How to Analyze time Complexity?

→ Running Time depends upon:

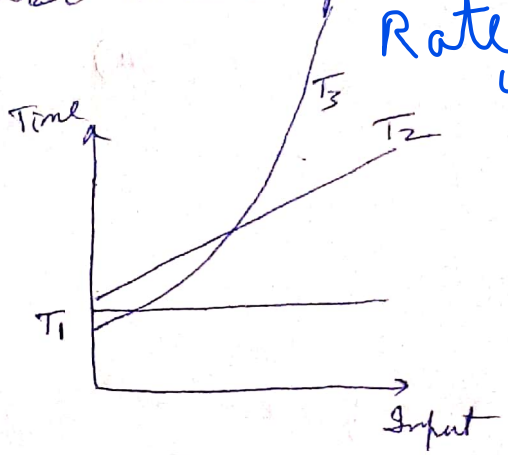
1) Single vs multi processor ×
2) Read/Write speed to memory ×
3) 32 bit or 64 bit ×
4) Inputs ✓

eg: 1+2 but 8×10^{10} + 10^{10}

do we don't see them when we talk about time Complexity.

but we see

As we see the rate of growth of time w.r.t to our inputs



eg:

T_1 $T_{sum\ of\ two\ nos}$ = Constant time or $O(1)$

T_2 $T_{sum\ of\ nos\ in\ array}$ = $O(n)$

T_3 $T_{sum\ of\ nos\ in\ matrix}$ = $O(n^2)$ so

This big O is called Asymptotic notation
we also have some other Asymptotic notions like Θ, Ω etc.

Suppose we have two algorithms

Algo 1: $T(n) = 5n^2 + 7$
Algo 2 $T(n) = 17n^2 + 6n + 8$

} Based on some Model Machine

Q) So why we only take n and not Constants in Asymptotic notations.

Ans) Generally we see for larger values of n say $n = \infty$

So that makes other constants and lesser power of n insignificant as compared to larger powers

eg: ① $17 \times n^2 + 6n + 8$ and ② $5n^2 + 7$

If $n = \infty$ this is very huge as compared to so we just consider n^2

The both equation will have some behaviour for bigger values of n.

We define the nature or behaviour of rate of growth of time in terms of some Asymptotic notations.

1) O - "big-oh" notation :=>

Suppose we have a non-negative function $g(n)$ which takes a non-negative argument say n.

then

$O(g(n))$ is defined as the set of all the functions $f(n)$

or $\{ f(n) :$ there exist constants c and n_0 such that $f(n) \leq c \times g(n)$, for $n \geq n_0 \}$

eg:

$f(n) = 5n^2 + 2n + 1$

$g(n) = n^2$

these two will always be lesser than n^2 so

$c = 5+2+1 = 8$

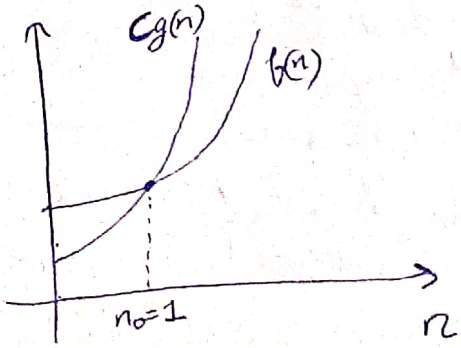
$n_0 = 1$

$8n^2 \geq f(n)$ ✓

which is $c \times g(n)$

then $O(n^2)$

$8n^3, 8 \times 2^n, \ldots$ but nearest/tight is $8n^2$ so



So this gives us the upper bound of rate of growth of a function i.e time can't grow faster than this

Note :-> We can choose c and n_0 but the conditions will still be valid

2) Ω - Omega notation:→

Suppose if we have +ve function $g(n)$ which takes positive argument n then

$\Omega(g(n)) = \{ f(n) :$ there exist constants c and n_0 such that

$cg(n) \leq f(n)$ for $n \geq n_0$ }

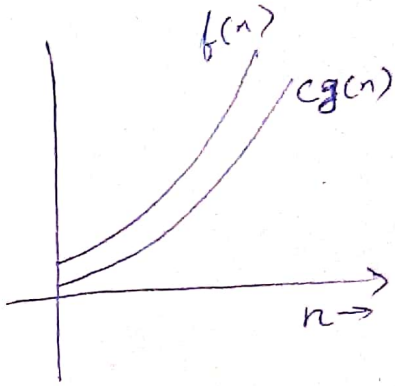
eg:

$f(n) = 5n^2 + 2n + 1$

$g(n) = n^2$

$c = 5$ $\quad 5n^2 \leq f(n)$ $\quad n \geq 0$

$n_0 = 0$



we may say $f(n)$ is $\Omega(n^2)$

Omega notations gives us lower bound of a function i.e. atleast the time will grow with this rate

3) Θ — Theta notation.

