the market could use a multifactor model similar to the one estimated by Burmeister and McElroy to find stocks with relatively small coefficients associated with the inflation factor (F_3) and coefficients that are approximately equal to one for the independent market factor (F_5). Similarly, by selecting stocks based on their sensitivities to a set of factors such as inflation, interest rates, and the market, an investor may be able to speculate on not only the market's future performance but also on inflation and interest rate changes.

As we conclude this chapter on equilibrium models, it is important to keep in mind that we are dealing with theory, and not a definitive explanation of what determines security returns. This, of course, is not comforting to portfolio and investment managers who can't afford to be too dogmatic or theoretical, but rather must find out what works. As theories supported by some empirical work, however, both the CAPM and the APT give us important insights into how the capital market functions: the importance of portfolio decisions, how diversification reduces the importance of firm and industry factors, how returns can be determined by certain factors, and the importance of macroeconomic factors. As the noted economist John Maynard Keynes said, “half-baked theory may not be much value in practice, but it may be halfway towards final perfection.”

Notes

1. The allocations needed to obtain portfolios that have a sensitivity to the first factor of b^*_{p1} and sensitivity to the second of b^*_{p2} can be found by substituting $1 - w_A - w_B$ for w_C in the equations for b_{p1} and b_{p2}, setting the equations equal to b^*_{p1} and b^*_{p2}, then solving the two equations simultaneously for w_A and w_B. That is:

2. The Fama and MacBeth study discussed in Chapter 9 could be viewed as an empirical test of the multifactor/APT model. Recall that Fama and MacBeth estimated the relation between the equilibrium return of portfolios and beta, beta squared, and the variance of the residual error.

3. Note that this was not the problem in the CAPM where we started with a theory that identified market return as the relevant factor to explain a security's equilibrium return.

Selected References

Arnott, Robert D., and Chris G. Luck. 2003. The many elements of equity styles: Quantitative management of core, growth, and value styles. *Handbook of Equity Style Management*. Edited by T. D. Coggin, F. J. Fabozzi, and R. D. Arnott. Hoboken, NJ: John Wiley & Sons.

Burmeister, Edwin, and Marjorie McElroy. 1988. Joint estimation of factor sensitivities and risk premia for the Arbitrage Pricing Theory. *Journal of Finance* 43 (July): 721–733.

Chen, Nai-fu. 1983. Some empirical tests of the theory of arbitrage pricing. *Journal of Finance* 38 (December): 1392–1414.

Chen, Nai-fu, RichardRoll, and StephenRoss. 1986. Economic forces and the stock market. *Journal of Business* 59 (July): 386–403.

Cho, D. Chinhyung, Edwin Elton, and Martin Gruber. 1984. On the robustness of the Roll and Ross Arbitrage Pricing Theory. *Journal of Financial and Quantitative Analysis* 24 (March 1984): 1–10.

Fama, Eugene, and Kenneth French. 1992. The cross section of expected stock returns. *Journal of Finance* 47 (June): 427–466.

Fama, Eugene, and James MacBeth. 1973. Risk, return, and equilibrium: Empirical test. *Journal of Political Economy* 38: 607–636.

Gibbons, M. 1982. Multivariate tests of financial models: A new approach. *Journal of Financial Economics* 10 (March): 3–27.

Huberman, G. 1982. A simple approach to Arbitrage Pricing Theory. *Journal of Economic Theory* 28 (October): 183–191.

Ingersoll, Jonathan E. 1984. Some results in the theory of arbitrage pricing. *Journal of Finance* 19: 1021–1039.

Jarrow, Robert, and Andrew Rudd. 1983. A comparison of the APT and SML. *Journal of Banking and Finance* 7 (June): 295–303.

Joreskog, K. J. 1963. *Statistical Estimation in Factor Analysis*. Stockholm: Almqvist & Wiksell.

Kim, Dongcheol. 1995. The errors-in-variables problem in the cross-section of expected stock returns. *Journal of Finance* 50: 1605–1634.