It says if we have a +ve function $g(n)$ that takes +ve argument (n) then

$\Theta(g(n)) = \{ f(n) :$ there exist constants c_1 and c_2 and n_0 such that

$c_1 g(n) \leq f(n) \leq c_2 g(n)$ for $n \geq n_0$ }

eg: $f(n) = 5n^2 + 2n + 1$

$g(n) = n^2$

$\Theta(n^2)$

$c_1 = 5$

$c_2 = 8$

$n_0 = 1$

If we choose them then conditions of Θ holds

Θ theta notation gives the best idea about the rate of growth of time because, it gives us a tight bound unlike Big Oh and Omega which gives us upper bound and lower bound

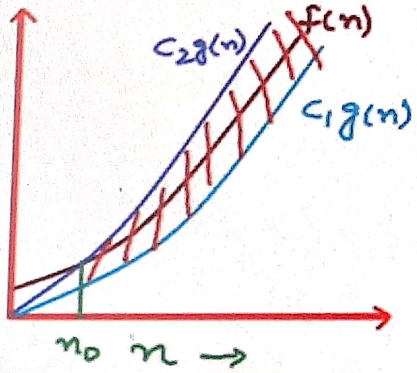
Time Complexity analysis - asymptotic notations

Θ - Theta notation - Tight bound

$\Theta(g(n)) = \{ f(n)$: there exist constants c_1, c_2 and n_0,

$c_1 g(n) \leq f(n) \leq c_2 g(n)$,

for $n \geq n_0$ $\}$



$f(n) = 5n^2 + 2n + 1 = \Theta(n^2)$

$g(n) = n^2$

$c_1 = 5, c_2 = 8, n_0 = 1$

Time Complexity Analyses - some general rules.

We analyze time complexity for :-

a) Very large input size

b) Worst case scenario.

Suppose this is the time function :- $T(n) = n^3 + 3n^2 + 4n + 2$

↑ for very large value of n
i.e $(n \xrightarrow{\text{tend to}} \infty)$
this portion will become insignificant and

$T(n) \approx n^3$
or
$C \times n^3$ or $O(n^3)$

So
Rule a) Drop all the lower order terms
b) Drop constant multiplier

eg: $T(n) = 17n^4 + 3n^3 + 4n + 8$

Rule a) $17n^4$
Rule b) n^4

So $O(n^4)$

eg: $16n + \log n$

Rule a) $16n$
Rule b) n

So $O(n)$ or $\Theta(n)$

Rule :- We can calculate the Running time of an algorithm by Suming the running time of all fragments

a) All the simple declaration run in constant time $O(1)$

eg: int a; } $O(1)$ ↳ +, −, =, ==
a = 5
a + i

b) If we have a loop then time Complexity will be the number of times the loop runs multiplied by the running time of statements inside

eg: for(i=0; i<n, i++) — n+1
{
//simple statement — n×1
}
So $O(n)$

c) If nested loops are there then also we multiply considering one at a time eg: for( )
{
for( )
{
statement → n × n × 1 = n^2
}
}
So $O(n^2)$

Suppose we have a program like this

```
int a;
a = 5;
a++;
```
} O(1)

```
for (i=0; i<n; i++)
{
  // simple statements
}
```
} O(n)

```
for(i=0; i<n; i++)
{
  for(j=0, j<n, j++)
  {
    // simple statements
  }
}
```
} O(n²)

$T(n) = O(1) + O(n) + O(n^2)$

$= O(n^2)$ because for very large values of n the bracketed part becomes insignificant.

Rule :→ Do the most will always be taken

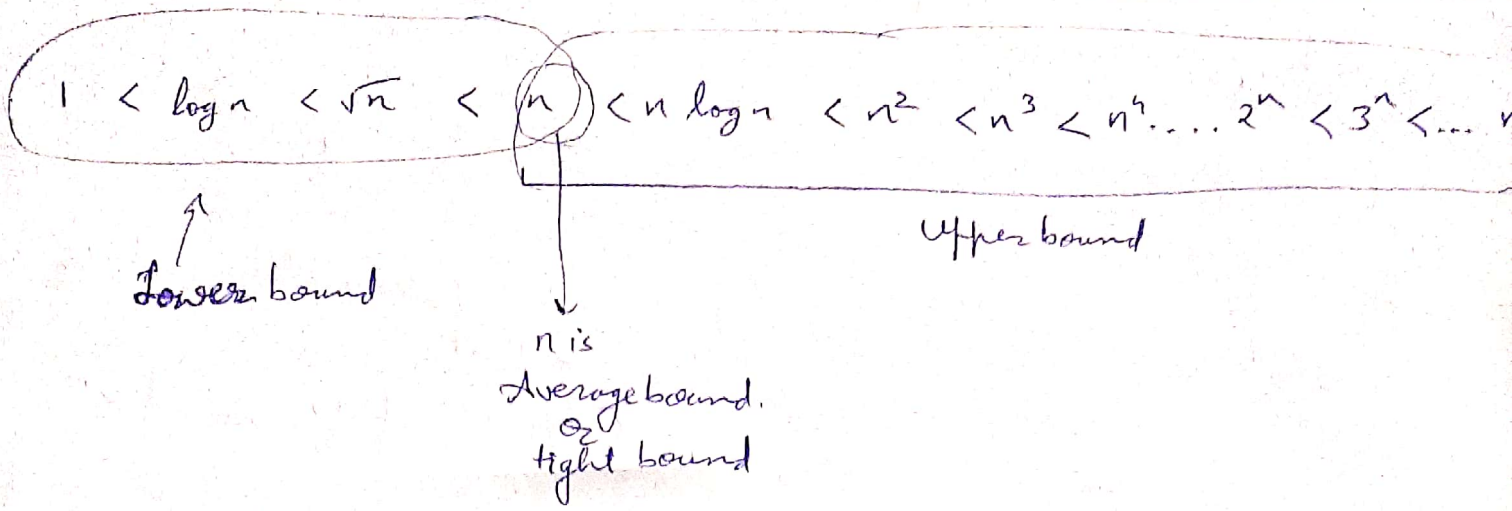
Suppose we have a program

```
if (some condition)
{
  for (i=0; i<n; i++)
  {
    // some statement
  }
}
else
{
  for (i=0; i<n; i++)
  {
    for (j=0, j<n, j++)
    {
      // some statement
    }
  }
}
```

for (i=0; i<n; i++) { // some statement } } O(n) ← Time Complexity will be this if the condition is true

for (i=0; i<n; i++) { for (j=0, j<n, j++) { // some statement } } } O(n²) ← Time complexity will be this else

Note :→ We always consider time complexity for worst case as discussed earlier



1) O - big - Oh

The function $f(n) = O(g(n))$ if and only if there exist a +ve constant C and n_0 such that

$$f(n) \leq C * g(n) \quad \forall \quad n \geq n_0$$

eg: $f(n) = 2n+3$ $\qquad n_0 = 1$

$$2n+3 \leq C * g(n)$$

* Note:→ Always consider the nearest value

1	*	n	✗
2	*	n	✗
3	*	n	✗
4	*	n	✗
5	*	n	✓ condition satisfies after this as we continue
6	*	n	✓

$5n^2$, $5n^2$, $5n^3$, $5n^{10}$
$5 2^n$, $5 3^n$ ....

So time complexity is: $f(n) = O(n)$

2) Ω - Omega notation :→

The function $f(n) = \Omega(g(n))$ if and only if there exist a +ve constant C and n_0 such that

$$f(n) \geq C * g(n) \quad \text{where} \quad n \geq n_0 \quad \text{and} \quad n_0 = 1$$