Lawley, D., and M. A. Maxwell. 1963. *Factor Analysis as a Statistical Method*. London: Butterworths.

McElroy, Marjorie, and Edwin Burmeister. 1988. Arbitrage Pricing Theory as a restrictive non-linear multivariate regression model: ITNLSUR estimate. *Journal of Business and Economics Statistics* 6 (January): 29–42.

Reinganum, M. 1981. The Arbitrage Pricing Theory: Some empirical results. *Journal of Finance* 36 (May): 313–321.

Roll, Richard. 1977. A critique of the asset-pricing theory's tests; Part I: On past and potential testability of the theory. *Journal of Financial Economics* 4 (March): 129–176.

Roll, Richard, and Stephen Ross. 1980. An empirical investigation of the Arbitrage Pricing Theory. *Journal of Finance* 35 (December): 1073–1103.

Ross, Stephen. 1976. The arbitrage pricing theory of capital asset pricing. *Journal of Economic Theory* 13 (December): 341–360.

Sharpe, W. F. 1970. *Portfolio Theory and Capital Markets*. New York: McGraw-Hill.

Sharpe, William. 1984. Factor models, SMLs, and the APT. *Journal of Portfolio Management* 11 (Fall): 21–25.

Sharpe, W. F., and G.M. Cooper. 1972. Risk-return class of New York Stock Exchange common stocks, 1931–1967. *Financial Analysts Journal* 28 (March–April): 46–52.

Bloomberg Exercises

1. French and Fama argue that firms with high ratios of book-to-market value are likely to be in more financial stress than firms with low ratios. Altman's *Z*-score) is a measure of a company's likelihood of default. Examine the significance of financial stress in determining equilibrium returns by conducting a second-pass test of stock returns explained by their betas and Altman's *Z*-scores.
   To run your regression, use the Bloomberg Excel Add-In to run the second-pass test in Excel. Consider the following: for your stocks, consider the stocks composing the DJIA or the S&P 100 or 500; for rates of return, the internal rates of return calculated from Bloomberg's DDM model; for betas, the stocks' adjusted betas; for risk, Altman's *Z*-score. For an example on using multiple regression analysis in Excel, see the Excel table in the Bloomberg Exhibit box, “Bloomberg Excel Add-In: Second-Pass Test for Multifactor Models.”

2. In an early study of the CAPM, Fama and MacBeth found the market risk premium to be significant in explaining equilibrium returns but found residual errors not significant. Their study provided support that systematic risk is the only relevant factor in determining an investment's equilibrium return. Using a second-pass test, reexamine Fama and MacBeth's findings.

   To run your regression, use the Bloomberg Excel Add-In to run the second-pass test in Excel. Consider: for your stocks, the stocks composing the DJIA, the S&P 100 or S&P 500; for your rates of return, the internal rates of return calculated from Bloomberg's DDM model; for your betas, the stocks' adjusted betas; and for your unsystematic risk, the standard deviation of the errors. For an example of using multiple regression analysis in Excel, see the Excel table in the Bloomberg Exhibit box, "Bloomberg Excel Add-In: Second-Pass Test for Multifactor Models."
3. William Sharpe in a 1970s study identified market returns and long-term bond rates as being significant in explaining equilibrium returns. Using Bloomberg's MRA screen, conduct a multiple regression analysis for several stocks using pro- portional changes in the S&P 500 (SPX) as a measure of the market rate and proportional changes in the yields on 30-year Treasuries (USGG30YR).

   To use MRA:

   1. MRA <Enter>; Click MRA Run.
   2. Fill in fields in the "Multiple Regression Creation Box:" for dependent variable stock ticker <Equity>; for independent variables: SPX <Index> and USGG30YR <Index>; for Column fields: Percent (P) for Data, Close (C) for Value, and None (N) for Log Type; Click Menu or Enter 1 <Go> to save.
   3. Click your saved MRA run to bring up the Multiple Regression Analysis Box.
   4. In the Multiple Regression Box, enter data range and period (e.g., Month), and hit <Enter> to see regression results.