eg: $f(n) = 2n+3$

$$2n+3 \geq C * g(n)$$

1	× n	✓
1	× log n	✓
1	× √n	✓

so least or nearest is $\log n$ so $\Omega(\log n)$

3) Θ - Theta notation :→

$$C_1 * g(n) \leq f(n) \leq C_2 * g(n)$$

$\Theta(n)$ so not always depend on upper and lower bound eg: n^2 n^3 2^n etc

general rules

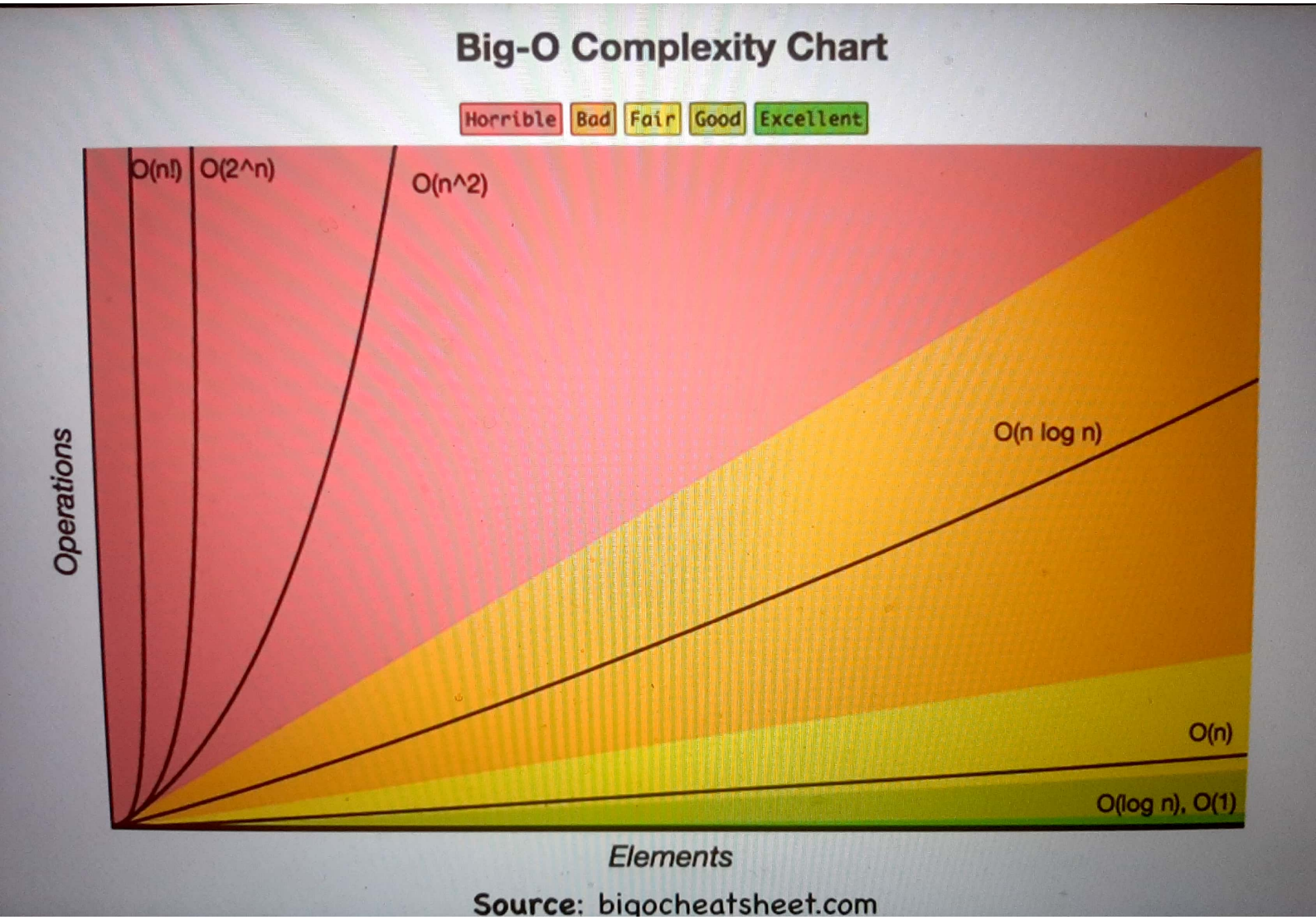
1. ignore constants

   $5n \rightarrow O(n)$

2. certain terms "dominate" others

   $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(2^n) < O(n!)$

   i.e., ignore low-order terms


Big-O Complexity Chart
Horrible
Bad
Fair
Good
Excellent
O(n!)
O(2^n)
O(n^2)
O(n log n)
O(n)
O(log n), O(1)
Operations
Elements
Source: bigocheatsheet.com

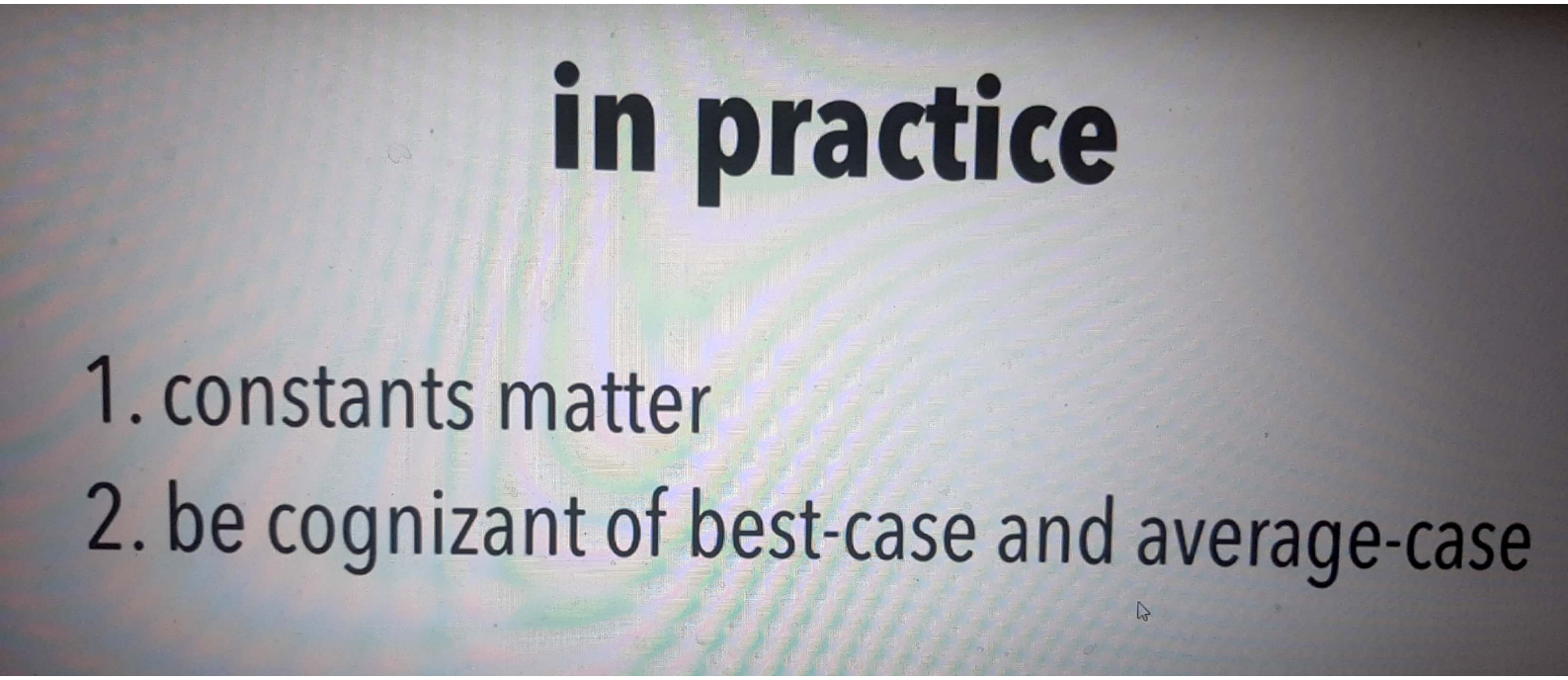
```
x = 5 + (15 * 20);
y = 15 - 2;
print x + y;
```

total time = O(1) + O(1) + O(1) = O(1)

3 * O(1)

```
y = 5 + (15 * 20);          O(1)
for x in range (0, n):   }  O(N)
    print x;
```

total time = O(1) + O(N) = O(N)


in practice
1. constants matter
2. be cognizant of best-case and average-case

#

Constant time :

```
x = 5 + (15 * 20);
```

1+1+1=3 is a const. So generally we can take 1 only

So O(1) = O(3)

This basic computer step doesn't depend on anything in any way and we say it gets computed in constant time.

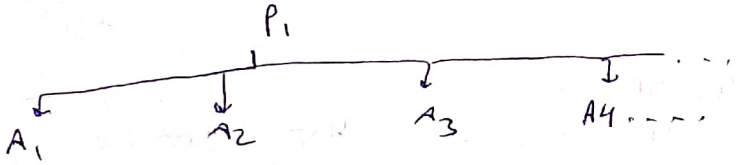
For Compt science we have some problems

↓ In order to solve them we write

Codes/programs in C, C++, python

but before directly writing the program we must write our ~~algo~~ logic in an informal language i.e making a blueprint which is called as algorithm.

Like there is some Problem then it can have many solutions or algorithm.



So how to choose the best Sol^n. It is chosen based on time and Memory.

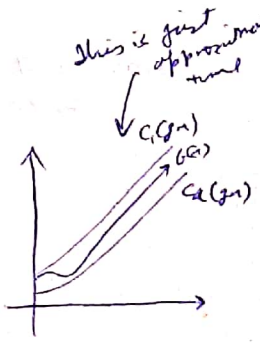
1) Time — Best Algorithm is one which takes less time and memory.
2) Memory

so Design and Analyse

Practical significance of Asymptotic notations

1) O — worst case
2) Ω — best case
3) Θ — Average

✱ We are always interested in the worst case of any algorithm.



Eg :

5	6	7	8	9	12	14	15

Let the size of this array be n.

We will do linear search

So in best case if x=5 we can find it one 1st location only. So Ω(1)

and worst case is O(n)

When best case and worst case are same then we use the Average case

There are three types of Algorithms as we will either use loop or recursion to solve problems

1) Iterative Algorithms ⟶ eg: for(i=1), while, do-while etc.
2) Recursive Algorithm ⟶ eg: A(n) { if ( ) i.e. If a function calling itself
A(n/2)
}

Note :- Any Iterative Algorithm can be written in Recursive Alg. and vice-versa
So both are equivalent in power

Analysis wise they both are different. i.e Iterative and Recursive.

Note :- If Algorithm doesnt contains either iteration or recursion then it means that there is no dependency on input size which means that running time for such algorithm will be a constant value.

eg:
```
A()
{
  int i;
  for(i=1 to n)
      Pf("Todeb")  → n
  }
```
So O(n)

eg:
```
A()
{ i=1; s=1;
  while (s<=n)
  { i++;
    s = s+i;
    Pf("Ravi");
  }
}
```
So O(√n)

s	i
1	1
3	2
6	3
10	4
15	5
21	6
⋮	⋮
k	n
(n+k)	

so increment in i is linear but s depends on i.

So $\frac{k(k+1)}{2}$

$\frac{k(k+1)}{2} > n$; $k = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

$k^2 + k > 2n$

k^2

$k^2 > n$

$k > \sqrt{n}$

$O(\sqrt{n})$

At k+k=2n
k
At s > n, it will stop

```
A()
{ i=1
  for(i=1; i^2<=n; i++)
     Pf("Todeb")
}
```

i
1^2
2^2
3
4
⋮
k

$k^2 \geq n$

$k = \sqrt{n}$

$O(\sqrt{n})$

or

$\Theta(\sqrt{n})$

#

```
A()
{
  int i,j,k,n;
  for(i=1; i<=n; i++)          // it is running from 1 to n
  {
    for(j=1; j<=i; j++)        // it depends on i
    {
      for(k=1; k<=100; k++)    // it is default
      {
        Pf("ravi")
      }
    }
  }
```

i	j	k
1 time	1 time	100 times
2	2 times	200 times
3	3 times	300 times
⋮	⋮	⋮
n	n times	n × 100 times

1×100
2×100
⋮
+

Note :- After full execution of loop and at termination ~~of~~ the value of j again becomes 1 or equal to what it was before ~~?~~

bracket

So total no of times this Pf("ravi") will get executed is sum of all k

$100 + 200 + 300 + \cdots n \times 100$

$\Rightarrow 1\times100 + 2\times100 + 3\times100 + \cdots n\times100$

$\Rightarrow 100(1+2+3+\cdots n)$

$\Rightarrow 100 \times \frac{n(n+1)}{2}$

$\Rightarrow 50(n^2+n)$

So $O(n^2)$

#

```
A()
{ int i,j,k,n;
  for(i=1; i<=n; i++)          // so it will run from 1 to n
  {                            // it depends on i
    for(j=1; j<=i^2; j++)      // it will run from 1 to n/2
    {
      for(k=1; k<=n/2; k++)
      {
        Pf("Todeeb");
      }
    }
  }
```

i	j	k
1	1 time	$n/2 * 1$
2	4 times	$n/2 * 4$
3	9 times	$n/2 * 9$
⋮	⋮	⋮
n	n^2 times	$n/2 * n^2$

+

k

$n/2 * 1 + n/2 * 4 + n/2 * 9 + \cdots n/2 * n^2$

$n/2 (1+4+9+16+\cdots n^2)$

$n/2 \times \frac{n(n+1)(2n+1)}{6}$

$O(n^3)$

A()

```
{ for(i=1; i<n; i=i*2)
    Pf ("rahul");
}
```

$O(\log_2 n)$

i
1 or 2^0
2 or 2^1
4 or 2^2
8 or 2^3
⋮
2^k

$2^k \geq n$

$k \log_2 2 \geq \log_2 n$

$k \geq \log n$

Note :-

So always remember whenever i is running from 1 to n and incrementation is in terms of 2 i.e $i = i*2$ in for loop

then time complexity will be $O(\log_2 n)$

if $i = i*3$

then time complexity will be $O(\log_3 n)$

and so on...

A()

```
int i, j, k;
for (i=n/2; i<=n; i++)
  { for (j=1; j<=n/2; j++)
      { for (k=1; k<=n; k=k*2)
          Pf ("ravi");
      }
  }
```

i — $n/2$ times; j — $n/2$ times; k — $\log_2 n$

So all loops are independent.

So no need to unroll the loops as we did in earlier questions. So

$n/2 \times n/2 \times \log_2 n$

$\Rightarrow \frac{n^2}{2} \log_2 n$

$O(n^2 \log_2 n)$

$n = 6$ so $n/2 = 3$

i	j	k
3 ✓	1 ✓	1 ✓
4 ✓	2 ✓	2 ✓
5 ✓	3 ✓	4 ✓
6	4 ✗	8 ✗
✗		

eg: 1 → 3 boxes, 2 → 3 boxes, 3 → 3 boxes ∴ 3×3 = 9 boxes

```
A()
{ int i,j,k;
  for (i=n/2; i<=n; i++)
  { for (j=1; j<=n; j=2*j)
    { for (k=1; k<=n; k=k*2)
        Pf("Todeb");
    }
  }
}
```

i	j	k
n/2 times	$\log_2 n$	$\log_2 n$

all are independent

so

$n/2 \times \log_2 n \times \log_2 n$

$O\left(\frac{1}{2}(\log_2 n)^2\right)$

Assuming $n \geq 2$

```
A()
{ while (n>1)
  { n=n/2
  }
}
```

n
2^1

Case 2) when n is increasing with powers of 2

2^2
twice

2^3
thrice

So when | 2^k | k times |

Say loop is stopping here, so

$2^k = n$

$k = \log_2 n$

Case 2) when n is not a power of 2

Say $n = 20$

then

20 ✓
10 ✓
5 ✓
2 ✓
1 ... ×

4 times so $\log_2 20$

So $O(\lfloor \log_2 20 \rfloor)$

Ask Sir ↓

```
# A()
{ for (i=1; i<=n; i++)
  { for (j=1; j<=n; j=j+i)
    { Pf("ravi"); }
  }
}
```

$O(n \log n)$

$i = k$

$j = 1$ to n

n/k

i	j
1	1
2	2
3	3
4	4
...	...
n+1	

so for each value of i it will run for n times then

n values of i it will run for

$O(n^2)$

$a_n = 1+(n-1)2$

$a_n = 2n-1$

$n = 1+(n-1)3$

$\frac{n-1}{3} + 1 = n$

$\frac{n+2}{3}$

$\frac{n^2(n-1)}{2}$

$O(n^2)$

$1+n^2 = 1+(n-1)d$

$\frac{n^2+n}{2}$

i = 1	i = 2	$\frac{n-3}{2}$	i = 3
j = 1 to n	j = 1, 3, 5, 7, ... 2n−1		j = 1, 4, 7, 10, ...
n times			

i = k

1, 1+k, 1+2k, 1+3k, ...

A()

```
{
int n = 2^k;
for(i=1; i<=n, i++)
{
    j=2
    while (j<=n)
    {
        j = j^2
        Pf("ravi");
    }
}
}
```

i	j
1	2^1
2	2^2
3	2^4
4	2^8
⋮	⋮
n	

$2^8 \times 8$

$2^5 6$

2 | 256
2 | 128
2 | 64
2 | 32
2 | 16
2 | 8
2 | 4
2 | 2
1

As it is depending on n so

$n = 2^k$ lets take

- $k=1$

  $n = 2^1 = 4$

  so for this value j will run 2 ✓ 4 ✓ two times.

  and because of for loop, it will get repeated n times so $n \times 2$

- if $k=2$

  $n = 2^2 = 16$

  j will run 2, 4, 16 → three times

  and $n \times 3$

- if $k=3$

  $n = 2^3 = 256$ so j will run

  2, 4, 16, 256 → 4 times

  $4 \times n$

So for $k=k$

$(k+1) \times n$

So $(k+1) \times n$ times this algorithm will run.

in terms of only n.

as $n = 2^k$ $\quad \log_2 n = k \log_2 2$

so $k = \log_2 n$ $\quad \boxed{\log_2(\log_2 n) = k}$

so

$\log_2(\log_2 n) + 1) \times n$

hence

$(\log_2(\log_2 n) + 1)\, n$

$n \log_2(\log_2 n) + n$

$O(n \log_2(\log_2 n))$ ✓

2^k

$2k \log_2 2$

$2k = \log_2 n$

Masters Theorem.

$T(n) = a(n/b) + \Theta(n^k \log^p n)$ ← If recurrence statement is like this and follows

where $a \geq 1, b > 1, k \geq 1$, p = Real Number ←

Case 1: If $a > b^k$, then $T(n) = \Theta(n^{\log_b a})$

Case 2: If $a = b^k$, then (i) If $p > -1$ then $T(n) = \Theta(n^{\log_b a} \cdot \log^{p+1} n)$
(ii) If $p = -1$ then $T(n) = \Theta(n^{\log_b a} \log\log n)$
(iii) If $p < -1$ then $T(n) = \Theta(n^{\log_b a})$

Case 3: if $a < b^k$ then
(i) if $p \geq 0$ then $T(n) = \Theta(n^k \log^p n)$
(ii) if $p < 0$ then $T(n) = \Theta(n^k)$

Limitations → Only Recurrence Statements following the condition can be solved using Masters theorem.

eg: $T(n) = 4T(n/2) + n$

$a = 4$
$b = 2$
$k = 1$
$p = 0$

now $a > b^k$

$4 > 2^1$ ✓

So Case 1 is satisfied then $T(n) = \Theta(n^{\log_2 4})$

$T(n) = \Theta(n^2)$

Q) $T(n) = 2T(n/2) + n\log n$

Ans.

$a = 2$
$b = 2$
$k = 1$
$p = 1$

Case 2 satisfies and $p > -1$ then $T(n) = \Theta(n^{\log_2 2} \cdot \log^2 n)$

$= \Theta(n \log^2 n)$

Q) $T(n) = 2T(n/2) + n/\log n$

$a = 2$
$b = 2$
$p = -1$
$k = 1$

Case 2: $p = -1$ so $T(n) = \Theta(n^{\log_2 2} \log\log n)$

$= \Theta(n \log(\log n))$

a) $T(n) = 2T(n/2) + n^2$

$a = 2$
$b = 2$
$p = 0$
$k = 2$

case 3
$p \geq 0$

$\Theta(n^k \log^p n)$

$(n^2 \log^0 n)$

$\Theta(n^2)$

$T(n) = T(n/2) + n$

$a = 0$
$p = 0$
$k = 1$
$b = 2$

case 3
$a < b^k$

$0 < 2^1$

$p \geq 0$

$n \log^0$

b) $T(n) = 3T(n/2) + \frac{n^2}{\log n}$

$a = 3$
$b = 2$
$p = -1$
$k = 2$

$3 < 2^2$

so
$a < b^k$

case 3
$p < 0$

$\Theta(n^2)$

$T(n) = 16^n T(n/3) + 1/n$

$a = 16^n$
$b = 3$
$k = -1$
$p = 0$

$16^n \quad 3^{-1}$

$16^n > 1/3$

$\Theta(\log_3 16^n)$

We can't solve it using Masters theorem as it is not in that form.

Recursive Algorithm :→

```
# A(n)
{ if( )
    return( A(n/2) + A(n/2) )
}
```

The way we found out time Complexity for Iterative Algorithms is very much different for Recursive Algorithm i.e. the reason is there is nothing to count here.

Let us Assume, in order to solve A(n) time taken is T(n), for if condition the time taken will be constant, and now the time taken for calling A(n/2) is 2T(n/2) as time taken for A(n) is T(n).

So

Total time taken for the function is

$T(n) = 2T(n/2) + 6$ → This is the recursive function now how to solve them.

1st Way) Back Substitution (It can be applied on all Recursive Algo. but it is slow)

eg:

```
# A(n)
{ if (n>1)
    return (A(n-1))
}
```

So $T(n) = T(n-1) + 1 \cdots ①$

now
$T(n-1) = T(n-2) + 1 \cdots ②$
$T(n-2) = T(n-3) + 1 \cdots ③$

now substitute eqn ② in ①

So n is starting and gradually decreasing

$T(n) = T(n-2) + 2$

now substitute eqn ③ in the above eqn so

$T(n) = T(n-3) + 3$

⋮ k times

$T(n) = T(n-k) + k$

So $n-k = 1$

$k = n-1$

So for this value of k it will become 1

Now where to stop, so see eqn 1 is valid only when $n > 1$

i.e. $T(n) = T(n-1) + 1 \quad n > 1$

So at some point it will reach 1 and we will stop.

$T(n) = T(n-1) + 1$

$= 1$